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C

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C 1 - TOPOLOGIE GÉNÉRALE ET ALGÉBRIQUE

K-THEORY, SIMPLICIAL COMPLEXES AND CATEGORIES

by D. W. ANDERSON

Recently, there has been quite a bit of activity centered around the problem of directly constructing spaces which are infinite loop spaces. Boardman and Vogt [2] constructed the classifying space BF for sphere fibrations, as well as other classifying spaces, in such a way that they were naturally infinite spaces. More recently, Barratt [1] and Quillen independently have shown how to construct $\Omega^\infty \Sigma^\infty X$, for any simplicial set X. Finally, G. Segal [3] has shown how to fit the Barratt-Quillen construction into the framework of a category-theoretic construction which is motivated by standard K-theoretic constructions. Segal obtains also the Boardman-Vogt results in a particularly simple fashion, and obtains some new results.

Our approach to the problem can be described as follows. We construct functors of the form $\Phi : \mathfrak{S}.\mathfrak{S}. \to \mathfrak{S}.\mathfrak{S}.$ where $\mathfrak{S}.\mathfrak{S}.$ is the category of simplicial sets (c.s.s. sets). These functors have the property that $X \mapsto \pi\ \Phi(X)$ is a homology theory. Furthermore, the simplicial sets $\Phi(X)$ are automatically infinite loop spaces, as $\Phi(X) = \Omega\Phi(\Sigma X)$. This is a major advantage over taking functors which define cohomology theories, where the relationship between Ω and Σ is reversed.

The functors Φ which we construct will give rise to most of the well known homology theories, except for bordism theory. As mentioned before, stable homotopy theory is of this form. Also, the homology theories associated to connective K-theory for real, complex, and PL bundles, as well as for sphere fibrations is of this form. Ordinary homology theory arises from a particularly degnerate type of functor. Other types of homology theories can also be constructed using functors arising from algebraic geometry. These may prove to be quite interesting, especially as some of them are closely related to the theories which arise out of Quillen's work on the Adams conjecture on the order of the image of the J-homomorphism.

Our method for producing the functors Φ breaks into steps as follows. First, we begin with a suitable simplicial category $\mathcal{C}$, which has a monoid structure of a suitable sort. Next. we define, for a simplicial set X, a new simplicial category $\mathcal{C}(X)$, which also has a suitable monoid structure. Next, we apply the "morphism complex" functor M and obtain a simplicial monoid $M(\mathcal{C}(X))$. Finally, we let $\Phi(X)$ be the group completion of $M(\mathcal{C}(X))$.

To obtain various homology theories, we choose the categories $\mathcal{C}$ as follows. For stable homotopy, $\mathcal{C}$ is the category with one object $[n]$ for each positive integer n, with $\mathrm{Hom}([n], [k]) = \emptyset$ if $n \neq k$, and $\mathrm{Hom}([n], [n])$ the symmetric group Σ_n on n letters.

If we replace Σ_n by the signed permutations (the wreath product $\Sigma_n \int (Z/2)$), we obtain the homology theory "stable homotopy with RP^∞ coefficients", where RP^∞ is the infinite dimensional real projective space.

If we replace Σ_n by the unimodular group of $n \times n$ matrices, we obtain a homology theory, which might be called the Whitehead homology theory for Z, as the O-dimensional group of a point is the group $K_0(Z[Z])$.

If we replace Σ_n by the singular complex of the general linear group $GL(n, \mathfrak{A})$ for a Banach algebra $\mathfrak{A}$ we obtain the homology $k_{\mathfrak{A}*}(X)$, the connective K-theory whose Spanier dual cohomology theory is the connective K-theory obtained from $\mathfrak{A}$-bundles with finitely generated projective fibers. Other forms of K-theory are defined analogously.

Finally, if we replace Σ_n by the group with one element, we obtain ordinary integral homology.

1. Simplicial Categories.

The category Δ of ordered simplicies has as its objects the sets $\underline{n} = \{0, 1, \ldots, n\}$ for $n \geqslant 0$, and as its morphisms the order preserving set maps. If $\mathcal{C}$ is any category, a simplicial $\mathcal{C}$-object is a contravariant functor $\Delta \to \mathcal{C}$. For example, a simplicial category is a contravariant functor $\Delta \to \mathcal{C}\mathfrak{A}\mathfrak{J} =$ the category of small categories and functors.

As an elementary example of a simplicial category, every simplicial monoid and every simplicial group may be considered as a simplicial category.

Notice that if X is a simplicial set and $\mathcal{C}$ is a category, one can easily define a simplicial category $X \times \mathcal{C}$, by letting the set of morphisms be given by

$$\mathrm{Mor}((X \times \mathcal{C})(\underline{n})) = X(\underline{n}) \times \mathcal{C},$$

and letting $(x_1, \alpha_1)(x_2, \alpha_2)$ be defined and equal to $(x_1, \alpha_1, \alpha_2)$ if and only if $x_1 = x_2$, and $\alpha_1 \alpha_2$ is defined. Similarly one can define the product $\mathcal{C}_1 \times \mathcal{C}_2$ of two simplicial categories $\mathcal{C}_1$ and $\mathcal{C}_2$. The product above is a special case of this product if we consider X to be a simplicial category in which all morphisms are identity maps, and $\mathcal{C}$ to be the same in each degree.

The n-simplex $\Delta(n)$ is defined by $\Delta(n)(\underline{k}) = \mathrm{Hom}_\Delta(\underline{k}, \underline{n})$.

Remark. – Notice that if M is a simplicial monoid, $\Delta(1) \times M$ is a simplicial category. A map of simplicial sets $\Delta(1) \times M \to N$ into a simplicial monoid N is what is called a loop homotopy if and only if it is a simplicial functor. Most ideas involving simplicial monoids can be organized to fit into the framework of simplicial categories in a reasonable way.

The $\overline{W}$ construction for simplicial monoids can be extended to simplicial categories. There are two forms of the $\overline{W}$ construction, one homogeneous and one inhomogeneous. (The homogeneous form is a fibering over the inhomogeneous form with contractible fiber for a simplicial group). In most accounts, $\overline{W}$ is used for the inhomogeneous construction, so we shall use M to denote the homogeneous form of this construction.

An elegant description of $M(\mathcal{C})$ for a simplicial category $\mathcal{C}$ can be given as follows. Regard n as the category with objects $0, . 1, \ldots, n$, and a morphism $i \to j$ if and only if $i \leqslant j$. Then Δ is the category of functors between the $\underline{n}$'s. Define a bisimplicial set (a $\Delta \times \Delta$-set) $M(\mathcal{C})$ by $M(\mathcal{C})\ (\underline{i}, \underline{j}) = \text{S.F.}\ [\Delta(i) \times \underline{j}, \mathcal{C}]$, where S.F. is the set of simplicial functors.

From the bisimplicial set $M(\mathcal{C})$, we can extract a simplicial set, which we also write as $M(\mathcal{C})$, by taking the diagonal $\Delta \to \Delta \times \Delta$.

Remark. – There are two obvious ways to obtain a simplicial set from a bisimplicial set (as well as some less obvious ways). One is to take the diagonal as we have done here. The second is condensation, where one takes the disjoint union of simplicial set $\Delta(i) \times \Delta(j)$, one for each bisimplex of bidegree (i, j), and make identifications by means of the horizontal and vertical face and degeneracy operators. It is an elementary, though tedious, matter to verify that the two resulting simplicial sets are the same.

Notice that $M(\mathcal{C})\ (-, 0)$ is the set of objects of $\mathcal{C}$, and $M(\mathcal{C})\ (-, 1)$ is the set of morphisms of $\mathcal{C}$. If $n \geqslant 1$, $M(\mathcal{C})\ (-, n)$ is the set of strings of morphisms $(\alpha_1, \ldots, \alpha_n)$ so that each $\alpha_i \alpha_{i+1}$ is defined. Notice that if $\alpha_i \alpha_{i+1}$ is defined, both α_i and α_{i+1} have the same degree.

If $\mathcal{C}$ and $\mathcal{D}$ are two simplicial categories, and $\Phi_0, \Phi_1 : \mathcal{C} \to \mathcal{D}$ are two simplicial functors, a simplicial natural transformation from Φ_0 to Φ_1 is a simplicial functor $\Phi : \mathcal{C} \times \underline{1} \to \mathcal{D}$, such that $(1 \times d_i)(\Phi) = \Phi_i$ for $i = 0, 1$. Since M carries products of simplicial categories into products of simplicial sets, and since $M(\underline{1}) = \Delta(1)$, $M(\Phi)$ defines a homotopy from $M(\Phi_0)$ to $M(\Phi_1)$. Not all such homotopies are of the form $M(\Phi)$ – for example, every homotopy of functors gives rise to a homotopy on M. Homotopies on the M's give the definition for natural transformation of simplicial functors, which generalizes the notion of simplicial natural transformation. We shall not study this more general concept here.

2. Multiplications on Categories.

If $\mathcal{C}$ is a category, a multiplication on $\mathcal{C}$ is a functor $\mu : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We shall only consider associative multiplications unless we specifically say otherwise. If $\mathcal{C}$ is a simplicial category, we assume that μ is a simplicial functor.

Notice that $M(\mu) : M(\mathcal{C}) \times M(\mathcal{C}) \to M(\mathcal{C})$ defines the structure of an associative simplicial monoid on $M(\mathcal{C})$.

Let $\mu_2 = \mu$, $\mu_n : \mathcal{C}^n \to \mathcal{C}$ for $n > 2$ be defined by $\mu_n = \mu(\mu_{n-1} \times 1)$. We will say that μ is n-isomorphism commutative if μ is $(n-1)$ isomorphism commutative and if there is a function θ_n from the symmetric group Σ_n on n letters into the natural transformations of μ_n to itself, satisfying the following properties

$$\theta_n(\pi_1 \pi_2) = \theta_n(\pi_1)\, \theta_n(\pi_2),\ \theta_n(1) = \text{identity} \tag{2.1}$$

if $\pi_1 \in \Sigma_n$, $\pi_2 \in \Sigma_k$, and $\pi_1 \times \pi_2$ (2.2)

is their product in Σ_{n+k}, $\theta_{n+k}(\pi_1 \times \pi_2) = \mu(\theta_n(\pi_1), \theta_k(\pi_2))$. We say that μ is isomorphism commutative if μ in n-isomorphism commutative for all n.

The most obvious examples of isomorphism commutative structures are provided by categories of sets with μ being either disjoint union of Cartesian product (defined in such a way as to be strictly associative). Then the permutation groups act either by interchanging the order of subsets (in the case of disjoint union) or interchange the coordinates (in the case of Cartesian product). More generally, if $\mathcal{C}$ is a category with either direct sums or direct products defined (in such a way as to be associative), the direct sum (resp. the direct product) are isomorphism commutative.

Finally, suppose $\mathcal{C}$ is a category with an isomorphism commutative product, $\mathcal{C}'$ is a category with multiplication, and $\Gamma : \mathcal{C}' \to \mathcal{C}$ is a functor of categories with multiplication. Then if Γ is faithful, and if every $\theta(\pi)$ lies in the image of Γ, $\mathcal{C}'$ inherits an isomorphism commutative structure.

As an example of this last phenomenon, let $\mathcal{J}$ be the category whose objects are the spheres S^n for $n \geqslant 1$, and whose morphisms are the basepoint homotopy equivalences $S^n \to S^n$. Then if we consider S^n to be the one point compactification of R^n, there is a faithful functor to sets given by $S^n \mapsto$ underlying set of R^n. If we define a multiplication in $\mathcal{J}$ by $S^n * S^k = S^{n+k}$, the functor to sets preserves products, if the product on sets is Cartesian product. However, the permutations of factors certainly define homotopy equivalences, so they lie in $\mathcal{J}$. Thus $\mathcal{J}$ has an isomorphism commutative multiplication.

3. Free Categories.

If $\mathcal{C}$ is a simplicial category with a multiplication, the set of morphism Mor($\mathcal{C}$) is a simplicial monoid. We shall say that $\mathcal{C}$ is a free category if Mor($\mathcal{C}$) is a free monoid (without neutral element), and if $\mathrm{Hom}(\xi, \zeta)$ is empty unless when ξ and ζ are written as a product of indecomposables, they differ only by order. The following two results are elementary.

PROPOSITION 3.1. — If $\mathcal{C}$ is a free category, the objects of $\mathcal{C}$ form a free monoid.

PROPOSITION 3.2. — If $\mathcal{C}$ is a free category, $M.(\mathcal{C})$ is a free monoid.

The concepts "free monoid" and "free simplicial monoid" agree for simplicial monoids without neutral element (for any element). To see this, we need to know that the set of indecomposable elements of a simplicial monoid which is free as a monoid is closed under face operators. However, if $s_i(\sigma) = \sigma_1 * \sigma_2$, $\sigma = d_i s_i(\sigma) = (d_i \sigma_1) * (d_i \sigma_2)$, so σ indecomposable implies $s_i(\sigma)$ indecomposable.

If $\mathcal{C}$ is a free category, and C_1, C_2 are two objects, then the product defines an injection $\mathrm{End}(C_1) \times \mathrm{End}(C_2) \to \mathrm{End}(C_1 * C_2)$, where $\mathrm{End}(C_i) = \mathrm{Hom}(C_i, C_i)$. To see this, suppose $\gamma_i, \gamma_i' \in \mathrm{End}(C_i)$, and that $\gamma_1 * \gamma_2 = \gamma_1' * \gamma_2'$. Since Mor($\mathcal{C}$) is free, either γ_1 divides γ_1' or γ_2 divides γ_2'. However,

$$C_1 = \text{source}(\gamma_1) = \text{source}(\gamma_1')$$

does not divide itself (as there is no identity for any element), so γ_1 does not divide γ_1'. Similarly, γ_2 does not divide γ_2', so $\gamma_i = \gamma_i'$ for $i = 1,2$.

If $\mathcal{C}$ is a free monoid category with an isomorphism commutative sum, we can define, for any set X a new category $\mathcal{C}(X)$ as follows. The objects of $\mathcal{C}(X)$ are the elements of the free monoid generated by the pairs (x_i, C_i), where $x_i \in X$, $C_i \in Ob(\mathcal{C})$, C_i indecomposable. If $x \in X$, $C \in Ob(\mathcal{C})$, $C = C_1 * \ldots * C_n$, with each C indecomposable, we write (x, C) for $(x, C_1) * (x, C_2) * \ldots * (x, C_n)$.

We define morphisms in $\mathcal{C}(X)$ as follows. If $\xi = (x, C)$, End$(x, C) =$ End( ). If $\xi = (x_1, C_1) * \ldots * (x_n, C_n)$, where all of the x_i are distincts,

$$\text{End}(\xi) = \text{End}(C_1) \times \ldots \times \text{End}(C_n)$$

(to be thought of as a subset of End$(C_1 * \ldots * C_n)$). If

$$\xi = (y_1, C_1) * \ldots * (y_n, C_n),$$

let π be a permutation of n objects which puts ξ into the form

$$\pi(\xi) = \zeta = (x_1, B_1) * \ldots * (x_k, B_k),$$

where the x_i are distinct. Then End(ξ) is the subset $\theta(\pi^{-1})$ End$(\zeta)\,\theta(\pi)$ of End$(C_1 * \ldots * C_n)$.

We let Hom(ξ_1, ξ_2) be empty, unless there is a permutation which transforms ξ_1, written as a product of indecomposables, into ξ_2. If $\pi(\xi_1) = \xi_2$, we let Hom(ξ_1, ξ_2) be the "coset" $\theta(\pi)$ End(ξ_1). If

$$\pi_1(\xi_1) = \xi_2 = \pi_2(\xi_1),\ \theta(\pi_1)^{-1}\ \theta(\pi_2) = \theta(\pi_1^{-1}\ \pi_2) \in \text{End}(\xi_1).$$

The category $\mathcal{C}(X)$ has an obvious multiplication, and is easily seen to be free. If $\mathcal{C}$ is a simplicial category, $\mathcal{C}(X)$ is also a simplicial category. Ignoring the components in X, we have a faithful functor $\mathcal{C}(X) \to \mathcal{C}$ whose image contains the $\theta(\pi)'s$. Thus $\mathcal{C}(X)$ also has an isomorphism commutative product.

PROPOSITION 3.3. – If X, Y are two sets, there is an isomorphism, natural in all three variables, between $\mathcal{C}(X)(Y)$ and $\mathcal{C}(Y \times X)$.

Proof. – The indecomposable objects in the first case have the form $(y, (x, \mathcal{C}))$, and in the second, $((y, x), \mathcal{C})$. The correspondence is obvious.

Given a free category $\mathcal{C}$, we can form a new category $\mathcal{C}_0$ obtained from $\mathcal{C}$ by adjoining a neutral object 0, with End(0) consisting of the identity object only, Hom$(C, 0 = \emptyset =$ Hom$(0, C)$ if $C \neq 0$. Products are defined by $C * 0 = C = 0 * C$. Then Mor$(\mathcal{C}_0)$ is just Mor$(\mathcal{C})$ with a neutral element, the identity of 0, adjoined.

If X is a set with basepoint x_0, we define $\widetilde{\mathcal{C}}(X) = \mathcal{C}(X, x_0)$ to be $\mathcal{C}(X - \{x_0\})_0$. Notice that $\mathcal{C}_0$ and $\mathcal{C}_0(X, x_0)$ are free categories with neutral objects, in the sense that they are the disjoint union of a free category with $\underline{0}$.

If X and Y are sets with basepoint, we can form the one point union $X \vee Y$. The inclusions $X, Y \to X \vee Y$ define a functor of categories with multiplication $\widetilde{\mathcal{C}}(X \vee Y) \to \widetilde{\mathcal{C}}(X) \times \widetilde{\mathcal{C}}(Y)$.

PROPOSITION 3.4. – If $\mathcal{C}$ is a free isomorphism commutative (simplicial) category, $\widetilde{\mathcal{C}}(X \vee Y) \to \widetilde{\mathcal{C}}(X) \times \widetilde{\mathcal{C}}(Y)$ is an equivalence of categories (resp. simplicial categories). Furthermore, the inverse functor and the natural transformations are natural in all three variables $\mathcal{C}$, X, and Y.

Proof. – Define $\widetilde{\mathcal{C}}(X) \times \widetilde{\mathcal{C}}(Y) \to \widetilde{\mathcal{C}}(X \vee Y)$ by $(\xi, \zeta) \to \xi * \zeta$, where $\widetilde{\mathcal{C}}(X)$ and $\widetilde{\mathcal{C}}(Y)$ are included in $\widetilde{\mathcal{C}}(X \vee Y)$ by means of the projections. Then the composition of this with the functor above is the identity, so that $\widetilde{\mathcal{C}}(X) \times \widetilde{\mathcal{C}}(Y)$ is a retract of $\widetilde{\mathcal{C}}(X \vee Y)$.

The composition $\widetilde{\mathcal{C}}(X \vee Y) \to \widetilde{\mathcal{C}}(X \vee Y)$ is given on objects by taking any sequence of indecomposables, and rearranging them so that all the terms involving X – (basepoint) occur first, followed by all the terms involving Y – (basepoint). As internal order of the X-terms and the Y-terms is preserved, there is a well defined permutation which does this. For any object ξ, let $\pi(\xi)$ be this permutation. Then $\xi \to \theta(\pi(\xi))$ defines a natural equivalence from the identity to the composition of the two functors above. Since $\xi \mapsto \pi(\xi)$ is simplicial, $\xi \to \theta(\pi(\xi))$ is simplicial.

There is an obvious map of simplicial categories $X \times \mathcal{C}_0 \to \mathcal{C}(X)_0$ given by $(x, C) \to (x, C)$ if x is not the basepoint, $C \neq 0$, $(x, C) \to 0$ if x is the basepoint or $C = 0$.

PROPOSITION 3.5. – If $f_0, f_1 : X \to Y$ are homotopic relative to the basepoint, where X, Y are simplicial sets $\mathcal{C}(f_0)$ is homotopic to $\mathcal{C}(f_1)$ as simplicial functors from $\mathcal{C}(X)_0$, to $\mathcal{C}(Y)_0$ by a product preserving homotopy.

Proof. – Let $I = \Delta(1) \amalg \Delta(0)$. Then there is a map $F : X \wedge I \to Y$ which provides a homotopy form f_0 to f_1. Composing $\mathcal{C}(F)$ with the functor

$$I \times \mathcal{C}(X)_0 \to \mathcal{C}(X)(I)_0 = \mathcal{C}(X \wedge I)_0 ,$$

we obtain the desired homotopy.

4. The Simplicial Groups $\Phi.(\mathcal{C}, X)$.

Let $\mathcal{C}$ be a free isomorphism commutative simplicial category. Then for any set X with basepoint, $M(\widetilde{\mathcal{C}}(X))$ is a free simplicial monoid with neutral element. If X is a simplicial set, $\Phi^+(\mathcal{C}, X) = M(\widetilde{\mathcal{C}}(X))$ is a trisimplicial free monoid with neutral element. We define $\Phi(\mathcal{C}, X)$ to be the group completion of $\Phi^+(\mathcal{C}, X)$. Since $\Phi^+(\mathcal{C}, X)$ is free, this is well defined. The simplicial group $\Phi.(\mathcal{C}, X)$ is the diagonal part of $\Phi(\mathcal{C}, X)$.

Recall that there was a functor $X \times \mathcal{C}_0 \mapsto \widetilde{\mathcal{C}}(X)$. Since $M(X) = X$, where X is considered as a cetegory with only identity maps, we have an induced map $X \times M.(\mathcal{C}_0) \to M.(\widetilde{\mathcal{C}}(X))$. Since $\mathcal{C}_0 = \widetilde{\mathcal{C}}(S^0)$, where $S^0 = \Delta(0) \amalg \Delta(0)$ is the 0-sphere, this defines a map $X \times \Phi.(\mathcal{C}, S_0) \to \Phi.(\mathcal{C}, X)$. A little observation will convince the reader that this map is constant on the axes, and so defines a map $X \wedge \Phi.(\mathcal{C}, S^0) \to \Phi.(\mathcal{C}, X)$. Thus we have a map

$$\Phi.(\mathcal{C}, S^0) \to \Omega\Phi.(\mathcal{C}, S^1).$$

THEOREM 4.1. – The map $\Phi.(\mathcal{C}, S^0) \to \Omega\Phi.(\mathcal{C}, S^1)$ is a homotopy equivalence.

To prove this theorem, we need a result which follows from 3.4, and some slight additional arguments. We will discuss the proof of this proposition later.

PROPOSITION 4.2. – If X, Y are two sets with basepoint, the natural map $\Phi(\mathcal{C}, X \vee Y) \to \Phi(\mathcal{C}, X) \times \Phi(\mathcal{C}, Y)$ is a homotopy equivalence.

To return to our theorem, observe that $S^1(\underline{n})$ contains $n + 1$ objects, one of which is the basepoint. Thus $\Phi.(\mathcal{C}, S^1(\underline{n})) \simeq \Phi.(\mathcal{C}, S^0(\underline{n}))^n$, since $S^0(\underline{n})$ contains two objects.

We can consider $\Phi(\mathcal{C}, S^1)$ to be a bisimplicial complex, with the vertical complexes the $\Phi.(\mathcal{C}, S^1(\underline{n}))$. Thus the vertical homotopy groups of $\Phi(\mathcal{C}, S^1)$ are the $(\pi_*(\Phi(\mathcal{C}, S^0)))^n$. The horizontal face operators correspond to the face operators in S^1.

It is not a difficult matter now to see that the horizontal homotopy groups of the vertical homotopy groups vanish, except for those whose horizontal degree is equal to 1. By a theorem of Quillen [5], this implies that

$$\pi_i(\Phi.(\mathcal{C}, S^1)) = \pi_{i-1}(\Phi.(\mathcal{C}, S^0)).$$

By checking the constructions, one sees that the maps

$$\pi_i(\Phi.(\mathcal{C}, S^0) \to \pi_i(\Omega\Phi.(\mathcal{C}, S^1))$$

are isomorphisms, so that $\Phi.(\mathcal{C}, S^0) = \Phi.(\mathcal{C}, S^1)$.

COROLLARY 4.3. – The spaces $\Phi.(\mathcal{C}, S^0), \ldots, \Phi.(\mathcal{C}, S^n), \ldots$ form an Ω-spectrum $\mathrm{Spec}(\mathcal{C})$.

If X is any space, we have maps $X \wedge \Phi.(\mathcal{C}, S^n) \to \Phi.(\mathcal{C}, S^n \wedge X)$. Thus, we have a natural transformation of functors

$$\widetilde{H}_*(X : \mathrm{Spec}(\mathcal{C})) \to \pi_*(\Phi(\mathcal{C}, X)) \tag{4.4}$$

where the left hand side is defined as in the paper of G. Whitehead [6]. If we knew that the right hand side defined a cohomology theory, since the map is an isomorphism for $X = S^0$, it is an isomorphism for all X.

PROPOSITION 3.5. – Implies that $\pi_*(\Phi(\mathcal{C}, X))$ satisfies the homotopy axiom for a homology theory. The homotopy exact sequence for a pair provides a suitable sort of definition for relative homology groups and for the long exact homology sequence of a pair. What remains to check is the excision axiom – that $\Phi.(\mathcal{C}, -)$ carries cofibrations into quasifibrations.

There are several approaches to the excision axiom. We outline two of them here.

The first approach was suggested to me by D. Kan. The functors $\Phi(\mathcal{C}, -)$ are degreewise convergent functors from simplicial sets to bisimplicial groups, in the sense of [4], which carry one point unions into products, up to homotopy type. If we argue as in [4], we see that such functors carry cofibrations into quasifibrations, and we are finished.

A second approach is suggested by Barratt's talk at the recent meeting in Madison [1]. Notice that $\Phi.(\mathcal{C}, \Delta(0)) = \Delta(0)$. Thus, if $X(\underline{j}) = \Delta(0)(\underline{j})$ for $j < n$, $\Phi(\mathcal{C}, X)(\underline{i}, \underline{j})$ contains only one element for $j < n$, so that

$$\Phi.(\mathcal{C}, X)(\underline{j}) = \Delta(0)(\underline{j}) \quad \text{for} \quad j < n.$$

Thus $\Phi.(\mathcal{C}, S^n)$ is $(n-1)$ connected. Indeed, we see that without changing its homotopy type, we way assume that $\mathcal{C}_0(S^n)(\underline{j}) = \Delta(\underline{0})$ for $j < n$.

By suitably adapting Barratt's argument, one can prove that for an n-connected category $\mathcal{C}$, $X \wedge \Phi.(\mathcal{C}, S^0) \to \Phi.(\mathcal{C}, X)$ is a homotopy equivalence in the stable range $(< 2n)$. This shows that (4.4) is an isomorphism.

THEOREM 4.4. – $\widetilde{H}_*(X : \mathrm{Spec}(\mathcal{C})) = \pi_*(\mathcal{C}, X))$.

We now consider 4.2. We know that $\Phi^+(\mathcal{C}, X) \times \Phi^+(\mathcal{C}, Y)$ is a deformation retract of $\Phi^+(\mathcal{C}, X \vee Y)$, though not necessarily as a simplicial monoid. Because $\Phi^+(\mathcal{C}, X)$ is homotopy abelian, $\pi = \pi_0(\Phi^+(\mathcal{C}, X))$ is abelian. Thus π is an abelian monoid, and can be considered to be a semi-directed set, by $x + y \geqslant x$ all x, y, and a suitable quotient π' a directed set. The homology groups of $\Phi^+(\mathcal{C}, X)$ are indexed by π, using representatives as basepoints and using the homotopy commutativity. In fact, they are indexed by π', because right translation by elements of π makes the homology groups into a semidirected set.

By a slight modification of a theorem of J. Moore,

$$\lim \{H_*(\Phi^+(\mathcal{C}, X), p) : p \in \pi'\} = H_*(\Phi(\mathcal{C}, X), \text{identity}).$$

Thus the map in 4.2 induces an isomorphism on reduced homology. It clearly induces an isomorphism on π_0. Thus it is a homotopy equivalence.

5. Examples.

Let $\mathfrak{Z}$ be the category whose objects are the positive integers, all of whose morphisms are identity maps. Then $\mathfrak{Z}$ has an obvious product : $n * k = n + k$. This is clearly isomorphism abelian.

The morphism complex $M(\mathfrak{Z})$ is the simplicial set with the positive integers in each degree, with the face operator s_0 always an isomorphism. Thus $\Phi(\mathfrak{Z}, S^0)$ is the simplicial group $Z \times \Delta(0)$, where Z denotes the integers. Thus

$$\pi_i(\Phi(\mathfrak{Z}, S^0)) = 0 \quad \text{if} \quad i \neq 0,$$

Z if $i = 0$. Thus the theory defined by $\mathfrak{Z}$ satisfies the dimension axiom, and so is ordinary integral homology. Notice that if $X \neq S^0$, $\Phi(\mathfrak{Z}, X)$ does not equal $\widetilde{C}_*(X; Z)$, but is a fibering over it with acyclic fiber.

If $\mathfrak{A}$ is Banach algebra, let $\mathcal{V}(\mathfrak{A})$ be the category whose objects are the $\mathfrak{A}$-modules $\mathfrak{A}, \mathfrak{A} \oplus \mathfrak{A}, \ldots$, etc, and whose morphisms are the singular simplices of the group of $\mathfrak{A}$-linear automorphisms of the objects. Then, up to homotopy type, $\Phi^+(\mathcal{V}(\mathfrak{A}), S^0)$ is the disjoint union of the classifying spaces $BGL(n, \mathfrak{A})$ for $n > 0$. Thus $\Phi(\mathcal{V}(\mathfrak{A}), S^0)$ can be seen to be $Z \times BGL(\infty, \mathfrak{A})$.

If $\mathcal{R}(\mathfrak{A})$ is the category whose objects are the projection operators on the $\mathfrak{A}, \mathfrak{A} \oplus \mathfrak{A}, \ldots$ etc., $\Phi(\mathcal{R}(\mathfrak{A}), S^0) = K_0(\mathfrak{A}) \times BGL(\infty, \mathfrak{A})$. Thus, for dimensions

> 0, both $\mathcal{V}(\mathcal{C})$ and $\mathcal{R}(\mathcal{C})$ define the same theory. However, $\mathcal{R}(\mathcal{C})$ will have a periodicity theorem, in the sense that $\pi_i(\Phi(\mathcal{R}(\mathcal{C}), X)) = \pi_{i+8}(\Phi(\mathcal{R}(\mathcal{C}), X))$ for $i \geqslant \dim(X)) =$ highest degree in which non-degenerate simplices occur. We define $k_{\mathcal{C}, i}(X) = \pi_i(\Phi(\mathcal{R}(\mathcal{C}), X^+))$. This gives us our usual connective K-theory based on $\mathcal{C}$.

Similarly, one can define connective K-homology based on the groups $PL(n)$, $F(n)$, and, assuming that they have the homotopy type of CW-complexes, (so that they can be replaced by their singular complexes), Top(n).

BIBLIOGRAPHY

[1] M. G. Barratt. — A Free Group Functor for Stable Homotopy, *Proc. A.M.S.*, Conference on Topology, 1970.

[2] J. M. Boardman and R. M. Vogt. — Homotopy Everything H-spaces, *Bull. A.M.S.*, v. 74 (1968), pp. 1117-1122.

[3] G. Segal. — *Homotopy Everything H-spaces* (to appear).

[4] D. M. Kan and G. W. Whitehead. — The Reduced Join of Two Spectra, *Topology*, v. 3, sup. 2 (1965), pp. 239-261.

[5] D. G. Quillen. — Spectral Sequences of a Double Semi-Simplicial Group, *Topology*, v. 5, No. 2 (1966), pp. 155-158.

[6] G. W. Whitehead. — Generalized Homology Theories, *Transactions A.M.S.*, v. 102, No. 2 (1962), pp. 227-283

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HOMEOMORPHISMS ON INFINITE-DIMENSIONAL MANIFOLDS

by R. D ANDERSON

Dedicated to Prof. R. L. Moore *

1. Introduction.

An infinite-dimensional (*I-D*) manifold is a paracompact Hausdorff space admitting an open cover of sets homeomorphic to open subsets of a given *I-D* homogeneous space called the model. In this paper we restrict ourselves to a survey of results in the set-theoretic topology of such manifolds. In other papers N.H. Kuiper and J. Eells discuss the differential topology of *I-D* manifolds. Both subjects have seen many new and striking results since the last Congress.

It is almost true that in *I-D* spaces or manifolds, every conjecture about homeomorphisms is true and can be proved unless it is reasonably obvious that it is false. The theorems quoted below support this assertion.

We shall state a number of theorems illustrating the nature and flavor of recent research. We concentrate on the manifold aspect of the theory and not on linear space theory or the Hilbert cube per se although results in these areas have been and are vital to the development of the subject. For simplicity, we restrict ourselves to shorter forms of somewhat more general theorems and, except in Section 9, to separable metric spaces. In several of the theorems to be cited below, for example, the homeomorphisms asserted to exist can be further specified to be the result of a small isotopic motion. A reasonably complete list of references would be longer than this paper. We give only a brief list and refer the reader to good lists in recent papers of several of the authors mentioned in Section 5. Many recent results are still in preprint form. Where feasible, theorems are attributed to the authors of the essential arguments, limitation of space prevents proper designation of authors of partial results or even of some of the important lemmas. Of necessity, in any short summary of a currently active field many valuable and impressive contributions must be omitted or mentioned only briefly.

2. Agreements.

Except for Section 9, all spaces are separable metric. Let

$$l^2 = \left\{(x_i) \mid \sum_{i=1}^{\infty} x_i^2 < \infty\right\}, \quad s = \prod_{i=1}^{\infty} (-1/i, 1/i) \quad \text{and} \quad Q = \prod_{i=1}^{\infty} [-1/i, 1/i].$$

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(1) All of the American mathematicians who have made recent contributions to the theory described below, are mathematical descendents of R.L. Moore.

Then $s \subset Q \subset l^2$. Q is the Hilbert cube and l^2 is Hilbert space. Let E denote an *I-D* topological vector space (TVS) or Q and let M, M_1 and M_2 denote manifolds modeled on E.

For $X = l^2$, s, or Q, we say that $K \subset X$ has infinite deficiency in X, if, in infinitely many coordinates, K projects onto a single point. Let "$\sim$" denote either "is homotopic to" or "is of the same homotopy type" and let "$\approx$" denote "is homeomorphic to", for single spaces or for pairs of spaces. Let $H(X)$ denote the space of all homeomorphisms of X onto X, $H_I(X)$ denote the space of all elements of $H(X)$ isotopic to the identity, id. Let $H(X$ into $Y)$ and $H(X$ onto $Y)$ denote the spaces of all homeomorphisms as indicated. Let X^ω be the countable infinite product of X by itself. Let $l_f = \{(x_i) \in l^2 |$ for all but finitely many coordinates $x_i = 0\}$. Let s_f and $(E^\omega)_f$ be similarly defined. Let $\sigma = l_f$ and let Σ be the linear span of Q in l^2. Then σ and Σ are dense σ-compact linear subspaces of l^2.

We denote a countable locally finite ($-$ dimensional) simplicial complex by $clfsc$ (or $clf-dsc$, using the metric derived from barycentric coordinates). Note that every $clfsc$ is automatically $clf-dsc$. For K a complex, we let $|K|$ be a geometric realization of K.

3. Methods.

The original methods of convexity and of renorming of $TVS's$ as well as the more recent method of (local) compactification have largely been superceded. In addition to coordinate juggling and the exploitation of the existence of infinitely many coordinates, a dominant procedure in homeomorphism theory in the past several years has been the development and use of convergence procedures for sequences $(f_i \cdot \cdots \cdot f_2 . f_1)$ with $f_i \in H(X)$ and with the limit homeomorphism f in $H(X)$ or in $H(Y$ onto $X)$ for some $Y \subset X$. Such procedures frequently involve the use of homeomorphisms close to the identity, i.e. within $\epsilon > 0$ for compact spaces and limited by an arbitrarily given cover for some more general spaces. For example, convergence in a complete metric space, X, can be guaranteed with $f \in H(X)$ if f_{i+1} can be inductively required to be close enough to the identity. In many cases, f_{i+1} is a homeomorphism defined geometrically by an isotopic motion on a product of a small number of coordinate lines or intervals. The exploitation of negligibility properties and techniques (see Section 7) continues to play a vital role. The use of selected finite-dimensional manifold techniques is becoming more important.

4. Two phenomena.

Among the phenomena that play especially useful roles in *I-D* topology and that have not been previously identified in finite-dimensional topology, are Z-sets and (f-d) *cap* sets.

A set K in a space X is a Z-set (has *Property Z*) if K is closed and for each non-empty homotopically trivial open set $U \in X$, $U \setminus K$ is non-empty and homotopically trivial. It is an important lemma that a closed set K in $X = l^2$, s, or Q is a Z-set in X iff there is an element $h \in H(X)$ such that $h(K)$ has infinite-deficiency. "Boundary" sets are Z-sets as are all compact sets in l^2 and all compact subsets of $Q \setminus s$ or of s in Q.

The next property was developed concurrently and independently by Bessaga and Pelczynski and by the author. The former used a more abstract treatment involving (G, K)-skeletons where G is a group of homeomorphisms and K a collection of compacta related to G. Here we use the author's terminology but a form of the property close in spirit to the Bessaga-Pelczynski treatment. In what follows two parallel properties are defined, the "finite-dimensional" condition being used throughout or not at all. A set $A \subset X$ has the (*finite-dimensional*) *compact absorption property* (the (*f-d*) *cap*) in X if $A = \bigcup_{i=1}^{\infty} A_i$ where for each $i > 0$, A_i is a (finite-dimensional) compact Z-set in X with $A_i \subset A_{i+1}$ and for any integer $m > 0$, any open cover U of X and any (finite-dimensional) compact Z-set $K \subset X$, there exist an integer n and an $h \in H(X)$ such that $h(K) \subset A_n$, $h|(K \cap A_m) = \mathrm{id}$ and h is limited by U.

The *cap* characterizes $Q \setminus s$ in Q and Σ in l^2, whereas the *f-d cap* characterizes Q_f or s_f in Q (or s_f in s) and $\sigma = l_f$ in l^2. The (*f-d*) *cap* is the basis for the development of the theory of σ and Σ manifolds by Chapman.

5. A brief history of homeomorphism theory of *I-D* spaces.

The history falls naturally into two main periods with an interim period between.

(I) *Prior to 1966.* – The main contributors in this period were O.H. Keller, V.L. Klee, M.I. Kadec, and C.M. Bessaga and A. Pelczynski. Keller (1931) and Klee (with occasional collaborators, 1953-1965) used convexity to get significant and useful results on the topological properties of subsets of (normed) *TVS's* and of Q. Kadec and Bessaga and Pelczynski attacked the problem of Fréchet and Banach on the topological classification of *TVS's* culminating in the theorem of Kadec (1965) that all separable *I-D* Banach spaces are homeomorphic. A principal tool used was the renorming of *TVS's*.

(II) *1966-67.* – In early 1966, the author showed that $l_2 \approx s$, thus completing the proof that all separable *I-D* Fréchet spaces are homeomorphic. In other related papers, the author studied the topology of s and of Q and of their relationship using homeomorphism convergence procedures and leading to Property Z and initial theorems about it. R.Y-T. Wong studied isotopies and wild sets in s and Q and J.E. West group actions on s.

(III) *1968-date.* – In this period the emphasis has shifted to manifolds, initially to those modeled on s or l^2 and later to more general ones. Some of the techniques are outgrowths of those of the second period. A number of very able younger mathematicians have joined the earlier researchers in making valuable contributions. Among these are D.W. Henderson, R.M. Schori, T.A. Chapman, H. Torunczyk, A. Szankowski, R. Geoghegan, W. Cutler, W.K. Mason, J. McCharen, D.E. Sanderson, R.A. McCoy, D. Curtis. This period has seen many problems on manifolds solved. The open embedding theorem, 6.3 below, and other results reduce many problems on manifolds to problems on *TVS's*. A number of useful N. and S. conditions have been established.

6. Manifolds.

The following representation and characterization theorems have been proved for manifolds modeled on $E \approx l^2$. These theorems, or combinations of them, seem to be definitive with respect to a wide class of questions.

6.1 (Anderson and Schori) $M \times E \approx M$

6.1A (Anderson and Schori) $M \times Q \approx M$

6.2 (Henderson) $M \times E \approx U^{\text{open}} \subset E$

6.3 (6.1 and 6.2, Henderson) $M \approx U^{\text{open}} \subset E$

6.4 (Kuiper and Burghelea, Moulis) Given $U_1^{\text{open}}, U_2^{\text{open}} \subset E$. $U_1 \approx U_2$ iff $U_1 \sim U_2$.

6.5 (6.3 and 6.4, Henderson) $M_1 \approx M_2$ iff $M_1 \sim M_2$.

Theorem 6.4 was originally proved in the domain of differential topology. The final two theorems relate l^2-manifolds with $clfsc$'s or $clf-dsc$'s. The proof of the second is technically very delicate.

6.6 (Henderson) For every M, there exists K^{clfsc} such that $|K| \times E \approx M$

6.7 (West) For every $K^{clf-dsc}$ there exists M such that $|K| \times E \approx M$.

Theorems 6.5, 6.6 and 6.7 give useful and essentially complete characterizations of l^2-manifolds.

Using the above results for $E \approx l^2$, Chapman has proved Theorems 6.1-6.7 for $E \approx \sigma, \Sigma$ except for 6.1A for $E \approx \sigma$, and with $- K^{clfsc.}$ The case for $E \approx Q$ is not as completely known. Theorems 6.1A and 6.7 (for K^{clfsc}) are known and Chapman has many partial results. The conjecture replacing 6.5 for $E \approx Q$ is "$M_1 \approx M_2$ iff M_1 and M_2 are *properly* homotopic".

Under any definition of "manifold - with - boundary" of which the author is aware, such a "manifold - with - boundary" is homeomorphic to a manifold. Furthermore, the "boundary" is homeomorphic to a Z-set in this manifold. Thus the study of "manifolds - with - boundary" becomes simply a special case of the study of manifolds and Z-sets contained in them. Theorems 7.1 and 8.1 below may be considered as part of such a study.

7. Negligibility.

Many of the proofs of the theorems in *I-D* topology employ negligibility considerations. Historically, negligibility questions were among the first considered by Klee and by the author. Theorem 7.1 below was a main result in the first of the current series of papers written on the set-theoretic topology of l^2-manifolds. A subset K of a space X is {*strongly*} *negligible* if {for each open cover U in X} there is an element h of $H(X$ onto $X \setminus K)$ {and h is limited by U}.

7.1 (Anderson - Henderson - West) *A* closed subset $K \subset M$ is strongly negligible iff K is a Z-set.

7.2 (Anderson) *A* subset $K \subset M$ is strongly negligible iff K is a countable union of Z-sets.

Chapman has proved 7.1 for $E = \sigma, \Sigma$ but 7.2 is obviously false for such manifolds.

8. Homeomorphism Extension Theorems.

Klee proved the basic lemma : Given K_1, K_2 closed sets in complementary subspaces of l^2 and $h \in H(K_1$ onto $K_2)$. There exists $h^* \in H_I(l^2)$ such that $h^* \mid K = h$. The theorem below has several previously known corollaries which have been used in other proofs.

8.1 (Anderson and Mc Charen for $E \approx l^2$). Let K be a Z-set in M and $h \in H(K_1$ into $M)$. Then there exists $h^* \in H_I(M)$ with $h^* \mid K = h$ iff $h(K)$ is a Z-set and $h \sim$ id.

Independently, Henderson proved 8.1 for the special case of K an *ANR*.

Chapman has proved the theorem for $E \approx \sigma$ or Σ and Chapman and the author have proved a similar theorem for $E \approx Q$ but with the necessarily altered condition that h be *properly* homotopic to id.

9. Non-Separable Manifolds.

Culter made the first useful generalization to non-separable manifolds while studying certain negligibility questions. Within the past year the results cited in Sections 6, 7 and 8 have all been generalized to manifolds modeled on any *I-D* Banach space E for which $E \approx E^\omega$. (It is conjectured that $E \approx E^\omega$ for every *I-D* Banach space and it is known, by Bessaga and Pelcynski, for every *I-D* Hilbert space). The use of the product structure for E^ω lets many of the processes of the separable case be applied to the other manifolds. Specifically, Schori and Henderson using some separate and some joint work have established 6.1 to 6.5 for all such Banach spaces. West has very recently established results like 6.6 and 6.7 for manifolds modeled on such Banach spaces but with K a metric locally finite - dimensional simplicial complex related to the weight of E. Using results of Cutler and Henderson, Torunczyk and Chapman have, independently, gotten results implying Theorems 7.1, 7.2 and 8.1 for E any such Banach space.

Henderson and West have also established results like those of Section 6 for certain incomplete *TVS's* E for which $E \approx (E^\omega)_f$, thus generalizing Chapman's results for $E \approx \sigma$ or Σ.

10. Miscellaneous other results.

Georghegan has recently proved that for any compact finite - dimensional manifold $Y, H(Y) \times l^2 \approx H(Y)$, thus giving a partial coordinatization to $H(Y)$. The general question as to whether $H(Y)$ is an l^2-manifold is open. It is known true for dim $Y = 1$ and recent results of Mason and Henderson strongly suggest that it is true for dim $Y = 2$. A positive solution of the general question would let the results and methods of infinite - dimensional manifolds be much more readily available for finite - dimensional manifold problems.

West has used the methods employed in the proof of Theorem 6.7 to show that for every finite contractible complex K, $|K| \times Q \approx Q$ and indced that all countable infinite products of such (non - degenerate) polyhedra are homeomorphic to Q. The question of whether $A \times Q \approx Q$ for each compact absolute retract A is still open. Recent results of West show that $2^I \times Q \approx Q$ where 2^I is the space of closed subsets of I. The question "Is $2^I \approx Q$?" remains open.

Chapman has proved that $(M, A_1) \approx (M, A_2)$ for A_1 and A_2 both *cap* or both *f-d cap* sets in M with $E \approx l^2$ or Q and that for any Z-set K or countable union of compact (finite - dimensional) Z-sets K^*,

$$(M, A_1) \approx (M, A_1 \setminus K) \approx (M, A_1 \cup K^*).$$

Torunczyk has slightly weaker results in a much more general setting.

Bessaga and Pelczynski have used their version of the (*f-d*) *cap* to show that all $\aleph_0$-dimensional locally convex metrizable $TVS's$ are homeomorphic, to give a new and easier proof that all separable *I-D* Fréchet spaces are homeomorphic and to show that the space S_Y of all measurable functions on [0 , 1] into any complete metric space Y is homeomorphic to l^2.

Many interesting questions remain open. The exploitation of the several useful topological models of a manifold should lead to significant further results However, more general and useful characterizations of l^2 and Q are needed. The identification of other important *I-D* phenomena could well open up new areas of activity.

REFERENCES

[1] ANDERSON R.D. — On topological infinite deficiency, *Mich. Math. Journal,* Vol. 14, 1967, p. 365-383.

[2] BESSAGA C. and PELCZYNSKI, A. — Estimated extension theorem, homogeneous collections and skeletons, and their applications to topological classification of linear metric spaces and convex sets, *Fundamenta Math.,* To appear.

[3] CHAPMAN T. A. — Four classes of separable metric infinite-dimensional manifolds, *Bull. A.M.S.,* Vol. 76, 1970, p. 399-403.

[4] HENDERSON David W. — Infinite-dimensional manifolds are open subsets of Hilbert space, *Bull. A.M.S.,* Vol. 75, 1969, p. 759-762.

[5] WEST James E. — Factoring the Hilbert Cube, *Bull. A.M.S.,* Vol. 76, 1970, p. 116-120.

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A SURVEY OF SOME RECENT ADVANCES IN GENERAL TOPOLOGY, OLD AND NEW PROBLEMS

by A. V. ARHANGELSKIJ

The four years after the Moscow Congress have brought advances in all main directions of General Topology. New lucky notions were introduced, new interesting classes of spaces were discovered. New general theorems were found, the formulations of which are as unexpected as simple, and the proofs are non-trivial and exquisite. Much of those will become a natural part of the courses of General Topology in seventeenth years. I must say also that some old problems were solved and many new challenging problems were posed.

Nearly without comments - because of the lack of the place, - I will mention here some results. Of course, the exposition is not full.

I – Besides paracompactness, all metric spaces and all bicompact Hausdorff spaces enjoyes being feathered (перистые) spaces (p-space). The notion of p-space was introduced in [1].

DEFINITION. – X is p-space if in βX (the Stone-Cech bicompactification of the space(*) X) there exists a family φ of coverings of X by sets open in βX such that $\cap \{\lambda(x) : \lambda \in \varphi\} \subset X$ for each $x \in X$

$$(\text{here } \lambda(x) = U\{\mathfrak{U} : \mathfrak{U} \in \lambda \quad \text{and} \quad \mathfrak{U} \ni x\}).$$

Among p-spaces we find some non-paracompact spaces - for example, all spaces complete in the sense of E. Cech are p-spaces. Paracompact p-spaces are characterized as preimages of metric spaces under perfect maps [1,7]. The product of a countable family of paracompact p-spaces is a paracompact p-space [7]. Clearly preimage of a p-space under a perfect map is again a p-space. V.V. Filippov was the first to prove that the image of a paracompact p-space under a perfect map is a paracompact p-space [24]. A little later, but independently, this result was obtained by K. Morita and T. Ishii [12]. Recently H.H. Wicke announced an analogous assertion for p-spaces. It is worth noticing that a closed map of a metric space preserves metrizability if and only if it preserves the property of being p-space [7]. Each p-space with a countable grid (network, net - see [6,7]), has a countable base, hence, it is metrizable [7]. A map of arbitrary topological space onto a p-space doesn't increase the weight [7]. If a paracompact p-space can be mapped onto a metric space by a one-to-one continuous map, then it

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(*) In what follows, "a space" means "completely regular space" – until anything different is explicitly stated. The term "map" means everywhere "continuous one-valued function".

is metrizable [7]. Each paracompact p-space with a point-countable base is metrizable [27]. Paracompact p-space is metrizable iff it is symmetrizable [6]. If an open finite-to-one map of a p-space X onto a metric space Y is given, then X is metrizable [30] (the conclusion doesn't hold for open countable-to-one maps). Among Hausdorff space explicitly those which are of point-countable type [6] can be represented as an image of a paracompact p-space under an open map (H.H. Wicke [19]).

Two theorems are to be distinguished. These are theorems-schemes about perfect maps. They constitute a base for unified approach to the proofs of many specific theorems, concerning with preservation - from image to preimage - of topological properties under such maps. If $\mathfrak{E}$ is a class of spaces such that : (α) each closed subspace of a space in $\mathfrak{E}$ belongs to $\mathfrak{E}$; (β) if $X \in \mathfrak{E}$ and Φ is bicompact, then $X \times \Phi \in \mathfrak{E}$. Then the preimage of a space in $\mathfrak{E}$ under a perfect map is in $\mathfrak{E}$ (up to a homeomorphism) ([21] - van der Slot).

The theorem may be applied to bicompact spaces, paracompact spaces, locally bicompact spaces, as well as to k-spaces, Lindelöf spaces, spaces complete in the sense of E. Cech and to many other classes of spaces.

Let $\mathfrak{R}$ be a class of Hausdorff spaces, such that : (α^*) the product of each two spaces in $\mathfrak{R}$ is in $\mathfrak{R}$; (β^*) each closed subspace of a space in $\mathfrak{R}$ is in $\mathfrak{R}$. Then each space which can be mapped by a perfect map onto a space belonging to $\mathfrak{R}$ and can also be mapped by a one-to-one map into a space in $\mathfrak{R}$ belongs to $\mathfrak{R}$ (Archangelskij [2]). For example, the conditions (α^*) and (β^*) are satisfied by the following classes of spaces (1) spaces with countable base ; (2) spaces with the first axiom of countability ; (3) metrizable spaces ; (4) finite - dimensional spaces ; (5) spaces with countable grid ; (6) developable spaces ; (7) quasidevelopable spaces (in the sense of H.R. Bennett) ; (8) all completely-regular spaces.

The last two theorems are of the mixed, topologically - cathegorical, nature ; we have few results of this type yet. Among purely topological theorems dealing with maps of general topological spaces we find many new interesting - often unexpected results ; more than we can mention here. First of all, the class of bifactor maps was discovered (it was done by E. Michael [15] - and soon afterwards and independently - by V.V. Filippov [26]). I dare say that bifactor maps will become a central notion of the theory of maps.

DEFINITION. – A map $f : X \to Y$ is called bifactor iff for each $y \in Y$ and and each covering γ of the set $f^{-1}y$ by sets open in X there exists a finite family $\lambda \subset \gamma$ such that $\operatorname{Int} (U\{f\mathfrak{U} : \mathfrak{U} \in \lambda\}) \ni y$.

All open maps and all perfect maps are bifactor maps, but there are closed maps which are not bifactor. Bifactor maps are always pseudoopen - hence, hereditarily factor [6]. The following fact is of particular importance : the product of arbitrary family of bifactor maps is bifactor map (Michael [15]). To feel all the sweetness of the assertion it is sufficient to remember that even the product of a factor map with identity may be a non-factor map. Moreover bifactor maps are exactly those, the product of which with each identity is a factor map (Michael [15]). Two hard and deep theorems on bifactor maps were proved by V.V. Filippov [26]. A. Let $f : X \to Y$ be a bifactor map, τ - a cardinal number, $\tau \geqslant \aleph_0$, and the weight of the space $f^{-1}y$ doesn't exceed τ for each $y \in Y$. Then if X has a point-countable base, a point countable base has the space Y also. B. Let $f : X \to Y$ be a factor s-map, X - a space with a point countable base, Y - a Hausdorff space of point-countable type [6] (for example, Y may be taken to be a p-space or even a bicompact) and $fX = Y$. Then f is bifactor map. The both theorems arose from the work on the solution of the following problem : is it true that each bicompact Hausdorff space Y which is an image of a metric space X under a factor S-map, is metrizable [6] ? As we see now, the answer is positive (even in the case if Y is a p-space), but for the proof we need the mentioned two theorems of V.V. Filippov, the theorem of A.H. Stone about paracompactness of metric space and the theorem of A.S. Mischenko about point countable bases of bicompact spaces. In particular, among bifactor maps we find all pseudoopen bicompact maps.

The following result I consider as a very refined one : if a paracompact space Y is an image of a metric space under a pseudoopen bicompact map, then Y is metrizable (М.М. Чобан - M.M. Coban). Here is the proof. The topology of the space agrees with the symmetric $d(y_1, y_2)$ on Y, defined by the formula : $d(y_1, y_2) = (f^{-1}y_1, f^{-1}y_2)$ (see [6]) - where ρ is the given metric on X. As the space X has a point-countable base (A.H. Stone), the space Y also has such a base (V.V. Filippov). But each symmetrizable space with a point countable base has a development (R.W. Heath). Hence we need only to use the following known theorem of R.H. Bing : each paracompact with a development is metrizable. The proof is complete. I don't know whether a direct proof of the assertion can be found. It would be fine to have it.

Some other results on maps. An image of a complete metric space under an open map has always a base of countable order (H. Wicke, J. Worrell [20]) – i.e. a base such that each decreasing sequence of its elements with non-empty intersection constitutes a base of some point. As each paracompact with a base of countable order is metrizable [6], the cited result due to H. Wicke give a new performance to the known theorem due to E. Michael about metrizability of each paracompact space, which is an image of a complete metric space under an open map. If $f : X \to Y$ is an openclosed finite-to-one map and $fX = Y$, then the weight of X equals to the weight of Y, and Y is metrizable iff X is metrizable [5]. But an open countable-to-one map of a nonmetrizable perfectly normal bicompact space onto a bicompactum with countable base exists (V.V. Filippov [25]). Let us mention, that there is a countable space X, all bicompact subsets of which are finite, and there is an open finite-to-one map of X onto the simplest countably infinite bicompactum. This means that the results cited above are in some sense conclusive. Besides, we see that an open finite-to-one map of a Lindelöf space onto a metric space may be not k-covering (compact-covering) (a map is called k-covering, iff each bicompact subspace of the image is contained in the image of some bicompact subspace of the preimage – see [6]). On the other hand each closed map of a paracompact space is k-covering (E. Michael [13]). If X is a space with countable grid (for example, with countable base), and $f : X \to Y$ – a closed map, then the set of all $y \in Y$ such that $f^{-1}y$ is not bicompact is countable (may be, finite, or empty) (Arhangel'skij [6]).

In conclusion of this most general part of the survey I wish to point on some new classes of spaces – which are very interesting in my opinion. These are : $\aleph_0$-spaces of E. Michael [14], Σ-spaces of K. Nagami, stratifiable and semistratifiable spaces (C.R. Borges , E. Michael, G. Creede , Я. Кофнер), spaces with countable quasidevelopment (H.R. Bennet) – a quasidevelopment differs from a development in that its elements needn't be coverings. Rather general method of defining natural classes of spaces is shown in [6] (see definitions of Mobi, Mobos and Fabos). The M-spaces introduced by K. Morita [16] are being investigated successfully (K. Morita, T. Ishii, A. Okuyama, J. Nagata and others). For paracompact spaces the notion of M-space is equivalent with the notion p-space. So in most important points the theories of p-spaces and of M-spaces closely correspond to each other (compare the results of V.V. Filippov and M.M. Čhoban with the results of Japanese mathematricians).

Fundamental works on symmetrizable spaces were fulfilled by S. Nedev (С. Недев), R. Heath, and S. Nedev jointly with M. Chöban. Beautiful results on k-spaces received N. Noble.

II – At last one of the central problems of general theory of dimension (posed by P.S. Alexandroff in 1935) was solved. V.V. Filippov has constructed bicompactum X such that $\operatorname{ind} X \neq \operatorname{Ind} X$.

I don't fear to call the result sensational. A serious corollary of it is evident : even for bicompact Hausdorff spaces we must construct two theories separately : the theory for ind and the theory for Ind. The result of Filippov was being improved afterwards by B.A. Pasynkov (Б. А. Пасынков), I.K. Lifanov (И. К. Лифанов) and by V.V. Filippov himself. Now we have a bicompactum X, satisfying the first axiom of countability such that $\dim X \neq \operatorname{ind} X \neq \operatorname{Ind} X$. It is well known, that for all perfectly normal bicompact spaces ind = Ind. But is ind = dim for these spaces ? This old question was also answered, in a negative way, by V.V. Filippov – but the continuum – hypothesis is assumed.

A new characterization of the dimension dim of metric spaces was established by V.V. Zolotarev (В. В. Золотарёв). His theorem : let X be a metric space. Then $\dim X \leqslant n$ iff the topology of X can be represented as intersection of a family F of topologies on X such that (α) the power of F is equal $n+1$; (β) $\dim(X, \mathfrak{T}) = 0$ for each $\mathfrak{T} \in F$, and (γ) the intersection of any subfamily of F is a metrizable topology. I don't know whether the result can be improved by inclusion instead of (γ) the following condition (γ') : $(X, \mathfrak{T})$ is metrizable for each $\mathfrak{T} \in F$.

III – I begin here with a brief consideration of some important and exquisite results obtained by Z. Frolik.

(A) The set of all fixed points of a homeomorphism of an extremally disconnected bicompactum into itself is open – closed [27].

(B) (a corollary from A) : if X is a subspace of an extremally disconnected bicompactum Y and X contains a topological copy of Y as a nowhere dense subspace then X is not homogeneous [27]. Particularly, $\beta N \setminus N$ contains a topological copy of βN, and βN is extremally disconnected. Hence $\beta N \setminus N$ is not homogeneous (the continuum hypothesis is not used in this argumentation of Z. Frolik – which differs it principally from the well known argumentation due to W. Rudin). Another method (also discovered by Z. Frolik) of investigation of the problem of homogeneity of extemally disconnected bicompacta is based on consideration of the set T_x of types of a point x related to arbitrary countable discrete subsets M, having x as a point of accumulation. Types are defined as equivalence classes of ultrafilters induced on M by the system of all neighborhoods of the point x. The remarkable fact discovered by Z. Frolik : the set T_x has a natural linear order [28]. Another fundamental and astonishing result in this area : each point $x \in \beta N \setminus N$, considered as an ultrafilter on N, is not equivalent to the ultrafilter, induced on a discrete countable set $M \subset \beta N \setminus N$ by the system of neighborhoods of x in βN – for each such M. Comparing the cardinalities of the sets T_x and $T = U\{T_x : x \in X\}$, Frolik proves that each infinite extremally disconnected bicompactum is not homogeneous – but here he uses essentially any from the following two nearly opposite suggestions : 1) $2^c > c^+$, or 2) $2^{\aleph_0} = \aleph_1$. It may be mentioned that nonhomogeneity of an infinite extremally disconnected bicompactum the weight of which is less or equal c, was proved without any suggestion of the sort (Arhangel'skij). Some fine results about the structure of $\beta N \setminus N$ belong to M.E. Rudin. It was established (under assumption that $2^{\aleph_0} = \aleph_1$), that some points of $\beta N \setminus N$ aren't accumulation points for any countable subset of $\beta N \setminus N$. Besides the points were found which are accumulation points for a countable set but aren't accumulation points for any countable discrete subset of $\beta N \setminus N$ (K. Kunen [11]). In a recent survey of M.E. Rudin two new interesting partial orderings on the set of types of the points in $\beta N \setminus N$ are described. An interesting article on $\beta N \setminus N$ was written by W.W. Comfort and S. Negrepontis. But the main problem in the area : of topological classification of points of $\beta N \setminus N$ is far from the solution. Let us mention that a separated extremally disconnected topological group exists the topology of which is non-trivial (S. Sirota – С. Сирота [18]). But each bicompact subspace of each extremally disconnected topological group is finite (Arhangelskij [8]).

In other aspects the extremal disconnectedness was investigated by B. Efimov (Б. Ефимов). Each nonmetrizable dyadic bicompactum contains a topological copy of βN – this assertion is equivalent to the continuum hypothesis [9]. Without any assumption of this sort B. Efimov proved that each bicompactum X the Souslin number of which is not greater than τ and the weight of which is bigger than expexpexp τ, contains all extremally disconnected spaces the weight of which is not greater than $(\exp \tau)^+$, hence contains the Stone-Cech bicompactification of the discrete space of cardinality τ [10].

IV – I. Juhasz and A. Hajnal derived from a fundamental theorem proved by Erdös, Rado and Hajnal, that the power of each Hausdorff space with Souslin number $\leqslant \tau$ and the character in each point less or eqial τ, isn't greater than 2^τ. This result is deep and useful ; some of its topological generalizations and applications see in [4], [29]. Another result of A. Hajnal and I. Juhasz : if the power of a Hausdorff space X is bigger than $2^{2^{\tau}}$ then a discrete subspace Y of the space X exists the power of which is bigger than τ. In the proofs of the following results very essential role plays the notion of a free sequence of the length α, introduced in [3]. Let A be a well ordered (by a relation $<$) subset of a topological space X such that

$$[\{a \in A : a < a^*\}] \cap [\{a \in A : a^* \leqslant a\}] = \Lambda .$$

for each $a^* \in A$. Then A is called a free sequence in X. The length of this sequence is the type of A. If the Souslin number of a sequential bicompactum X equals $\aleph_0$ then the power of X (i.e. $|X|$) is less or equal $2^{\aleph_0}$ (by a sequential bicompactum I mean bicompact Hausdorff space which is sequential space (in the sense of S.P. Franclin) also). If $2^{\aleph_1} > 2^{\aleph_0}$, then each sequential bicompactum X satisfies the first axiom of countability on a set of points dense in X (Arhangelskij [4]). If sequential bicompactum X is homogeneous then either it is finite, or its power equals $2^{\aleph_0}$. The proof of the last assertion heavily depends of a bunch of serious theorems. It would be nice to find an elementary or more direct proof. In 1922 the following problem was posed by P.S. Alexandroff: is it true that the power of each bicompactum satisfying the first axiom of countability at each point is less or equal than $2^{\aleph_0}$? I have proved in [3] — using free sequences and ramifications, — that the power of each Lindelöf space satisfying the first axiom of countability at each point is less or equal $2^{\aleph_0}$. The generalizations of the result dealing with arbitrary cardinal number τ are also received [3].

V — Problems (unsolved so far as I know).

(1) Is each completely regular metacompact space an image of a paracompact space under an open bicompact (continuous) mapping ? (Arhangelskij).

(2) Let f be a k-covering map of a complete separable metric space X onto a metrizable space Y. Is Y metrizable by a complete metric then ? (E. Michael).

(3) Suppose that an image of a metric space under an open bicompact (and continuous) mapping is normal space. Is this space metrizable then ? (P.S. Alexandroff).

(4) Is it true that $\operatorname{ind} X = \operatorname{Ind} X = \dim X$ for each regular space with a countable grid ? (one need only to verify the inequality $\operatorname{ind} X \leqslant \dim X$ for such spaces) (Arhangelskij).

(5) Does there exist an infinite homogeneous extremally disconnected bicompactum ? (without using the continuum hypothesis or its negation) (Arhangelskij).

(6) Is each regular countable space X imbeddable into a bicompactum (depending from X) the power of which is less or equal than $2^{\aleph_0}$? (Arhangelskij).

(7) Is it true that each Hausdorff space with the power greater than $2^{\aleph_0}$ contains an uncountable discrete subspace ? (A. Hajnal, I. Juhasz).

(8) Is it true that the power of each hereditarily separable bicompactum is less or equal than $2^{\aleph_0}$? (A. Hajnal, I. Juhasz).

(9) Let us say that a topological space X is of countable density, if for each $x \in X$ and for each $M \subset X$ from $x \in [M]$ it follows that $x \in [M']$ for some countable subset M' of the set M. Is it true that each bicompactum of countable density satisfies the first axiom of countability at some point ? (even in the assumption that $2^{\aleph_0} = \aleph_1$?) (A. Arhangelskij, B. Efimov).

(10) Is it true that each bicompactum of countable density contains a nontrivial convergent sequence of points ? (A. Arhangelskij, B. Efimov).

(11) Is there a homogeneous bicompactum of countable density the power of which is greater than $2^{\aleph_0}$? (A. Arhangelskij).

(12) Prove that the power of each bicompactum of countable density satisfying the Souslin condition (i.e. $cX = \aleph_0$) is not greater than $2^{\aleph_0}$ (A. Arhangelskij).

(13) Suppose that the generalized continuum hypothesis is fulfilled. Let X be a bicompactum, τ-a cardinal number, and the character of each point $x \in X$ in X is strictly less than τ. Is it true then that $|X| \leqslant \tau$? (A. Arhangelskij). (The answer is "yes" when τ is a regular cardinal number).

(14) Let X be a Lindelof space of countable density and each $x \in X$ is an intersection of countably many open sets in X. Is it true than that $|X| \leqslant 2^{\aleph_0}$? (or at least is it true that $|X| \leqslant 2^c$?) (*).

(15) Suppose a completely regular space Y is an image of a completely regular space X having a uniform base under an open (continuous bicompact mapping. Is it true than that Y has also a uniform base ? (A Arhangel'skij)(*).

REFERENCES

[1-5] Архангельский А. В. — ДАН СССР т. 151, № 4, 1963, p. 751; т. 176, № 5, 1967, p. 983; т. 187, № 5, 1969; т. 192, № 2, 1970.

[6] Архангельский А. В. — УМН т. XXI, в. 4 (130), 1966, p. 133.

[7] Архангельский А. В. — Матем. Сб. 67 (109) : I 1965, p. 55.

[8] Arhangelskij A. — *C. R. Acad. Sci. Paris*, 265, 1967, p. 822.

[9-10] Ефимов Б. А. — ДАН СССР т. 189, 2, 1969, p. 244.

[11] Kunen K. — *Notices Amer. Math. Soc.*, Vol. 17, No. 1, 1970, p. 299.

[12] Ishii T. — *Proc. Japan Acad.*, 43, 1967, p. 752-61.

[13-15] Michael E. — *Journ. of Math. and Mech.*, Vol. 15, No. 6, 1966, p. 983; *Israel J. Math.*, 2, 1964, p. 173;
Ann. Inst. Fourier, Grenoble, Vol. 18, No. 2, 1968, p. 287.

[16] Morita K. — *Math. Ann.*, 154, 1964, p. 365.

[17] Rudin M.E. — *Topology Seminar*, Wisconsin, 1965, p. 147.

[18] Сирота С. — Матем. Сб. т. 79 (121), № 2 (6), 1969, p. 179.

[19] Wicke H.H. — *Proc. Amer. Math. Soc.*, Vol. 22, No. 1, 1969, p. 136.

[20] Wicke H.H. and Worrell J.M. Jr. — *Duke Math. Journ.*, Vol. 34, No. 2, 1967, p. 255.

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(*) These problems (14 and 15) were added when correcting proofs. At the time was already known that the question (6) has negative ans'wer – this was proved by B. Efimov .

[21] VAN DER SLOT J. — *Stichting Math. Centrum,* Amsterdam, ZW, 1966, 11, p. 1-9.
[22-25] Филиппов В. В. — ДАН СССР т. 187, № 5, 1969, p. 1020; т. 178, № 3, 1968, p. 555; т. 176, № 3, 1967, p. 533; т. 189, № 4, 1969, p. 736.
[26] Филиппов В. В. — Матем. Сб., н. с., т. 80 (122), в. 4, 1969, p. 521.
[27] FROLIK Z. — *Bull. Acad. Polon.,* XVI, 1968, p. 269.
[28] FROLIK Z. — *Comment. Math. Univ. Carolinae,* Vol. 8, No. 4, 1967, p. 757.
[29] HAJNAL A. and JUHASZ I. — *Proc. Koninkl. Nederl. Akad. Wet.,* A 70, No. 3, 1967, p. 343.
[30] Чобан М. М. — ДАН СССР, т. 174, № 1, 1967, p. 41.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 27 à 32.

FONDEMENTS DE LA K-THEORIE

par Max KAROUBI

0. Introduction.

Depuis le travail de Grothendieck sur le théorème de Riemann-Roch en géométrie algébrique, la K-théorie a connu un développement intensif, marqué essentiellement par des applications nombreuses dans divers domaines des mathématiques. Elle s'est même divisée en deux branches essentielles : la "K-théorie topologique" dont une idée peut être donnée dans le livre connu d'Atiyah [1] et la "K-théorie algébrique" exposée par exemple dans le livre de Bass [2]. Ces deux livres et bien d'autres publications contiennent évidemment des résultats importants dont je ne parlerai pas ici. Mon but est essentiellement théorique : on va tâcher d'unifier les deux "K-théories" en les intégrant dans la perspective générale de l'algèbre homologique.

De manière plus précise, considérons un anneau A avec élément unité (pour l'instant) et la catégorie $\mathcal{P}(A)$ des A-modules (1) projectifs de type fini. Soit G un groupe abélien et soit

$$f : Ob\, \mathcal{P}(A) \to G$$

une application qui satisfait à la propriété suivante : si

$$0 \to P' \to P \to P'' \to 0$$

est une suite exacte de A-modules projectifs (nécessairement scindée), on a $f(P) = f(P') + f(P'')$. Parmi les couples (G, f) il en existe évidemment un d'universel : on le notera $(K(A), \gamma)$. Un homomorphisme $\epsilon : A \to B$ induit un foncteur "extension des scalaires" $M \to M \otimes_A B$ de $\mathcal{P}(A)$ dans $\mathcal{P}(B)$, d'où un homomorphisme $K(\epsilon)$ de $K(A)$ dans $K(B)$. Il est clair que $K(A)$ devient ainsi un foncteur *covariant* de l'anneau A. Si A n'a pas nécessairement d'élément unité, considérons l'ensemble $A^+ = A \times \mathbf{Z}$ muni des deux lois de composition suivantes

$$(a, \lambda) + (a', \lambda') = (a + a', \lambda + \lambda')$$
$$(a, \lambda) \cdot (a', \lambda') = (aa' + \lambda' a + \lambda a', \lambda\lambda').$$

Alors A^+ est un anneau avec élément unité et A s'identifie au noyau de "l'homomorphisme d'augmentation" $\epsilon : A^+ \to \mathbf{Z}$ où $\epsilon(a, \lambda) = \lambda$. On définit alors $K(A)$ comme le noyau de $K(\epsilon) : K(A^+) \to K(\mathbf{Z})$. Il est facile de voir que cette définition est cohérente avec la définition antérieure dans le cas où A a déjà un élément unité . . .

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(1) à droite pour fixer les idées.

Considérons une suite d'anneaux et d'homomorphismes

$$\text{(S)} \qquad 0 \to A' \to A \xrightarrow{f} A'' \to 0$$

Cette suite est dite *exacte* si elle est exacte en tant que suite de groupes abéliens (ainsi A' s'identifie à l'idéal noyau de f et n'a pas en général d'élément unité).

THEOREME 0. — (Bass-Schanuel). *La suite*

$$K(A') \to K(A) \to K(A'')$$

obtenue à partir de la suite (S) *en appliquant le foncteur K est une suite exacte.*

Le premier réflexe d'un spécialiste d'algèbre homologique ou d'un topologue est évidemment de chercher à construire les foncteurs "satellites" du foncteur "semi-exact" K. En d'autres termes, on aimerait pouvoir définir des foncteurs $K^n(A)$ (1), $n \in \mathbf{Z}$, tels que $K^0(A) = K(A)$ et tels qu'on ait une suite exacte infinie :

$$\cdots \to K^{n-1}(A) \to K^{n-1}(A'') \to K^n(A') \to K^n(A) \to K^n(A'') \to \cdots$$

Nous allons voir que, sous certaines hypothèses restrictives sur les suites (S), il est effectivement possible de définir des foncteurs K^n. Pour cela, nous allons adopter la définition de Villamayor et de l'auteur qui est présentée dans [6]. Des définitions différentes ont été proposées par d'autres auteurs (avec de moins bonnes propriétés formelles en général). Faute de place, nous nous bornerons à les mentionner au passage.

1. Anneaux de Banach.

L'originalité de la K-théorie dans la présentation adoptée réside dans le fait que la définition des groupes $K^n(A)$ va dépendre du choix d'une topologie (plus précisément d'une norme) sur l'anneau A. Ainsi, si l'anneau A est discret, on obtiendra des foncteurs K^n intéressants pour les algébristes ; si A est une algèbre de Banach réelle ou complexe, les foncteurs K^n obtenus seront intéressants pour les topologues. De manière plus précise, posons la définition suivante :

DEFINITION 1. — *Un "anneau de Banach" est un anneau A (non nécessairement unitaire) muni d'une "norme"* $p : A \to \mathbf{R}^+$ *satisfaisant aux axiomes suivants* :

(1) $p(x) = 0 \Leftrightarrow x = 0$

(2) $p(x + y) \leqslant p(x) + p(y)$

(3) $p(-x) \;\; = p(x)$

(4) $p(xy) \;\; \leqslant p(x)\, p(y)$

(5) *A est complet pour la distance* $d(x, y) = p(x - y)$.

Il est clair que les anneaux discrets, les algèbres de Banach ordinaires ou ultramétriques sont des exemples d'anneaux de Banach. Pour simplifier l'écriture, on notera $\|x\|$ l'expression $p(x)$ comme il est d'usage.

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(1) Dans la littérature on écrit aussi K_{-n} au lieu de K^n. Nous nous conformons ici à la tradition de la K-théorie topologique.

Si A est un anneau de Banach, $A<x>$ est le sous-anneau de $A[[x]]$ formé des séries formelles $S = S(x) = \sum_{i=0}^{+\infty} a_i x^i$ telles que $\sum_{i=0}^{+\infty} \|a_i\| < +\infty$; $A<x>$ est évidemment un anneau de Banach pour la norme $\|S\| = \sum_{i=0}^{+\infty} \|a_i\|$ (si A est discret, on a $A<x> = A[x]$). Plus généralement, le sous-anneau $A<x_1, \ldots, x_n>$ de $A[[x_1, \ldots, x_n]]$ formé des séries S telles que la somme des normes des coefficients soit finie est un anneau de Banach. Un homomorphisme borné $f : A \to B$ induit un homomorphisme borné $f_n : A<x_1, \ldots, x_n> \to B<x_1, \ldots, x_n>$. Pour tout anneau C, posons

$$GL(C, p) = \mathrm{Ker}[GL(C^+, p) \to GL(\mathbf{Z}, p)] \quad \text{et} \quad GL(C) = \varinjlim GL(C, p).$$

Alors f_n induit un homomorphisme de groupes

$$GL(A<x_1, \ldots, x_n>) \to GL(B<x_1, \ldots, x_n>)$$

que nous noterons encore f_n.

DEFINITION 2. – *L'homomorphisme $f : A \to B$ est une "fibration" si, pour tout élément $\beta = \beta(x_1, \ldots, x_n)$ de $GL(B<x_1, \ldots, x_n>)$ tel que $\beta(0, \ldots, 0) = 1$, il existe un élément α de $GL(A<x_1, \ldots, x_n>)$ tel que $f_n(\alpha) = \beta$. L'homomorphisme f est une "cofibration" si f est surjectif et si la norme de B est équivalente à la norme quotient de A.*

Exemples. – Si A et B sont des algèbres de Banach sur $\mathbf{R}$ ou $\mathbf{C}$, tout homomorphisme surjectif est à la fois une fibration et une cofibration. Il en est de même si f est surjectif et si B est un anneau noethérien régulier discret.

Soit

$$\text{(S)} \qquad 0 \to A' \to A \xrightarrow{f} A'' \to 0$$

une suite exacte d'anneaux de Banach et d'homomorphismes bornés. Par abus de langage, on dit que (S) est une fibration (resp. une cofibration) si la norme de A' est équivalente à la norme induite par A et si f est une fibration (resp. une cofibration).

2. Définition des foncteurs K^n.

Soit $\mathcal{B}$ la "catégorie" des anneaux de Banach, les morphismes étant les homomorphismes bornés. Une "*théorie de la cohomologie positive*" (resp. "*négative*") sur $\mathcal{B}$ est la donnée de foncteurs $K^n, n \geqslant 0$ (resp. $n \leqslant 0$) de $\mathcal{B}$ dans la catégorie des groupes abéliens ainsi que d'opérateurs de connexion naturels

$$\partial^{n-1} : K^{n-1}(A'') \to K^n(A') \quad , \quad n \geqslant 1 \text{ (resp. } n \leqslant 0)$$

définis pour toute cofibration (S) (resp. toute fibration (S)). On suppose en outre que la suite

$$K^{n-1}(A') \to K^{n-1}(A) \to K^{n-1}(A'') \to K^n(A') \to K^n(A) \to K^n(A'')$$

est exacte pour les valeurs de n où elle est définie.

DEFINITION 3. – *Soit A un anneau de Banach et soient $q_i : A < x > \to A$, $i = 0, 1$, les homomorphismes définis par $q_i(S) = S(i)$. On dit que A est "contractile" s'il existe un homomorphisme borné $h : A \to A < x >$ tel que $q_0 \bullet h = 0$ et $q_1 \bullet h = \mathrm{Id}$.*

Exemple. – L'anneau $EA = \operatorname{Ker} q_0$ est contractile.

THEOREME 4. – *Il existe une théorie de la cohomologie négative et une seule à isomorphisme près sur $\mathcal{B}$ qui satisfait aux axiomes suivants :*

(1) $K^n(A) = 0$ *pour $n < 0$ si A est contractile.*

(2) $K^0(A) = K(A)$.

Cette définition est évidemment à rapprocher de celle des groupes d'homotopie d'un espace topologique. La définition des groupes K^n pour n positif va nécessiter quelques préliminaires techniques qui trouvent leur origine dans la théorie des opérateurs de Fredholm dans un espace de Hilbert (cf. [4]).

Soit $M = (a_{ji})$ une matrice infinie à coefficients dans A. On pose

$$\|M\| = \operatorname{Sup}_i \sum_{j=0}^{\infty} \|a_{ji}\| .$$

Les matrices M telles que $\|M\| < +\infty$ forment un anneau de Banach B. Une matrice diagonale M est dite de type fini si elle ne contient qu'un nombre fini d'éléments de A différents. Le "*cône*" CA de A est le plus petit anneau de Banach contenu dans B qui contient les matrices diagonales de type fini et les matrices de permutation. La limite inductive $A(\infty) = \varinjlim A(n)$ suivant les inclusions

$$M \longrightarrow \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$$

est un sous-anneau de CA. Son adhérence $\widetilde{A}$ est "*l'anneau stabilisé*" de A (dans le cas discret on a $\widetilde{A} = A(\infty)$). L'anneau stabilisé est en fait un idéal dans CA et l'anneau de Banach quotient $SA = CA/\widetilde{A}$ est la "*suspension*" de A.

Un anneau de Banach unitaire C est dit "*flasque*" s'il existe un bimodule de Banach M sur C, projectif de type fini à droite, tel que $M \oplus C$ soit isomorphe à M en tant que bimodule (exemples : le cône CA d'un anneau de Banach unitaire A ; l'algèbre des endomorphismes d'un espace de Hilbert de dimension infinie).

THEOREME 5. – *Il existe une théorie de la cohomologie positive et une seule à isomorphisme près sur $\mathcal{B}$ qui satisfait aux axiomes suivants :*

(1) *L'inclusion naturelle $A \to \widetilde{A}$ induit un isomorphisme $K^n(A) \overset{\approx}{\to} K^n(\widetilde{A})$.*

(2) $K^n(A) = 0$ *si A est un anneau flasque.*

(3) $K^0(A) = K(A)$.

3. Comparaison avec d'autres définitions.

THEOREME 6. – *Soit A une algèbre de Banach sur* **R** (resp. **C**). *Alors les groupes $K^n(A)$ définis ici coïncident avec les groupes K^n de la catégorie de Banach $\mathcal{P}(A)$ définis dans* [3]. *En particulier, ils sont périodiques de période* 8 (resp. 2). *Si A est l'algèbre de Banach des fonctions continues sur un espace compact X, on retrouve les groupes $K^n(X)$ introduits par Atiyah et Hirzebruch* [1].

THEOREME 7. – *Soit A un anneau discret. Alors, pour $n \geqslant 0$, $K^n(A)$ coincide avec le groupe $K_{-n}(A)$ défini par Bass* [2]. *En particulier $K^n(A) = 0$ pour $n > 0$ si A est un anneau noethérien régulier. Enfin, on a la formule*

$$K_n(A) = K^{-n}(A) \oplus \binom{n-1}{1} K^{-n+1}(A) \oplus \binom{n-1}{2} K^{-n+2}(A) \oplus \cdots \oplus K^{-1}(A)$$

où K_n est le foncteur introduit par Nobile et Villamayor [8].

THEOREME 8. – *Soit A un anneau de Banach. On a alors des homomorphismes naturels*

$$h_1 : K_1(A) \to K^{-1}(A)$$
$$h_2 : K_2(A) \to K^{-2}(A)$$

où K_1 et K_2 sont les foncteurs introduits par Bass et Milnor respectivement [2] [7]. *L'homomorphisme h_1 est toujours surjectif. Si A est noethérien régulier discret, h_1 est bijectif et h_2 est surjectif.*

4. Interprétation de la périodicité de Bott.

La périodicité de Bott "naïve" $K^n(A) \approx K^{n+\alpha}(A)$, $\alpha \neq 0$, pour tout anneau de Banach A est fausse en général (considérer par exemple un anneau noethérien régulier). Cependant, Bass a montré dans [2] que la "bonne" généralisation de la périodicité s'exprime par une formule du type "$LK^n \approx K^{n+1}$". Avec nos notations, ceci peut se formuler de la manière suivante. Soit $A < t, t^{-1} >$ l'anneau des séries formelles $\sum_{i=-\infty}^{+\infty} a_i t^i$ telles que $\sum_{i=-\infty}^{+\infty} \|a_i\| < +\infty$. Si F est un foncteur quelconque de $\mathcal{B}$ dans la catégorie des groupes abéliens, on pose

$$(LF)(A) = \operatorname{Coker}\,[F(A<t>) \oplus F(A<t^{-1}>) \to F(A<t, t^{-1}>)].$$

THEOREME 9. – *Pour tout entier $n \geqslant 0$, on a un isomorphisme naturel de foncteurs $K^{n+1} \approx LK^n$.*

Le théorème analogue pour $n < 0$ va nécessiter quelques hypothèses restrictives sur l'anneau de Banach A. On a par exemple le résultat suivant :

THEOREME 10. – *Soit A une algèbre de Banach sur* **R** *ou* **C** *ou un anneau noethérien régulier discret. Pour $n < 0$, on a alors un isomorphisme naturel de foncteurs $K^{n+1} \approx LK^n$* (voir [5] pour un résultat de portée plus générale).

Remarque. – Notons ΓA l'idéal de $A < t, t^{-1} >$ formé des séries $S(t)$ telles que $S(1) = 0$. Alors le théorème précédent peut s'écrire aussi $K^n(\Gamma A) \approx K^{n+1}(A)$. Dans le cas où A est une algèbre de Banach complexe, $K^n(\Gamma A)$ est isomorphe à $K^n(\Omega A) \approx K^{n-1}(A)$, ΩA désignant l'idéal de $A < x >$ formé des séries $S(x)$ telles que $S(0) = S(1) = 0$. La périodicité de Bott classique (dans le cas complexe) en résulte.

Les techniques permettant de démontrer le théorème précédent servent aussi à démontrer le résultat suivant sur le foncteur K_2 de Milnor :

THEOREME 11. – *Soit A un anneau discret. Alors $K_2(A[t, t^{-1}])$ peut s'écrire de manière naturelle sous la forme $K_2(A) \oplus K_1(A) \oplus X$ où X est un groupe en général inconnu* (1).

BIBLIOGRAPHIE

[1] ATIYAH M.F. — *K-theory,* Benjamin, 1967.

[2] BASS H. — *Algebraic K-theory*. Benjamin, 1968.

[3] KAROUBI M. — Algèbres de Clifford et K-theorie, *Annales Scientifiques de l'Ecole Normale Supérieure,* 1968, p. 161-270.

[4] KAROUBI M. — Foncteurs dérivées et K-théorie, Séminaire Heidelberg-Saarbrücken-Strasbourg, 1967-68. *Exposé 4. Springer Lecture Note* n° 136.

[5] KAROUBI M. — La périodicité de Bott en K-théorie générale. *C.R. Acad. Sci.,* Paris, t. 270, 1970, p. 1305-1307.

[6] KAROUBI M. et VILLAMAYOR O. — Foncteurs K^n en algèbre et en topologie. A paraître; en attendant *cf. C.R. Acad. Sci.* Paris, 269, 1969, p. 416-419.

[7] MILNOR J. — *Notes on algebraic K-theory* (preprint).

[8] NOBILE A et VILLAMAYOR O. — Sur la K-théorie algébrique. *Ann. Scient. Ec. Norm. Sup.,* 1968, p. 581-616.

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(1) Ce résultat a été aussi prouvé indépendamment par Farrell et Wagoner et, dans le cas où A est noethérien régulier, par Gersten.

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NORMALITY OF PRODUCTS

by Keiô NAGAMI

1. – Finite products. All spaces in this address are assumed to be Hausdorff and all mappings continuous. Let spaces X_α, $\alpha \in A$, be given. Then the product ΠX_α is regular or completely regular according as each X_α is respectively regular or completely regular. In other words the regularity and the complete regularity are productive. A celebrated example due to Michael [9] shows that :

THEOREM 1. – *There exists a hereditarily paracompact space whose product with a separable metric space is not normal.*

Thus we know that the normality is not even finitely productive. Let X be a normal space and I the unit closed interval. Is $X \times I$ normal ? This is Dowker's problem in 1951. Dowker [6] shows that :

THEOREM 2. – *$X \times I$ is normal if and only if X is normal and countably paracompact (i.e. every countable open covering is refined by a locally finite open covering).*

Dowker's problem is related to Souslin's problem [19] in 1920 : Does there exist a linearly ordered space which is not separable and in which every collection of disjoint segments is countable ? Mary Rudin [18] shows that :

THEOREM 3. – *If there exists such a space, then Dowker's question has a negative answer.*

It is known that Souslin's problem is independent of our axioms for set theory ; yet the negative answer of Dowker's problem may still be obtained in the usual set theory. From the following two theorems we can realize that the normality of the product has a great influence upon the factors.

THEOREM 4 (*Tamano* [21]). – *If $X \times \beta X$ is normal, then X is paracompact, where βX is the Stone-Čech compactification of X.*

THEOREM 5 (*Morita* [11]). – *Let m be an infinite power. Then X is normal and m-paracompact (i.e. every open covering consisting of at most m elements is refined by a locally finite open covering) if and only if $X \times I^m$ is normal, where I^m is the product of m copies of I.*

The following are some of problems which are naturally raised.

PROBLEM 1 (Morita). – Let $X \times Y$ be normal and X compact. Let Z be the image of Y under a closed mapping f. Is $X \times Z$ normal ? Catch the nice property of $l_X \times f$, where l_X is the identity transformation of X to X, which may assure the normality of $X \times Z$.

PROBLEM 2. – Let $X \times Y$ be normal and Y metric. Let Z be the image of Y under a closed mapping. Is $X \times Z$ normal ?

All of the preceding theorems are concerned with the influence of the normality of $X \times Y$, with Y fixed, upon X. If Y ranges in a class of spaces, say $\mathcal{C}$, the normality of $X \times Y$, $Y \in \mathcal{C}$, will characterize the feature of X. Morita did the characterization for the class of metric spaces using the idea of Morita space. A space is said a Morita space if for each index set Ω and for each open collection $\{G(\alpha_1 \ldots \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega\}$ with

$$G(\alpha_1 \ldots, \alpha_n) \subset G(\alpha_1 \ldots \alpha_{n+1}),\ n = 1, 2, \ldots,$$

there exists a closed collection $\{F(\alpha_1 \ldots \alpha_n)\}$ such that

$$F(\alpha_1 \ldots \alpha_n) \subset G(\alpha_1 \ldots \alpha_n)$$

and such taht $\bigcup_{n=1}^{\infty} G(\alpha_1 \ldots \alpha_n) = X$ implies $\bigcup_{n=1}^{\infty} F(\alpha_1 \ldots \alpha_n) = X$.

THEOREM 6 (*Morita* [12]). – *$X \times Y$ is normal (paracompact) for each metric space Y if and only if X is a normal (paracompact) Morita space.*

A space is said compact-dispersed if every closed set has a point one of whose relative neighborhood is compact. Then the following gives a nice sufficient condition on X assuring the paracompactness of $X \times Y$, where Y ranges in the class of paracompact spaces.

THEOREM 7 (*Telgarsky* [22]). – *Let X be a paracompact space which is the countable sum of closed compact-dispersed spaces. Then $X \times Y$ is paracompact for each paracompact space Y.*

2. – Countably productive class. Let $\mathcal{C}$ be a class of paracompact spaces which is productive. Then by Stone [20], $\mathcal{C}$ has to be a subclass of the class of compact spaces. If we are interested in a class of spaces which are not necessarily compact, it is the best possible for such a class to be countably productive. I think that the discovery of countably productive classes of normal spaces, containing non-compact spaces, is one of the main event in the history of general topology. Frolík [8] is the first to find such a class : The class of paracompact absolute G_δ (i.e. being G_δ in its Stone-Čech compactification) spaces is countably productive. The concept of absolute G_δ space, originally due to Čech [4], is extrinsic. Frolík [7] gave an equivalent intrinsic definition. As a generalization of absolute G_δ space Arhangelskii [1] obtained the concept of p-space along the line of extrinsic definition. Morita [12] also got the concept of M-space as a generalization of absolute G_δ space along the line of intrinsic definition. Both concepts coincide for paracompact spaces.

DEFINITION 1. – A space X is a p-space if there exists a sequence

$$\mathcal{U}_i,\ i = 1, 2, \ldots,$$

of open collections of βX such that for each point x of X the intersection $\cap S(x, \mathfrak{U}_i)$ is in X, where $S(x, \mathfrak{U}_i)$ is the sum of all elements of $\mathfrak{U}_i$ which contain x.

DEFINITION 2. – A space X is an M-space if there exists a normal sequence $\mathfrak{U}_i$, $i = 1, 2, \ldots$, of open coverings of X satisfying the condition : If $K_1 \supset K_2 \supset \ldots$ is a sequence of non-empty closed sets such that $K_i \subset S(x, \mathfrak{U}_i)$ for some fixed point x of X and for each i, then $\cap K_i$ is not empty.

THEOREM 8 (*Arhangelskii* [1] *and Morita* [12]). – *The class of paracompact p-spaces is countably productive.*

Let us list up other countably productive classes of normal spaces obtained during the last ten years.

DEFINITION 3 (Ceder [5] and Borges [3]). – A space X is said a stratifiable (or an M_3) space if each open set U of X, one can assign a sequence U_i, $i = 1, 2, \ldots$, of open sets of X such that $\overline{U_i} \subset U$, $\cup U_i = U$ and $U_i \subset V_i$ whenever $U \subset V$.

THEOREM 9 (*Ceder* [5]. – *Each stratifiable space is paracompact and the product of a sequence of stratifiable spaces is again stratifiable.*

DEFINITION 4 (Arhangelskii [2] and Okuyama [17]). – A collection $\mathfrak{S}$ of sets of X is said a net if for each point x of X and for each neighborhood U of x one can find an element S of $\mathfrak{S}$ with $x \in S \subset U$. If X has a σ-locally finite net, then X is said a σ-space.

THEOREM 10 (Okuyama [16] and [17]). – The product of a sequence of paracompact σ-spaces is again a paracompact σ-space. The intersection of the class of paracompact p-spaces and the class of paracompact σ-spaces is precisely the class of metric spaces.

Quite recently it is reported that Heath proved the following :

THEOREM 11. – *Every stratifiable space is a σ-space.*

DEFINITION 5 (Nagami [13]). – We say that X is a Σ-space provided there exists a sequence $\mathfrak{F}_i$, $i = 1, 2, \ldots$, of locally finite closed coverings of X such that if $K_1 \supset K_2 \supset \ldots$ is a sequence of non-empty closed sets with

$$K_i \subset \cap\{F \in \mathfrak{F}_i : x \in F\}, \quad \text{alors} \quad \cap K_i \neq \emptyset .$$

THEOREM 12 (*Nagami* [13]). – *The product of a sequence of paracompact Σ-spaces is again a paracompact Σ-space. Both every σ-space and every M-space are Σ-spaces.*

We obtain thus the following diagram of implications.

metric space ↗ stratifiable space → σ-space ↘ Σ-space
metric space ↘ M-space ↗ Σ-space

Just recently Michael [10] has gotten an interesting example as follows :

THEOREM 13. – *There exists a space Y such that Y^i is paracompact for $i = 1, 2, \ldots$, but the countable product Y^ω is not normal.*

It is to be noticed that if Y^i is normal for each i and Y^ω is countably paracompact, then Y^ω is also normal by Nagami [14]. We have had no countably productive class of normal spaces containing non-paracompact spaces. So we want to know the answer of the following :

PROBLEM 3. – If X^ω is normal, is X paracompact ?

3. – Perfect class. Consider the following five conditions which may be satisfied by a class of spaces $\mathcal{C}$.

(1) If $X \in \mathcal{C}$, then X is normal.

(2) If $X \in \mathcal{C}$ and $Y \subset X$, then $Y \in \mathcal{C}$.

(3) If $X_i \in \mathcal{C}$, $i = 1, 2, \ldots$, then $\Pi X_i \in \mathcal{C}$.

(4) If $X \in \mathcal{C}$, then there exists an element $Z \in \mathcal{C}$ with $\dim Z \leqslant 0$ such that X is the image of Z under a perfect mapping (i.e. a closed mapping each of whose point-inverse is compact).

(5) If $X \in \mathcal{C}$ and Y is the image of X under a perfect mapping, then $Y \in \mathcal{C}$.

If $\mathcal{C}$ satisfies these five conditions, then $\mathcal{C}$ is said perfect. Indeed $\mathcal{C}$ with the five conditions is worth saying perfect : The classes of metric spaces and of separable metric spaces are perfect and only these two classes are known to be perfect. The following is a natural question in this respect.

PROBLEM 4. – Let X be a paracompact σ-space (a stratifiable space). Then is X the image of a paracompact σ-space (a stratifiable space) Z with $\dim Z \leqslant 0$ under a perfect mapping ?

I think an appropriate perfect class may be a rich ground where we can build harmonious dimension theory. Let us present a perfect class. A space is said σ-metric by Nagami [15] if it is the countable sum of closed metric subsets. A space is said a μ-space if it is a subset of the countable product of paracompact σ-metric spaces. A space is said a ν-space if it is the image of a μ-space under a perfect mapping.

THEOREM 14. – *The class of ν-spaces is perfect.*

PROBLEM 5. – Is every ν-space a μ-space ?

PROBLEM 6. – Find another perfect class containing all metric spaces.

REFERENCES

[1] ARHANGELSKII A. — On a class of spaces containing all metric and all bicompact spaces, *Dokl. Akad. Nauk S.S.S.R.*, 151, 1963, p. 751-754.

[2] ARHANGELSKII A. — On closed mappings, bicompact spaces, and a problem of P. Alexandroff, *Pacific J. Math.*, 18, 1966, p. 201-208.

[3] BORGES C. — On stratifiable spaces, *Pacific J. Math.*, 17, 1966, p. 1-16.

[4] ČECH E. — On bicompact spaces, *Ann. Math.*, 38, 1937, p. 823-844.

[5] CEDER J. — Some generalizations of metric spaces, *Pacific J. Math.*, 11, 1961, p. 105-126.
[6] DOWKER C.H. — On countably paracompact spaces, *Canad. J. Math.*, 1, 1951, p. 219-224.
[7] FROLIK Z. — Generalizations of the Gδ property of complete metric spaces, *Čech. Math. J.*, 10, 1960, p. 359-378.
[8] FROLIK Z. — On the topological product of paracompact spaces, *Bull. Acad. Polon. Sci.*, 8, 1960, p. 747-750.
[9] MICHAEL E. — The product of a normal space and a metric space need not be normal, *Bull. Amer. Math. Soc.*, 69, 1963, p. 375-376.
[10] MICHAEL E. — *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, forthcoming.
[11] MORITA K. — Paracompactness and product spaces, *Fund. Math.*, 50, 1962, p. 223-236.
[12] MORITA K. — Products of normal spaces with metric spaces, *Math. Ann.*, 154, 1964, p. 365-382.
[13] NAGAMI K. — Σ-spaces, *Fund. Math.*, 65, 1969, p. 169-192.
[14] NAGAMI K. — *Countable paracompactness of inverse limits and products*, forthcoming.
[15] NAGAMI K. — *Dimension for σ-metric spaces*, forthcoming.
[16] OKUYAMA A. — On metrizability of M-spaces, *Proc. Japan Acad.*, 40, 1964, p. 176-179.
[17] OKUYAMA A. — Some generalizations of metric spaces, their metrization theorems and product spaces, *Sci. Rep. Tokyo Kyoiku Daigaku* sect. A, 9 1967, p. 236-254.
[18] RUDIN Mary. — Countable paracompactness and Souslin's problem, *Canad. J. Math.*, 7, 1955, p. 543-547.
[19] SOUSLIN M.M. — Problem 3, *Fund. Math.*, 1, 1920, p. 223.
[20] STONE A.H. — Paracompactness and product spaces, *Bull. Amer. Math. Soc.*, 54, 1948, p. 977-982.
[21] TAMANO H. — On paracompactness, *Pacific J. Math.*, 10, 1960, p. 1043-1047.
[22] TELGARSKY R. — On the product of paracompact spaces, *Bull. Acad. Polon. Sci.*, 17, 1969, p. 533-536.

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ANALOGUES HERMITIENS DE LA K-THÉORIE

par S.P. NOVIKOV

1 – Le développement rapide de l'algèbre stable, ces dernières années, est un fait bien connu. Ce développement est dû, surtout, à trois facteurs : aux succès de la K-théorie en théorie de l'homotopie (et à ses applications), à la brillante application de la K-théorie algébrique (pure) à la théorie des variétés non simplement connexes, et, enfin, aux liens avec l'algèbre et la théorie des nombres. Les notions fondamentales de la K-théorie algébrique sont les groupes $K^0(A)$ et $K^1(A)$ pour un anneau A, leurs propriétés, et leurs extensions $K^i(A)$ $(i \geqslant 2)$.

On ne pourra pas donner ici une présentation de ce vaste thème ; son histoire et ses résultats sont l'œuvre de beaucoup de mathématiciens remarquables appartenant à des domaines différents.

Néanmoins, je vais attirer l'attention sur une lacune de la K-théorie algébrique dans son état actuel : il n'y a pas d'analogue algébrique de la théorie des classes caractéristiques (Pontryaguine et Chern) qui sont l'un des objets importants de la K-théorie habituelle (topologique). Ceci n'est pas un hasard. Je ne connais aucun exemple de problème naturel, ou de théorème, qui fasse appel à un foncteur du type Chern-Pontryaguine, défini sur le groupe de Grothendieck $K^0(A)$, ou sur les groupes de Dieudonné-Whitehead $K^1(A)$ ou sur ceux de Milnor $K^2(A)$.

Une autre lacune, moins évidente, est l'absence d'un analogue algébrique de la périodicité de Bett. C'est une lacune assez compréhensible, vu que, déjà dans le cas topologique, la périodicité n'est pas une conséquence des propriétés homotopiques générales de la K-théorie, mais un théorème difficile qu'on démontre après, séparément.

2 – On va s'occuper maintenant de l'aspect algébrique des problèmes de classification en topologie (différentielle ou P.L.), liés à la technique de chirurgie (surtout dans le cas non simplement connexe).

Déjà, dans les années 1965-66, on avait remarqué (Novikov [3], [7], Wall [5]) que le formalisme général des obstructions pour la chirurgie, dans le cas des dimensions paires, conduit à des analogues du groupe $K^0(A)$, construits avec des formes hermitiennes ou hermitiennes-gauches (skew-hermitian) sur des modules libres (ou projectifs) avec la forme

$$\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$$

comme objet trivial ; ces formes sont à valeurs dans l'anneau à involution $A = Z[\pi]$. Les groupes obtenus sont respectivement $K_H^0(A)$ et $K_{SH}^0(A)$. L'involution sur A est liée à "l'orientation" $\pi \to Z_2$, nous ne mentionnons pas l'invariant d'Arf ; le groupe K_H^0 apparait dans le cas des dimensions $4k$ et K_{SH}^0 apparait dans le cas des dimensions $4k + 2$. Essentiellement, K_H^0 et K_{SH}^0 sont des objets très classiques (les classes stables de formes quadratiques).

Une belle découverte géométrique (qui est justifiée a posteriori, de manière naturelle, du point de vue algébrique) est faite par Wall [6] en 1968 : la théorie de la chirurgie pour les variétés de dimension impaire conduit, pour $4k + 1$ (resp. pour $4k + 3$) à des groupes du type K_H^1, K_{SH}^1, construits pour $A = Z[\pi]$, à partir d'automorphismes qui conservent un certain produit scalaire (il est nécessaire de multiplier A par $Z\left[\frac{1}{2}\right]$ puisqu'on ne parle pas de l'invariant de Arf).

On savait déjà qu'en général un problème de chirurgie se présente plus ou moins de la même façon pour toutes les dimensions congruentes mod. 4. Serait-ce ici un analogue de la périodicité de Bett ? Dans les problèmes de chirurgie, il y a un mélange de concepts algébriques de deux types différents : les objets "hermitiens" K_H^0, K_H^1 et les objets "hermitiens –gauches" K_{SH}^0, K_{SH}^1. D'autre part, ces objets sont définis, pas directement sur les variétés, mais à partir de $A = Z[\pi]$. On devrait se demander s'il existe un formalisme algébrique, dans le cadre des analogues hermitiens de la K-théorie, utilisant la suite des dimensions (croissantes), et qui donnerait une relation entre K_{SH}^0 et l'objet hermitien "K_H^2" qui reste à construire. La collection K_H^i ($i = 0, 1, 2, 3$) devrait être une "théorie homologique". On devrait au moins pouvoir faire ce qui suit. On s'intéresse à $A = Z[\pi]$ et "l'extension de Laurent" $A \to A[Z, Z^{-1}]$ conserve la classe des algèbres de groupe $Z[\pi] \to Z[\pi \times Z]$; on devrait donc construire un opérateur de Bass, reliant $K_{SH}^{i+1}(A[Z, Z^{-1}])$ et $K_{SH}^i(A)$, $K_H^{i+1}(A[Z, Z^{-1}]) \rightleftharpoons K_H^i(A)$, dans un certain cadre algébrique, en particulier pour $i = 1$, où $K_H^2(A)$ devrait coincider essentiellement avec $K_{SH}^0(A)$ (comme le montre la théorie de la chirurgie).

Notons tout de suite qu'un énoncé sur un "théorème d'existence" non effectif pour un opérateur du type Bass peut être extrait des travaux de Browder (1966, $\pi_1 = Z$) et de Shaneson (1968, $\pi_1 = G \times Z$), quoiqu'il n'y ait pas de constructions algébriques dans ces travaux ni de définition algébrique de l'obstruction à la chirurgie non simplement connexe.

Dans le travail récent [4], l'auteur a étudié le problème de la construction algébrique d'un analogue hermitien à la K-théorie, sur un anneau avec involution A, du point de vue de ce qu'on appelle le formalisme hamiltonien. En parlant de façon plutôt vague (puisque, par exemple, dans [4] on introduit plusieurs K-théories hermitiennes U^*, V^*, W^*), l'auteur a réussi à construire l'opérateur de Bass dans le cas crucial

$$K_{SH}^0(A[Z, Z^{-1}]) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} K_H^1(A)$$

et

$$K_H^0(A[Z, Z^{-1}]) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} K_{SH}^1(A),$$

où

$$K_H^2 \cong K_{SH}^0, \quad K_{SH}^2 \cong K_H^0,$$

C'est plus facile de comprendre la construction de l'"opérateur de Bass"

$$K^1_{SH}(A[Z,Z^{-1}]) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} K^0_{SH}(A)$$

$$K^1_H(A[Z,Z^{-1}]) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} K^0_H(A),$$

par analogie avec la construction classique (de Bass). Par contre, en ce qui concerne les opérateurs

$$K^0_H(A[Z,Z^{-1}]) \to K^1_{SH}(A)$$

et

$$K^0_H(A[Z,Z^{-1}]) \to K^1_{SH}(A),$$

leur construction est faite de manière algébrique et elle est difficile à deviner par des considérations de topologie différentielle. Toujours par voie algébrique, on montre que si $K^0(A[Z,Z^{-1}]) = K^0(A)$, alors on a

$$K^0_H(A[Z,Z^{-1}]) = K^0_H(A) + \bar{B}K^1_{SH}(A)$$

et

$$K^0_{SH}(A[Z,Z^{-1}]) = K^0_{SH}(A) + \bar{B}K^1_H(A)$$

C'est montré seulement pour l'une des K-théories hermitiennes de [4] ; mais pour nos autres K-théories hermitiennes, on peut définir des homomorphismes naturels (qui "changent la symétrie") $K^2_H \to K^0_{SH}$, $K^2_{SH} \to K^0_H$. Cela permet d'affirmer que, du point de vue algébrique, dans la catégorie des modules avec produit scalaire hermitien, le rôle des "éléments de K^2_H" est joué par les modules avec produit scalaire hermitien-gauche. En fait, tous les résultats de [4] sont dans la théorie $\otimes Z\left[\frac{1}{2}\right]$. Si on ne multiplie pas (tensoriellement) par $Z\left[\frac{1}{2}\right]$, le formalisme hamiltonien de l'auteur doit être remanié d'une manière assez délicate.

Encore une remarque. Quand on a plusieurs variables $z_1, \ldots, z_k \in \pi_1 = G \times Z^k$, la composition des opérateurs de Bass

$$B(z_1)\cdot\ldots B(z_k)$$

dépend seulement du produit extérieur $z_1^* \wedge \ldots \wedge z_k^* \in \Lambda^k \pi_1^*$. Il en résulte, en particulier, une construction algébrique de l'"homomorphisme de la haute signature" :

$$\sigma = \sum \sigma_q : K^i_H(A) \to \sum_q \Lambda^{i-4q} \pi_1^{**} = \sum H_{i-4q}(\pi_1^{**}),\ \pi^* = \mathrm{Hom}_Z(\pi, Z),$$

où $\pi_1^{**} = Z^k$ (abélien libre) et

$$\Lambda^* \pi_1^{**} = H^*(\pi_1^{**}, Q),$$

Q ensemble des rationnels. Il s'agit du fait que, ici, $\pi_1^{**} = \pi_1 = Z^k$ et l'homomorphisme $\sigma : K^{4q}_H \to Z$ induit un produit scalaire symétrique sur $M \otimes_A R$ (où $A = Z[\pi]$). Par définition

$$< \sigma_q(x), z^*_{j_1} \wedge \ldots \wedge z^*_{j_{l-4q}} > = \sigma \cdot B(z^*_{j_1} \wedge \ldots \wedge z^*_{j_{l-4q}})\,[x].$$

L'existence de cet "homomorphisme des hautes signatures" (pour i pair) a été établie par l'auteur dans [3], [7] tandis que le fait que σ est un isomorphisme $\left(\text{quand on tensorise par } Z\left[\frac{1}{2}\right]\right)$ est démontré, dans un autre langage, par Shaneson [8]. Tout ceci étant fait comme un théorème non effectif "d'existence et unicité" sans constructions algébriques.

4 – Ayant montré le formalisme des théories hermitiennes et ayant construit l'opérateur de Bass, on peut dire que le problème d'un analogue de la périodicité de Bett a un sens pour les K-théories hermitiennes sur un anneau à involution.

Essayons de comprendre la relation de la périodicité hermitienne avec celle de la K-théorie habituelle $K(X)$. Dans [1], Gelfand et Mischenko ont montré que, dans l'anneau des fonctions complexes $A = C(X)$, les groupes $K^0_H(A) = K^0_{SH}(A)$ (on a $i = \sqrt{-1} \in A$) sont canoniquement isomorphes aux groupes habituels $K^0(X)$. Ils ont montré aussi que, pour π commutatif, les groupes $K^0_H(A)$ et $K^0_{SH}(A)$ se ramènent, par le passage $X = \text{char } \pi$, au foncteur $K(X)$ habituel (on plonge $A \to C(X)$ où $X = \text{char } \pi$, ramenant ainsi $K^0_H(A)$ à $K(X)$).

Voici quelques résultats simples et importants qui ne sont pas mentionnés dans [1] :

a) En appliquant l'homomorphisme de Gelfand-Mischenko : $K^0_H \to K(X)$ et le caractère de Chern ch : $K(X) \to H^*(X, Q)$, on obtient, précisément l'homomorphisme des "hautes signatures" pour $X = \text{char } \pi$, $\pi = Z^k$. Ceci m'a été communiqué par Mischenko.

b) Il est utile de remarquer que $K^0_H(A) = K^0(X)$, $A = Z[\pi]$ $\left(\text{modulo} \otimes Z\left[\frac{1}{2}\right]\right)$.
Dans la théorie habituelle, on a $\widetilde{K}^0(A) = 0$, $\widetilde{K}^0(X) \neq 0$. Ici

$$A = Z[\pi],\ X = \text{char } \pi = T^k.$$

Tout ce qu'on vient de dire sur $K^0_H(A)$ et $K^0(x)$ (pour $A = C(X)$) reste vrai quand on passe à $K^1_H(A) = K^1(X)$.

Dans le cas réel, $A = R(X)$, on a aussi l'égalité $K^*_H(A) = KO^*(X)$ module $\otimes Z\left[\frac{1}{2}\right]$.

Ainsi, l'examen des anneaux de fonctions montre que les théories hermitiennes K^*_H se présentent comme une autre forme de la K-théorie classique où, dans une certaine mesure, on a la périodicité de Bett.

Il est difficile de juger ici dans quelle mesure l'auteur a réussi (ou pas) avec son formalisme hamiltonien, et on renvoie à [4] pour plus d'information. Dans le même travail, on montre les relations entre les constructions algébriques et la topologie différentielle, ainsi que les notions d'analyse qui entrent en jeu.

Remarquons que la bonne présentation des notions fondamentales du formalisme hamiltonien (la classe des variétés lagrangiennes, les particularités de la projection sur X, l'index de Maslov et ses relations avec la théorie de Morse, le rôle du hessien dans la théorie lagrangienne) a été donnée pour la première fois par Maslev [9] ; elle a beaucoup influencé l'auteur.

5 – Revenons au problème des classes caractéristiques. Existe-t-il un foncteur du type "caractère de Chern" pour les anneaux à involution A, défini dans $K_H^*(A)$ ou $K_{SH}^*(A)$? (et où prendrait-il ses valeurs ?). Dans quels problèmes serait-il nécessaire ?

Pour l'anneau $A' = C(X)$ ou $A'' = R(X)$, il existe un tel "caractère de Chern-Pontryaguine" vu les isomorphismes

$$K_H^*(A') = K_{SH}^*(A') = K(X)$$

$$K_H^*(A'') \overset{1/2}{=} KO^*(X)$$

et prend ses valeurs dans $H^*(X, Q)$. De même ch : $K_H^*(A) \to H^*(X, Q)$ pour $A = C(X)$.

Pour l'anneau de groupe $A = Z[\pi]$ (où $\pi = Z^k$), on a l'homomorphisme des hautes signatures

$$\sigma : K_H^*(A) \to H_*(\pi, Q),$$

qui, comme l'a montré Mischenko, devient D. ch quand on passe à $K(X)$, pour $X = \text{char } \pi = T^k$.

Hypothèse : Il existe un homomorphisme généralisé des signatures

$$\sigma : K_H^*(A) \to H_*(\pi, Q),$$

pour tout $A = Z[\pi]$, où π est un groupe de représentation finie (l'auteur ne sait pas s'il faut se limiter au cas où l'homologie est de type fini ou même, peut-être, au cas où $K(\pi, 1)$ est une variété compacte).

Une telle construction introduirait les classes caractéristiques en algèbre.

Pour certains π (par exemple les groupes abéliens libres), un tel "caractère de Chern-Pontryaguine" existe et peut être construit par voie algébrique. Il est facile de faire la même construction pour les groupes fondamentaux des nil-variétés ou des solv-variétés. Néanmoins le formalisme général d'une telle construction n'apparaît pas clairement à l'auteur.

Un tel "homomorphisme de la signature généralisée" σ, ou un "caractère de Chern-Pontryaguine" ch pour $A = Z[\pi]$, jouent un rôle fondamental dans la topologie des variétés non simplement connexes. Pour Z^k, l'auteur l'avait déjà rencontré, pour la première fois, en 1965-66. Un problème d'actualité, comme celui des "formules de Hirzebruch non-simplement connexes" ou de la classification des invariants homotopiques issus des classes de Pontryaguine, peut se formuler comme suit : Si $L = \sum_k L_k(p_1, \ldots, p_k)$ est un polynôme de Hirzebruch, M^n une variété fermée de groupe fondamental π_1, on définit l'homomorphisme naturel

$$H^*(\pi_1, Q) \overset{\varphi}{\to} H^*(M^n, Q),$$

et la forme linéaire $(DL(M^n), x)$, $x = \varphi(y)$, sur $H^*(\pi_1, Q)$; c'est un élément de $H_*(\pi_1, Q)$ désigné par $\langle L, M^n \rangle$. Cette quantité $\langle L, M^n \rangle$ est-elle un invariant homotopique ? comment peut-on la calculer ?

Récemment, Mischenko [2] a trouvé la construction d'un invariant homotopique qui associe à M^n un élément $\tau(M^n)$ de $K^n_H(A)$ $(A = Z[\pi])$, déterminé modulo $\otimes Z\left[\frac{1}{2}\right]$. Cette construction définit une représentation de la théorie du SO-bordisme (comme pour $\pi = 1$) :

$$\Omega_*(\pi_1) \underset{\tau}{\to} K^*_H(A) \otimes Z\left[\frac{1}{2}\right].$$

Si l'homomorphisme

$$\sigma : K^*_H(A) \to H_*(\pi, Q)$$

existait, alors la "formule de Hirzebruch non simplement connexe" serait

$$\sigma \cdot \tau[M^n] = \langle L, M^n \rangle.$$

Pour $\pi = Z^k$, une telle formule a été établie par l'auteur.

Notons que, par le passage de $A = Z[\pi]$ à $A = R(X)$, l'homomorphisme r de Mischenko s'identifie à la transformation bien connue de Riemann-Roch :

$$\Omega^*(X) \overset{1/2}{\to} KO^*(X),$$

construite à partir de l'isomorphisme spécial de Thom en $KO^* \otimes Z\left[\frac{1}{2}\right]$ - théorie, lié au L-genre (ou à un élément de $KO^*(MSO)$). On sait aussi que σ devient ch. La seule existence de l'homomorphisme

$$\sigma : K^*_H(A) \to H_*(\pi, Q)$$

implique déjà que

(1) Si $K(\pi, 1)$ est une variété compacte, toutes les classes caractéristiques sont formellement définie par π.

(2) Si σ est monomorphe (modulo des groupes finis, le nombre des variétés compactes (lisses ou P.L.) du type d'homotopie de $K(\pi, 1)$ est fini.

(3) Si la structure de la cohomologie $H^*(M^n, Q)$ est telle que toutes les classes de Pontryaguine rationnelles se calculent par la "formule de Hirzebruch" non simplement connexe, et si

$$H^{n-4k-1}(\pi, Q) \to H^*(M^n, Q)$$

est un monomorphisme, alors il n'y a qu'un nombre fini de variétés du même type d'homotopie que M^n (en suppose que $\sigma \otimes Q$ est injectif).

Pour $\pi = Z^k$ les classes caractéristiques sont toujours des invariants d'homotopie (Novikov [3], [7]). et le théorème de finitude a été démontré en 1969 par Wall et Hsiang-Shaneson (à paraître). Dans [7], l'auteur a montré que tous ces résultats sont des exemples d'un analogue non-simplement connexe de la formule de Hirzebruch. Pour $\pi = 1$, le théorème de finitude correspondant (quand $b_{4k} = 0$, $0 < 4k < n$) était déjà un exemple connu.

En conclusion, toute une série de problèmes sur les classes caractéristiques des variétés différentiables conduisent à la nécessité de la construction d'un analogue du caractère de Chern (la haute signature) pour les K-théories hermitiennes sur les anneaux de groupes.

ЛИТЕРАТУРА

[1] Гельфанд, Мищенко. — Функ. Анализ, 1969, т. 3, № 4.
[2] Мищенко. — Известия АН, 1970, т. 34, № 3.
[3] Новиков. — Известия АН, 1966, т. 30, № 1.
[4] Новиков. — Известия АН, 1970, т. 34, № 2, 3.
[5] Wall. — *Ann. of Math.*, 1966, t. 82, p. 217-276.
[6] Wall. — *Surgery of compact manifolds*, 1968.
[7] Novikov. — *Ess. on Top. and Rel. Topics* (Memoires dediés a Georges de Rham),
[8] Shaneson. — *Bull of A.M.S.*, 1968, 74, No. 3.
[9] Маслов. — 1965, Москва, Изд-во МГУ.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 47 à 51.

COHOMOLOGY OF GROUPS

by Daniel QUILLEN *

This is a report of research done at the Institute for Advanced Study the past year. It includes some general results on the structure of the ring $H^*(BG, \mathbf{Z}/p\mathbf{Z})$ when G is a compact Lie group, a theorem computing this ring for a large number of interesting finite groups, and applications to algebraic K-theory consisting of a definition of K-groups $K_i A$ for $i \geqslant 0$ agreeing with those of Bass and Milnor and their computation when A is a finite field.

1. The spectrum of $H^*(BG, \mathbf{Z}/p\mathbf{Z})$.

Let G be a compact Lie group (e.g. a finite group) and let $H^*(BG)$ be the cohomology ring of its classifying space with coefficients in $\mathbf{Z}/p\mathbf{Z}$ where p is a fixed prime number. According to Venkov (and Evens for finite G) the ring $H^*(BG)$ is finitely-generated, hence its Poincaré series $\Sigma\ (\dim_{\mathbf{Z}/p\mathbf{Z}} H^n(BG))\, t^n$ is a rational function of t and one may define the dimension $\dim H^*(BG)$ to be the order of the pole of this function at $t = 1$. For example if $A = (\mathbf{Z}/p\mathbf{Z})^r$ is an elementary abelian p-group ($[p]$-group for short) of rank r, then

$$\dim H^*(BA) = r .$$

The following for finite G has been conjectured independently by Atiyah and Swan.

PROPOSITION 1. – $\dim H^*(BG) =$ *the maximum rank of a* $[p]$*-subgroup of* G.

To prove this one follows the method used by Atiyah-Segal to prove the completion theorem in equivariant K-theory and first generalizes it to G-spaces, which for the sake of simplicity I suppose to be smooth compact G-manifolds with boundary. Let X_G be the associated fibre space over BG with fibre X and set $H_G^*(X) = H^*(X_G)$.

PROPOSITION 1′ – $\dim H_G^*(X) =$ *the maximum rank of a* $[p]$*-subgroup of* G *fixing some point of* X.

To prove this one can replace the pair (G, X) by (U, Y) where U is a unitary group containing G and $Y = U \times {}^G X$; then one can reduce to the case (A, Y) where A is the subgroup of elements of order p in a maximal torus of U, because $H_A^*(Y)$ is a finitely-generated free $H_U^*(Y)$-module. Hence one can

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suppose that G is a $[p]$-group, in which case the result can be checked by using the spectral sequence

$$E_2^{st} = H^s(X/G,\ Gx \to H_G^t(Gx)) \Rightarrow H_G^{s+t}(X).$$

The same technique can be used to prove the following result.

THEOREM – *Consider the $[p]$-subgroups A of G as the objects of a category in which a morphism from A to A' is a component of the set of g such that $gAg^{-1} \subset A'$, and let*

$$u : H^*(BG) \to \varprojlim H^*(BA)$$

be the homorphism induced by restriction. Then every element of Ker (u) *is nilpotent and if z is an element of the inverse limit then $z^{p^n} \in$* Im (u) *for large n.*

In other words 'up to extraction of p-th roots' a cohomology class of BG is the same as a family of cohomology classes for each $[p]$-subgroup compatible with conjugation and restriction. One should compare this result with Brauer's theorem asserting that the analogous map with character rings and the category of elementary subgroups is an isomorphism when G is finite.

This theorem and some commutative algebra permit one to deduce the following description of the space Spec $H^*(BG)$ of prime ideals in $H^*(BG)$ (i.e. inverse images of prime ideals in the commutative ring $H^*(BG)_{\mathrm{red}} = H^*(BG)/$ideal of nilpotent elements). If A is a $[p]$-subgroup of G, let

$$\mathfrak{p}_A = \mathrm{Ker}\,\{H^*(BG) \to H^*(BA)_{\mathrm{red}}\}.$$

Then $A \to \mathfrak{p}_A$ gives an order-reversing bijection between conjugacy classes of $[p]$-subgroups and those homogeneous prime ideals of $H^*(BG)$ which are closed under the Steenrod operations. In particular the irreducible components of Spec $H^*(BG)$ are in one-one correspondence with maximal $[p]$-subgroups up to conjugacy. If T_A is the s[illegible]set of prime ideals containing $\mathfrak{p}_A$ but not $\mathfrak{p}_{A'}$ for $A' < A$, then there is a s[illegible]tification

$$\mathrm{Spec}\, H^*(BG) = \coprod T_A$$

into irreducible locally closed subspaces indexed by the conjugacy classes of $[p]$-subgroups. Moreover

$$T_A = (\mathrm{Spec}\, S(A^\vee)\,[e_A^{-1}])/N(A)$$

where $N(A)$ is the finite group of components of the normalizer of A in G, where $S(A^\vee) = H^*(BA)_{\mathrm{red}}$ is the symmetric algebra of the dual of A over $\mathbf{Z}/p\mathbf{Z}$, and e_A is the product of the non-zero elements of A.

2. Computations using etale cohomology and the Lang isomorphism.

One knows (Chevalley, Steinberg) that a large number of interesting finite groups occur as the group G^σ of fixpoints of an endomorphism σ of a connected algebraic group G defined over an algebraically closed field k. For example if G is defined over a finite subfied k_0 of k then the group of rational points $G(k_0)$ is the group of fixpoints of the Frobenius endomorphism associated to this finite field of definition. Since G^σ is finite there is an inseparable isogeny

$$G/G^\sigma \to G$$
$$gG^\sigma \to g(\sigma g)^{-1}$$

(the Lang isomorphism when σ is a Frobenius endomorphism), hence G/G^σ and G are homeomorphic for the etale topology. This suggests that $H^*(BG^\sigma)$ (coefficients in $\mathbf{Z}/l\mathbf{Z}$ where l is a prime number different from the characteristic of k) might be computed by using the analogue in etale cohomology of the Leray spectral sequence of the "fibration" $(G/G^\sigma, BG^\sigma, BG)$, because the rings $H^*(BG)$ and $H^*(G)$ are usually known, e.g. by lifting G to characteristic zero.

Before going on I should explain what is meant by BG in this context. Let $\mathcal{C}$ be the topos of sheaves for the etale topology on the category of all algebraic k-schemes. Identifying a k-scheme with the sheaf it represents, G becomes a group object of $\mathcal{C}$ and so it has a "classifying topos" $\mathcal{C}_G$ consisting of objects of $\mathcal{C}$ endowed with G-action (Grothendieck, reedition of *SGAA*). If X is a k-scheme endowed with a G-action, let X_G be the object of $\mathcal{C}_G$ it gives rise to, and denote by $H_G^*(X)$ the cohomology of X_G with coefficients in the constant sheaf $\mathbf{Z}/l\mathbf{Z}$; write BG instead of e_G where $e = \operatorname{Spec} k$. The Leray spectral sequence for the map $X_G \to BG$, or as I shall say of the fibration (X, X_G, BG) takes the form

(1) $$E_2 = H^*(BG) \otimes H^*(X) \Rightarrow H_G^*(X)$$

provided the map $X \to e$ is cohomologically proper, which is the case for $X = G$ because the map factors into a sequence of principal $\mathbf{G}_a$ and $\mathbf{G}_m$ bundles and the proper map $G/B \to e$.

Taking X to be G acting on itself by left translations gives a spectral sequence

(2) $$E_2 = H^*(BG) \otimes H^*(G) \Rightarrow H^*(e)\,.$$

Assume that this spectral sequence has the nice form studied by Borel in his thesis, namely $H^*(G)$ has a simple system of transgressive generators, whence $H^*(BG) = S(V)$ is a polynomial ring and the transgression sets up an isomorphism of the primitive subspace P of $H^*(G)$ and $V[-1]$ (the $[-1]$ means degrees are shifted down by one). When $X = G^t$, the G-scheme obtained by letting G act on itself by the rule $g(g_1) = gg_1(\sigma g)^{-1}$, (1) takes the form

(3) $$E_2 = H^*(BG) \otimes H_G^*(G) \Rightarrow H^*(G/G^\sigma) = H^*(BG^\sigma)\,.$$

on account of the Lang isomorphism. To determine the differentials in (3), let G^s be the $(G \times G)$-scheme obtained by letting $G \times G$ act on G by the rule $(g_1, g_2) = g_1 g g_2^{-1}$ and consider the map of spectral sequences associated to the map $(G, (G^t)_G, BG) \to (G, (G^s)_{G\times G}, B(G \times G))$. In the latter spectral sequence a primitive element z of $H^*(G)$ transgresses to $\nu \otimes 1 - 1 \otimes \nu$ if z transgresses to ν in (2), consequently in (3) z transgresses to $\nu - \sigma^*(\nu)$. Thus the spectral sequence (3) can be determined completely and it yields the following.

THEOREM – *Let G be a connected algebraic group defined over an algebraically closed field k, and let σ be an endomorphism of G such that G^σ is finite. Assume that the etale cohomology $H^*(G)$* (coefficients in $\mathbf{Z}/\ell\mathbf{Z}$, *$\ell$ prime* $\neq$ char (k)) *has a simple system of transgressive generators for the spectral sequence* (2) (*e.g. if $H^*(G)$ is an exterior algebra with odd degree generators*), *and that the* *subspace V of generators for the polynomial ring $H^*(BG)$ can be chosen so as to be stable under σ^*. Let V^σ and V_σ be the kernel and cokernel of the endomorphism* id $- \sigma^*$ *of V. Then there is an isomorphism of graded $\mathbf{Z}/\ell\mathbf{Z}$ – vector spaces*

$$H^*(BG^\sigma) = S(V_\sigma) \otimes \Lambda\ (V^\sigma[-1])$$

which is an algebra isomorphism if l is odd.

This theorem may be used to determine the mod ℓ cohomology rings of the classical groups over a finite field k_0 (at least additively when $\ell = 2$) provided ℓ is different from the characteristic. For example if k_0 has q elements and r is the order of q mod ℓ, then $H^*(BGL_n(k_0), \mathbf{Z}/\ell\mathbf{Z})$ is the tensor product of a polynomial ring with one generaror of degree $2jr$ and an exterior algebra with one generator of degree $2jr - 1$ for $l \leqslant j \leqslant [n/r]$, except that this is only true additively when $\ell = 2$.

3. Applications to algebraic K-theory

Let A be a ring, $GL(A)$ its infinite general linear group, and $E(A)$ the subgroup generated by elementary matrices. As $E(A)$ is perfect, by attaching 2- and 3-cells to $BE(A)$ to kill its kundamental group without changing homology one can construct a map $f : BGL(A) \to BGL(A)^+$ such that $\pi_1(f)$ kills $E(A)$ and such that f as a map in the homotopy category of pointed spaces is universal with this property. Set $K_i A = \pi_i BGL(A)^+$ for $i \geqslant 1$; it is not hard to show that this definition agrees with those of Bass and Milnor.

The representable functor on the homotopy category

$$K(X\ ; A) = [X\, , K_0 A \times BGL(A)^+]$$

deserves to be called K-theory with coefficients in A, because it enjoys many of the properties of topological K-theory. For example it is the degree zero part of a connected generalized cohomology theory. Indeed Graeme Segal has recently associated such a cohomology theory to any category with a coherent commutative associative composition law, and the cohomology theory in question come from the additive category of finitely-generated projective A-modules . Also $K(X\ ; A)$ is naturally a λ-ring when A is commutative.

If one wants to compute the groups $K_i A$ by standard techniques of homotopy theory (e.g. unstable Adams spectral sequence for the H-space $BGL(A)^+$) it is necessary to know the homology of $BGL(A)^+$, which is the same as that of $BGL(A)$. For example $K_i A \otimes \mathbf{Q}$ is isomorphic to the primitive subspace of $H_i(BGL(A), Q)$. For a finite field enough is known about the homology to do the computation :

Let k_0 be a finite field with q elements, let k be an algebraic closure of k_0, and let $\phi : k^* \to C^*$ be an embedding. By modular character theory one knows how to associate to a representation of a group G over k_0 a virtual complex representation fixed under the Adams operation Ψ^q by using ϕ to lift eigenvalues. Lifting the standard representation of $GL_n(k_0)$ on k_0^n, one obtains a map $BGL_n(k_0) \to E\Psi^q$, where the latter space is the fibre of the endomorphism $\Psi^q -$ id of BU. This map kills elementary matrices, hence gives rise to a map in the pointed homotopy category

$$BGL(k_0)^+ \to E\Psi^q$$

which depends only on the choice of ϕ.

THEOREM 1. — *The map (*) is a homotopy equivalence.*

This is proved by showing that the map induces an isomorphism on homology and using the Whitehead theorem. The homotopy groups of $E\Psi^q$ may be computed by using Bott periodicity, so one obtains the formulas

$$K_{2i}(k_0) = 0 \qquad i \geqslant 1$$
$$K_{2i-1}(k_0) \cong \mathbf{Z}/(q^i - 1)\,\mathbf{Z} \qquad i \geqslant 1 \quad .$$

The functorial behavior of these groups as the finite field varies may be determined in similar fashion, and it leads to the following :

THEOREM 2. — *If k is an algebraic closure of $\mathbf{F}_p$, then*

$$K_{2i}(k) = 0 \qquad i \geqslant 1$$
$$K_{2i-1}(k) \cong \bigoplus_{\ell \neq p} \mathbf{Q}_\ell/\mathbf{Z}_\ell \qquad i \geqslant 1$$

and the Frobenius automorphism of k over $\mathbf{F}_p$ acts on $K_{2i-1}(k)$ by multiplying by p^i. If k_1 is any subfield of k, then the extension-of-scalars homomorphism induces an isomorphism

$$K_{2i-1}(k_1) \cong K_{2i-1}(k)^{\mathrm{Gal}(k/k_1)}$$

in terms of which the restriction-of-scalars homomorphism

$$u_* : K_{2i-1}(k_2) \to K_{2i-1}(k_1)$$

associated to a finite extension $u : k_1 \to k_2$ is given by the norm from $\mathrm{Gal}(k/k_2)$-*invariants to* $\mathrm{Gal}(k/k_1)$-*invariants.*

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ON A CERTAIN CLASS OF EQUIVARIANT MAPS

by Helmut RÖHRL (*)

Let M be a topological monoid and denote by $\underline{\mathrm{Top}}_M$ the category of left M-spaces and continuous M-equivariant maps. For a given M-space B, we denote by $\underline{\mathrm{Top}}_M \mid B$ the category of M-spaces over B. Both $\underline{\mathrm{Top}}_M$ and $\underline{\mathrm{Top}}_M \mid B$ are complete as well as cocomplete. Given the object $\mathcal{C}$ in $\underline{\mathrm{Top}}_M \mid B$ we denote by $\Gamma(B, \mathcal{C})$ the set of global sections of $\mathcal{C}$. Such a section s is called an equivariant section if it is an equivariant map. The set of global equivariant sections of $\mathcal{C}$ is denoted by $\Gamma(B, \mathcal{C})^M$. The corresponding functors $\Gamma(B, \ )$ and $\Gamma(B, \ )^M$ both possess a coadjoint.

Let $\varphi : M_1 \to M_2$ be a morphism of topological monoids, i.e. a continuous homomorphism that preserves the neutral element. Then there is an obvious functor $\varphi_* : \underline{\mathrm{Top}}_{M_2} \to \underline{\mathrm{Top}}_{M_1}$ which describes the "restriction of operators by φ". φ_* has as a coadjoint the functor φ^* which describes the "extension of operators by φ". φ_* gives rise to a functor $\varphi_* \mid B_2 : \underline{\mathrm{Top}}_{M_2} \mid B_2 \to \underline{\mathrm{Top}}_{M_1} \mid \varphi_* B_2$. Denoting the counit $\varphi^* \varphi^* \to \underline{\mathrm{Top}}_{M_2}$ that is associated with the above adjunction by β, one sees that $\varphi_* \mid B_2$ possesses the functor $\underline{\mathrm{Top}}_{M_2} \mid \beta_{B_2} \cdot \varphi^* \mid \varphi_* B_2$ as a coadjoint.

For the morphism $\varphi : M_1 \to M_2$ of topological monoids with kernel N consider the following conditions

(o) φ is a surjection

(i) there are maps $\psi : M_1 \to M_1$ and $\chi : M_1 \times N \to N$ such that for all $m \in M_1$ and $n \in N$, $nm = \chi(m, n)\, \psi(m)$ holds.

For the M_1-space B consider the condition

(ii) for every $b \in B$ and for every $m', m'' \in M_1$ satisfying $\varphi m' = \varphi m''$ there is a $c \in B$ and $n', n'' \in N$ with $m'b = n'c$ and $m''b = n''c$.

It should be remarked that conditions (o) – (ii) are satisfied for both the identity morphism $M : M \to M$ and the terminal morphism $\tau : M \to 1$. Moreover, if $\varphi' : M_1' \to M_2'$ and $\varphi'' : M_1'' \to M_2''$ are surjective morphisms and if B is a $M_1' \times M_1''$-space such that (o) – (ii) are satisfied for φ' and φ'', with M_1' resp. M_1'' operating on B through $M_1' \to M_1' \times \{1\} \hookrightarrow M_1' \times M_1''$ resp. $M_1'' \to M_1'' \times \{1\} \hookrightarrow M_1' \times M_1''$, then $\varphi' \times \varphi'' : M_1' \times M_1'' \to M_2' \times M_2''$ again satisfies (o) – (ii). It should alos be noted that (i) is satisfied if M_1 is either a group or an abelian monoid.

Calling a continuous map h from the M_1-space B_1 to the M_2-space B_2 a φ-equivariant map if $h(m_1 b_1) = \varphi(m_1)\, h(b_1)$ holds, one has the following

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LEMMA 1. — Suppose that for the morphism $\varphi : M_1 \to M_2$ of topological monoids and for the M_1-space B the conditions (o) — (ii) are satisfied, and that the kernel of φ is denoted by N. Then there is a φ-equivariant map $cl_B : B \to N \backslash B$ sucht that for every φ-equivariant map $h : B \to B'$ there is a unique M_2-equivariant map $h' : N \backslash B \to B'$ such that $h = h' . cl_B$ holds. The set underlying $N \backslash B$ is the set of equivalence classes generated by the operation of N on B.

Let $\underline{\mathrm{Top}}_{M_1,\varphi}$ denote the full subcategory of $\underline{\mathrm{Top}}_{M_1}$ defined by those objects for which (ii) is satisfied (note that $\underline{\mathrm{Top}}_{M,\tau} = \underline{\mathrm{Top}}_M$). If (o) and (i) are also satisfied then Lemma 1 gives rise to a unique functor $N \backslash : \underline{\mathrm{Top}}_{M_1,\varphi} \to \underline{\mathrm{Top}}_{M_2}$ for which the assignmant $B \to cl_B$ is a morphism of functors from $\mathrm{Top}_{M_1,\varphi}$ to $\varphi_* \cdot N \backslash$. Similarly, $N \backslash$ induces a functor from $\underline{\mathrm{Top}}_{M_1,\varphi} | B$ to $\underline{\mathrm{Top}}_{M_2} | N \backslash B$ which shall be denoted by $N \backslash_B$ .This functor, in turn, gives rise to an obvious functor $\Gamma(cl_B, \ )^{\varphi} : \Gamma(B, \ )^{M_1} \to \Gamma(N \backslash B, \ )^{M_2} \cdot N \backslash_B$.

If (o) — (ii) are satisfied then we can form the composite of functors

$$\underline{\mathrm{Top}}_{M_2} | N \backslash B \xrightarrow{\varphi_* | N \backslash B} \underline{\mathrm{Top}}_{M_1} | \varphi_* (N \backslash B) \xrightarrow{B \times_{\varphi_*(N \backslash B)}} \underline{\mathrm{Top}}_{M_1,\varphi} | B .$$

Since both functors involved preserve limits and since, evidently, a solution set exists, this composite possesses a coadjoint. Specifically, one obtains the

THEOREM 1. — *Suppose that for the morphism $\varphi : M_1 \to M_2$ of topological monoids and for the M_1-space B the conditions* (o) — (ii) *are satisfied, and that the kernel of φ is denoted by N. Then $B \times_{\varphi_*(N \backslash B)} \cdot \varphi_* | N \backslash B$ has $N \backslash_B$ as a coadjoint.*

A closer scrutinity of the associated counit b leads to

PROPOSITION 1. — Suppose that $cl_B : B \to N \backslash B$ is an open map. Then the counit b associated with the adjunction expressed in Theorem 1 is an isomorphism of functors, that is for every object $\mathfrak{S}$ of $\underline{\mathrm{Top}}_{M_2} | N \backslash B$, $b_{\mathfrak{S}}$ is a homeomorphism.

Using a well-known theorem of Gabriel [1] we therefore have the

COROLLARY. — If Σ denotes the set of morphisms in $\underline{\mathrm{Top}}_{M_1,\varphi} | B$ that are made invertible by $N \backslash_B$, then $\underline{\mathrm{Top}}_{M_2} | N \backslash B$ is equivalent to $\underline{\mathrm{Top}}_{M_1,\varphi} | B [\Sigma^{-1}]$.

On account of this Corollary it is of some interest to obtain conditions — necessary or sufficient — for a morphism of $\underline{\mathrm{Top}}_{M_1,\varphi} | B$ to be made invertible by $N \backslash_B$. For that purpose we formulate the following condition

(A) for every $n_1, n_2 \in N$ there exist $n'_1, n'_2 \in N$ with $n'_1 n_1 = n'_2 n_2$.

It should be noted that condition (A) is satisfied if N is either a group or an abelian monoid. Moreover, if both N' and N'' satisfy condition (A) then so does $N' \times N''$.

PROPOSITION 2. — Suppose that the conditions (o) — (ii) and (A) are satisfied. Suppose furthermore that $N \backslash_B g$ is an isomorphism. Then

(1) g is N-finally surjective, i.e. for every element t in the codomain T of g there is an element $n \in N$ and an element t' in the domain T' of g with $nt = g(t')$,

(2) g is N-finally injective, i.e. for $t_1', t_2' \in T'$ with $g(t_1') = g(t_2')$ there are elements $n_1, n_2 \in N$ with $n_1 t_1' = n_2 t_2'$; moreover, t_1' and t_2' belong to the same fiber

(3) for every open subset V of $N \setminus T'$, $g(cl_{T'}^{-1}(V))$ is the intersection of $g(T')$ with an open, N-saturated subset of T.

COROLLARY. – Assumptions as in Proposition 2. Assume furthermore that N operates on $T' \to B$ as a monoid of fiber bijections and on $T \to B$ as a monoid of fiber injections. Then $N \setminus_B g$ being an isomorphism implies that g is a bijection that preserves open, N-saturated sets.

There is a partial converse of Proposition 2, namely

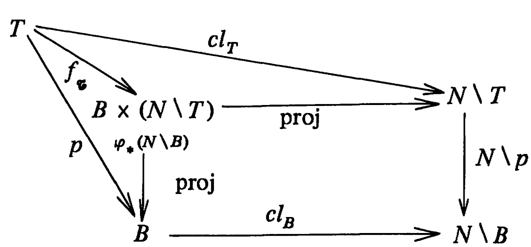
PROPOSITION 3. – Suppose that the conditions (o) – (ii) and (A) are satisfied. Suppose furthermore that the statements (1) – (3) of Proposition 2 are fulfilled and that, in addition,

(4) $N \setminus T$ is the quotient space of T under cl_T.

Then $N \setminus_B g$ is an isomorphism.

The unit f associated with the adjunction expressed in Theorem 1 fails to be an isomorphism of functors. Yet, it is of interest to obtain conditions – necessary or sufficient – for an object $\mathscr{C}$ of $\underline{\text{Top}}_{M_1, \varphi} \mid B$ to render $f_{\mathscr{C}}$ an isomorphism, i.e. a homoemorphism. For that purpose we assert the

LEMMA 2. – Given the object $\mathscr{C} = T \overset{p}{\to} B$ of $\underline{\text{Top}}_{M_1, \varphi} \mid B$, the unit $f_{\mathscr{C}}$ is the unique map rendering the following diagram commutative



From it one concludes

PROPOSITION 4. – Suppose that $f_{\mathscr{C}}$ is an isomorphism. Then

$$\Gamma(cl_B, \mathscr{C})^{\varphi} : \Gamma(B, \mathscr{C})^{M_1} \to \Gamma(N \setminus B, N \setminus_B \mathscr{C})^{M_2}$$

is a bijection.

Another consequence of Lemma 2 is the

PROPOSITION 5. – Suppose that $f_{\mathscr{C}}$ is an isomorphism. Then N operates on $\mathscr{C}$ as a monoid of fiber bijections.

Proposition 5 suggests to call an object $\mathscr{C}$ of $\text{Top}_M | B$ a N-regular object, N being a submonoid of M, if N operates on $\mathscr{C}$ as a monoid of fiber bijections.

The full subcategory, for instance, of $\underline{\mathrm{Top}}_M \mid B$ that is defined by the N-regular objects shall be denoted by $\underline{\mathrm{Top}}_M^{N\text{-reg}} \mid B$.

There is a partial converse of Proposition 5 (see also [3], Proposition 2.1), namely

PROPOSITION 6. – Suppose that the conditions (o) – (ii) and (A) are satisfied and that N operates on B freely. Then for every object $\mathscr{C} = T \overset{p}{\to} B$ of $\underline{\mathrm{Top}}_{M_1,\varphi}^{N\text{-reg}} \mid B$, $f_{\mathscr{C}}$ is a bijection. If, in addition,

(a) both $p : T \to B$ and $cl_T : T \to N \setminus T$ are open maps

(b) for every $t_0 \in T$ there are neighborhoods W of t_0 and V of $p(t_0)$ such that for any $t', t'' \in T$ and any $n', n'' \in N$, the relations $n't' = n''t''$, $p(t') \in V$, $t' \in W$ imply $t'' \in W$.

Then $f_{\mathscr{C}}$ is an isomorphism.

It should be noted that assumption (b) of Proposition 6 is satisfied if for every $b_0 \in B$ there is a neighborhood V of b_0 such that for any $b', b'' \in V$ and any $n', n'' \in N$, $n'b' = n''b''$ implies $b' = b''$.

Let $T \overset{p}{\to} B$ be a principal G-bundle in the sense of Steenrod and let the topological monoid M operate equivariantly on it. Then we speak of a M-equivariant principal bundle if

(α) $m(tg) = (mt)g$ holds for all $m \in M$, $t \in T$, $g \in G$

(β) in suitable fibercoordinates, the map $m_b : p^{-1}(b) \to p^{-1}(mb)$ induced by the operation of $m \in M$, is the left multiplication with some element of the structural group G.

We regard a M-equivariant principal G-bundle as a $M \times G^{op}$-space over B, with G operating trivially on B. With this in mind we denote the full subcategory of $\underline{\mathrm{Top}}_{M \times G^{op}} | B$ defined by the M-equivariant principal G-bundles by $\underline{\mathrm{Bun}}_{M,G} | B$. Evidently we have $\underline{\mathrm{Bun}}_{M,G} | B \subset \underline{\mathrm{Top}}_{M \times G^{op}}^{M \times G^{op}\text{-reg}} \mid B$. Similarly, one defines M-equivariant bundles with a given fiber, in particular, $\underline{\mathrm{Vec}}_M | B$ and $\underline{\mathrm{Vec}}_M^{M\text{-reg}} | B$. (See also [3]).

Let $\pi : M \times G^{op} \to G^{op}$ be the projection. Then the functor $B \times_{\pi_*(M \setminus B)} \cdot \pi_* | M \setminus B$ maps $\underline{\mathrm{Bun}}_G | M \setminus B$ into the full subcategory $\underline{\mathrm{Bun}}_M^{M-lt} | B$ of $\underline{\mathrm{Bun}}_{M,G} \mid B$ that is defined by those objects that are M-locally trivial, i.e. are trivial over $cl_B^{-1} V$ with V a suitable neighborhood of any given element $cl_B b \in M \setminus B$. Conversely, one wishes to have conditions under which for an object $\mathscr{C}$ of $\underline{\mathrm{Bun}}_{M,G} | B$, $M \setminus \mathscr{C}$ is in $\underline{\mathrm{Bun}}_G | M \setminus B$. Here we have

PROPOSITION 7. – Let $\mathscr{C}$ be an object of $\underline{\mathrm{Bun}}_{M,G} | B$ such that $M \setminus \mathscr{C}$ is an object of $\underline{\mathrm{Bun}}_G | M \setminus B$. Then $\mathscr{C}$ is M-locally trivial, and thus an object of $\underline{\mathrm{Bun}}_{M,G}^{M-lt} | B$.

A partial converse (see also [3]) of this proposition is

THEOREM 2. – *Suppose that the condition (A) is satisfied by M, that M operates freely on B, and that $cl_B : B \to M \setminus B$ has local corss-sections. Then the functor $M \setminus_B$ maps $\underline{Bun}_{M,G}^{M-lt} \mid B$ into $\underline{Bun}_G \mid M \setminus B$. Moreover, the unit f restricted to the objects of $\underline{Bun}_{M,G}^{M-lt} \mid B$ is an isomorphism ; in particular, $\underline{Bun}_{M,G}^{M-lt} \mid B$ is equivalent to $\underline{Bun}_G \mid M \setminus B$ and the morphism*

$$\Gamma(cl_B, \quad)^\tau : \Gamma(B, \quad)^M \to \Gamma(M\backslash B, \quad)\cdot M\backslash_B$$

is an isomorphism.

COROLLARY. – Under the assumptions of Theorem 2 and the additional assumption that $M\backslash B$ is paracompact, there is a bijection

$$\text{Iso. classes } (\text{Ob } \underline{\text{Bun}}_{M,G}^{M-lt} \mid B) \cong [M\backslash B, X_G]$$

– X_G being the classigying space for G – that is natural in both B and G.

COROLLARY. – Under the assumptions of Theorem 2 and the additional assumption that $M\backslash B$ is paracompact, there is a bijection

$$\text{Iso. classes } (\text{Ob } \underline{\text{Vec}}_M^{M\text{-reg}, M-lt} \mid B) \cong \coprod_{k \geqslant 0} [M\backslash B, G_k],$$

with G_k the Grassmannian.

Clearly, the results concerning equivariant principal bundles carry over to arbitrary equivariant fiber bundles.

By a difference equation we mean a M-equivariant fiber bundle over B such that

(i) M is a submonoid of $\mathbf{Z}^k$

(ii) M operates on B freely

(iii) for every element of $M\backslash B$ there is an open neighborhood V such that $cl_B^{-1} V$ is the disjoint union of contractible open sets each of which is mapped topologically into V by cl_B.

If $M \subset \mathbf{Z}^k$ operates on the Euclidean space E by translation and if B is an open subset of E with $MB \subset B$, then the concept of a trivial M-equivariant fiber bundle over B coincides with the classical concept of a (system of) difference equations on B. One checks easily that for such a fiber bundle $\mathcal{C}$, $\Gamma(B, \mathcal{C})^M$ coincides with the set of solutions of the difference equation $\mathcal{C}$.

If G is the structural group and F is the fiber involved then the corresponding category of equivariant fiber bundles is denoted by $\Delta_{G,F} \mid B$ and is called the category of difference equations of type G, F over B. Also, this instance of $\underline{\text{Vec}} \mid B$ is denoted by $\Delta\underline{\text{Vec}} \mid B$ and is called the category of linear difference equations over B. Evidently we have $\Delta_{G,F}^{M-lt} \mid B = \Delta_{G,F} \mid B$. Thus Theorem 2 leads to

PROPOSITION 8. – The category $\Delta_{G,F}^{M-\text{reg}} \mid B$ of regular difference equations is equivalent to the category of fiber bundles over $M\backslash B$ with structural group G and fiber F. Moreover, for every such difference equation $\mathcal{C}$ there is a bijection $\Gamma(B, \mathcal{C})^M \cong \Gamma(M\backslash B, M\backslash_B \mathcal{C})$ that is natural in both B and $\mathcal{C}$.

The bijection given in Proposition 8 can be used to obtain existence theorems for solutions of difference equations. It can also be employed to study boundary value problems and initial value problems.

COROLLARY. – Suppose that the group G operates effectively on F. If $M\backslash B$ is paracompact then the set of isomorphy classes of M-regular difference equations of type G, F over B are in bijection with the set $[M\backslash B, X_G]$. This bijection is natural in both B and G.

Proposition 8, together with [4], furnishes the

COROLLARY (Linearization Theorem). – Every M-regular difference equation of type Diff $(\mathbf{R}^n)$, $\mathbf{R}^n$ over B is isomorphic to a linear one.

Using [2] we obtain.

COROLLARY. – Let **H** be a separable Hilbert space. Then every M-regular difference equation of type $GL(\mathbf{H})$, **H** over B is isomorphic to the one given by $f(b+m) = f(b)$.

Proposition 8 implies that the equivariant K-theory based on M-regular linear difference equations is isomorphic with $K(\ )\cdot M\backslash$. Yet the equivariant K-theory based on all linear difference equations has some interest. In general, they will be different from each other.

Finally it should be remarked that these ideas could be used in studying difference approximations of completely integrable systems of total differential equations.

LITERATURE

[1] GABRIEL P. and ZISMAN M. — Calculus of Fractions and Homotopy Theory. *Erg. d. Math. u. Grenzgeb.*, Springer Verlag, 1967.

[2] KUIPER N.H. — The Homotopy Type of the Unitary Group of Hilbert Space. *Topology*, 3, 1964, p. 19-31.

[3] SEGAL G. — Equivariant K-Theory. *Publ. I.H.E.S.*, 34, 1968, p. 129-151.

[4] STEWART T.E. — On Groups of Diffeomorphisms. *Proc. A.M.S.*, 11, 1960, p. 559-563.

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EQUIVARIANT STABLE HOMOTOPY THEORY

by G. B. SEGAL

Equivariant maps between spheres.

Let G be a finite group.

A finite-dimensional real vector-space V on which G acts linearly will be called a *G-module.* Its one-point compactification S^V is a sphere with G-action, in which we shall regard ∞ as a base-point. Our object is to describe the homotopy-classes of equivariant maps between such spheres.

For each G-module V there is a concept of *suspension* : if X is a G-space with base-point x_0 we define the suspension $S^V X$ as

$$S^V \wedge X = (S^V \times X)/((\infty \times X) \cup (S^V \times x_0)).$$

If X and Y are G-spaces with base-points then $[X ; Y]_G$ denotes the set of homotopy-classes of base-point-preserving G-maps $X \to Y$. There is a suspension-map $[X ; Y]_G \to [S^V X ; S^V Y]_G$ for any G-module V. One can order the isomorphism-classes of G-modules by

$$V \leqslant V' \Longleftrightarrow V \text{ is isomorphic to a submodule of } V' \ ;$$

then one defines the set of stable equivariant maps

$$\{X ; Y\}_G = \varinjlim_{V} \ [S^V X \ ; \ S^V Y] \, ,$$

which is an abelian group. (Strictly speaking the limit is taken over the category of G-modules and embeddings).

PROPOSITION 1. – $[S^{V \oplus W} ; S^W]_G$ is independent of W if W is sufficiently large, and can be identified with the set of cobordism-classes of compact V-framed G-manifolds.

The terminology in the proposition is explained by

DEFINITION 1. – If V is a G-module, a compact G-manifold M is called *V-framed* if there is given a stable G-isomorphism φ_M of its tangent bundle T_M with $M \times V$, i.e. if there is given a G-module W and an isomorphism of G-vector-bundles $T_M \oplus (M \times W) \cong M \times (V \oplus W)$. Such a manifold is said to bound if there is a G-manifold N with boundary M and a stable isomorphism of T_N with $N \times (V \oplus \mathbf{R})$ wich induces φ_M.

The proof of Proposition 1 depends essentially on the concept of "consistent transversality" introduced by Wasserman [2]. Details can be found in [1].

Proposition 1 makes clear in particular what plays the role of the *degree* of a map in the equivariant theory. Recall that one defines for any group G its *Burnside* ring $A(G)$ as the Grothendieck group of the category of finite G-sets, i.e. $A(G)$ is the free abelian group on the set of conjugacy-classes of subgroups of finite index in G. Then we have

COROLLARY. – For large W, $[S^W;S^W]_G \cong A(G)$ as rings, where the multiplication in $[S^W;S^W]_G$ is composition of maps, and that in $A(G)$ corresponds to forming the product of G-sets.

Thus the equivariant homotopy class of a map $S^W \to S^W$ is determined by the degrees of its restrictions to the fixed-point subsets of the subgroups H of G ; and the diagram

$$\begin{array}{ccc} [S^W ; S^W]_G & \longrightarrow & A(G) \\ \downarrow & & \downarrow \epsilon_H \\ [(S^W)^H ; (S^W)^H] & \xrightarrow{\text{degree}} & \mathbf{Z} \end{array}$$

commutes, where ϵ_H assigns to a G-set S the cardinal of S^H.

PROPOSITION 2. – If $V = \mathbf{R}^n$ with trivial G-action then

$$[S^{V\oplus W} ; S^W]_G \cong \bigoplus_H \pi_n^S(BW_H) ,$$

where the sum is taken over the conjugacy classes of subgroups H of G, π_*^S denotes stable homotopy, and $W_H = N_H/H$, where N_H is the normalizer of H in G.

Proof. – If M is a V-framed G-manifold then the isotropy-group must be constant on each component of M, for if g is an element of the isotropy group at x then g acts trivially on the tangent-space to M at x, and so leaves fixed all the geodesics through x. But if M has all its isotropy-groups conjugate to H one can write it as $(G/H) \times_{W_H} M^H$, where M^H, the H-invariant part of M, is a free W_H space. Thus a general V-framed manifold can be written $\coprod_H (G/H) \times_{W_H} M_H$, where M_H is a V-framed free W_H-manifold. The cobordism-classes of such M_H can be identified with $\pi_n^S(BW_H)$, and Proposition 2 follows.

Equivariant stable cohomology theory

For any pair $Y \subset X$ of compact G-spaces and any virtual G-module α (i.e. any $\alpha \in RO(G)$) let us define

$$\omega_G^\alpha(X, Y) = \varinjlim_V [S^V(X/Y) ; S^{V+\alpha}] = \{X/Y ; S^\alpha\}.$$

This is a generalized cohomology theory in the sense that it satisfies obvious homotopy, exactness, and excision axioms (for any pair (X, Y) there is a boundary homomorphism $\omega_G^\alpha(Y) \to \omega_G^{\alpha+1}(X, Y)$). It has the additional stability property that $\tilde{\omega}_G^\alpha(X) \cong \tilde{\omega}_G^{\alpha+V}(S^V X)$ for any X and V. Furthermore it is universal among cohomology theories with those four properties.

On free G-spaces and trivial G-spaces one can express ω_G^α in terms of ordinary stable homotopy, at least when $\alpha \in Z \subset RO(G)$, as follows.

PROPOSITION 3. – If X is a free compact G-space, then

$$\omega_G^n(X) \cong \omega^n(X/G) = \pi_S^n(X/G).$$

PROPOSITION 4. – If G acts trivially on X then

$$\omega_G^n(X) \cong \bigoplus_H \{X \, ; S^n BW_H^+\},$$

where BW_H^+ is the union of BW_H with a disjoint base-point.

As ordinary stable homotopy coincides with homology when tensored with the rationals, and as classifying-spaces for finite groups have trivial rational homology, one deduces from Proposition 4 that $\omega_G^n(X) \otimes \mathbf{Q} \cong A(G) \otimes H^n(X\,;\mathbf{Q})$ when G acts trivially on X. More generally one has

PROPOSITION 5. – For any compact G-space X, and any $\alpha \in RO(G)$,

$$\omega_G^\alpha(X) \otimes \mathbf{Q} \cong \bigoplus_H H^{\alpha_H}(X^H \, ; \mathbf{Q})^{W^H}$$

where, if $\alpha = V - W \in RO(G)$, $\alpha_H = \dim V^H - \dim W^H$.

It is easy to see that $\{S^V; S^W\}_G$ is a finitely generated abelian group, so Proposition 5 implies the

COROLLARY. – $\{S^V; S^W\}_G$ is finite unless $\dim V^H = \dim W^H$ for some subgroup H of G.

The equivariant J-homomorphism (1).

The relationship between equivariant stable cohomotopy as defined here and equivariant K-theory is precisely analogous to that in the classical case. There is a J-homomorphism

$$J : KO_G^{-1}(X) \to \omega_G^0(X)$$

(from the additive group KO_G^{-1} to the multiplicative group of ω_G^0) defined by the usual Hopf construction. Its image can be determined in the following way.

The Adams operations ψ^k act on $KO_G(X)$, and hence on the profinite completion $KO_G(X)^\wedge$. They define an action of $\mathbf{Z}$ on $KO_G(X)^\wedge$ which is continuous when $\mathbf{Z}$ is given the profinite topology, and so the action extends to an action of the profinite completion $\hat{Z}$. The group of units $\hat{Z}^*$ of this ring is the product of the subgroup (± 1) with a topologically cyclic group Γ. Let α be a generator of Γ. Then $\psi^\alpha : KO_G(X)^\wedge \to KO_G(X)^\wedge$ extends to a transformation of multiplicative cohomology theories, and so one can define a new multiplicative cohomology theory J_G^* with a multiplicative transformation $J_G^* \to KO_G^{*\wedge}$ fitting into an exact triangle

$$\cdots \to J_G^* \to KO_G^{*\wedge} \xrightarrow{\psi^\alpha - 1} KO_G^{*\wedge} \to \cdots .$$

---

(1) The proofs of the results in this section depend on the work of Sullivan on the Adams conjecture.

Thus there is a short exact sequence

$$0 \to \operatorname{coker}(\psi^a - 1) \to J_G^* \to \ker(\psi^a - 1) \to 0. \qquad \ldots (\dagger)$$

In terms of the theory J_G^* one can describe the J-homomorphism as follows. The Hurewicz homomorphism $\omega_G^* \to KO_G^{*\wedge}$ factorizes through J_G^*, giving a multiplicative transformation $h : \omega_G^* \to J_G^*$. In view of the exact sequence (†) one sees that this assigns to an element of stable cohomotopy its d- and e-invariants in the sense of Adams. The J-homomorphism $J : KO_G^{-1}(X) \to \omega_G^0(X)$ factorizes through $\widetilde{J}_G^0(X)$ to give an exponential map $J : \widetilde{J}_G^0(X) \to \omega_G^0(X)$. If G is a p-group the composite $hJ : \widetilde{J}_G^0(X) \to J_G^0(X)$ is an isomorphism between the additive group $\widetilde{J}_G^0(X)$ and the multiplicative group $1 + \widetilde{J}_G^0(X)$. Then $\widetilde{J}_G^0(X)$ is a direct summand in the multiplicative group $1 + \widetilde{\omega}_G^0(X)$.

The definition of an equivariant cohomology theory.

In conclusion I shall mention two facts which tend to support the use of all real representations for suspending in equivariant stable homotopy theory, and the indexing of equivariant cohomology theories by $RO(G)$.

The first is the generalization of the construction of Eilenberg-Maclane spaces as the infinite symmetric products of spheres.

PROPOSITION 6. – If A is a topological abelian group with G-action, and V is a G-module, there is a G-space $B^V A$ and a G-homotopy-equivalence

$$A \to \operatorname{Map}(S^V ; B^V A) ;$$

and if $A = \mathbf{Z}$ with trivial G-action one can take $B^V A = F(S^V)$, the free abelian group on S^V.

The second is the generalization of a theorem of Barratt and Quillen. If S is a finite G-set let us write Σ_S for the group of G-automorphisms of S. One can form an associative monoid $\Gamma_G = \coprod_S B\Sigma_S$, the sum being over all finite G-sets S. The monoid can also be written $\prod_H \coprod_{n \in \mathbf{N}} B(\Sigma_n \int W_H)$, where H runs through the conjugacy-classes of subgroups G, and $\Sigma_n \int W_H$ denotes the semi-direct product

$$\Sigma_n \widetilde{\times} (\overleftarrow{W_H \times \overset{n}{\ldots} \times W_H}).$$

PROPOSITION 7. – The classifying-space for Γ_G is homotopy-equivalent to that of $\varinjlim_V \operatorname{Map}_G(S^V ; S^V)$, where V runs through all G-modules.

The theorem of Barratt and Quillen tells one that

$$B(\coprod_n B(\Sigma_n \int W_H)) \simeq B(\Omega^\infty S^\infty (BW_H^+)),$$

so one deduces.

COROLLARY. – $\varinjlim_V \operatorname{Map}_G(S^V ; S^V) \simeq \prod_H \Omega^\infty S^\infty (BW_H^+)$.

This is of course just a restatement of Proposition 4, but it provides a completely different proof of it.

REFERENCES

[1] O'CONNOR J.J. — *D. Phil. thesis, Oxford,* (in preparation).
[2] WASSERMAN A.G. — Equivariant differential topology. *Topology,* 8, 1969, p. 127-150.

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C 2 - TOPOLOGIE DES VARIÉTÉS

ESPACE DE PLONGEMENTS

par A. V. ČERNAVSKII

1. Au cours des quelques dernières années on étudiait extensivement les propriétés locales des espaces de plongements. Bien que ce travail soit encore en progrés on peut néanmoins présenter aujourd'hui un tableau de résultats assez complet.

Les problèmes fondamentaux ayant ici un caractère local, c'est de plongements dans l'espace euclidien R^n qu'il s'agira plus loin. Aussi nous nous limiterons, à quelques exceptions près, au cas de variétés de dimension $k \leqslant n - 3$. Le cas $k = n - 2$ a, comme on le sait bien, ses traits spécifiques, liés au groupe fondamental local non-trivial.

Dans le cas $k = n - 1$ on peut pousser l'étude plus loin, mais la situation touche ici au problème de base, tel que l'approximation des homéomorphismes ; on n'y doit espérer avancer qu'à l'aide des résultats récents et encore peu accessibles de Siebenman, Wall, Kirby etc.

Je vais maintenant rappeler les définitions usuelles :

Le plongement $q : u^k \to R^n$ est dit localement plat, s'il est possible de donner dans le voisinage de chaque point $q\, x \in R^n$ des coordonnées locales telles que q devienne linéaire à proximité de x.

Le plongement $q : u^k \to R^n$ est dit localement 1-connexe $(1 - \mathrm{LC})$, s'il est possible, pour chaque point $x \in q\, u$ et pour chaque voisinage V_x de contracter dans $v \backslash qu$ en un point tout contour fermé pris dans $u \backslash qu$, u un autre voisinage de x suffisamment petit. D'après un résultat récent ces deux classes coïncident pourvu que $k \leqslant n - 3$.

Etant donnés deux plongements $q\, , q' : u \to R^n$. l'isotopie qui fait passer q à q' est une famille h_t à un paramètre d'homéomorphismes de R^n tel que $h_0 = 1$ et $h_1 q = q'$.

2. Les problèmes fondamentaux dont il s'agira ici sont : l'isotopie locale, l'approximation et les propriétés locales des espaces de plongements.

Quant à l'isotopie locale je veux citer deux résultats :

Soit q et $q' : u^k \to R^n$ deux plongements.

THEOREME 1. – *Si q' est assez proche de q, les deux étant localement plats et $k \neq n - 2$, alors il y a quel que soit ϵ une ϵ-isotopie qui fait passer q' à q* (A. Černavski, Doklady 187, (1969)).

THEOREME 2. — *Si q et q' sont linéaires par morceaux et $k \leqslant n - 3$ le même reste vrai, en admettant l'isotopie linéaire par morceaux* (Miller, USA, thèse à paraître).

Le premier résultat s'obtient par le même raisonnement (légèrement généralisé) à l'aide duquel j'ai prouvé le théorème sur la contractibilité locale d'espace des homéomorphismes d'une variété. Cependant la situation est ici plus compliquée et il faut se servir d'une gamme plus large de moyens pour avoir des résultats plus faibles. On doit s'appuyer ici sur un argument de type d'"Engulfing lemma", notamment sur une modification du résultat de J. Bryant et C. Seebeck (*Quart. j. Math.* 19 (1968), 275) que l'on a vu déjà plus d'une fois comme un moyen technique très utile :

Pour chaque plongement $1 - LC, q : u^k \to R^n$, $k \neq n - 2$, et chaque $\epsilon > 0$ il existe un $\delta > 0$ tel que pour un plongement $1 - LC, q' : u^k \to R^n$ δ-proche de q et un voisinage v de $q'u$, il existe une ϵ-isotopie h_t telle que $h_0 = 1, h_1 v \subset qu$ et $h_1 = 1/q'u$ (voir A. Černavski, Doklady, 181 (1968), 290).

Quelques autres formes de ce théorème sont utiles dans de différentes situations.

Ces résultats s'étendent aussi au cas $k = n - 2$ mais leur présentation devient alors trop compliquée. Nous indiquerons des généralisations possibles du théorème 1 à la fin de cette communication.

3. Passons maintenant aux approximations. Le premier résultat à citer ici est un théorème de M. Štanko :

THEOREME 3. — *Le plongement q d'un espace K localement compact de dimension $k \leqslant n - 3$* dans *R^n pour lequel $q(K)$ est localement fermé peut être approché par un plongement $1 - LC$. Cela reste encore vrai pour les variétés de dimension $n - 1$.*

A l'aide de ses résultats précédents Štańko déduit de là une solution complète du célèbre problème de Menger : Le compact de Menger M_n^k est universel pour les sous-compacts k-dimensionnels de R^n. Une autre conséquence, c'est que les plongements des variétés $u^k, k \leqslant n - 3$, sont approximables à l'aide de plongements localement plats.

En effet, chaque plongement $1 - LC$ est localement plat pour $k \leqslant n - 3$. Ceci est une conséquence presque immédiate du théorème 1 et du théorème suivant :

THEOREME 4. — *Chaque plongement d'une boule $\mathcal{B}^k$ de dimension $k \leqslant n - 3$ peut être approché par un plongement linéaire par morceaux (Černavski, Matem. Sborn., 80 (1969))* (*).

Une autre démonstration de ce résultat a été donnée récemment par Miller.

Avec ce résultat local on peut obtenir par des voies différentes une démonstration du théorème général que voici :

THEOREME 5. — *Chaque plongement d'une variété linéaire par morceaux u^k dans une autre w^n, $k \leqslant n - 3$, peut être approché par un plongement linéaire par un plongement linéaire par morceaux.*

(*) Cet article contient une erreur, signalée par C.D. Edwards ; une rectification sera publiée ultérieurement.

Ou bien on doit aller du local au global en utilisant des approximations locales à l'aide des théorèmes d'isotopie locale (Théorème 1 et 2). Cette voie a été choisie par Miller, Bryant, Connely et par moi-même (voir *Matem. Sbornik* 1970, Juli).

Ou bien on peut se baser sur le théorème de M. Štańko qui donne une approximation localement plate. Alors il ne reste qu'à appliquer un résultat récent de Rushing sur l'isotopie d'un plongement localement plat d'une variété linéaire par morceaux ($k \leqslant n - 3$) à un plongement linéaire par morceaux.

Par ailleurs ce résultat de Rushing est une conséquence directe des théorèmes 1 et 5.

Des résultats analogues pour les plongements de polyèdres ont été démontrés tout récemment par Cobb, Bryant etc.

4. Aujourd'hui c'est l'étude de l'espace total $\mathcal{E} = \mathcal{E}mb(\mathcal{B}^k, R^n)$ qui est à l'ordre du jour. Soit $\mathcal{E}_0 \subset \mathcal{E}$ le sous-espace des plongements localement plats, H l'espace des homéomorphismes de R^n et $f : H \to \mathcal{E}$ une fonction telle que $fh = hq_0$ et q_0 un plongement localement plat quelconque.

THÉOREME 6. – *f est un fibré au sens de J.P. Serre, $k \neq n - 2$.*

Essentiellement, c'est une conséquence du théorème 1 et du travail de Michael sur les "continuous selections".

Certainement, il serait désirable de prouver que f est un fibré localement trivial. Ce serait possible si l'on pourrait construire un voisinage de l'unité en H qui soit contractible. Ce fait serait aussi utile (comme D. Henderson l'affirme) dans une démonstration que H est une variété modelée sur l'espace l_2.

On peut aussi déduire des résultats précédents que pour $k \leqslant n - 3$ l'espace $\mathcal{E}$ de tous plongements est localement connexe dans toutes les dimensions.

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THE OBSTRUCTION TO FIBERING A MANIFOLD OVER A CIRCLE

by F. Thomas FARRELL

Let M be a compact, connected, smooth manifold whose dimension is greater than five, and let f be a continuous map from M to the circle, which we denote by S^1. Suppose that f restricted to the boundary of M, denoted by ∂M, is a smooth fibration. We note that a map h from a smooth manifold N to S^1 is a smooth fibration if h is smooth, and for each point x of N the derivative of h maps the tangent plane to N at x onto the tangent plane to S^1 at $h(x)$. We wish to address ourselves in this talk to the following problem.

Fibering Problem.

When does there exist a smooth fiber map g from M to S^1 which agrees with f when restricted to the boundary of M and which is homotopic to f relative to the boundary of M ?

Before we can state our solution to this problem, we must introduce some notation and make some constructions. Denote the additive groups of real numbers by **R**, the additive group of integers by Z, and the exponential map from **R** to S^1 by exp. Then, $\mathbf{R} \overset{\exp}{\to} S^1$ is a principal Z bundle. Pull this bundle back over M via f, and denote the pullback by $X \overset{p}{\to} M$. When the answer to the cartesian product of the fiber of g and **R**. Hence, whenever the answer to the fibering problem is affirmative, X is homotopically equivalent to a finite C.W. complex. Denote the principal **R** fibration associated to $X \overset{p}{\to} M$ by $X \times_Z \mathbf{R} \overset{q}{\to} M$. Since **R** is contractible, q is a homotopy equivalence, and $X \times_Z \mathbf{R}$ is diffeomorphic to $M \times \mathbf{R}$.

Under the assumption that X has the homotopy type of a finite C.W. complex, we formulate a torsion obstruction, denoted by $\tau(f)$, which is an element of $Wh\Pi_1 M$. Before the smooth fiber map sought can exist, it is necessary that $\tau(f)$ vanish. $\tau(f)$ is defined as the Whitehead torsion of the composite of q and F where F is any admissible homotopy equivalence. In order for this definition to be meaningful, we must formulate what constitutes an admissible homotopy equivalence. Before we can do this, we need the concept of mapping torus. Let Y be a topological space and φ a continuous map from Y to itself. The mapping torus of φ, denoted by Y_φ, is the quotient space of $Y \times [0, 1]$ obtained by introducing the identifications $(y, 1) = (\varphi(y), 0)$ for each y in Y. We will consider Y as identified with $Y \times 0$ inside of Y_φ. Recall that the mapping cylinder of φ, denoted by $Y(\varphi)$, is the quotient space of $Y \times [0, 1] \cup Y$ obtained by identifying $(y, 1)$ to $\varphi(y)$ for each y in Y. By construction, Y is identified with a subspace of $Y(\varphi)$. A C.W. complex structure on Y_φ which has Y

as a subcomplex, induces a C.W. complex structure on $Y(\varphi)$ so that Y is a subcomplex. Let T be the self diffeomorphism of X corresponding to the action of the integer 1 on X. Then, $X \times_Z \mathbf{R}$ is cannonically identified with X_T.

A map F from Y_φ to X_T is an admissible homotopy equivalence if 1) F is a homotopy equivalence ; 2) $F^{-1}(X) = Y$; 3) $F : Y \to X$ is a homotopy equivalence, and 4) the domain of F, Y_φ, is a finite C.W. complex having Y as a subcomplex such that the inclusion map of Y into the mapping cylinder of φ is a simple homotopy equivalence.

Next, we show that admissible homotopy equivalences always exist. Assume that K is a finite C.W. complex which is homotopically equivalent to X via continuous maps l from K to X and k from X to K. Define φ' from K to K to be $k \circ T \circ l$, and let φ be a cellular map which is homotopic to φ'. Then, K_φ has a natural C.W. complex structure with the required properties. $T \circ l$ is homotopic to $l \circ \varphi$, and we denote the homotopy by h. Define $F : K_\varphi \to X_T$ by

$$F(k, t) = (h(k, t), t),$$

then F is an admissible homotopy equivalence. If F and F' are two admissible homotopy equivalences, we can show that the Whitehead torsion of $g \circ F$ equals the torsion of $g \circ F'$. Hence, we can define an element $\tau(f)$ in $Wh\,\Pi_1 M$ as the torsion of $g \circ F$ where F is an arbitrary admissible map. We note that by an extension of the above technique we can define a torsion $\tau(f)$ in $Wh\,\Pi_1 M$ under the weaker assumption that X is dominated by a finite C.W. complex. Now, we state our solution to the fibering problem.

Fibering Theorem.

A smooth fiber map g exists if and only if 1/ X is homotopically equivalent to a finite C.W. complex, and 2/ $\tau(f) = 0$.

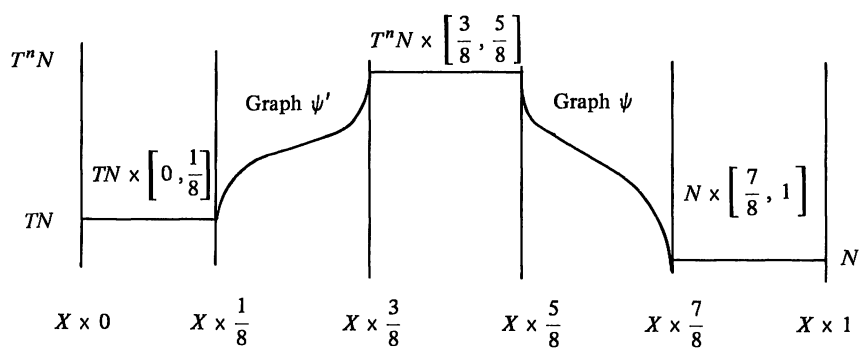
We note that conditions 1/ and 2/ of the fibering theorem can be replaced by different conditions I and II. Namely, I : X is dominated by a finite C.W. complex, and II : $\tau(f) = 0$.,

We have already demonstrated the necessity of condition 1/. We now show that condition 2/ of the fibering theorem is necessary. Suppose that a smooth fibration g exists, then we can construct a cross section F to the fibration $q : X_T \to M$ such that $F^{-1}(X)$ is a fiber of g, and F maps this fiber homotopically equivalently to X. F is an admissible map, and since $g \circ F$ is the identity, $\tau(f) = 0$. Now, we will indicate why conditions 1/ and 2/ are sufficient to produce a smooth fibration g. In order to facilitate our discussion, we will assume that M has no boundary, and that X is connected. The second assumption is equivalent to the assertion that $f^*(\alpha)$ is an indivisible element of $H^1(M, Z)$ where α is a generator of $H^1(S^1, Z)$. Let $F : K_\varphi \to X_T$ be an admissible map, then $g \circ F : K_\varphi \to M$ ia an homotopy equivalence, and the covering space induced from $X \xrightarrow{p} M$ via $g \circ F$ is properly homotopically equivalent to X and to $K \times \mathbf{R}$. Hence, X is properly homotopically equivalent to $K \times \mathbf{R}$. Since the Whitehead torsion of $g \circ F$ is zero, we can show that the splitting obstruction of Siebenmann [7] vanishes and therefore by the splitting theorem of Siebenmann [7] and Novikov [8], there exists a closed smooth manifold N such that X is diffeomorphic to $N \times \mathbf{R}$.

We say that a closed, smooth manifold M' prefibers a circle if there exists a closed co-dimension one submanifold N' of M' with trivial normal bundle such that after deleting a small open tubular neighborhood of N' from M' the resulting cobordism is an h-cobordism. If this h-cobordism is trivail, then M' fibers a circle. Next, we indicate how to construct an h-cobordism between M and a manifold which prefibers a circle. Since N is compact, some translate of it, say T^nN, is disjoint from both N and TN. Let W and W' denote the regions between N and T^nN, and between T^nN and TN respectively. Both W and W' are h-cobordisms. Let $\psi : W \to [5/8, 7/8]$ and $\psi' : W' \to [1/8, 3/8]$ be a smooth Urysohn functions such that ψ is identically 5/8 and 7/8 on neighborhoods of T^nN and N respectively while ψ' is identically 1/8 and 3/8 on neighborhoods of TN and T^nN respectively. M' denotes the union of the following five subspaces of $X \times_{Z} \mathbf{R}$:

$$T^nN \times [3/8, 5/8],\ TN \times [0, 1/8],\ N \times [7/8, 1],$$

graph of ψ, and graph of ψ'. M' is a closed codimension one submanifold of $X \times_Z \mathbf{R}$, and the inclusion map of M' into $X \times_Z \mathbf{R}$ is a homotopy equivalence. The diagram below illustrates the construction of M'.



M' would be a smooth manifold which prefibers a circle except that it has corners at $X \times [0, 1/8] \cap$ graph ψ', graph $\psi' \cap T^nN \times [3/8, 5/8]$, $T^nN \times [3/8, 5/8] \cap$ graph ψ, and graph $\psi \cap N \times [7/8, 1]$. But, by a techniquely more complicated construction, we could have avoided these corners. Assuming that we have done this and recalling that $X \times_Z \mathbf{R}$ is diffeomorphic to $M \times \mathbf{R}$, we see that M' is disjoint from Mxt for t small enough, and that the region between M' and Mxt is an h-cobordism. By analyzing the torsion of this h-cobordism utilizing the generalization of the formula of Bass, Heller, and Swan [1] found in [10], we verify the existence of a smooth fiber map $g : M \to S^1$ homotopic to f.

Finally, we wish to make a few remarks about the history of the problem we have been discussing. John Stallings was the first person to consider this problem. He solved it for 3-manifold in [3]. W. Browder and J. Levine solved it for manifolds of dimension greater than five whose fundamental group is infinite cyclic. (See [4]). Our original solution, given in [9], to the problem proceeded by extending the techniques of Browder and Levine. The approach outlined above was partially motivated by the work of C.T.C. Wall in [5]. The algebraic formula due to Bass, Heller, and Swan (See [1]) as generalized by W.C. Hsiang and the author is needed in this approach as well as in applications and extensions of the fibering theorem due to W.C. Hsiang and the author. (See [11] and [12]). The example of an h-cobordism which is not topologically a product, given in [11], also makes crucial use of the work of H. Bass and M.P. Murthy. (See [2]). In my first approach to the fibering problem (See [9]), I defined two torsion obstructions. L. Siebenmann (See [6]) was the first person to combine them into a single obstruction, but the method of combining the two obstructions presented in this talk differs from that found in [6].

REFERENCES

[1] Bass H., Heller A. and Swan R. — The Whitehead group of a polynomial extension, *Publ. de l'Inst. des Hautes Etude Sie.*, 22.

[2] Bass H. and Murthy M.P. — Grothendieck groups and Picard groups of abelian group rings, *Ann. of Math.*, 86, p. 16-73.

[3] Stallings J. — *On fibering certain 3-manifolds, Topology of 3-manifolds,* Prentice Hall, 1962, p. 95-100.

[4] Browder W. and Levine J. — Fibering manifolds over a circle, *Commentarii Mathematici Helevetici.*, 40, p. 153-160.

[5] Wall C.T.C. — *Surgery on compact manifolds,* book to appear.

[6] Siebenmann L. — The total obstruction to fibering a manifold over a circle, *Commentarii Mathematici Helvetici.*

[7] Siebenmann L. — The obstruction to finding a boundary for an open manifold of dimension greater than five, *Ph. D. thesis,* Princeton University.

[8] Novikov S.P. — *Manifolds with Free Abelian fundamental groups and their applications.*

[9] Farrell F.T. — The obstruction to fibering a manifold over a circle, *Ph. D. thesis,* Yale University, 1967, See also Yale notes, 1968.

[10] Farrell F.T. and Hsiang W.C. — A formula for $K_1 R_\alpha$ [T], *Proceedings of Symposia in Pure Mathematics,* Vol. 17, Applications of Categorical Algebra, *Amer. Math. Soc.*, 1970.

[11] Farrell F.T. and Hsiang W.C. — h-cobordant manifolds are not necessarily homeomorphic, *Bull. of A.M.S.*, 73, p. 741-744.

[12] Farrell F.T. and Hsiang W.C. — A Geometric interpretation of the Künneth formula for algebraic K-theory, *Bull. of A.M.S.*, 74, p. 548-553.

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DIFFERENTIABLE ACTIONS OF COMPACT CONNECTED LIE GROUP ON R^n

by Wu-Chung HSIANG (1)

I. Introduction.

In this lecture, I shall summarize some joint work on differentiable actions (unpublished yet) with my brother Wu-yi Hsiang. Let $\Psi : G \times R^n \to R^n$ be a differentiable action of a compact connected Lie group G on R^n. Ψ defines a representation of G into Diff (R^n) which we also denote by $\Psi : G \to \text{Diff}(R^n)$. For a fixed inner product structure on R^n, we have an inclusion $\text{SO}(n) \subset \text{Diff}(R^n)$. We say that Ψ is linear if, up to conjugacy in Diff (R^n), if factors through SO(n). Even though most actions are non-linear, we may still find many features of an action resembling a linear one. Therefore, we shall follow the following guiding principle in our study : *Compare the behaviour of general differentiable actions with that of linear ones.* At the end, I shall also discuss actions on homotopy spheres. Although the result summarized here are extensions of $[2^{\text{I, II}}]$, the proofs are actually independent of the previous works. We make use of the weight system of [9] and the group generated by differentiable reflections [6] as our new ingredients.

II. Geometric weight system and a fundamental fixed point theorem :

Let Ψ be a differentiable action of a compact connected Lie group G on a Q-acyclic manifold X. Let T be a fixed maximal torus of G and it follows from P.A. Smith theory [1] that the fixed point set of T, $F(T, X) = M$ is also Q-acyclic and consequently, connected. Hence, the local representation (2), $(\Psi|T)_x$, is independent of the choice of x. It is an invariant of Ψ, and shall denote it by $\Omega(\Psi)$. We may split the representation of T as a sum of 2-dim representations and some trivial representations. As usual, we write a non-trivail 2-dim representation of T as exp $(\pm\, 2i\alpha\pi)$ and identify the corresponding weights in $\Omega(\Psi)$ by $\pm\,\alpha$. We shall identify the trivial representations with the zero weights in $\Omega(\Psi)$ and denote the subset of non-zero weight in $\Omega(\Psi)$ by $\Omega'(\Psi)$. $\Omega(\Psi)$ is symmetric with respect to $W(G) = N(T)/T$ the Weyl group of G. The weights in $\Omega'(\Psi)$ appear in pairs $\pm\,\alpha$. $\Omega(\Psi)$ was first introduced in [9] for studying effective actions of Spin (m) on acyclic manifolds, and was used to determine the identity

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(2) After we give an invariant metric on X, the local representation $(\Psi|X)_x$ is just the induced action of T on the tangent space at x.

component of the principal isotropy subgroup(1) of a classical group acting on acyclic manifolds [2^{II}]. $\Omega(\Psi)$ is not a complete invariant of Ψ, but it is a rather good book-keeping device. Our first problem is to find possible patterns of $\Omega(\Psi)$, and then determine whether it resembles the weight of some linear representation of R^n. If G itself has a fixed point, $\Omega(\Psi)$ coïncides with the character of the local representation Φ of G at the fixed point. But unfortunately, G does not always have a fixed point [21], [9]. So we would like to find a large maximal rank subgroup K such that $\Psi|K$ has fixed points. For this purpose, let us introduce the following subsets of the root system $\Delta(G)$ of G relative to $\Omega(G)$: $\Sigma_j(\Psi)$ is the subset of α in $\Delta(G)$ such that the integral multiples of α in $\Omega'(\Psi)$ form exactly a j-string, i.e., $\pm\,\alpha,\ \pm\,2\alpha,\ldots,\ \pm\,j\alpha$, Note that most of $\Sigma_j(\Psi)$ are empty, $\Sigma_0(\Psi) = \alpha\in\Delta(G),\alpha\in\Omega'(\Psi)$ and $\Sigma_1(\Psi)$ is the subset of α in $\Delta(\Psi)$ such that $\Omega'(\Psi)$ contains only one pair of integral multiples of α, $\pm\,\alpha$.

THEOREM 1. – *Let X be a Z_2-acyclic manifold*(2) *and Ψ be a differentiable action of a compact connected Lie group G on X. Then, there is a maximal rank subgroup K of G such that $F(K,X)$ is also Z_2-acyclic and*

$$\Delta(K)\supset\Sigma_0(\Psi)\cup\Sigma_1(\Psi)\cup\Sigma_3(\Psi).$$

Theorem 1 seems technical, but it is rather strong. As we shall see in the next section, it gives a strong hold of the principal isotropy subgroup of the action. The following results are also consequences of Theorem 1. (A) If Ψ has at most 3 types of orbits(3), then G has a fixed point. One may eventually classify actions on R^n with up to 3 types of orbits. (B) If the dimension of the orbit space of Ψ is less then or equal to 6, then G has a fixed point(4). Therefore, we shall call Theorem 1 the *fundamental fixed point theorem.* It was proved by a combination of weight system, the fixed point theorem of differentiable reflections and an analysis of SO(3) actions on Z_2-acyclic manifolds.

III. Determination of principal isotropy subgroups and a reduction theorem.

For a differentiable action Ψ of G on a manifold M, there is an absolute minimum among the conjugate classes of isotropy subgroups under the partial ordering by inclusion. Denite it by (H_Ψ). For $H_\Psi\in(H_\Psi)$, G/H_Ψ is called the principal orbit of Ψ. The Montgomery-Samelson-Yang theorem [13] [14] asserts that the union of all principal orbits in M is an everywhere dense open submanifold. From [2^{II}] [3], we see that (H_Ψ) has a strong influence on other isotropy subgroup classes and it is desirable to determine (H_Ψ). We say that (H_Ψ) is non-trivial if

(1) For the definition of principal isotropy subgroups, see § III.

(2) Z_2-acyclicity implies Q-acyclicity.

(3) I. e., there are at most three conjugate classes of isotropy subgroups. For results on actions with up to 2 types of orbits, see [I, Ch. XIV].

(4) Montgomery-Samelson-Yang had results for actions with the orbit space of the dim less or equal to 2 [15].

H_Ψ is not equal to the kernel of the representation $\Psi : G \to \mathrm{Diff}(M)$. For determining (H_Ψ) of a differentiable action Ψ of G on an acyclic manifold, it would be desirable, of course, if G had a fixed point whenever (H_Ψ) was non-trivial. Then the classification of (H_Ψ) would be reduced to the linear actions. Unfortunately, there are actions of F_4 on euclidean spaces with (Spin (5)) and (Spin (2)) as the principal isotropy subgroups without a fixed point. So, we can only expect the next best thing.

THEOREM 2. – *Let Ψ be a differentiable action of a simple compact connected Lie group G on R^n. Suppose that (H_Ψ) is non-trivial. Then, we have either*

(1) $F(G, R^n)$ *is* $\mathbf{Z}_2$*-acyclic,* or

(2) $G = F_4$, $(H_\Psi) = (\mathrm{Spin}\,(5))$ *and*

$$\Omega'(\Psi) = 2 \cdot \left\{\frac{1}{2}\,\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4), \pm \theta_1, \pm \theta_2, \pm \theta_3, \pm \theta_4\right\}, \text{ or}$$

(3) $G = F_4$, $(H_\Psi) = (\mathrm{Spin}\,(2))$ *and*

$$\Omega'(\Psi) = 3 \cdot \left\{\frac{1}{2}\,(\pm\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4), \pm \theta_1, \pm \theta_2, \pm \theta_3, \pm \theta_4\right\}.$$

In fact, the cases (2), (3) *do occur.*

We can extend the result of Theorem 2 to semi-simple connected compact Lie groups, but the statement becomes a little technical due to the possible normal factors of F_4-type. However, we can still show that for a differentiable action Ψ of a compact connected Lie group G on R^n, if (H_Ψ) is not-trivial, then there is a linear representation Φ such that $(H_\Phi) = (H_\Psi)$. If G is simple, it is a consequence of Theorem 2 that we may choose Φ such that $(H_\Phi) = (H_\psi)$ and $\Omega'(\Phi) = \Omega'(\Psi)$. In any case, we complete the determination of principal isotropy subgroups of actions on R^n. (Cf. [2^{II}] [7] [12]). The basic reason why we can do this is because of the fundamental fixed point theorem (Theorem 1).

Of course, we recover all the regularity theorems of [$2^{\mathrm{I},\mathrm{II}}$] for euclidean spaces as we did in [2^{II}]. In fact, we have the following reduction theorem motivated by [2^{I}], [10], [11].

THEOREM 3. – *Let Ψ be a differentiable action of a compact connected Lie group G on R^n. Let H_Ψ be a fixed principal isotropy subgroup, i.e., a fixed element in (H_Ψ). Set $W(\Psi) = N(H_\Psi)/H_\Psi$. Then Ψ induces a differentiable action Φ of $W(\Psi)$ on $M = F(H_\Psi, R^n)$ and Φ determines Ψ.*

For example, if H_Ψ is a maximal torus of G, then M is $\mathbf{Z}$-acyclic and $W(\Psi) = W(G)$ acts on M as a group generated by differentiable reflections. Using [6], we have a complete understanding of this case. In fact, if G is a classical group, then it follows from Theorem 2 that either Ψ is a regular action in the sense of [2^{II}], [11] or the induced action Φ is generated by reflections. For this case, we also have a fairly good understanding by [$2^{\mathrm{I},\mathrm{II}}$] [6].

IV. Concluding Remarks.

When we started our work [2I], we made use of the dimension restriction of the total space relative to the group and the property of 'vanishing first Pontrjagin class' to nail down the identity components of the isotropy subgroups. We then applied P.A. Smith theory and a formula of Borel [I, pp. 175-179] to get the structure of isotropy subgroups. Under this approach, euclidean space and homotopy spheres are completely parallel. But now, we use weight system and the group generated by reflections as our tools. They depend on the fact that the fixed point set of the restriction of the action to a maximal torus is acyclic. The situation becomes somewhat different for these two cases. However, it seems to us that we still have all the parallel results if we use Borel's formula quoted above carefully and systematically. The interest in working out the homotopy sphere case is because of the existence of various differentiable structure on spheres. One expects to have more refined and interesting results on the 'degree of symmetry of spheres' [4] [5] [8] when the corresponding results for spheres are obtained.

Finally, let us pose two rather important problems from the present point of view :

PROBLEM 1. – *For a given differentiable action* Ψ *of* G *on* R^n, *is there a linear representation* Φ *such that* $\Omega'(\Phi) = \Omega'(\Psi)$?

PROBLEM 2. – *Classify all the differentiable actions of* $SO(3)$, $Sp(1)$ *on* R^n *and write down their weight system.*

REFERENCES

[1] BOREL A. et al. — Seminar on transformation groups, *Ann. of Math. Studies,* 46, Princeton University Press, 1961.

[2] HSIANG W.C. and HSIANG W.Y. — Differentiable actions of compact connected classical groups : I, *Amer. J. Math.,* 89, 1967, p. 705-786; II, *Ann. of. Math.,* 92, No. 2, p. 189-223.

[3] HSIANG W.C. and HSIANG W.Y. — Classification of differentiable action on S^n, R^n and D^n with S^k as the principal orbit type, *Ann. of Math.,* 82, 1965, p. 421-433.

[4] HSIANG W.C. and HSIANG W.Y. — On compact subgroups of diffeomorphism group of Kervaire spheres, *Ann. of Math.,* 85, 1967, p. 359-369.

[5] HSIANG W.C. and HSIANG W.Y. — The degree of symmetry of homotopy spheres, *Ann. of Math.,* 89, 1969, p. 52-67.

[6] HSIANG W.C. and HSIANG W.Y. — A fixed point theorem for finite diffeomorphism group generated by reflections (to appear in *Notes of Springer-Verlag,* the volume in honor of Professor N.E. Steenrod for his sixtieth birthday).

[7] HSIANG W.Y. — On the principal orbit type and P.A. Smith theory of $SU(p)$-actions, *Topology,* 6, 1967, p. 125-135.

[8] HSIANG W.Y. — On the bound of the dimensions of the isometry groups of all the possible riemannian metric on an exotic sphere, *Ann. of Math.,* 85, 1967, p. 351-358.

[9] HSIANG W.Y. — *On the geometric weight system of differentiable compact transformation group on acyclic manifolds,* (to appear, memeo. at University of California at Berkeley).

[10] HIRZEBRUCH F. — Singularities and exotic spheres, *Séminaire Bourbaki,* 19, 1966-67.

[11] JANICH K. — On the classification of $O(n)$-manifold, *Math. Annalen,* 176, 1968, p. 53-76.

[12] KRAMER K. — *Hauptisotropie gruppen bei endlich dimensionalen Darstellungen kompacter halbeinfacher Lie gruppen,* Diplomarbeit, Bonn, 1966.

[13] MONTGOMERY D. and YANG C.T. — Orbits of the highest dimension, *Trans. Amer. Math. Soc.,* 87, 1958, p. 284-293.

[14] MONTGOMERY D. and SAMELSON H. and YANG C.T. — Exceptional orbits of highest dimension, *Ann. of Math.,* 64, 1957, p. 131-141.

[15] MONTGOMERY D., SAMELSON H. and YANG C.T. — Groups on E^n with $(n\text{-}2)$-dimensional orbits. *Proc. AMS,* 7, 1956, p. 719-728.

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SOME CONJECTURES ABOUT FOUR-MANIFOLDS

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The theory of topological manifolds and their piecewise linear structures has reached a certain level of completeness in dimensions not equal to four, (see references [2]-[10]). We list below some theorems on existence and uniqueness of PL structures, TOP transversality, and TOP handlebodies which are unknown in dimension 4. There are some natural conjectures about these and other theories for dimension 4 ; we show that they are all related, and that in fact the Product Structure Theorem in dimension 4 would imply all of them.

The theory presented here is rather simple ; basically it assumes that the higher dimensional theory holds as much as possible in dimension 4, given Rohlins theorem [12] and the uniqueness of PL structures on 3-manifolds [11], both of which are anomalies. If this theory does not hold, it would seem that the correct theory must be much more complicated.

Q will always refer to a TOP manifold of dimension q, and C will be a closed subset. A property is said to hold *near* a closed set if it holds on a neighborhood of the closed set. "CAT" refers to either the PL or DIFF category. A CAT structure on a manifold is denoted by a capital Greek letter, e.g. Q_Σ, except that $I = [0, 1]$ and R^s denote the unit interval and Euclidean s-space *with* their usual linear structures.

Two PL structures Σ and Θ on Q which agree near C are equivalent if there exists a PL homeomorphism $h : Q_\Sigma \to Q_\Theta$ with h = identity near C. Σ and Θ are equivalent up to isotopy (resp. homotopy) if h is isotopic (resp. homotopic) modulo C to the identity homeomorphism.

Let $h : B^k \times R^n \to V^{k+n}$ be a homeomorphism of the unit k-ball cross n-space onto a PL manifold V^{k+n} such that h is PL near the boundary $S^{k-1} \times R^n$. "Straightening the k-handle h" means finding an isotopy $h_t : B^k \times R^n \to V$, $t \in [0, 1]$, such that $h_0 = h$, h_1 is PL near $B^k \times B^n$, and $h_t = h$ near $S^{k-1} \times R^n$ and outside a compact set.

A central theorem in the theory of TOP manifolds is the.

PRODUCT STRUCTURE THEOREM [5]. — *Let $q \geqslant 5$ or $q = 5$ if $\partial Q \subset C$. Let Σ_0 be a* CAT *structure near C. Let Θ be a* CAT *structure on $Q \times R^s$ which agrees with $\Sigma_0 \times R^s$ near $C \times R^s$. Then Q has a* CAT *structure Σ, extending Σ_0 near C, and $\Sigma \times R^s$ is concordant to Θ modulo $C \times R^s$.*

In fact, there is an ϵ-isotopy $h_t : Q_\Sigma \times R^s \to (Q \times R^s)_\Theta$ with h_0 = identity, h_1 a CAT *homeomorphism, and h_t = identity near $C \times R^s$, where $\epsilon : Q \times R^s \to (0, \infty)$ is a continuous function.*

This theorem is easy to prove for $q \leqslant 2$, and $q + s \geqslant 6$ or $q + s = 3$ or $q + s = 5$ if $\partial Q \subset C$, for then Q and $Q \times R^s$ have unique PL structures up to isotopy.

The Product Structure Theorem is known to fail for closed 3-manifolds ; the PL structures up to isotopy on $Q^3 \times R^n$, Q closed and $n \geqslant 2$, are classified by $H^3(Q ; Z_2) = Z_2$ but there is only one PL structure up to isotopy on Q^3. Moreover, these counterexamples are not valid just for equivalence *up to isotopy*, because $S^3 \times R^2$ has two PL structures which are not equivalent (PL homeomorphic) [2], [8].

Conjecture A_1. The Product Structure Theorem holds for $q = 5$ and $\partial Q \not\subset C$ or for $q = 4$ and $\partial Q \subset C$.

Conjecture A_2. The Product Structure Theorem holds for $q = 4$ and $\partial Q \not\subset C$ (respectively for $q = 3$) if ∂Q (respectively Q) has a handlebody decomposition with no 3-handles which are not in C.

Conjecture A_1 implies Conjecture A_2 (see Theorem 6). In fact Conjecture A_1 implies all the other conjectures in this paper. Note that Conjectures A_1 and A_2 imply that the Product Structure Theorem holds for all open manifolds with $C = \emptyset$.

To prove A_1, it would suffice to know that several 5-dimensional relative CAT s-cobordisms were CAT products [5]. Consider the CAT s-cobordism $(Z ; Y_0, Y_1)$ where $(Z ; Y_0, Y_1)$ is homeomorphic to either $(I ; 0, 1) \times B^k \times T^{4-k}$ or

$$(I ; 0, 1) \times B^k \times S^{3-k} \times R$$

and the homeomorphism is CAT near $1 \times Y_1$ and near $I \times \partial Y_1$. Then Conjecture A_1 is equivalent to knowing $(Z ; Y_0, Y_1)$ is CAT homeomorphic to $(I ; 0, 1) \times Y_1$ relative to $1 \times Y_1$ and $I \times \partial Y_1$.

CONCORDANCE-IMPLIES-ISOTOPY THEOREM [2], [5], [8]. — *Let $q \geqslant 6$ or $q = 5$ if $\partial Q \subset C$. Let Γ be a* CAT *structure on $I \times Q$, and $0 \times \Sigma$ its restriction to $0 \times Q$. Suppose $\Gamma = I \times \Sigma$ near $I \times C$. Then there exists an ϵ-isotopy $h_t : I \times Q_\Sigma \to (I \times Q)_\Gamma$, $t \in [0, 1]$, such that $h_0 =$ identity, $h_t =$ identity on $0 \times Q$ and near $I \times C$, and h_1 is a* CAT *homeomorphism, where $\epsilon : I \times Q \to (0, \infty)$ is continuous.*

This theorem is well known [11] for $q \leqslant 2$. It also fails in dimension 3 or 4 because if it was true in all dimensions, it would imply the Product Structure Theorem in all dimensions [5].

Conjecture B_1. Concordance implies isotopy for $q = 5$ and $\partial Q \not\subset C$ or $q = 4$ and $\partial Q \subset C$.

Conjecture B_2. Concordance implies isotopy for $q = 4$ and $\partial Q \not\subset C$ (respectively for $q = 3$) if ∂Q (respectively Q) has a handlebody decomposition with no 2-handles that are not in C.

THEOREM 1. — *Conjecture A_1 is equivalent to Conjecture B_1.*

Proof. — In [5], it is shown that the Concordance-implies-isotopy Theorem plus the Annulus Theorem are together equivalent to the Product Structure Theorem. The same method of proof gives Theorem 1 since Conjecture B_1 implies the Annulus Conjecture in dimension 4.

CLASSIFICATION THEOREM [3] [6] [8] [10]. — *Let $q \geqslant 6$, or $q = 5$ and $\partial Q \subset C$. The homotopy classes of reductions of the stable tangent bundle of Q to a* CAT *bundle, modulo C, correspond bijectively to isotopy classes of* CAT *structures on Q agreeing with a given* CAT *structure Σ_0 near C.*

This theorem fails for closed 3-manifolds, for there exist two reductions but only one CAT structure on Q^3.

However the non-stable Classification Theorem [6], [8], [9], (for reductions of the tangent bundle itself) holds for $q \leqslant 3$ as well as $q \geqslant 6$, or $q = 5$ and $\partial Q \subset C$.

THEOREM 2. — *Conjecture A_1 implies that the stable Classification Theorem holds also for $q = 5$ or $q = 4$ and $\partial Q \not\subset C$, and that the non-stable Classification Theorem holds without dimensional restriction.*

Proof. — The only ingredients in the proof of the stable Classification Theorem requiring dimensional restrictions are the Product Structure Theorem and the Concordance-implies-isotopy Theorem ; the dimensional restrictions are lowered by one in Conjectures A_1 and B_1. The non-stable version uses immersion theory and requires only Conjecture B_1 to hold in all dimensions.

THEOREM 3. — *Conjecture A_1 implies*

(i) *$h : B^k \times R^{4-k} \to V^4$ can be "straightened" if $k \neq 3$,*

(ii) *there exists $h' : B^3 \times R \to V$ which cannot be straightened, and*

$$h' \times \mathrm{id} : B^3 \times R^2 \to V \times R$$

corresponds to the non-zero element in $\pi_3(\mathrm{TOP}_5, \mathrm{PL}_5) = Z_2$.

(iii) $\pi_4(\mathrm{TOP}_4, \mathrm{PL}_4) = \begin{cases} 0 & k \neq 3 \\ Z_2 & k = 3 \end{cases}$

(iv) $\pi_3(\mathrm{TOP}_3, \mathrm{PL}_3) = 0 \to \pi_3(\mathrm{TOP}_4, \mathrm{PL}_4) \xrightarrow[s]{\cong} \pi_3(\mathrm{TOP}_5, \mathrm{PL}_5) \xrightarrow{\cong} \cdots$ *where s is the stabilization map.*

Note that similar statements are true in dimensions > 4.

Proof. — (i) h pulls back a PL structure from V onto $B^k \times R^{4-k}$, say Σ, which agrees with the standard structure near the boundary. $\Sigma \times R$ is equivalent (modulo boundary) up to isotopy with the standard structure because $h \times \mathrm{id}$ can be straightened [5]. By Conjecture A_1, Σ is equivalent (modulo boundary) up to isotopy with the standard structure, so we compose this isotopy with h to straighten h.

(ii) $B^3 \times R^2$ has an exotic PL structure which agrees with the standard one near the boundary, so by Conjecture A_1, $B^3 \times R$ has an exotic PL structure Σ' which is standard near $S^2 \times R$. Let $h' = \mathrm{id} : B^3 \times R \to (B^3 \times R)_{\Sigma'}$.

(iii) and (iv) follow as in [2], or [3] or [7].

It is known that if Σ_0 is a PL structure near C, then Q has a PL structure Σ extenting Σ_0 near C if $q \leqslant 3$ or if $q \geqslant 6$ and $H^4(Q, C; Z_2) = 0$ or if $q = 5$, $\partial Q \subset C$ and $H^4(Q, C; Z_2) = 0$. The remaining cases are taken care of by

THEOREM 4. — *Conjecture A_1 implies*

(i) *If $q = 4$ and $\partial Q \subset C$ or $q = 5$ and $\partial Q \not\subset C$, then Q has a* PL *structure Σ extending Σ_0 near C if $H^4(Q, C; Z_2) = 0$,*

(ii) *If $q = 4$ and $\partial Q \not\subset C$ then Σ exists if $H^4(Q, C \cup \partial Q; Z_2) = 0$.*

Proof. — (i) The theorem follows immediately because the stable Classification Theorem holds in these dimensions (see Theorem 2).

(ii) ∂Q has a unique PL structure [11] so we extend Σ_0 to a neighborhood of $\partial Q \cup C$ and use (i).

Also it is known that if Q has a PL structure Σ, then the PL structures, agreeing with Σ near C, are unique up to isotopy if $q \leqslant 3$, and are classified up to isotopy by $H^3(Q, C; Z_2)$ if $q \geqslant 6$, or if $q = 5$ and $\partial Q \subset C$.

THEOREM 5. — *Conjecture A_1 implies*

(i) *the isotopy classes of* PL *structures* mod *C are still classified by* $H^3(Q, C; Z_2)$ *if* $q = 5$ *and* $\partial Q \not\subset C$ *or if* $q = 4$ *and* $\partial Q \subset C$.

(ii) *the isotopy classes of* PL *structures* mod *C are classified by*

$$H^3(Q, C \cup \partial Q; Z_2)$$

if $q = 4$ *and* $\partial Q \not\subset C$.

Proof. — See the proof of Theorem 4.

THEOREM 6. — *Conjecture A_1 implies Conjecture A_2 and Conjecture B_1 implies Conjecture B_2.*

Thus we have

$$\begin{array}{ccc} \text{Conjecture } A_1 & \Leftrightarrow & \text{Conjecture } B_1 \\ \Downarrow & \searrow & \Downarrow \\ \text{Conjecture } A_2 & \Rightarrow & \text{Conjecture } B_2 \end{array}$$

Proof. — These Conjectures depend on the following conjecture which is a handle version of concordance-implies-isotopy (see §§ 3, 4, 5 of [5]).

Let $H : (I, 0) \times B^k \times R^n \to (X, V)$ be a homeomorphism, which is CAT near $(1 \times B^k \times R^n) \cup (I \times S^{k-1} \times R^n)$, onto a CAT manifold X where V is a codimension one, CAT locally flat, submanifold. Then there exists a pairwise isotopy $H_t : (I, 0) \times B^k \times R^n \to (X, V)$, $t \in [0, 1]$, with $H_0 = H$, $H_1 =$ CAT homeomorphism on $I \times B^k \times B^n$, and $H_t = H$ near $(1 \times B^k \times R^n) \cup (I \times S^{k-1} \times R^n)$ and outside a compact set.

The Concordance-implies-isotopy Theorem gives an H_t if $k + n \neq 3, 4$. Conjecture A_1 implies the cases $k + n = 4$, and $k + n = 3$ with $k \neq 2$. But if $k = 2, n = 1$, then some H cannot be straightened. The implications in Theorem 5 can be derived from this.

THEOREM 7 (TOP transversality). — *Let $\xi^n = (E(\xi^n) \xrightarrow{\pi} X)$ be an n-plane bundle over a topological space X and let $f : M^m \to E(\xi^n)$ be a continuous function. Then if $m \neq 4$ and $m - n \neq 4$, f is homotopic to a map f_1 which is transverse to the 0-section of ξ (this means that f^{-1} (0-section) is an $(m - n)$-manifold P with normal bundle in M equal to $(\pi f_1 | P)^*(\xi)$). Moreover, if f is already transverse near a closed set C in M, then the homotopy equals f near C.*

THEOREM 8. — *Conjecture A_1 implies that* TOP *transversality holds in all dimensions.*

Proof. — The proof of Theorem 7 (see [2]) uses only the Product Structure Theorem for open manifolds with $C = \emptyset$ which follows in all cases from Conjecture A_1.

THEOREM 9 (TOP handlebody structures). — *If* $m \geqslant 6$, *then* M^m *is a* TOP *handlebody* (*if* $m = 6$ *and* $\partial M \neq \emptyset$, *then we obtain* M *by adding handles to* ∂M). *Equivalently,* M *admits a Morse function* $f : M \to R$ (*that is,* f *is locally of the form* $x_1^2 + \cdots + x_\lambda^2 - x_{\lambda+1}^2 - \cdots - x_m^2$.

THEOREM 10. — *Conjecture* A_1 *implies that all* 5*-manifolds are* TOP *handlebodies. It follows that there is a* 4*-manifold which is not a* TOP *handlebody.*

Proof. — See [2] [13]. Note that if dim $(\partial M) = 4$, then we give M a handlebody structure by adding handles to ∂M. The boundary of a 5-dimensional TOP handlebody is not necessarily a TOP-handlebody.

It is known [7] that $Z = \pi_4(B_{PL}) \to \pi_4(B_{TOP}) = Z$ is multiplication by two. If ξ is the generator of $\pi_4(B_{TOP})$, then ξ represents a TOP n-plane bundle

$$\xi^n = (E(\xi^n) \xrightarrow{\pi} X)$$

which is fiber homotopy trivial. If $f : E(\xi^n) \to R^n$ is the trivialization, then using TOP transversality (Theorem 8), $M^4 = f^{-1}(0)$ is a closed, almost parallelizable 4-manifold. Furthermore, the identification of $\pi_4(B_{TOP})$ with Z tells us that index $(M^4) = 8$, because the generator of $\pi_4(B_{PL})$ corresponds to a PL 4-manifold of index 16.

M^4 cannot be PL by Rohlins Theorem [12]. Therefore M^4 is not a TOP handlebody. (Any 4-dimensional TOP handlebody is PL since the attaching maps are 3-dimensional imbeddings and can be straightened [11]). A more explicit construction of such an M^4 appears in [13].

REFERENCES

[1] KIRBY R. — Stable homeomorphisms and the annulus conjacture, *Ann. of Math.*, 89, 1969, p. 575-582.

[2] KIRBY R. — Lectures on triangulations of manifolds, *Mimeo. notes,* U.C.L.A., Los Angeles, 1969.

[3] KIRBY R. and SIEBENMANN L.C. — On the triangulation of manifolds and the Hauptvermuntung, *Bull. Amer. Math. Soc.*, 75, 1969, p. 742-749.

[4] KIRBY R. and SIEBENMANN L.C. — *Notices Amer. Math. Soc.*, 16, 1969, p. 587, 695, 848.

[5] KIRBY R. and SIEBENMANN L.C. — *Deformation of smooth and piecewise linear manifold structures,* to appear.

[6] KIRBY R. and SIEBENMANN L.C. — *Classification of smooth and piecewise linear manifold structures using the product structure theorem,* to appear.

[7] KIRBY R. and SIEBEMMANN L.C. — To appear (sequel to [5], [6]).

[8] KIRBY R. and SIEBENMANN L.C. — Some theorems on topological manifolds, to appear in Proceedings of summerschool on manifolds, Amsterdam, 1970, *Lecture notes on Math.*, Springer-Verlag.

[9] LASHOF R.K. — The immersion approach to triangulation, Topology of Manifolds, (edited by Cantrell and Edwards), Markham, Chicago, 1970, p. 52-57.

[10] Lashof R.K. and Rothenberg M. — Triangulation of manifolds I, II, *Bull. Amer. Math Soc.*, 75, 1969, p. 750-757.
[11] Moise E. — Affine structures on 3-manifolds, *Ann. of Math.*, 56, 1957, p. 96-114.
[12] Rohlin V.A. — A new result in the theory of 4-dimensional manifolds, *Doklady*, 8, 1957, p. 221-224.
[13] Siebenmann L.C. — Disruption of low dimensional handlebody theory by Rohlins Theorem, (edited by Cantrell and Edwards), Markham, Chicago, 1970, p. 57-77.

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THE DIFFERENTIAL TOPOLOGY OF SEPARABLE BANACH MANIFOLDS

by Nicolaas H. KUIPER *

1. Introduction.

The differential topology of separable metrisable Banach manifolds has recently made considerable advances. Five years ago Bessage proved (following an analogous homeomorphy statement of Klee) that two specific homotopy trivial manifolds, namely Hilbert space H and $H \setminus \{0\}$ are in fact diffeomorphic. At present, the main, interesting but not so stimulating, conjecture or conclusion is that manifolds of infinite dimension and their embeddings have not more structure than that given by homotopic invariants, like the tangential homotopy type, easily suggested by a topologist concerned with finite dimensional manifolds. The same general conclusion holds for the topology of topological manifolds modelled on possibly non-separable Fréchet spaces (compare the lecture of R.D. Anderson), and also for differentiable Hilbert manifolds with Fredholm or layer structure (compare the lecture of J. Eells in which he announces that Fredholm Hilbert manifolds are completely classified by their Fredholm reduced tangential homotopy type). It is easy to see that a smooth Hilbert manifold carries many non-equivalent complex analytic structures. Nothing, except the equivalence of H and $H \setminus \{0\}$, is known yet concerning real analytic structures, however. Also non-separable smooth Banach manifolds have not been touched upon.

This paper gives only a survey of some pure differential topological aspects of Banach manifolds without applications to analysis on finite dimensional manifolds. Such analysis will need consideration of additional geometric structure in the manifold as space of maps.

2. Tangential homotopy types.

Let E be a *separable Banach space of infinite dimension* with a C^k-norm n : $E \to [0, \infty)$, $n(x) = \|x\|$, i.e. n has continuous *m-th* derivatives for $m \leqslant k \leqslant \infty$ at all $x \neq 0 \in E$. We call E a C^k-Banach space. Recall that for example the Banach space l_p, $0 < p \notin 2\mathbf{Z}$, has a C^k-norm for $k \leqslant p$, but not for $k > p$. It is not known whether the existence of a non-zero C^k-function of bounded support implies the existence of a C^k-norm. An *E-chart* (κ, U) for a topological space M is a homeomorphism κ between open sets $U \subset M$ and $\kappa(U) \subset E$. A metrisable topo-

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logical space M covered by an atlas of E-charts is called a topological E-manifold. It is called *a* Λ-E-manifold or short Λ-manifold in case for any two charts (κ_1, U_1) and (κ_2, U_2) the restriction of $\kappa_2 \kappa_1^{-1}$ to any component of $\kappa_1(U_1 \cap U_2)$ belongs to Λ, a subpseudo group of the pseudo group C^0 of homeomorphisms between open sets of E.

For example : Λ can be the pseudo group of diffeomorphisms Φ that are of class C^q in E, $1 \leqslant q \leqslant k$; or, real analytic (C^ω) ; complex analytic (C) ; or Fredholm ($d\Phi_x$ = identity + ρ_x, ρ_x a compact operator).

If $\Lambda \subset C^1$, then the derivatives $d\psi_x$ of elements $\psi \in \Lambda$ generate a subgroup G_Λ of $GL(E)$ the group of invertable linear operators from E to E with the norm topology. It may be of interest to consider the group generated by 2-jets $j_x^2(\psi)$ for $\psi \in \Lambda \subset C^2$, for example if Λ is the affine group, or if Λ consists of elements that locally differ from the identity by a map with image in a finite dimensional subspace (Layer maps).

$\underline{\Lambda}$ will be the set of equivalence classes of Λ-E-manifolds. If $\Lambda \subset \Lambda'$ then there is a *natural forget map* $\alpha = \alpha(\Lambda, \Lambda') : \underline{\Lambda} \to \underline{\Lambda'}$.

Let CW be the class of locally finite CW complexes, and $\underline{CW}$ the set of homotopy equivalence classes or *homotopy types* in CW. Homotopy equivalence will be denoted by $\sim$. Two E-vector bundles ξ and η over X and Y in CW, are called equivalent ($\sim$) if there is a bundle map, which is an isomorphism on each fibre, $f^* : \xi \to \eta$, that covers some homotopy equivalence $f : X \to Y$. The equivalence classes are E-bundle *types*. They form a set $E(\underline{CW})$. If the bundle is reduced to some subgroup $G \subset GL(E)$ we get a set of equivalence classes $E_G(\underline{CW})$.

Every Λ-E-manifold ($\Lambda \subset C^q$) has the *homotopy type* $\in \underline{CW}$ of a locally finite countable CW-complex for $q \geqslant 0$, and its tangent E-bundle for $q \geqslant 1$ has an E-bundle type, called its *tangential homotopy type*, $\in E(\underline{CW})$ and (if $G_\Lambda \neq GL(E)$) a *G_Λ-reduced tangential homotopy type* $\in E_{G_\Lambda}(CW)$.

Here is a diagram of maps :

$$
\begin{array}{ccccccc}
\underline{\Lambda} & \longrightarrow & \underline{C^q} & \longrightarrow & \underline{C^1} & \longrightarrow & \underline{C^0} \\
\downarrow & & & \searrow^{\alpha_q} & \downarrow{\scriptstyle \alpha_1} & & \downarrow \\
E_G(\underline{CW}) & & \longrightarrow & & E(\underline{CW}) & \xrightarrow{\beta} & \underline{CW}
\end{array} \tag{1}
$$

The values are *invariants*, that can distinguish certain manifolds. No counter-example has been found to the following :

MAIN CONJECTURE 1. — *If E has a C^q-norm, then C^r-E-manifolds are completely classified by their tangential homotopy type, and α_r is bijective for* $1 \leqslant r \leqslant q$. (Also in case $q < r = \infty$, there are no counter-examples). One can prove :

THEOREM 1. — *α_∞ is surjective in case there is a splitting $E = E' \oplus E''$ with* $\dim E'' = \infty$, *inducing a homotopy equivalence $g' \to g' \oplus id_{E''} : GL(E') \to GL(E)$.* (Every E seems to have such a splitting). β is bijective if and only if $GL(E)$, the space of invertable linear operators from E to E with the norm topology, *is contractible.* Since my proof for $E = H$(= Hilbert space), such contractibility was established for c_0 (Arlt) ; $l_p(1 \leqslant p < \infty)$ (Neubauer) ; C, the space of continuous functions on a compact metric space with more than a countable number of points (Edelstein, Mitjagin, Semenov) ; $L_p([0,1])$ (isometric to $L_p(\mu)$ for μ non atomic measure on a space) (Mitjagin). On the other hand Douady found that $GL(c_0 \oplus H) \sim \mathbf{Z} \times BO$ is not contractible, hence β is not bijective for $E = c_0 \oplus H$ ($c_0 \oplus H$ as well as c_0 has a compatible C^∞-norm).

R.C. James constructed long ago (1951) the interesting Banach space

$$J = J_p = \{x = (x_1, x_2, \dots) : x_i \in \mathbf{R},\ \lim_{j=\infty} x_j = 0\}$$

with norm $\|x\| = \sup |(x_{k_1} - x_{k_2})^p + \cdots (x_{k_n} - x_{k_1})^p|^{1/p} < \infty$, where sup is the least upper bound over finite sequences $1 \leqslant k_1 < k_2 \dots < k_n$, for $p = 2$.

Crucial is that J embeds naturally with codimension one in its double dual J^{**}. Then for any bounded linear operator $\sigma : J^n \to J^n$ we obtain Elworthy's invariant $\delta(\sigma)$ in the commutative diagram with exact rows :

$$(2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & J^n & \longrightarrow & J^{**n} & \longrightarrow & \mathbf{R}^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma^{**}} & & \downarrow{\scriptstyle \delta(\sigma)} & & \\ 0 & \longrightarrow & J^n & \longrightarrow & J^{**n} & \longrightarrow & \mathbf{R}^n & \longrightarrow & 0 \end{array}$$

Mitjagin and Edelstein proved :

(3) $\delta : GL(J^n) \to GL(\mathbf{R}^n)$ is a homotopy equivalence.

(Observe (Dieudonné) that J^{2m+1} has no complex structure $\sigma : \sigma^2 = -1$, because $(\delta(\sigma))^2 = -1$ is impossible).

The (fairly simple) most complicated homotopy type for $GL(E)$ known is obtained with Douady's method applied to James spaces J_p for different values of p $(1 < p < \infty)$:

$$GL(E) \sim GL(\mathbf{R}^{n_0}) \times \cdots \times GL(\mathbf{R}^{n_m}) \times (BO \times \mathbf{Z})^m \quad \text{for}$$

$$E = J_{p_0}^{n_0} \oplus \dots \oplus J_{p_m}^{n_m} \quad \text{(Mitjagin – Edelstein).}$$

3. Hilbert manifolds.

The first main result was.

THEOREM 2. – α_q *(and* β *of course) is bijective for separable Hilbert manifolds and* $0 \leqslant q \leqslant \infty$.

The history of the proof for $q = \infty$ was as follows. Burghelea and Kuiper proved :

(1) If a complete Riemannian manifold M has a Morse function giving a handle decomposition, then M is Palais stable : $M \simeq M \times H$ (conjectured in general by Palais ; $\simeq$ means diffeomorphic) ;

(2) $M \sim M'$ implies $M \times H \simeq M' \times H$. Nicole Moulis exhibited such a function for any open set in H. Consequently homotopy equivalent open sets of H are

diffeomorphic. Next Eells and Elworthy used a theory of Fredholm maps f for example into H, earlier developed by Elworthy, and a transversality of Mukherjea with respect to a dense flag embedding $\mathbf{R}^1 \subset \mathbf{R}^2 \subset \mathbf{R}^3 \cdots \subset H$, to show that M is a union of nicely nested tubes around compact manifolds $f^{-1}(\mathbf{R}^n)$. This led to an open embedding $M \hookrightarrow H$ and direct proofs of all statements required here for $q = \infty$. Nicole Moulis proved that C^1-functions on H with values in a linear space can be C^1-approximated by C^∞-functions, whence a C^q-embedded image in H can be approximated by a C^∞-image. The case $q \geqslant 1$ then follows. The case $q = 0$ is due to Henderson (See Anderson's lecture).

Eells and Elworthy obtained considerable *generalisations.* I mention :

THEOREM 3. – *If E has a C^∞-norm, a Schauder basis, and for some F, $E \simeq E \oplus F$, $\dim F = \infty$, and the map $g \to g \oplus id_F : GL(E) \to GL(E \oplus F) = GL(E)$ is a homotopy equivalence, then $\beta \circ \alpha_\infty$ is bijective for parallelisable E-manifolds.*

This applies to $E = c_0 \oplus H$ for example. The main conjecture is still open for non parallelisable manifolds and for $1 \leqslant q < \infty$, but the most interesting theorems may have been obtained now.

4. Embedding an E-manifold in F.

Following finite dimensional immersion theory we propose the *conjecture* 2. *There is a closed split C^q-embedding f of a separable C^q-E-manifold M into a Banach space F, if and only if there is a bundle map for the tangent bundle $\tau_M \to F$ which is injective on each fibre.* If $E \simeq E \oplus E = F$ (e.g. $= c_0 \oplus H$), then f exists always (Kuiper-Terpstra). If M is parallelisable and $F = E \oplus E'$, $\dim E' = \infty$, then also (Elworthy). A more interesting illustration concerns J^n-manifolds :

THEOREM 4. – *The following conditions are equivalent for a J^n-manifold M :*

(a) *M has a closed split embedding into J^k ;*

(b) *There is a bundle map, monomorphic on each fibre :*

$$\alpha : \tau_M \to J^k \ ,$$

(c) *There is a bundle map, monomorphic on each fibre :*

$$\alpha' : \delta(\tau_M) \to \mathbf{R}^k$$

Proof. – Clearly (a) $\Rightarrow$ (b), and (b) $\Rightarrow$ (c) with the *definitions* of the $\mathbf{R}^n$-bundle $\delta(\tau_M)$ and of $\alpha' = \delta(\alpha)$ by the commutative diagram with exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_M & \longrightarrow & \tau_M^{**} & \longrightarrow & \delta(\tau_M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha^{**}} & & \downarrow{\scriptstyle \delta(\alpha)} & & \\ 0 & \longrightarrow & J^k & \longrightarrow & J^{**k} & \longrightarrow & \mathbf{R}^k & \longrightarrow & 0 \end{array}$$

Given (c) we find a normal $\mathbf{R}^{k-n}$-bundle ζ over M, such that

$$\delta(\tau_M) \oplus \zeta \cong \mathbf{R}^k \qquad \text{(trivial bundle)}$$

hence $\tau_M \oplus (\zeta \oplus J) \cong \mathbf{R}^k \oplus J = J^k$ (trivial bundle ; see (3)).

As $J \cong J \oplus H$ (Jameson), Elworthy's embedding theorem above gives an embedding of the parallelisable (!) total space of $\zeta \oplus J$ (with 0-section M) into J^k. A closed split embedding for M is obtained with a suitable additional C^1-function whose values space $\mathbf{R}$ can be absorbed into J^k, to obtain (a).

COROLLARY. – (Elworthy, modified). If η_n is the universal $\mathbf{R}^n$-bundle over BO_n, then $\xi = \eta_n \circ J$ is a J^n-bundle over BO_n. By theorem 1 there is a *J^n-manifold* M with tangential homotopy type ξ. *M has no C^1-embedding in J^k for $1 \leqslant k < \infty$.* For the proof observe that $\delta(\tau_M) \sim \eta_n$.

5. Hilbert manifold pairs, knots.

Following finite dimensional theory (e.g. W. Browder) one can guess and prove with Elworthy's results :

THEOREM 5. – *A complete set of invariants for a pair (X, Y) consisting of a connected separable C^∞-H-manifold X and a connected closed C^∞-H-submanifold Y is :*

(a) *the homotopy type $h(X, Y)$ of (X, Y) ;*

(b) *the homotopy type $h(X \setminus Y)$ of the complement ;*

(c) *$h\nu(X, Y)$ the bundle type of the normal bundle $\nu(X, Y)$ of Y in X, and*

(d) *$\gamma\nu(X, Y)$, the homotopy class of the normal exponential embedding of a small normal sphere bundle of Y in X into $X \setminus Y$.*

COROLLARY 6. – *All knots (H, S^{-k}), where S^{-k} is a closed embedded Hilbert-space of codimension k in H, are trivial for $k \neq 2$. For $k = 2$ the knot is characterised by the homotopy type $h(H \setminus S^{-2})$ and some element $\gamma \in \pi_1(H \setminus S^{-2})$ representing $\gamma\nu(H, S^{-2})$.* Sufficient for the existence of a knot with given homotopytype $L = h(H \setminus S^{-2})$ is that $H_*(L) \cong H_*(S^1) \cong \mathbf{Z}$ and $\pi_1(L)$ is generated by elements conjugate to some element $\gamma \in \pi_1(L)$ which generates $H_*(S^1)$ (More complicated is the situation in finite dimensions (Kervaire)). Finally we observe that the cone from 0 on an essential knot of codimension 2 in the unitsphere of H, gives an essential isolated singularity on a manifold of codimension 2 in H (a question suggested by R.D. Anderson).

REFERENCES

Can be found in the report of R.D. Anderson, Eells and Palais in this congress and in the following papers and others forthcoming by the same authors.

BURGHELEA, KUIPER. — *Ann. of Math.*, 90, 1969, p. 379-417.
EELLS, ELWORTHY. — *Ann. of Math.*, 91, 1970, p. 465-485.
EELLS. — *Bull. A.M.S.*, 72, 1966, p. 751-807.
JAMES R.C. — *Proc. Nat. Acad. of Sc.*, 37, 1951, p. 174-177.

PALAIS. — *Topology*, 5, 1966, p. 1-16.
MOULIS. — Thèse, Paris, 1970, *Ann. Inst. Fourier*, 1971.
EDELSTEIN, MITJAGIN, SEMENOV. — *Bull. Acad. Sci. Pol.*, 1970.
MITJAGIN, EDELSTEIN. — Homotopy type of the linear groups of two classes of Banach spaces (Russian), *Functional analysis*, 4, No. 3, 1970.
KUIPER, BESSELINE TERPSTRA. — *Essais dédiés à G. de Rham*, Springer Verlag, 1970, p. 118-125.

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THE IMMERSION APPROACH TO TRIANGULATION AND SMOOTHING

by R. LASHOF

That immersion theory should be useful in triangulation and smoothing problems is indicated by the following (where we abbreviate piccewise linear by *PL*) :

LEMMA 1. – Let $f : M^n \to Q^n$ be an immersion (i.e. a local homeomorphism) of a topological manifold into a smooth (*PL*) manifold ; then f induces a smooth (*PL*) structure on M.

Topological immersion theory was developed by Jack Lees [8] using a result of R. Kirby on stable homeomorphisms. Namely, Lees proved an isotopy extension theorem for topological manifolds using the result that a homeomorphism of the n-torus induces a stable homeomorphism of R^n by passing to the universal cover [3].

The following lemma is a direct result of immersion theory :

LEMMA 2. – Let K be a smooth (*PL*) compact n-manifold with boundary embedded topologically in Euclidean n-space E^n as a locally flat submanifold. Let τK be the tangent microbundle of K. Suppose the reduction of τK to a smooth (*PL*) microbundle, defined by the smooth (*PL*) structure of K, entends to a reduction of τE^n to a smooth (*PL*) microbundle. Then the smooth (*PL*) structure on K extends to a smooth (*PL*) structure on $E^n - P$, where P is a finite subset.

Further one may show that the reduction of $\tau(E^n - P)$ defined by the smooth (*PL*) structure is the assumed reduction.

An immediate consequence of Lemma 2 is :

THEOREM 1. – *Let M^n be an open topological manifold, and suppose τM has a smooth (PL) reduction. Then, for any n, M^n admits a smooth (PL) structure.*

COROLLARY. – Every contractible open topological manifold is smoothable.

To prove uniqueness we use engulfing techniques (and hence require $n \geqslant 5$) and the following :

PROPOSITION. – (Haefliger and Poenaru [3] : Let $f : I \times M^n \to I \times Q^n$ be a regular homotopy (i.e. a level preserving local homeomorphism). Let $K \subset M$ be compact. Then for any $t_0 \in I$, there is an $\epsilon > 0$, and a neighbourhood U of K, such that $f_t \mid U$ can be factored

$$f_t = f_{t_0} \circ h_t \ , \ |t - t_0| < \epsilon \ ,$$

where h_{t_0} is the inclusion and $h_t : U \to M$ is an isotopy.

The uniqueness theorem implies existence for closed manifolds of dimension at least five, and we obtain :

THEOREM 2. – *Let M^n, $n \geqslant 5$, be a topological manifold without boundary. Then the isotopy classes of smooth (PL) structures on M are in one to one correspondence with the equivalence classes of reductions of τM to a smooth (PL) microbundle.*

Similar results hold for manifolds with boundary. Details appear in [6]. See also [5].

Theorem 2 has also been proved by Kirby and Siebenmann using other techniques [4].

For $n = 4$ it can be shown that if M^4 is a closed manifold and τM reduces to a smooth microbundle then $M \# k(S^2 \times S^2)$, (connected sum with k copies of $S^2 \times S^2$) is smoothable, for some k. Further if two smoothings M_α, M_β of M^4 correspond to equivalent reductions, then

$M_\alpha \# k(S^2 \times S^2)$ is isotopic to $M_\beta \# k(S^2 \times S^2)$ for some k.

(*PL* and smooth are equivalent for $n = 4$). This will appear in a forthcoming paper with J. Shaneson.

Reductions of τM correspond to homotopy classes of lifts of the classifying map

$$\tau : M^n \to B\,\mathrm{Top}_n$$

to BPL_n or BO_n ; and hence are determined by the homotopy type of the fibres Top_n/PL_n and Top_n/O_n

Using a result of Kirby [3] on stable homeomorphism and Lees' immersion theorem one may prove [7] :

THEOREM 3. – $\pi_i(\mathrm{Top}_n/PL_n) \simeq \pi_i(\mathrm{Top}/PL)$, $i \leqslant n$, $n \geqslant 5$.

$$\pi_i(\mathrm{Top}_n/O_n) \cong \pi_i(\mathrm{Top}/O),\ i \leqslant n,\ n \geqslant 5\,.$$

Using non-simply connected surgery results of Hsiang, Shaneson and Wall [2] and [9], Kirby and Siebenmann [4] have shown

THEOREM 4. – $\pi_i(\mathrm{Top}/PL) = 0\ ,\ i \neq 3$

$$Z_2\ ,\ i = 3\,.$$

Thus the only obstruction to putting a *PL* structure on M^n, $n \geqslant 5$, lies in $H^4(M\,;Z_2)$; and the only obstruction to equivalence of *PL* structures lies in $H^3(M\,;Z_2)$.

REFERENCES

[1] HAEFLIGER A. and POCNARU V. — La classification des immersions combinatories, *I.H.E.S.*, No. 23, 1964.

[2] ILSIANG W.C. and SHANESON J. — Fake Torl, etc., *Proc. Nat. Acad. Sci.* (U.S.A.), 1969, p. 687-691.

[3] KIRBY R — Stable homeomorphisms and the annulus conjecture, *Ann. of Math.*, 2, 90, 1969.

[4] KIRBY R. and SIEBENMANN L. — On the triangulation of manifolds and the haup-rermuting, *Bull. AMS*, 1969, 75, p. 742-749.

[5] LASHOF R. — Lees' immersion theorem and the triangulation of manifolds, *Bull AMS*, 75 1969, p. 535-538.

[6] LASHOF R. — The immersion approach to triangulation and smoothing. *Proc. Adv. Study Inst. Alg. Top.*, 1970, Aarhus, Denmark.

[7] LASHOF R. and ROTHENBERG M. — Triangulation of manifolds, I and II, *Bull. AMS*, 75, 1969, p. 750-757.

[8] LEES J. — Immersions and surgeries on topological manifolds, *Bull AMS*, 75, 1969, p. 529-534.

[9] WALL C.T.C. — Homotopy theory, *Bull. London Match. Soc.*, 1, 1969, p. 95-97.

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THE ROLE OF THE SEIFERT MATRIX IN KNOT THEORY

by J. LEVINE

1. I will be interested in the following situation.

$K^n \subset \mathbf{R}^{n+2}$ is an imbedded oriented sphere (topological) – the imbedding may be smooth or *PL* locally flat. If n is odd, say $n = 2q - 1$, one can associate to K^n a matrix, called the *Seifert matrix,* in the following way. Let $M^{n+1} \subset \mathbf{R}^{n+2}$ be a submanifold bounded by K^n – one always exists. Then define a pairing Φ : $Hq(M) \otimes Hq(M) \to Z$ as follows. If α, $\beta \in Hq(M)$, choose representative cycles α', β'. Translate α' off M in the positive normal direction (defined from the orientation of K) and define $\Phi(\alpha \otimes \beta)$ to be the linking number of the translated α' with β'. Φ is bilinear and satisfies : $\Phi + (-1)^q \Phi^T = - B$ where Φ^T is the transpose of Φ and B is the intersection pairing of M. Any representative matrix A of Φ is called a Seifert matrix of K. $A \pm A^T$ is unimodular, since B is. Conversely any A satisfying this property is the Seifert matrix of a knot – unless $n = 3$, in which case the additional condition – signature $(A + A^T) \equiv 0$ mod 16 – is needed and even then it is not quite true. See [1] for details.

Naturally K does not determine a unique Seifert matrix – any congruent matrix is clearly also a Seifert matrix. But even more so, the size of A can be changed by altering M. For example adding a handle to M of index q which links the other q-cycles in M will enlarge A to one of the forms :

$$\left(\begin{array}{c|c} A & \xi\,0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} A & 0 \\ \hline \xi & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{array}\right)$$

where ξ is, respectively, a column or row vector. We call these, respectively, right and left enlargements.

These types of enlargements, together with congruence, generate an equivalence relation we call *S-equivalence* (in [1], it is called *equivalence*). This was first considered by Trotter [3] and Murasugi [2].

THEOREM [1]. – *Any two Seifert matrices of a knot are S-equivalent.*

This was proved by Murasugi [2], when $n = 1$. The proof proceeds by considering a cobordism $V \subset I \times \mathbf{R}^{n+2}$ between two choices of M, stationary on K. By considering a handle-body decomposition of V, the transition between the Seifert matrices can be broken down into steps of the sort used to define S-equivalence.

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(1) This work was done while the author was partially supported by NSF GP 21510.

Thus S-equivalence is the correct relation on Seifert matrices. It cannot be expected that the S-equivalence class of its Seifert matrices is a complete invariant of knot type, since it contains no information pertaining to other than the middle dimension. On the other hand, it seems to contain all the information on the middle dimensional behavior.

THEOREM [1]. – *Two knots with S-equivalent Seifert matrices are of the same knot type if they satisfy the conditions :* (1) $\Pi_i(\mathbf{R}^{n+2} - K) \approx \Pi_i(S^1)$ *for* $i < q$, (i.e. *the complement looks like a circle up to dimension* q) *and* (2) $n > 1$.

2. We now look at the algebraic problem.

One problem is the variable size of the matrix. There are two ways of dealing with such a situation. The usual approach is to *stabilize* ie. consider infinite matrices. This does not seem to be of much use here. Another approach is to find minimal representatives and restrict attention to these. This seems to be more fruitful.

LEMMA [3]. – Any matrix, satisfying $A \pm A^T$ unimodular, is S-equivalent to a non-singular matrix. Moreover the rank and determinant are invariants of the S-equivalence class.

This is the algebraic analogue of a minimal spanning surface of a knot of dimension one, ie. a surface of minimal genus. Actually a minimal spanning surface need not give rise to a non-singular Seifert matrix.

We may now restrict our attention to non-singular matrices of a fixed rank. One question that arises is whether S-equivalence may coincide with congruence (among non-singular matrices). This corresponds algebraically to asking whether minimal spanning surfaces are unique. We shall see the answer is *NO*. One approach to this problem is to find ways of generating all the matrices S-equivalent to a given one (or rather congruence classes) in a finite number of steps, and to recognize when we are through. We show how this can be done. Proofs will appear in a future work.

The first step is :

THEOREM 1. – *Any two S-equivalent non-singular matrices can be joined by a sequence of the following two types of moves :*

(i) *right enlargement, then left reduction*

(ii) *left enlargement, then right reduction.*

Moreover, we can do all of type (i) *first and then all of type* (ii).

Thus we never have to deal with matrices much larger than the original one.

The next step would be to examine a single move of the type (i) or (ii) and be able to write down all the matrices obtained by such a move from a given one. A priori this may seem improbable since the vectors ξ used in the enlargment may vary over an infinite number of choices. But, in fact, only a finite number of distinct (up to congruence) enlargements occur and these can be constructed in a finite number of steps.

Suppose A has rank r. Consider the free abelian group of rank r, written as column vectors, and the subgroup generated by the columns of A. Let the quotient group be denoted $V(A)$. Then $V(A)$ is a finite group with det A elements. Let $O(A) = \{P \text{ unimodular} : PAP' = A\}$ be the orthogonal group of A. Then $O(A)$ acts on $V(A)$ by left multiplication. Given A clearly one can completely write down this situation.

THEOREM 2. – *If ξ, η are two column vectors, the right enlargements of A given by ξ and η are congruent if and only if the corresponding elements of $V(A)$ lie in the same orbit of $O(A)$.*

Thus we can write down all the enlargements of A. To handle the reductions :

LEMMA. – Two non-singular right (left) reductions of a matrix are congruent.

Thus we can effectively write down all the matrices obtained from A by one step of the form (i) or (ii) in Theorem 1.

One useful observation one can already make at this stage is :

COROLLARY [3]. – Two *unimodular* matrices are S-equivalent if and only if they are already congruent.

This follows immediately from the above results and $V(A) = 0$. For example fibered knots have unimodular Seifert matrices.

As illustration I would like to give some examples.

Example 1. – $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ – this seems to be the simplest non-trivial example.

$V(A) \approx Z_6$ generated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $O(A) = \pm I$. Thus there are four orbits and right enlargement gives four different matrices. Upon left reduction this reduces to three : A together with $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = A'$ and $\begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix}$, all of which are non-congruent. Repeating these steps always yields the same three matrices. Since A' appears here, we have also considered left enlargement and right reduction of A. It is easy to then conclude that these matrices comprise the S-equivalence class of A (up to congruence).

Example 2. – $\begin{pmatrix} a\lambda^2 & 1 \\ 0 & b \end{pmatrix}$ is S-equivalent to $\begin{pmatrix} a & 1 \\ 0 & b\lambda^2 \end{pmatrix}$ – by right enlargement by $\begin{pmatrix} 0 \\ \lambda \end{pmatrix}$ – but they are congruent only if $\lambda = \pm 1$ or $a = b = 1$. This gives examples of S-equivalence classes containing arbitrarily large numbers of congruence classes. For example $\begin{pmatrix} x^{2i} & 1 \\ 0 & x^{2j} \end{pmatrix}$ for all $i + j = k$ gives k non-congruent, but S-equivalent, matrices, for $x > 1$.

Finally we would like to know how many steps are needed to obtain *all* matrices S-equivalent to A. We must first ask whether it is finite i.e. are there only a finite number of matrices (up to congruence) S-equivalent to A ? If so, how many ?

Of course we can tell when we are done by noticing that no new matrices are produced at a given step, but it would be nicer to have a number given a priori from A which would serve as an upper bound for the number of steps required. This would also give an upper bound a priori for the number of matrices (up to congruence) S-equivalent to A.

THEOREM 3. – If B is obtained from A by a sequence of steps of type (i) in Theorem 1, then no more than (rank A) of such steps are needed. Similarly for steps of type (ii).

COROLLARY. – If $d = |\det A|$ and $r = \text{rank } A$, then there are at most d^{2r} congruence classes of non-singular matrices S-equivalent to A.

Question. – Is A always S-equivalent to A' ?

This is related to the existence of non-invertible knots of dimensions > 1.

BIBLIOGRAPHY

[1] LEVINE J. — An algebraic classification of some knots of codimension two, *Comm. Math. Helv.*, 45, 1970, p. 185-198.
[2] MURASUGI K. — On a certain numerical invariant of link types, *Trans. A.M.S.*, 117, 1965, p. 387-422.
[3] TROTTER H. — Homology of group systems with applications to knot theory, *Annals of Math.*, 76, 1962, p. 464-498.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 99 à 112.

INVARIANT KNOTS AND SURGERY IN CODIMENSION 2

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In the first part of this paper we study the problem of finding an invariant knot for an involution of a homotopy sphere Σ^{4k}. By an involution (T, Σ^n) we understand a fixed point free involution $T : \Sigma^n \to \Sigma^n$, smooth or p.l., of a homotopy sphere Σ^n. Reference [17] contains the properties of these involutions that will be needed. In the second part we use the experience obtained in the study of invariant knots to suggest the lines along which future research in the study of codimension 2 problems could be carried out, and we state a few results, which are only the initial steps in this direction.

Conversations with Drs. F. González Acuña and Mauricio Gutiérrez were very helpful in the elaboration of the ideas presented in this paper.

1. Invariant Knots.

An *invariant knot* for an involution (T, Σ^n) is an embedded (locally flat, in the p.l. case) homotopy sphere $\Sigma^{n-2} \subset \Sigma^n$ which is invariant under T(i.e. $T(\Sigma^{n-2}) = \Sigma^{n-2}$),

Σ^{n-2}), and a *trivial invariant knot* is one that is trivial as a knot, i.e. one that bounds an embedded disc $D^{n-1} \subset \Sigma^n$. In this last definition no relation between D and T is required, but it can be assumed that $D \cap TD = \Sigma^{n-2}$ if $n \geqslant 6$, by the fibering theorem ([5]).

We want to consider the problem of finding an invariant knot for a given involution (T, Σ^n). For $n \geqslant 7$, n not a multiple of 4, this can be solved using the Browder-Livesay theory and its developments ([6], [17]), and for $n \geqslant 7$ we can solve the problem of finding trivial invariant knots. Browder and Livesay defined an invariant $\sigma(T, \Sigma^n)$ which lies in the following groups :

$$\sigma(T, \Sigma^n) \in \begin{cases} \mathbf{Z} & \text{for} \quad n \equiv 3 \text{ mod. } 4 \\ \mathbf{Z}_2 & \text{for} \quad n \equiv 1 \text{ mod. } 4 \\ 0 & \text{for} \quad n \text{ even.} \end{cases}$$

and using this invariant and some of its properties, another invariant $\rho(T, \Sigma^n)$ can be defined for $n \neq 4$ with values in the groups

$$\rho(T, \Sigma^n) \in \begin{cases} \mathbf{Z}_2 & \text{for} \quad n \equiv 3 \text{ mod. } 4 \\ 0 & \text{for} \quad n \equiv 3 \text{ mod. } 4 \end{cases}$$

The results are :

THEOREM 1. ([17]). — *For* $n \geqslant 7$, $n \not\equiv 0$ (mod. 4), (T, Σ^n) *admits an invariant knot if, and only if* $\rho(T, \Sigma^n) = 0$. *For* $n \geqslant 7$, (T, Σ^n) *admits a trivial invariant knot if, and only if,* $\sigma(T, \Sigma^n) = 0$ *and* $\rho(T, \Sigma^n) = 0$.

All values of the invariant σ can be realized both in the p.l. and in the smooth cases, and all values of the invariant ρ can be realized in the p.l. case and for n odd in the smooth case, but known examples with non-zero value of ρ in the smooth case are scarce for n even. In any case, this shows that there are plenty of examples of involutions that do not admit invariant knots, and, for $n \equiv 3$ mod. 4, of involutions that admit invariant knots but do not admit trivial ones.

The case $n = 4k$ is the only one that cannot be reduced to the Browder-Livesay theory, and is the one that we shall study in this section. We shall present all the ideas and proofs, including a direct definition of the invariant ρ for this case, so that only ocassional references to the theory of involutions are needed. These ideas appear also in [17], but have been refined and simplified for this presentation to make it as self-contained as possible, and in view of the generalization given in section 2.

So far we know that (T, Σ^{4k}) admits a trivial invariant knot if, and only if, $\rho(T, \Sigma^{4k}) = 0$. The general form of Theorem 1 suggests that this condition is also necessary for the existence of an invariant knot, but it could still be possible that (T, Σ^{4k}) admits an invariant knot, even if it doesn't admit a trivial one, just as in the case mentioned above of an involution (T, Σ^{4k+3}). We shall see what happens.

It is convenient to rephrase the problem in terms of the quotient spaces : if (T, Σ^n) is an involution, the quotient $Q^n = \Sigma^n / T$ is called a *homotopy projective space*. As the terminology suggests, it can be shown ([17], IV.3.1) that Q^n is homotopy equivalent to real projective space P^n, and the homotopy equivalence is essentially unique. We can reformulate the problem of finding an invariant knot as follows : given a homotopy projective space Q^n, find an embedded homotopy projective space $Q^{n-2} \subset Q^n$, such that the embedding induces an isomorphism of fundamental groups. From Levine's unknotting theorem ([13]) it follows that the problem of finding a trivial invariant knot for (T, Σ^n) is equivalent to that of finding an embedded $Q^{n-2} \subset Q^n$ so that the complement $Q^n - Q^{n-2}$ has the homotopy type of S^1, as is case for the standard embedding $P^{n-2} \subset P^n$.

Browder's embedding theorem

The best way to attack the problem is to use the methods of the proof of Browder's embedding theorem (in fact, there is a theorem that says that this is the best possible way : [17], Theorem VI.1) which we proceed to describe.

Let M^m be a closed manifold (smooth or p.l.) and $N^n \subset M^m$ a submanifold with normal bundle ξ. Then, given a homotopy equivalence $f : M' \to M$ we would like to find inside M' a manifold N' homotopy equivalent to N. We have to state this problem in a more precise form, and sometimes we have to consider also the complements of the submanifolds. For this purpose, it is natural to introduce the following definitions :

DEFINITION. — Let $f : M' \to M$ be a homotopy equivalence and N a submanifold of M. We say that f is *weakly h-regular at* N if

(i) f is t-regular at N, and

(ii) if $N' = f^{-1}(N), f|N' : N' \to N$ is a homotopy equivalence.

If, further, we have

(iii) $f|M' - N' : M' - N' \to M - N$ is a homotopy equivalence, then we say that f is *strongly h-regular at* N.

("Homotopy equivalence" will mean "simple homotopy equivalence", whenever the distinction is relevant).

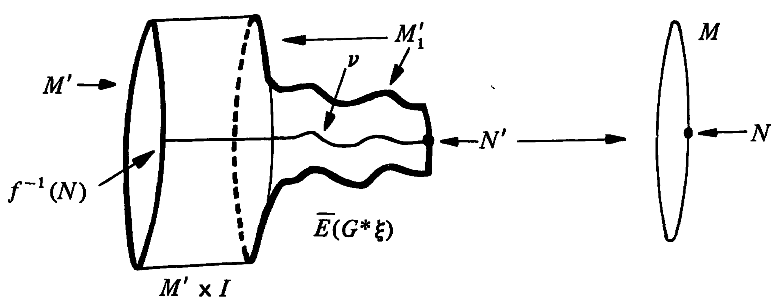
The problem now is, when is a homotopy equivalence $f : M' \to M$ homotopic to one that is weakly, or strongly, h-regular at N? If we make f t-regular at N, and we consider the map $g = f|f^{-1}(N) : f^{-1}(N) \to N$, it is easy to see that g is a normal map in a natural way, whose normal cobordism class depends only on the homotopy class of f, and defines a surgery obstruction $\theta(g)$ in the appropriate group. $\theta(g)$ is the obstruction to obtaining a homotopy equivalence, normally coborant to g, so $\theta(g) = 0$ is a necessary condition for making f weakly h-regular at N. Browder's embedding theorem says that, under some circumstances, this condition is sufficient for making f strongly h-regular at N.

Browder's Embedding Theorem ([3]). *Assume that both* M *and* $M - N$ *are* 1*-connected and* $n \geqslant 5$. *Then, if* $\theta(g) = 0$, f *is homotopic to a map strongly h-regular at* N.

Actually a more general situation is covered by this theorem, where instead of the pair (M, N) one gives only the homotopy theoretical information which is called a "normal system" or a "Poincaré embedding", and the manifold N' can be specified from the begining within its normal cobordism class. Also, if instead of assuming $\theta(g) = 0$, one assumes that g is normally cobordant to a homotopy equivalence to cover the small dimensions, we only have to ask $m \geqslant 5$. Wall has generalized this theorem to the case where $\pi_1(M - N) \approx \pi_1(M)$ (induced by the inclusion), which is always the case when $m \geqslant n + 3$, and has described the obstruction groups in the general situation ([21]). In all these results, the final conclusion is strong h-regularity, which is more than we can hope for in our problem when $\rho \neq 0$.

We describe the proof of this theorem only for $m = 4k$, for simplicity, the other cases requiring only minor modifications. Since $\theta(g) = 0$, g is normally cobordant to a homotopy equivalence $g_1 : N' \to N$. If $G : V \to N$ is the normal cobordism, we can glue $M' \times I$ and $\overline{E}(G^*\xi)$ along $\overline{E}(g^*\xi) \times \{1\}$, where $\overline{E}(G^*\xi)$ denotes the total space of the closed disc bundle of $G^*\xi$, etc., and where $\overline{E}(g^*\xi) \times \{1\}$ has been identified with a tubular neighborhood of $f^{-1}(N) \times \{1\}$ in $M' \times \{1\}$, thus obtaining a normal cobordism between f and a new normal map $f_1 : M_1' \to M$, such that $f^{-1}(N) = N'$. (This trick will be refered to as the normal cobordism extension lemma).

Now f_1 restricts to the homotopy equivalence $g_1 : N' \to N$, but is not itself a homotopy equivalence. We correct this by doing surgery on the complement of N' in M_1'. Let $X = M - U$, where U is an open tubular neighborhood of N in M and $X_1' = M_1' - U'$, where U' is an open tubular neighborhood of N' in M_1'.



Since we can assume that f_1 sends U_1' onto U as a bundle map, and X_1' onto X, we have a normal map $h = f_1|X_1' : X_1' \to X$, and since $h|\partial X_1'$ is a homotopy equivalence, we can try to make h a homotopy equivalence, by doing surgery on the interior of X_1'. The obstruction to doing this, being the index of the intersection form on ker h_*, can be identified with the obstruction to making f_1 a homotopy equivalence. But this obstruction is 0, since f_1 is normally cobordant to the homotopy equivalence f. Therefore we can find a normal cobordism, rel. boundary, between h and a homotopy equivalence, and this cobordism, together with $U_1' \times I$, gives a normal cobordism between f_1 and a homotopy equivalence $f_2 : M_2' \to M$ which is strongly h-regular at N. Since f and f_2 are normally cobordant and the normal cobordism is odd dimensional, we can turn it into an h-cobordism, and therefore $M' = M_2'$ and f is homotopic to f_2, so the theorem is proved.

The invariant ρ.

We want to consider the case $M = P^{4k}, N = P^{4k-2}$. In this case $\pi_1(M) = \mathbf{Z}_2$ and $X = M - U$ is a closed tubular neighborhood of the P^1 that links P^{4k-2} in P^{4k}. Therefore X is the total space of the non-orientable $(4k-1)$-disc bundle over $S^1 = P^1$, so it is non-orientable and $\pi_1(X) = \mathbf{Z}$. In another description, X is the mapping torus of the orientation reversing diffeomorphism $D^{4k-1} \to D^{4k-1}$.

Let $f : Q^{4k} \to P^{4k}$, $k > 1$, be a homotopy equivalence, t-regular at P^{4k-2} and $g = f|f^{-1}(P^{4k-2})$. It is shown in [17], Theorem 1, IV.3.3, that $\theta(g) = 0$ (and this is the only place where we shall use the Browder-Livesay theory ; there is a cohomological proof of the same fact in [20]), so we can apply the normal cobordism extension lemma to obtain a normal map $f_1 : M_1 \to P^{4k}$, normally cobordant to f, such that $f_1^{-1}(P^{4k-2}) = Q^{4k-2}$ and $g_1 = f_1|Q^{4k-2} : Q^{4k-2} \to P^{4k-2}$ is a homotopy equivalence, and such that f_1 sends a tubular neighborhood U_1 of Q^{4k-2} in M_1 onto U as a bundle map, and $X_1 = M_1 - U_1$ onto X. Let $h = f_1|X_1$. To carry out the next step in the proof of Browder's embedding theorem in our case, we should have $\theta(h) = 0$, but this will not always be the case. Therefore, we define.

$$\rho(Q^{4k}) = \theta(h)$$

To show ρ is well defined, let $f_1' : M_1' \to P^{4k}$ be another normal map with the same properties as f_1, and let h' be the corresponding map. If $F : W \to P^{4k}$ is a normal cobordism between f_1 and f_1', t-regular at P^{4k-2}, and $V = F^{-1}(P^{4k-2})$, we can turn $F|V$ into an h-cobordism because $L_{4k-1}(\mathbf{Z}_2, -) = 0$ ([20], [21]). But, by the normal cobordism extension lemma (for manifolds with boundary this time) we can assume that V itself is an h-cobordism, by changing F through a normal cobordism, rel. boundary. Since we can further assume that F sends a tubular neighborhood of V in W onto U by a bundle map, and Y, the complement of that neighborhood, onto X, $F|Y : Y \to X$ is a normal cobordism, rel. boundary, between h an h', so $\theta(h) = \theta(h')$ and ρ is well defined.

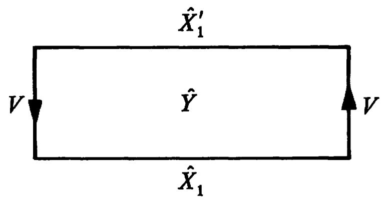
Therefore, if $\rho(Q^{4k}) = 0$ we can proceed as in the proof of Browder's embedding theorem, and obtain a homotopy equivalence $f_2 : Q_2^{4k} \to P^{4k}$ which is normally cobordant to f and strongly h-regular at P^{4k-2}. Since $L_{4k+1}(\mathbf{Z}_2, -) = 0$ ([20], [21]) we can turn a normal cobordism between f and f_2 into an h-cobordism, and therefore $Q_2^{4k} = Q^{4k}$ and f_2 is homotopic to f. In other words, (T, Σ^{4k}) admits the trivial invariant knot Q^{4k-2}. It is not difficult to see that a trivial invariant knot for (T, Σ^{4k}) induces a homotopy equivalence $f : Q^{4k} \to P^{4k}$, strongly h-regular at P^{4k-2} ([17], Theorem VI.1) and therefore $\rho(T, \Sigma^{4k}) = \rho(Q^{4k})$, being the obstruction to strong h-regularity, is the obstruction to the existence of a trivial invariant knot for (T, Σ^{4k}). We have then proved the second part of Theorem 1 for $n = 4k$ with our new definition of ρ, and also that this definition must coincide with the original one. To study the case $\rho \neq 0$ we need a detailed description of the surgery obstruction $\theta(h)$.

The surgery obstruction.

The surgery obstruction $\theta(h)$ can be described using the methods of [2] (see also [21]). Let $h : X_1 \to X$ be a normal map such that $h|\partial X_1$ is a homotopy equivalence, and let $D = D^{4k-1}$ be a fibre of $X \to S^1$. By the fibering theorem ([5]) we can assume that $h^{-1}(\partial D)$ is a homotopy sphere. Make h t-regular at D and let $W = h^{-1}(D)$.

W is a framed manifold with boundary $h^{-1}(\partial D)$, so it is framed cobordant, rel. boundary, to a disc D', and by the normal cobordism extension lemma we can assume that $h^{-1}(D) = D'$. Let $\hat{X}_1$ and $\hat{X}$ be the manifolds obtained from X_1 and X by cutting along (i.e. by removing a tubular neighborhood of) D' and D, respectively. Then h induces a normal map $\hat{h} : \hat{X}_1 \to \hat{X}$. Since $\hat{X}$ is a disc, $\theta(\hat{h}) = 1/8$ (Index X_1). We claim that the mod. 2 class of $\theta(\hat{h})$ is the surgery obstruction of h. This is because :

(a) $\theta(\hat{h})$ mod. 2 depends only on the normal cobordism class of h. For if $H : Y \to X$ is a normal cobordism, rel. boundary, between h and another normal map $h' : X_1' \to X$ such that $h'^{-1}(D)$ is a disc, we can again assume that $H^{-1}(\partial D)$ is an h-cobordism. If $V = H^{-1}(D)$ and $\hat{Y}$ is obtained from Y by cutting along V, then $\hat{Y}$ can be considered as a (normal) cobordism, rel. boundary between $\hat{X}_1'$ and $V \cup \hat{X}_1 \cup V$. (V gets the same orientation twice, because Y is non-orientable).



Therefore $\theta(\hat{h}') = \theta(\hat{h}) + 2\theta(H \mid V)$.

(b) If $\theta(\hat{h})$ is even h is normally cobordant, rel. boundary, to a homotopy equivalence. This is because we can construct a cobordism like the above Y with any value of $\theta(H \mid V)$ (using the normal cobordism extension lemma), and by choosing it properly we can assume $\theta(\hat{h}') = 0$. But that means that we can perform surgery on the interior of $\hat{X}_1'$ to obtain a disc, which amounts to performing surgery on the interior of X_1' to make it homotopy equivalent to X.

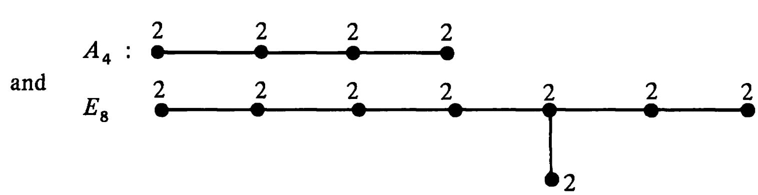
(c) If h is a homotopy equivalence, then $\theta(\hat{h}) = 0$. Because we can assume from the beginning that $h^{-1}(D) = D'$, by the fibering theorem ([5]), and then $\hat{X}_1$ is a disc.

We can further say that a normal map with non-zero obstruction is normally cobordant to one with $X_1 = X \# M_0$ (connected sum along the boundary), where M_0 is the Milnor manifold obtained by plumbing along E_8 ([4], [10]).

Now let $f : Q^{4k} \to P^{4k}$ be a homotopy equivalence, weakly h-regular at P^{4k-2}. $Q^{4k} - f^{-1}(P^{4k-2})$ is not necessarily homotopy equivalent to X, i.e., to S^1, but anyway it must be quite simple ; in particular, it must have the same homology groups as X. The question now is whether such a simple manifold can carry a non-zero surgery obstruction or not ; or in other words, whether we can or cannot "simplify" $X \# M_0$ enough. Now, the fact that the surgery obstruction of a normal map $X_1 \to X$ doesn't change if we add to X_1 two copies of M_0 can be interpreted as follows : we can move one of the copies around an orientation reversing loop, and it will come back as $-M_0$, so we can cancel it with the other copy of M_0 by surgery. For the map $X \# M_0 \to X$, if we could somehow split M_0 into two equal parts, and move one of the parts around the loop so it comes back with the opposite orientation, we could expect to simplify $X \# M_0$ by surgery, and hopefully get something that looks like the complement of a Q^{4k-2} in a Q^{4k}. This is in principle what we shall do next.

Cracking.

We now describe a process that is, in a sense, the opposite of plumbing. Recall ([4], [10]) that by the process of plumbing we can associate to a weighted graph, such as



a parallelizable $4k$-manifold with boundary, as follows : for every vertex (with weight 2) take a copy of the tangent closed disc bundle of S^{2k}, and plumb two of these copies together if the corresponding vertices are joined by an edge in the graph. This plumbing of two copies consist in identifying product neighborhoods $D^{2k} \times D^{2k}$ – one in each bundle, and disjoint from any other such neighborhoods where plumbing has been done at a previous stage – with each other by an identification that interchanges the base and fibre factors. The manifold constructed from A_4 will be denoted by A_4 again, and the one constructed from E_8 is, by definition, the Milnor manifold M_0. Let $L = \partial A_4$, and W the manifold obtained from L by removing an open disc. $\Sigma_0 = \partial M_0$ is the generator of $\theta^{4k-1}(\partial\pi)$. The homology groups of these manifolds can be computed : $H_i(A_4) = 0$ for $i \neq 2k$, and $H_{2k}(A_4)$ is free on 4 generators, represented by the 0-sections of the bundles, with respect to which the intersection form has as matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

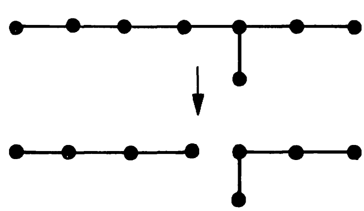
which has index 4 and determinant 5. This last fact implies that

$$H_{2k-1}(L) \approx H_{2k-1}(W) = \mathbf{Z}_5 .$$

All the other homology groups of L and W are trivial, except the top dimensional for L, being an orientable closed manifold. Similarly, $H_i(M_0) = 0$ for $i \neq 2k$, and $H_{2k}(M_0)$ is free on 8 generators $e_1, \ldots, e_8$ with respect to which the intersection from has as matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

We want to show that M_0 is the union of two copies of A_4, glued along W. Symbolically, the proof of this can be viewed as the process of cracking the E_8 into two copies of A_4, by breaking one of the links :



In precise terms, let $e'_1, \ldots, e'_8$ be the elements of $H_{2k}(M_0)$ given by

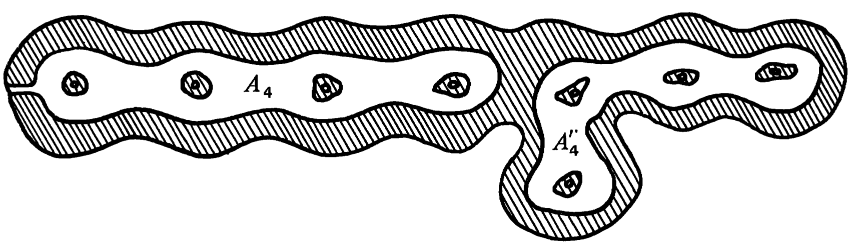
$$e'_i = e_i \quad i \neq 5$$

$$e'_5 = -e_1 + 2e_2 - 3e_3 + 4e_4 - 5e_5 + 4e_6 - 2e_7 + 3e_8$$

These elements do not form a basis of the group ; in fact they generate a subgroup of index 5 of $H_{2k}(M_0)$. The interesting thing about them is that the matrix of intersection numbers $e'_i \,.\, e'_j$ is

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

which is clearly equivalent to the block sum of two copies of the matrix of A_4. That is, the link between the fourth and fifth rows and columns has disappeared! If we represent these elements by embedded spheres whose only intersections with each other are those given by this matrix and are transversal, then a regular neighborhood of the union of the spheres representing $e'_1, \ldots, e'_4$ is easily seen (by choosing an adequate Riemannian metric near the intersection points) to be diffeomorphic to A_4 (and we will call it A_4). So is a regular neighborhood of the spheres representing $e'_5, \ldots, e'_8$, and we will denote it by A'_4. We can assume A_4 and A'_4 are disjoint and contained in the interior of M_0, but we will take a small tube joining the boundary of A_4 to the boundary of M_0, and we will consider it as also forming part of A_4.



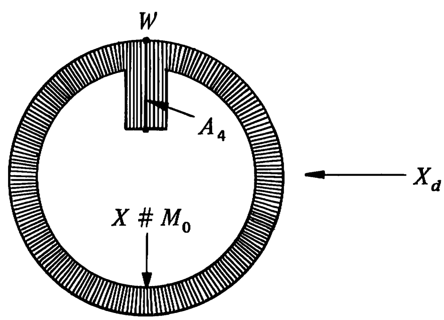
(This picture can be misleading ; the "tube" representing e'_5 really goes all over M_0, but missing the "tubes" representing $e'_1, \ldots, e'_4$ and e'_7).

Let $K = \overline{M_0 - A_4}$. We now show that the inclusion $A'_4 \subset K$ induces an isomorphism of homology groups, which implies that $\overline{K - A'_4}$ is an h-cobordism and, everything being simply connected, that K is diffeomorphic to A'_4. To prove this, first one can see, using Lefschetz duality, excision and universal coefficients, that $H_{2k-1}(K) = 0$. Then the Mayer-Vietoris sequence of $(M_0\ ; A_4, K)$,

$$0 \to H_{2k}(A_4) \oplus H_{2k}(K) \to H_{2k}(M_0) \to H_{2k-1}(W) \to 0$$

shows that $H_{2k}(A_4) \oplus H_{2k}(K)$ can be identified with a subgroup of $H_{2k}(M_0)$ of index 5. Since $H_{2k}(A_4) \oplus H_{2k}(A'_4)$ is contained in this subgroup, and has also index 5 in $H_{2k}(M_0)$, being the subgroup generated by the $\{e_i'\}$, it follows that these two subgroups are equal, and that the inclusion induces an isomorphism $H_{2k}(A_4') \approx H_{2k}(K)$. Since all other groups are trivial, this proves our assertion. Therefore M_0 can be expressed as the union of two copies of A_4, glued along W by an orientation reversing diffeomorphism d.

We shall be interested in the mapping torus of d, which we shall denote by X_d, whose boundary is the mapping torus of a diffeomorphism of S^{4k-2} representing Σ_0. We clearly have a normal map $h_d : X_d \to X$, obtained by collapsing the complement of a collar neighborhood of ∂X_d fibrewise to an S^1. (There is no obstruction to making this map normal, because all the homology of X_d comes from S^1 ; see below). Now $X \# M_0$ has the same boundary as X_d, and in fact it is normally cobordant, rel. boundary, to X_d, since the framed cobordism A_4 from W to a disc induces, by the normal cobordism extension lemma, a normal cobordism, rel. boundary, from X_d to $X \# M_0$ (which now appears as the union of two copies of A_4, joined by a tube, and then glued along W by d).



Therefore $h_d : X_d \to X$ represents the normal cobordism class with non-zero surgery obstruction.

The only thing left to do is to see if X_d looks like the complement of a tubular neighborhood of a Q^{4k-2} in a Q^{4k}. For this to be true it is necessary that the double cover $\widetilde{X}_d$ looks like the complement of a knot. Now $\widetilde{X}_d$ is the mapping torus of d^2, so it can be described as the union of two copies of $W \times I$ glued along one end by d^2 and along the other one by the identity. Therefore we have a Mayer-Vietoris sequence

$$0 \to H_{2k}(\widetilde{X}_d) \to H_{2k-1}(W \times \dot{I}) \to H_{2k-1}(W \times I) \oplus H_{2k-1}(W \times I) \to H_{2k-1}(\widetilde{X}_d) \to 0$$

Identifying both middle groups with $\mathbf{Z}_5 \oplus \mathbf{Z}_5$, it follows that the central homomorphism has as matrix

$$\begin{pmatrix} 1 & 1 \\ d_*^2 & 1 \end{pmatrix}$$

so we have to compute d_*^2. d_* itself must be multiplication by a certain number m. Let $x, y \in H_{2k-1}(W)$ be such that $L(x, y) = 1$, where

$$L : H_{2k-1}(W) \times H_{2k-1}(W) \to \mathbf{Z}_5$$

is the non-degenerate bilinear pairing given by linking numbers ([12]). Since d is orientation reversing we have

$$1 = L(x, y) = -L(d_* x, d_* y) = -L(mx, my) = -m^2 L(x, y) = -m^2$$

Therefore d_*^2 is multiplication by $m^2 = -1$, the above matrix in non-singular, and the central map in the Mayer-Vietoris sequence is an isomorphism. Therefore we have

$$\pi_1(\widetilde{X}_d) = H_1(\widetilde{X}_d) = \mathbf{Z}$$

$$H_i(\widetilde{X}_d) = 0 \quad , \quad i > 1 \quad .$$

(and since m must equal ± 2, the same holds for X_d), and also $\pi_i(X_d) = 0$ for $1 < i < 2k - 1$. So we have shown that we can represent the normal map into X with non-zero surgery obstruction by X_d, which has very little homology, and looks like the complement of a Q^{4k-2} in a Q^{4k}. In fact we can now prove :

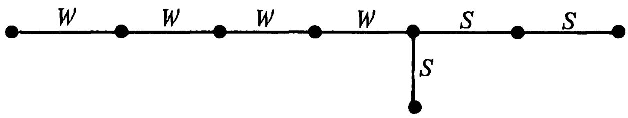
THEOREM 2. — *Every involution* $(T, \Sigma^{4k}), k > 1$, *admits an invariant knot. In fact, it admits one that is simple and equivarantly fibered.*

For the proof, we only have to do a weak version of the last steps of the proof of Browder's embedding theorem. We had arrived before at a normal map $f_1 : M_1 \to P^{4k}$, such that $f_1^{-1}(P^{4k-2}) = Q^{4k-2}, f_1 | Q^{4k-2}$ is a homotopy equivalence and $f_1 | X_1 = h : X_1 \to X$. The case $\theta(h) = 0$ has already been considered. If $\theta(h) \neq 0$, we know that h is normally cobordant to $h_d : X_d \to X$, rel. boundary, so we get a new normal map $f_2 : Q'^{4k} \to Q^{4k}$, where $Q' = U_1 \cup X_d$. Now $\widetilde{Q}' = \widetilde{U}_1 \cup \widetilde{X}_d$ is clearly a homotopy sphere, because it is simply connected and it is easy to see from the pıoperties of $\widetilde{X}_d$ that it has no homology below the top dimension, so f_2 is a homotopy equivalence, weakly h-regular at P^{4k-2}. The rest of the proof follows as in the case $\theta(h) = 0 : Q' = Q^{4k}$ and f is homotopic to f_2, so (T, Σ^{4k}) admits the invariant knot $\widetilde{Q}^{4k-2}$. The exterior of this knot is $\widetilde{X}_d$, and since $\pi_i(\widetilde{X}_d) \approx \pi_i(S^1)$ for $1 < i < 2k - 1$, the knot is simple, by definition ([14]), and $\widetilde{X}_d / T = X_d$ fibers over S^1, which can be taken as a definition of an "equivariantly fibered" knot.

Remarks. — The proof of this theorem gives us a direct geometric way of computing the surgery group $L_{4k}(\mathbf{Z}_2, -) = \mathbf{Z}_2$, since it can be used to prove [17] Theorem 1, IV.3.3 without having to appeal to this computation. Also, it can be used to construct very simple examples of non-standard p.l. involutions : In P^{4k} substitute X by X_d (their boundaries are p.l. homeomorphic) and the involution obtained has $\rho \neq 0$.

The decomposition $M_0 = A_4 \cup_d A'_4$ is interesting in itself, since it shows that M_0 (and also the closed p.l. manifold $\overline{M}_0$, obtained from M_0 by attaching to it the cone on its boundary) is a "twisted double". This is a case not covered by the theorems of Smale [18], Barden [1], Levitt [16] and Winkelnkemper [22], which show that under certain, quite general conditions, a manifold must be a twisted double. Our example is more twisted than any of those covered by these theorems, in the sense that d is orientation reversing.

The process of cracking can be applied to other situations. For example, the E_8 graph can be cracked at other links, giving a decomposition of M_0 as the union of the manifolds obtained by plumbing according to the subgraphs into which E_8 is divided. In the following diagram those links at which this cracking process can be carried out are labeled W(eak), and those at which it cannot be done are labeled S(trong) :



For the weak links, formulas giving the e_i' are very similar to the ones we have given here.

This gives several relations between the boundaries of the plumbed manifolds. For example, we have shown that $L \# \Sigma_0$ is diffeomorphic to $-L$. It is possible that this process could be exploited to complete the classification of highly connected odd dimensional manifolds up to diffeomorphism ([19]).

Another remark can be made about the comparison with the situation of a knot $\Sigma^{4k-2} \subset S^{4k}$. It is proved in [11] that every such knot is cobordant to the trivial knot. If one tried to carry over the proof to the equivariant case, one would have to carry out Kervaire's proof, which can be done, and then apply some equivariant version of the engulfing theorem, as in [14] Lemma 4. But since we know that there are involutions (T, Σ^{4k}) which admit invariant knots, but do not admit trivial ones, it is not true that every invariant knot for a (T, Σ^{4k}) is equivariantly cobordant (with the obvious definition of this term) to a trivial invariant knot. Therefore, there must be something wrong with equivariant engulfing (as could be expected from the fact that the connectivity conditions on the quotient spaces are as bad as possible).

2. Surgery in Codimension 2.

The proof of theorem 2 suggest the general philosophy for dealing with surgery problems in codimension 2 : do not insist on obtaining homotopy equivalences when you are doing surgery on the complement of a submanifold, be happy if you can obtain the correct homology conditions. This has relevance both in the existence problems, as in the existence of invariant knots, and in the classification problems, as in the cobordism classification of knots.

In its simplest form, this approach suggests the following definitions and problems :

A map $f : X \to Y$ is a *homology equivalence* (H-equivalence) if it satisfies the following conditions :

(i) $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

(ii) $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for all i.

A cobordism $(W ; M_0, M_1)$ is an H-cobordism if both inclusions $M_i \subset W$ are H-equivalences. Two H-equivalences $f_i : M_i \to M$ between manifolds are *H-cobordant* if they extend to a map $F : W \to M$, where W is an H-cobordism between the M_i.

PROBLEM 1. — When is a normal map $M' \to M$ normally cobordant to an H-equivalence ?

PROBLEM 2. — When are two normally cobordant H-equivalences H-cobordant ?

Problem 1 is equivalent to the question of which elements in the Wall group can be represented by H-equivalences, so this problem is in a certain sense simpler that the standard surgery problem, since its obstruction cannot be stronger than the standard surgery obstruction. On the other hand Problem 2 is much more complicated than the standard problem of obtaining h-cobordisms, since in the only known non-simply-connected example, that of cobordism of knots, the obstruction groups are not finitely generated ([14]).

In the applications the problems are more complicated to formulate. First of all, we are really interested in the relative case, where manifolds have a boundary, and the restrictions of the maps and cobordisms to the boundaries are homotopy equivalences and h-cobordisms. This is the situation when we consider cobordism classes of knots : two knots are cobordant if, and only if, their exteriors are H-cobordant, rel. boundary, when we consider them together with their normal maps onto the exterior of the trivial knot. This example also suggests that condition (i) in the definition of an H-equivalence could and should be weakened, if not totally forgotten, in the sense that the solutions to Problems 1 and 2 will probably be unaffected by this modification of the definitions. This also seems to be the case in other situations, like in the study of H-cobordism classes of homology spheres ([8]).

The other complication has been already found in the proof of Theorem 2 : we had to make sure that the double covering of the map $h_d : X_d \to X$, and not only h_d itself, was an H-equivalence. In general we can say that $f : X \to Y$ is an H-equivalence with respect to a subgroup G of $\pi_1(Y)$ if the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is an H-equivalence, where $\widetilde{Y} \to Y$ is the covering corresponding to G. (If $G = 0$, this means that f is a (weak) homotopy equivalence). In the applications G is the kernel of $\pi_1(M - N) \to \pi_1(M)$. Another interesting case is when G is the kernel of the orientation map.

When M is orientable, the best possible solution of Problem 1 would be that a normal map is normally cobordant to an H-equivalence if its surgery obstruction lies in the kernel of the homomorphism $L_m(\pi_1(M)) \to L_m(0)$ induced by the orientation map, that is, if its good old index or Kervaire invariant is 0. It this were true the weak h-transversality problem in codimension 2 would be solved whenever the ambient manifold is simply connected. For other forms of Problem 1 there are similar conjectures with equally nice consequences. For the moment we can prove some of these conjectures when the fundamental group is $\mathbf{Z}$, obtaining the following theorem on weak h-regularity :

THEOREM 3. — *Assume (M^m, N^{m-2}) is such that $\pi_1(M - N) = \mathbf{Z}$ and either $\pi_1(M) = 0$ or $\pi_1(M) = \mathbf{Z}_2$. Then, if $m - 2 \geqslant 5$, a homotopy equivalence*

$$f : M' \to M$$

is normally cobordant to a homotopy equivalence weakly h-regular at N if, and only if, the surgery obstruction $\theta(g) = 0$, where $g = f \mid f^{-1}(N)$.

The proof rests on the knowledge of a good number of examples from knot theory and the theory of involutions. When one is trying to do surgery to make the complement of the inverse image of N H-equivalent to $M - N$, one can make it a homotopy equivalence outside the inverse image of a tube representing a generator of $\pi_1(M - N)$. Then one can use these examples to substitute this inverse image by something H-equivalent (with respect to the kernel of $\pi_1(M - N) \to \pi_1(M)$) to the tube, just as we did in the proof of Theorem 2. In this way we get a homotopy equivalence, weakly h-regular at N and normally cobordant to f. When $\pi_1(M) = \mathbf{Z}_2$ there are a few cases when we cannot conclude that this homotopy equivalence is h-cobordant (and therefore homotopic) to f, but under extra hypotheses, which are probably irrelelevant, we can obtain this stronger result. When $\pi_1(M) = 0$ there is no problem.

About Problem 2 we have very little to say. One would hope that there are obstruction groups, similar to Levine's knot cobordism groups, and that these groups depend only on the fundamental group. If this were the case, there would be nice consequences again : many problems of classification of embeddings in codimension 2 up to concordance would be reduced in a large measure to knot cobordism theory, and there would be a geometric interpretation of the periodicity of Levine's groups.

The methods of knot cobordism theory are in most cases too specific to be directly helpful in the general situation. One such method is the use of engulfing to show that every knot is cobordant to a simple knot ([14], Lemma 4) since we have shown in particular that this method cannot work for the case of invariant knots. We have found a proof of this result that only uses surgery (similar proofs have been found independently by Kervaire and Ungoed-Thomas) which works also for invariant knots :

THEOREM 4. — *Every invariant knot for (T, Σ^n) is equivariantly cobordant to a simple invariant knot.*

The proof consists in constructing an (equivariant) H-cobordism between the complement of the knot and the complement of a simple knot, which gradually kills the homotopy groups. The general step goes as follows : If X is the complement of the knot and if we assume $\pi_1(X) \approx \pi_1(S^1)$ for $i < q$ and q is below the middle dimension, we can perform equivariant surgery on the generators of $\pi_q(X)$, obtaining a cobordism W between X and X', rel. boundary. Now both X' and W have some unwanted homology in dimension $q + 1$. However, since $\pi_{q+1}(X') \to H_{q+1}(X')$ is onto, because $H_{q+1}(\mathbf{Z}) = 0$ (See [7], p. 483) we can kill this homology by doing surgery on X', which kills automatically also the extra homology in W, thus obtaining an H-cobordism W' between X and X'', where $\pi_i(X'') = \pi_i(S^1)$ for $i \leqslant q$. This type of proof also works when we consider knots which are invariant under other group actions, and for links in codimension 2 ([9]).

The next step would be to compute the equivariant cobordism classes of invariant knots, which means that we should identify the obstruction to doing the last step (the middle dimension) of the homology surgery process described above. There are further complications because there are in some dimensions examples of two trivial invariant knots which are not equivariantly cobordant ([17], VI.3, Corollary), and in the cases where there is no trivial invariant knot, we don't know if there is a simplest invariant knot to which we could refer all the others. There is the nice circumstance, however, that the equivariant H-cobordism class of the exterior of an invariant knot, and the involution restricted to the invariant knot itself, determine completely the involution.

There is another problem, even more difficult than Problem 2, namely that of deciding when two H-equivalences are h-cobordant. This has to do with the problem of isotopy of embeddings, and one case has been solved in [15].

REFERENCES

[1] Barden D. — The structure of manifolds, *Ph. D. Thesis*, Cambridge, 1963.
[2] Browder W. — Manifolds with $\pi_1 = \mathbf{Z}$, *Bull. A.M.S.*, 72, 1966, p. 238-244.
[3] Browder W. — Embedding smooth manifolds, *Proc. I.C.M.*, Moskow, 1966.
[4] Browder W. — *Surgery on simply connected manifolds* (to appear).
[5] Browder W. and Levine J. — Fibering manifolds over a circle, *Comment. Math. Helv.*, 40, 1966, p. 153-160.
[6] Browder W. and Livesay G.R. — Fixed point free involutions on homotopy spheres, *Bull. Amer. Math. Soc.*, 73, 1967, p. 242-245.
[7] Eilenberg S. and MacLane S. — Relations between homology and homotopy groups of spaces, *Ann. of Math.*, 46, 1945, p. 480-509.
[8] González Acuña F. — *On homology spheres*, Princeton Ph. D. Thesis, 1970.
[9] Gutiérrez M. — *Links in codimension 2*, Brandeis Ph. D. Thesis, 1970.
[10] Hirzebruch F. — *Differentiable manifolds and quadratic forms*, Mimeographed notes, Berkeley 1962.
[11] Kervaire M. — Les nœuds de dimensions supérieures, *Bull. Soc. Math. France*, 93, 1965, p. 225-71.
[12] Kervaire M. and Milnor J. — Groups of homotopy spheres I, *Ann. of Math.*, 77, 1963, p. 504-537.
[13] Levine J. — Unknotting spheres in codimension 2, *Topology*, 4, 1965, p. 9-16.
[14] Levine J. — Knot cobordism groups in codimension two, *Comment. Math. Helv.*, 44, 1969, p. 229-244.
[15] Levine J. — An algebraic classification of some knots of codimension two, *Comment. Math. Helv.*, 45, 1970, p. 185-198.
[16] Levitt N. — Applications of Engulfing, Princeton Ph. D. Thesis, 1967.
[17] López de Medrano S. — Involutions on manifolds, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 59, Berlin-Götingen-Heidelberg, Springer, 1971.
[18] Smale S. — On the structure of manifolds, *Amer. Jour. of Math.*, 84, 1962, p. 387-399.
[19] Wall C.T.C. — Classification problems in differential topology - VI, Classification of $(s-1)$-connected $(2s+1)$-manifolds, *Topology*, 6, 1967, p. 273-296.
[20] Wall C.T.C. — Free piecewise linear involutions on spheres. *Bull. Amer. Math. Soc.*, 74, 1968, p. 554-558.
[21] Wall C.T.C. — Surgery of compact manifolds, *London Mathematical Society Monographs*, No. 1, Academic Press, 1971.
[22] Winkelnkemper E.H. — On equators for manifolds and the action of Θ^n, Princeton, Ph. D. Thesis, 1970.

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EXTRAORDINARY HOMOLOGY THEORIES : BORDISM AND K - THEORY

by A. S. MISHCHENKO

The modern position of algebraic topology is characterized by appearance of new, so called extraordinary, homology theories.

While the reduction of a problem of analysis to a homotopy problem was formerly considered as the best achievement, now the main interest is concentrated in making computations in various domains of algebraic topology. The effective method of spectral sequences is now an usual instrument of algebraic topology. On one hand, the extraordinary homology theories enlarged the number of useful spectral sequences, on the other hand they are the simplest instrument for working with spectral sequences.

Moreover, we tend to consider that the extraordinary homology theories are not only useful methods for computing various topological invariants, but to a greater extend they are a new language able to describe these very invariants in a more adequate fashion.

This report is devoted to several problems dealing with two well known extraordinary homology theories : K-theory and bordism.

The results which have been reported here were obtained by Bukhshtaber and the reporter partly jointly and partly independently.

I. K-theory on the category of infinite complexes.

In 1956 appeared the Milnor construction of functorial filtration, which is in a sense the geometric realization of the bar-construction. The spectral sequence corresponding to this filtration is called the Milnor spectral sequence. However, it was useless for classical homology because the E_2-term has a simple algebraic form only when the homology groups of the loop space ΩX are torsion-free. There is a wider supply of such spaces for K-theory but there a new difficulty appears : one needs a reasonable extension of the definition of $K(X)$ to infinite complexes X. Now one may give two definitions of the groups $K(X)$:

(a) $k(X) = [X, BU]$

(b) $\mathcal{K}(X) = \varprojlim K(X_n)$

where X_n is the n-skeleton of X. The groups $k(X)$ define a homology theory but it is difficult to compute them, whereas the groups $\mathcal{K}(X)$ can be computed but they do not constitute a homology theory.

THEOREM 1 [1]. – *The following statements are equivalent :*

(i) $\mathcal{K}(X) = k^*(X)$

(ii) *The Chern character*

$$ch : \mathcal{K}^*(X) \otimes \mathbf{Q} \to H^{**}(X, \mathbf{Q})$$

is an isomorphism

(iii) *In the Atiyah-Hirzebruch spectral sequence, for any p and q, there exists* $r_0 = r_0(p, q)$ *such that* $E_{r_0}^{p,q} = E_\infty^{p,q}$.

Each of these conditions implies that the Atiyah-Hirzebruch spectral sequence is strongly converging to $\mathcal{K}^*(X)$.

There exist examples where the Atiyah-Hirzebruch spectral sequence does not converge to $\mathcal{K}^*(X)$.

Theorem 1 and its analogues for K-theory mod p provide the extension of the Milnor spectral sequence method to the category of infinite complexes.

THEOREM 2 ([2]). – *If* $n \geqslant 3$, *then*

$$k^*(K(\mathbf{Z}, n), \mathbf{Z}_p) = 0 \quad \text{for} \quad p \geqslant 2,$$

$$\mathcal{K}^*(K(\mathbf{Z}, n)) = 0 \ , \ k^n(K(\mathbf{Z}, n)) = 0,$$

$$k^{n+1}(K(\mathbf{Z}, n)) = \hat{\mathbf{Z}}/\mathbf{Z} \quad \text{for} \quad n \text{ odd},$$

$$k^{n+1}(K(\mathbf{Z}, n)) = \hat{\mathbf{Z}}[[t]]/\mathbf{Z}[[t]] \quad \text{for } n \text{ even},$$

where $\hat{\mathbf{Z}}$ *is the completion of* $\mathbf{Z}$ *under the topology for which all non-trivial subgroups are neighbourhoods of zero.*

By means of theorem 2 it is possible to compute the groups $\mathcal{K}^*(X)$ and $h^*(X)$ for any Eilenberg-Mac-Lane complex and for spaces with finite Postnikov system as well.

The computation of cohomology operations in K-theory mod. p, $p \geqslant 2$, is another application of the Milnor spectral sequence.

THEOREM 3 ([2]). – *The cohomology operations ring for K-theory mod p is isomorphic to* $\mathbf{Z}_p[[\Phi_p^k]] \hat{\otimes} \Lambda_p [[\lambda_p^k]]$ *where k is prime to p,*

$$\lambda_p^1 \Phi_p^k = \lambda_p^k \quad \text{and} \quad \Phi_p^k \lambda_p^1 = k \lambda_p^1 (\Phi_p^k + \Phi_p^{k-1}).$$

THEOREM 4 ([2]). – *The group of all stable operations from stable K-theory mod p to* $\mathbf{Z}_2$*-graded K-theory mod p is isomorphic to* $\mathbf{Z}_p[[\theta^q]] \otimes \Lambda(\sigma_p)$ *where q and p are prime,* $q \neq p$, *where*

$$\theta^q = \left\{ \frac{(-1)^{q+1}}{s^n} \sum_{j=0}^{s-1} C_s^j \Phi_p^{q-j} \right\}, \ \theta^q \sigma_p = s \sigma_p \theta^q$$

for $q = kp + s$.

II. The Atiyah-Hirzebruch spectral sequence.

Let h be an extraordinary cohomology theory, A_h be the ring of stable operations. It is useful to notice that the $A - H$ spectral sequence is a differential A_h-module. The action of A_h on E_r is completely defined by the representation of A_h in the cohomology group of a point. This representation, in contrast to the classical case, is non-trivial, as it is for K-theory and complex cobordism.

Let us define, for any differential $d_r^t = \Sigma\, d_r^{s,t}$, the A_h-module $\mathcal{D}_r^t$ with generators d_r^t and relations $a \,.\, d_r^t = 0$, where $a \in A_h$ is such that for any complex X and $x \in E_r^{s,t}(X)$ it annules $d_r^{s,t}(x)$.

Let $\Lambda = \Sigma\, \Lambda_t$ be the cohomology ring of a point, let $A_h(t) \subset A_h$ be the set of operations that kill the elements of the group Λ_t.

THEOREM 5 ([4]). – *If the $A - H$ spectral sequence in the theory h for the stable spectrum which represents the theory of cohomology h is trivial, then*

$$\mathcal{D}_r^0 = A_h \,/\, (A_h(0) + A_h(1 - r)) \,.$$

Theorem 5 provides the algebraic description of the differential module $\mathcal{D}_r^0$ for cobordism theory which correspond to the SO, U, Sp and trivial groups.

COROLLARY 1. – *For the U-theory, one has the following formula :*

$$\text{ord}\ \ d^0_{2r+1} = \text{ord}\ \ \text{Ext}^{1,2r}_{A_U}(\Lambda\,,\,\Lambda) = \begin{cases} 12 & \text{for} \quad r = 2 \\ k^N (k^r - 1) & \text{for} \quad r \neq 2\,. \end{cases}$$

Using the Adams operations in K-theory and the Riemann-Roch transformation from U-theory to K-theory we can get a similar result for K-theory. Especially for any prime p, $d_r \equiv 0$ mod. p if $r \neq 2s(p-1) + 1$.

THEOREM 6 ([3]). – *There exist a $2(p-1)$ periodical cohomology theory : $h_p^* = \Sigma\, h_p^q$ such that*

$$\sum_{i=0}^{p-1} h_p^{-2i}(X) \approx k^0(X\,,\,\mathbf{Z}_p)\,.$$

An analogous theorem holds for K-theory with p-adic coefficients.

The known problem of the realization of (co) cycles in U-theory by submanifolds (and accordingly in K-theory by Chern classes) has an useful interpretation in terms of $A - H$ spectral sequences. These two problems, and the problem of the denominators of Chern characters as well, are reduced to computations of differentials of a spectral sequence. Namely, for any complex X, number q and cocycle $a \in H^{2(q+r)}(X)$, there exist a new complex $V(X)$ and a cocycle $\nu(a)$ and this correspondance is functorial.

THEOREM 7 ([4]). – *Let λ_0 be the smallest number such that $\lambda_0\, \nu(a)$ is a cycle for all the differentials. Then λ_0 is maximal among the denominators of the rational cocycles of the following form :*

$$ch_{q+r}(\xi) = \frac{\lambda_1}{\lambda_2}\, a\ , \quad for \quad \xi \in K^0_{2q}(X)\,.$$

Due to theorems 6 and 7 there exist an estimate of the multiples of cycles realisable by submanifolds ; one can get it from information on the homology groups. If one uses information about the Steenrod operations it is possible to improve this estimate.

THEOREM 8. — *For K-theory, one has the following formulas* :

$$d_{2r(p-1)+1}\,(p^{r-1}x)_p = \epsilon_r\,(\beta P^r)(x)\,,$$

where $d\;(\;)_p$ *is the p-component of a differential,* $\epsilon_r \not\equiv 0 \bmod p$, (βP^r) *is the integral Steenrod operation.*

III. The Chern-Dold character and formal groups.

The natural transformation of cohomology theories

$$ch_h : h^*(X) \to H^*(X\,,\Lambda^* \otimes \mathbf{Q})\,,$$

where Λ^* is the cohomology ring of a point, is called the Chern-Dold character. If X is a point, then ch_h is the canonical homomorphism $\Lambda^* \to \Lambda^* \otimes \mathbf{Q}$. It is also a useful fact that ch_h is a homomorphism of A_h-modules. The following theorem gives useful formulae for the Chern-Dold character for unitary and symplectic cobordism.

THEOREM 9 ([5]). — (a) *Let* $u \in U^2(\mathbf{C}P^\infty)$ *be the geometrical cobordism. Then*

$$ch_U(u) = x + \Sigma\,[M^{2n}]\,x^{n+1}/(n+1)\,!$$

where $x \in H^2(\mathbf{C}P^\infty, \mathbf{Z})$ *is the generator,* $[M^{2n}] = \sigma_1(\xi_{n+1})$, *where* ξ_{n+1} *is the generator of* $K^0\,(S^{2(n+1)})$ *and* σ_1 *is the first Chern class in cobordism. The elements* $[M^{2n}]$ *are undivisible because the Todd genus,* $Td\,(M^{2n})$, *is equal to* $(-1)^n$. *They are completely defined by the following conditions :*

$s_\omega(-\tau\,(M^{2n})) = 0,\ \omega \neq (n), s_{(n)}(-\tau\,(M^{2n})) = -(n+1)\,!$, *where* s_ω *are the Chern numbers.*

(b) *Let* $\sigma \in Sp^4\,(\mathbf{H}P^\infty)$, *be the geometrical cobordism, then*

$$ch_{Sp}\,(v) = z + \Sigma\,a_{n-1}\,c_{n-1}\,z^n/(2n)\,!\,,$$

where $z \in H^4(\mathbf{H}P^\infty, \mathbf{Z})$. *is the generator,* $c_{n-1} \in \Omega_{Sp}^{-4n+4}$, $c_{n-1} = (-1)^n\,p_1(\xi_n)$ *where* p_1 *is the first Pontrjagin class in cobordism,* $a_n = 1$ *if n is odd,* $a_n = 2$ *if n is even.*

There are formulas which express C_n *in function of the* M^{2n}'s.

COROLLARY 2.— $ch_U\,\sigma_1(\xi) = ch_1(\xi) + \Sigma\,[M^{2n}]\,ch_{n+1}(\xi)$.

Theorem 9 and Corollary 2 give the connection between the functions $\varphi(n)$ and the multiplicative homomorphisms $\varphi^* : \Omega_U \to \Omega_U$:

COROLLARY 3. — *If* $ch_U\,\varphi(u) = x + \Sigma\,\alpha_n\,x^n$, *then* $\varphi^*([M^{2n}]) = (n+1)!\;\alpha_n$.

The Chern-Dold character is the inverse series for the logarithm of the formal group of the geometrical cobordisms, which is the universal formal group according to what Quillen has shown not long ago. Namely,

$$(ch_U(u))^{-1} = g(u) = n + \Sigma\,[CP^n]\,u^{n+1}/(n+1).$$

This formula allows us to produce the Adams projectors and some other projectors.

Let $\Omega_U(\mathbf{Z})$ be the logarithm coefficient ring of the universal group. Then the Chern-Dold character coefficients generate $\Omega_U(\mathbf{Z})$. That is, the ring $\Omega_U(\mathbf{Z})$ consists in all bordisms of $\Omega_U \otimes \mathbf{R}$ for which all Chern numbers are integral. Finally $\Omega_U(\mathbf{Z})$ can be viewed as the integral homology of MU. The Chern-Dold character defines the following natural transformations :

(1) $H_*(X) \to \mathrm{Hom}_{A_U}(U^*(X)\,,\,\Omega_U(\mathbf{Z}))$,

(2) $H_*(X, \Omega_U(\mathbf{Z})) \to \mathrm{Hom}_{\Omega_U}(U^*(X)\,,\,\Omega_U(\mathbf{Z}))$;

they give rise to new spectral sequences which connect cobordism and homology. The exact sequence

$$0 \to \Omega_U \to \Omega_U(\mathbf{Z}) \to \Omega_U(\mathbf{Z})/\Omega_U \to 0$$

produces an interesting connection between the first spectral sequence and the Adams spectral sequence in U^*-theory.

Due to Conner and Floyd, the bordism class of a manifold with action on the group $\mathbf{Z}_p$ is defined by the set of bundle-bordisms of fixed submanifolds with normal $\mathbf{Z}_p$-bundles. The family of all sets of bundle bordisms which have a realisation as fixed submanifolds for some actions of $\mathbf{Z}_p$ can be described by mean of the universal formal group in a useful way.

Let P be a fixed point of the periodic transformations $T, x_1, \ldots, x_n$, be the weights of dT at the point p. Then the "Conner-Floyd invariant",

$$\alpha(x_1, \ldots, x_n) \in U_{2n-1}(B\mathbf{Z}_p),$$

is defined.

THEOREM 10 (Novikov, Kasparov, Mischenko ([6])). –

$$\alpha(x_1, \ldots, x_n) = \prod_{j=1}^{n} \frac{u}{g^{-1}(x_j\, g(u))} \cap \alpha(1, \ldots, 1).$$

In the general case the fixed submanifolds have more complicated descriptions by the series of pairs $(x_1, k_1), \ldots, (x_n, k_n)$. Their geometrical interpretation is given by $CP^{k_1} \times \ldots \times CP^{k_n}$ with the normal bundle $\xi_1 \times \ldots \times \xi_n$, with the weight x_m on the Hopf bundle ξ_m.

THEOREM 11 ([7]). – *We have the following formula :*

$$\alpha((x_1, k_1), \ldots, (x_n, k_n)) = \prod_{j=1}^{n} \frac{u\, G_{K_j}(g^{-1}(x_j\, g(u)))}{g^{-1}(x_j\, g(u))} \cap \alpha(1, \ldots, 1)$$

where

$$1 + \Sigma\, G_n(u)\, t^n = \frac{\partial/\partial t\,(g(ut))}{g^{-1}(g(u) - g(ut))}$$

Theorems 10 and 11 give the additive basis of the Ω_U-module of fixed submanifolds for all actions of the group Z_p.

IV. Bordism of Eilenberg-Mac-Lane complexes and homotopy invariants of non simply connected manifolds.

S. Novikov raised a problem, (see his report at I.C.M., 1966), on homotopical invariants of non-simply-connected manifolds which are responsible for surgery to homotopy equivalences. In the case of simply-connected manifolds, the signature is the only such invariant. We give a new interpretation of Wall groups $L_n(\pi)$, $L_n^s(\pi)$ as some version of bordism of the Eilenberg Mac-Lane complexes.

Let Λ be the group ring of the group π over the dyadic numbers ring. Let $C = \{C_i, d_i\}$ be a Λ-free chain complex and $\xi_i : C_{n-i}^* \to C_i$ be homomorphisms such that $d_i \xi_i = (-1)^i \xi_{i-1} d_{n-i+1}^*$, $\xi_i = (-1)^{(n-i)i} \xi_{n-i}^*$, where $\xi_* : H(C^*) \to H(C)$ are isomorphisms. The triad $\alpha = (C, d, \xi)$ is called an algebraic Poincaré complex. Similarly there is a definition of an algebraic Poincaré pair β and a notion of a boundary $\partial\beta$ of the pair β. If $\alpha = \partial\beta$, then α is considered to be equivalent to zero. Then $\Omega_n(\Lambda)$ is the set of bordism classes of algebraic Poincaré complexes. According to Wall, $\Omega_n^s(\Lambda)$ is defined in a similar way.

THEOREM 12 ([8]). – *Let $L_n^Q(\pi)$, $L_n^{Q,s}(\pi)$ be the Wall groups for the ring Λ. Then $\Omega_n(\Lambda) \approx L_n^Q(\pi)$ and $\Omega_n^s(\Lambda) \approx L_n^{Q,s}(\pi)$. If two algebraic Poincaré complexes are (simply) homotopically equivalent, then they define the same element in the group $\Omega_n(\Lambda)$ (resp. in the group $\Omega_n^s(\Lambda)$).*

COROLLARY 4. – *Let $q : L_n^s(\pi) \to L_n^{Q,s}(\pi)$ be the natural homomorphism. There exist homomorphisms $\sigma : \Omega_n(K(\pi, 1)) \to L_n^{Q,s}(\pi)$ such that :*

(a) *If M_1 and M_2 are simply homotopically equivalent manifolds then*

$$\sigma(M_1) = \sigma(M_2).$$

(b) *If $(M_1, f, \varphi) \in \Omega_n(M_2, \eta)$, $\theta(M_1, f, \varphi) \in L_n(\pi)$ is the obstruction to surgery M_1 to simple homotopy equivalence, then $q(\theta(M_1, f, \varphi)) = \sigma(M_1) - \sigma(M_2)$. Such a statement is still true if one omits the word "simple".*

THEOREM 13. – *There exist an exact sequence*

$$\ldots \to L_{n+1}^{OTH}(\pi) \to L_n^Q(\pi) \to L_n^{Q,s}(\pi) \to L_n^{OTH}(\pi) \to \cdots$$

where $L_n^{OTH}(\pi)$ consists of elements of order smaller than four.

Corollary 4 gives a method of classification of smooth structures of non-simply-connected manifolds with fundamental group π, provided the Wall groups $L_n(\pi)$ are known. For instance the $L_n(\pi)$ are known if π is free abelian, if $\pi = Z_p$, if π is the fundamental group of a surface or of a so called solvable manifold. Theorem 13 gives the way that one can follow to extend the Farrel-Hsiang method to a wide class of groups such as the free product $G * Z$. Let $G_0 \subset G$, $\varphi : G_0 \to G$ be any monomorphism. Let us consider the group $G_1 = G \odot_\varphi Z$, which is the quotient group of the free product $G * Z$ by the relations $gt = t\varphi(g)$, $g \in G_0$; then the following sequences

$$0 \to \operatorname{Ker} \alpha \to L_n(G_1) \xrightarrow{\alpha} L_{n-1}(G_0) \to 0$$

$$0 \to \operatorname{Ker} \alpha \to L_n(G)$$

are exact modulo elements of finite order.

Another problem arises from the analysis of the Hirzebruch formula which expresses the signature of a manifold by means of the characteristic numbers of the tangent bundle. C.G. Kasparov suggested the following generalisation of the notion of "higher signatures" for non-simply connected manifolds. Let $x \in H^*(K(\pi, 1), Q)$, $f: M \to K(\pi, 1)$ be the canonical mapping. Let us write $\sigma_x(M) = \langle L(M) f^*(x), [M] \rangle$. If $\sigma_x(M) = 0$ for any x, then $\sigma(M) = 0$. The converse statement is true for the above mentioned class of groups Π.

Conjecture. (G.G. Kasparov). The "higher signatures" $\sigma_x(M)$ are homotopy invariants if $K(\pi, 1)$ is an oriented manifold.

REFERENCES

[1] Bukhstaber V.M., Mischenko A.S. — *Dokl. A.N. U.S.S.R.*, T. 178, No 6, 1968.

[2] Bukhstaber V.M., Mischenko A.S. — *Izv. A.N. U.S.S.R. Mat. Ser.*, T. 32, No 3, 1968.

[3] Bukhstaber V.M. — *Dokl. A.N. U.S.S.R.*, T. 186, No 3, 1969.

[4] Bukhstaber V.M. — *Matem. Sb.*, T. 78, No 2, 1969; T. 82, No 1, 1970.

[5] Bukhstaber V.M. — *Izv. A.N. U.S.S.R. Mat. Ser.*, T. 34, 1970.

[6] Mischenko A.S. — *Matem. Zametki*, T. 4, No 4, 1968.

[7] Mischenko A.S. — *Matem. Sb.*, T. 80, No 3, 1969.

[8] Mischenko A.S. — *Izv. A.N. U.S.S.R. Mat. Ser.*, T. 34, No 3, 1970.

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CHARACTERISTIC CLASSES AND COBORDISM

by F. P. PETERSON

1. Coalgebras over Hopf algebras.

Let A be a connected Hopf algebra over Z_p. Let M be a connected coalgebra over A. Let $\phi : A \to M$ be given by $\phi(a) = a(1)$. Let B be a Hopf subalgebra of A such that there exist differentials $Q_i \in B$ such that a B-module P is free if and only if $H(P, Q_i) = 0$ all i. If N is an A-module, let $N^{(n)}$ = the A-submodule generated by all elements of dimension $\leqslant n$.

THEOREM 1.1. – *Assume* Ker $\phi = A\overline{B}$ *where B is finite. Assume there is an A-module N and an A-map $\theta : N \to M$ such that*

$$\theta_* : H(N, Q_i) \to H(M, Q_i)$$

is an isomorphism for all i. Assume $N^{(0)} = A/AB$ and that if $x \in N^{(n)}/N^{(n-1)}$ with $|x| = n$, then there is a non-zero $b \in B$ such that $bx = 0$. Then θ is a monomorphism and Coker θ is free, i.e.

$$M \cong N \oplus \text{a free } A\text{-module}.$$

This is a generalization of an algebraic theorem in our Spin cobordism paper [2] and for a proof, see [10]. See Margolis [7] for a related theorem.

Example 1. – Adams and Margolis [1] have shown that all Hopf subalgebras of the mod 2 Steenrod algebra $\mathfrak{A}$ satisfy the condition on differentials. The case $B = \{1, Sq^1\}$ gives the structure of $H^*(MSO ; Z_2)$ and we now show how it applies to $H^*(M\,\text{Spin} ; Z_2)$ using $B = \mathfrak{A}_1 = \{1, Sq^1, Sq^2\}$.

Let $J = (j_1, \ldots, j_r)$ with $r \geqslant 0$, $j_i > 1$. Let

$$P_J = P_{j_1} \ldots P_{j_r} \in H^{4n(J)}(B\,\text{Spin} ; Z_2).$$

THEOREM 1.2. – *If $n(J)$ is even, there exist classes u_J such that*

$$x_J = P_J + Q_0 Q_1 (u_J) \in \text{Ker } Sq^1 \cap \text{Ker } Sq^2.$$

If $u(J)$ is odd, then there exist classes y_J such that

$$Sq^2(y_J) = P_J.$$

This was proved in our paper [2] using KO-theory and a lot of work. An elementary proof can be constructed as follows. If $n(J)$ is even, by induction it is enough to find $u_{(2k)}$ and $u_{(2k+1, 2l+1)}$. Let

$$z_k = \sum_{i=0}^{\infty} W_{k-i} \cdot W_{k+2+i}\,.$$

Let

$$u_{(2k)} = z_{4k-3} \quad \text{and} \quad u_{(2k+1,2m+1)} = z_{4k} \cdot z_{4m}$$

and compute. If $n(J)$ is odd, let $J = (j_1, J')$ with $n(J')$ even and $j_1 = 2k \pm 1$. Then let

$$y_J = z_{4_k} \,.\, x_{J'} + Sq^2 \,(P_{2k+1} \,.\, u_{J'})$$

and compute.

To apply Theorem 1.1 to M Spin we define

$$N = \sum_{\substack{n(J)\\ \text{even}}} \mathcal{A} / \mathcal{A}(Sq^1, Sq^2) \oplus \sum_{\substack{n(J)\\ \text{odd}}} \mathcal{A}/\mathcal{A}(Sq^3)$$

and define θ by sending

$$J \to x_J \,.\, U \quad \text{if} \quad n(J) \quad \text{is even}$$

and

$$J \to y_J \,.\, U \quad \text{if} \quad n(J) \quad \text{is odd.}$$

By Theorem 1.2, θ defines an $\mathcal{A}$-map. One must, of course, prove that θ_* is an isomorphism. Finally, one reads off Ω_*^{Spin} from the Adams spectral sequence.

Example 2. Let $A = \mathcal{A}_p$, the mod p Steenrod algebra, and B be an exterior subalgebra, for example

$$B = E(Q_0, Q_1)$$

where

$$Q_0 = \beta \quad \text{and} \quad Q_1 = P^1 \beta - \beta P^1.$$

It is reasonable to expect that

$$M = H^*(MSPL \,;\, Z_p)$$

satisfies the hypothesis of 1.1 with this B ; in particular it is known that

$$Q_0, Q_1 \in \text{Ker}\, \phi \quad \text{and} \quad Q_2 \notin \text{Ker}\, \phi$$

and my computations produce N and θ for a range of dimensions. $H^*(BSPL\,;\, Z_p)$ is "known" by Madsen and May, but at present it is not known in a strong enough form to compute $Q_i(U)$. I have computed the p-torsion of $\Omega_*^{PL} = \Omega_*^{\text{Top}}$ for some range, say in dimensions $\leqslant p^2\,(2p-2)$. An interesting case is

$$\Omega^{PL}_{(2p+1)\,(2p-2)-1}$$

which contains Z_{p^2} and p times the generator is not detected by ordinary cohomology characteristic numbers (see [9] for details).

2. Secondary characteristic classes

Let

$$I^n = \bigcap_{M^n} \operatorname{Ker}(\nu^* : H^*(BO) \to H^*(M^n)) \subset H^*(BO\,;Z_2)$$

be the ideal of relations among Stiefel-Whitney classes. In [6] we proved that

$$I_n = \sum_{2j > n-i} H^i(BO)\,Sq^j\,,$$

where right operations by $\mathcal{A}$ on $H^*(BO)$ are defined by

$$(u)a = \Phi^{-1}(\chi(a)\,\phi(u))\,,$$

where

$$\Phi : H^*(BO) \to H^*(MO)$$

is the Thom isomorphism. Furhtermore, we can write down an additive basis for I_n and a minimal generating set as a right $\mathcal{A}$-module, but we do not know the structure of I_n as an ideal over $\mathcal{A}$. Let $\{y_i\}$ be a minimal generating set for I_n as an ideal over $\mathcal{A}$. Let $|y_i| = r_i$. Construct

$$\begin{array}{c} B_n \\ \downarrow \\ BO \longrightarrow \Pi\, K(Z_2\,, r_i) \end{array}$$

where

$$Y^*(\iota_i) = y_i\,.$$

Note that $r_1 = \left[\frac{n}{2}\right] + 1$ and $y_1 = \nu_{[\frac{n}{2}]+1}$. If

$$\nu : M^n \to BO$$

is the normal map to a closed, C^∞-manifold, then

$$\nu \cong \pi\tilde{\nu}\,,$$

where

$$\tilde{\nu} : M^n \to B_n\,.$$

Hence any element $x \in H^*(B_n)$ gives $\tilde{\nu}^*(x) \in H^*(M)$, a secondary characteristic class defined for all n-manifolds. For example, right $\mathcal{A}$-relations give rise to such elements $x \in H^*(B_n)$ which are not in Im π^*. E.g.,

$$(1)\,Sq^3Sq^2 = 0 \quad \text{and} \quad (1)\,Sq^3 \in I_5\,.$$

This gives an $x \in H^4(B_5)$. (Many such examples were noted in [8]). We also note that if we form cobordism with respect to B_n we get a theory where the Arf invariant is defined [4]. This may give an interesting multiplicative cobordism theory.

Let M_n be the Thom spectra for B_n and let F_n be the fibre of

$$M_n \xrightarrow{T(\pi)} MO.$$

As a corollary of a theorem of Browder [3], we have the following result.

THEOREM 2.1. – *In dimensions* $< n$, F_n *is a wedge of* $K(Z_2)$'s, *one for each element of*

$$\{y_i\} \otimes \{\text{additive basis of } H^*(BO)\}.$$

The following result has been checked by explicit computation by E.H. Brown and myself in a range of dimension and we believe it to be true in general.

STATEMENT 2.2. – In dimensions $< n$,

$$\tau : H^*(F_n) \to H^{*+1}(MO)$$

is a monomorphism on the $\mathcal{A}$-generators of $H^*(F_n)$.

The following are corollaries of 2.2.

COROLLARY 2.3. – In dimensions $< 3n/4$,

$$H^*(B_n) = \operatorname{Im} \pi^* = H^*(BO)/I_n.$$

This follows because I_n is a free right $\mathcal{A}$-module in that range.

COROLLARY 2.4. – If $x \in H^*(B_n)$, $|x| < n$, then $\tilde{\nu}^*(x)$ is independent of the lifting $\tilde{\nu}$ of ν.

COROLLARY 2.5. – If $x \in H^*(B_n)$, $|x| < n$, then there exists $\omega \in H^*(BO)$ such that

$$\tilde{\nu}^*(x + \pi^*(\omega)) = 0 \quad \text{for all } M^n \text{ and } \tilde{\nu}.$$

2.4 and 2.5 follow from the arguments given in [5].

COROLLARY 2.6. – $H^*(B_n)$ can be computed in dimensions $\leqslant n$.

COROLLARY 2.7. – Let

$$c_k = \text{the number of } y_i \text{ with } r_i = \left[\frac{n}{2}\right] + 1 + k.$$

Then c_k is independent of n and

$$c(t) = \Sigma c_k t^k$$

is given by the polynomial

$$c(t) = \frac{\sum_{m \in L} t^{m/2} + \sum_{m_1,m_2,m_3 \in L} t^{(m_1+m_2+m_3)/2} + \sum_{m_1,\ldots,m_5 \in L} t^{(m_1+\ldots+m_5)/2} + \ldots}{t^{\frac{1}{2}} \prod_{m \in L} (1 - t^m)}$$

where L is the set of odd positive integers not of the form $2^r - 1$ and 1.

BIBLIOGRAPHY

[1] ADAMS J.F. and MARGOLIS H. — Modules over the Steenrod algebra, to appear.
[2] ANDERSON D.W., BROWN E.H. Jr. and PETERSON F.P. — The structure of the Spin cobordism ring, *Ann. of Math.*, 86, 1967, p. 271-298.
[3] BROWDER W. — The Kervaire invariant of framed manifolds and its generalization, *Ann. of Math.*, 90, 1969, p. 157-186.
[4] BROWN E.H. Jr. — The Kervaire invariant of a manifold, *Proc. of Symposia in Pure Math.*, Madison, 1970.
[5] BROWN E.H. Jr., and GITLER S. — The dual and excess of higher order cohomology operators, *Proc. Adv. Study Inst. Alg. Top.*, Aarhus, 1970.
[6] BROWN E.H. Jr., and PETERSON F.P. — Relations among characteristic classes, I, *Topology*, 3 suppl. 1, 1964, p. 39-52.
[7] MARGOLIS H. — to appear.
[8] PETERSON F.P. — Relations among Stiefel-Whitney classes of manifolds, *Colloquium on Alg. Top.*, Aarhus, 1962, p. 10-15.
[9] PETERSON F.P. — Some results on PL-cobordism, *Jour. of Math. of Kyoto Univ.*
[10] PETERSON F.P. — Cobordism theory, *Proc. of Symposia in Pure Math.*, Madison, 1970.

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BLOCK STRUCTURES IN GEOMETRIC AND ALGEBRAIC TOPOLOGY

by C. P. ROURKE

I want to describe some homotopy functors which are defined on the category of *CW* complexes but which are naturally defined on the *PL* category first. The point being that the 'kinky' nature of *PL* topology turns out to be a positive advantage in the definition : we use the description of a polyhedron as an equivalence class of simplicial complexes under the relation of common subdivision. I shall give two examples of such functors and then a general recipe which includes both examples.

Example 1. – "Block bundles".

The first example is the classical example constructed by Kato [4], Morlet [7] and Rourke-Sanderson [10] ; it belongs in the realm of geometric topology.

Let K be a *PL* cell complex, i.e. a polyhedron $|K|$ and a collection $\{\sigma\}$ of *PL* balls contained in $|K|$, the cells, which cover $|K|$ and satisfy

(i) the interiors of the cells are disjoint

(ii) the boundary of a cell or the intersection of two cells is a union of cells.

A *q-block bundle* ξ^q/K is a total space $E(\xi) \supset |K|$ and for each i-cell $\sigma \in K$ a *block* $\beta_\sigma \subset E(\xi)$ such that (β_σ , σ) is an unknotted $(q + i , i)$-ball pair. The blocks cover $E(\xi)$ and satisfy axioms which can be summarised by saying that they "fit together like the cells of K".

The crucial theorem for turning block bundles into a functor is the subdivision theorem :

Let K' be a subdivision of K. Then there is a natural $1 : 1$ *correspondance between isomorphism classes of block bundles over K and block bundles over K'.*

The correspondence is established by subdividing the bundle itself, that is by finding blocks of ξ'/K' inside blocks of ξ. In the next example I shall give a general proof of subdivision which includes this theorem. We can now define pullbacks for a *PL* map $f : |L| \to |K|$ by constructing the bundle $\xi \times L/K \times L$ with total space $E(\xi) \times |L|$ and block $\beta_\sigma \times \tau$ over $\sigma \times \tau$ and then subdividing and restricting to $\Gamma f \subset |K \times L|$. And we get a homotopy functor as stated earlier (for details see [10 ; § 1]). It is worth noting that there is also a natural notion of Whitney sum given by restricting $\xi \times \eta/K \times K$ to Δ_K. Block bundles were invented to give a 'normal bundle' theory for the *PL* category.

Example 2. – "Cobordism".

The second example belongs to algebraic topology. For details of this example, see Rourke-Sanderson [12]. A *q-mock bundle* ξ^q/K consists of a total space $E(\xi)$ and for each i-cell $\sigma \in K$ a $(q + i)$-manifold $M_\sigma \subset E(\xi)$, the *block* over σ. The blocks cover $E(\xi)$ and satisfy two axioms :

(i) $\mathring{M}_\sigma$ are disjoint

(ii) $\partial M_\sigma = \cup \{M_\tau \mid \tau \subset \dot{\sigma}\}$

$$M_\sigma \cap M_\tau = \cup \{M_\rho \mid \rho \subset \sigma \cap \tau\}$$

Note the similarity of these axioms to those for a cell complex ; so we can again summarise the definition by saying that the "blocks fit together like the cells of K".

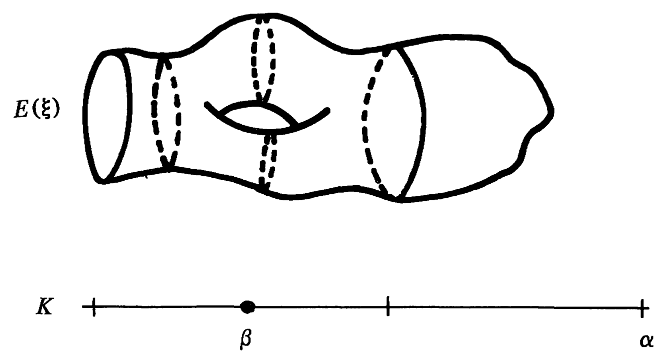

Figure 1 – Picture of a 1-mock bundle ; the block over α is empty. Possible subdivisions corresponding to the new vertex β are shown dotted.

Subdivision theorem. – *Let K' be a subdivision of K and ξ/K a mock bundle. Then there is a mock bundle ξ'/K' with $E(\xi') = E(\xi)$ and $M_\sigma(\xi) = \cup\{M_\tau(\xi') \mid \tau \subset \sigma\}$.*

In other words we cut up the blocks of ξ over cells of K'. As figure 1 illustrates, this theorem is a kind of transversality theorem ; so I am going to include a sketch of proof to stress the elementary nature of the method, which *uses only collars.*

Sketch of proof. By induction we can assume ξ already subdivided over the $(n-1)$–skeleton of K and we have to extend over one n-cell $\sigma \in K$. If we can subdivide ξ over a further subdivision σ'' of σ', then on taking unions of blocks (amalgamating) we get a subdivision over σ'. So we can assume that σ' has a top dimensional cell $\sigma_1 \subset \mathring{\sigma}$ and $\sigma' - \sigma_1$ is a 'cylindrical triangulation', using a *PL* isomorphism of $\sigma' - \mathring{\sigma}_1$ with $\dot{\sigma}' \times I$. Choose a collar on M_σ and define the blocks over cells of $\sigma' - \sigma_1$ by identifying the two collar parameters, and finally define $M_{\sigma_1} = M_\sigma$-collar.

Now subdivisions are not unique, but they are unique up to *cobordism* by the same proof, where mock bundles ξ_0, ξ_1 are *cobordant* if they are restrictions of a mock bundle over $K \times I$. And the cobordism classes of mock bundles define a contravariant functor $T^q(\ )$ on polyhedra.

THEOREM. – *There is a natural equivalence*

$$T^q(\ ) \simeq \mathfrak{N}_{PL}^{-q}(\ )$$

where $\mathfrak{N}_{PL}^r$ *denotes the r-th unoriented PL cobordism group.*

Sketch of proof : It suffices to construct an Alexander duality isomorphism between mock bundles and bordism. As usual this follows from Poincaré duality. So let ξ^q/M^n be a mock bundle over a manifold. Then $E(\xi)$ is an $(n+q)$–manifold (for proof see [3 ; 1-2]) and $p : E(\xi) \to M$ (a projection constructed inductively over cells) is a bordism class which defines the Poincaré dual to ξ. Conversely, given $f : W \to M$ then make f simplicial and consider dual cells in M. Then f^{-1}(cell) is a manifold by [2] and all the manifolds give a mock bundle structure with total space W.

Finally to end this example we observe that the various operations in cobordism have easy geometric pictures in terms of mock bundles :

Addition = disjoint union

Cup product = Whitney sum (defined as for block bundles)

Cap product = Amalgamated pull back : i.e. given $f : W \to K$ and ξ/k, form $f^*(\xi)/W$, then $f \cdot p : E(f^*(\xi)) \to K$ is the required bordism class.

General recipe.

Let $\mathfrak{M}$ be a category of "pseudo-manifolds" and inclusions in the boundary, where a pseudo-manifold is an object with a virtual dimension and a boundary of one dimension lower. For a cell complex K define the associated category denoted K, as in [11 ; § 1], to have objects the cells of K and morphisms the face inclusions in K.

Then an $(\mathfrak{M}, q)$-bundle, ξ^q/K, is a functor

$$\xi : K \to \mathfrak{M}$$

which raises dimension by q and such that the *blocks* $\xi(\sigma)$ "fit together like the cells of K" i.e.

(i) $\partial \xi(\sigma) = \cup \{\xi(\tau) \mid \tau \subset \dot{\sigma}\}$

(ii) $\xi(\sigma) \cap \xi(\tau) = \cup \{\xi(\rho) \mid \rho \in \sigma \cap \tau\}$

Example (i)

$\mathrm{Ob}(\mathfrak{M}) = \{D^p \times D^q \mid p, q \geqslant 0\}$

$\dim(D^p \times D^q) = p + q$

$\partial(D^p \times D^q) = \partial D^p \times D^q$

(N.B. ∂ not necessarily in $\mathfrak{M}$!)

Mor $(\mathcal{M}) = \{f : D^{p'} \times D^q \hookrightarrow \partial D^p \times D^q \mid p' < p$ and $f(D^{p'} \times 0) \subset D' \times 0\}$

Then an $(\mathcal{M}, q)$-bundle is a q-block bundle.

Ex (ii). – (Block bundles with arbitrary fibre)

Ob $(\mathcal{M}) = \{D^p \times F\}$ virtual dim. p

$\partial (D^p \times F) = \partial D^p \times F$

Mor $(\mathcal{M}) = \{f : D^{p'} \times F \hookrightarrow \partial D^p \times F$ *blockwise i.e.* $\mathrm{Im}(f) = X \times F$, some $X\}$

Then an $(\mathcal{M}, 0)$-bundle is a block bundle with fibre F in the sense of [1,11] *with charts.*

Ex (iii). – Homology cell analogue of (i) (Martin-Maunder [6]), where a homology cell is the cone on a homology manifold which is a homology sphere. This theory is the normal bundle theory for the "homology" category.

Ex (iv). – $\mathcal{M}$ = all manifolds (graded by dimension) then an $(\mathcal{M}, q)$-bundle is a q-mock bundle.

Ex (v). – $\mathcal{M}$ = {manifolds with restriction on normal block bundle}. Then the corresponding mock bundle theory gives a more general cobordism theory.
E.g. (a) normal bundle smooth oriented ; result smooth oriented cobordism.

(b) normal bundle trivialised ; result stable cohomotopy.

Another direction to generalise example (iv) is to introduce singularities. For example if we introduce all possible singularities :

Ex (vi). – $\mathcal{M}$ = {principal n-polyhedra}, $\partial = Z_2$-boundary. Then the resulting theory is Z_2-cohomology (same proof as for mock bundles).

Ex (vii). – $\mathcal{M}$ = {Poincaré duality spaces}. Then the resulting theory is the cohomology theory corresponding to Levitt's "transversal subcomplex" of MG [5].

Axioms for a theory.

We now axomatise the properties of $\mathcal{M}$ which are needed to set up the theory :

Axiom 1. – Objects of $\mathcal{M}$ have collars up to cobordism.

Axiom 2. – $M \in \mathcal{M} \Rightarrow M \times I \in \mathcal{M}$.

Axioms 1 and 2 allow the proof of the subdivision theorem to work, to provide subdivisions up to cobordism.

Axiom 3 (amalgamation). – Suppose $M_1, M_2, M_1 \cap M_2 \in \mathcal{M}$, where M_1 and M_2 have "dim" n, and $M_1 \cap M_2$ has "dim" $n-1$, and that the inclusions $M_1 \cap M_2 \subset M_i$, $i = 1, 2$, are in $\mathcal{M}$. Then $M_1 \cup M_2 \in \mathcal{M}$.

Axiom 3 is necessary to pass from a bundle over K' to one over K by "amalgamating blocks", Axioms 1, 2 and 3 imply independence of the cell structure of K and that we have a homotopy functor by the proof outlined for mock bundles.

Remark. – There are significant cases where axioms 1 and 2 are satisfied but not axiom 3, for example $\mathfrak{M} =$ unions of discs . In this case we can again define a functor by letting an object "over K" be an object over *some subdivision* of K. Amalgamation is then formal and the proof goes through. The functor corresponding to unions of discs , in codim 3 at least, is dual to "immersed bordism theory", for details see [14].

Axiom 4. – $\mathfrak{M}$ closed under disjoint union.

Axiom 4 implies that we have an abelian semigroup functor under disjoint union ; in most natural cases, an abelian group.

Axiom 5. – $\mathfrak{M}$ closed under cartesian product.

Axiom 5 gives an external product and by restriction an internal product (cup product, Whitney sum).

Two final remarks

(1) The general description of an $\mathfrak{M}$-bundle applies directly for a Δ-set, as in [11 ; § 1], so that our functors are defined for the *CW* category.

(2) There is a universal bundle $\gamma_{\mathfrak{M}}/G_{\mathfrak{M}}$ where $G_{\mathfrak{M}}$ is the "Grassmannian" of $\mathfrak{M}$-bundles over Δ^k embedded in $\Delta^k \times R^\infty$ and $\gamma_{\mathfrak{M}}$ is the obvious functor (compare [11 ; § 1]). See also [8 ; § 1].

Credits

Sanderson and myself were awakened to the possibility of more general "block bundles" by the work of Martin and Maunder [6] ; however the construction has strong relations with the ideas of Casson and Sullivan, as exposited by myself [9], and, in a more general setting, by Quinn [8]. The terminology "mock bundle" is due to Cohen.

REFERENCES

[1] Casson A. — *Block bundles with manifolds as fibres,* (to appear).
[2] Cohen M.M. — Simplicial structures and transverse cellularity, *Ann. of Math.,* 85 (1967), p. 218-245.
[3] Cohen M.M. and Sullivan D.P. — On the regular neighbourhood of a 2-sided submanifold, *Topology,* 9 (1970), p. 141-147.
[4] Kato M. — *Combinatorial prebundles* : I.
[5] Levitt N. — *Poincaré duality bordism,*
[6] Martin N. and Maunder C.R.F. — Homology cobordism bundles, *Topology* (to appear).
[7] Morlet C. — *Thesis.*
[8] Quinn. — *Thesis.*
[9] Rourke C.P. — *Block bundles and Sullivan theory,* (to appear).
[10] Rourke C.P. and Sanderson B.J. — Block bundles : I, *Ann. of Math.,* 87 (1968), p. 1-28.

[11] Rourke C.P. and Sanderson B.J. — Δ-sets : II, *Quart. Jour. Math.* (Oxford), (to appear).
[12, 13, 14] Rourke C.P. and Sanderson B.J. — *Mock bundles I, II and III,* (to appear).

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TOPOLOGICAL MANIFOLDS *

by L. C. SIEBENMANN

0. Introduction.

Homeomorphisms – topological isomorphisms – have repeatedly turned up in theorems of a strikingly conceptual character. For example :

(1) (19^{th} century). There are continuously many non-isomorphic compact Riemann surfaces, but, up to homeomorphism, only one of each genus.

(2) (B. Mazur 1959). Every smoothly embedded $(n - 1)$-sphere in euclidean n-space R^n bounds a topological n-ball.

(3) (R. Thom and J. Mather, recent work). Among smooth maps of one compact smooth manifold to another the topologically stable ones form a dense open set.

In these examples and many others, homeomorphisms serve to reveal basic relationships by conveniently erasing some finer distinctions.

In this important role, PL (= piecewise-linear)(**) homeomorphisms of simplicial complexes have until recently been favored because homeomorphisms in general seemed intractable. However, PL homeomorphisms have limitations, some of them obvious ; to illustrate, the smooth, non-singular self-homeomorphism $f : R \to R$ of the line given by $f(x) = x + \frac{1}{4} \exp(-1/x^2) \sin(1/x)$ can in no way be regarded as a PL self-homeomorphism since it has infinitely many isolated fixed points near the origin.

Developments that have intervened since 1966 fortunately have vastly increased our understanding of homeomorphisms and of their natural home, the category of (finite dimensional) topological manifolds(***). I will describe just a few of them below. One can expect that mathematicians will consequently come to use freely the notions of homeomorphism and topological manifold untroubled by the frustrating difficulties that worried their early history.

(*) This report is based on theorems concerning homeomorphisms and topological manifolds [44] [45] [46] [46 A] developed with R.C. Kirby as a sequel to [42]. I have reviewed some contiguous material and included a collection of examples related to my observation that $\pi_3(\mathrm{TOP/PL}) \neq 0$. My oral report was largely devoted to results now adequately described in [81], [82].

(**) A continuous map $f : X \to Y$ of (locally finite) simplicial complexes is called PL if there exists a simplicial complex X' and a homeomorphism $s : X' \to X$ such that s and fs each map each simplex of X' (affine) linearly into some simplex.

(***) In some situations one can comfortably go beyond manifolds [82]. Also, there has been dramatic progress with infinite dimensional topological manifolds (see [48]).

1. **History.**

A topological (= TOP) m-manifold M^m (with boundary) is a metrizable topological space in which each point has an open neighborhood U that admits an open embedding (called a *chart*) $f : U \to R^m_+ = \{(x_1, \ldots, x_m) \in R^m | x_1 \geq 0\}$, giving a homeomorphism $U \approx f(U)$.

From Poincaré's day until the last decade, the lack of techniques for working with homeomorphisms in euclidean space R^m (m large) forced topologists to restrict attention to manifolds M^m equipped with atlases of charts $f_\alpha : U_\alpha \to R^m_+$, $\cup U_\alpha = M$, (α varing in some index set), in which the maps $f_\beta f_\alpha^{-1}$ (where defined) are especially tractable, for example all DIFF (infinitely differentiable), or all PL (piecewise linear). Maximal such atlases are called respectively DIFF or PL manifold structures. Poincaré, for one, was emphatic about the importance of the naked homeomorphism – when writing philosophically [68, §§ 1, 2] – yet his memoirs treat DIFF or PL manifolds only.

Until 1956 the study of TOP manifolds as such was restricted to sporadic attempts to prove existence of a PL atlas (= *triangulation conjecture*) and its essential uniqueness (= *Hauptvermutung*). For $m = 2$, Rado proved existence, 1924 [70] (Kerékjártó's classification 1923 [38] implied uniqueness up to isomorphism). For $m = 3$, Moise proved existence and uniqueness, 1952 [62], cf. a misproof of Furch 1924 [21].

A PL manifold is easily shown to be PL homeomorphic to a simplicial complex that is a so-called combinatorial manifold [37]. So the *triangulation conjecture* is that any TOP manifold M^m admits a homeomorphism $h : M \to N$ to a combinatorial manifold. The *Hauptvermutung* conjectures that if h and $h' : M \to N'$ are two such, then the homeomorphism $h'h^{-1} : N \to N'$ can be replaced by a PL homeomorphism $g : N \to N'$. One might reasonably demand that g be *topologically isotopic* to $h'h^{-1}$, or again *homotopic* to it. These variants of the Hauptvermutung will reappear in §5 and §15.

The Hauptvermutung was first formultated in print by Steinitz 1907 (see [85]). Around 1930, after homology groups had been proved to be topological invariants without it, H. Kneser and J.W. Alexander began to advertise the Hauptvermutung for its own sake, and the triangulation conjecture as well [47] [2]. Only a misproof of Nöebling [66] (for any m) ensued in the 1930's. Soberingly delicate proofs of triangulability of DIFF manifolds by Cairns and Whitehead appeared instead.

Milnor's proof (1956) that some 'well-known' S^3 bundles over S^4 are homeomorphic to S^7 but not DIFF isomorphic to S^7 strongly revived interest. It was very relevant ; indeed homotopy theory sees the failure of the Hauptvermutung (1969) as quite analogous. The latter gives the first nonzero homotopy group $\pi_3(\mathrm{TOP/O}) = Z_2$ of TOP/O ; Milnor's exotic 7-spheres form the second $\pi_7(\mathrm{TOP/O}) = Z_{28}$.

In the early 1960's, intense efforts by many mathematicians to unlock the geometric secrets of topological manifolds brought a few unqualified successes : for example the generalized Shoenflies theorem was proved by M. Brown [7] ; the tangent microbundle was developed by Milnor [60] ; the topological Poincaré conjecture in dimensions ≥ 5 was proved by M.H.A. Newman [65].

Of fundamental importance to TOP manifolds were Černavskii's proof in 1968 that the homeomorphism group of a compact manifold is locally contractible [10] [11], and Kirby's proof in 1968 of the stable homeomorphism conjecture with the help of surgery [42]. Key geometric techniques were involved – a meshing idea in the former, a particularly artful torus furling and unfurling idea(*) in the latter. The disproof of the Hauptvermutung and the triangulation conjecture I sketch below uses neither, but was conceived using both. (See [44] [44 B] [46 A] for alternatives).

2. Failure of the Hauptvermutung and the triangulation conjecture.

This section presents the most elementary disproof I know. I constructed it for the Arbeitstagung, Bonn, 1969.

In this discussion $B^n = [-1, 1]^n \subset R^n$ is the standard PL ball ; and the sphere $S^{n-1} = \partial B^n$ is the boundary of B^n. $T^n = R^n/Z^n$ is the standard PL torus, the n-fold product of circles. The closed interval $[0, 1]$ is denoted I.

As starting material we take a certain PL automorphism α of $B^2 \times T^n$, $n \geqslant 3$, fixing boundary that is constructed to have two special properties (1) and (2) below. The existance of α was established by Wall, Hsiang and Shaneson, and Casson in 1968 using sophisticated surgical techniques of Wall (see [35] [95]). A rather naive construction is given in [80, § 5], which manages to avoid surgery obstruction groups entirely. To establish (1) and (2) it requires only the s-cobordism theorem and some unobstructed surgery with boundary, that works from the affine locus $Q^4 : z_1^5 + z_2^3 + z_3^2 = 1$ in C^3. This Q^4 coincides with Milnor's E_8 plumbing of dimension 4 ; it has signature 8 and a collar neighborhood of infinity $M^3 \times R$, where $M^3 = \mathrm{SO}(3)/A_5$ is Poincaré's homology 3-sphere, cf. [61, § 9.8].

(1) *The automorphism β induced by α on the quotient T^{2+n} of $B^2 \times T^n$ (obtained by identifying opposite sides of the square B^2) has mapping torus*

$$T(\beta) = I \times T^{2+n}/\{(0, x) = (1, \beta(x))\}$$

not PL *isomorphic to T^{3+n} ; indeed there exists(**) a* PL *cobordism $(W ; T^{n+3}, T(\beta))$ and a homotopy equivalence of W to $\{I \times T^3 \# Q \cup \infty\} \times T^n$ extending the standard equivalences $T^{3+n} \simeq 0 \times T^3 \times T^n$ and $T(\beta) \simeq 1 \times T^3 \times T^n$. The symbol # indicates (interior) connected sum* [41].

(2) *For any standard covering map $p : B^2 \times T^n \to B^2 \times T^n$ the covering automorphism α_1 of α fixing boundary is* PL *pseudo-isotopic to α fixing boundary. (Covering means that $p\alpha_1 = \alpha p$). In other words, there exists a* PL *automorphism H of $(I ; 0, 1) \times B^2 \times T^n$ fixing $I \times \partial B^2 \times T^n$ such that $H | 0 \times B^2 \times T^n = 0 \times \alpha$ and $H | 1 \times B^2 \times T^n = 1 \times \alpha_1$.*

(*) Novikov first exploited a torus *furling* idea in 1965 to prove the topological invariance of rational Pontrjagin classes [67]. And this led to Sullivan's partial proof of the Hauptvermutung [88]. Kirby's *unfurling* of the torus was a fresh idea that proved revolutionary.

(**) This is the key property. It explains the exoticity of $T(\beta)$ – (see end of argument), and the property (2) – (almost, see [80, § 5]).

In (2) choose p to be the 2^n-fold covering derived from scalar multiplication by 2 in R^n. (Any positive integer would do as well as 2.) Let $\alpha_0 (= \alpha), \alpha_1, \alpha_2, \ldots$ be the sequence of automorphisms of $B^2 \times T^n$ fixing boundary such that α_{k+1} covers α_k, i.e. $p\alpha_{k+1} = \alpha_k p$. Similarly define $H_0 (= H), H_1, H_2, \ldots$ and note that H_k is a PL concordance fixing boundary from α_k to α_{k+1}. Next define a PL automorphism H' of $[0, 1) \times B^2 \times T^n$ by making $H' | [a_k, a_{k+1}] \times B^2 \times T^n$, where $a_k = 1 - \frac{1}{2^k}$, correspond to H_k under the (oriented) linear map of $[a_k, a_{k+1}]$ onto $[0, 1] = I$. We extend H' by the identity to $[0, 1) \times R^2 \times T^n$. Define another self-homeomorphism H'' of $[0, 1) \times B^2 \times T^n$ by $H'' = \varphi H' \varphi^{-1}$ where

$$\varphi(t, x, y) = (t, (1-t)x, y) \quad .$$

Finally extend H'' by the identity to a bijection

$$H'' : I \times B^2 \times T^n \to I \times B^2 \times T^n \quad .$$

It is also continuous, hence a homeomorphism. To prove this, consider a sequence $q_1, q_2, \ldots$ of points converging to $q = (t_0, x_0, y_0)$ in $I \times B^2 \times T^n$. Convergence $H''(q_j) \to H''(q)$ is evident except when $t_0 = 1, x_0 = 0$. In the latter case it is easy to check that $p_1 H''(q_j) \to p_1 H''(q) = 1$ and $p_2 H''(q_j) \to p_2 H''(q) = 0$ as $j \to \infty$, where p_i, $i = 1, 2, 3$ is projection to the i-th factor of $I \times B^2 \times T^n$. It is not as obvious that $p_3 H''(q_j) \to p_3 H''(q) = y_0$. To see this, let

$$\widetilde{H}_k : I \times B^2 \times R^n \to I \times B^2 \times R^n$$

be the universal covering of H_k fixing $I \times \partial B^2 \times R^n$. Now

$$\sup \{ |p_3 z - p_3 \widetilde{H}_k z| \quad ; \quad z \in [0, 1] \times B^2 \times R^n \} \equiv D_k$$

is finite, being realized on the compactum $I \times B^2 \times I^n$. And, as $\widetilde{H}_k$ is clearly $\theta_k^{-1} \widetilde{H}_0 \theta_k$, where $\theta_n(t, x, y) = (t, x, 2^n y)$, we have $D_k = \frac{1}{2^k} D_0$. Now D_k is $\geqslant$ the maximum distance of $p_3 H_k$ from p_3, for the quotient metric on $T^n = R^n / Z^n$; so $D_k \to 0$ implies $p_3 H''(q_j) \to p_3 H''(q) = y_0$, as $j \to \infty$.

As the homeomorphism H'' is the identity on $I \times \partial B^2 \times T^n$ it yields a self-homeomorphism g of the quotient $I \times T^2 \times T^n = I \times T^{2+n}$. And as

$$g | 0 \times T^{2+n} = 0 \times \beta \quad ,$$

and $g | 1 \times T^{2+n} =$ identity, g gives a homeomorphism h of $T(\beta)$ onto

$$T(\mathrm{id}) = T^1 \times T^{2+n} = T^{3+n}$$

by the rule sending points (t, z) to $g^{-1}(t, z)$ – hence $(0, z)$ to $(0, \beta^{-1}(z))$ and $(1, z)$ to $(1, z)$

The homeomorphism $h : T^{3+n} \approx T(\beta)$ *belies the Hauptvermutung.* Further, (1) offers a certain PL cobordism $(W ; T^{3+n}, T(\beta))$. Identifying T^{3+n} in W to $T(\beta)$ under h we get a closed topological manifold

$$X^{4+n} \simeq \{S^1 \times T^3 \# Q \cup \infty\} \times T^n$$

($\simeq$ indicating homotopy equivalence).

If it had a PL manifold structure the fibering theorem of Farrell [19] (or the author's thesis) would produce a PL 4-manifold X^4 with $w_1(X^4) = w_2(X^4) = 0$ and signature $\sigma(X^4) \equiv \sigma(S^1 \times T^3 \# Q \cup \infty) \equiv \sigma(Q \cup \infty) \equiv 8 \bmod. 16$, cf. [80, § 5]. Rohlin's theorem [71] [40] cf. § 13 shows this X^4 doesn't exist. *Hence X^{4+n} has no* PL *manifold structure.*

Let us reflect a little on the generation of the homeomorphism $h : T(\beta) \approx T^{3+n}$. The behaviour of H'' is described in figure 2-a (which is accurate for B^1 in place of B^2 and for $n = 1$) by partitioning the fundamental domain $I \times B^2 \times I^n$ according to the behavior of H''. The letter α indicates codimension 1 cubes on which H'' is a conjugate of α.

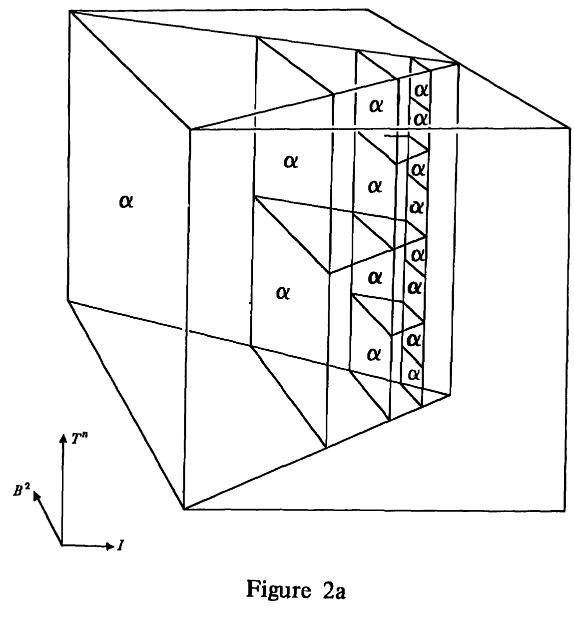


Figure 2a

Observe the infinite ramification (2^n-fold) into smaller and smaller domains converging to all of $1 \times 0 \times T^n$. In the terminology of Thom [92, figure 7] this reveals the failure of the Hauptvermutung to be a *generalized catastrophe* !

Remark 2.1. — Inspection shows that $h : T(\beta) \approx T^{3+n}$ is a Lipschitz homeomorphism and hence X^{4+n} is a Lipschitz manifold as defined by Whitehead [98] for the pseudogroup of Lipschitz homeomorphisms — see § 4. A proof that $T(\beta) \approx T^{3+n}$ (as given in [44]) using local contractibility of a homeomorphism group would not reveal this as no such theorem is known for Lipschitz homeomorphisms. Recall that a theorem of Rademacher [69] says that every Lipschitz homeomorphism of one open subset of R^m to another is almost everywhere differentiable.

3. The unrestricted triangulation conjecture.

When a topological manifold admits no PL manifold structure we know it is not homeomorphic to a simplicial complex which is a combinatorial manifold [37]. But it may be homeomorphic to *some* (less regular) simplicial complex — i.e. triangulable in an unrestricted sense, cf. [79]. For example $Q \cup \infty$ (from §2) is triangulable and Milnor (Seattle 1963) asked if $(Q \cup \infty) \times S^1$ is a topological manifold even though $Q \cup \infty$ obviously is not one. If so, the manifold X^{4+n} of § 2 is easily triangulated.

If all TOP manifolds be triangulable, why not conjecture that that every locally triangulable metric space is triangulable ?

Here is a construction for a compactum X that is *locally triangulable* but is *non-triangulable.* Let L_1, L_2 be closed PL manifolds and

$$(W\,; L_1 \times R\,, L_2 \times R)$$

an invertible(*) PL cobordism that is not a product cobordism. Such a W exists for instance if $\pi_1 L_i = Z_{257}$ and $L_1 \cong L_2$, compare [78]. It can cover an invertible cobordism $(W', L_1 \times S^1, L_2 \times S^1)$ [77, § 4]. To the Alexandroff compactification $W \cup \infty$ of W adjoin $\{(L_1 \times R) \cup \infty\} \times [0\,,1]$ identifying each point $(x\,,1)$ in the latter to the point x in $W \cup \infty$. The resulting space is X. See Figure 3-a. The properties of X and of related examples will be demonstrated in [83]. They complement Milnor's examples [57] of homeomorphic complexes that are PL (combinatorially) distinct, which disproved an *unrestricted Hauptvermutung.*

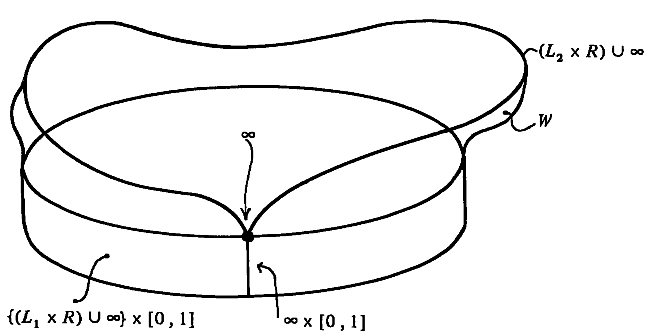


Figure 3a

(*) This means that W can be expressed as a union $W = C_1 \cup C_2$, where C_i is a closed collar neighborhood of $L_i \times R$ in W.

4. Structures on topological manifolds.

Given a TOP manifold M^m (without boundary) and a pseudo-group G of homeomorphisms(*) of one open subset of R^m to another, the problem is to find and classify G-structures on M^m. These are maximal "G-compatible" atlases $\{U_\alpha, f_\alpha\}$ of charts (= open embeddings) $f_\alpha : U_\alpha \to R^m$ so that each $f_\beta f_\alpha^{-1}$ is in G. (Cf. [29] or [48].)

One reduction of this problem to homotopy theoretic form has been given recently by Haefliger [28] [29]. Let $G(M^m)$ be the (polyhedral quasi-) space (**) of G-structures on M. A map of a compact polyhedron P to $G(M)$ is by definition a G-foliation $\mathfrak{F}$ on $P \times M$ transverse to the projection $p_1 : P \times M \to P$ (i.e. its defining submersions are transverse to p_1)(***). Thus, for each $t \in P$, $\mathfrak{F}$ restricts to a G-structure on $t \times M$ and, on each leaf of $\mathfrak{F}$, p_1 is an open embedding. Also note that $\mathfrak{F}$ gives a G_P-structure on $P \times M$ where G_P is the pseudo-group of homeomorphisms of open subsets of $P \times R^m$ locally of the form $(t, x) \to (t, g(x))$ with $g \in G$. If G consists of PL or DIFF homeomorphisms and $P = [0, 1]$, then $\mathfrak{F}$ gives (a fortiori) what is called a *sliced concordance* of PL or DIFF structures on M (see [45] [46]).

We would like to analyse $G(M^m)$ using Milnor's tangent R^m-microbundle $\tau(M)$ of M, which consists of total space $E(\tau M) = M \times M$, projection $p_1 : M \times M \to M$, and (diagonal) section $\delta : M \to M \times M$, $\delta(x) = (x, x)$. Now if ξ^m is *any* R^m microbundle over a space X we can consider $G^\perp(\xi)$ the space of G-foliations of $E(\xi)$ transverse to the fibers. A map $P \to G^\perp(\xi)$ is a G-foliation $\mathfrak{F}$ defined on an open neighborhood of the section $P \times X$ in the total space $E(P \times \xi) = P \times E(\xi)$ that is transverse to the projection to $P \times X$. Notice that there is a natural map

$$d : G(M^m) \to G^\perp(\tau M^m) \quad ,$$

which we call the *differential*. To a G-foliation $\mathfrak{F}$ of $P \times M$ transverse to p_1, it assigns the G-foliation $d\mathfrak{F}$ on $P \times M \times M = E(P \times \tau(M))$ obtained from $\mathfrak{F} \times M$

(*) e.g. the PL isomorphisms, or Lipschitz or DIFF or analytic isomorphisms. Do not confuse G with the stable monoid $G = \cup G_n$ of § 5.5.

(**) Formally such a space X is a contravariant functor $X : P \to [P, X]$ from the category of PL maps of compact polyhedra (denoted P, Q etc.) to the category of sets, which carries union to fiber product. Intuitively X is a space of which we need (or want or can) only know the maps of polyhedra to it.

(***) A *G-foliation* on a space X is a maximal G-compatible atlas $\{V_\alpha, g_\alpha\}$ of topological submersions $g_\alpha : V_\alpha \to R^m$. (See articles of Bott and Wall in these proceedings.) A map $g : V \to W$ is a *topological submersion* if it is locally a projection in the sense that for each x in V there exists an open neighborhood W_x of $g(x)$ in W a space F_x and an open embedding onto a neighborhood of x, called a *product chart* about x, $\varphi : F_x \times W_x \to V$ such that $g\varphi$ is projection $p_2 : F_x \times W_x \to W_x \subset W$. One says that g is *transverse* to another submersion $g' : V \to W'$ if for each x, φ can be chosen so that $F_x = W'_x \times F'_x$ and $g'\varphi$ is projection to W'_x an open subset of W'. This says roughly that the leaves (= fibers) of f and g intersect in general position. Above they intersect in points.

by *interchanging* the factors M. If P is a point, the leaves of $d\mathfrak{F}$ are simply

$$\{P \times M \times x \mid x \in M\} \quad .$$

Clearly $d\mathfrak{F}$ is transverse to the projection $P \times p_1$ to $P \times M$.

THEOREM 4.1. CLASSIFICATION BY FOLIATED MICROBUNDLES. — *The differential*

$$d : G(M^m) \to G^{\perp}(\tau M^m)$$

is a weak homotopy equivalence for each open (metrizable) m-manifold M^m with no compact components.

Haefliger deduces this result (or at least the bijection of components) from the topological version of the Phillips-Gromov transversality theorem classifying maps of M transverse to a TOP foliation. (See [29] and J.C. Hausmann's appendix).

As formulated here, 4.1 invites a direct proof using Gromov's distillation of immersion theory [25] [26]. This does not seem to have been pointed out before, and it seems a worthwhile observation, for I believe the transversality result adequate for 4.1 requires noticeably more geometric technicalities. In order to apply Gromov's distillation, there are two key points to check. For any $C \subset M^m$, let $G_M(C) = \text{inj lim}\,\{G(U) \mid C \subset U \text{ open in } M\}$.

(1) *For any pair $A \subset B$ of compacta in M, the restriction map $\pi : G_M(B) \to G_M(A)$ is micro-gibki* — i.e., given a homotopy $f : P \times I \to G_M(A)$ and $F_0 : P \times 0 \to G_M(B)$ with $\pi F_0 = f \mid P \times 0$ there exists $\epsilon > 0$ and $F : P \times [0, \epsilon] \to G_M(B)$ so that $\pi F = f \mid P \times [0, \epsilon]$. Chasing definitions one finds that this follows quickly from the TOP isotopy extension theorem (many-parameter version) or the relative local contractibility theorem of [10] [17].

(2) *d is a weak homotopy equivalence for $M^m = R^m$.* Indeed, one has a commutative square of weak homotopy equivalences

$$\begin{array}{ccc} G(R^m) & \xrightarrow{d} & G^{\perp}(\tau R^m) \\ \simeq\downarrow & & \downarrow\simeq \\ G_{R^m}(0) & \xleftarrow{\simeq} & G^{\perp}(\tau R^m \mid 0) \end{array}$$

in which the verticals are restrictions and the bottom comes from identifying the fiber of $\tau R^m \mid 0$ to R^m, cf. [27].

Gromov's analysis applies (1) and (2) and more obvious properties of G, $G^{\perp}$ to establish 4.1. Unfortunately, M doesn't always have a handle decomposition over which to induct ; one has to proceed more painfully chart by chart.

We can now pass quickly from a bundle theoretic to a homotopy classification of G-structures. Notice that if $f : X' \to X$ is any map and ξ^m is a R^m microbundle over X equipped with a G-folitation $\mathfrak{F}$, transverse to fibers, defined on an open neighborhood of the zero section X, then $f^*\xi$ over X' is similarly equipped with a pulled-back foliation $f^*\mathfrak{F}$. This means that equipped bundles behave much like bundles. One can use Haefliger's notion of "gamma structure" as in [29] to deduce for numerable equipped bundles the existence of a universal one $(\gamma^m_G, \mathfrak{F}_G)$ over a base space $B_{\Gamma(G)}$(*). There is a map $B_{\Gamma(G)} \to B_{\mathrm{TOP}(m)}$ classifing γ_G^m as an R^m-microbundle ; we make it a fibration. Call the fiber $\mathrm{TOP}(m)/\Gamma(G)$. One finds that there is a weak homotopy equivalence $G^{\perp}(\xi) \simeq \mathrm{Lift}\,(f \text{ to } B_{\Gamma(G)})$, to the space of liftings to $B_{\Gamma(G)}$ of a fixed classifying map $f : X \to B_{\mathrm{TOP}(m)}$ for ξ^m. Hence one gets

THEOREM 4.2. — *For any open topological m-manifold M^m, there is a weak homotopy equivalence $G(M) \simeq \mathrm{Lift}\,(\tau \text{ to } B_{\Gamma(G)})$ from the space of G-structures $G(M)$ on M to the space of liftings to $B_{\Gamma(G)}$ of a fixed classifying map map $\tau : M \to B_{\mathrm{TOP}(m)}$ for $\tau(M)$.*

Heafliger and Milnor observe that for $G = \mathrm{CAT}^m$ the pseudo-group of CAT isomorphisms of open subsets of R^m — CAT meaning DIFF (= smooth C^∞), or PL (= piecewise linear) or TOP (= topological) — one has

$$\pi_i\,(\mathrm{CAT}(m)/\Gamma\,(\mathrm{CAT}^m)) = 0 \quad , \quad i < m \quad . \tag{4.3}$$

Indeed for CAT = TOP, 4.2 shows this amounts to the obvious fact that $\pi_0(G\,(S^i \times R^{m-i})) = 0$. Analogues of 4.2 with DIFF or PL in place of TOP can be proved analogously(**) and give the other cases of (4.3). Hence one has

THEOREM 4.4. — *For any open topological manifold M^m, there is a natural bijection $\pi_0 \mathrm{CAT}^m(M^m) \simeq \pi_0 \mathrm{Lift}\,(\tau \text{ to } B_{\mathrm{CAT}(m)})$.*

This result comes from [44] for $m \geqslant 5$. Lashof [50] gave the first proof that was valid for $m = 4$. A stronger and technically more difficult result is sketched in [63] [45]. It asserts a weak homotopy equivalence of a "sliced concordance" variant of $\mathrm{CAT}^m(M^m)$ with $\mathrm{Lift}\,(\tau \text{ to } B_{\mathrm{CAT}(m)})$. This is valid without the openness restriction if $m \neq 4$. For open M^m (any m), it too can be given a proof involving a micro-gibki property and Gromov's procedure.

5. The product structure theorem.

THEOREM 5.1 (*Product structure theorem*). — *Let M^m be a* TOP *manifold, C a closed subset of M and σ_0 a* CAT (= DIFF *or* PL) *structure on a neighborhood of C in M. Let Σ be a* CAT *structure on $M \times R^s$ equal $\sigma_0 \times R^s$ near $C \times R^s$. Provide that $m \geqslant 5$ and $\partial M \subset C$.*

Then M has a CAT *structure σ equal σ_0 near C. And there exists a* TOP *isotopy (as small as we please) $h_t : M_\sigma \times R^s \to (M \times R^s)_\Sigma$, $0 \leqslant t \leqslant 1$, of h_0 = identity, fixing a neighborhood of $C \times R^s$, to a* CAT *isomorphism h_1.*

It will appear presently that this result is the key to TOP handlebody theory and transversality. The idea behind such applications is to reduce TOP lemmas to their DIFF analogues.

(*) Alternatively, for our purpose, $B_{\Gamma(G)}$ can be the ordered simplicial complex having one d-simplex for each equipped bundle over the standard d-simplex that has total space in some $R^n \subset R^\infty$.

(**) The forgetful map $\varphi : B_{\Gamma(\mathrm{PL}^m)} \to B_{\mathrm{PL}(m)}$ is more delicate to define. One can make $B_{\Gamma(\mathrm{PL}^m)}$ a simplicial complex, then define φ simplex by simplex.

It seems highly desirable, therefore, to prove 5.1 as much as possible by pure geometry, without passing through a haze of formalism like that in § 4. This is done in [46]. Here is a quick sketch of proof intended to advertise [46].

First, one uses the CAT s-cobordism theorem (no surgery !) and the handle-straightening method of [44] to prove – without meeting obstructions –

THEOREM 5.2 (*Concordance implies isotopy*). — *Given* M *and* C *as in* 5.1, *consider a* CAT *structure* Γ *on* $M \times I$ *equal* $\sigma_0 \times I$ *near* $C \times I$, *and let* $\Gamma | M \times 0$ *be called* $\sigma \times 0$. (Γ is called a concordance of σ rel C).

There exists a TOP *isotopy (as small as we please)* $h_t : M_\sigma \times I \to (M \times I)_\Sigma$, $0 \leqslant t \leqslant 1$, *of* h_0 = *identity, fixing* $M \times 0$ *and a neighborhood of* $C \times I$, *to a* CAT *isomorphism* h_1.

Granting this result, the Product Structure Theorem is deduced as follows.

In view of the relative form of 5.2 we can assume $M = R^m$. Also we can assume $s = 1$ (induct on s !). Thirdly, it suffices to build a concordance Γ (= structure on $M \times R^s \times I$) from $\sigma \times R^s$ to Σ rel $C \times R^s$. For, applying 5.2 to the concordance Γ we get the wanted isotopy. What remains to be proved can be accomplished quite elegantly. Consider Figure 5-a.

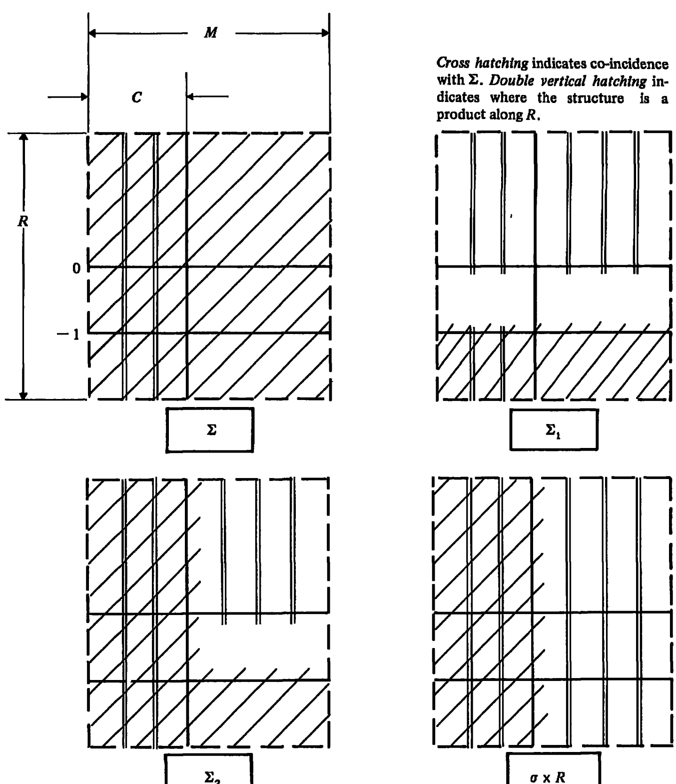


Figure 5a

We want a concordance rel $C \times R$ from Σ to $\sigma \times R$. First note it suffices to build Σ_2 with the properties indicated. Indeed Σ_2 admits standard (sliced) concordances rel $C \times R$ to $\sigma \times R$ and to Σ. The one to $\sigma \times R$ comes from sliding R over itself onto $(0, \infty)$. The region of coincidence with $\sigma \times R$ becomes total by a sort of window-blind effect. The concordance to Σ comes from sliding R over itself onto $(-\infty, -1)$. (Hint : The structure picked up from Σ_2 at the end of the slide is the same as that picked up from Σ).

It remains to construct Σ_2. Since $M \times R = R^{m+1}$, we can find a concordance (not rel $C \times R$) from Σ to the standard structure, using the STABLE homeomorphism theorem(*) [42]. Now 5.2 applied to the concordance gives Σ_1, which is still standard near $M \times [0, \infty)$. Finally an application of 5.2 to $\Sigma_1 | N \times [-1, 0]$, where N is a small neighborhood of C, yields Σ_2. The change in $\Sigma_1 | M \times 0$ (which is standard) on $N \times 0$ offered by 5.2 is extended productwise over $M \times [0, \infty)$. This completes the sketch.

It is convenient to recall here for later use one of the central results of [44]. Recall that $\mathrm{TOP}_m/\mathrm{PL}_m$ is the fiber of the forgetful map $B_{\mathrm{PL}(m)} \to B_{\mathrm{TOP}(m)}$. And TOP/PL is the fiber the similar map of stable classifying spaces $B_{\mathrm{PL}} \to B_{\mathrm{TOP}}$. Similarly one defines $\mathrm{TOP}_m/\mathrm{DIFF}_m \equiv \mathrm{TOP}_m/\mathrm{O}_m$ and TOP/DIFF $\equiv$ TOP/O.

THEOREM 5.3(**) (Structure theorem).— *TOP/PL $\simeq K(Z_2, 3)$ and*

$$\pi_k(\mathrm{TOP}_m/\mathrm{CAT}_m) = \pi_k(\mathrm{TOP}/\mathrm{CAT})$$

for $k < m$ and $m \geqslant 5$. Here CAT = PL *or* DIFF.

Since $\pi_k(\mathrm{O}_m) = \pi_k(\mathrm{O})$ for $k < m$, we deduce that $\pi_k(\mathrm{TOP}, \mathrm{TOP}_m) = 0$ for $k < m > 5$, a weak stability for TOP_m.

Consider the second statement of 5.3 first. Theorem 4.4 says that

$$\pi_k(\mathrm{TOP}_m/\mathrm{CAT}_m) = \pi_0(\mathrm{CAT}^m(S^k \times R^{m-k})) \equiv S_k^m$$

for $k < m \geqslant 5$. Secondly, 5.1 implies $S_k^m = S_k^{m+1} = S_k^{m+2} = \cdots, m \geqslant k$. Hence $\pi_k(\mathrm{TOP}_m/\mathrm{CAT}_m) = \pi_k(\mathrm{TOP}/\mathrm{CAT})$.

We now know that $\pi_k(\mathrm{TOP}/\mathrm{PL})$ is the set of isotopy classes of PL structures on S^k if $k \geqslant 5$. The latter is zero by the PL Poincaré theorem of Smale [84], combined with the stable homeomorphism theorem [42] and the Alexander isotopy. Similarly one gets $\pi_k(\mathrm{TOP}/\mathrm{DIFF}) = \Theta_k$ for $k \geqslant 5$. Recall $\Theta_5 = \Theta_6 = 0$ [41].

The equality $\pi_k(\mathrm{TOP}/\mathrm{PL}) = \pi_k(\mathrm{TOP}/\mathrm{DIFF}) = \pi_k(K(Z_2, 3))$ for $k \leqslant 5$ can be deduced with ease from local contractibility of homeomorphism groups and the surgical classification [35] [95], by $H^3(T^5; Z_2)$, of homotopy 5-tori. See [43] [46 A] for details.

Combining the above with 4.4 one has a result of [44].

(*) Without this we get only a theorem about compatible CAT structures on STABLE manifolds (of Brown and Gluck [8]).

(**) For a sharper result see [63] [45], and references therein.

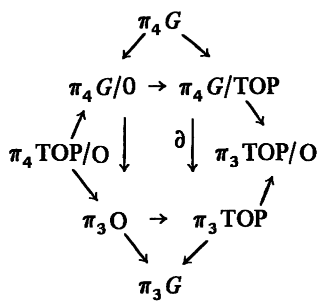
CLASSIFICATION THEOREM 5.4. — *For $m \geqslant 5$ a* TOP *manifold M^m (without boundary) admits a* PL *manifold structure iff an obstruction $\Delta(M)$ in $H^4(M; Z_2)$ vanishes. When a* PL *structure Σ on M is given, others are classified (up to concordance or isotopy) by elements of $H^3(M\,; Z_2)$.*

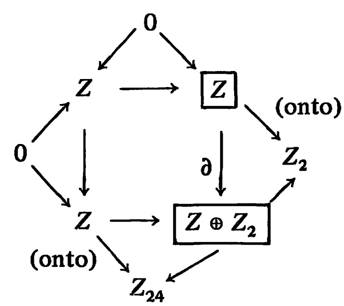
Complement. — Since π_k(TOP/DIFF) = π_k(TOP/PL) for $k < 7$ (see above calculation), the same holds for DIFF in low dimensions.

Finally we have a look at low dimensional homotopy groups involving

$$G = \lim \{G_n \,|\, n \geqslant 0\}$$

where G_n is the space of degree± 1 maps $S^{n-1} \to S^{n-1}$. Recall that $\pi_n G = \pi_{n+k} S^k$, k large. G/CAT is the fiber of a forgetful map $B_{\text{CAT}} \to B_G$, where B_G is a stable classifying space for spherical fibrations (see [15], [29]).





The left hand commutative diagram of natural maps is determined on the right. Only π_3TOP is unknown(*). So the exactness properties evident on the left leave no choice. Also ∂ must map a generator of $\pi_4 G$/TOP $= Z$ to $(12, 1)$ in $Z \oplus Z/2Z = \pi_3$TOP.

The calculation with PL in place of O is the same (and follows since π_i(PL/O) $=$ $\Gamma_i = 0$ for $i \leqslant 6$).

6. Simple homotopy theory [44] [46 A].

The main point is that every compact TOP manifold M (with boundary ∂M) has a preferred simple homotopy type and that two plausible ways to define it are equivalent. Specifically, a handle decomposition of M or a combinatorial triangulation of a normal disc-bundle to M give the same simple type.

The second definition is always available. Simply embed M in R^n, n large, with normal closed disc-bundle E [31]. Theorem 5.1 then provides a small homeomorphism of R^n so that $h(\partial E)$, and hence $h(E)$, is a PL submanifold.

(*) That $\pi_4 G$/TOP is Z (not $Z \oplus Z_2$) is best proved by keeping track of some normal invariants in disproving the Hauptvermutung, see [46A]. Alternatively, see 13.4 below.

Working with either of these definitions, one can see that the preferred simple type of M and that of the boundary ∂M make of $(M, \partial M)$ a finite Poincaré duality space in the sense of Wall [95], a fact vital for TOP surgery.

The Product structure theorem 5.1 makes quite unnecessary the bundle theoretic nonsense used in [44] (cf. [63]) to establish preferred simple types.

7. **Handlebody theory** (statements in [44 C] [45], proofs in [46 A]).

7.1. The main result is that handle decompositions exist in dimension $\geqslant 6$. Here is the idea of proof for a closed manifold M^m, $m \geqslant 6$. Cover M^m by finitely many compacta $A_1, \ldots, A_k$, each A_i contained in a co-ordinate chart $U_i \approx R^m$. Suppose for an inductive construction that we have built a handlebody $H \subset M$ containing $A_1 \cup \ldots \cup A_{i+1}$, $i \geqslant 0$. The Product Structure Theorem shows that $H \cap U_i$ can be a PL (or DIFF) m-submanifold of U_i after we adjust the PL (or DIFF) structure on U_i. Then we can successively add finitely many handles onto H in U_i to get a handlebody H' containing $A_1 \cup \ldots \cup A_i$. After k steps we have a handle decomposition of M.

A TOP Morse function on M^m implies a TOP handle decomposition (the converse is trivial) ; to see this one uses the TOP isotopy extension theorem to prove that a TOP Morse function without critical points is a bundle projection. (See [12] [82, 6.14] for proof in detail).

Topological handlebody theory as conceived of by Smale now works on the model of the PL or DIFF theory (either). For the sake of those familiar with either, I describe simple ways of obtaining transversality and separation (by Whitney's method) of attaching spheres and dual spheres in a level surface.

LEMMA 7.2. (Transversality). — *Let $g : R^m \to R^m, m \geqslant 5$, be a* STABLE *homeomorphism. In R^m, consider $R^p \times 0$ and $0 \times R^q, p + q = m$, with 'ideal' transverse intersection at the origin. There exists an ϵ-isotopy of g to $h : R^m \to R^m$ such that $h(R^p \times 0)$ is transverse to $0 \times R^q$ is the following strong sense. Near each point $x \in h^{-1}(0 \times R^q) \cap R^p \times 0$, h differs from a translation by at most a homeomorphism of R^m respecting both $R^p \times 0$ and $0 \times R^q$.*

Furthermore, if C is a given closed subset of R^m and g satisfies the strong transversality condition on h above for points x of R^m near C, then h can equal g near C.

Proof of 7.2 — For the first statement $\epsilon/2$ isotop g to diffeomorphism g' using Ed Connell's theorem [14] (or the Concordance-implies-epsilon-isotopy theorem 5.2), then $\epsilon/2$ isotop g' using standard DIFF techniques to a homeomorphism h' which will serve as h if $C = \emptyset$.

The further statement is deduced from the first using the flexibility of homeomorphisms. Find a closed neighborhood C' of C near which g is still transverse such that the frontier $\dot{C}'$ misses $g^{-1}(0 \times R^q) \cap (R^p \times 0)$ — which near C is a discrete collection of points. Next, find a closed neighborhood D of $\dot{C}'$ also missing $g^{-1}(0 \times R^q) \cap (R^p \times 0)$, and $\delta : R^m \to (0, \infty)$ so that $d(gx, 0 \times R^q) < \delta(x)$ for x in $D \cap (R^p \times 0)$. If $\epsilon : R^m \to (0, \infty)$ is sufficiently small, and h' in the first paragraph is built for ϵ, Cernavskii's local contractibility theorem [11] (also [17] and [82 , 6.3]) says that there exists a homeomorphism h equal g on C' and equal h' outside $C' \cup D$ so that $d(h', g) < \delta$. This is the wanted h.

7.3. THE WHITNEY LEMMA.

The TOP case of the Whitney process for eliminating pairs of isolated transverse intersection points (say of M^p and N^q) can be reduced to the PL case [99] [37]. The Whitney 2-disc is easily embedded and a neighborhood of it is a copy of R^m, $m = p + q$. We can arrange that either manifold, say M^p, is PL in R^m, and(*) N^q is PL near M^q in R^m. Since $5 \leqslant m = p + q$, we can assume $q \leqslant m/2$; so N^q can now be pushed to be PL in R^m by a method of T. Homma, or by one of R.T. Miller [54 A], or again by the method of [44], applied pairwise [44 A] (details in [73]). Now apply the PL Whitney lemma [37]. On can similarly reduce to the original DIFF Whitnev lemma [99].

7.4. CONCLUSION.

The *s*-cobordism theorem [37] [39], the boundary theorem of [76], and the splitting principle of Farrell and Hsiang [20] can now be proved in TOP with the usual dimension restrictions.

8. **Transversality** (statements in [44 C] [45], proofs in [46 A]).

If $f : M^m \to R^n$ is a continuous map of a TOP manifold without boundary to R^n and $m - n > 5$, we can homotop f to be transverse to the origin $0 \in R^n$. Here is the idea. One works from chart to chart in M to spread the transversality, much as in building handlebodies. In each chart one uses the product structure theorem 5.1 to prepare for an application of the *relative* DIFF transversality theorem of Thom.

Looking more closely one gets a *relative* transversality theorem for maps $f : M^m \to E(\xi^n)$ with target *any* TOP R^n-microbundle ξ^n over *any* space. It is parallel to Williamson's PL theorem [100], but is proved only for $m \neq 4 \neq m - n$. It is indispensible for surgery and cobordism theory.

9. **Surgery.**

Surgery of compact manifolds of dimension $\geqslant 5$ as formulated by Wall [95] can be carried out for TOP manifolds using the tools of TOP handlebody theory. The chief technical problem is to make the self-intersections of a framed TOP immersion $f : S^k \times R^k \to M^{2k}$ of S^k, $k \geqslant 3$, transverse (use Lemma 7.2 repeatedly), and then apply the Whitney lemma to find a regular homotopy of f to an embedding when Wall's self-intersection coefficient is zero.

In the simply connected case one can adapt ideas of Browder and Hirsch [4].

Of course TOP surgery constantly makes use of TOP transversality, TOP simple homotopy type and the TOP *s*-cobordism theorem.

(*) Use of the strong transversality of 7.2 makes this trivial in practice.

10. Cobordism theory : generalities.

Let Ω_n^{TOP} [respectively Ω_n^{STOP}] be the group of [oriented] cobordism classes of [oriented] closed n-dimensional TOP manifolds. Thom's analysis yields a homomorphism

$$\theta_n : \Omega_n^{\text{TOP}} \to \pi_n(\text{MTOP}) = \lim_{\overrightarrow{k}} \pi_{n+k}(\text{MTOP}(k)) \quad .$$

Here MTOP(k) is the Thom space of the universal TOP R^k-bundle γ_{TOP}^k over $B_{\text{TOP}(k)}$ – obtained, for example, by compactifying each fiber with a point (cf. [49]) and crushing these points to one. The Pontrjagin Thom definition of θ_n uses a stable relative existence theorem for normal bundles in euclidean space – say as provided by Hirsch [30] and the Kister-Mazur Theorem [49].

Similarly one gets Thom maps

$$\theta_n : \Omega_n^{\text{STOP}} \to \pi_n(\text{MSTOP}), \quad \text{and} \quad \theta_n : \Omega_n^{\text{SPINTOP}} \to \pi_n(\text{MSPINTOP}) ,$$

and more produced by the usual recipe for cobordism of manifolds with a given, special, stable structure on the normal bundle [86, Chap. II].

THEOREM 10.1. — *In each case above the Thom map $\theta_n : \Omega_n \to \pi_n(M)$ is surjective for $n \neq 4$, and injective for $n \neq 3$.*

This follows immediately from the transversality theorem.

PROPOSITION 10.2. — $B_{\text{SO}} \otimes Q \simeq B_{\text{STOP}} \otimes Q$, *where Q denotes the rational numbers.*

Proof. — $\pi_i(\text{STOP/SO}) = \pi_i(\text{TOP/O})$ is finite for all i by [40] [44] cf. § 5, STOP/SO being fiber of $B_{\text{SO}} \to B_{\text{STOP}}$. (See § 15 or [90] for definition of $\otimes Q$).

PROPOSITION 10.3. — $\pi_*\text{MSO} \otimes Q \cong \pi_*\text{MSTOP} \otimes Q$.

Proof. — From 10.2 and the Thom isomorphism we have

$$H_*(\text{MSO} ; Q) \cong H_*(\text{MSTOP} ; Q) \quad .$$

Now use the Hurewicz isomorphism (Serre's from [75]).

PROPOSITION 10.4. — $\Omega_*^{\text{SO}} \otimes Q \cong \Omega_*^{\text{STOP}} \otimes Q$ *each being therefore the polynomial algebra freely generated by* $\text{CP}_{2n}, n \geqslant 1$.

Proof of 10.4. — The uncertainty about dimensions 3 and 4 in 10.1 cannot prevent this following from 10.2. Indeed, $\Omega_3^{\text{STOP}} \to \pi_3\text{MSTOP}$ is injective because every TOP 3-manifold is smoothable (by Moise et al., cf. [80, § 5]). And

$$\Omega_4^{\text{STOP}} \to \pi_4\text{MSTOP}$$

is rationally onto because $\Omega_4^{\text{SO}} \to \pi_4\text{MSTOP}$ is rationally onto.

Since $\pi_i(\text{STOP/SPL}) = \pi_i(\text{TOP/PL})$ is Z_2 for $i = 3$ and zero for $i \neq 3$ the above three propositions can be repeated with SPL in place of SO and dyadic rationals $Z[\frac{1}{2}]$ in place of Q. The third becomes :

PROPOSITION 10.5. — $\Omega_*^{SPL} \otimes Z[\frac{1}{2}] \cong \Omega_*^{STOP} \otimes Z[\frac{1}{2}]$.

Next we recall

PROPOSITION 10.6. — (S.P. Novikov). $\Omega_*^{SO} \to \Omega_*^{STOP}$ *is injective.*

This is so because every element of Ω_*^{SO} is detected by its Stiefel-Whitney numbers (homotopy invariants) and its Pontrjagin numbers (which are topological invariants by 10.2).

In view of 10.2 we have canonical Pontrjagin characteristic classes p_k in

$$H^{4k}(B_{STOP}\,;Q) = H^{4k}(B_{SO}\,;Q)$$

and the related Hirzebruch classes $L_k = L_k(p_1, \dots, p_k) \in H^{4k}$. Hirzebruch showed that $L_k : \Omega_{4k}^{SO} \otimes Q \to Q$ sending a $4k$-manifold M^{4k} to its characteristic number $L_k(M^{4k}) = L_k(\tau(M^{4k}))\,[M^{4k}] \in Q$ is the signature (index) homomorphism. From 10.2 and 10.4, it follows that the same holds for STOP in place of SO. Hence we have

PROPOSITION 10.7. — *For any closed oriented* TOP $4k$*-manifold* M^{4k} *the signature* $\sigma(M^{4k})$ *of the rational cohomology cup product pairing* $H^{2k} \otimes H^{2k} \to H^{4k} = Q$ *is given by* $\sigma(M^{4k}) = L_k(\tau M^{4k})\,[M^{4k}] \in Z$.

11. Oriented cobordism.

The first few cobordism groups are fun to compute geometrically — by elementary surgical methods, and the next few pages are devoted to this.

THEOREM 11.1. — $\Omega_n^{STOP} \simeq \Omega_n^{SO} \oplus R_n$ *for* $n \leqslant 7$, *and we have* $R_n = 0$ *for* $n \leqslant 3$, $R_4 \leqslant Z_2, R_5 = 0$, $R_6 = Z_2$, $R_7 \leqslant Z_2$.

Proof of 11.1. — For $n = 1, 2, 3$, $\Omega_n^{STOP} = \Omega_n^{SO} = 0$ is seen by smoothing.

For $n = 4$, first observe that $Z = \Omega_4^{SO} \to \Omega_4^{STOP}$ maps Z to a summand because the signature of a generator CP_2 is 1 which is indivisible. Next consider the Z_2 characteristic number of the first stable obstruction $\Delta \in H^4(B_{STOP}\,;Z_2)$ to smoothing. It gives a homomorphism $\Omega_4^{STOP} \to Z_4$ killing Ω_4^{SO}. If

$$\Delta(M^4) \equiv \Delta(\tau(M^4))\,[M^4] = 0 \quad ,$$

then, by 5.3, $M^4 \times R$ has a DIFF structure Σ. Push the projection $(M^4 \times R)_\Sigma \to R$ to be transversal over $0 \in R$ at a DIFF submanifold M' and behold a TOP oriented cobordism M to M'. Thus $R_4 \leqslant 0$.

For $n \geqslant 5$ note that any oriented TOP manifold M^n is oriented cobordant to a simply connected one M' by a finite sequence of 0 and 1-dimensional surgeries. But, for $n = 5, H^4(M'\,;Z_2) \cong H_1(M'\,;Z_2) = 0$ so M' is smoothable. Hence $R_5 = 0$.

For $n = 6$ we prove

PROPOSITION 11.2. — *The characteristic number* $\Delta w_2 : \Omega_6^{STOP} \to Z_2$ *is an isomorphism.*

Proof. — It is clearly non-zero on any non-smoothable manifold $M^6 \simeq CP_3$, since $w_2(M^6) = w_2(CP_3) \neq 0$, and we will show that such a M^6 exists in 15.7 below.

Since $\Omega_6^{SO} = 0$ it remains to prove that Δw_2 is injective. Suppose $\Delta w_2(M^6) = 0$ for oriented M^6. As we have observed, we can assume M is simply connected. Consider the Poincaré dual $D\Delta$ of $\Delta = \Delta(\tau M)$ in

$$H_2(M^6\,;Z_2) = H_2(M^6\,;Z) \otimes Z_2 = \pi_2(M^6) \otimes Z_2$$

and observe that it can be represented by a locally flatly embedded 2-sphere $S \subset M^6$. (Hints : Use [24], or find an immersion of $S^2 \times R^4$ [52] and use the idea of Lemma 7.1).

Note that $\Delta|(M-S) = \Delta(M-S)$ is zero because $\Delta[x] = x \cdot D\Delta$ (the Z_2 intersection number) for all $x \in H_2(M\,;Z_2)$. Thus $M-S$ is smoothable.

A neighborhood of S is smoothed, there being no obstruction to this ; and S is made a DIFF submanifold of it. Let N be an open DIFF tubular neighborhood of S. Now $0 = \Delta w_2[M] = w_2[D\Delta] = w_2[S]$ means that $w_2(\tau M)|S$ is zero. Hence $N = S^2 \times R^4$. Killing S by surgery we produce M', oriented cobordant to M, so that, writing $M_0 = M - S^2 \times \overset{o}{B}{}^4$, we have $M' = M_0 + B^3 \times S^3$ (union with boundaries identified). Now M' is smoothable since M_0 is and there is no further obstruction. As $\Omega_6^{SO} = 0$, Proposition 11.2, is established.

PROPOSITION 11.3. — *The characteristic number* $(\beta\Delta)w_2 : \Omega_7^{STOP} \to Z_2$ *is injective, where* $\beta = Sq^1$.

Proof of 11.3. — We show the $(\beta\Delta)w_2[M] = 0$ implies M^7 is a boundary. Just as for 11.2, we can assume M is simply connected. Then $\pi_2 M = H_2(M\,;Z)$ and we can kill any element of the kernel of $w_2 : H_2(M\,;Z) \to Z_2$, by surgery on 2-spheres in M. Killing the entire kernel we arrange that w_2 is injective.

We have $0 = (\beta\Delta)w_2[M] = w_2[D\beta\Delta]$. So the Poincaré dual $D\beta\Delta$ of $\beta\Delta$ is zero as w_2 is injective.

Now $\beta\Delta = 0$ means Δ is reduced integral ; indeed β is the Bockstein

$$\delta : H^4(M\,;Z_2) \to H^5(M\,,Z)$$

followed by reduction mod 2. But

$$H^5(M\,;Z) \cong H^5(M\,;Z_2), \text{ since } H_2(M\,;Z) \cong H_2(M\,;Z_2)$$

(both isomorphisms by reduction). Thus $\beta\Delta = 0$ implies $\delta\Delta = 0$, which means Δ is reduced integral. Hence $D\Delta$ is reduced integral. Since the Hurewicz map $\pi_3 M \to H_3(M\,;Z)$ is onto, $D\Delta$ is represented by an embedded 3-sphere S. Following the argument for dimension 6 and recalling $\pi_2 O = 0$, we can do surgery on S to obtain a smoothable manifold.

12. Unoriented cobordism (*).

Recalling calculations of Ω_i^{O} and Ω_i^{SO} from Thom [91] we get the following table

i	4	5	6	7
Ω_i^{SO}	Z	Z_2	0	0
$\Omega_i^{\mathrm{STOP}}/\Omega_i^{\mathrm{SO}} = R_i$	$\leqslant Z_2$	0	Z_2	$R_7 \leqslant Z_2$
Ω_i^{O}	$Z_2 \oplus Z_2$	Z_2	$Z_2 \oplus Z_2 \oplus Z_2$	Z_2
$\Omega_i^{\mathrm{TOP}}/\Omega_i^{\mathrm{O}}$	$\leqslant R_4 \leqslant Z_2$	Z_2	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2 \oplus R_7$

The only non zero-entry for $0 < i < 4$ would be $\Omega_2^{\mathrm{O}} = Z_2$.

To deduce the last row from the first three, use the related long exact sequences (from Dold [16])

$$(12.1)\qquad \begin{array}{ccccccc} \ldots \to \Omega_i^{\mathrm{SO}} & \to \Omega_i^{\mathrm{O}} & \overset{(\partial,d)}{\to} & \Omega_{i-1}^{\mathrm{SO}} \oplus \Omega_{i-2}^{\mathrm{O}} & \overset{j}{\to} \Omega_{i-1}^{\mathrm{SO}} & \to \ldots \\ \downarrow & \downarrow & & \downarrow & \downarrow & \\ \ldots \to \Omega_i^{\mathrm{STOP}} & \to \Omega_i^{\mathrm{TOP}} & \overset{(\partial,d)}{\to} & \Omega_{i-1}^{\mathrm{STOP}} \oplus \Omega_{i-2}^{\mathrm{TOP}} & \overset{j}{\to} \Omega_{i-1}^{\mathrm{STOP}} & \to \ldots \end{array} .$$

If transversality fails $\pi_4(M\ ?)$ should replace $\Omega_4^?$ in the TOP sequence. (See [93, §6], [3] for explanation).

All the maps are forgetful maps except those marked j and (∂, d). The map j kills the second summand, and is multiplication by 2 on the first summand (which is also the target of j).

At the level of representatives, ∂ maps M^i to a submanifold M^{i-1} dual to $w_1(M^i)$, and d maps M^i to $M^{i-2} \subset M^{i-1}$ dual to $w_1(M^i)|M^{i-1}$.

The map d is onto with left inverse φ defined by associating to M^{i-2} the RP^2 bundle associated to $\lambda \oplus \epsilon^2$ over M^{i-2}, where λ is the line bundle with

$$w_1(\lambda) = w_1(M^{i-2})$$

and ϵ^2 is trivial.

The diagram (12.1) gives us the following generators for $S_i \equiv \Omega_i^{\mathrm{TOP}}/\Omega_i^{\mathrm{O}}$.

$S_4 \leqslant Z_2$: Any M^4 with $\Delta(M) \neq 0$ - - - if it exists.

$S_5 = Z_2$: Any M^5 detected by Δw_1.

$S_6 = Z_2 \oplus Z_2$: Non-smoothable $M_1^6 \simeq \mathrm{CP}_3$, detected by Δw_2 ;

$$M_2^6 \simeq \mathrm{RP}_2 \times (Q \cup \infty)$$

(*) Added in proof : A complete calculation of Ω_*^{TOP} has just been announced by Brumfiel, Madsen and Milgram (Bull. AMS to appear).

detected by Δw_1^2. $M_2^6 \times R$ can be $RP_2 \times \widetilde{X}^5$, where $\widetilde{X}^5$ is the universal covering of a manifold constructed in [80, § 5].

$S_7 = 2Z_2 \oplus R_7 : M_1^7 = \varphi M_1^5$ detected by Δw_1^3; M_2^7 is detected by $\Delta w_2 w_1$, $M_2^7 = T(\rho)$, the mapping torus of an orientation reversing homeomorphism of $M_1^6 \simeq CP_3$ homotopic to complex conjugation in CP_3. Such a ρ exists because conjugation doesn't shift the normal invariant for $M_1^6 \simeq CP_3$ in $[CP_3, G/TOP]$. Finally M_3^7 a generator of R_7 detected by $(\beta\Delta) w_2$ (if it exists).

13. Spin cobordism.

The stable classifying space $B_{SPINTOP}$ is the fiber of $w_2 : B_{STOP} \to K(Z_2, 2)$. So $\pi_i B_{SPINTOP}$ is 0 for $i \leqslant 3$ and equals $\pi_i B_{TOP}$ for $i \geqslant 3$. Topological spin cobordism is defined like smooth spin cobordism Ω_*^{SPIN} but using TOP manifolds. Thus $\Omega_*^{SPINTOP}$ is the cobordism ring for compact TOP manifolds M equipped with a spin structure – i.e. a lifting to $B_{SPINTOP}$ of a classifying map $M \to B_{TOP}$ for $\tau(M)$ – or equivalently for the normal bundle $\nu(M)$.

THEOREM 13.1. — *For* $n \leqslant 7$, $\Omega_n^{SPINTOP}$ *is isomorphic to* Ω_n^{SPIN}, *which for* $n = 0, 1, \ldots, 8$ *has the values* $Z, Z_2, Z_2, 0, Z, 0, 0, 0, Z \oplus Z$ [59] [86]. *The image of the forgetful map* $Z = \Omega_4^{SPIN} \to \Omega_4^{SPINTOP} = Z$ *is the kernel of the stable triangulation obstruction* $\Delta : \Omega_4^{SPINTOP} \to Z_2$.

The question whether Δ is zero or not is the question whether or not Rohlin's congruence for signature $\sigma(M^4) \equiv 0 \bmod 16$ holds for all topological spin manifolds M^4. Indeed $\sigma(M^4) \equiv 8\Delta(M^4) \bmod 16$, $\Delta(M^4)$ being 0 or 1.

Proof of 13.1. — The isomorphism $\Omega_n^{SPINTOP} \cong \Omega_n^{SPIN}$ for $n \leqslant 3$ comes from smoothing.

Postponing dimension 4 to the last, we next show $\Omega_n^{SPINTOP}/\Omega_n^{SPIN} = 0$ for $n = 5, 6, 7$. Note first that a smoothing and a topological spin structure determine a unique smooth spin structure. The argument of § 11 shows that the only obstruction to performing oriented surgery on M^n to obtain a smooth manifold is a characteristic number, viz. 0, Δw_2, $(\beta\Delta) w_2$ for $n = 5, 6, 7$ respectively. But $w_2(M^n) = 0$ for any spin topological manifold. It remains to show that the surgeries can be performed so that each one, say from M to M', thought of as an elementary cobordism $(W^{n+1}; M^n, M'^n)$, can be given a topological spin structure extending that of M. The only obstruction to this occurs in $H^2(W, M; Z_2)$, which is zero except if the surgery is on a 1-sphere. And in that case we can obviously find a possibly different surgery on it (by spinning the normal bundle !) for which the obstruction is zero.

Finally we deal with dimension 4. If $\Delta(M^4) = 0$ for any spin 4-manifold, then M^4 is spin cobordant to a smooth spin manifold by the proof of 11.1. Next suppose M^4 is a topological spin manifold such that the characteristic number $\Delta(M^4)$ is not zero. If we can show that $\sigma(M^4) \equiv 8\Delta(M^4) \bmod 16$ the rest of 13.1 will follow, including the fact that $\Omega_4^{SPINTOP} \cong Z$ rather than $Z \oplus Z_2$. We can assume M^4 connected (by surgery).

LEMMA 13.2. — *For any closed connected topological spin 4-manifold M^4, there exists a (stable)* TOP *bundle ξ over S^4 and a degree* 1 *map $M \to S^4$ covered by a* TOP *bundle map $\nu(M) \to \xi$. This ξ is necessarily fiber homotopically trivial. A similar result (similarly proved) holds for smooth spin manifolds.*

Proof of 13.2. — Since any map $M_0 \equiv M -$ (point) $\to B_{\text{SPINTOP}}$ is contractible, $\nu(M)|M_0$ is trivial, and so $\nu(M) \to \xi$ exists as claimed. Now ξ is fiberhomotopically trivial since it is — like $\nu(M)$ — reducible, hence a Spivak normal bundle for S^4. (Cf. proof in [40].)

LEMMA 13.3. — *A fiber homotopically trivialized* TOP *bundle ξ over the 4-sphere is (stably) a vector bundle iff* $\frac{1}{3} p_1(\xi)\,[S^4] \equiv 0$ mod. 16.

Proof of 13.3. — Consider the homomorphism $\frac{1}{3} p_1 : \pi_4 G/\text{TOP} \to Z$ given by associating the integer $\frac{1}{3} p_1(\xi)\,[S^4]$ to a such a bundle ξ over S^4. The composed map $\frac{1}{3} p_1 : \pi_4 G/0 \to Z$ sends a generator η to $\pm$ $16 \in Z$. Indeed, by Lemma 13.2 , DIFF transversality, and the Hirzebruch index theorem, $\frac{1}{3} p_1(\eta)$ is the least index of a closed smooth spin 4-manifold, which is $\pm$ 16 by Rohlin's theorem [40]. The lemma follows if we grant that $\pi_4 G/\text{TOP} = Z$ (not $Z \oplus Z_2$).

Now we complete 13.1. In $Z/16Z$ we have

$$\sigma(M^4) = \frac{1}{3} p_1(\tau M)\,[M] = \frac{1}{3} p_1(\xi)\,[S^4] = 8\Delta(\xi)\,[S^4] = 8\Delta(\tau(M))\,[M^4] = 8\Delta(M^4)$$

the third equality coming from the last lemma.

π_4G/TOP $= Z$ is used in 13.3 and in all following sections. So we prove it as

PROPOSITION 13.4. — *The forgetful map* π_4 G/O $\to \pi_4$ G/TOP *is* $Z \overset{\times 2}{\to} Z$.

Proof of 13.4. — (cf. naïve proof in [46A]). Since the cokernel is π_3TOP/O $=$ π_3TOP/PL $= Z_2$, it suffices to show that $\frac{1}{3} p_1$: π_4G/TOP $\to Z$ in the proof of 13.3 sends some element ζ to $\pm$ $8 \in Z$.

Such a ζ is constructed as follows. In §2, we constructed a closed TOP manifold X^{4+n} with $w_1(X) = w_2(X) = 0$ and a homotopy equivalence $f : N^4 \times T^n \simeq X^{4+n}$, where N^4 is a certain homology manifold (with one singularity) having $\sigma(N^4) = \pm$ 8. Imitating the proof of 13.2 with N^4 and $\nu' = f^*\nu(X^{4+n})|\,N^4$ in place of M^4 and $\nu(M^4)$ we construct ξ over S^4 and $\nu' \to \xi$ over the degree 1 map $N^4 \to S^4$. This ξ is fiber homotopically trivial because $\nu(X), f^*\nu(X)$ and ν' are Spivak normal bundles. Let ξ represent ζ in π_4G/TOP.

It remains to show $\frac{1}{3} p_1(\zeta) = \pm$ 8. First we reduce n to 1 in $f : N^4 \times T^n \simeq X^{4+n}$ by using repeatedly a splitting principle valid in dimension ≥ 6. (eg. use the TOP version of [76], or just the PL or DIFF version as in the latter part of 5.4 (a) in [80]). Consider the infinite cyclic covering

$$\bar{f} : N^4 \times R \simeq \bar{X}^5 \quad \text{of} \quad f : N^4 \times T^1 \simeq X^5 .$$

Splitting as above, we find that $CP_2 \times \bar{X}^5 \approx Y^8 \times R$ for some 8-manifold Y^8. Thus using the index theorem 10.7, and the multiplicativity of index and L-classes we have

$$\pm 8 = \sigma(N^4) = \sigma(CP_2 \times N) = \sigma(Y^8) = L_2(Y^8) = L_1(CP_2)\, L_1(\bar{X}^5)\, [CP_2 \times N] =$$

$$L_1(\bar{X}^5)\,[N^4] = - L_1(\nu')\,[N^4] = -\frac{1}{3} p_1(\nu')\,[N^4] = -\frac{1}{3} p_1(\xi)\,[S^4] = -\frac{1}{3} p_1(\zeta).$$

(We have suppressed some natural (co)homology isomorphisms).

14. The periodicity of Casson and Sullivan.

A geometric construction of a "periodicity" map

$$\pi : G/PL \to \Omega^4 G/PL$$

was discovered by A.J. Casson in early 1967 (unpublished)(*).

He showed that the fiber of π is $K(Z_2, 3)$, and used this fact with the ideas of Novikov's proof of topological invariance of the rational Pontrjagin classes to establish the Hauptvermutung for closed simply connected PL manifolds $M^m, m \geqslant 5$, with $H^3(M^m, Z_2) = 0$. (Sullivan had a slightly stronger result [88]).

Now precisely the same construction produces a periodicity map π' in a homotopy commutative square

(14.1)
$$\begin{array}{ccc} G/PL & \xrightarrow{\pi} & \Omega^4 G/PL \\ \varphi \downarrow & & \downarrow \Omega^4\varphi \\ G/TOP & \xrightarrow{\pi'} & \Omega^4 G/TOP \end{array}$$

The construction uses TOP versions of simply connected surgery and transversality. Recalling that the fiber of φ is $K(Z_2, 3)$ we see that $\Omega^4\varphi$ is a homotopy equivalence. Hence π' must be a homotopy equivalence. Thus $(\pi')^{-1} \circ (\Omega^4\varphi)$ gives a homotopy identification of π to φ ; and an identification of the fiber of π to the fiber TOP/PL of φ. Thus TOP/PL had been found (but not identified) in 1967 !

The perfect periodicity $\pi' : G/TOP \simeq \Omega^4(G/TOP)$ is surely an attractive feature of TOP. It suggests that topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing.

(*) Essentially the same construction was developed by Sullivan and Rourke later in 1967-68, see [72]. The "periodicity" π is implicit in Sullivan's analysis of G/PL as a fiber produc of $(G/PL)_{(2)}$ and $B_O \otimes Z[\frac{1}{2}]$ over $B_O \otimes Q$, [88] [89].

15. Hauptvermutung and triangulation for normal invariants ; Sullivan's thesis(*).

Since $\mathrm{TOP/PL} \xrightarrow{j} \mathrm{G/PL} \xrightarrow{\varphi} \mathrm{G/TOP} \xrightarrow{\Delta} K(Z_2, 4)$ is a fibration sequence of H-spaces see 15.5 we have an exact sequence for any complex X

$$H^3(X ; Z_2) = [X , \mathrm{TOP/PL}] \to [X , \mathrm{G/PL}] \xrightarrow{\varphi_*} [X , \mathrm{G/TOP}] \xrightarrow{\Delta_*} H^4(X ; Z_2)$$

Examining the kernel and cokernel of φ using Sullivan's analysis of $\mathrm{G/PL})_{(2)}$(**) we will obtain

THEOREM 15.1. — *For any countable finite dimensional complex X there is an exact sequence of abelian groups* :

$$H^3(X ; Z_2)/\mathrm{Image}\, H^3(X ; Z) \xrightarrow{j_*} [X , \mathrm{G/PL}] \xrightarrow{\varphi_*} [X , \mathrm{G/TOP}] \xrightarrow{\Delta_*} \{\mathrm{Image}\,(H^2(X ; Z) + \mathrm{Sq}^2 H^2(X ; Z_2))\}$$

The right hand member is a subgroup of $H^4(X ; Z_2)$, and j^* comes from

$$K(Z_2, 3) \simeq \mathrm{TOP/PL} \xrightarrow{j} \mathrm{G/PL}$$

In 1966-67, Sullivan showed that φ_* is injective provided that the left hand group vanishes. Geometrically interpreted, this implies that a homeomorphism $h : M' \to M$ of closed simply connected PL manifolds of dimension $\geqslant 5$ is homotopic to a PL homeomorphism if $H^3(M ; Z_2)/\mathrm{Image}\, H^3(M ; Z) = 0$, or equivalently if $H^4(M ; Z)$ has no 2-torsion [88]. Here $[M , \mathrm{G/PL}]$ is geometrically interpreted as a group of *normal invariants*, represented by suitably equipped degree 1 maps $f : M' \to M$ of PL manifolds to M, cf. [95]. The relevant theorem of Sullivan is :

(15.2) *The Postnikov K-invariants of* G/PL, *except for the first, are all odd ; hence*

$$(\mathrm{G/PL})_{(2)} = \{K(Z_2 ; 2) \times_{\delta \mathrm{Sq}^2} K(Z_{(2)}, 4)\} \times K(Z_2, 6) \times K(Z_{(2)}, 8) \times K(Z_2, 10) \times K(Z_{(2)}, 12) \times \dots ,$$

where $Z_{(2)} = Z\left[\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right]$ *is Z with $\frac{1}{p}$ for an odd primes p adjoined.* This is one of the chief results of Sullivan's thesis 1966 [87]. For expositions of it see [72] [13] [74] [89].

(*) Section 15 (indeed §§10-16) discusses corollaries of $\pi_3(\mathrm{TOP/PL}) = Z_2$ collected in spring 1969. For further information along these lines, the reader should see work of Hollingsworth and Morgan (1970) and S. Morita (1971) (added in proof).

(**) The localisation at 2, $A_{(2)} = A \otimes Z_{(2)}$ of a space A will occur below, only for countable H-spaces A such that, for countable finite dimensional complexes X, $[X , A]$ is an abelian group (usually a group of some sort of stable bundles under Whitney sum). Thus E.H. Brown's representation theorem offers a space $A_{(2)}$ and map $A \to A_{(2)}$ so that $[X , A] \otimes Z_{(2)} = [X , A_{(2)}]$. For a more comprehensive treatment of localisation see [89]. The space $A \otimes Q$ is defined similarly.

Sullivan's argument adapts to prove

(15.3) *The Postnikov K-invariants of* G/TOP *are all odd ; Hence*

$$(\mathrm{G/TOP})_{(2)} = K(Z_2, 2) \times K(Z_{(2)}, 4) \times K(Z_2, 6) \times K(Z_{(2)}, 8) \times \dots$$

Indeed his argument needs only the facts that (1) TOP surgery works, (2) the signature map $Z = \pi_{4k}(\mathrm{G/TOP}) \to Z$ is $\times\, 8$ (even for $k = 1$, by 13.4), and (3) the Arf invariant map $Z_2 = \pi_{4k+2}(\mathrm{G/TOP}) \to Z_2$ is an isomorphism.

Alternatively (15.2) $\Rightarrow$ (15.3) if we use $\Omega^4(\mathrm{G/PL}) \simeq \mathrm{G/TOP}$ from §14.

Remark 15.4.

It is easy to see directly that the 4-stage of G/TOP must be $K(Z_2, 2) \times K(Z, 4)$. For the only other possibility is the 4 stage of G/PL with K-invariant $\delta \mathrm{Sq}^2$ in $H^5(K(Z_2, 2), Z) = Z_2$. Then the fibration $K(Z_2, 3) = \mathrm{TOP/PL} \to \mathrm{GL/PL} \to \mathrm{G/TOP}$ would be impossible. (Hint : Look at the induced map of 4 stages and consider the transgression onto $\delta \mathrm{Sq}^2$). This remark suffices for many calculations in dimension $\leqslant 6$. On the other hand it is not clear to me that (15.2) $\Rightarrow$ (15.3) without geometry in TOP.

Proof that Kernel $\varphi \cong H^3(X ; Z_2)/\mathrm{image}\ H^3(X ; Z)$.

This amounts to showing that for the natural fibration

$$\Omega \mathrm{G/TOP} \to \mathrm{TOP/PL} \to \mathrm{G/PL}$$

the image of $[X, \Omega\mathrm{G/TOP}]$ in $[X, \mathrm{TOP/PL}] = H^3(X ; Z_2)$, consists of the reduced integral cohomology classes. Clearly this is the image of $[X, \Omega(\mathrm{G/TOP})_{(2)}]$ under $\Omega(\mathrm{G/TOP})_{(2)} \xrightarrow{j_{(2)}} (\mathrm{TOP/PL})_{(2)} = \mathrm{TOP/PL}$. Now $j_{(2)}$ is integral reduction on the factor $K(Z_{(2)}, 3)$ of $\Omega(\mathrm{G/TOP})_{(2)}$ because $\pi_4(\mathrm{G/TOP}) \to \pi_3(\mathrm{TOP/PL})$ is onto, and it is clearly zero on other factors. The result follows. The argument comes from [13] [72].

Proof that $\mathrm{Coker}(\varphi) = \{\mathrm{Image}\ H^4(X ; Z) + \mathrm{Sq}^2 H^2(X ; Z_2)\}$.

The following lemma is needed. Its proof is postponed to the end.

LEMMA 15.5. — *The triangulation obstruction* $\Delta : B_{\mathrm{TOP}} \to K(Z_2, 4)$ *is an H-map.*

Write $\varphi : A \to B$ for $\varphi : \mathrm{G/PL} \to \mathrm{G/TOP}$ and let $\varphi_4 : A_4 \to B_4$ be the induced map of Postnikov 4-stages, which have inherited H-space structure. Consider the fibration $A_4 \xrightarrow{\varphi_4} B_4 \xrightarrow{\Delta_4} K(Z_2, 4)$.

Assertion (1). — $(\Delta_4)_*[X, B_4] = \{\mathrm{Image}\, H^4(X ; Z) + \mathrm{Sq}^2 H^2(X ; Z_2)\}$.

Proof of (1). — Since $B_4 = K(Z_2, 2) \times K(Z, 4)$ and

$$[X, B_4] = H^2(X ; Z_2) \oplus H^4(X ; Z)$$

what we have to show is that the class of Δ_4 in

$$[B_4, K(Z_2, 4)] = H^4(K(Z_2, 2) \times K(Z, 4) ; Z_2) =$$
$$= H^4(K(Z_2, 2), Z_2) \oplus H^4(K(Z, 4) ; Z_2)$$

is (Sq^2, ρ) where ρ is reduction mod 2.

The second component of Δ_4 is $\Delta_4 | K(Z, 4)$ which is indeed ρ since

$$Z = \pi_4 \mathrm{G/TOP} \twoheadrightarrow Z_2 = \pi_4 K(Z_2, 4) \quad .$$

The first component $\Delta_4 \mid K(Z_2, 2)$ can be Sq^2 or 0 a priori, but it cannot be 0 as that would imply $A_4 \simeq K(Z_2, 2) \times K(Z, 4)$. This establishes Assertion (1).

Assertion (2). — $(\Delta_4)_* [X, B_4] = \Delta_* [X, B]$ by the projection $B \to B_4$.

Proof of (2). — In view of 15.5, localising B_4 and B at 2 does not change the left and right hand sides. But after localization, we have equality since $B_{(2)}$ is the product (15.3).

The theorem follows quickly

$$[X, A]/\varphi_* [X, A] = \Delta_* [X, B] = (\Delta_4)_* [X, B_4] = \{\mathrm{Image}\, H^4(X, Z) + \mathrm{Sq}^2 H^2 (X ; Z_2)\}$$

The three equalities come from Lemma 15.5 and (1) and (2) respectively.

It remains now to give

Proof of Lemma 15.5 (S. Morita's, replacing something more geometrical).

We must establish homotopy commutativity of the square

$$\begin{array}{ccc} B_{\mathrm{TOP}} \times B_{\mathrm{TOP}} & \xrightarrow{\sigma} & B_{\mathrm{TOP}} \\ {\scriptstyle \Delta \times \Delta} \downarrow & & \downarrow {\scriptstyle \Delta} \\ K(Z_2, 4) \times K(Z_2, 4) & \xrightarrow{\alpha} & K(Z_2, 4) \end{array}$$

where σ represents Whitney sum and α represents addition in cohomology.

Now $\alpha \circ (\Delta \times \Delta)$ represents $\Delta \times 1 + 1 \times \Delta$ in $H^4(B_{\mathrm{TOP}} \times B_{\mathrm{TOP}} \; ; \; Z_2)$. Also $\Delta \circ \sigma$ certainly represents something of the form $\Delta \times 1 + 1 \times \Delta + \Sigma$, where Σ is a sum of products $x \times y$ with x, y each in one of $H^i(B_{\mathrm{TOP}} ; Z_2) = H^i(B_{\mathrm{PL}} \; ; Z_2)$, for $i = 1,2$ or 3. Since $\Delta \circ \sigma$ restricted to $B_{\mathrm{PL}} \times B_{\mathrm{PL}}$ is zero, Σ must be zero.

Theorem 15.1 is very convenient for calculations. Let M be a closed PL manifold, m-manifold $m \geqslant 5$, and write $\mathcal{S}_{\mathrm{CAT}}(M)$, CAT = PL or TOP, for the set of h-cobordism classes of closed CAT m-manifolds M' equipped with a homotopy equivalence $f : M' \to M$. (See [95] for details).

There is an exact sequence of pointed sets (extending to the left) :

$$\ldots \to [\Sigma M, \mathrm{G/CAT}] \to L_{m+1}(\pi, w_1) \to \mathcal{S}_{\mathrm{CAT}}(M) \xrightarrow{\nu} [M, \mathrm{G/CAT}] \to L_m(\pi, w_1) \quad .$$

It is due to Sullivan and Wall [95]. The map ν equips each $f : M' \to M$(above) as a CAT normal invariant. Exactness at $\mathcal{S}_{\mathrm{CAT}}(M)$ is relative to an action of $L_{m+1}(\pi, w_1)$ on it. Here $L_k(\pi, w_1)$ is the surgery group of Wall in dimension k for fundamental group $\pi = \pi_1 M$ and for orientation map $w_1 = w_1(M) ; \pi \to Z_2$. There is a generalisation for manifolds with boundary. Since the PL sequence maps naturally to the TOP sequence, our knowledge of the kernel and cokernel of

$$[M, \mathrm{G/PL}] \to [M, \mathrm{G/TOP}]$$

will give a lot of information about $\mathcal{S}_{PL}(M) \to \mathcal{S}_{TOP}(M)$. Roughly speaking failure of triangulability in $\mathcal{S}_{TOP}(M)$ is detected by non triangulability of the TOP normal invariant ; and failure of Hauptvermutung in $\mathcal{S}_{PL}(M)$ cannot be less than its failure for the corresponding PL normal invariants.

In case $\pi_1 M = 0$, one has $\mathcal{S}_{CAT}(M) \cong \mathcal{S}_{CAT}(M_0) \cong [M_0, G/CAT]$ where M_0 is M with an open m-simplex deleted, and so Theorem 15.1 here gives complete information.

Example 15.6. — The exotic PL structure Σ on $S^3 \times S^n$, $n \geqslant 2$, from

$$1 \in H^3(S^3; Z_2) = Z_2$$

admits a PL isomorphism $(S^3 \times S^n)_\Sigma \cong S^3 \times S^n$ homotopic (not TOP isotopic) to the identity.

Example 15.7. — For $M = CP_n$ (= complex projective space), $n \geqslant 3$, the map $[M_0, G/PL] \to [M_0, G/TOP]$ is injective with cokernel $Z_2 = H^4(M_0, Z_2)$. This means that 'half' of all manifolds $M' \simeq CP_n$, $n \geqslant 3$, have PL structure. Such a PL structure is unique up to isotopy, since $H^3(CP_n, Z_2) = 0$.

16. Manifolds homotopy equivalent to real projective space P^n.

After sketching the general situation, we will have a look at an explicit example of failure of the Hauptvermutung in dimension 5.

From [54] [94] we recall that, for $n \geqslant 4$,

(16.1) $$[P^n, G/PL] = Z_4 \oplus \sum_{i=6}^{n} \pi_i(G/PL) \otimes Z_2 \quad .$$

This follows easily from (15.2). For G/TOP the calculation is only simpler. One gets

(16.2) $$[P^n, G/TOP] = \sum_{i=2}^{n} \pi_i(G/TOP) \otimes Z_2 \quad .$$

Calculation of $\mathcal{S}_{PL}(P^n) \equiv I_n$ is non-trivial [54] [94]. One gets (for $i \geqslant 1$)

(16.3) $$I_{4i+2} = I_{4i+1} = [P^{4i}, G/PL] \,; I_{4i+3} = I_{4i+2} \oplus Z \,; I_{4i+4} = I_{4i+2} \oplus Z_2 \,.$$

The result for $\mathcal{S}_{TOP}(P^n)$ is similar, when one uses TOP surgery. Then

$$\mathcal{S}_{PL}(P^n) \to \mathcal{S}_{TOP}(P^n)$$

is described as the direct sum of an isomorphism with the map

$$Z_4 = [P^4, G/PL] \to [P^4, G/TOP] = Z_2 \oplus Z_2 \quad ,$$

which sends Z_4 onto $Z_2 = \pi_2 G/TOP$.

Remark 16.4. — When two distinct elements of $\mathcal{S}_{PL}(P^n)$, $n \geqslant 5$, are topologically the same, we know already from 15.1 that their PL normal invariants are distinct since $H^3(P^n; Z_2)$ is not reduced integral. This facilitates detection of examples.

Consider the fixed point free involution T on the Brieskorn-Pham sphere in C^{m+1}

$$\Sigma_d^{2m-1} : z_0^d + z_1^2 + z_2^2 + \cdots z_m^2 = 0 \quad , \quad |z| = 1 \quad ,$$

given by $T(z_0, z_1, \ldots, z_m) = (z_0, -z_1, \ldots, -z_m)$. Here d and m must be odd positive integers, $m \geqslant 3$, in order that Σ_d^{2m-1} really be topologically a sphere [61].

As T is a fixed point free involution the orbit space $\Pi_d^{2m-1} \equiv \Sigma_d^{2m-1}/T$ is a DIFF manifold. And using obstruction theory one finds there is just one oriented equivalence $\Pi_d^{2m-1} \to P^{2m-1}$ (Recall $P^\infty = K(Z_2, 1)$). Its class in $\mathcal{S}_{\mathrm{CAT}}(P^{2m-1})$ clearly determines the involution up to equivariant CAT isomorphism and conversely.

THEOREM 16.5. — *The manifolds Π_d^5, d odd, fall into four diffeomorphism classes according as $d \equiv 1, 3, 5, 7$ mod 8, and into two homeomorphism classes according as $d = \pm 1, \pm 3$ mod 8. Π_1^5 is diffeomorphic to P^5.*

Remark 16.6. — With Whitehead C^1 triangulations, the manifolds Π_d^5 have a PL isomorphism classification that coincides with the DIFF classification (§5, [9] [64]). Hence we have here rather explicit counterexamples to the Hauptvermutung. One can check that they don't depend on Sullivan's complete analysis of $(G/PL)_{(2)}$. The easily calculated 4-stage suffices. Nor do they depend on topological surgery.

PROBLEM. — Give an explicit homeomorphism $P^5 \approx \Pi_7^5$.

Remark 16.7. — Giffen states [23] that (with Whitehead C^1 triangulations) the manifolds Π_d^{2m-1}, $m = 5, 7, 9, \ldots$ fall into just four PL isomorphism classes $d \equiv 1, 3, 5, 7$ mod 8. In view of theorem 16.4., these classes are already distinguished by the restriction of the normal invariant to P^5 (which is that of Π_d^5). So Giffen's statement implies that the homeomorphism classification is $d \equiv \pm 1$, ± 3 mod 8.

Proof of 16.5.

The first means of detecting exotic involutions on S^5, was found by Hirsch and Milnor 1963 [32]. They constructed explicit(*) involutions (M_{2r-1}^7, β_r), r an integer $\geqslant 0$, on Milnor's original homotopy 7-spheres, and found invariant spheres $M_{2r-1}^7 \supset M_{2r-1}^6 \supset M_{2r-1}^5$. They observed that the class of M_{2r-1}^7 in $\Gamma_{28}/2\Gamma_{28}$ is an invariant of the DIFF involution (M_{2r-1}^5, β_r) — (consider the suspension operation to retrieve (M_{2r-1}^7, β) and use $\Gamma_6 = 0$). Now the class of M_{2r-1}^7 in $Z_{28} = \Gamma_7$ is $r(r-1)/2$ according to Eells and Kuiper [18], which is odd iff $r \equiv 2$ or 3 mod 4. So this argument shows (M_{2r-1}^5, β_r) is an exotic involution if $r \equiv 2$ or 3 mod 4.

Fortunately the involution (M_{2r-1}^5, β_r) has been identified with the involution (Σ_{2r-1}^5, T).

(*) β_r is the antipodal map on the fibers of the orthogonal 3-sphere bundle M_{2r-1}^7.

There were two steps. In 1965 certain examples (X^5, α_r) of involutions were given by Bredon, which Yang [101] explicitly identified with (Σ^5_{2r-1}, T). Bredon's involutions extend to O(3) actions, α_r being the antipodal involution in O(3). And for any reflection α in O(3) $\alpha\alpha_r = \alpha_r\alpha$ has fixed point set diffeomorphic to $L^3(2r+1, 1) : z_0^{2r+1} + z_1^2 + z_2^2 = 0 ; |z| = 1$. This property is clearly shared by (Σ^5_{2r+1}, T), and Hirzebruch used this fact to identify (Σ^5_{2r+1}, T) to (X, α_r) [33, §4] [34]. The Hirsch-Milnor information now says that Π^5_d is DIFF exotic. if $d \equiv 5{,}7 \bmod 8$.

Next we give a TOP invariant for Π_d in Z_2. Consider the normal invariant ν_d of Π_d in $[P^5, G/O] = Z_4$. Its restriction $\nu_d|P^2$ to P^2 is a TOP invariant because $[P^2, G/O] = [P^2, G/TOP] = Z_2$.

Now Giffen [22] shows that $\nu_d|P^2$ is the Arf invariant in Z_2 of the framed fiber of the torus knot $z_0^d + z_1^2 = 0, |z_0|^2 + |z_1|^2 = 1$ in $S^3 \subset C^2$. This turns out to be 0 for $d \equiv \pm 1 \bmod 8$ and 1 for $d \equiv \pm 3 \bmod 8$, (Levine [53], cf. [61, § 8]).

We have now shown that the diffeomorphism and homeomorphism classifications of the manifolds Π^5_d are *at least* as fine as asserted. But there can be at most the four diffeomorphism classes named, in view of 16.3. (Recall that the PL and DIFF classifications coincide since $\Gamma_i = \pi_i(PL/O) = 0, i \leqslant 5$). Hence, by Remark 16.4, there are exactly four – two in each homeomorphism class.

REFERENCES

[1] ALEXANDER J.W. — On the deformation of an n-cell, *Proc. Nat. Acad. Sci. U.S.A.*, 9, 1923, p. 406-407.

[2] ALEXANDER J.W. — Some problems in topology, *Proc. I.C.M.*, Zürich, 1932, Vol. 1, p. 249-257.

[3] ATIYAH M. — Bordism and cobordism, *Proc. Camb. Phil. Soc.*, 57, 1961, p. 200-208.

[4] BROWDER W. and HIRSCH M. — Surgery in piecewise linear manifolds and applications, *Bull. Amer. Math. Soc.*, 72, 1966, p. 959-964.

[5] BROWDER W., LIULEVICIUS A. and PETERSON F. — Cobordism theories, *Ann. of Math.*, 84, 1966, p. 91-101.

[6] BROWN Edgar H. — Abstract homotopy theory, *Trans. Amer. Math. Soc.*, 119, 1965, p. 79-85.

[7] BROWN M. — A proof of the generalized Shoenflies theorem, *Bull. Amer. Math. Soc.*, 66, 1960, p. 74-76.

[8] BROWN M. and GLUCK H. — Stable structures on manifolds, *Bull. Amer. Math. Soc.*, 69, 1963, p. 51-58.

[9] CERF J. — Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$), *Springer Lecture Notes in Math.*, No. 53, 1968.

[10] ČERNAVSKII A.V. — Local contractibility of the homeomorphism group of a manifold, *Soviet Math. Doklady*, 9, 1968, p. 1171-1174; *Dokl. Akad. Nauk. S.S.S.R.*, 182, No. 3, 1968,

[11] Černavskii A.V. — Local contractibility of the homeomorphism group of a manifold (Russian), *Math. Sbornik,* Vol. 79 (121), 1969, p. 307-356.
[12] Cheeger J. and Kister J. — Counting topological manifolds, *Topology,* 9, 1970, p. 149-151.
[13] Cooke G. — On the Hauptvermutung according to Sullivan, *Mimeo. I.A.S.,* Princeton, 1968.
[14] Connell E.H. — Approximating stable homeomorphisms by piecewise-linear ones, *Ann. of Math.,* 78, 1963, p. 326-338.
[15] Dold A. — Halbexacte Homotopiefunktoren, *Springer Lecture Notes,* 12, 1966.
[16] Dold A. — Structure de l'anneau de cobordisme Ω_*, d'après Rohlin V.A. et Wall C.T.C., *Séminaire Bourbaki,* 1959-60, n° 186.
[17] Edwards R.D. and Kirby R.C. — Deformations of spaces of imbeddings, *Ann. of Math.,* 93, 1971, p. 63-88.
[18] Eells J. and Kuiper N. — An invariant for smooth manifolds, *Ann. Math. Pura Appl.,* 60, 1962, p. 93-110.
[19] Farrell F.T. — These proceedings.
[20] Farrell F.T. and Hsiang W.C. — *Manifolds with* $\pi_1 = G \times_\alpha T$ (to appear).
[21] Furch R. — Zur Grundlegung der kombinatorischen Topologie, *Hamburger,* Abh. 3, 1924, p. 69-88.
[22] Giffen C.H. — Desuspendability of free involutions on Brieskorn spheres, *Bull. Amer. Math. Soc.,* 75, 1969, p. 426-429.
[23] Giffen C.H. — Smooth homotopy projective spaces, *Bull. Amer. Math. Soc.,* 75, 1969, p. 509-513.
[24] Gluck H. — Embeddings in the trivial range, *Bull. Amer. Math. Soc.,* 69, 1963, p. 824-831; *Ann. of Math.,* 81, 1965, p. 195-210.
[25] Gromov M.L. — Transversal maps of foliations, *Dokl. Akad. Nauk. S.S.S.R.,* 182, 1968, p. 225-258; *Soviet Math. Dokl.,* 9, 1968, p. 1126-1129. MR 38-6628.
[26] Gromov M.L. — Thesis, *Izv. Akad. Nauk. S.S.S.R.,* 33, 1969, p. 707-734.
[27] Haefliger A. and Poenaru V. — Classification des immersions, *Publ. Math. Inst. Hautes Etudes Sci.,* 23, 1964, p. 75-91.
[28] Haefliger A. — Feuilletages sur les variétés ouvertes, *Topology,* 9, 1970, p. 183-194.
[29] Haefliger A. — Homotopy and integrability, *Manifolds Conf.,* Amsterdam, 1970, editor N. Kuiper, Springer Lecture Notes, 1971, No. 179.
[30] Hirsch M. — On normal microbundles, *Topology,* 5, 1966, p. 229-240.
[31] Hirsch M. — On nonlinear cell-bundles, *Ann. of Math.,* 84, 1966, p. 373-385.
[32] Hirsch M. and Milnor J. — Some curious involutions of spheres, *Bull. Amer. Math. Soc.,* 70, 1964, p. 372-377.
[33] Hirzebruch F. — Singularities and exotic spheres, *Séminaire Bourbaki,* n° 314, 1966-67.
[34] Hirzebruch F. and Mayer K.H. — O(n)-Mannifaltigkeiten, exotische Sphären und Singularitäten, *Lecture notes in Math.,* No. 57, 1968, Springer.
[35] Hsiang W.-C. and Shaneson J.L. — Fake tori, pages 18-51, in *Topology of Manifolds,* editors J.C. Cantrell and C.H. Edwards, Markham, Chicago 1970.
[36] Hsiang W.-C. and Wall C.T.C. — On homotopy tori II, *Bull. London Math. Soc.,* 1, 1969, p. 341-342. (Beware the homotopy invariance p. 342.)
[37] Hudson J.F.P. — *Piecewise-linear topology,* W.A. Benjamin, Inc., New York, 1969.
[38] Kerékjártó B. v. — *Vorlesungen über Topologie, I, Flächentopologie,* Springer, 1923.
[39] Kervaire M. — Le théorème de Barden-Mazur-Stallings, *Comment. Math. Helv.,* 40, 1965, p. 31-42.

[40] KERVAIRE M. and MILNOR J. — Bernoulli numbers, homotopy groups and a theorem of Rohlin, *Proc. I.C.M.*, Edinborough 1958, p. 454-458.

[41] KERVAIRE M. and MILNOR J. — Groups of homotopy spheres, I, *Ann. of Math.*

[42] KIRBY R.C. — Stable homeomorphisms and the annulus conjecture, *Ann. of Math.*, 89, 1969, p. 575-582.

[43] KIRBY R.C. — Lectures on triangulation of manifolds, *Mimeo. U. of Calif.*, Los Angeles, 1969.

[44] KIRBY R.C. and SIEBENMANN L.C. — On the triangulation of manifolds and the Hauptermutung, *Bull. Amer. Math. Soc.*, 75, 1969, p. 742-749.

[44 A] KIRBY R.C. and SIEBENMANN L.C. — A straightening theorem and a Hauptvermutung for pairs, *Notices Amer. Math. Soc.*, 16, 1969, p. 582.

[44 B] KIRBY R.C. and SIEBENMANN L.C. — For manifolds the Hauptvermutung and the triangulation conjecture are false, *Notices Amer. Math. Soc.*, 16, 1969, p. 695.

[44 C] KIRBY R.C. and SIEBENMANN L.C. — Foundations of topology, *Notices Amer. Math. Soc.*, 16, 1969, 698.

[45] KIRBY R.C. and SIEBENMANN L.C. — Some theorems on topological manifolds, in *Manifolds*, editor N. Kuiper, Springer Lecture Notes, 1971.

[46] KIRBY R.C. and SIEBENMANN L.C. — Deformation of smooth and piecewise-linear manifold structures, *Ann. of Math.* (to appear).

[46 A] KIRBY R.C. and SIEBENMANN L.C. — To appear.

[47] KNESER H. — Die Topologie der Mannigfaltigkeiten, *Jahresbericht der D. Math. Ver.*, 34, 1925-26, p. 1-14.

[48] KUIPER N.H. — *These proceedings.*

[49] KUIPER N.H. and LASHOF R.K. — Microbundles and bundles I, *Invent. Math.*, 1, 1966, p. 1-17.

[50] LASHOF R. — The immersion approach to triagulation, pages 52-56 in *Topology of Manifolds*, editors J.C. Cantrell and C.H. Edwards, Markham, Chicago 1970, in extenso pages 282-355 in *Proc. Adv. Study Inst. Alg. Top.*, Aarhus, Denmark, 1970.

[51] LASHOF R. and ROTHENBERG M. — Triangulation of manifolds I, II, *Bull. Amer. Math. Soc.*, 75, 1969, p. 750-757.

[52] LEES J.A. — Immersions and surgeries of topological manifolds, *Bull. Amer. Soc.*, 75, 1969, p. 529-534.

[53] LEVINE J. — Polynomial invariants of knots of codimension two, *Ann. of Math.*, 84, 1966, p. 537-554.

[54] LOPEZ DE MEDRANO S. — Involutions on manifolds, *Ergebnisse Math.*, No. 59, Springer 1971.

[54 A] MILLER R.T. — *Trans. Amer. Math. Soc.* (to appear).

[55] MILNOR J. — On manifolds homeomorphic to the 7-sphere, *Ann. of Math.*, 64, 1956, p. 399-405.

[56] MILNOR J. — A proceedure for killing homotopy groups of differentiable manifolds, *Proc. Symposia in Pure Math.*, Vol. 3; Differentiable Geometry, *Amer. Math. Soc.*, 1961, p. 39-55.

[57] MILNOR J. — Two complexes that are homeomorphic but combinatorially distinct, *Ann. of Math.*, 74, 1961, p. 575-590.

[58] MILNOR J. — Topological manifolds and smooth manifolds, *Proc. Int. Congr. Math.*, Stockholm, 1962, p. 132-138.

[59] MILNOR J. — Spin structures on manifolds, *Enseignement Math.*, 9, 1963, p. 198-203.

[60] MILNOR J. — Microbundles, Part I, *Topology*, Vol. 3, 1964, Sup. 1, p. 53-80.

[61] MILNOR J. — Singular points of complex hypersurfaces, *Ann. of Math. Studies*, No. 61, Princeton U. Press, 1968.
[62] MOISE E. — Affine structures on 3-manifolds, *Ann. of Math.*, 56, 1952, p. 96-114.
[63] MORLET C. — Hauptvermutung et triangulation des variétés, *Sém. Bourbaki*, 1968-69, exposé 362.
[64] MUNKRES J.R. — Concordance of differentiable structures: two approaches, *Michigan Math. J.*, 14, 1967, p. 183-191.
[65] NEWMAN M.H.A. — The engulfing theorem for topological manifolds, *Ann. of Math.*, 84, 1966, p. 555-571.
[66] NÖEBLING G. — Zur Topologie der Manigfaltigkeiten, *Monatshefte für Math. und Phys.*, 42, 1935, p. 117-152.
[67] NOVIKOV S.P. — On manifolds with free abelian fundamental group and applications (Pontrjagin classes, smoothings, high dimensional knots), *Izvestia Akad. Nauk. S.S.S.R.*, 30, 1966, p. 208-246.
[68] POINCARÉ H. — *Dernières pensées*, E. Flammarion 1913, Dover (English translation), 1963.
[69] RADEMACHER H. — Partielle und totale Differentierbarkeit von Funktionen mehrer Variabeln, *Math. Annalen*, 79, 1919, p. 340-359.
[70] RADO T. — Ueber den Begriff der Riemannschen Fläche, *Acta Litt. Scient. Univ. Szegd*, 2, 1925, p. 101-121.
[71] ROHLIN V.A. — A new result in the theory of 4-dimensional manifolds, *Soviet Math. Doklady*, 8, 1952, p. 221-224.
[72] ROURKE C. and SANDERSON B. — The Hauptvermutung according to Sullivan, Part I, *I.A.S. mimeo*, 1967-68.
[73] ROURKE C. and SANDERSON B. — On topological neighborhoods, *Compositio Math.*, 22, 1970, p. 387-424.
[74] ROURKE C. and SULLIVAN D. — *On the Kervaire obstruction* (to appear).
[75] SERRE J.P. — Groupes d'homotopie et classes de groupes abéliens, *Ann. of Math.*, 58, 1953, p. 258-294.
[76] SIEBENMANN L.C. — The obstruction to finding a boundary for an open manifold of dimension > 5, *thesis*, Princeton U., 1965.
[77] SIEBENMANN L.C. — A total Whitehead torsion obstruction, *Comment. Math. Helv.*, 45, 1970, p. 1-48.
[78] SIEBENMANN L.C. — Infinite simple homotopy types, *Indag. Math.*, 32, 1970, p. 479-495.
[79] SIEBENMANN L.C. — Are non-triangulable manifolds triangulable? p. 77-84, *Topology of Manifolds*, ed. J.C. Cantrell and C.H. Edwards, Markham, Chicago 1970.
[80] SIEBENMANN L.C. — Disruption of low-dimensional handlebody theory by Rohlin's theorem, p. 57-76, *Topology of Manifolds*, ed. J.C. Cantrell and C.H. Edwards, Markham, Chicago 1970.
[81] SIEBENMANN L.C. — Approximating cellular maps by homeomorphisms, *Topology*.
[82] SIEBENMANN L.C. — Deformation of homoemorphisms on stratified sets, *Comment. Math. Helv.*
[83] SIEBENMANN L.C. — *Some locally triangulable compact metric spaces that are not simplicial complexes.*
[84] SMALE S. — On the structure of manifolds, *Amer. J. of Math.*, 84, 1962, p. 387-399.
[85] STEINITZ E. — Beiträge zur analysis situs, *Sitzungsberichte der Berliner Math. Gesellschaft*, 7, 1907, p. 29-49.

[86] STONG R.L.E. — Notes on cobordism theory, *Math. Notes,* Princeton U. Press, 1968.
[87] SULLIVAN D. — Triangulating homotopy equivalences, *Ph. D. thesis,* Princeton, 1966.
[88] SULLIVAN D. — On the Hauptvermutung for manifolds, *Bull. Amer. Math. Soc.* 73, 1967, 598-600.
[89] SULLIVAN D. — *Triangulation and smoothing homotopy equivalences and homeomorphisms* (seminar notes), Princeton U., 1967.
[90] SULLIVAN D. — Geometric Topology, Part I, *Mimeo., M.I.T.,* 1970.
[91] THOM R. — Quelques propriétés globales des variétés différentiables, *Comment. Math. Helv.,* 28, 1954, p. 17-86.
[92] THOM R. — Topological models in biology, *Topology,* 8, 1969, p. 313-336.
[93] WALL C.T.C. — Differential topology, Part V A, Cobordism : geometric theory, *Mineographed notes U. of Liverpool,* 1966.
[94] WALL C.T.C. — Free piecewise linear involutions on spheres, *Bull. Amer. Math. Soc.,* 74, 1968, p. 554-558.
[95] WALL C.T.C. — *Surgery on compact manifolds,* Academic Press, 1971.
[96] WEBER C. — Elimination des points doubles dans le cas combinatoire, *Comment. Math. Helv.,* 41, 1966-67, p. 179-182.
[99] WHITNEY H. — The self-intersections of a smooth n-manifold in $2n$-space, *Ann. of Math.,* 45, 1944, p. 220-246.
[97] WHITEHEAD J.H.C. — On C^1 complexes, *Ann. of Math.,* 41, 1940, p. 809-824.
[98] WHITEHEAD J.H.C. — Manifolds with transverse fields in euclidean space, *Ann. of Math.,* 73, 1961, 154-212.
[100] WILLIAMSON Jr. R.E. — Cobordism of combinatorial manifolds, *Ann. of Math.,* 83, 1966, p. 1-33.
[101] YANG C.T. — On involutions of the five sphere, *Topology,* 5, 1966, p. 17-19.

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GROUP THEORY AND 3-MANIFOLDS

by John R. STALLINGS

The Sphere Theorem of Papakyriakopoulos [1] gives an algebraic criterion ($\pi_2(M) \neq 0$) which implies a topological decomposition of the 3-manifold M (M is either a non-trivial connected sum, or $S^1 \times S^2$, or something analogous where the real projective plane P^2 takes the role of S^2). This topological decomposition determines a similar decomposition of the fundamental group.

Conversely, we have results [2] about group theory which from an algebraic criterion (the finitely generated group G has more than one end) derives an algebraic decomposition (G is a non-trivial free product with finite amalgamated subgroup, or else a group with a sort of finite handle attached). It is possible to use this group theory plus Dehn's Lemma to give an independent proof of the Sphere Theorem.

In the course of this investigation we have noticed certain general situations which may be of interest.

1. Bipolar structures.

A group G is said to have a *bipolar structure over* F, if F is a subgroup of G, and $G \backslash F$ is partitioned into four sets A_{ij}, $i, j = 0, 1$, satisfying these four axioms :

(1) $A_{ij}^{-1} = A_{ji}$

(2) $A_{ij} \bullet A_{1-j,k} \subset A_{ik}$

(3) $F \bullet A_{ij} = A_{ij}$.

For the fourth axiom, we need to define an *indecomposable* element to be an element of A_{ik} which cannot be written as a product as in axiom 2. Then let P denote F plus all indecomposable elements.

(4) G is generated by P.

The bipolar structure is said to be *non-trivial* if $A_{01} \neq \emptyset$.

In our group-theoretic investigation, we showed, using graph-theory, that a finitely generated group with more than one end has a non-trivial bipolar structure over a finite subgroup F.

Bipolar structures are closely related to free products with amalgamation. Let

$$G_i = F \cup \{\text{indecomposable elements of } A_{ii}\}.$$

Then if there are no indecomposable elements of A_{01}, we have $G = G_0 *_F G_1$. If there is, however, an indecomposable element of A_{01} then G can be described as the result of adding an F-handle to G_0.

Conversely, suppose $G = G_0 *_F G_1$, then every element of $G \backslash F$ can be written, according to Schreier, uniquely in the form

$$h = f\alpha_1\alpha_2 \ldots \alpha_n$$

where $f \in F$, and α_i are coset representatives of the non-trivial cosets of the form Fg , $g \in G_0 \cup G_1$, the sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ alternating between G_0 and G_1. We say $h \in A_{ij}$ if and only if $\alpha_1 \in G_i$, $\alpha_n \in G_j$. This gives G a bipolar structure over F.

It is worth noting that if G has a bipolar structure over F, then any subgroup H inherits a bipolar structure over $H \cap F$, by simply defining $B_{ij} = H \cap A_{ij}$. Indecomposable elements of H may well be decomposable in G. The nature of H as a subgroup of G can perhaps be studied by investigating the inherited bipolar structures of all the conjugates of H.

2. Pregroups.

A group G with a bipolar structure over F defines in particular the subset P consisting of F together with all indecomposable elements. It is a theorem then that every element $g \in G$ can be written as

$$g = p_1 \ldots p_n \quad \text{where} \quad p_i \in P \ , \ p_i p_{i+1} \notin P$$

and that if there is another such expression for g :

$$g = q_1 \ldots q_m \quad , \quad q_i \in P \ , \ q_i q_{i+1} \notin P \ ,$$

then $m = n$ and $p_i = f_{i-1}^{-1} q_i f_i$ for some $f_0, \ldots, f_n \in F$, $f_0 = f_n = 1$. In other words, every element of G is expressible as a reduced word in P, and two such words for the same element are equivalent modulo interleaving of elements of F.

This situation has an interesting generalization which, like bipolar structures, may provide insight into the nature of free products with amalgamation. It is as follows.

A *pregroup* P is a set having a distinguished element 1, a unary operation $x \to x^{-1}$, and a binary operation partly defined $(x, y) \to xy$, satisfying the following five axioms.

(1) $x1$ and $1x$ are always defined and equal to x.

(2) xx^{-1} and $x^{-1}x$ are always defined and equal to 1.

(3) If xy is defined, then $y^{-1}x^{-1}$ is defined and equal to $(xy)^{-1}$.

(4) If xy and yz are defined, then, $x(yz)$ is defined if and only if $(xy)z$ is defined, in which case $x(yz) = (xy)z$.

(5) If xy , yz , zw are defined, then either $x(yz)$ or $(yz)w$ is defined.

The principal theorem about pregroups is that there exists, for any pregroup P, a largest group $U(P)$ generated by P, and that the elements of $U(P)$ are expressible as reduced words in P, two reduced words giving the same element of $U(P)$ if and only if they have the same length and are related by an interleaving of elements of P.

Pregroups P and their universal groups $U(P)$ generalize the amalgamated free product situation, in which $P = A \cup B$, where A and B are two groups intersecting along a common subgroup C, and $U(P) = A *_C B$. There are examples, however, of pregroups which do not seem to be in any sense reducible to amalgamated free products.

The study of pregroups, we feel, offers a rich prospect for investigating free products and amalgamated free products.

3. Topological implications and analogues.

If $X \subset Y$ are path-connected spaces, and X is bicollared in Y, and $\pi_1(X)$ injects, monomorphically, to $\pi_1(Y)$, then any closed loop based at $y_0 \in X$ is homotopic to a kind of irreducible loop whose start is to one particular side of X and whose end is to a particular side. For any element of $\pi_1(Y)\backslash\pi_1(X)$, these beginning and ending sides are well-determined ; thus $\pi_1(Y)$ has a certain structure, which turns out to be a bipolar structure over $\pi_1(X)$. Furthermore, every bipolar structure has this sort of topological reflection.

Our proof of the Sphere Theorem is based on this circumstance. If, say, M is a closed 3-manifold with $\pi_2(M) \neq 0$, then Poincaré duality in the universal cover shows $\pi_1(M)$ has more than one end, and thus $\pi_1(M)$ has a non-trivial bipolar structure over a finite group. This allows us to find X and Y as above, with $\pi_1(X)$ finite, and a map $\varphi : M \to Y$ inducing a π_1-isomorphism. We now perform surgery of φ, using Dehn's Lemma and the Loop Theorem, so that φ will have the property that $\varphi^{-1}(X)$ is a two-sided 2-manifold, each component of which, having finite π_1, is either S^2 or P^2. The non-triviality of the bipolar structure will then imply that at least one of these components carries a non-trivial element of $\pi_2(M)$.

It would be interesting to know whether pregroups have some kind of topological picture, but we conjecture that pregroups are too general and rich for this to happen.

REFERENCES

[1] PAPAKYRIAKOPOULOS C.D. — On Dehn's Lemma and the asphericity of knots, *Annals of Mathematics*, 66, 1957, p. 1-26.

[2] STALLINGS John R. — *Group theory and three-dimensional manifolds, Lectures at Yale University* 1969, to appear.

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GALOIS SYMMETRY IN MANIFOLD THEORY AT THE PRIMES

by Dennis SULLIVAN

Let $\mathfrak{M}$ denote the category generated by compact simply connected manifolds(1) and homeomorphisms. In this note we consider certain formal manifold categories related to $\mathfrak{M}$. We have the profinite category $\hat{\mathfrak{M}}$, the rational category $\mathfrak{M}_Q$ and the adele category $\mathfrak{M}_A$. The objects in these categories are CW complexes whose homotopy groups are modules over the ground ring of the category ($\hat{Z} = \varprojlim Z/n$, Q, and $A = Q \otimes \hat{Z}$), and which have certain additional manifold structure.

From these formal manifold categories we can reconstruct $\mathfrak{M}$ up to equivalence. For example a classical manifold M corresponds to a profinite manifold $\hat{M}$, a rational manifold M_Q, and an equivalence between the images of $\hat{M}$ and M_Q in $\mathfrak{M}_A$. In fact $\mathfrak{M}$ is the fibre product of $\hat{\mathfrak{M}}$ and $\mathfrak{M}_Q$ over $\mathfrak{M}_A$.

Thus we can study $\mathfrak{M}$ by studying these related categories. Here we find certain advantages.

– the structure of $\mathfrak{M}$ finds natural expression in the related categories.

– these categories are larger and admit more examples – manifolds with certain singularities and more algebraic entities than topological spaces.

– there is a pattern of symmetry not directly observable in $\mathfrak{M}$.

For the last point consider the collection of all non-singular algebraic variétiés over C. The Galois group of C over Q permutes these variétiés (by conjugating the coefficients of the defining relations) and provides certain (discontinuous) self maps when these coefficients are fixed.

As far as geometric topology is concerned we can restrict attention to the field of algebraic numbers $\bar{Q}$ (for coefficients) and its Galois group Gal $(\bar{Q}/Q)$. Conjugate variétiés have the same profinite homotopy type (canonically) so Gal $(\bar{Q}/Q)$ permutes a set of smooth manifold structures on one of these profinite homotopy types. [S3]

If we pass to the topological category $\hat{\mathfrak{M}}$ we find this galois action is abelian and extends to a natural group of symmetries on the category of profinite manifolds ;

$$\begin{matrix}\text{abelianized} \\ \text{Gal }(\bar{Q}/Q)\end{matrix} \simeq \left\{\begin{matrix}\text{group of} \\ \text{units} \\ \text{of } \hat{Z}\end{matrix}\right\} \text{ acts on } \hat{\mathfrak{M}}.$$

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(1) The case $\pi_1 \neq 0$ can be treated to a considerable extent using families (see S3).

First description.

The possibility of defining formal manifold categories arises from the viewpoint begun by Browder and Novikov. For example, Browder observes that

(i) a manifold has an underlying homotopy type satisfying Poincaré duality.

(ii) there is a Euclidean bundle over this homotopy type – the normal bundle in R^n (which is classified by a map into some universal space B)

(iii) the one point compactification of the bundle has a certain homotopy theoretical property – a degree one map from the sphere S^n. (the normal invariant)

Novikov used the invariant of (iii) to classify manifolds with a fixed homotopy type and tangent bundle, while Browder constructed manifolds from the ingredients of (i), (ii), and (iii).

We propose tensoring such a homotopy theoretical description of a simply connected manifold with a ring R. For appropriate R we will obtain formal manifold categories $\mathfrak{M}_R$.

To have such a description of $\mathfrak{M}_R$ we assume there is a natural construction in homotopy theory $Y \to Y_R$ which tensors the homotopy groups with R (under the appropriate hypotheses) and that a map $X \xrightarrow{\nu} B_R$ has an associated sphere fibration (sphere $= S^n_R$).

There are such constructions for R any of the subrings of Q, Z_l, any of the non-Archimedean completions of Q, Q_p, the arithmetic completions of Z, $\hat{Z}_p$ and $\hat{Z}$, the finite Adeles $Q \otimes \hat{Z}$ (see S1).

The fibre product statement

$$\mathfrak{M} \sim \mathfrak{M}_Q \times_{\mathfrak{M}_A} \hat{\mathfrak{M}} \qquad (\hat{\mathfrak{M}} = \mathfrak{M}_{\hat{Z}})$$

follows from the Browder surgery theorem and analogous decomposition of ordinary (simply connected) homotopy theory (see [B] and [S1] chapter 3).

Second description.

If one pursues the study of Browder's description of classical manifolds in a more intrinsic manner – internal to the manifolds studied – certain transversality invariants occur in a natural way. These signature and arf invariants of quadratic forms on submanifolds control the situation and the structure which accrues can be expressed in the formal manifold categories see [S2] and [S3].

In the "rational manifold theory", a manifold is just a rational homotopy type satisfying homological duality over Q together with a preferred characteristic class

$$l_n + l_{n-4} + \ldots + l_{n-4i} + \ldots = l_X \in H_{n-4*}(X, Q), \; n = \dim X.$$

Here l_n is an orientation class and l_0 is the signature of X (if $n \equiv 0$ (4)).

To pursue a more precise discussion we should regard X as a specific CW complex endowed with specific chains representing the characteristic class.

Then a homotopy equivalence $X \xrightarrow{f} Y$ between two such complexes and a chain ω_f so that $f_\# l_X - l_Y = \partial \omega_f$ determines a "homeomorphism" up to concordance.

There is an analogous "homological" description for $\mathfrak{M}_A$ if we replace Q by $A = Q \otimes \hat{Z}$, (or by any field of characteristic zero).

The profinite manifold theory has a more intricate structure. First of all there is a complete splitting into p-adic components

$$\hat{\mathfrak{M}} \sim \prod_p \hat{\mathfrak{M}}_p$$

where the product is taken over the set of prime numbers and $\hat{\mathfrak{M}}_p$ is the formal manifold category based on the ring $R = \hat{Z}_p$, the p-adic integers.

For the odd primes we have a uniform structure. Let (k_*, k^*) denote the cohomology theory constructed from the p-adic completion of real K-theory by converting the filtration into a grading. Then the p-adic manifolds are just the k-duality spaces at the prime p. That is, we have a CW complex X (with p-adic homotopy groups) and a k-homology class.

$$\mu_X \in k_m(X) \qquad m = \dim X \text{ (defn)}$$

so that forming cap products with the orientation class gives the Poincaré duality

$$k^i(X) \sim k_{m-i}(X)(^1)$$

The homeomorphisms in $\hat{\mathfrak{M}}_p$ correspond to the maps $X \to Y$ giving an isomorphism of this natural duality in k-theory

$$\begin{array}{ccc} k_* X & \xrightarrow{\simeq} & k_* Y \\ {\scriptstyle \cap \mu_X} \uparrow \simeq & & {\scriptstyle \cap \mu_Y} \uparrow \simeq \\ k^* Y & \xleftarrow{\simeq} & k^* Y, \end{array} \qquad \text{"homeomorphism condition"}$$

ie f is a homotopy equivalence and $f_* \mu_X = \mu_Y$.

Again a more precise discussion (determining a concordance class of homeomorphisms. . .) requires the use of cycles (analogous to the chains above) and a specific homology producing the relation $f_* \mu_X = \mu_Y$.

Note that $K(X)^*$, the group of units in $k^0(X)$, acts bijectively on the set of all orientations of X. Thus the set of all manifold structures (up to equivalence) on the underlying homotopy type of X is parametrized exactly by this group of units.

Also note that a homotopy type occurs as that of a p-adic manifold precisely when there is a k-duality in the homotopy type : (see [S2] and [S3]).

At the prime 2 the manifold category is not as clear. To be sure the 2-adic manifolds have underlying homotopy types satisfying homological duality (coefficients $\hat{Z}_2$). Thus we have the natural (mod 2) characteristic classes of *Wu*.

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(1) We could reformulate this definition of a k-duality space at the prime p in terms of homological Poincaré duality and the existence of an orientation class in "periodic" K-homology. Using the connective k-theory seems more elegant and there is a natural cycle interpretation of k_* in terms of manifolds with signature free singularities. (see [S2])

We also note that the Pontryagin character of μ_X would be compatible with the rational characteristic class of a classical manifold determining X.

$$v_X = v_1 + v_2 + \ldots + v_i + \ldots \quad i \leqslant \frac{\dim X}{2}$$

where $v_i \in H^i(X, Z/2)$ is defined by duality and the Steenrod operations

$$v_i \cup x = S_q^i x \qquad \dim x + i = \dim X.$$

The square of this class only has terms in dimensions congruent to zero mod 4

$$v_2^2 + v_4^2 + v_6^2 + \ldots$$

and the "manifold structure" on X defines a lifting of this class to $\hat{Z}_2$ coefficients.

(1) $$\mathcal{L}_X = l_1 + l_2 + \ldots + l_i + \ldots \quad \text{in } H^{4*}(X, \hat{Z}_2),$$

The possible manifold structures on the homotopy type of X are acted on bijectively by a group constructed from the cohomology algebra of X. We take inhomogeneous cohomology classes,

$$u = u_2 + u_4 + u_6 + \ldots + u_{2i} + \ldots$$

using $Z/2$ or $\hat{Z}_2$ coefficients in dimensions congruent to 2 or 0 mod 4 respectively. We form a group G from such classes by calculating in the cohomology ring using the law,

$$u \cdot v = u + v + 8\,uv.$$

Note that G is the product of the various vector spaces of (mod 2) cohomology (in dimensions $4i + 2$) and the subgroup G_8 generated by inhomogeneous classes of $H^{4*}(X, \hat{Z}_2)$.

If we operate on the manifold structure of X by the element u in G_8 the characteristic class changes by the formula

$$\mathcal{L}_{Xu} = \mathcal{L}_X + 8\,u\,(1 + \mathcal{L}_X).$$

For example, the characterictic class mod 8 is a homotopy invariant.(see S3)

Local Categories

If l is a set of primes, we can form a local manifold category $\mathfrak{M}_l$, by constructing the fibre product

$$\mathfrak{M}_l \equiv \mathfrak{M}_Q \times_{\mathfrak{M}_A} \Big(\prod_{p \in l} \hat{\mathfrak{M}}_p \Big)$$

The objects in $\mathfrak{M}_l$ satisfy duality for homology over Z_l plus the additional manifold condition imposed at each prime in l and at Q.

For example we can form $\mathfrak{M}_2$ and $\mathfrak{M}_\theta$ the local categories corresponding to $l = \{2\}$ and $l = \theta = \{\text{odd primes}\}$. Then our original manifold category $\mathfrak{M}$ satisfies

$$\mathfrak{M} \cong \mathfrak{M}_2 \times_{\mathfrak{M}_Q} \mathfrak{M}_\theta$$

and we can say

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(1) Again, the Poincaré dual of $\mathcal{L}_X$ would be compatible with the rational characteristic class of a classical manifold determining X.

$\mathfrak{M}$ is built from $\mathfrak{M}_2$ and $\mathfrak{M}_\theta$ with coherences in $\mathfrak{M}_Q$

$\mathfrak{M}_2$ is defined by homological duality spaces over $Z_{(2)}$ satisfying certain homological conditions and having homological invariants (at 2).

$\mathfrak{M}_\theta$ is defined by homological duality spaces over $Z[1/2]$ with the extra structure of a $KO \otimes Z[1/2]$ orientation.

$\mathfrak{M}_Q$ is defined by homological duality spaces over Q with a rational charactéristic class.

Examples

(1) Let V be a polyhedron with the local homology properties of an oriented manifold with R coefficients. Then V satisfies homological duality for R coefficients.

If $R = Q$, the rational characteristic class can be constructed by transversality. (Thom) and we have a rational manifold

$$V \in \mathfrak{M}_Q$$

The Thom construction can be refined to give more information. The characteristic class l_V satisfies a canonical integrality condition. At 2 l_V can be lifted to an integral class. At $p > 2$ l_V can be lifted (via the Chern character) to a canonical K-homology class [S1].

So if V also satisfica Z/p – duality (p > 2) we have a k-duality space and a local manifold at odd primes, $V \in \mathfrak{M}_\theta$.

If V satisfies $Z/2$ – duality we have a good candidate for a manifold at 2. ($V \in \mathfrak{M}_2$?)

Note that such polyhedra are readily constructed by taking the orbit space of an action of a finite group π on a space W(1). For example if the transformations of π are orientation preserving then W/π is a Z/p homology manifold if W is and p is prime to the order of π. W/π is a Q homology manifold if W is.

(2) Now let V be a non-singular algebraic variety over an algebraically closed field k of characteristic p. Then the complete etale type of V determines a q-adic homological duality space at each prime q not equal to p (See [AM] and [S1]).

V has an algebraic tangent bundle T. Using the etale realization of the projective bundle of T one can construct a complex K-duality for V. To make this construction we have only to choose a generator μ_k of

$$H^1(k - \{0\}, \hat{Z}_q) \simeq \hat{Z}_q$$

This K-duality is transformed using the action of the Galois group to the appropriate (signature) duality in real K-theory, $q > 2$. If $\pi_1 V = 0$, we obtain a q-adic manifold for each $q \neq 2$ or $q \neq p$

$$[V] \in \hat{\mathfrak{M}}_q \quad (^2)$$

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(1) More generally with finite isotropy groups.

(2) The prime 2 can also be treated. [S3].

Now suppose that V is the reduction mod p of a variety in characteristic zero. Let $V_{\mathbf{C}}$ denote the manifold of complex points for some embedding of the new ground ring into $\mathbf{C}$

Of course $V_{\mathbf{C}}$ determines q-adic manifolds for each q, $[V_{\mathbf{C}}] \in \hat{\mathfrak{M}}_q$.

We have the following comparison. If μ_k corresponds to the natural generator of $H^1(\mathbf{C}-0, \hat{Z}_q)$ then

$$[V] \simeq [V_{\mathbf{C}}] \quad \text{in } \hat{\mathfrak{M}}_q.$$

The Galois symmetry

To construct the symmetry in the profinite manifold category $\mathfrak{M}$ we consider the primes separately.

For $p > 2$ we have the natural symmetry of the p-adic units $\hat{Z}_p^*$ in isomorphism classes in $\mathfrak{M}_p$. If M is defined by the homotopy type X with k-orientation μ_X, define M^α by X and the k-orientation μ_X^α using the Galois action of $\alpha \in \hat{Z}_p^*$ on k-theory.($q \in Z_p^*$ acts by the Adams operation ψ^q when q is an ordinary integer). Note that M and M^α have the same underlying homotopy type(1).

For $p = 2$ we proceed less directly. Let M be a manifold in $\mathfrak{M}_2$ with characteristic class $\mathcal{L}_M = l_1 + l_2 + \dots$. If $\alpha \in Z_2^*$ define $u_\alpha \in G_8(M)$ by the formula

$$1 + 8\,u_\alpha = \frac{1 + \alpha^2 l_1 + \alpha^4 l_2 + \dots}{1 + l_1 + l_2 + \dots}$$

ie

$$u_\alpha = \left(\frac{\alpha^2 - 1}{8}\right) l_1 + \left(\frac{\alpha^4 - 1}{8} l_2 + \frac{1 - \alpha^2}{8} l_1^2\right) + \dots$$

Define M^α by letting u_α act on the manifold structure of M. An interesting calculation shows that we have an action of $\hat{Z}_2^*$ on the isomorphism classes of 2-adic manifolds – again the underlying homotopy type stays fixed(1)

We have shown the

THEOREM. – *The profinite manifold category $\mathfrak{M}$ possesses the symmetry of the subfield of* $\mathbf{C}$ *generated by the roots of unity.*

The compatibility of this action of $\hat{Z}^*$ on $\mathfrak{M}$ with the Galois action on complex variéties discussed above is clear at $p > 2$, and at $p = 2$ up to the action of elements of order 8 in the underlying cohomology rings of the homotopy types. We hope to make the more precise calculation in [S3].

REFERENCES

[S1] SULLIVAN. — *Geometric Topology Part I*, Localization, Periodicity, Galois Symmetry, MIT preprint, Cambridge, Mass.

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(1) This connection proves the Adams conjecture for vector bundles ([S1] chapters 4 and 5), an extension to topological euclidean bundles (chapter 6), and finally an analogue in manifold theory [S3].

[AM] ARTIN and MAZUR. — Etale Homotopy, *Springer Verlag Lecture Notes.*
[B] BROWDER W. — *Surgery on simply connected manifolds,* a book to appear.
[N] NOVIKOV S.P. — Homotopy Equivalent Smooth Manifolds, 1964. *Translations AMS.*
[S2] SULLIVAN. — Geometric Periodicity and the Invariants of Manifolds, *Proceedings of the Amsterdam conference on Manifolds,* 1970.
[S3] SULLIVAN. — *Geometric Topology,* Part II... (to appear).

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C3 – GÉOMÉTRIE DIFFÉRENTIELLE

MANIFOLDS OF NONNEGATIVE CURVATURE

by Detlef GROMOLL

Let M be a complete riemannian manifold of dimension n. It is a classical problem to study geometrical and topological properties of M if the curvature is nonnegative. Here curvature means either the sectional curvature K or the Ricci (mean) curvature Ric.

Until a few years ago, the efforts of geometers had been directed almost entirely towards compact manifolds of positive curvature. Open manifolds seemed to be much less promising to deal with. This has rapidly changed during the last three years. It became apparent that the most significant aspect of the problem is a strong interaction of nonnegative sectional curvature and convexity in the large, which gave rise to some new techniques and surprising results. The structure of complete manifolds of nonnegative sectional curvature is now known to a very large extent, modulo the structure of compact manifolds of that type. Some of the new results even have generalizations in the case of nonnegative Ricci curvature. There are also applications in the compact case as far as the fundamental group is concerned.

The purpose of my talk was to outline these recent developments. I refer to joint work of J. Cheeger, resp. W. Meyer, and myself in [1], [2], [3], and [4]. All definitions, details, and references not given here explicitly can be found there.

A set $\emptyset \neq C \subset M$ is called totally convex if for all geodesics $c : [0,1] \to M$ with $c(0), c(1) \in C$, also $c[0,1] \subset C$. A point can only be totally convex in case M is contractible. A closed totally convex set C carries the structure of a k-dimensional topological submanifold of M with totally geodesic interior and not necessarily smooth boundary $\partial C, 0 \leqslant k \leqslant n$.

Let M be noncompact. Then given any $p \in M$, there is a ray $c : [0,\infty) \to M$ with $c(0) = p$. The union of all open metric balls of radius $t > 0$ centered at $c(t)$ is called the open half-space B_c with respect to c. Now let $K \geqslant 0$.

THEOREM 1 (Basic construction). – *For any half-space B_c, the complement $M - B_c$ is totally convex.*

The proof is based on a limiting argument involving Toponogov's angle comparison theorem for generalized triangles. The case $K > 0$ can be handled more easily by direct second variation techniques. Using Theorem 1 it is not hard to construct compact totally convex sets in M. Moreover :

THEOREM 2 (Expansion principle). – *There exists a continuous filtration of M by compact totally convex sets C_t, $t \geqslant 0$, such that whenever $t_1 \leqslant t_2$, C_{t_1} is* *the subset of all points in* C_{t_2} *having distance at least* $t_2 - t_1$ *from the boundary* ∂C_{t_2}.

Doing the basic construction at an arbitrary point p, one may choose $C_t = \cap (M - B_{c_t})$, where the intersection is taken over all rays emanating from p, and c_t is the restricted ray with $c_t(s) = c(t + s)$. The structure of compact totally convex sets can be studied by means of the following result.

THEOREM 3 (Contraction principle). — *Let C be compact totally convex,* $\partial C \neq \emptyset$. *Then the set* C^a *of all points in C at distance* $\geqslant a \geqslant 0$ *from the boundary* ∂C *is totally convex. In particular, the set of furthest points* $C^{\max} = \cap C^a$ *(intersection over all a with* $C^a \neq \emptyset$*) is totally convex, and* $\dim C^{\max} < \dim C$. *But* $C^{\max}$ *may have boundary again.*

The argument uses Rauch comparison techniques. In a certain sense, the expansion principle may be viewed as an extension of the contraction principle for the noncompact manifold M having convex boundary at infinity. Both principles together have very strong implications.

THEOREM 4. — *There exists a compact totally convex submanifold S of M without boundary, the soul of M, such that M is diffeomorphic to the normal bundle* $\nu(S)$ *of S in M. If* $K > 0$, *then S is a point and M is diffeomorphic to euclidean space* $\mathbf{R}^n$.

The exponential map $\nu(S) \to M$ is not a diffeomorphism in general. The soul of M need not be unique. Up to diffeomorphism, complete manifolds of nonnegative curvature are vector bundles over compact manifolds of nonnegative curvature. For some time we had conjectured that M might even be a locally isometrically trivial bundle over its soul S, so in particular, $\nu(S)$ would be flat. But there seem to be nontrivial counterexamples now. The rigidity conjecture is true in many interesting cases, say when S has dimension or codimension 1, or when M is locally homogeneous. Very recently we were able to show in addition : If M is diffeomorphic to $S^2 \times \mathbf{R}^2$, then M is an isometric product of its soul S and some complete open surface of nonnegative curvature. In dimensions $\leqslant 3$, complete open manifolds with $K \geqslant 0$ can be classified up to isometry.

We mention some further applications. The global behavior of geodesics in M can be described, though not completely yet. For example, if $K > 0$, then for any $p \in M$, the exponential map $\exp_p : M_p \to M$ is proper. Both branches of every nonconstant geodesic $c : \mathbf{R} \to M$ go to infinity. M does not contain lines. If $K \geqslant 0$, then every line splits off M isometrically as a factor (Toponogov ; [5]), which is an easy conclusion within our methods. Hence :

THEOREM 5. — *There is a unique isometric decomposition* $M = M_0 \times \mathbf{R}^k$, *where* M_0 *does not contain a line and* $\mathbf{R}^k$ *is flat euclidean space.*

Using this result and some arguments derived from an equivariant basic construction we obtain information about the isometry group $I(M)$ of M.

THEOREM 6. — $I(M) = I(M_0) \times I(\mathbf{R}^k)$, *where* $I(M_0)$ *is compact.*

$I(M)$ is always compact if $K > 0$ and has a fixed point. To study the fundamental group π of a complete manifold M of nonnegative curvature, it suffices to consider the compact case, in view of Theorem 4. π acts as a group of isometries on the universal riemannian covering $\widetilde{M}$ of M. By compactness of M there exist lines in $\widetilde{M}$ when π is infinite. The main results are :

THEOREM 7. — *There is a diagram of covering maps*

$$\begin{array}{ccccc} M_0 & \to & \widetilde{M} = M_0 \times \mathbf{R}^k & \to & \mathbf{R}^k \\ \downarrow & & \downarrow & & \downarrow \\ M_1 & \to & M & \longrightarrow & T^k \\ & & \downarrow & \searrow & \\ & & M & & M_1 \times T^k \end{array}$$

where the vertical maps are isometric coverings, the horizontal maps locally isometrically trivial fibrations, and the diagonal map is a diffeomorphism. T^k is a flat torus.

THEOREM 8. — *There exists an invariant finite subgroup $\varphi \subset \pi$ such that $\pi^* = \pi/\varphi$ is isomorphic to a crystallographic group, so π^* contains a free abelian normal subgroup Γ of rank k, $0 \leqslant k \leqslant \dim M$, π^*/Γ finite.*

We give only one immediate corollary : A compact $K(\pi, 1)$ – manifold with $K \geqslant 0$ is flat.

We conclude with some remarks on complete manifolds M of nonnegative Ricci curvature. The interesting fact is that the splitting Theorem 5 and all direct consequences remain true. In particular, the structure Theorem 8 for the fundamental group of M holds, if M is compact with $\mathrm{Ric} \geqslant 0$. The crucial step was inspired by the basic construction in Theorem 1.

THEOREM 9. — *Let $c : [0, \infty) \to M$ be a ray. The functions g_t on M with*

$$g_t(p) = \rho(p, c(t)) - t$$

converge pointwise to a continuous function g on M for $t \to \infty$. Now g is superharmonic.

The proof combines geometric and analytic arguments and is not easy.

Our last application concerns the holonomy group φ of an arbitrary compact riemannian manifold M. If in the de Rham decomposition of the universal riemannian covering $\widetilde{M} = \hat{M} \times \mathbf{R}^k$ either $\hat{M}$ compact or $k \leqslant 1$, then φ is compact. The classical results had some gap related to questions about manifolds of zero Ricci curvature.

REFERENCES

[1] CHEEGER J. and GROMOLL D. — The structure of complete riemannian manifolds of nonnegative curvature, *Bull. A.M.S.*, 74, 1968, p. 1147-1150.

[2] CHEEGER J. and GROMOLL D. — On the structure of complete manifolds of nonnegative curvature, Preprint (1970), State University of New York at Stony Brook; to appear in *Ann. of Math.*

[3] CHEEGER J. and GROMOLL D. — The splitting theorem for manifolds of nonnegative Ricci curvature, Preprint (1970), State University of New York at Stony Brook; to appear in *J. of Differential Geometry*.
[4] GROMOLL D. and MEYER W. — On complete open manifolds of positive curvature, *Ann. of Math.*, 90, 1969, p. 75-90.
[5] TOPONOGOV V. — Spaces with straight lines, *A.M.S. Translations*, 37, 1964, p. 287-290.

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PSEUDO-DISTANCES INTRINSÈQUES SUR LES ESPACES COMPLEXES

by Shoshichi KOBAYASHI

Rappelons d'abord le lemme classique de Schwarz-Pick. Soit D le disque de rayon unité, $D = \{z \in \mathbf{C} \; ; \; |z| < 1\}$, muni de la métrique de Poincaré-Bergman

$$ds^2 = \frac{dz\, d\bar{z}}{(1 - |z|^2)^2} \tag{1}$$

de courbure -4. Alors toute application holomorphe $f : D \to D$ est décroissante pour la métrique ds^2, c'est-à-dire,

$$f^* ds^2 \leqq ds^2 \, . \tag{2}$$

Si l'on désigne par ρ la distance définie par ds^2 dans D, l'inégalité (2) est équivalente à

$$\rho(f(a), f(b)) \leqq \rho(a, b) \qquad a, b \in D. \tag{3}$$

C'est M. Ahlfors qui a démasqué le caractère vraiment géométrique du lemme de Schwarz-Pick en démontrant en 1938 la généralisation suivante :

Soit M une surface de Riemann munie d'une métrique $ds_M^2 = 2g dw d\overline{w}$ de courbure $k \leqq -4$. Alors toute application holomorphe $f : D \to M$ satisfait à l'inégalité

$$f^* ds_M^2 \leqq ds^2 . \tag{4}$$

D'autre part, Carathéodory a trouvé en 1926 une autre généralisation du lemme de Schwarz-Pick. Soit M un domaine borné dans $\mathbf{C}^n$. Carathéodory n'est parti d'aucune métrique donnée sur M, mais a construit une distance intrinsèque c_M sur M (que l'on appelle *la distance de Carathéodory*) telle que toute application holomorphe $f : M \to D$ satisfasse à

$$c_M(p, q) \geqq \rho(f(p), f(q)) \qquad p, q \in M. \tag{5}$$

Soit F la famille des applications holomorphes $f : M \to D$. Alors la distance c_M est définie par

$$c_M(p, q) = \sup_{f \in F} \rho(f(p), f(q)). \tag{6}$$

Cette définition s'applique évidemment à tous les espaces complexes M. Si M est un espace complexe quelconque, c_M est seulement une pseudo-distance ; car $c_M(p, q) = 0$ n'implique pas nécessairement $p = q$. En particulier, si M est compact, on a $c_M(p, q) = 0$ pour $p, p \in M$, (d'après le principe du maximum). Pour le plan complexe C, on a aussi $c_{\mathbf{C}}(p, q) = 0$ identiquement d'après le théorème de Liouville). La pseudo-distance de Carathéodory possède les deux propriétés suivantes :

(7) $$c_D = \rho \ ;$$

(8) Si $f : D \to X$ est une application holomorphe, on a

$$c_Y(f(p), f(q)) \leqq c_X(p, q) \qquad p, q \in X.$$

Tandis que (7) est équivalente au lemme classique de Schwarz-Pick, (8) est une conséquence immédiate de la définition (6). Le lemme de Schwarz-Pick généralisé (8) s'applique aux applications holomorphes entre espaces complexes arbitraires. Cependant c_X n'est qu'une *pseudo*-distance pour la plupart des espaces complexes X.

En réunissant l'idée de Carathéodory et celle de Ahlfors, nous allons maintenant construire une pseudo-distance d_X sur X qui sera caractérisée par les trois propriétés suivantes :

(9) $$d_D = \rho \ ;$$

(10) Si $f : D \to X$ est une application holomorphe, on a

$$d_D(a, b) \geqq d_X(f(a), f(b)) \qquad a, b \in D \ ;$$

(11) d_X est la pseudo-distance la plus grande possédant les propriétés (9) et (10).

Etant donnés deux points p et q de X, nous choisissons des points

$$p = p_0, p_1, \ldots, p_{k-1}, p_k = q$$

de X, des points $a_1, \ldots, a_k, b_1, \ldots, b_k$ de D et des applications holomorphes $f_1, \ldots, f_k$ de D dans X satisfaisant aux conditions suivantes :

(12) $$p_0 = f_1(a_1), p_1 = f_1(b_1) = f_2(a_2), \ldots,$$
$$p_{k-1} = f_{k-1}(b_{k-1}) = f_k(a_k), p_k = f_k(b_k).$$

(Nous pouvons considérer $f_1(D), \ldots, f_k(D)$ comme des disques holomorphes qui joignent p à q). La distance $d_X(p, q)$ est alors définie par

(13) $$d_X(p, q) = \inf \sum_{i=1}^{k} \rho(a_i, b_i),$$

où l'infimum est pris sur tous les choix possible de points et d'applications. On voit immédiatement que

(14) si $f : X \to Y$ est une application holomorphe, on a

$$d_Y(f(p), f(q)) \leqq d_X(q, q) \qquad p, q \in X.$$

Cette définition de d_X ressemble à celle de la distance sur une variété riemannienne. (M. Roydon a annoncé récemment que d_X peut être définie par une métrique finslérienne pris dans le sens le plus général de la même manière qu'est définie la distance riemannienne). A cause de cette ressemblance, la pseudo-distance d_X possède de nombreuses propriétés des distances riemanniennes. Par exemple,

(15) Si $\pi : \widetilde{X} \to X$ est un revêtement de X, on a

$$d_X(p\,,q) = d_X(\pi^{-1}(p)\,,(\pi^{-1}(q)) \qquad p\,,q \in X\,,$$

(16) Dans le cas où d_X est une distance, X est complet dans le sens de Cauchy si est seulement si tout ensemble fermé et borné dans X est compact.

Nous disons qu'un espace complexe X est *hyperbolique* si d_X est une (vraie) distance et qu'il est *hyperbolique-complet* si d_X est une distance complète. Voici quelques exemples d'espaces complexes hyperboliques.

(i) *Une variété hermitienne X à courbure holomorphe sectionnelle $\leqq -A < 0$ est hyperbolique. Si la métrique hermitienne est complète, X est hyperbolique-complet.* En particulier une surface de Riemann compacte de genre $\geqq 2$ ou $X = \mathbf{C} - \{2 \text{ points}\}$ est hyperbolique-complète.

(ii) Puisque $d_X \geqq c_X$, *tout domaine borné X dans $\mathbf{C}^n$ est hyperbolique.*

(iii) *Si $\widetilde{X}$ est un revêtement de X, il suit de* (15) *que $\widetilde{X}$ est hyperbolique (-complet) si est seulement si X l'est.*

(iv) *Un sous-espace complexe (fermé) d'un espace hyperbolique (-complet) est hyperbolique (-complet)*, (c'est une conséquence de (14))

(v) M. Royden a montré que *l'espace de Teichmüller $X = T^g$ des surfaces de Riemann compactes de genre $g \geqq 2$ est hyperbolique-complet.* En effet, il a montré que *la distance d_X coïncide avec la distance de Teichmüller sur $X = T^g$.*

Par contre, $d_{\mathbf{C}}$ s'annule identiquement. *Si f est une application holomorphe du plan complexe $\mathbf{C}$ dans un espace complexe hyperbolique X, f est une application constante d'après* (14). En particulier, une application $f : \mathbf{C} \to \mathbf{C} - \{2 \text{ points}\}$ se réduit à une application constante, (c'est le petit théorème de Picard).

Le grand théorème de Picard établit que si une fonction holomorphe dans le disque épointé $D^* = \{0 < |z| < 1\}$ a deux valeurs lacunaires, elle est méromorphe dans $D = \{|z| < 1\}$. Je préfère l'interpréter comme un théorème d'extension d'applications holomorphes. C'est-à-dire, si f est une application holomorphe de D dans $P_1(\mathbf{C})$. Voici notre généralisation du grand théorème de Picard. *Soit M un espace complexe hyperbolique contenu dans un espace complexe Y et tel que $\overline{M}$ soit compact. Soit X une variété complexe sans singularité et A une sous-variété sans singularité. Supposons que, pour tout point $p \in \partial M$ $(= \overline{M} - M)$ et pour tout voisinage U de p dans Y, il y ait un voisinage $V \subset U$ de p tel que $d_M(V \cap M, (Y - U) \cap M) > 0$. Alors toute application holomorphe $f : X - A \to M$ s'étend à une application holomorphe $f : X \to Y$.* En particulier, on en déduit, un résultat de Kwack : toute application holomorphe f de $X - A$ dans un espace complexe hyperbolique compact M s'étend en une application holomorphe $f : X \to M$.

Citons deux applications de notre généralisation du grand théorème de Picard :

(i) *$Y = P$ (C) et $M = P$ (C) $- Q$, où Q est un quadrilatère complet.* (Par un quadrilatère complet, on entend la réunion des six lignes complexes joignant quatre points indépendants dans $P_2(\mathbf{C})$).

(ii) *Soit $\mathcal{D}$ un domaine borné symétrique et Γ un groupe arithmétique discontinu d'automorphismes de $\mathcal{D}$* (ou, un peu plus généralement, un groupe normal discontinu dans le sens de Pyatetzkii-Shapiro). *Soit $M = \mathcal{D}/\Gamma$, et Y le compactifié de M dans le sens de Satake.* Par soucis de brièveté, supposons que tout élément de Γ ayant des points fixes soit l'identité. En utilisant la théorie de Pyatetzkii-Shapiro sur les domaines de Siegel du troisième type et la distance $d_{\mathcal{D}}$, on peut montrer que la paire (M, Y) satisfait aux conditions du grand théorème de Picard généralisé (résultat d'Ochiai et de l'auteur). On obtient ainsi une démonstration simple d'un résultat de Borel (non-publié) sur l'extension d'une application $D^* \to M \subset Y$.

Au lieu de donner d'autres applications de pseudo-distances intrinsèques, nous allons esquisser une théorie semblable pour des éléments de volume et des mesures. Soit M une variété complexe de dimension n possédant un élément de volume $v_M = K \,.\, i^n dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n$, où $z^1, \ldots, z^n$ est un système de coordonnées locales de M et K une fonction positive. A cet élément de volume v_M est associée une forme hermitienne

$$(17) \qquad \varphi = \Sigma\, i\, h_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta, \quad \text{où} \quad h_{\alpha\bar{\beta}} = \partial^2 \log K/\partial z^\alpha \partial \bar{z}^\beta .$$

Supposons que $(h_{\alpha\bar{\beta}})$ soit définie positive et que $(\varphi^n/v_M) \geqq c > 0$. Sous cette hypothèse, nous avons le lemme de Schwarz-Pick généralisé comme suit. Si v_{D_n} est un élément de volume invariant dans une boule D_n de rayon unité de $\mathbf{C}^n$, on a

$$(18) \qquad f^* v_M \leqq a \,.\, v_{D_n}$$

pour toute application holomorphe $f : D_n \to M$. On peut toujours normaliser v_M de sorte que $a = 1$, c'est-à-dire, f est décroissante en volume. Par exemple, une variété complexe compacte M, à première classe de Chern négative (c'est-à-dire, à fibre canonique ample), possède un élément de volume v_M satisfaisant aux conditions ci-dessus. On en déduit que *si A est un sous-espace complexe d'une variété complexe X de dimension n et si f est une application holomorphe, non identiquement dégénérée, de $X - A$ dans une variété complexe compacte M* à $c_1(M) < 0$, *f s'étend en une application méromorphe de X dans M,* (résultat d'Ochiai et de l'auteur). Cela est aussi un théorème de type du grand théorème de Picard. Le lemme de Schwarz-Pick est valable dans le cas un peu plus général où l'élément de volume v_M est de la forme $v_M = |\alpha|^2 K \,.\, i^n dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n$, où K est une fonction positive et α est une fonction holomorphe éventuellement avec zéros. Dans ce cas, la forme φ est encore définie par (17) et on a l'inégalité (18). *Cette généralisation s'applique aux variétés algébriques de type général dans le sens de Kodaira,* c'est-à-dire, à presque toutes les variétés algébriques. On en déduit le théorème de type de Picard pour ces variétés aussi.

De la même manière qu'on a construit la pseudo-distance d_M, on peut définir une mesure intrinsèque μ_M sur M. Soit B un ensemble borélien dans M. Choisissons des ensembles boréliens E_i dans la boule $D_n \subset \mathbf{C}^n$ et des applications holomorphes $f_i : D_n \to M$ de manière que $\bigcup_{i=1}^{k} f_i(E_i) \supset B$. La mesure $\mu_M(B)$ est définie par

$$\mu_M(B) = \inf \sum_{i=1}^{k} \mu(E_i),$$

où μ désigne la mesure définie par la métrique invariante de Bergman dans D_n. Nous disons qu'une variété M est *hyperbolique en mesure* si $\mu_M(B) > 0$ pour tout ouvert B non-vide de M.

Un analogue du petit théorème de Picard dit que *toute application holomorphe* f de $\mathbf{C}^n$ *dans une variété M hyperbolique en mesure et de dimension n est partout dégénérée.*

Du lemme de Schwarz-Pick généralisé il résulte que *toute variété algébrique de type général est hyperbolique en mesure.*

La classe des variétés hyperboliques contient toutes les variétés hermitiennes à courbure holomorphe sectionnelle assez négative ainsi que tous les quotients $\mathcal{D}/\Gamma$ de domaines bornés par des groupes discontinus opérant librement. La classe des variétés hyperboliques en mesure contient toutes les variétés hermitiennes à courbure de Ricci assez négative, toutes les variétés algébriques de type général et toutes les variétés hyperboliques. J'espère que les distances et les mesures intrinsèques seront utiles en géométrie algébrique et en géométrie différentielle ainsi que dans la théorie géométrique des fonctions.

Pour plus de détails et des références, voir ma monographie *"Hyperbolic manifolds and holomorphic mappings"*, (1970) ; Marcel Dekker, New-York.

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THE RIGIDITY OF LOCALLY SYMMETRIC SPACES

by G. D. MOSTOW *

1. Introduction

I shall outline in this talk a proof of the following rigidity theorem.

THEOREM. – *Let X and X' be simply connected symmetric Riemannian spaces of negative curvature having no factors of rank one. Let Γ and Γ' be discrete subgroups of isometries on X and X' respectively such that $\Gamma\backslash X$ and $\Gamma'\backslash X'$ are compact. If Γ and Γ' are isomorphic, then $\Gamma\backslash X$ and $\Gamma'\backslash X'$ are isometric with respect to suitably selected invariant metrics on X and X'.*

The conclusion of this theorem is equivalent to the assertion :

The isomorphism $\theta : \Gamma \to \Gamma'$ *extends to an analytic isomorphism* of the group G of isometries of X onto the group G' of isometries of X'.

The principal strategy of our proof pursues an idea first introduced in [4d] and employed in [4e] to treat the case of real hyperbolic space of dimension greater than 2. (In dimension 2 it is false). The idea is to find a homotopy equivalence $\overline{\varphi}$ of $\Gamma\backslash X$ into $\Gamma'\backslash X'$, to lift $\overline{\varphi}$ to a map $\varphi : X \to X$, and then to study φ at infinity. That is, the symmetric space X has a compactification $\overline{X}$ introduced independently by Satake ([5]) and Furstenberg ([2]) on which G operates, and which is a finite union of G-orbits. In any such compactification X, there is in $\overline{X} - X$ a unique compact G-orbit X_0. Now $\overline{X}$ and X_0 are not unique, but there is a certain maximal compactification for which the stabilizer of a point in X_0 is a minimal parabolic subgroup. This X_0 has been called by Furstenberg the *maximal boundary* of X. In [4d] I proved in case $X' = X$:

If the map φ induces a *differentiable* map $\varphi_0 : X_0 \to X_0$, then $\theta : \Gamma \to \Gamma'$ extends to an analytic automorphism of G. There are no restrictions on the rank of the rank of the factors for this result.

On the other hand, for the case of real hyperbolic space, if the map

$$\varphi : \Gamma\backslash X \to \Gamma'\backslash X$$

is a diffeomorphism, it was possible to show that φ_0 exists and is in fact analytic (cf. [4e].

In the case at hand, the proof is divided into two diverse steps.

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(I) The proof that φ_0 exists and is a homeomorphism.

(II) The proof that $\varphi_0 \circ G \circ \varphi_0^{-1} = G'$ where we have identified G with its action on the maximal boundary X_0.

The proof of part I rests on the introduction of a new class of mappings called *pseudo-isometries* (cf. Section 5) and rests on a study of pseudo-isometries of flat spaces into symmetric spaces.

The proof of part II rests on the study of the action of Γ on X_0. The special fixed point properties of the elements of G operating on X_0, together with various density properties of Γ permit us to prove that φ_0 sends orbits of parabolic groups to orbits of parabolic groups. This leads us into a situation that has been axiomatized by Tits, and the desired conclusion for φ_0 rests on Tits' generalization to the geometry of parabolic subgroups of a theorem of W.L. Chow generalizing the fundamental theorem of projective geometry.

Any mapping of real projective space taking planes to planes is a projective mapping.

The weakening of the hypothesis that $\Gamma \backslash X$ is compact to the assumption that $\Gamma \backslash X$ has finite measure is discussed in our concluding remarks (Section 13).

2. Preliminaries

Let X be a simply connected symmetric Riemannian space and let G be the connected component of the identity in the group of isometries of X. To say that X is of negative curvature is equivalent to assuming that G is semi-simple and has no compact factors. *That is the case that we shall deal with exclusively.* In that case, G has no center. Let K be the stabilizer of a point in G. Then K is a maximal compact subgroup of G, and we can identify X with G/K. The group G has a faithful linear representation - the adjoint representation for example, and we can choose coordinates in a representation space for G so as to have

$$G = G \cap P(n, \mathbf{R}) \times G \cap 0(n, \mathbf{R})$$

where $P(n, \mathbf{R})$ denotes the space of positive definite real hermitian $n \times n$ matrices, and $K = G \cap 0(n, \mathbf{R})$. The map $\mu : g \to g^t g$ can be identified with the canonical projection of G onto G/K. Set $P = G \cap P(n, \mathbf{R})$. The tangent space to X at $\mu(1)$ can be identified with the tangent space P_1 to P at 1.

The invariant metric on P is given by

$$ds^2 = Tr(p^{-1}\dot{p})^2$$

and this in turn gives an invariant metric on X. The space X has no product decomposition if and only if G does not; that is to say G is simple if X is irreducible. In that case, any invariant metric is unique up to a constant factor.

The image under μ of a maximal *abelian* subgroup A of P is a maximal *flat* subspace F of G. Their common dimension is called the **R**-*rank* of G and also the *rank* of X. We call A and F *r-flats* in G and X respectively, and any image of these under an automorphism is also called an r-flat.

We consider the adjoint action of A on the Lie algebra $\dot{G}$ of G. Then Ad A is diagonalizable and its irreducible one dimensional subrepresentations are called the *roots* on A. The set of roots Σ forms a root system just as in the case of roots on Cartan subalgebras of complex semi-simple Lie algebras, except that 2α may be a root if α is. In any case, there is a fundamental system of r roots Δ such that $\Sigma = \Sigma^+ \cup (-\Sigma^+)$ where Σ^+ lies in free abelian semi-group generated by Δ. We call the r roots in Σ^+ *positive* roots. If $\Delta_1 \subset \Delta$, we denote by $\{\Delta_1\}$ the set of linear combinations of Δ_1.

The connected components of $A - \bigcup_{\alpha\in\Sigma} \operatorname{Ker}\alpha$ are called *chambers*. The subgroup $W = N(A)/Z(A)$ opérates simply transitively on the chambers, where $N(\ )$ denotes normalizer and $Z(\ )$ denotes centralizer in G.

The image of a chamber in A under $\mu : G \to X$ is called a chamber in X with origin $\mu(1)$. Geodesics in X lying on a wall of a chamber are called *singular* geodesics. They can be described equally well as geodesics lying in more than one r-flat. The roots of G have a simple interpretation is in terms of the curvature tensor R of X : for any Y_1, Y_2, Y_3 in P_1

$$R(Y_1, Y_2, Y_3) = -[[Y_1, Y_2], Y_3]$$

The image of the r-flat A under conjugations by K covers P. Correspondingly $KF = X$ for any r-flat F and the stabilizer K of a point in F.

Let A' be a chamber in A. Then $K[A'] = (K/M) \times A'$ (direct) where $x[y]$ denotes xyx^{-1}, and $M = Z(A) \cap K$.

Let F' be a chamber in X with origin x_0, and let K denote the stabilizer of x_0. The map $(K/M) \times F' \to KF'$ given by $(k, x) \to kx$ is called *orbital coordinates* with respect ot x_0 and F'.

The metric on X can be represented in orbital coordinates as :

$$ds^2 = \sum_\alpha \sinh^2 u_\alpha \, d\theta_\alpha^2 + da^2$$

where we identify A' with F' via μ, and $u_\alpha = \log \alpha(h)$ for $h \in A'$, and $d\theta_\alpha^2$ is a K-invariant quadratic differential on K/M with support in the K-orbit of the α^2-eigenspaces of $(\operatorname{ad} \log h)^2$.

There is also another type of subgroup that we shall consider. Let Δ_1 be a subset of the fundamental system of roots Δ. We set

$$\Delta_1^\perp = \bigcap_{\alpha\in\Delta_1} \operatorname{Ker}\alpha\ ,\ G(\Delta_1) = Z(\Delta_1^\perp)\ ,\ N(\Delta_1) = \prod_{\substack{\alpha\notin\Delta_1 \\ \alpha>0}} G_\alpha$$

where G_α is the analytic group whose Lie algebra is the set of eigenvectors belongs to α,

$$P(\Delta_1) = G(\Delta_1)\, N(\Delta_1)\,.$$

Then $N(\Delta_1)$ is a unipotent group, that is its elements are unipotent in any representation of G. $P(\Delta_1)$ is the normalizer of $N(\Delta_1)$ and $G = K P(\Delta_1)$. In particular $G/P(\Delta_1)$ is compact. We shall call any subgroup of G conjugate to $P(\Delta_1)$ for some $\Delta_1 \subset \Delta$ a *parabolic* subgroup.

3. Polar decomposition

Let g be an invertible real matrix. Then there exist unique commuting elements m, s, u such that their eigenvalues are respectively of modulus 1, positive, and equal to 1, and such that m and s are semi-simple (that is diagonalizable over C). One calls ms and u the semi-simple and unipotent *Jordan* parts of g. We call s the *polar* part of g and denote it by pol g. If G is an algebraic linear group, then the connected component of the identity contains pol g whenever G contains g. In particular, semi-simple analytic linear groups have this property.

An element g of G is called **R**-*split* if its semi-simple part is pol g ; it is called *polar regular* if $\dim Z(\mathrm{pol}\, g) \leqslant \dim Z(\mathrm{pol}\, x)$ for all $x \in G$.

LEMMA 3.1. – An element of G is polar regular if and only if it leaves invariant a unique r-flat in X. For suitable choice of an invariant metric on X we have for all $g \in G$

$$\inf_{x \in X} d(x, gx)^2 = \mathrm{Tr}\,(\log \mathrm{pol}\, g)^2$$

4. The maximal boundary X_0

As noted above, we can assume that $G = G \cap P(n, \mathbf{R}) \times G \cap O(n, \mathbf{R})$, and thus X may be identified with the subset $P = G \cap P(n, \mathbf{R})$. Let $\mathfrak{S}$ denote the linear space of all $n \times n$ real matrices, and let β denote the projection of $\mathfrak{S} - (0)$ onto the projection space $[\mathfrak{S}]$ of lines through the origin. Then the map $\beta \circ \mu$ yields an injection of X into $[\mathfrak{S}]$. The G-action on X goes into the action

$$p \to g p\, {}^t g$$

which is linear on $\mathfrak{S}$ and thus passes to an action on $[\mathfrak{S}]$. The closure of $\beta \circ \mu(X)$ is denoted $\bar{X}$. It is the *Furstenberg-Satake compactification* of X([2], [5]). (It depends on the representation of G as a linear group.) The function $\bar{X} - X$ consists of a finite number of G orbits and among these, there is a *unique* compact G-orbit, which we denote by X_0 ; it is also characterized as the orbit of least dimension. The stabilizer of a point in X_0 is a parabolic subgroup. For a suitable representation of G, the stabilizer is a minimal parabolic subgroup P. It that is the case, X_0 is called the *maximal* boundary. The maximal boundary can be defined intriniscally as follows :

Let $\mathfrak{X}_0$ denote the set of chambers in X with arbitrary origins. Define two chambers F_1 and F_2 to be equivalent if and only if $\delta(F_1, F_2) < \infty$, where δ denotes Hausdorff distance. Let X_0 denote the quotient of $\mathfrak{X}_0$ by this equivalence relation. We take as topology on X_0 the quotient of convergence in compact sets of X. A set of representatives for X_0 is provided by the set of all chambers with common origin. Thus $X_0 = K F'$ where F' is a chamber with origin fixed by K, and $X_0 = K/M = G/MAN = G/P$ where $M = Z(A) \cap K$ and $N = N(\phi)$ and $P = P(\phi)$.

5. **Pseudo-isometries**

DEFINITION. – Let X and X' be metric spaces. Let k and b be non-negative numbers, A continuous map $\varphi : X \to X'$ is called a (k, b) pseudo-isometry if

(1) $$d(\varphi(x), \varphi(y)) \leqslant k\, d(x, y) \quad \text{for all} \quad x, y \text{ in } X$$

(2) $$d(\varphi(x), \varphi(y)) \geqslant k^{-1}\, d(x, y) \quad \text{if} \quad d(x, y) \geqslant b.$$

A *pseudo-isometry* is a map which is a (k, b) pseudo-isometry for some k and b.

LEMMA 5.1. – Let G and G' be semi-simple analytic groups without compact factors, and let Γ and Γ' be discrete subgroups such that G/Γ and G/Γ' are compact. Let $\theta : \Gamma \to \Gamma'$ be an isomorphism. Let X and X' be the associated symmetric spaces. Assume Γ and Γ' have no elements of finite order. Then there is a pseudo-isometry $\varphi : X \to X'$ equivariant with respect to θ. Moreover, let F and F' be r-flats such that $\Gamma\backslash\Gamma F$ and $\Gamma'\backslash\Gamma' F'$ are compact. Then φ may be selected so as to be linear on F.

1. **Some inequalities**

Let F be an r-flat in X and let $x \in F$. Let $F_x^{\perp}$ denote the union of geodesics through x perpendicular to F at x. Then each point of X lies in a unique set $F_x^{\perp}$ [4b]. Let $\pi : X \to F$ denote the map sending each point of F_x into x.

For any $p \in X$, let $\dot{X}_p$ denote the tangent space to X at p, and $\dot{\pi}_p : \dot{X}_p \to \dot{F}_{\pi(p)}$ the differential of π at p.

LEMMA 6.1. – Let $p \in X$. Let Y be the unique element of $\dot{P}_1$ such that exp $Y(\pi(p)) = p$. Let $C \in X_p$ and set $A = \dot{\pi}_p(C)$.

Then

$$|C|^2 \geqslant \Sigma\, w_i A_i^2$$

where $A = \sum_i A_i \eta_i$, the set of tangent vectors $\eta_1, \ldots, \eta_n$ in $X_{\pi(p)}$ is an orthonormal set of vectors for $(\text{ad } Y)^2$, $w_i = \nu_i \cosh \nu_i/2\ (2 \sinh \nu_i/2)^{-1}$ and $(\text{ad } Y)^2\ \eta_i = \nu_i^2\, \eta_i\ (i = 1, \ldots, n)$.

LEMMA 6.2. – Continue the above notation. Then

$$\frac{|C|}{|A|} \geqslant (2n)^{-1/4} |Y|^{1/2} \left\| \left[\frac{Y}{|Y|}, \frac{A}{|A|} \right] \right\|$$

This lemma implies that if Y is large and $|A| / |C|$ not small, then A lies close to a singular element A' and Y lies close to the centralizer of A' (identified with and element in the Lie algebra of G)

7. Application to Γ-compact *r*-flats

Let Γ be a discrete subgroup of G. An r-flat F of X (resp. A of G) is called Γ-compact if $\Gamma\backslash\Gamma F$ (resp. $\Gamma\backslash\Gamma A$) is compact. This is the case if and only if there is a free abelian subgroup Δ in Γ of rank r which keeps F invariant and such that $\Delta\backslash\Delta F$ is compact.

LEMMA 7.1. – Let X and X' be symmetric spaces, Γ and Γ' discrete subgroups of isometries on X and X', $\theta : \Gamma \to \Gamma'$ an isomorphism, and $\varphi : X \to X'$ a pseudo-isometry equivariant with respect to θ. Then there is a number ν_0 such that :

For any free abelian subgroup Δ of Γ and for any Δ-compact r-flat F, $\varphi(F)$ lies within a distance ν_0 of a flat space F' which is invariant under $\theta(\Delta)$.

LEMMA 7.2. – Let X_i be a symmetric space of rank r, Δ_i a free abelian group of isometries of X_i, F_i a Δ_i-compact r-flat in $X_i (i = 1, 2)$. Let $\theta : \Delta_1 \to \Delta_2$ be an isomorphism and $\varphi : X_1 \to X_2$ a pseudo-isometry which carries F_1 linearly into F_2. Then φ sends singular geodesics of F_1 to singular geodesics of F_2.

The proof is based on a chain of consequences of Lemma 6.2.

LEMMA 7.3. – Let G_i be a semi-simple analytic group, Γ_i a discrete subgroup such that $\Gamma_i\backslash G_i$ is compact, Δ_i an abelian group of rank r in Γ, pol Δ_i the set of polar parts of elements of Δ_i, and j_i the canonical injection of pol $\Delta_i \otimes_{\mathbf{R}} \mathbf{R}$ onto the r-flats A_i containing pol Δ_i ($i = 1, 2$). Let $\theta : \Gamma_1 \to \Gamma_2$ be an isomorphism and let θ denote the induced map of pol $\Delta_1 \otimes_{\mathbf{R}} \mathbf{R}$ to pol $\Delta_2 \otimes_{\mathbf{R}} \mathbf{R}$. Then $j_2\ \theta\ j^{-1}$ sends singular elements of A_1 to singular elements of A_2.

The proof of Lemma 7.3 comes from constructing a pseudo-isometry of the associated symmetric spaces X_1 onto X_2 sending F_1 linearly onto F_2 as in Lemma 5.1, and then applying Lemma 7.2.

8. Density properties of Γ

LEMMA 8.1. – Let X be a symmetric space of rank r and let Γ be a discrete group of isometries on X such that $\Gamma\backslash X$ is compact. Let B be a ball of positive radius in X. Then the union of all Γ-compact r-flats meeting B is dense in X.

This lemma is based on a conjugacy theorem for polar regular elements announced in [4d] (where we used the term **R**-regular for polar-regular) and a proof of the conjugacy theorem can be found in [4f].

LEMMA 8.2. – Let G be a semi-simple analytic group having no compact factors and Γ a discrete subgroup such that G/Γ be finite measure. Then

(1) Ad Γ is Zariski-dense in Ad G ([1])

(2) Any **R**-split semi-simple element b of G operates ergodically on $H/H \cap \Gamma$ where H is the smallest normal analytic subgroup of G such that ΓH is closed ([4g])

(4) $\overline{\Gamma P} = G$ for any parabolic subgroup of G ([4d, f])

In the special case that the H of assertion 2) is the smallest analytic normal subgroup of G containing b, the result is proved by C. Moore ([3]) and generalizes a result of F. Mautner.

9. The boundary map φ_0.

THEOREM 9.1. – *Let Q and Q' be compact locally symmetric spaces of negative curvature, let X and X' be their simply connected covers, and let $\Gamma = \pi_1(Q)$, $\Gamma' = \pi_1(Q')$. Assume there is an isomorphism $\theta : \Gamma \to \Gamma'$. Then there is a pseudo-isometry $\varphi : X \to X'$ equivariant with respect to θ. Such a φ induces a homeomorphism $\varphi_0 : X_0 \to X'_0$, where X_0 and X'_0 denote the maximal boundaries of X and X'.*

Sketch of proof – The ranks of X and X' are equal since they are given by the maximal rank r of abelian subgroups of Γ. Let $p \in X$, let $\mathfrak{F}_r$ be the set of all Γ-compact r-flats through some ball containing p, and let $\mathfrak{F}_r^*$ denote the union of all the F in $\mathfrak{F}_r$.

The existence of φ is given by Lemma 5.1. By Lemma 7.1, there is a number ν_0 such that $\varphi(F)$ lies within a distance ν_0 of a Γ-compact r-flat F' for every $F \in \mathfrak{F}_r$. Let Δ and Δ' denote the stabilizers in Γ and Γ' respectively of F and F'.

Applying Lemma 7.3, we can see that singular lines of F are mapped by φ to within a distance ν_0 of singular lines in F'. Let F be an r-flat through p and let L be a ray well within a chamber (say for definiteness all the fundamental roots are equal along L). Then φ carries L into a region bounded away from S', the union of singular lines through $\varphi(p)$. Using the orbital coordinate form of the metric (cf. § 2) we see that the K-orbital distance (i.e. the "angle") diminishes exponentially along a path receding to infinity so long as the chamber coordinates stays away from the walls. Thus given a family of nearby chambers with origin p, the K'-orbital coordinates of their images diminishes exponentially as one passes to infinity. Therefore φ induces a continuous map $\varphi_0 : X_0 \to X'_0$.

Clearly φ_0 is equivariant with respect to θ. Similarly, we can get a continuous θ^{-1}-equivariant map $\psi_0 : X'_0 \to X_0$. Without loss of generality we can assume that $\psi \circ \varphi$ is the identity on some r-flat F (Lemma 5.1)

Therefore $\psi_0 \circ \varphi_0$ leaves fixed a point x_0 of X_0 and hence each point of Γx_0. Sonce the stabilizer of x_0 is a parabolic subgroup we get $\overline{\Gamma} x_0 = X_0$ by Lemma 8.2(3). Hence $\psi_0 \circ \varphi_0$ is the identity and similarly $\varphi_0 \circ \psi_0$ is the identity. Thus φ_0 is a homeomorphism.

10. Stable fixed points

Let A be a semi-group operating via diffeomorphisms on a manifold X_0. We denote by $f(A)$ the set of fixed points of A in X_0.

DEFINITION. – A point $p \in f(A)$ is of A-type (m, n) if there is a neighborhood of p of the form $\mathbf{R}^m + \mathbf{R}^n + \mathbf{R}^k$ with each element of A contracting on $(\mathbf{R}_m, \eta, 0)$ for all $\eta \in \mathbf{R}^n$, fixing $(0, \mathbf{R}^n, 0)$, and expanding in the complement of $(\mathbf{R}^m, \mathbf{R}^n, 0)$, that is $\{a^n(\xi, \eta, \zeta) ; n = 1, 2, \ldots\}$ has no point of accumulation in $\mathbf{R}^m + \mathbf{R}^n + \mathbf{R}^k$ if $\zeta \neq 0$ for any $a \in A$.

DEFINITION. – Type $(m, \dim X_0 - m)$ is called *stable type.* The subset of $f(A)$ of stable type is denoted by $f(A)^s$ and is called the stable part of $f(A)$.

The type of fixed point of a diffeomorphism does not vary along a connected component of the fixed point set. In particular $f(A)^s$ consists of connected components of $f(A)$.

LEMMA 10.1. – Let G be a semi-simple analytic group and X_0 the maximal boundary of its associated symmetric space. Let A be a cyclic semi-group of $\mathbf{R}$-split semi-simple elements. Then $f(A)^s$ is connected and its stabilizer is a parabolic subgroup which operates transitively in $f(A)^s$.

11. Approximating sequences

Let G be a semi-simple analytic group and Γ a discrete subgroup such that G/Γ has finite measure. Let A be a cyclic semi-group of G.

DEFINITION. – A_Γ is the set of all sequences $\{\gamma_k\}$ of elements of Γ satisfying the condition : there is an increasing sequence of positive integers such that

$$\lim_{k \to \infty} \gamma_k a^{-n_k} = 1$$

where a is a generator of A. If the element a is $\mathbf{R}$-split and semi-simple, and if A_Γ is not empty, we call the semi-group *Γ-approximable,* and elements in A_Γ are called A-approximations.

DEFINITION. – Let $\{\gamma_n\}$ be a sequence of elements in Γ. Set

$$f(\vec{\gamma}_n) = \{p \ ; p \in X_0 \ , \lim_{n \to \infty} \gamma_n p = p\}$$

Let $p \in f(\vec{\gamma}_n)$. We say that p is of $\vec{\gamma}$-type (m, n) if there is a neighborhood of the form $\mathbf{R}^m + \mathbf{R}^n + \mathbf{R}^k$ satisfying

(1) $\lim_{n \to \infty} \gamma_n(\xi, \eta, 0) = (0, \eta, 0)$ for $\xi \in R^m, \eta \in R^n$

(2) $\gamma_n(\xi, n, \zeta)$ has no accumulation point in $\mathbf{R}^m + \mathbf{R}^n + \mathbf{R}^k$ if $\zeta \neq 0$.

We call a point of $\vec{\gamma}_n$-type $(m, \dim X_0 - m)$ a point of *stable* type. We denote by $f(\vec{\gamma}_n)^s$ the points of stable type in $f(\vec{\gamma}_n)$. If C is a collection of sequences of Γ, we set

$$f(C) = \bigcap_{\{\gamma_n\} \in C} f(\vec{\gamma}_n)$$

$$f(C)^s = \bigcap_{\{\gamma_n\} \in C} f(\vec{\gamma}_n)^s$$

LEMMA 11.1. – Let A be a cyclic semi-group of R-regular semi-simple elements. Then gAg^{-1} is Γ-approximable for almost all $g \in G$.

This follows form Lemma 8.2 (2).

LEMMA 11.2.– Let $\{\gamma_n\}$ be a sequence on Γ. Then $f(\vec{\gamma}_n) = f(A)$ where A is the subgroup of G fixing the points of $f(\vec{\gamma}_n)$.

LEMMA 11.3.– Let A be a cyclic semi-group of R-split semi-simple elements which is Γ-approximable. Then $f(A_\Gamma) = f(A)$ and $f(A_\Gamma)^s = f(A)^s$.

LEMMA 11.4. – Let X, X', and $\theta : \Gamma \to \Gamma'$ be as above, and let $\varphi_0 : X_0 \to X'_0$ be the induced θ-equivariant homeomorphism. Let A be an abelian semi-group of R-split semi-simple elements of G. Then $\varphi_0(f(A)^s) = f(A')^s$ for some abelian semi-group of R-split semi-simple elements in G'. If Λ lies in a chamber or chamber wall, then so also does A'.

12. Tits geometries and the main theorem

Let τ denote the set of all fixed point parts $f(A)^s$ as A varies over cyclic semi-group of R-split semi-simple elements of G. Let $P(A)$ denote the stabilizer of $f(A)^s$. Two elements of τ are called *incident* if they have a non-empty intersections, this is equivalent to saying that their stabilizers intersect in a parabolic subgroup. We have thus arrived at a Tits geometry !

In his famous Erlanger program, Felix Klein urged that the properties of a geometry be translated into properties of its automorphism group. Tits has used this principle in reverse to define a geometry on the set of parabolic subgroups of any algebraic group ; in the classical geometries, this amounts merely to labelling every point, line, plane etc., by its stabilizer. In particular, Tits defines two parabolic subgroups as incident if their intersection is a parabolic subgroup. From Lemma 11.4 we see that the boundary map φ_0 induces an isomorphism also called $\varphi_0 : \tau(G) \to \tau(G')$ of the associated Tits geometries.

A basic result of Tits, specialized to the case of semi-simple real analytic groups G having no center, no compact factors, and no factors of R-rank one :

$$G = (\mathrm{Aut}\ \tau(G))^0$$

where the superscript denote the connected component of the identity (cf [7]). Since φ_0 is incidence preserving, we get

$$\varphi_0 \circ \mathrm{Aut}\ \tau(G) \circ \varphi_0^{-1} = \mathrm{Aut}\ \tau(G')$$

and therefore for any $g \in G$

$$\varphi_0 \circ g \circ \varphi_0^{-1} \in G'$$

where each elements of G are identified with its action on X_0. Set

$$\hat{\theta}(\gamma) = \varphi_0 \circ g \circ \varphi_0^{-1}.$$

Then $\hat{\theta}(\gamma) = \theta(\gamma)$ for $\gamma \to \Gamma$. Thus $\hat{\theta}$ is an analytic isomorphism of G to G' extending θ and therefore induces an isometry of $\Gamma\backslash X$ onto $\Gamma'\backslash X'$ with respect to suitably normalized metrics.

13. Concluding remarks

To sum up. we have

THEOREM 13.1. – *Let X and X' be simply connected symmetric Riemannian spaces of negative curvature having no factors of rank 1. Let Γ and Γ' be discrete groups of isometries on X and X' respectively operating freely. Assume $\Gamma\backslash X$ and $\Gamma'\backslash X'$ are compact. If $\theta : \Gamma \to \Gamma'$ is an isomorphism, then θ extends to an analytic isomorphism of the group G of isometries of X onto the group G' of isometries of X'.*

The groups G and G' above have no centers. According to a result of Selberg ([6]), any finitely generated matrix group has a subgroup of finite index having no elements of finite order. Such a subgroup of G operates freely on the associated symmetric space X and conversely, a subgroup operating freely on X has no elements of finite order. Thus we get

THEOREM 13.2. – *Let G and G' be semi-simple analytic groups with no center, no compact factors, and no factors of* R*-rank 1. Let Γ and Γ' be discrete subgroups of G and G' respectively such that G/Γ and G'/Γ' are compact. Let $\theta : \Gamma \to \Gamma'$ be an isomorphism. Then θ extends to an analytic isomorphism of G onto G'.*

In the hypotheses of Theorem 13.2, we have dropped the assumption that Γ has no elements of finite order because each element of Γ is uniquely determined by its action on a normal subgroup Γ_1 of finite index in Γ – this follows at once from the Zariski-density of Γ_1 in G.

It is natural to wonder about the validity of Theorem 13.2 for groups of R-rank 1. If $G = PL(2, \mathbf{R})$, the result is false since two compact Riemann surfaces of the same genus need not be conformally equivalent. For the case of the group of isometries on real hyperbolic space, the boundary map turns out to be quasi-conformal; (this result has been recently announced by G.A. Margulies, Dokl. Akad. Nauk SSSR v. 192, No. 4 (1970)) and therefore φ_0 is conformal by [4e].

As for generalization of Theorem 13.2 to the case where G/Γ has finite measure, such an extension is possible as soon as one can establish

(1) A pseudo-isometry of $\Gamma\backslash X$ to $\Gamma'\backslash X'$.

(2) The existence of sufficiently many Γ-compact r-flats.

An affirmative solution to the questions above in rank 1 would lead to the following generalization of the theorem on solvmanifolds in [4a].

THEOREM. – *Let G_i be a simply connected analytic group having no compact factors and let Γ_i be a discrete subgroup such that G_i/Γ_i is compact ($i = 1, 2$). Let $\theta : \Gamma_i \to \Gamma_2$ be an isomorphism. Then there exist subgroups of finite index Γ_1 and Γ_2 in Γ_1 and Γ_2 respectively since that θ is induced by a homeomorphism of G_1/Γ_1 onto G_2/Γ_2.*

If the questions relating to the finite measure case have affirmative answers, then the above result should remain true if G/Γ has finite measure.

BIBLIOGRAPHY

[1] BOREL A. — Density properties for certain subgroups of semi-simple groups without compact components, *Ann. of Math.*, v. 72, 1960, p. 179-188.

[2] FURSTENBERG P. — A Poisson formula for semi-simple groups, *Ann. of Math.*, v. 77, 1963, p. 335-386.

[3] MOORE C.C. — *Ergodicity of flows on homogeneous spaces.*

[4] MOSTOW G.D. — (a) Factor spaces of solvable groups, *Ann. of Math.*, v. 60, 1954, p. 1-27;

(b) Some new decomposition theorems for semi-simple groups, *Memoirs A.M.S.*, No. 14, p. 31-54;

(c) Homogeneous spaces of finite invariant measure, *Ann. of Math.*, v. 75, 1962, p. 17-37.

(d) On the conjugacy of subgroups of semi-simple groups, *Proc. of Symposia in Pure Math.*, v. 9, 1966, p. 413-419.

(e) Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, *Publ. I.H.E.S.*, v. 34, p. 53-104;

(f) *Lectures*, Tata Inst. of Fundamental Research Bombay, 1970;

(g) Intersections of discrete subgroups with Cartan subgroups, *Indian J. of Math*, 1970.

[5] SATAKE I. — On representations and compactifications of symmetric Riemannian spaces, *Ann. of Math.*, v. 71, 1960, p. 77-110.

[6] SELBERG A. — On discontinuous groups in higher dimensional symmetric spaces, *Internat. Colloq. Function Theory*, p. 147-164, Tata Inst. of Fundamental Research, Bombay, 1960.

[7] TITS J. — Lectures on buildings of spherical type and finite B-N pairs, *Springer Lecture Notes* (to appear).

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GEOMETRY OF MODULI SPACES OF VECTOR BUNDLES

by M. S. NARASIMHAN

Moduli spaces of vector bundles.

I shall speak about certain varieties which arise in the moduli problem for holomorphic vector bundles on a compact Riemann surface.

Let X be a compact Riemann surface of genus $g \geqslant 2$. By a vector bundle we shall always mean a holomorphic vector bundle. If W is a vector bundle on X we shall denote by $d(W)$ its degree and by $n(W)$ its rank.

As is well known, the classification of line bundles of degree zero on X is achieved by the Jacobian J of X. The underlying differentiable manifold for J is the space of characters of the first homology group $H_1(X, \mathbf{Z})$. Moreover, the holomorphic tangent space to J at any point is identified with the cohomology space $H^1(X, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X. We remark that there is a natural positive definite hermitian form on $H^1(X, \mathcal{O})$: the space $H^1(X, \mathcal{O})$ is identified with the space H of closed $(0, 1)$ forms on X and if $\omega, \eta \in H$ we set

$$(\omega, \eta) = \frac{1}{i} \int_X \omega \wedge \overline{\eta} .$$

Passing on to vector bundles of higher rank, to obtain good moduli varieties one has to restrict the class of vector bundles. A vector bundle W on X is said to be stable if for every proper subbundle V of W one has $d(V)/n(V) < d(W)/n(W)$. D. Mumford proved that the isomorphism of classes of stable bundles of rank n and degree d form a non-singular quasi-projective variety.

Let π_1 be the fundamental group of X. If ρ is a representation of π_1 in the unitary group $\mathrm{U}(n)$, ρ defines a holomorphic vector bundle $W(\rho)$ of rank n and degree 0 on X. Moreover if ρ_1 and ρ_2 are unitary representations of π_1, then the holomorphic vector bundles $W(\rho_1)$ and $W(\rho_2)$ are isomorphic if and only if ρ_1 and ρ_2 are equivalent representations. It was proved in [2] that a vector bundle of degree 0 on X is stable if and only if it arises from an *irreducible* unitary representation of π_1. This result suggests a natural compactification for the space of stable bundles of degree 0. In fact C.S. Seshadri proved [4] that there is a natural structure of a *projective* variety on the space equivalence classes of all n-dimensional unitary representations of π_1. This structure depends in general on the complex structure on X. In general this space has singularities and the singular points have been determined in [3]. The singular points correspond precisely to reducible representations except when $g = 2$, $n = 2$, in which case the variety is non-singular. Other moduli spaces (corresponding to vector bundles of degree $\neq 0$) are obtained by considering unitary representations of suitabley defined Fuchsian groups [2].

Non-singular moduli spaces.

The moduli spaces are non-singular for vector bundles whose degree and rank are coprime. These varieties are constructed in the following way. Let π be a discrete group acting effectively, properly and holomorphically on the unit disc Y such that $Y/\pi = X$ and such that the natural projection $p : Y \to X$ is ramified over a single point $x_0 \in X$ with ramification order n. Let $y_0 \in p^{-1}(x_0)$ and π_{y_0} be the isotropy group at y_0. Let τ be a character of π_{y_0} such that τ is an isomorphism of π_{y_0} onto the n^{th} roots of unity. A representation ρ of π into unitary group $\mathrm{U}(n)$ is said to be of type τ if $\rho \mid \pi_{y_0} = \tau \,.\, I_n$, where I_n is the $n \times n$ identity matrix. Such a representation is irreducible. To each ρ of type τ we can associate a holomorphic vector bundle $W(\rho)$ of rank n and degree d on X (Here d is an integer coprime to n, associated with τ). There is a natural structure of a compact complex manifold ([1], [2, Remark 10.1]) on the set M of equivalence classes of n-dimensional unitary representations of type τ of π. (In fact M is a projective variety).

Let $m \in M$ and ρ a representation of type τ in the class m. Let $\mathrm{Ad}\rho$ denote the representation of π in $\mathfrak{gl}(n, \mathrm{C})$ obtained by composing ρ and the adjoint representation of $U(n)$ in $\mathfrak{gl}(n, C)$. $\mathrm{Ad}\rho$ is a representation of π_1. Let $W(\mathrm{Ad}\rho)$ be the holomorphic vector bundle on X associated with $\mathrm{Ad}\rho$. Then the holomorphic tangent space to M at m is naturally identified with the cohomology space $H^1(X, W(\mathrm{Ad}\rho))$ [1].

Canonical hermitian metrics

We now introduce a hermitian metric on M. To do this it is sufficient to introduce a positive definite hermitian form on $H^1(X, W(\mathrm{Ad}\rho))$. Since $W(\mathrm{Ad}\rho)$ is given by a local system, the operator of exterior differentiation, d, is well defined on C^∞ differential forms with values in $W(\mathrm{Ad}\rho)$. Let $T(\rho)$ denote the space of d-closed C^∞ forms of type $(0, 1)$ with coefficient in $W(\mathrm{Ad}\rho)$. Then $T(\rho)$ is canonically isomorphic to $H^1(X, W(\mathrm{Ad}\rho))$. So it suffices to introduce a positive definite hermitian form on $T(\rho)$. If $\omega \in T(\rho)$, let $\omega^\#$ denote the $(1, 0)$ form with coefficients in $W(\mathrm{Ad}\rho)$ obtained by using the conjugation $A \mapsto A^*$ in $\mathfrak{gl}(n, C)$. (A^* denotes the conjugate transpose of A. Locally, if $\omega = A(z)\,d\bar{z}$, $\omega^\# = A^*(z)\,dz$). Define the hermitian scalar product in $T(\rho)$ by

$$(\omega_1, \omega_2) = \frac{1}{i} \int_X \mathrm{Trace}\,(\omega_1, \omega_2^\#),\ \omega_1, \omega_2 \in T(\rho),$$

where the exterior product is taken with respect to the multiplication

$$\mathfrak{gl}(n, C) \times \mathfrak{gl}(n, C) \to \mathfrak{gl}(n, C).$$

This defines a hermitian metric on M. This metric is Kählerian. In fact, using the forms in $T(\rho)$ we can construct at each point P of M a geodesic holomorphic coordinate system (i.e., one in which all first derivatives of the components of the metric tensor are zero at P).

REFERENCES

[1] Narasimhan M.S. and Seshadri C.S. — Holomorphic vector bundles on a compact Riemann surface, *Math. Ann*, 155, 1964, p. 69-80.

[2] Narasimhan M.S. and Seshadri C.S. — Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.*, 82, 1965, p. 540-567.

[3] Narasimhan M.S. and Ramanan S. — Moduli of vector of bundles on a compact Riemann surface, *Ann. of Math.*, 89, 1969, p. 19-51.

[4] Seshadri C.S. — Space of unitary vector bundles on a compact Riemann surface. *Ann. of Math.*, 85, 1967, p. 303-336.

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SOME GENERALIZATIONS OF CHRISTOFFEL AND MINKOWSKI'S THEOREMS

by A. V. POGORELOV

This lecture deals with the problem of existence of the closed convex surface with the given function of the main radii of curvature. I mean the surface whose main radii of curvature are R_1, R_2 and the unit vector of normal in each point satisfies the condition

$$f(R_1, R_2) = \varphi(n) \tag{1}$$

Here f and φ are the given functions.

In the case when $f(R_1, R_2) = R_1 R_2$ it is the well-known problem of Minkowski. It is known that the problem of Minkowski is solvable if the given function φ is twice differentiable, positive and satisfies the condition

$$\iint n\varphi(n)\, d\omega = 0 \tag{2}$$

Here the integration extends over the unit sphere. The condition (2) is not only sufficient but also necessary.

In the other particular case when $f = R_1 + R_2$ we obtain the problem of Christoffel. It is known that this problem is solved in compact form. As far as I know these two results are all that is known about the solution of the general problem.

About the uniqueness of the solution of the problem mentioned here we have a rather general theorem of A.D. Alexandrov. According to this theorem the closed convex surface is exactly defined to the parallel transfer by the condition (1), if the function f is strictly monotonous over both variables, that is

$$\partial f/\partial R_1 > 0 \,, \partial f/\partial R_2 > 0 \tag{3}$$

The main difficulty of the solution of the general problem in my opinion is to find the necessary conditions which in the case of Minkowski's problem are reduced to the system of three equations (2). There is no hope to find these conditions in the general case. I think. Here we hope to solve the problem using some sufficient conditions.

One of these sufficient conditions of common character is the requirement of the symmetry of the function $\varphi(n)$, that is

$$\varphi(n) = \varphi(-n) \tag{4}$$

This condition restricts our consideration to the case of the central symmetrical surfaces. It is clear that if the condition (4) is realized, the condition (2) for the solution of Minkowski's problem is fulfilled in the trivial way.

For the solution of the problem we use the continuity method. That is why we include the function $\varphi(n)$ in the family of continuous functions :

(5) $$\varphi_\lambda (n) = \lambda\varphi(n) + (1 - \lambda) f(1,1), \ 0 \leqslant \lambda \leqslant 1$$

It is clear that the problem can be solved for $\lambda = 0$. The solution is given by the sphere of a unit radius.

For the complete solution of the problem it is sufficient to prove two statements :

a) If the problem is solvable for some value of the parameter $\lambda = \lambda_0$ then it is solvable for all the values near this parameter ;

b) If the problem is solvable for the values of the parameter λ convergent to some λ_1, it is also solvable for $\lambda = \lambda_1$.

Let us consider the statement a). The equality (1) is the equation in partial derivatives of the second order for the surface support function. If the function f is symmetrical $f(R_1, R_2) = f(R_2, R_1)$ and satisfies the condition (3), this equation will be of elliptical type. In the following discussion this condition is supposed to be realized.

Consider the equation in variations for the equation (1). This equation will be a linear one of the elliptical type

(6) $$L(u) = \psi(n)$$

It appears that the gomogeneous equation $L(u) = 0$ in the class of the center symmetrical infinitesimal surface deformations has no other solutions except the trivial $u = 0$. From this we conclude that the nonhomogeneous equation (6) is solvable for any symmetrical function $\psi(n)$, that is $\psi(n) = \psi(-n)$.

Now applying the method of successive approximation the statement a) is proved.

To prove the statement b) it is sufficient to establish the a priori estimates for the normal curvatures of the unknown surface. The proof for the existence of such estimates is based on the following theorem.

In the point P_1 of surface F satisfying the equation (1) where the larger of the main radii of the curvature R_1 reaches its maximum, we have

(7_1) $$(R_2 - R_1)\frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2}\frac{\varphi_S^2}{(\partial f/\partial R_2)^2} \geqslant \varphi_{SS}$$

In the point P_2 of surface F where R_2 reaches its minimum, we have

(7_2) $$(R_1 - R_2)\frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2}\frac{\varphi_S^2}{(\partial f/\partial R_1)^2} \leqslant \varphi_{SS}$$

Let us give an example. Let $f(R_1, R_2) = R_1 R_2$. The unequality (7_1) takes the form

$$(R_2 - R_1) R_1 \geqslant \varphi_{SS}$$

From this as $R_1 R_2 = \varphi$ we obtain the estimate

$$R_1^2 \leqslant \varphi - \varphi_{SS}$$

Now let the functions f and φ satisfy the following conditions

$$(8_1) \qquad \overline{\lim_{\substack{R_2 = R_2(R_1, n) \\ R_1 \to \infty}}} \left\{ (R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\varphi_S^2}{(\partial f/\partial R_2)^2} \right\} < \varphi_{SS}$$

$$(8_2) \qquad \underline{\lim_{\substack{R_1 = R_1(R_2, n) \\ R_2 \to 0}}} \left\{ (R_1 - R_2) \frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2} \frac{\varphi_S^2}{(\partial f/\partial R_1)^2} \right\} > \varphi_{SS}$$

Then there exists $\epsilon > 0$, depending only on functions and φ such that on the surface

$$(9) \qquad \epsilon < R_2 \leqslant R_1 < \frac{1}{\epsilon}$$

Thus the condition (8_1) and (8_2) make it possible to obtain the a priori estimates we need.

So the surface saticfying the Eq. (1) always exists if the functions f and φ are twice differentiable and satisfy the conditions (3), (4), (8_1), (8_2).

It is known that Minkowski came to the solution of the problem stated by him starting from the corresponding problem for the closed convex polyhedra. The problem is to prove the existence of the closed polyhedron with given directions faces and their areas. In the case of the general problem I think it is possible to state the corresponding problem for polyhedra. Here the role of the area takes the arbitrary function defined on the faces and having the properties similar to the area.

Let α be the arbitrary plane and ω_a the function defined on the closed convex polygons lying in the planes parallel to α plane. Let this function satisfy the following conditions.

(1) Function ω_a is positive and continuous.

(2) Function ω_a is invariant with respect to parallel transfer. It means that if the polygons P and Q are superimposed by parallel transfer the values of the function ω_a on these polygons are equal :

(3) The function ω_a is strictly monotonous, that is, if the polygon Q is a part of the polygon P then $\omega_a(Q) < \omega_a(P)$.

(4) If the polygon P changes so that its area $S(P) \to \infty$ then $\omega_a(P) \to \infty$. If the polygon P degenerates into a segment, then $\omega_a(P) \to 0$.

We have the following theorem.

Let $\alpha_1, \alpha_2, \ldots \alpha_n$ $(n \geqslant 3)$ be a system of planes not parallel to one straight line : $\omega_1, \ldots \omega_n$ be the system of functions defined in polygons parallel to planes $\alpha_1, \ldots \alpha_n$ satisfying the conditions (1) – (4).

Then for any positive numbers $\varphi_1, \ldots \varphi_n$ there exists a closed convex polyhedron with $2n$ faces parallel to planes $\alpha_1, \ldots \alpha_n$ and the values of the functions ω on those faces equal to $\varphi_1, \ldots \varphi_n$.

This polyhedron has a centre of symmetry and is defined uniquely to the parallel transfer.

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C 4 - ANALYSE SUR LES VARIÉTÉS

ELLIPTIC OPERATORS AND SINGULARITIES OF VECTOR FIELDS

by M. F. ATIYAH

Introduction.

On a compact smooth manifold X there is a well-known theorem of *H.* Hopf which asserts that the number of zeros of a smooth field ν of tangent vectors depends only on X (not on ν) and is equal to the Euler-Poincaré characteristic $E(X)$. Here we assume of course that the number of zeros of ν is finite (which is true in general) and that each is counted with an appropriate multiplicity. If we consider now r vector fields $\nu_1, \ldots, \nu_r$, and denote by $S(\nu_1, \ldots, \nu_r)$ their singular set (i.e. the points P at which $\nu_1(P), \ldots, \nu_r(P)$ become linearly dependent) we have, in general, dim $S = r - 1$. The homology class of S (with appropriate coefficient group) turns out to be independent of $\nu_1, \ldots, \nu_r$: this is called the Stiefel-Whitney class of X. The study of these invariants is part of the general theory of characteristic classes. However this theory does not give complete topological information about the singular sets S. Thus if the Stiefel-Whitney class vanishes the theory merely tells us that we can modify $\nu_1, \ldots, \nu_r$ so that dim $S \leqslant r - 2$.

At the opposite extreme from the theory of characteristic classes we can, following E. Thomas [4], attempt to study the case of *finite* singularities (i.e. S a finite set) and ask for generalizations of Hopf's theorem. At each singular point P we have a local obstruction

$$o_P(\nu_1, \ldots, \nu_r) \in \pi_{n-1}(V_{n,r})$$

(where $V_{n,r}$ is the Stiefel manifold $SO(n)/SO(n-r)$). This obstruction is, by definition, the homotopy class of the map $x \to \nu_1^P(x), \ldots, \nu_r^P(x)$ where $x \in S^{n-1}$ is a point on a small sphere centre P and $\nu_i^P(x)$ is the parallel through P to $\nu_i(x)$- using a local coordinate system. For $r = 1$ we have

$$V_{n,r} = S^{n-1} \qquad \text{and} \qquad \pi_{n-1}(S^{n-1}) \cong \mathbf{Z}$$

gives the multiplicity used in the Hopf theorem. We can now form the global obstruction $\sum_P o_P(\nu_1, \ldots, \nu_r)$ and ask how far this is independent of X.

There are now many results in this direction (see [4]) and the invariants of X which appear in these results (as global obstructions) are the Euler characteristic, the signature and the Kervaire semi-characteristic $k(X)$ (defined (for $\dim X$ odd) to be $\sum_p \dim_R H^{2p}(X, R) \bmod 2$). As an explicit example we mention here the following.

THEOREM 1. – *Let X be oriented and of* $\dim 4q + 1$, *and let v_1, v_2 be 2 vector fields with finite singularities. Then*

$$\sum_P o_P(v_1, v_2) = k(x) \in Z_2$$

Note. – $\pi_{n-1}(V_{n,2}) \cong Z_2$ for n odd > 3.

Now the invariants of X mentioned above all occur as indices of elliptic operators on X or as mod 2 analogues of such indices. Because of this it is reasonable to expect some interesting connection between Theorem 1 and elliptic operators. In fact it turns out that Theorem 1, and others like it, can be proved using analysis and the topology appropriate to elliptic operators, namely K-theory. As an indication in this direction I will briefly indicate how to prove, by analysis, the weak form of Theorem 1 – namely its corollary.

COROLLARY. – *If X is as in Theorem* 1 *and if there exist vector fields v_1, v_2 everywhere independent, then $k(X) = 0$.*

First of all we can construct on X an elliptic operator D which is real and skew-adjoint and such that

$$\operatorname{Ker} D \cong \sum_p H^{2p}(X, R)$$

This is done by choosing a Riemannian metric on X and using the Hodge theory of harmonic forms. Explicitly D is an operator defined on all even-dimensional differential forms by

$$D\phi = (-1)^p \, d * \phi + (-1)^{p+1} * d\phi \qquad \phi \in \Omega^{2p}$$

where $*$ is the duality operator defined by the metric.

Next we define for any 1-form v an operation $R(v)$ on forms by

$$R(v)\,\phi = \phi \wedge v - \phi \vee v$$

(where $\phi \vee v$ denotes the interior product with v, adjoint of the exterior product). If v_1, v_2 are 2 independent vector fields, we may suppose them orthonormal for convenience and (using the metric) we may also identify them with 1-forms. If we then define the composite operation $R(v) = R(v_1) \circ R(v_2)$ on forms we can verify the following :

(1) $R(v)^2 = -$ Identity

(2) $RD - DR$ has order zero

(D is a 1st order differential operator, R has o-order so (2) asserts that the highest order terms of RD and DR cancel).

Using (1) and (2) we shall now prove the corollary. First we replace D by D' defined by

$$D' = \frac{1}{2}(D + RDR^{-1})$$

Clearly D' now commutes with R, so that by (1) Ker D' admits a complex structure and so has even dimension. On the other hand (2) implies that D and D' agree in 1st order and hence

$$D_t = tD + (1-t)D' \qquad 0 \leqslant t \leqslant 1$$

is a family of real skew elliptic operators connecting D and D'. But it is not difficult to show (see [2]) that this implies

$$\dim \operatorname{Ker} D \equiv \dim \operatorname{Ker} D' \bmod 2 \ ;$$

since $\dim \operatorname{Ker} D = \sum_p \dim H^{2P}(X, R)$ this completes the proof.

Note. – dim Ker D mod 2 for D real skew elliptic is a kind of mod 2 index – having the basic property of homotopy invariance.

To prove Theorem 1 by similar analysis one would have to excise small balls around the singular points P and set up a suitable boundary value problem. It is however technically easier to pass at this stage to K-theory which is essentially the topological machinery for dealing with elliptic problems (see [3]). The details are reasonably standard but cannot be described here.

For a general survey concerning vector fields with such singularities we refer to [4] while the material described here is explained in more detail in [1]. My purpose here has simply been to illustrate the interaction between the geometry of vector fields and index theory.

REFERENCES

[1] Atiyah M.F. — Vector fields on manifolds, *Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen*, 200, 1970.

[2] Atiyah M.F. and Singer I.M. — Index theory for skew adjoint Fredholm operators, *Publ. Math. Inst. Hautes Etudes Sci. Paris*, No. 37, 1969.

[3] Atiyah M.F. and Singer I.M. — The index of elliptic operators I, *Ann. of Math.*, 87, 1968, p. 484-530.

[4] Thomas E. — Vector fields on manifolds, *Bull. Amer Math. Soc.*, 75, 1969, p. 643-683.

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ON THE MOTION OF INCOMPRESSIBLE FLUIDS

by D. G. EBIN and J. E. MARSDEN

We are concerned with the initial value problem for an incompressible inviscid fluid. Specifically, given a bounded domain M in $\mathbf{R}^3$ (or any compact Riemannian manifold which may have boundary) and a smooth vector field V_0 tangent to bdy(M), we seek a time dependent vector field $V(t)$ satisfying :

$$(E) \qquad \begin{array}{ll} \dfrac{\partial V(t)}{\partial t} + \nabla_{V(t)} V(t) = - \operatorname{grad} p & V(t) \text{ tangent to bdy} M \\ \operatorname{div}(V) = 0 & V(0) = V_0 \end{array}$$

Where ∇ is the affine connection in M $\left(\text{so in } \mathbf{R}^3 \nabla_V V = \sum_{j=1}^{3} V_j \dfrac{\partial V_i}{\partial X_j}\right)$ div means divergence, and grad p is the gradient of a time dependent function on M, which is determined implicitly.

We have shown [2] that given V_0, there exists a unique V satisfying (E), defined on a time interval depending on V_0. In this report we will discuss the methods of [2].

Our approach to the problem is patterned after the work of Arnold, [1] ; that is, we translate the problem into a Hamiltonian system on a certain non-linear infinite dimensional space. We show that this space has a natural Riemannian structure and we solve the problem by finding geodesics on this space.

We first present a typical situation from mechanics : Let X be differential manifold with Riemannian structure (,) ; let TX be its tangent bundle and T^*X the cotangent bundle. (,) defines an isomorphism between TX and T^*X, and T^*X has a natural sympectic two form Ω (cf. [3], p. 86). By means of the isomorphism, we can consider Ω as a sympectic form on TX. We define Kinetic energy $K : TX \to \mathbf{R}$ by $K(V) = \frac{1}{2} (V, V)$. Then there exists a unique vector field Z on TX which satisfies the equation : $\Omega(Z, Y) = - Y(K)$, for Y any vector field on TX. The integral curves of Z project to geodesics on X, and also they are the curves of motion of the Hamiltonian system with energy K.

We proceed to find X for the problem of fluid motion. We assume that the manifold M is filled with fluid and let $\varphi_t : M \to M$ be the map which takes each particle of fluid from its position p at time zero to its position $\varphi_t(p)$ at time t. Since the fluid is incompressible φ_t will preserve the volume element of M. (For a domain in $\mathbf{R}^3$, this means that the Jacobian of φ_t is everywhere 1). From this and the assumption that φ_t is onto, it follows that φ_t must be a volume-preserving diffeomorphism of M. Thus, we let X equal $\mathcal{D}_\mu$ the set of all volume preserving diffeomorphisms of M (μ being the volume element), and our next task is to endow $\mathcal{D}_\mu$ with a differentiable structure. For simplicity, we shall assume that M has no boundary.

First consider $\mathcal{D}$, the set of smooth diffeomorphisms of M, with the C^∞ topology. This space is locally like $C^\infty(T)$, the Frechet space of smooth vector fields on M. Since one cannot in general solve ordinary differential equations on Frechet spaces, we enlarge $\mathcal{D}$ so that it is locally a Hilbert space.

Specifically, we let $\mathcal{D}^s$ be the set of bijective maps $\eta : M \to M$ such that η and η^{-1} are both of class H^s. That is, when written in local coordinates, η(and η^{-1}), together with all partial derivatives up to order s, are square integrable. For $s > n/2 + 1$, ($n = \dim M$) the smoothness of η does not depend on the choice of coordinates ; furthermore, $\mathcal{D}^s$, with the H^s topology, is a topological group which is continuously included in the group of C^1-diffeomorphisms.

Now we construct a differentiable structure for $\mathcal{D}^s$. Let $H^s(T_\eta)$ be the space H^s vector fields over η ; i.e.

$$H^s(T_\eta) = \{V : M \to TM \mid V \in H^s \quad \text{and} \quad \pi \circ V = \eta\}$$

($\pi : TM \to M$ is the bundle projection). It is a Hilbert space with the H^s topology. Let $e : TM \to M$ be the exponential map of M coming from its Riemannian Structure. Then $\Omega_e : H^s(T_\eta) \to \mathcal{D}^s$, defined by $\Omega_e(V) = e \circ V$ has domain a neighborhood of the origin of $H^s(T_\eta)$ and is a homeomorphism from some neighborhood of 0 in $H^s(T_\eta)$ to a neighborhood of η.

For each η, $\Omega_e : H^s_\infty(T_\eta) \to \mathcal{D}^s$ provides a chart about η and using the fact that $e : TM \to M$ is C^∞, one can show that the transitions between charts are smooth. Thus $\mathcal{D}^s$ is a C^∞-manifold and for each η, the tangent space to $\mathcal{D}^s$ at η (denoted $T_\eta \mathcal{D}^s$) is identified with $H^s(T_\eta)$.

Using the manifold structure of $\mathcal{D}^s$ we will derive a manifold structure for $\mathcal{D}^s\mu = \{\eta \in \mathcal{D}^s \mid \eta^*(\mu) = \mu\}$, where μ is the volume element of M and $\eta^*(\mu)$ is the usual pull-back of an n-form μ by η.

Let $H^{s-1}(\Lambda^n)$ be the space of H^{s-1} n-forms of M and let

$$A = \left\{ \omega \in H^{s-1}(\Lambda^n) \,\middle|\, \int_M \omega = \int_M \mu \right\}.$$

A is clearly a closed linear subspace of $H^{s-1}(\Lambda^n)$ of co-dimension 1.

Let $\Psi : \mathcal{D}^s \to A$ by $\Psi(\eta) = \eta^*(\mu)$. Ψ is a smooth map and is a surjection ; i.e., the tangent map $T_\eta\Psi : T_\eta\mathcal{D}^s \to T_{\Psi(\eta)}A$ is onto. From the implicit function theorem it follows that $\Psi^{-1}(\mu) = \mathcal{D}^s_\mu$ a submanifold of $\mathcal{D}^s$. It is also a subgroup. Furthermore, at the identity $id \in \mathcal{D}^s$, one computes that $T_{id}\Psi : T_{id}\mathcal{D}^s \to T_\mu A$ satisfies $T_{id}\Psi(V) = L_V(\mu)$ where "L" means Lie derivative. Therefore,

$$T_{id}\mathcal{D}^s_\mu = \{V \in H^s(T) \mid L_V(\mu) = 0\},$$

the set of divergence free vector fields. Also,

$$T_\eta\mathcal{D}^s_\mu = \{V \in H^s(T_\eta) \mid \operatorname{div}(V \circ \eta^{-1}) = 0\}.$$

Our next step is to define a Riemannian structure on $\mathcal{D}^s\mu$. On $T_\eta \mathcal{D}^s = H^s(T_\eta)$, we define $(V, W) = \int_M \langle V, W \rangle \mu$ where $\langle \, , \, \rangle$ is the Riemannian metric of M. This gives a weak Riemannian metric on $\mathcal{D}^s$. That is $(\, , \,)$ has all the usual properties except that on each tangent space $T_\eta \mathcal{D}^s$, $(\, , \,)$ does not induce the H^s-topology. However, $(\, , \,)$ is invariant under right multiplication by elements of $\mathcal{D}^s_\mu$. $\mathcal{D}^s_\mu$ inherits a weak Riemannian structure because it is a submanifold of $\mathcal{D}^s$. Also the right invariance of $(\, , \,)$ means that $\mathcal{D}^s_\mu$ is actually a weak Riemannian homogeneous space.

Our final task is to find the geodesics on $\mathcal{D}^s_\mu$ To do this we look for geodesics of $\mathcal{D}^s$ and these can be found virtually by inspection. Indeed to find a geodesic $\eta(t)$ from η_0 to η_1 in $\mathcal{D}^s$ we want to minimize the energy :

$$\int_M \left\{ \int_0^1 \langle \eta'(t), \eta'(t) \rangle \, dt \right\} \mu,$$

and here the integral in braces is, for each $p \in M$, the energy of the path $t \to \eta(t)(p)$. If each such path is a geodesic, the integral will be minimal at each point of M and hence the total energy will be minimal. Thus, the geodesics of $\mathcal{D}^s$ are those curves $\eta(t)$ in $\mathcal{D}^s$ which have the property that for each $p \in M$, $t \to \eta(t)(p)$ is a geodesic in M.

Of course, the geodesics on $\mathcal{D}^s$ are related to an affine connection on $\mathcal{D}^s$ which we will call $\overline{\nabla}$, and also to a Hamiltonian vector field $\overline{Z}$ on $T\mathcal{D}^s$. Furthermore, since $\mathcal{D}^s_\mu$ is a Riemannian submanifold of $\mathcal{D}^s$, it inherits a connection $\widetilde{\nabla} = P \circ \overline{\nabla}$ where at each tangent space $T_\eta(\mathcal{D}^s)$, P is the orthogonal projection

$$P : T_\eta \mathcal{D}^s \to T_\eta \mathcal{D}^s_\mu \quad .$$

Similarly $T\mathcal{D}^s\mu$ gets a Hamiltonian vector field $\widetilde{Z} = TP(\overline{Z})$.

Since our Riemannian structure is weak, it is not clear that P is a smooth map. However, at $id \in \mathcal{D}^s$, $P : T_{id} \mathcal{D}^s \to T_{id} \mathcal{D}^s_\mu$ is simply the projection onto the first summand of the well-known decomposition :

$$H^s(T) = \operatorname{div}^{-1}(0) \oplus \operatorname{grad} \mathfrak{F}^{s+1}$$

where the first summand is the set of divergence free H^s vector fields and the second is the set of gradients of H^{s+1} functions on M. This direct sum is topological and P is in fact smooth.

Thus $\mathcal{D}^s_\mu$ has a smooth Hamiltonian vector field which can of course, be integrated to give the required geodesics.

If $\eta(t)$ is such a geodesic and $W(t) = \dfrac{d}{dt}(\eta(t)) \in T_{\eta(t)} s_\mu$, then $V(t) = W(t) \circ \eta(t)^{-1}$ is a time dependent vector field which satisfies our original system (E).

Given V_0 a vector field on M, $V_0 \in T_{id} \mathcal{D}^s_\mu$ and there exists a unique geodesic $\eta(t)$ starting at id in direction V_0. By the above each $\eta(t)$ corresponds to a solution of (E).

REFERENCES

[1] ARNOLD V. — Sur la géometric differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits, *Ann. Inst. Grenoble,* 16(1), 1966, p. 319-361.
[2] EBIN D.G. and MARSDEN J.E. — Groups of Diffeomorphisms and the motion of Incompressible Fluids, *Annals of Math. Vol.,* 92, No. 1, 1970.
[3] LANG S. — Introduction to Differentiable Manifolds, *Interscience,* New York, 1962.

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ON FREDHOLM MANIFOLDS

by J. EELLS and K. D. ELWORTHY

1. Introduction.

Spaces of maps arising in analysis often are smooth manifolds modeled on Banach spaces. As such they have very regular structure : their algebraic topological invariants are especially easy to handle ; and in many instances their homeomorphism and diffeomorphism types are effectively describable through homotopy type. See [5] for general background, as well as [1] and [14] for recent developments in the topology of C^0- and C^∞-manifolds of infinite dimension.

Certain concrete problems of global analysis (e.g., variational and elliptic boundary value problems) not only provide a manifold of maps but also a more refined structure (called a Fredholm structure) on that manifold. Fredholm manifolds are rich in topology, for we can find non-trivial analogues of many of the notions of finite dimensional theory (e.g., Brouwer degree, Poincaré and Alexander-Pontrjagin duality, characteristic classes). We refer to [6, 12] and their bibliography for background.

Here we describe a classification theorem for Fredholm structures on a smooth Hilbert manifold (for simplicity of exposition – our theorem is valid for a somewhat broader class of Banach manifolds). We exhibit the close relationship between these structures and Fredholm maps (a class of maps which arise naturally in elliptic problems). In fact, our theorem can be viewed (see Example 2) as a contribution to the structure theory of Fredholm maps. Also, we indicate briefly how our theorem provides new proofs of the basic results on the differential topology of smooth Hilbert manifolds ; and, by way of illustration, how Fredholm structures arise naturally in certain path spaces.

2. Fredholm structures.

Let E denote the infinite dimensional separable real Hilbert space, $L(E)$ the Banach algebra of bounded endomorphisms of E (with its norm topology), and $GL(E)$ its Banach Lie group of units, with identity operator I. If $C(E)$ is the closed ideal in $L(E)$ of compact endomorphisms, let

$$GL_c(E) = \{ I + u \in GL(E) \ : \ u \in C(E)\}.$$

Then $GL_c(E)$ is a closed Banach Lie subgroup of $GL(E)$, and the coset map $p : GL(E) \to GL(E)/GL_c(E)$ is a locally C^0-trivial fibration (it is not locally C^∞-trivial).

Let X be a metrizable C^∞-manifold modeled on E. Its principal $GL(E)$-bundle ξ has the associated bundle η with fibre the homogeneous space $GL(E)/GL_c(E)$:

$$\begin{array}{ccc} P(X) & \searrow & \\ \xi \downarrow & & P(X)/GL_c(E) \\ X & \swarrow_{\eta} & \end{array}$$

It is elementary that the reductions of ξ to $GL_c(E)$-bundles correspond bijectively to the sections of the associated bundle η.

Remark. – A theorem of Palais-Švarc asserts that $GL_c(E)$ has the homotopy type of $\varinjlim GL(\mathbf{R}^n)$, where the limit is defined through the standard inclusions $\mathbf{R}^n \subset \mathbf{R}^{n+1} \subset \ldots$ It follows (a) that $GL_c(E)$-bundles are classified by real K-theory, and (b) that they have characteristic (Stiefel-Whitney, Pontrjagin) classes.

A *Fredholm structure on* X is an integrable $GL_c(E)$-reduction of ξ ; otherwise said, a maximal atlas $\mathfrak{A} = \{(\theta_i , U_i)\}$ for the differential structure of X such that the transition maps $\theta_j \circ \theta_i^{-1}$ have differentials belonging to $GL_c(E)$ at every point. A *Fredholm manifold* is a Hilbert manifold with a specified Fredholm structure. If X and Y are Fredholm manifolds with tangent vector bundles $T(X)$ and $T(Y)$, a map $f : X \to Y$ is a *tangential homotopy equivalence (of Fredholm manifolds)* if f is a homotopy equivalence such that $T(X)$ and $f^*T(Y)$ are isomorphic as vector bundles with their natural $GL_c(E)$-structure. A *Fredholm diffeomorphism* $f : X \to Y$ is a diffeomorphism which is an isomorphism of Fredholm structures.

THEOREM. – *Let X and Y be two Fredholm manifolds, and $f : X \to Y$ a tangential homotopy equivalence. Then f is homotopic to a Fredholm diffeomorphism of X onto Y.*

COROLLARY. – *Every $GL_c(E)$-reduction of ξ is homotopic to one and (up to diffeomorphism) only one integrable reduction.*

Any Hilbert manifold admits a parallelizable Fredholm structure [9, 12]. Therefore, *two such manifolds are diffeomorphic if and only if they have the same homotopy type.* That result is discussed in [2 § 4] ; the methods described below provide a different and more direct proof, which moreover can be applied to manifolds modeled on more general Banach spaces. Another consequence is that *any Hilbert manifold admits an embedding onto an open subset of E.* The proof below is similar to that given in [8].

3. Some methods.

Let us now indicate the main steps in establishing the above-mentioned results. Our primary aim is to illustrate the role played by Fredholm maps.

(1) Since $GL(E)$ *is an absolute retract* (a theorem of Kuiper), the principal bundle ξ is trivial. Consequently, by choosing a parallelization of X we can identify the sections of η with the maps $X \to GL(E)/GL_c(E)$.

(2) Let $p : L(E) \to L(E)/C(E)$ be the coset projection. If $\Phi_0(E)$ denotes the subset of $L(E)$ consisting of all Φ_0-operators (i.e., those $u \in L(E)$ with dim Ker u = dim Coker $u < \infty$), then it is elementary that *p induces a homotopy equivalence*

$\Phi_0(E) \to GL(E)/GL_c(E)$. In particular, p induces a bijection between the spaces of homotopy classes

$$[X, \Phi_0(E)] \to [X, GL(E)/GL_c(E)].$$

(3) A *Fredholm map of index* 0 (briefly, a Φ_0*-map*) $f : X \to E$ is a smooth map such that its differential at each point is a Φ_0-operator. Let $\Phi_0[X, E]$ denote the space of homotopy classes of these Φ_0-maps. It is established in [9, 12] that *a choice of parallelization of* X *determines a bijection* $\Phi_0[X, E] \to [X, \Phi_0(E)]$.

(4) For technical reasons it is important now to refine our notion of Fredholm structure : A *layer structure* on X is a maximal atlas $\mathfrak{A} = \{(\theta_i, U_i)\}$ such that the transition maps have the form $\theta_j \circ \theta_i^{-1} = I + \alpha$, where α is locally finite dimensional (i.e. every point of its domain has a neighborhood whose α-image lies in a finite dimensional subspace of E). Then [9, 12] *every* Φ_0*-map* $X \to E$ *determines a unique layer (and hence Fredholm) structure on* X ; *every layer structure on* X *determines a* Φ_0*-map* $X \to E$, *unique up to locally finite dimensional perturbations.* Thus we now have relations between $[X, \Phi_0(E)]$, $\Phi_0[X, E]$, and layer structures on X. These steps give the existence part of the Corollary.

(5) [17] If E_n is the subspace of E spanned by the first n vectors of a base for E, then we can choose a Φ_0-map $f : X \to E$ which is transversal to each E_n. Therefore the $X_n = f^{-1}(E_n)$ *form a nested sequence of finite dimensional closed submanifolds of* X *whose union is dense in* X. Furthermore, the natural map of the inductive limit space $\varinjlim X_n \to X$ is a homotopy equivalence.

(6) *Any two open discs in E are layer diffeomorphic* (i.e., by a diffeomorphism of the form $I + \alpha$, as above) ; *and are ambient layer isotopic* [4]. Any two closed layer tubular neighborhoods of a layer submanifold of the layer manifold X are ambient layer isotopic. This refines [2, § 4].

(7) *There are nested open neighborhoods* U_n *of* X_n *in* X *(which are rather like layer tubular neighborhoods) such that* $\cup U_n = X$.

(8) Induction on n and repeated use of Step (6) are used to prove that *there is a layer diffeomorphism of* X *onto* $X \times E$ *(with its product layer structure)* ; here a layer diffeomorphism is locally of the form $T + \alpha$ with T a fixed linear isomorphism of E onto $E \times E$). That X and $X \times E$ are diffeomorphic (conjecture of Palais) was first established in [2, 16, 7] via Morse-Smale handlebody theory ; then more directly in [10] for open subsets $X \subset E$.

(9) *A layer manifold* X *has* $X \times E$ *layer diffeomorphic to the total space of a layer vector bundle over an open subset of* E. This follows immediately from the existence of closed layer embeddings and tubular neighborhoods.

(10) A version of a theorem of Mazur [15, 2] asserts that *if two layer manifolds are tangentially homotopy equivalent (e.g., as Fredholm manifolds) by a map* $f : X \to Y$, *then* f *is homotopic to a layer diffeomorphism* $f_1 : X \times E \to Y \times E$. Because of (9) it suffices to prove this in the case of parallelizable layer manifolds.

4. Illustrations and applications.

Example 1.

If U is a contractible open subset of E, then its layer (or Fredholm) structure – determined by the single chart U – is unique. In particular, there is a layer diffeomorphism of U onto E. By way of contrast, there are many distinct real analytic layer structures on E, for there is no real analytic bounded layer map $E \to E$. Similarly, some other Banach spaces (e.g., the space $C[0,1]$) possess distinct smooth Fredholm structures.

Example 2.

A Φ_0-map $f : E \to E$ induces a layer structure on E. The uniqueness part of the Corollary implies that there is a diffeomorphism d on E such that $f \circ d = I + \alpha$, where α is locally finite dimensional. Equivalently, any such Φ_0-map differs from a diffeomorphism by a locally finite dimensional map.

Example 3.

Let M be a complete finite dimensional Riemannian manifold, and $a \in M$. Let $P_a(M)$ denote the space of all paths $x : I = [0,1] \to M$ with $x(0) = a$, and x absolutely continuous with square integrable derivative. Then $P_a(M)$ is a separable smooth Hilbert manifold ; a complete Riemannian structure on $P_a(M)$ is given by the inner product

$$<u, v>_x = \int_I \left\langle \frac{Du(t)}{dt}, \frac{Dv(t)}{dt} \right\rangle_{x(t)} dt,$$

where D/dt denotes covariant differentiation along the path x. If $M(a)$ denotes the tangent Euclidean space to M at a, then *E. Cartan's development* $\mathfrak{b}$ *induces a natural diffeomorphism of* $P_a(M)$ *onto the Hilbert space* $P_0(M(a))$. Thus $\mathfrak{b}$ exhibits the unique Fredholm structure on the contractible manifold $P_a(M)$. (Here $\mathfrak{b}$ is defined as follows : Let τ_0^s denote parallel translation along x from $M(x(s))$ to $M(a)$. Then

$$t \to \mathfrak{b}(x)t = \int_0^t \tau_0^s \, x'(s)\, ds$$

defines a path in $P_0(M(a))$.

Let us remark in passing that $\mathfrak{b}$ plays a basic role in the theory of Wiener measure w_a on the Banach manifold $C_a(M)$ of continuous paths on M starting at a. (w_a is defined via the Riemannian heat kernel of M). If w_0 denotes the Wiener measure on the Banach space $C_0(M(a))$ (using the Euclidean heat kernel of $M(a)$), then *stochastic integration determines an extension of* $\mathfrak{b}$ *such that* $\mathfrak{b}_* w_a = w_0$. This is a reformulation of a theorem of Gangolli [13]. It can be viewed as a transformation of integral formula, presumably related to that given by Cameron-Martin [3] in the case $M = \mathbf{R}$. Indeed, various types of layer structures arise naturally in attempting to establish a differentiable measure theory on Banach manifolds, owing to the restrictive nature of the transformation of integral formulae.

Example 4.

Let $\pi : P_a(M) \to M$ be defined by $\pi(x) = x(1)$. Let $P_{aB}(M) = \pi^{-1}(B)$, where B is a closed submanifold of M of codimension q. Then $P_{aB}(M)$ is a closed q-codimensional submanifold of $P_a(M)$, with normal bundle

$$\mathfrak{N}(P_a(M), P_{aB}(M)) = \pi^* \mathfrak{N}(M, B).$$

The Fredholm structure of $P_a(M)$ induces a Fredholm structure on $P_{aB}(M)$. Its Stiefel-Whitney class $w(P_{aB}(M))$ can be identified as the inverse of $\pi^* w(M, B)$, where $w(M, B)$ is the normal class of B in M.

REFERENCES

[1] ANDERSON R.D. — *Homeomorphisms of infinite-dimensional manifolds,* I.C.M. Nice, 1970.

[2] BURGHELEA D. and KUIPER N.H. — Hilbert manifolds, *Annals of Math.*, 90, 1969, p. 379-417.

[3] CAMERON R. and MARTIN W.T. — Transformations of Wiener integrals by non-linear transformations. *Trans. Amer. Math. Soc.*, 66, 1949, p. 385-392.

[4] DOUADY A. — *Equivalence de Fredholm entre les boules d'un espace de Hilbert.*

[5] EELLS J. — A setting for global analysis. *B.A.M.S.*, 72, 1966, p. 751-807.

[6] EELLS J. — Fredholm structures. *Proc. Symp. Non-linear Functional Analysis,* Chicago A.M.S., 1968.

[7] EELLS J. and ELWORTHY K.D. — On the differential topology of Hilbertian manifolds. *Proc. Summer Institute on Global Analysis,* Berkeley A.M.S., 1968.

[8] EELLS J. and ELWORTHY K.D. — Open embeddings of certain Banach manifolds. *Annals of Math.*, 91, 1970, p. 465-485.

[9] ELWORTHY K.D. — Fredholm maps and GL_c (E)-structures. Oxford Thesis (1967). *B.A.M.S.*, 74, 1968, p. 582-586.

[10] ELWORTHY K.D. — Embeddings, isotopy, and stability of Banach manifolds. *Comp. Math.* (to appear).

[11] ELWORTHY K.D. and TROMBA A.J. — Degree theory on Banach manifolds, *Proc. Symp. Non-linear Functional Analysis,* Chicago, A.M.S. 1968.

[12] ELWORTHY K.D. and TROMBA A.J. — *Fredholm maps and differential structures on Banach manifolds.* Summer Institute on Global Analysis. Berkeley. A.M.S. 1968.

[13] GANGOLLI R. — On the construction of certain diffusions on a differentiable manifold. *Z. Wahrscheinlichkeitstheorie,* 2, 1964, p. 406-419.

[14] KUIPER N.H. — *The differential topology of separable Banach manifolds,* I.C.M. Nice, 1970.

[15] MAZUR B. — Stable equivalence of differentiable manifolds. *B.A.M.S.*, 67, 1961, p. 377-384.

[16] MOULIS N. — Sur les variétés hilbertiennes et fonctions non-dégénérées. *Proc. Ned. Ak. v. W. Amsterdam,* LXXI-5, 1968, p. 497-512.

[17] MUKHERJEA K.K. — Cohomology Theory for Banach Manifolds, *Jour. of Math. and Mech.*, 19, 1970, p. 731-744.

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A TOPOLOGICAL TECHNIQUE FOR THE CONSTRUCTION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS AND INEQUALITIES

by M.L. GROMOV

1.— For smooth fibering $\alpha = \{X \to M\}$ we denote by X^r the manifold of r-jets of cross-sections $M \to X$ and by $\alpha^r = \{X^r \to M\}$ the natural fibering. Consider a set $\Omega \subset X^r$ and let us call it a r-order differential condition. We denote by $B = B^s(\Omega)$, $s = 0, 1, \ldots, \infty$, a (a is the real analyticity) the space of C^s-cross-sections

$$M \to \Omega \subset X^r$$

and by $A = A^{S+r}(\Omega)$ the space of C^{S+r} cross-sections $\gamma : M \to X$ with $J^r_\gamma \in B$ (i.e. the jet $J^r_\gamma : M \to X^r$ maps M in Ω). Consider the map $J : A \to B$, $J : \gamma \mapsto J^r_\gamma$.

If the map $J_* : \pi_0(A) \to \pi_0(B)$ induced on 0 dimensional homotopies is surjective we say that for Ω and for cross-section from A e-principle is true. If all homomorphisms $J_* : \pi_i(A) \to \pi_i(B)$ are bijective we say that w.h.e. -principle is true.

2. — On the examples 1.–7. the space A appears without describing the corresponding condition Ω which is of the second order in the examples 2, 3, 6 b and of the first order in the rest.

In all the examples except 7, we deal with trivial fibering $\alpha = \{M \times N \to M\}$ and so we consider maps $M \to N$ instead of cross-sections $M \to M \times N$.

(1) *If M is open or if $K < \dim N$, then for smooth maps $M \to N$ having rank $\geqslant K$ at each point $m \in M$ w.h.e. -principle is true* (S. Feit [3]). For immersions ($K = \dim M$) it has been proved by M. Hirsch and for submersions ($K = \dim N$) by A. Phillips [14] proved also w.h.e. -principle for maps $M \to N$ (M is open) which are transversal to leaves of given foliation on N and pointed out the corollary : *A plane field ξ on an open manifold M is homotopic to an integrable field if the structural group of the factor bundle $\tau(M)/\xi$ can be reduced to a discrete group.*

(2) A C^2 – map $f : M \to \mathbf{R}^q$ is called a free map (see [10]) if at each $m \in M$ vectors $\frac{\partial f}{\partial x_i}(m), \frac{\partial^2 f}{\partial x_i \, \partial x_j}(m) \in \mathbf{R}^q$, $1 \leqslant j \leqslant i \leqslant \dim M = n$ are linearly independent.

If $q > \frac{n^2}{2} + \frac{3}{2} n$ *or if M is open then for free maps w.h.e. -principle is true* [8].

(3) Consider a connected manifold M (non-empty !) closed manifold $C \subset M$ with codim $C = 1$ and a manifold N with dim N = dim M.

For smooth maps $M \to N$ with a crease at $C \subset M$ and without other singularities Σ^I (see [1]) *e – principle is true* (Eliashberg [2]).

COROLLARY. – *For any closed surface $C \subset S^3$ there exists a map $S^3 \to \mathbf{R}^3$ whose restrictions to C and to $M \backslash C$ are immersions.*

(4) Let M and N be complex manifolds and M a Stein manifold.

a) *If N is a complex Lie group then for holomorphic maps $M \to N$ w.h.e. -principle is true* (Grauert [4]).

b) *For holomorphic immersions (regular maps) $M \to \mathbf{C}^q$ w.h.e. -principle is true if $q > \dim_{\mathbf{C}} M$*([9]).

COROLLARY. – *If $2q \geqslant 3 \dim_{\mathbf{C}} M - 1$, $q > 1$, then there exists a holomorphic immersions $M \to \mathbf{C}^q$ (If $q > 2 + \dim_{\mathbf{C}} M (1 + \ln 2)$, then there exists a proper regular holomorphic embedding $M \to \mathbf{C}^q$* ([9])).

c) *If M is a complexification of a real manifold $M_0 \subset M$, then for holomorphic immersions $M \to N$ w.h.e. – principle is true near M_0 (i.e. in an inductive limit over neigbourhoods of $M_0 \subset M$* (Author, unpublished).

COROLLARY. – *A (real) n-dimensional π-manifold has a complexification which can be holomorphicly immersed in $\mathbf{C}^n$.*

(5) Let M and N be symplectic manifolds i.e. exact nondegenerate 2-forms ω_1 on M and ω_2 on N are given. As a rule for symplectic immersions $f: M \to N$ (i.e.

$$f^*(\omega_2) = \omega_1)$$

even e-principle is not true (An obvious obstruction is given by

$$f^* : H^2(N ; \mathbf{R}) \to H^2(M ; \mathbf{R}))$$

a) *If $\dim M < \dim N - 2$, then for symplectic immersions $f : M \to N$ with permissible homomorphisms $f^* : H^2(N ; \mathbf{R}) \to H^2(M ; \mathbf{R})$ w.h.e. – principle is true.* (Author, unpublished).

COROLLARY. – *If the periods of ω_1 are integrals then ω_1 can be induced by some map $M \to CP^q, q = \dim M$, from the standard 2-form on CP^q.*

b) *If M and N possess the standard symplectic structures of cotangent bundles of $M_0 \subset M$ and $N_0 \subset N$, then for symplectic immersions $M \to N$ w.h.e – principle is true near M_0* (Author, unpublished).

COROLLARY. – *If M_0 is an n-dimensional π-manifold then there exists a Lagrangian* (i.e. $f^*(\omega_2) = 0$) *immersion $M_0 \to \mathbf{R}^n \times \mathbf{R}^n$, $\omega_2 = \sum_{i=1}^{n} dx_i \wedge dy_i$ (compare with 4.c).*

(6) Let M be an n-dimensional Riemannian manifold.

a) *For isometric C^1 – immersions $M \to \mathbf{R}^q$ w.h.e. – principle is true if $q > \dim M$* (Nash [11]).

b) *If* $q \geqslant \frac{n^2}{2} + \frac{7n}{2} + 5$, *then for free* (see (2)) *isometric* $C^\infty (C^a)$ – *immersions* $M \to \mathbf{R}^q$ *w.h. e. – principle is true* ([6]).

COROLLARY. – *An n-dimensional Riemannian* $C^\infty (C^a)$ – *manifold can be isometricaly* C^∞ (C^a) *imbedded in* $\mathbf{R}^q$ *with* $q = \frac{n^2}{2} + \frac{7}{2} n + 5$ ([6], [10]).

c) *For free isometric* $C^\infty (C^a)$ – *immersions* $M \to \mathbf{R}^q$ *w.h.e. – principle is true near any submanifold* $M_0 \subset M$ *with* codim $M_0 \geqslant 2$ (*q is arbitrary !*) ([6]).

d) *Theorems similar to a., b., c. are true for isometric immersions* $M \to N$ *where M and N are pseudoriemannian manifolds* ([10]).

7) Consider an n-dimensional manifold M, an l-dimensional (real) vector bundle ψ over M and a k-dimensional vector bundle ξ over M with an Euclidean connection. A monomorphism $\psi \to \xi$ is called a regular one if (first) covariant derivatives of cross-sections of ξ which came from ψ generate in ξ a subbundle of dimension $(n + 1) l$.

If $k > (n + 1) l$, *then for regular monomorphisms* $\psi \to \xi$ *which induce a given Euclidean connection on* ψ *w.h. e. – principle is true.* (Author, unpublished).

COROLLARY. – *A bundle* ψ *with an arbitrary Euclidean connection can be induced by a map* $M \to G_{l,q}$ *with* $q = ln + n + l$ *from the standard l-dimensional bundle with the standard connection over Grassmanian manifold* $G_{l,q}$.

3. – Using the notation of 1. let us formulate an abstract fact which generalizes 2.(1) and partly 2.(2). Let M_0 be a smooth manifold with dim $M_0 \leqslant$ dim M and $\Psi : M_0 \to M$ a smooth map. Consider the induced fibrations $\alpha_0 = \varphi^*(\alpha) = \{X_0 \to M_0\}$ and $\beta = \varphi^*(\alpha^r) = \{Y \to M_0\}$. Let $\phi : Y \to X^r$ be the fibrewise map associated with $\Psi, \Pi : Y \to X_0^r$ the natural map and $\varphi^\#(\Omega) = \Pi(\phi^{-1}(\Omega)) \subset X_0^r$. Let the pseudogroup of the local diffeomorphisms of M fibrewise act on X and so on X^r (For example α is the tangent bundle $\tau(M)$ or α is associated to $\tau(M)$). Let $\Omega \subset X^r$ be invariant and open.

A. *Let* φ *be a generic map (for example* φ *is an immersion or* φ *is locally structurally stable)* :

(1) *If M is open then for* $\varphi^\#(\Omega)$ *e-principle is true.*

(2) *If* M_0 *is open or if dim* $M_0 <$ *dim M, then for* $\varphi^\#(\Omega)$ *w.h.e. – principle is true* (Author, unpublished).

Examples 2.4.c. and 2.5.b. illustrate the complex and symplectic generalisations of A. Let us give some other examples connected with A.

B. *Let M be a connected manifold with a non-empty boundary* $\mathcal{D}$ *and* $V \subset M$ *be a closed tubular neighbourhood of* $\mathcal{D}$. *Let a finite group G free and smoothly acts on V. If* $\mathcal{D}$ *is invariant then there exists a Riemannian metric of positive (negative) curvature in M for which acting G (on V) is isometrical.*

C. *If M is open then for free maps* $M \to \mathbf{R}^p \times \mathbf{R}^q$ *which induce from the form* $\sum_{i=1}^{p} dx_i^2 - \sum_{i=1}^{q} dy_i^2$ *the zero form on M w.h.e. – principle is true* ([7]).

4. — Let ψ and ξ be (real or complex) vector bundles over M and $\alpha = \underbrace{\psi \oplus \ldots \oplus \psi}_{q}$.
Let $\mathcal{D} : \Gamma^\infty(\psi) \to \Gamma^\infty(\xi)$ be a differential operator. We say that a cross-section

$$\gamma = \{\gamma_1, \ldots, \gamma_q\} \in \Gamma^\infty(\alpha), \quad \gamma_i \in \Gamma^\infty(\psi)$$

is k-regular if the cross-sections $\mathcal{D}(\gamma_i) \in \Gamma^\infty(\xi)$, $i = 1, \ldots, q$, generate in any fibre of ξ a space of dimension $\geqslant k$.

If $k < q$ then for k-regular cross-sections w.h.e. — principle is true ([8]).

The maps $M \to \mathbf{R}^q$ (i.e. cross-sections $M \to M \times \mathbf{R}^q$ from 2.(1) and 2.(2) give examples of k-regular sections.

5. — Besides *e*— and *w.h.e.* — principles there are approximation theorems of the following types :

(1) *Let M be an n-dimensional π-manifold $l \geqslant 0$ be an integer and $p \geqslant 1$ a real number. If $(l-1)\,p < q - n$ then any smooth map $M \to \mathbf{R}^q$ can be $W^{l,p}$-approximated by immersions $M \to \mathbf{R}^q$ and if $(l-2)\,p < q - \frac{n^2}{2} - \frac{3}{2}n$ it can be approximated by free maps $M \to \mathbf{R}^q$.* ($\|f\|_{W^{l,p}} = \int_M |J_f^l|^p$) ([8]).

(2) *Let M be a compact n-dimensional Riemannian C^∞ (C^a)-manifold. If*

$$q \geqslant \frac{n^2}{2} + \frac{7}{2}n + 5,$$

then any isometric C^1-immersion $M \to \mathbf{R}^q$ can be C^1-approximated by free isometric C^∞ (C^a)-immersions ([6], [10]).

Methods.

M_1 : Smale's covering homotopy property ([5]) M_1 proves 2.(1), 2.(4)c., 2.(5)b., 3 A.(2).
M_2 : Nash's implicit function theorem ([12], [13], [10], [7].).
M_3 : Nash's twisting ([11], [12], [10], [15]) M_3 proves 2.(1), 2.(5)a., 2.(6)a.
M_4 : The elimination of singularities ([8], [9]) M_4 proves 2.(1), 2.(2), 2.(4)b., 4, 5.1.
M_5 : Eliashberg's model ([2]). M_5 proves 2.(3).

Besides, $M_1 + M_2$ proves 3.C. ; $M_1 + M_5$ proves 3 A.(1), 3 B. ; $M_2 + M_3$ proves 2.(6)b. — d, 5.(2).

REFERENCES

[1] Boardman J. M. — Singularities of differentiable maps, *Univ. of Warwick*, 1965, p. 1-105.

[2] Eliashberg Ja. M. — Ob osobennostjah tıpa skladki, *Isv. Acad. Nauk S. S. S. R.*, 34, 1970, 1111-1127.

[3] FEIT S. D. — K-mersions of manifolds, *Act. math.*, 122, 1969, No. 3, 4, p. 173-195.

[4] GRAUERT H. — Holomorphe Functionen mit Werten in Komplexen Lieschen Gruppen, *Math. Ann.*, 133, 1957, 450-472.

[5] GROMOV M. L. — Stabil'niye otobrazhenija slojenii v mnopoobrazija, *Isv. Acad. Nauk S. S. S. R.*, 33, No. 4, 1969, p. 707-734.

[6] GROMOV M. L. — Isometritsheskie vlozhenija i pogruzhenija, *Dokl. Akad. Nauk S. S. S. R.*, 192, No. 6, 1970, p. 1206-1209.

[7] GROMOV M. L. — Sglazhivanie i obrastshenie differencialnih operatorov (to appear).

[8] GROMOV M. L., ELIASHBERG Ja. M. — Ustranenie osobennostei gladkih otobrazhenii, *Isv. Akad. Nauk S. S. S. R.* (to appear).

[9] GROMOV M. L., ELIASHBERG Ja. M. — *Neosobie otobrazhenija mnogoobrazii Steina* (to appear).

[10] GROMOV M. L., ROCHLIN V. A. — Vlozhenija i progruzhenija v rimanovoi geometrii, *Uspechi mat. Nauk* (S. S. S. R.), Vol. XXV, 5, 155, 1970, p. 3-62.

[11] NASH J. — C^1 isometric imbeddings, *Ann. of Math.*, 2, 60, No. 3, 1954, p. 383-396.

[12] NASH J. — The imbedding problem for Riemannia manifolds, *Ann. of. Math.*, 63, 1965, p. 20-63.

[13] NASH J. — Analyticity of the solutions of implicit function problems with analytic data, *Ann. of Math.*, 84, 1966, No. 2, p. 345-355.

[14] PHILLIPS A. — Foliations on open manifolds, II, *Comm. Math. Helv.*, 44, 1969, p. 367-370.

[15] KNIPER N. H. — On C^1-isometric imbeddings, *Nederl. Akad. Wet. Proc.*, Ser. A-58, 1955, No. 5.

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ON SUBELLIPTIC ESTIMATES FOR COMPLEXES

by Victor GUILLEMIN

Let E^0, E^1, E^2, etc. be vector bundles on a manifold X. Let

$$(*) \qquad 0 \to E^0 \xrightarrow{D} E^1 \xrightarrow{D} \ldots \xrightarrow{D} E^N \to 0$$

be a complex of differential operators. For simplicity we will assume all the D's are of order one. Given a complex covector $\zeta \in T_x^* \otimes C$, we get the symbol sequence attached to (*)

$$(**) \qquad 0 \to E_x^0 \xrightarrow{\sigma(D)(\zeta)} E_x^1 \to \ldots \xrightarrow{\sigma(D)(\zeta)} E_x^N \to 0$$

which is a complex of linear mappings. We will say $\zeta \in T_x^* \otimes C$ is *characteristic* if this sequence fails to be exact. The set of characteristic vectors forms an algebraic subvariety of $T_x^* \otimes C$, which we will denote by $\mathfrak{U}_x$. This is the zero variety of an ideal of polynomial functions on $T_x^* \otimes C$ which we will call the *characteristic ideal.* The characteristic ideal is defined as follows : Consider all homotopy operators for the sequence (**), i.e. all polynomial mappings

$$E_x^N \xrightarrow{A(\zeta)} E_x^{N-1} \xrightarrow{A(\zeta)} \ldots \xrightarrow{A(\zeta)} E_x^0 \to 0$$

such that :

$$\sigma(D)\,(\zeta)\,A(\zeta) + A(\zeta)\,\sigma(D)\,(\zeta) = p(\zeta)\ \text{Identity}$$

$p(\zeta)$ being a polynomial in ζ. The set of all p occuring on the RHS of this equation form an ideal. This is the characteristic ideal of (*), denoted by I_x.

Example – Let E^0 and E^1 be trivial line bundles over R^n, and let D be the Laplacian squared. The characteristic variety is the set of points where $\zeta_1^2 + \ldots + \zeta_n^2 = 0$ and the characteristic ideal is generated by $(\zeta_1^2 + \ldots + \zeta_n^2)^2$.

As we vary x, $\mathfrak{U}_x$ and I_x get deformed. If (*) is a Spencer complex, one can show :

(i) dim $\mathfrak{U}_x$ is the same for all x.

(ii) degree $\mathfrak{U}_x$ is a lower semi-continuous function of x and hence is constant on an open dense set. From now on we will assume (*) is a Spencer complex and that *both* dimension and degree are the same for all x. (This is to avoid pathological examples like the Tricomi operator).

DEFINITION 1. – We will say that a characteristic, ζ, is generic if :

(a) $\mathfrak{U}_x$ is non-singular at ζ.

(b) dimension $H(x, \zeta)$ is as small as possible, $H(x, \zeta)$ being the homology of the symbol sequence at the characteristic x, ζ.

DEFINITION 2. – We will say ζ is simple if :

(a) There exist $p_1, \ldots, p_q \in I_x$ such that $(\partial p_i/\partial \zeta_j)$ is of rank q and $\mathfrak{U}_x$ is locally the locus of points where $p_1 = \ldots = p_q = 0$.

(b) dimension $H(x, \zeta)$ is as small as possible.

Note. – Definition 2 is much stronger than definition 1. If D is the Laplacian squared, every non-zero characteristic is generic but no characteristic is simple.

PROPOSITION. – (Generic characteristic parameterization theorem).

Let $\zeta_0 \in \mathfrak{U}_{x0}$ be a generic characteristic. Then there exists a neighborhood, U, of (x_0, ζ_0) in the complex cotangent bundle and smooth functions $f_1, \ldots, f_q$ on U such that :

(a) $f_i(x, \zeta)$ is holomorphic and homogeneous of degree one in ζ.

(b) The Jacobian criterion rank $(\partial f_i/\partial \zeta_j) = q$ is satisfied.

(c) $\mathfrak{U} = \bigcup_{x \in X} \mathfrak{U}_x$ is just the locus of points where $f_1, \ldots, f_n$ are zero on U.

(d) $\{f_i, f_j\} = 0$.

The proof of this theorem is rather complicated, and we won't attempt to describe it here.

Let $\zeta_0 \in \mathfrak{U}_{x0}$ be a *real* generic characteristic. Let $f_1, \ldots, f_q$ be a parameterization of the characteristic set around x_0, ζ_0 satisfying (a), (b), and (c). We will call the Hermetian form $\frac{1}{\sqrt{-1}}\{f_i, \overline{f_j}\}$ the *Levi* form at x_0, ζ_0. It can be regarded as a Hermetian form on the normal space to the set of characteristics at x_0, ζ_0 providing we use as a basis for this space $df_1, \ldots, df_q$.

Before stating our main results we need one further notion, that of subellipticity for a complex of differential operators.

DEFINITION 3. – Let X be a compact manifold without boundary. The complex (*) is subelliptic at its ith position if, there is an estimate of the form :

$$C(\|D\varphi\| + \|D^*\varphi\| + \|\varphi\|) \geqslant \|\varphi\|_{1/2}$$

for all sections φ of E^i.

One can show by standard techniques :

PROPOSITION. – If X is a compact manifold without boundary and (*) is subelliptic in its ith position, then its ith homology group is finite dimensional.

The main results about subellipticity which we will quote here are due to Hörmander. At each characteristic $\zeta \in \mathfrak{U}_x$ Hörmander associates a certain test operator on R^n of the form :

$$A + B^i \frac{\partial}{\partial y^i} + C^i y_i$$

whose coefficients are functions of x and ζ. The Hörmander localization theoren says that (*) is subelliptic if and only if, for every characteristic x , ζ the test operator satisfies an L^2 estimate uniformly in a neighborhood of x , ζ. We will call this condition the Hörmander localization condition.

DEFINITION 4. – If the Hörmander localization condition is satisfied at x , ζ we will say that the complex (*) is subelliptic at x , ζ.

THEOREM 1. – *If ζ is a simple characteristic, then (*) is subelliptic at (x , ζ) in its ith position, $0 \leqslant i \leqslant q$, if and only if the Levi form had $i + 1$ positive eigenvalues or $q - i + 1$ negative eigenvalues.*

COROLLARY. – If ζ is simple, then (*) is subelliptic at (x , ζ) in all positions except $i = 0$ iff and only if the Levi form is negative definite.

This corollary has a kind of converse :

THEOREM 2. – *If ζ is a generic characteristic and the Levi form is negative definite, then in order for (*) to be subelliptic at (x , ζ) in all positions except $i = 0$, ζ must be a simple characteristic.*

Conjecture – If ζ is generic, then a neccessary and sufficient condition for (*) to be subelliptic at (x , ζ) in all positions except $i = 0$ is that the Levi form be negative definite and that ζ be simple.

Let X be a compact manifold with a smooth boundary. Suppose that the boundary is non-characteristic. Then one gets a differential complex induced on the boundary, called the boundary complex.

Example – Let D be the $\overline{\partial}$ complex and X a smooth domain in C^n. The boundary complex is the complex of Cauchy-Riemann equations tangent to ∂X. This has been studied by Kohn and Rossi.

At $x \in \partial X$ the characteristic variety of the boundary complex is the image of $\mathcal{U}_x$ under the projection $T^*_x \otimes C \rightarrow T^*_x(\partial X) \otimes C$. This variety can be highly singular ; however, we will show that the boundary complex is subelliptic providing the *complex* characteristics of the D complex satisfy certain conditions.

THEOREM 3. – *Let ξ be a real characteristic of the boundary complex. We will assume :*

(a) *the complex characteristics in $\mathcal{U}_x$ lying above ξ are simple* ;

(b) *(Calderon condition) the normal vector $n_x \in T^*_x$ is not tangent to $\mathcal{U}_x$ at the characteristics indicated in* (a).

Then we can conclude :

(i) the characteristic variety of the boundary complex at ξ is locally the union of a finite number of complex submanifolds, $\mathcal{W}_1 \cup \mathcal{W}_2 \ldots \cup \mathcal{W}_k$.

(ii) To each $\mathcal{W}_s$ we can associate a Levi form L_s. The boundary complex is subelliptic at (x , ξ) in its ith position if and only if, for all s, L_s has $i + 1$ positive eigenvalues or $q - i$ negative eigenvalues.

We will say that $\mathfrak{W}_s$ is *positive* if it is the image of a sheet of the characteristic variety $\mathfrak{U}_x$ passing through a characteristic of the form $\xi + \tau n_x$ with $\operatorname{Im} \tau > 0$.

Conjecture – Suppose the complex (*) is elliptic and suppose conditions (a) and (b) of theorem 3 are satisfied at all real characteristics of the boundary complex. Then the ith homology group of (*) is finite dimensional providing the Levi forms L_s associated with positive $\mathfrak{W}_s$ have either $i + 1$ positive eigenvalues or $q - i$ negative eigenvalues.

We have been able to prove the following slightly weaker assertion :

THEOREM 4. – *The ith homology group is finite dimensional if each of the indicated L_s's has either $i + 1$ positive eigenvalues or $q - i + 1$ negative eigenvalues.*

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CONVEXITY CONDITIONS RELATED TO 1/2 ESTIMATE ON ELLIPTIC COMPLEXES

by Masatake KURANISHI

Let $\mathcal{E}$ be an elliptic complex defined on a manifold $Y^{\#}$, i.e. we are given a pair consisting of a sequence of vector bundles $E^0, E^1, \ldots, E^j, \ldots$ on $Y^{\#}$ and a sequence of C^∞ differential operators $\mathbf{D}^j : C^\infty(Y^{\#}, E^j) \to C^\infty(Y^{\#}, E^{j+1})$, $j = 0, 1, 2, \ldots$, satisfying the following conditions :

(1) $\mathbf{D}^{j+1} \circ \mathbf{D}^j = 0$ for all j

(2) for each non-zero cotangent vector ξ at $x \in Y^{\#}$ the sequence of linear mappings $\sigma(\mathbf{D}^j, \xi) : E^j_x \to E^{j+1}_x$ is exact. In the above, $C^\infty(Y^{\#}, E)$ denotes the vector space of C^∞ sections of E over $Y^{\#}$, E_x the fiber over x of the vector bundle E, and $\sigma(\mathbf{D}, \xi)$ denotes the symbol of the differential operator $\mathbf{D}$ at x. For simplicity we assume that each $\mathbf{D}^j$ is of the first order. Since it is usually clear from the context we omit the index j in $\mathbf{D}$. We consider only C^∞ category here, so the adjective C^∞ will be omited.

Let Y be an open submanifold of $Y^{\#}$ with compact closure Y^- such that its boundary M is a submanifold. Then the complex on $Y^{\#}$ induces a complex

$$(*) \qquad C^\infty(Y^-, E^0) \to C^\infty(Y^-, E^1) \to \ldots$$

where the arrows are induced by $\mathbf{D}$. The j-th homology of the complex $(*)$ is denoted by $H^j(Y^-, \mathcal{E})$. The problem we discuss here is to find conditions which guarantee the finite dimensionality of $H^j(Y^-, \mathcal{E})$ for a given j. The problem is also discussed by D.C. Spencer and Victor Guillemin in this Congress. There are a number of methods to attack the problem. For example, to solve Neuman-Spencer boundary value problem is one of them. Another method is the one developed by Calderon, Hörmander, Seeley, and others, by which the problem is transformed to one on the boundary. Namely, for a given j we can construct vector bundles E, F on M and a pseudo-differential operator $A : C^\infty(M, E) \to C^\infty(M, F)$ of order 1 such that the finite dimensionality of $H^j(Y^-, \mathcal{E})$ is equivalent to that of the kernel of A. Since we can calculate the symbol of A in terms of the given datum, to find an answer to our original problem it would be enough to find conditions on the symbol of A so that the kernel is finite dimensional. This, in turn, follows if we have an estimate

$$(**) \qquad \|Au\|^2 + \|u\|^2 \geqslant c(\|u\|_{1/2})^2 \qquad (u \in C^\infty(M, E))$$

for a constant c, where $\| \;\|$ (resp. $\| \;\|_{1/2}$) denotes L_2-norm (resp. Sobolev 1/2 norm). This is due to the theory developed by Morrey, Kohn, and Nirenberg. Neuman-Spencer procedure leads to an estimate of the same type. So we will discuss conditions on the symbol of A which imply the estimate $(**)$. We set

$$Q(u) = \|Au\|^2 + \|u\|^2 .$$

It is easy to see that in order to have the estimate (**) it is necessary and sufficient that each point of M has a neighborhood U such that (**) holds for all u with support in U. So we may assume that A is a pseudo-differential operator $C^{\infty}(\mathbf{R}^n, E) \to C(\mathbf{R}^n, F)$, and we wish to have the estimate (**) for u with support in a sufficiently small neighborhood of origin. We generally denote by $x = (x_1, \ldots, x_n)$ a point in $\mathbf{R}^n$ and (x, ξ), $\xi = (\xi_1, \ldots, \xi_n)$, the cotangent vector $\xi_1 dx_1 + \ldots + \xi_n dx_n$ at x. We denote by $a(x, \xi)$ the symbol of A. If $p(x, \xi)$ is a symbol of pseudo-differential operator, $p(x, D)$ denotes the pseudo-differential operator defined by it.

Let $a^1(x, \xi)$ be the homogenous order 1 part of $a(x, \xi)$. If $a^1(x, \xi)$ is injective for all non-zero cotangent (x, ξ) with x in the closure of a neighborhood U of origin, we have a stronger estimate $Q(u) \geqslant c \, \|u\|_1^2$ for all u with support in U. In the following U generally denotes a neighborhood of origin which we may shrink if necessary. Thus, it is natural to consider the set $\mathcal{C}$ of non-zero (x, ξ) such that $a^1(x, \xi)$ is not injective. An element in $\mathcal{C}$ is called a characteristic of A. Then we have the following.

LEMMA 1. – Let $f(x, \xi)$ be a symbol of pseudo-differential operator of order o such that $\operatorname{Supp} f \cap \mathcal{C}$ is compact. Then

$$Q(u) \geqq c \, \|f(x, D) u\|_1^2 .$$

Since we can construct partitions of unity by symbols of p.d. operators of order o and apply the corresponding operators, it is left to analyze $f(x, D)u$ where $\operatorname{Supp} f \cap \mathcal{C}$ is not compact. To proceed further along this line, we introduce the following conditions. We say that characteristics of A are smooth if each point x in M has a neighborhood U such that $\mathcal{C}$ over U is a disjoint union of a finite number of (non-closed) submanifold $\mathcal{C}^1, \ldots, \mathcal{C}^k$ such that

(1) the projection $\pi : T^*M \to M$ induces a map with constant rank of $\mathcal{C}^\lambda$ onto a submanifold $'\mathcal{C}^\lambda$ of M and

(2) the dimension of the kernel of $a^1(x, \xi)$ for $(x, \xi) \in \mathcal{C}^\lambda$ is a constant for each λ.

Choose a cone neighborhood $\mathcal{U}^\lambda$ of $\mathcal{C}^\lambda$ in T^*U such that the closures of them are still disjoint except origin. If we choose $\mathcal{U}^\lambda$ sufficiently small, there will be $\epsilon > 0$ such that the dimension of the sum $W(x, \xi)$ of the eigen-spaces of $a^1(x, \xi)^* \, a^1(x, \xi)$ with eigen-values less than ϵ is independent of (x, ξ) in $\mathcal{U}^\lambda$, and such that $W(x, \xi) = \ker \, a^1(x, \xi)$ for $(x, \xi) \in \mathcal{C}^\lambda$. Denote by $\rho_1^\lambda(x, \xi)$ a symbol of p.d. operator which (when restricted to $\mathcal{U}^\lambda$) coincides with the projection to $W(x, \xi)$ outside a bounded neighborhood of the set of zero cotangents of T^*U. We define $\rho_2^\lambda(x, \xi)$ by the condition : $\rho_1^\lambda + \rho_2^\lambda = 1$.

LEMMA 2. – Assume that characteristics of A are smooth. Let $f(x, \xi)$ be a symbol of p.d. operator of order o such that $\operatorname{Supp} f \subset \mathcal{U}^\lambda$. Then

$$Q(u) \geqslant c \, \|f(x, D) \, \rho_2^\lambda(x, D) u\|^2$$

for all u with support in U.

Thus we reduced the problem to analyse $\rho_1^\lambda(x, D)u$. So, we try to extract the essential (as far as our estimate is concerned) part of $a^1(x, \xi)\ \rho_1^\lambda(x, \xi)$. Since the matter is fairly complicated, we introduce the following simplyfing conditions : We say that the fiber dimension of characteristics of A at origin is zero when $\pi : \mathcal{C}^\lambda \cap S^*U \to {}'\mathcal{C}^\lambda$ is bijective (where S^*U is the bundle of unit co-tangent vectors). We say that a characteristic ζ of A is non-degenerate, when (assuming $\zeta \in \mathcal{C}^\lambda$) we choose a submanifold $\mathfrak{M}^\lambda$ of S^*U transversal to $\mathcal{C}_1^\lambda = \mathcal{C}^\lambda \cap S^*U$ which intersects $\mathcal{C}_1^\lambda$ only at $\zeta/|\zeta|$, ζ is a non-degenerate critical point of the function $|a^1(x, \xi)\ \rho_1^\lambda(x, \xi)u|^2$ restricted to $\mathfrak{M}^\lambda$ for each $u \in W(\xi)$.

From now on we assume that characteristics of A over U are smooth, fiber dimension zero, and non-degenerate. Then over each $x \in {}'\mathcal{C}^\lambda$ there is a unique characteristic $\zeta^\lambda(x)$ of unit length. We extend the vector field $\zeta^\lambda(x)$ over ${}'\mathcal{C}^\lambda$ to that over U and still denote it by $\zeta^\lambda(x)$. Denote by $\chi(x, \xi)$ the projection of (x, ξ) to the orthogonal complement of $\zeta^\lambda(x)$. Thus we have

$$(x, \xi) = \langle \zeta^\lambda(x), \xi \rangle\ \zeta^\lambda(x) + \chi(x, \xi).$$

We choose a real valued symbol of p.d. operator $\varphi^\lambda(x, \xi)$ (resp. $\varphi(x, \xi)$) of order o with Supp $\varphi^\lambda \subset \mathfrak{U}^\lambda \cup \pi^{-1}$ (complement of U) (resp. Supp $\varphi \cap \mathcal{C}^\lambda$ bounded) such that

$$1 = \varphi(x, \xi) + \varphi^1(x, \xi) + \ldots + \varphi^k(x, \xi).$$

We choose a sufficiently small submanifold N^λ of U which is transversal to ${}'\mathcal{C}^\lambda$ at origin such that the map $N^\lambda \times {}'\mathcal{C}^\lambda \ni (w, y) \to w + y$ is a diffeomorphism onto a neighborhood U^λ of origin. Thus we can write

$$x = w(x) + y(x) \qquad (x \in U^\lambda)$$

where w and y are considered (as we will do in the following) as maps $U^\lambda \to N^\lambda$ and $U^\lambda \to {}'\mathcal{C}^\lambda$, respectively. We may choose $\zeta^\lambda(x)$ so that $\zeta^\lambda(x) = \zeta^\lambda(y)$. Note that $|\xi|^{-1}\chi(x, \xi)$ on Supp φ^λ can be made arbitrary small by choosing φ^λ so that its support lies very close to $\mathcal{C}^\lambda$. This suggests that we may, for each $u \in W(\zeta^\lambda(y))$, expand $a^1(x, \xi)\ \rho_1^\lambda(x, \xi)u$ in Taylor series in (w, χ) at $(y, \langle \zeta^\lambda(y), \xi \rangle \zeta^\lambda(y))$, and examine the contribution of each term in $\|\varphi^\lambda(x, D)\, a(x, D)\ \rho_1^\lambda(x, D)u\|^2$. This approach leads to the following results : We set

$$F^\lambda(x, \xi) = \Sigma\ f^j(x^0)\ \langle \zeta^\lambda(y), \xi \rangle\ w_j + \Sigma\ g^j(x^0)\ \chi_j(x, \xi)$$

where x^0 is origin and

$$f^j(y) = a^{1(j)}(y, \zeta^\lambda(y))\ \rho_1^\lambda(y, \zeta^\lambda(y)) + a(y, \zeta^\lambda(y))\ \rho_2^{\lambda(j)}(y, \zeta^\lambda(y))$$

$$g^j(y) = a^{1\langle j \rangle}(y, \zeta^\lambda(y))\ \rho_1^\lambda(y, \zeta^\lambda(y)) + a(y, \zeta^\lambda(y))\ \rho_2^{\lambda\langle j \rangle}(y, \xi^\lambda(y))$$

where the upper indexs (j) (resp. the upper indexes $\langle j \rangle$) denote the partial derivatives $\partial/\partial x_j$ (resp. $\partial/\partial \xi_j$).

THEOREM. – *Assume that characteristics of A over a neighborhood of origin are smooth, non-degenerate, and of fiber dimension zero. Then the estimate* (**) *holds for any u with support in a sufficiently small neighborhood of origin* *provided we have the following estimate for each* λ : *There is an integer* $d > 0$ *such that for any sufficiently small* $\theta > 0$ *there is a constant* C_θ *with an estimate of the form*

$$(***) \quad \|F^\lambda(x, D) v\|^2 + C_\theta \|v\|^2 \geqslant c\,\theta^d \langle\langle \zeta^\lambda(x), D\rangle v, v\rangle - c\theta^{d+1/2}\|v\|^2_{1/2} + \mathcal{R}\langle S(x, D) v, v\rangle$$

for all functions v *values in* $W(\zeta^\lambda(x^0))$ *with support in a sufficiently small neighborhood of origin* x^0. *In the above,* $S(x, \xi)$ *is the sum of a term linear in* w *and* χ *(with* C^∞ *coefficient) and a term which is zero at* $\zeta^\lambda(x^0)$.

In order to find algebraic conditions on $f^j(x^0)$ and $g^j(x^0)$ so that the above estimate holds for $F^\lambda(x, \xi)$ we introduce skow-symmetric c_{st} $(s, t = 1, 2, \dots, 2n)$ as follows : $c_{jk} = 0 \quad j, k = 1, \dots, n$,

$$dw_j = \Sigma\, c_{n+j\,k}\, \chi_k(y, dx) \qquad (\mathrm{mod}\ \langle \zeta^\lambda(y), dx\rangle),$$

$$d\zeta^\lambda = \Sigma\, c_{n+k\ n+j}\, \chi_j(y, dx) \wedge \chi_k(y, dx) \quad (\mathrm{mod}\ \langle \zeta^\lambda(y), dx\rangle),$$

where the cotangent vector field ζ^λ is considered as a differential form.

For $\eta = (\eta_1, \dots, \eta_{2n}) \in \mathbf{R}^{2n}$ we set

$$h(\eta) = \Sigma\, f^j(x^0)\, \eta_j + \Sigma\, g^j(x^0)\, \eta_{n+j} = \Sigma\, h^s\, \eta_s .$$

PROPOSITION 1. – Under the assumption on A mentioned in the theorem, $F(x, D)$ satisfies the estimate (***) if we can find $h^s_\theta \in \mathrm{Hom}(W, G)$, where G is a vector space, depending differentiably in θ and satisfying the following :

$$h^{s*} h^t = h^{s*}_0 h^t_0 ,$$

and for any sufficiently small $\theta > 0$

$$(\#) \qquad h(\eta)^* h(\eta) - h_\theta(\eta)^* h_\theta(\eta) \equiv 0 \qquad (\mathrm{mod}\ \theta^{d+1}),$$

$$\Sigma\, \langle i c_{ts}(x^0)\, (h^{s*} h^t - h_\theta^{s*} h^t_\theta)\, u, u\rangle \geqslant c\,\theta^d\, |u|^2$$

for all $u \in W(\zeta^\lambda(x^0)) = W$.

It is fairly complicated to find good algebraic conditions on $h(\eta)$ so that the above conditions are satisfied. We will state here a necessary and sufficient conditions in the case $d \leqslant 2$: Let $\mathbf{h} : W \otimes \mathbf{R}^{2n} \to G$ be defined by $\mathbf{h}(w \otimes \eta) = h(\eta) w$, $w \in W$. For γ in $\mathrm{Hom}(W \otimes \mathbf{R}^{2n}, W \otimes \mathbf{R}^{2n})$ we define $\mathrm{tr}_w(\gamma) \in \mathrm{Hom}(W, W)$ by the formula $\langle \mathrm{tr}_w(\gamma)\, w, w'\rangle = \Sigma\, \langle \gamma(w \otimes r_s), w' \otimes r_s\rangle$ where $r_1, \dots, r_{2n}$ is the standard base of $\mathbf{R}^{2n}$. We define $J : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ by

$$J(r_s) = \Sigma\, c_{st}\, r_t .$$

We also define τ $\mathrm{Hom}(W \otimes \mathbf{R}^{2n}, W \otimes \mathbf{R}^{2n}) \to \mathrm{Hom}(W \otimes \mathbf{R}^{2n}, W \otimes \mathbf{R}^{2n})$ by

$$\langle \gamma^\tau(w \otimes r_s), w' \otimes r_t\rangle = \langle \gamma(w \otimes r_t), w' \otimes r_s\rangle .$$

PROPOSITION 2. – For a given $h(\eta)$, h^s_θ satisfying the condition (#) for $d \leqslant 2$ exist if and only if we can find a self-adjoint

$\beta : W \otimes \mathbf{R}^{2n} \to W \otimes \mathbf{R}^{2n}$ such that

$$\beta^2 = -\beta ,$$

$$\rho_k \circ \beta \circ \rho_k \geqslant 0 ,$$

$$\mathrm{tr}_w (i \beta \circ (I \otimes J)) > 0$$

where K is the kernel of h , ρ_k is the orthogonal projection to K, and I is the identity map of W.

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SUR LES ENSEMBLES SEMI-ANALYTIQUES

par S. LOJASIEWICZ

1. Soit M une variété analytique réelle. Pour chaque $a \in M$ notons avec Σ_a la plus petite famille de germes de sous-ensembles de M en a contenant tous les $\{f > 0\}_a$ avec f analytique au voisinage de a, et vérifiant

$$u , v \in \Sigma_a \Rightarrow u \cup v , u \cap v \in \Sigma_a \quad .$$

Un sous-ensemble E de M est dit semi-analytique si $E_a \in \Sigma_a$ pour tout $a \in M$.

Une autre description équivalente est la suivante. On dit qu'un $A \subset M$ est décrit dans un $U \subset M$ par une famille F de fonctions réelles définies dans U, si $A \cap U = \bigcup_1^s \bigcap_1^T A_{ij}$ avec des A_{ij} de la forme $\{f_{ij} > 0\}$ ou $\{f_{ij} = 0\}$ ou $\{f_{ij} < 0\}$, $f_{ij} \in F$. Un sous-ensemble A de M est semi-analytique si et seulement s'il est décrit par des fonctions analytiques dans un voisinage de chaque point de M.

La définition entraîne trivialement que le complémentaire, l'intersection finie, la réunion localement finie, le produit et l'image inverse par une application analytique d'ensembles semi-analytiques est semi-analytique. Si N est une sous-variété analytique de M et $E \subset N$, alors la semi-analyticité de E dans M entraîne celle dans N, et réciproquement pourvu que $\overline{E} \subset N$.

2. On appelle système normal (dans $\mathbf{R}^n$) une famille $\{H_j^i(x_1, \ldots , x_i ; x_j)\}_{0 \leq i < j \leq n}$ de polynômes distingués, ayant des discriminants $D_j^i(x_1, \ldots , x_i) \not\equiv 0$, des coefficients analytiques au voisinage de 0 et vérifient

$$\left.\begin{array}{l}(1)\ H_k^{k-1} = 0 , H_l^k = 0 \Rightarrow H_l^{k-1} = 0 \\ (2)\ D_l^k = 0 \Rightarrow H_k^{k-1} = 0\end{array}\right\} \text{ sur un voisinage complexe de } 0 .$$

Un voisinage $Q = \{|x_i| < \delta_i\}$ s'appelle normal, si :

(a) les H_j^i sont holomorphes ; (1) et (2) subsistent au voisinage de

$$\{z \in \mathbf{C}^n : |z_i| \leq \delta_i\} \quad ,$$

(b) $H_j^i(u , z) = 0$, $|u_i| < \delta_i \Rightarrow |z| < \delta_j$ (dans le complexe).

Alors les

$$V^k = \{x \in Q : H_n^{n-1} = \cdots = H_{k+1}^k = 0 , H_k^{k-1} \neq 0\} ,$$

$$(V^n = \{H_n^{n-1} \neq 0\} , V^0 = \{H_n^{n-1} = \cdots = H_0^1 = 0\})$$

forment une partition de Q ; on montre que V^k est à la fois ouvert et fermé dans $\{x \in Q : H^k_{k+1} = \cdots = H^k_n = 0 , H^{k-1}_k \neq 0\}$, et est donc une sous-variété analytique de dimension k. On appelle la partition

$$\mathfrak{N} = \bigcup_{k=0}^{n} \{\text{la famille des composantes connexes de } V^k\}$$

une stratification normale de Q selon $\{H^i_j\}$.

Les voisinages normaux (selon $\{H^i_j\}$) forment une base de voisinages de 0.

On montre les propriétés suivantes :

(1) Une stratification normale est toujours finie.

(2) Pour chaque strate $\Gamma \in \mathfrak{N}$, $(\overline{\Gamma} \setminus \Gamma) \cap Q$ est une réunion de strates de $\mathfrak{N}$ de dimension inférieure à celle de Γ (propriété de la frontière).

(3) Chaque strate $\Gamma \in \mathfrak{N}$ de dimension k est le graphe d'une application analytique d'un ouvert de $\mathbf{R}^k$ dans $\mathbf{R}^{n-k}$.

(4) Si $\Gamma_0, \Gamma \in \mathfrak{N}$ avec $\Gamma_0 \subset \overline{\Gamma}$, alors chaque sous-variété (différentiable) transversale à Γ_0 intersecte Γ.

(5) Soit $\pi : (x_1, \ldots, x_n) \to (x_1, \ldots, x_m)$; alors $\pi(Q)$ est un voisinage normal pour $\{H^i_j\}_{0 \leq i < j \leq m}$, et $\pi(\Gamma)$ est une strate ou une réunion de strates de la partition normale de $\pi(Q)$ pour $\Gamma \in \mathfrak{N}$ selon que $\dim \Gamma \leq m$ ou $> m$; dans le premier cas on a $\pi((\overline{\Gamma} \setminus \Gamma) \cap Q) = (\overline{\pi(\Gamma)} \setminus \pi(\Gamma)) \cap \pi(Q)$.

Soit M une variété analytique. Une stratification normale en $a \in M$ est l'image d'une stratification normale par une carte g telle que $g(a) = 0$.

3. On montre que les strates d'une stratification normale sont semi-analytiques.

On dit que $\mathfrak{N}$ est compatible avec une famille d'ensembles F si pour chaque $\Gamma \in \mathfrak{N}, E \in F$ on a $\Gamma \subset E$ ou $\Gamma \subset M \setminus E$. On montre que :

(1) Pour chaque $a \in M$ et $E_1, \ldots, E_s$ semi-analytiques il existe une stratification normale en a, compatible avec $E_1, \ldots, E_s$, d'un voisinage arbitrairement petit de a.

(2) Un ensemble $E \subset M$ est semi-analytique si et seulement si pour chaque $a \in M$ il existe une stratification normale en a, compatible avec E.

Ceci entraîne que : chaque composante connexe d'un semi-analytique est semi-analytique ; la décomposition en composantes connexes d'un ensemble semi-analytique est localement finie ; chaque semi-analytique est localement connexe ; l'adhérence (donc l'intérieur et la frontière) d'un ensemble semi-analytique est semi-analytique ; enfin ce critère utile : un sous-ensemble F d'un semi-analytique E est semi-analytique si et seulement s'il en est de même de $\overline{F} \cap E \setminus F$ et $F \setminus \mathrm{int}_E F$.

4. Un point a d'un ensemble semi-analytique A est dit régulier de dimension k, si $V \cap A$ est une sous-variété analytique de dimension k pour un voisinage V de a. On montre que l'ensemble des points réguliers de dimension k d'un ensemble semi-analytique est semi-analytique.

5. Une notion utile est celle de la dimension d'un ensemble semi-analytique : $\dim_a E$ = maximum des dim. des points réguliers dans un voisinage suffisamment petit de $a = \max(\dim \Gamma : \Gamma \in \mathfrak{N}, \Gamma \subset E)$ où $\mathfrak{N}$ est une stratification normale en a compatible avec E ; $\dim E = \max \dim_a E$. Une propriété utile (dans les raisonnements qui procèdent par récurrence sur la dimension) : $\dim(\overline{E} \setminus E_0) < \dim \overline{E}$, où E_0 est l'ensemble des points réguliers de dimension maximale de E.

6. Soit M un espace euclidien.

Deux ensembles semi-analytiques compacts A et B tels que $A \cap B \neq \emptyset$ jouissent de la propriété de séparation régulière : $\rho(x, A) \geqslant d\rho(x, A \cap B)^N$ lorsque $x \in B$ avec certaines constantes $d, N > 0$.

Si f est analytique dans G, E compact $\subset G$, $Z = \{f = 0\}$, alors on a

$$|f(x)| \geqslant d\rho(x, Z)^N$$

pour $x \in E$, avec certaines constantes $d, N > 0$.

Si $f(a) = 0$ (f analytique), alors $|\operatorname{grad} f(x)| \geqslant |f(x)|^\theta$ dans un voisinage de a avec un $0 < \theta < 1$. (Cette inégalité peut servir à démontrer que $\{f = 0\}$ est rétracte fort par déformation de son voisinage).

Si $f(a) = 0$ (f analytique), $0 < \theta < 1$, alors $|\operatorname{grad} f(x)|\ |x| \geqslant \theta |f(x)|$ dans un voisinage de a. (Cette inégalité est utile par exemple pour démontrer qu'une condition de Kuiper-Kuo caractérise les germes f de classe C^r, qui sont C^0-équivalents à leur développement de Taylor d'ordre r, voir [12](1)).

Pour démontrer ces inégalités on utilise le "curve sellecting lemma" de Bruhat-Cartan-Wallace : Si A est semi-analytique et si $a \in \overline{A}$ n'est pas un point isolé de A, alors A contient un arc semi-analytique qui aboutit à a(2).

Théorème (P. Lelong [16], M. Herrera [17]). La mesure de dimension k d'un relativement compact semi-analytique de dimension $\leqslant k$ est toujours finie.

THEOREME ([1] et [14]). – Un semi-analytique compact possède toujours la propriété de Whitney. (Majoration de la longueur d'un arc qui permet de joindre deux points de l'ensemble dans cet ensemble, en fonction d'une puissance positive de la distance de ces points).

Pour démontrer ces faits on utilise le lemme de Rham : tout semi-analytique relativement compact est une réunion finie de variétés analytiques semi-analytiques chacune étant le graphe d'une application φ vérifiant $|d_z \varphi| \leqslant K$ (constante) dans un système de coordonnées.

En utilisant les propriétés métriques on montre que tout ouvert (resp. fermé), semi-analytique est localement de la forme $\bigcup_i \bigcap_j \{f_{ij} > 0\}$ (resp. $\bigcup_i \bigcap_j \{f_{ij} \geqslant 0\}$) avec f_{ij} analytiques.

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(1) Ces deux inégalités restent vraies dans le cas complexe. (On considère $z \to |f(z)|^2$).

(2) C'est-à-dire un arc un semi-analytique λ relativement compact qui est l'image de $(0, 1]$ par un plongement analytique $(0, 2) \to M$, et tel que $a = \overline{\lambda} - \lambda$; le fait important est que $\overline{\lambda}$ est toujours un arc simple de classe C^1.

7. On peut répéter toute la partie précédente de la théorie en remplaçant la classe des fonctions analytiques par celle des analytiques-algébriques c'est-à-dire vérifiant de plus $w(x, \varphi(x)) \equiv 0$ avec un polynôme $w \not\equiv 0$ (dépendant de φ) ; alors on a des variétés de Nash au lieu de variétés analytiques et des ensembles localement semi-algébriques au lieu d'ensembles semi-analytiques. On montre que dans $\mathbf{R}^n$ ce sont précisément les ensembles localement décrits par des polynômes ; ceux qui sont décrits globalement s'appellent semi-algébriques ; si $\mathbf{P}_n$ est l'espace projectif considéré comme $\mathbf{R}^n$ complété par "l'hyperplan à l'∞", alors, dans la classe des sous-ensembles de $\mathbf{R}^n$, les semi-algébriques de $\mathbf{R}^n$ coïncident avec les localement semi-algébriques de $\mathbf{P}_n$. Un sous-ensemble E de $M \times N$, avec N affine, s'appelle N-semi-algébrique si chaque $x \in M$ possède un voisinage U tel que E soit décrit dans $U \times N$ par des fonctions analytiques qui sont des polynômes par rapport à la variable qui parcourt N.

8. Théorème de Seidenberg. Soient M, N des espaces affines, $\pi : M \times N \to M$ la projection naturelle. Si $E \subset M \times N$ est semi-algébrique alors $\pi(E)$ est semi-algébrique ; dans le cas plus général où M est une variété analytique, si E est N-semi-algébrique alors $\pi(E)$ est semi-analytique.

Dans le cas général où M et N sont des variétés analytiques on a encore le théorème suivant : si E est un semi-analytique relativement compact de $M \times N$ et si l'on admet que $\dim E \leqslant 1$ ou $\dim M \leqslant 2$, alors $\pi(E)$ est semi-analytique (cf. [1]). Mais il y a un exemple d'une sous-variété analytique compacte de dimension 2 de $P_3 \times P_1$ dont la projection (par $P_3 \times P_1 \to P_3$) n'est pas semi-analytique.

Si M est un espace vectoriel, P l'ensemble des droites dans M, $\pi_\lambda : M \to M/\lambda$, pour $\lambda \in P$, la projection naturelle, on a le théorème de Koopman-Brown : Si $E \subset M$ est un semi-analytique relativement compact alors $\pi_\lambda(E)$ est semi-analytique sauf quand λ appartient à un fermé rare de P.

9. Soit M une variété analytique, F une famille localement finie de semi-analytiques de M ; alors il existe une stratification semi-analytique $\mathfrak{N}$ de M (partition localement finie de M en sous-variétés analytiques, semi-analytiques, avec la propriété de la frontière), compatible avec F (on a $\Gamma \subset E$ ou $\Gamma \subset M \setminus E$ quels que soient $\Gamma \in \mathfrak{N}$ et $E \in F$) et jouissant des propriétés (A) et (B) de Whitney (si $\Gamma_0, \Gamma \in \mathfrak{N}$ et $a \in \Gamma_0 \subset \overline{\Gamma}$, alors : (A) tous les sous-espaces-limites en a des espaces tangents de Γ contiennent ceux de Γ_0 on a ; (B) si $z \in \Gamma$ et $x \in \Gamma_0$ tendent vers a alors l'"angle" entre $z - x$ et l'espace tangent de Γ en z tend vers zéro).

10. Soit M une variété analytique de type dénombrable, F une famille localement finie de semi-analytiques de M. Alors il existe un complexe simplicial localement fini K dans un espace affine L et un homéomorphisme $h : |K| \to M$ tel que :

(a) le graphe de h soit L-semi-algébrique (donc l'image par h d'un semi-algébrique est semi-analytique) ;

(b) pour tout $s \in K$, $h(s)$ est une sous-variété analytique (et semi-analytique) et $h_s : s \to h(s)$ est un isomorphisme analytique ;

(c) la famille $\{h(s) : s \in K\}$ est compatible avec F.

BIBLIOGRAPHIE

[1] Łojasiewicz S. — *Ensembles semi-analytiques,* Institut des Hautes Etudes Sci., Bures-sur-Yvette, France, 1964.

[2] Łojasiewicz S. — Sur les ensembles semi-algébriques, *Symposia Mathematica,* 3, 1970; *Ist. Naz. Alta Matem.,* p. 233-239.

[3] Seidenberg A. — A new decision method for elementary algebra, *Ann. of Math.,* 2, 60, 1954, p. 365-374.

[4] Thom R. — La stabilité topologique des applications polynominales, *Ens. Math.,* 8, 1962, p. 24-33.

[5] Thom R. — Ensembles et morphismes stratifiés, *Bull. Amer. Math. Soc.,* 75, 1969, p. 240-284.

[6] Giesecke B. — Simpliziale Zerlegungenbzahlbarer kemplexer Raume, *Thesis,* A. Schubert, Munchen, 1963.

[7] Łojasiewicz S. — Triangulation of semi-analytie sets, *Annali Scuola Norm. Sup. Pisa,* (3) 18, 1964, p. 449-474.

[8] Sate K. — Local triangulation of real analytic varieties, *Osaca Math. J.,* 15, 1963, p. 109-125.

[9] Hörmander L. — On the division of distributions by polynomials, *Arkiv for Mat.,* 3, 1958, p. 555-568.

[10] Łojasiewicz S. — Division d'une distribution par une fonction analytique des variables réelles, *C. R. Acad. Sci. Paris,* 246, 1958, p. 683-686.

[11] Łojasiewicz S. — Une propriété topologique des sous-ensembles analytiques réels, *Coll. du C.N.R.S. sur les équations aux dérivées partielles,* Paris, 1962, p. 87-89.

[12] Bochnak J., Łojasiewicz S. — A convers of Kuiper-Kuo theorem, Symposium en Singularities, Liverpool, *Lecture Note,* 192, Springer 1971.

[13] Whitney H. — Elementary structure of real algebraic varieties, *Ann. of Math..* (2) 66, 1957, p. 545-556.

[14] Whitney H. — Local properties of Analytic Varieties, *Differential and Combinatorial Topology,* Princeton, 1965, p. 205-244.

[15] Whitney H. — Tangents to an Analytic Variety, *Ann. of Math.,* (2) 81, 1965, p. 496-549.

[16] Lelong P. — *Bull. Soc. Math.,* 85, 1954, p. 239-252.

[17] Herrera. — *Bull. Soc. Math.,* 94, 1966, p. 141-180.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 243 à 249.

BANACH MANIFOLDS OF FIBER BUNDLE SECTIONS

by Richard S. PALAIS *

1. Introduction.

In the past several years significant progress has been made in our understanding of infinite dimensional manifolds. Research in this area has split into two quite separate branches ; first a study of the theory of abstract Banach manifolds, and secondly a detailed study of the properties of certain classes of concrete manifolds that arise as spaces of differentiable maps or more generally as spaces of sections of fiber bundles. A survey of the remarkable progress made in the first mentioned area, (i.e. infinite dimensional differential topology) will be found in the reports to this Congress by N. Kuiper and R. Anderson. Here I would like to survey a part of the recent work in the second area, which for obvious reasons (made explicit in § 1 of [12]) has come to be called non-linear global analysis. I shall also attempt to indicate what in my opinion are fruitful directions of current research and hazard a few guesses for the near future. An attempt to be comprehensive would be futile since the subject shades off imperceptibly into many extremely active classical fields of mathematics for which in fact it plays the role of "foundations" (e.g. non-linear partial differential equations and continuum mechanics). I shall therefore concentrate on those few topics which have most engaged my personal interest, particularly the intrinsic structures of manifolds of sections and applications to the calculus of variations. I shall also not attempt to cover research prior to 1966 which is surveyed in the excellent and comprehensive review article [4] of James Eells Jr.

2. Manifold structures for spaces of bundle sections.

If ξ is a smooth ($= C^\infty$) vector bundle over a smooth compact manifold M we can define the Banach spaces $C^k(\xi)$ of C^k sections of ξ as well as many more exotic Banach spaces of distributional sections of ξ, such as the Sobolev spaces $L^p_k(\xi)$. Let us use the symbol Γ to denote a generic "differentiability class" such as C^k of L^p_k. We can regard Γ as a functor defined on the category $VB(M)$ of smooth vector bundles over M and taking values in the category of Banach spaces and continuous linear maps (if $f : \xi \to \eta$ is a smooth vector bundle morphism then $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$ is of course just $s \to f \circ s$). We shall assume that we have a continuous inclusion $\Gamma(\xi) \subset C^0(\xi)$ (e.g. if $\Gamma = L^p_k$ the condition for this is $k > n/p$ where $n = \dim M$). A central foundational question for many

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problems of non-linear analysis is : when can we "extend" Γ to a functor from the category $FB(M)$ of smooth fiber bundles over M to the category of smooth Banach manifolds ? This is an abstract and general version of the question, "when is the technique of linearizing non-linear problems meaningful and "natural"? If E and E' are smooth fiber bundles over M, a smooth map $f : E \to E'$ is a morphism of $FB(M)$ if for each $x \in M$ $f(E_x) \subset E'_x$. A smooth vector bundle ξ over M is called an open vector sub bundle of E if ξ is open in E and the inclusion map $\xi \to E$ is a morphism of $FB(M)$. If $s_0 \in C^0(E)$ then such a ξ is called a vector bundle neighborhood (VBN) of s_0 in E if $s_0 \in C^0(\xi)$. The existence of such a ξ is a basic lemma [12, Theorem 12.10]. Let us say $s_0 \in \Gamma(E)$ provided $s_0 \in \Gamma(\xi)$. It is easily seen that a sufficient condition for this to be independent of the choice of such a ξ is that :

> $FB(\Gamma)$: Given objects ξ and η of $VB(M)$ and a morphism $f : \xi \to \eta$ of $FB(M)$, $s \to f \circ s$ defines a continuous map $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$.

Equally obvious is the fact that if we define the "natural atlas" for $\Gamma(E)$ to be the collection $\{\Gamma(\xi)\}$ of Banach spaces indexed by the open vector subbundles ξ of E, then this same condition $FB(\Gamma)$ is just what is required to make these charts C^0-related and hence for the natural atlas to define $\Gamma(E)$ as a C^0 Banach manifold. More surprising perhaps is the observation that $FB(\Gamma)$ implies that the maps $\Gamma(f)$ of its statement are C^∞, whence the natural atlas defines $\Gamma(E)$ as a smooth Banach manifold and we have our desired extension of Γ. Given a smooth fiber bundle $\pi : E \to M$ its "tangent bundle along the fiber" is a smooth vector bundle over E, $p : TF(E) \to E$, but may also be regarded as a smooth fiber bundle $\widetilde{p} = \pi \circ p : TF(E) \to M$ over M and it is easily seen that there is a canonical identification of $\Gamma(p) : \Gamma(TF(E)) \to \Gamma(E)$ with the tangent bundle of $\Gamma(E)$. If $f : E \to E'$ is a smooth fiber bundle morphism then its "differential along the fiber" is a smooth fiber bundle morphism $\delta f : TF(E) \to TF(E')$ over M and with the above identification $\Gamma(\delta f)$ is the differential of $\Gamma(f)$. For further details see [12]. A similar treatment will be found in Eliasson [5], [6]. Recently J.P. Penot has given a detailed and comprehensive treatment of this problem including several new approaches to the manifold structure of $\Gamma(E)$, [13], and Mike Field has shown that when G is a compact Lie group, M a G-manifold and E is a G-fiber bundle over M, then $\Gamma_G(E)$, the equivariant G-sections of E, is a smooth submanifold of $\Gamma(E)$ [8]. One should also mention here the important related work of A. Douady [1] and Kijowski [9] concerning manifold structures for spaces of submanifolds of a given manifold.

3. Extra structures for manifolds of sections.

The manifolds $\Gamma(E)$ have aside from their differentiable structure much added structure whose properties are of the utmost importance in dealing with concrete problems in non-linear analysis. The "essence" of this extra structure is as yet not fully understood and manifests itself in differing though related guises in varying circumstances. The elucidation and axiomatization of this additional structure I regard as one of the most intriguing and important foundational questions of non-linear global analysis and I shall remark here on the current status of such research.

In dealing with the manifolds $\Gamma(E)$ it is appropriate to use only the charts $\Gamma(\xi)$ of the natural atlas ; the "extra structure" whatever it is, gets lost in passing to the maximal atlas. We should therefore look at the coordinate transformations between two such charts. These are of the form $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$ where $f : \xi \to \eta$ is a fiber bundle morphism of vector bundles. It makes sense to speak of bounded sets in the Banach spaces $\Gamma(\xi)$ and $\Gamma(\eta)$ and with mild conditions on Γ one can prove that the following condition holds :

$BP(\Gamma)$: If $f : \xi \to \eta$ is a fiber bundle morphism of vector bundles over M , $\Gamma(f)$ maps bounded sets to bounded sets.

(see e.g. [12, 19.12] for the case $\Gamma = L^p_k$). From this observation Karen Uhlenbeck in her thesis [17] developed a notion of intrinsically bounded (*IB*) *subsets of* $L^p_k(E)$ and used them very effectively to prove that certain wide classes of calculus of variations problems satisfied Condition (*C*) (See § 6 below). Perhaps the simplest of many diverse descriptions of *IB* sets is "a finite union of subsets of $L^p_k(E)$, each a bounded set in some $L^p_k(\xi)$". What gives them their usefulness (aside from their being preserved by induced morphisms $L^p_k(f)$) is that they are relatively compact in $C^0(E)$, by Rellich's theorem. U. Koschorke has investigated an abstract axiomatic notion of "boundedness structure" suggested by *IB* sets and made several interesting applications (unpublished). About a year ago J. Dowling and K. Uhlenbeck independently made what I consider a very surprising and important observation ; namely that if $f : \xi \to \eta$ is a smooth fiber bundle morphism of vector bundles over M, then for $p > 1$ $L^p_k(f)$ maps weakly convergent sequences to weakly convergent sequences, or equivalently $L^p_k(f)$ is weakly continuous on bounded sets. What makes this so remarkable is that $L^p_k(f)$ is highly non-linear and usually even the mildest non-linearity destroys weak continuity. [For example, consider the quadratic map φ of Hilbert space H to itself, $\varphi(x) = x + \|x\|^2 e$ where e is a non-zero vector in H. $D\varphi_0$ = identity so φ maps some ball (say of radius $2r$) diffeomorphically. If $\{e_n\}$ is an orthonormal base then $re_n \to 0$ weakly, $\varphi(re_n) \to r^2 e$ weakly, but $\varphi(0) = 0 \neq r^2 e$]. As a result it makes no sense to speak of the "weak topology" of an infinite dimensional manifold in general, yet the theorem of Dowling and Uhlenbeck shows that it does make sense for the $L^p_k(E)$, and moreover the *IB* sets turn out to be just the relatively compact sets of this topology. Quite recently Richard Graff has found a simple and elegant proof of this theorem which moreover works whenever $BP(\Gamma)$ is satisfied, Γ is reflexive (i.e. each $\Gamma(\xi)$ is), and Γ satisfies "Rellich's condition" (i.e. $\Gamma(\xi) \subset C^0(\xi)$ is a compact map), hence for any such Γ one can define the "weak topology" for the manifolds $\Gamma(E)$. This weak topology is certainly a part of the extra structure we seek. How big a part is not yet clear.

Another approach to "extra structure" starts with the observation that the functors Γ do not exist in isolation ; there is a vast collection of them (the various $L^{p'}_k$s, $C^{k+\alpha'}$s etc.) related by various "embedding theorems" (e.g. $L^p_k \subset C^r$ if $k > \frac{n}{p} + r$; $L^p_k \subset L^q_l$ for $k > l$ and $k - \frac{n}{p} > l - \frac{n}{p}$). These relationships are known to be absolutely crucial in the analysis of concrete linear and non-linear problems and it is quite plausible to me that it is to this family of relationships that we must look to fully understand the "extra structure". H. Omori has axiomatized at least part of this structure with his notion of ILH and ILB manifolds [10]. While too complicated to explain here it is clear from the applications already made by Omori and others that this is an important concept ; it is also probably the natural setting for some eventual abstract form of the Nash-Moser implicit function theorem.

Additional structures for the $\Gamma(E)$ deserving of special attention are the geometric structures (Finsler metrics, affine connections etc.) induced from similar structures for E and M. These are of course intimately related to numerous classical non-linear problems, particularly in the calculus of variations. Interesting work has been done in this area by Eliasson ([5], foundations), Dowling ([2], Hopf-Rinow theorem) and Ebin ([3], differential geometry of manifolds of Riemannian metrics). One should mention also Uhlenbeck's theorem [17] that IB sets in $L^p_k(E)$ are just those which are bounded sets for any one of a certain natural class of "admissible" Finsler structures. For a while there was a hope that the $\Gamma(E)$ would carry "natural" layer or Fredholm structures (see [7] and also the report to this congress by J. Eells). This could have important consequences (e.g. a degree theory and Leray-Schauder type fixed point theorems). Unfortunately, despite considerable effort there is little evidence to support such a conjecture.

4. Partial differential operators.

Let E be a C^∞ fiber bundle over M. Then $J^r(E)$, the bundle of r-jets of sections of E, is a C^∞ fiber bundle over E ; if $s \in C^\infty(E)$ then $j_r(s)_x$; its r-jet at $x \in M$ lies in $J^r(E)_{s(x)}$. Let F be another C^∞ fiber bundle over E and let $\Phi : J^r(E) \to F$ be a fiber bundle morphism over E. Given $s \in C^\infty(E)$ define $\Phi_*(s) \in C^\infty(s^*F)$ by

$$\Phi_*(s)\ (x) = \Phi\,(j_r(s)_*) \in F_{s(x)} = (s^*F)_x .$$

If we define $F_E(C^\infty, C^\infty)$ to be

$$\{(\sigma , s) \in C^\infty(M , F) \times C^\infty(E) \mid \sigma(x) \in F_{s(x)} \text{ all } x \in M\}$$

then $F_E(C^\infty, C^\infty)$ is a fiber bundle over $C^\infty(E)$ whose fiber over $s \in C^\infty(E)$ is just $C^\infty(s^*F)$, and Φ_* is a section of this bundle. For this reason K. Uhlenbeck, who introduced such operators in [17], called them differential section operators. This concept seems to capture the notion of partial differential operator in its full generality. Consider the special case when $F = \pi^*E'$ is induced from a bundle E' over M ($\pi : E \to M$ being the projection). Then we may regard $J^r(E)$ as a bundle over M also and Φ then becomes a bundle morphism $J^r(E) \to E'$ over M. Moreover $s^*F = s^*\pi^*E' = (\pi s)^*E' = E'$ for any $s \in C^\infty(E)$ so $F_E(C^\infty, C^\infty)$ is the trivial bundle $C^\infty(E') \times C^\infty(E)$ and Φ_* a map $C^\infty(E) \to C^\infty(E')$. This is just the class of non-linear partial differential operators defined in [12, § 15]. As a natural example of a differential section operator D which is not a partial differential operator in the latter more restricted sense, let $M = I = [0 , 1]$, $E = W \times I$ where W is a Riemannian manifold, and let $F = TW \times I$. Given $\sigma \in C^\infty(E) = C^\infty(I , W)$ let $D\sigma \in \sigma^*(TW) = \sigma^*(F)$ denote the covariant derivative of σ' along σ (so $D\sigma = 0$ is the condition that σ be a geodesic). More generally the Euler-Lagrange operator for a calculus of variations problem can in general only be interpreted globally as a differential section operator.

The elucidation of the general properties and structure of differential section operators is clearly a very difficult problem but deserves considerable effort since it is the very core of the foundations of global non-linear analysis.

One of the first natural questions that comes to mind, and an extremely important one for applications, is the following. Given E and F as above and section functors Γ and Γ' we can define analogously to $F_E(C^\infty, C^\infty)$ the bundle $F_E(\Gamma', \Gamma) = \{(\sigma, s) \in \Gamma'(M, F) \times \Gamma(E) \mid \sigma(x) \in F_{s(x)},\ \text{all}\ x \in M\}$ over $\Gamma(E)$ whose fiber over s is $\Gamma'(s^*F)$. Given a fiber bundle morphism $\Phi : J^r(E) \to F$ when does the differential section operator Φ_* extend to a continuous or differentiable section of $F_E(\Gamma', \Gamma)$. For the case that Γ and Γ' are Sobolev functors (L^p_r) and Φ_* a "polynomial" differential operator (almost the only case ever arising in practice) this is now fairly well understood ([12, § 16], [6], [17] and the thesis of Mark Schmidt [15] which is devoted to this question).

5. The Calculus of variations.

Let M have a smooth measure μ and let $\mathcal{L} : C^\infty(E) \to C^\infty(M, \mathbf{R})$ be a differential operator which extends to a smooth map $\mathcal{L} : \Gamma(E) \to L^1(M, \mathbf{R})$. Then we have a smooth real valued function $J : \Gamma(E) \to \mathbf{R}$ defined by $J(s) = \int \mathcal{L}(s)(x)\, d\mu(x)$. The calculus of variations is concerned with the study of the "extremals" or critical points of functionals such as J on certain submanifolds of $\Gamma(E)$ (where in general Γ is a Sobolev functor). Given $f \in C^\infty(E)$ let $\Gamma_{\partial f}(E)$ denote the closure in $\Gamma(E)$ of the set of $s \in \Gamma(E)$ which agree with f in a neighborhood of ∂M. Then $\Gamma_{\partial f}(E)$ is a smooth submanifold of $\Gamma(E)$ called the Dirichlet space of f and of particular interest is the "Dirichlet problem" of describing the critical points of $J \mid \Gamma_{\partial f}(E)$. An account of this will be found in [12, § 19] with the simplifying assumption that E is a sub-bundle of a trivial vector bundle and that $\mathcal{L}$ is of a special form relative to this embedding. Two more general intrinsic treatments will be found in [6] and [17]. The major concern of this research has been two-fold. First to find conditions for $\mathcal{L}$ that will guarantee $J \mid \Gamma_{\partial f}(E)$ satisfies Condition (C) of [11] and [16] (which in turn implies existence theorems for extremals) and secondly to prove that extremals have greater smoothness than is a priori evident ; under appropriate conditions on $\mathcal{L}$. The restrictions on $\mathcal{L}$ assume the form of "coerciveness" or "ellipticity" conditions familiar from linear theory. While the present state of affairs is far from definitive and much remains to be done there has been considerable progress in this area.

An important question in the case $\Gamma = L^2_k$ is "when are all the critical points of $J \mid \Gamma_{\partial f}(E)$ non-degenerate for most choices of f". For the case of geodesics on a Riemannian manifold V, where $M = I$, $E = V \times I$, $\mathcal{L}(\sigma) = \frac{1}{2} \|\sigma'\|^2$, $\Gamma = L^2_1$. Morse showed this was so. In this case $\Gamma_{\partial f}(E)$ consists of all $g \in \Gamma(E)$ having the same endpoints (p, q) as f and Morse's theorem is equivalent to the statement that for almost all $(p, q) \in V \times V$, q is not a conjugate point of p, a result which follows fairly directly from the Sard-Brown theorem or the more general Thom transversality theorem. For calculus of variations problems with several independent variables it has long been suspected that with appropriate conditions on $\mathcal{L}$ similar results could be proved, however only very recently have such conditions been found [18] by Uhlenbeck. Her proof involves regarding $\Gamma(E)$ as a smooth bundle of Hilbert manifolds with the $\Gamma_{\partial f}(E)$ as fibers and depends on F. Quinn's generalization of Smale's transversality theorem for Fredholm maps [14].

There is an application of the above results which should be fairly easy to carry out and would have interesting connections with topology. Assume M is a compact Riemannian symmetric space of rank one and V another Riemannian symmetric space of rank one. Among the natural Lagrangians for maps $M \to V$ there is the well-known higher order "energy" function whose extremals are the so called polyharmonic maps. Condition (C) and the regularity theorem are satisfied for this functional and moreover, because of the high degree of symmetry involved, the problem of finding explicitly the critical submanifolds and their indices should reduce to reasonably straightforward calculations. Using standard results of Morse theory this would lead to information about the homotopy type of $C^0(M, V)$ and in particular, taking M to be S^n, of the higher loop spaces of V.

Let me close by saying that I have only been able to give a small sample of the many promising lines of current research in non-linear global analysis. In particular I have not even mentioned here what I consider one of the most interesting and promising such programs, namely that initiated by Arnold in continuum mechanics and developed considerably in the past several years. For this I refer to *D*. Ebin's report to this Congress.

REFERENCES

[1] DOUADY A. — Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, *Sem. Collège de France,* 1964-65.

[2] DOWLING J. — Finsler geometry on Sobolev manifolds, *Proc. of Symposia in Pure Math.,* Vol. 15, Amer. Math. Soc., 1970, p. 1-10.

[3] EBIN D. — The manifold of Riemannian metrics, *Proc. of Symposia in Pure Math* Vol. 15, Amer. Math. Soc., 1970, p. 11-40.

[4] EELLS J. — A setting for Global Analysis, *Bull. of A.M.S.,* Vol. 72. No. 5, 1966, p. 751-807.

[5] ELIASSON H. — Geometry of manifolds of maps, *J. Differential Geometry,* Vol. 1, 1967, p. 169-194.

[6] ELIASSON H. — Variational integrals in fiber bundles, *Proc. of Symposia in Pure Math.* Vol. 16, Amer. Math. Soc., 1970, p. 67-90.

[7] ELWORTHY K.D. and TROMBA T. — Differentiable structures and Fredholm maps on Banach manifolds, *Proc. of Symposia in Pure Math.,* Vol. 15, Amer. Math. Soc., 1970, p. 45-94.

[8] FIELD M. — *Thesis,* Univ. of Warwick, England, 1970.

[9] KOJOWSKI J. — Existence of differentiable structures in the set of submanifolds, *Studia Math.,* Vol. 33, p. 93-108.

[10] OMORI H. — On the group of diffeomorphisms of a compact manifold, *Proc. of Symposia in Pure Math.,* Vol. 15, Amer. Math. Soc., 1970, p. 167-184.

[11] PALAIS R.S. — Morse theory in Hilbert manifolds, *Topology,* Vol. 2, 1963,, p. 299-340.

[12] PALAIS R.S. — *Foundations of global non-linear analysis,* W. A. Benjamin Inc., New York, 1968.
[13] PENOT J.P. — *Diverse methodes de construction des varietes des applications,* (in preparation).
[14] QUINN F. — Transversal approximations on Banach manifolds, *Proc. of Symposia in Pure Math.,* Vol. 15, Amer. Math. Soc., 1970, p. 213-222.
[15] SCHMIDT M. — Extension of differential operators to Sobolev spaces, *Thesis,* Brandeis Univ., 1969.
[16] SMALE S. — Morse theory and a non-linear generalization of the Dirichlet Problem, *Ann. of Math.,* Sol. 80, 1964, p. 382-396.
[17] UHLENBECK K. — The Calculus of variations and global analysis, *Thesis,* Brandeis University, 1968.
[18] UHLENBECK K. — Integrals with non-degenerate critical points, *Bull. A.M.S.,* Vol 76, 1970, p. 125-128.

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OVERDETERMINED OPERATORS : SOME REMARKS ON SYMBOLS

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The purpose of this note is to exhibit explicitly the operator δ (which defines the "δ-cohomology") in the symbol of a differential operator, in order to try eventually to relate its properties to the characteristic ideal of the operator. The symbol sequences associated with an arbitrary (formally integrable) operator can be given the same form as those of a flat operator, and we therefore include a discussion of flat operators.

1. Flat operators.

We begin by defining the notion of a flat operator, and we use the notation and terminology of the summary article [6] (in which, in particular, the same notation is used for a vector bundle and its sheaf of sections).

Let E, F be vector bundles over the differential manifold X (dim $X = n$), and let $\varphi : J_k(E) \to F$ be a morphism of vector bundles where J_k is the functor (from the category of vector bundles and their morphisms into itself) which associates to E the vector bundle $J_k(E)$ of k-jets of sections of E. The l-th prolongation $p_l(\varphi) : J_{k+l}(E) \to J_l(F)$ of $\varphi = p_0(\varphi)$ is the restriction to $J_{k+l}(E) \subset J_l(J_k(E))$ of the map $J_l(\varphi) : J_l(J_k(E)) \to J_l(E)$. Recalling that $S^{k+l}T^* \otimes E$ is the kernel of the projection $\pi_{k+l-1} : J_{k+l}(E) \to J_{k+l-1}(E)$, we define $\sigma_l(\varphi) : S^{k+l}T^* \otimes E \to S^l T^* \otimes F$ to be the morphism obtained by restriction of $p_l(\varphi)$. Now let $\Phi : E \to F$ be a differential operator of order k, i.e., there exists a morphism $\varphi : J_k(E) \to F$ such that $\Phi = \varphi \circ j_k$ where $j_k : E \to J_k(E)$ is the sheaf morphism induced by forming, by differentiation, jets of order k of sections. We let $R_{k+l} = \ker p_l(\varphi) \subset J_{k+l}(E)$, $g_{k+l} = \ker \sigma_l(\varphi) \subset S^{k+l}T^* \otimes E$, and observe that g_{k+l} is the kernel of the porjection $\pi_{k+l-1} : R_{k+l} \to R_{k+l-1}$. The differential operator Φ is said to be *formally integrable* if, for $l \geqslant 0$, R_{k+l} is a vector bundle and $\pi_{k+l} : R_{k+l+1} \to R_{k+l}$ is an epimorphism. If Φ is formally integrable, as we assume henceforth, the operator can be extended to a complex of which the initial portion is

$$E \xrightarrow{\Phi} F \xrightarrow{\Phi'} G, \tag{1.1}$$

where Φ' is a differential operator from F to the vector bundle G, and this complex is formally exact in the sense of Goldschmidt [1].

Next, we have a morphism $\delta : \Lambda^r T^* \otimes g_{s+1} \to \Lambda^{r+1} T^* \otimes g_s$ of vector bundles where $\delta^2 = 0$, and we call the cohomology of the resulting complex the "δ-cohomology" (of Φ). There exists an integer μ depending only on n (dim X), k (order of Φ) and the fibre dimension of E (if X is connected, as we suppose) such that the δ-cohomology vanishes at $\Lambda^r T^* \otimes g_s$ for $s \geqslant \mu - n$ and all r. Moreover, writing Θ (solution sheaf of Φ) for the kernel of $\Phi : E \to F$, we have the sheaf complex

$$(1.2) \qquad 0 \to \Theta \to C^0 \xrightarrow{D} C^1 \xrightarrow{D} C^2 \xrightarrow{D} \ldots \xrightarrow{D} C^n \to 0$$

where D is a first-order differential operator and

$$C^r = C^r_m = (\Lambda^r T^* \otimes R_{m+1}) / \delta(\Lambda^{r-1} T^* \otimes g_{m+2}),$$

where m is a fixed integer, $m \geqslant \mu$, and C^r is a vector bundle. The cohomology of this complex at C^1 is isomorphic to the cohomology of (1.1) at F.

Now let $P : R_m \to R_{m+1}$ be a splitting of the sequence of vector bundles $0 \to g_{m+1} \to R_{m+1} \to R_m \to 0$ and denote by $Q : R_{m+1} \to g_{m+1}$ the corresponding projection. The splitting induces an isomorphism

$$(1.3) \qquad C^r \cong (\Lambda^r T^* \otimes R_m) \otimes \delta(\Lambda^r T^* \otimes g_{m+1}),$$

and defines a connection for R_m with differential operator

$$D_0 : \Lambda^r T^* \otimes R_m \to \Lambda^{r+1} T^* \otimes R_{m+1}$$

where $D_0 = \hat{D} \circ P$ and

$$\hat{D} : \Lambda^r T^* \otimes R_{m+1} \to \Lambda^{r+1} T^* \otimes R_m$$

is the so-called "naive" operator (see [1], [6]). Identifying C^r with the direct sum in (1.3), the operator takes the form

$$(1.4) \quad D(\sigma, \rho) = (D_0 \sigma - \rho,\ D_0(D_0 \sigma - \rho)),\ (\sigma, \rho) \in (\Lambda^r T^* \otimes R_m) \oplus \delta(\Lambda^r T^* \otimes g_{m+1}).$$

The operator $\Phi : E \to F$ is *flat* if it is formally integrable and there exists a splitting P such that D_0^2 (the curvature of the connection) vanishes. We say that the operator is completely integrable if, for $l \geqslant k$ (order of Φ), R_l is the vector bundle of l-jets of its solution sheaf Θ. It is easily verified that an operator is flat if and only if it is completely integrable. For example, any formally integrable operator with analytic coefficients is flat ; in particular, the famous operator of H. Lewy [4] is flat.

From (1.3) and (1.4) we obtain for a flat operator the following commutative diagram :

$$(1.5) \quad \begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Theta' & \xrightarrow{j_m} & R_m & \xrightarrow{D_0} & T^* \otimes R_m & \xrightarrow{D_0} & \Lambda^2 T^* \otimes R_m & \xrightarrow{D_0} \ldots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Theta & \longrightarrow & C^0 & \xrightarrow{D} & C^1 & \xrightarrow{D} & C^2 & \xrightarrow{D} \ldots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Theta'' & \xrightarrow{\delta Q j_{m+1}} & \delta(g_{m+1}) & \xrightarrow{-D_0} & \delta(T^* \otimes g_{m+1}) & \xrightarrow{-D_0} & \delta(\Lambda^2 T^* \otimes g_{m+1}) & \xrightarrow{-D_0} \ldots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Here $\Theta' = \{\theta \in \Theta \mid P j_m(\theta) = j_{m+1}(\theta)\}$ and $\Theta'' = \delta Q j_{m+1}(\Theta)$. The columns of (1.5) are exact, and the rows are exact at R_m, C^0 and $\delta(g_{m+1})$.

The exactness of the first row of (1.4) is trivial and is equivalent to the Poincaré lemma for the exterior differential operator d. Hence the second row is exact if and only if the third row is. However its exactness (which implies that of (1.1)) does not hold in general. Since the curvature vanishes, the operator D_0 reduces to d (by introducing flat frames), but d operates on the image of δ and exactness is therefore equivalent to a Poincaré lemma for a restriction of d.

By introducing a splitting P the symbol sequences of diagram (1.5) remain valid for an arbitrary (formally integrable) operator (see (1.4)); their consideration is then reduced to the third row.

2. **Structure coefficients.**

We first introduce some notation. Let $q_{(s)}$ be the fibre dimension of R_s, set

$$(s) = \{l \in \mathbf{Z} \mid 1 \leqslant l \leqslant q_{(s)}\},\ (s-1\,,s) = \{l \in \mathbf{Z} \mid q_{(s-1)} + 1 \leqslant l \leqslant q_{(s)}\}$$

and, if $A = (a_{\nu\rho})$ is a matrix whose coefficients $a_{\nu\rho}$ are functions, write

$$A_{(s')}^{(s''-1,s'')} = \{a_{\nu\rho} \mid \nu \in (s'),\ \rho \in (s''-1\,,s'')\},$$

$$A_{(s'-1,s')}^{(s''-1,s'')} = \{a_{\nu\rho} \mid \nu \in (s'-1,s')\,,\ \rho \in (s''-1\,,s'')\}.$$

Multiplication is the usual matrix one, for example

$$A_{(s')}^{(s'',s''-1)}\ A_{(s''-1,s'')}^{(s-1,s)} = \sum_{\rho\,\in\,(s''-1,s'')} a_{\nu\rho}\ a_{\rho\tau}\,,\ \nu \in (s'), \tau \in (s-1,s).$$

A flat frame for the connection on R_m has the form $J_m^{(m)} = \{j_m(\vartheta_\nu) \mid \nu \in (m)\}$ where the ϑ_ν are local sections of Θ' and the $j_m(\theta_\nu)$ are independent and span R_m locally over the functions. The structure matrix $A_{(m-1)}^{(m-1,m)}$ is defined by $J_{m-1}^{(m-1,m)} = J_{m-1}^{(m-1)}\ A_{(m-1)}^{(m-1,m)}$, where $J_{m-1}^{(m-1,m)} = \{j_{m-1}(\theta_\nu) \mid \nu \in (m-1\,,m)\}$ and similarly for $J_{m-1}^{(m-1)}$. Applying to this identity the (naive) operator $\hat{D}$ which annihilates $J_{m-1}^{(m-1,m)}$, we obtain the structure equation

$$dA_{(m-2)}^{(m-1,m)} + A_{(m-2)}^{(m-2,m-1)} \cdot dA_{(m-2,m-1)}^{(m-1,m)} = 0. \tag{2.1}$$

Let $\bar{J}_m^{(m)}$ be another flat frame for R_m ; then $\bar{J}_m^{(m)} = J_m^{(m)}\, B_{(m)}^{(m)}$ where $B_{(m)}^{(m)}$ is a constant matrix (as is seen by applying again the operator $\hat{D}$).

Next, let $J_{m+1}^{(m+1)}$ be a frame for R_{m+1} which is obtained by first prolonging $J_m^{(m)}$ to $J_{m+1}^{(m)} = \{j_{m+1}(\theta_\nu) \mid \nu \in (m)\}$ and then adjoining the set

$$J_{m+1}^{(m,m+1)} = \{j_{m+1}(\theta_\nu) \mid \nu \in (m, m+1)\}$$

to make a frame, where the θ_ν, $\nu \in (m\,, m+1)$, are local sections of Θ. If $\bar{J}_{m+1}^{(m+1)}$ is another frame for R_{m+1} obtained by adjoining to $J_{m+1}^{(m)}$ a different set $\bar{J}_{m+1}^{(m,m+1)}$, then $\bar{J}_{m+1}^{(m+1)} = J_{m+1}^{(m+1)}\ B_{(m+1)}^{(m+1)}$ where $B_{(m+1)}^{(m+1)}$ is a constant non-singular matrix and $B^{(m)}_{(m)} = I^{(m)}_{(m)}$ (identity matrix). Writing $J^{(m,m+1)}_m = J^{(m)}_m A^{(m,m+1)}_{(m)}$, $\bar{J}^{(m,m+1)}_m = J^{(m)}_m \bar{A}^{(m,m+1)}_{(m)}$, and substituting into $\bar{J}^{(m+1)}_m = J^{(m+1)}_m B^{(m+1)}_{(m+1)}$, we obtain $\bar{A}^{(m,m+1)}_{(m)} = A^{(m,m+1)}_{(m)} B^{(m,m+1)}_{(m,m+1)}$ and hence $d\bar{A}^{(m,m+1)}_{(m)} = dA^{(m,m+1)}_m \cdot B^{(m,m+1)}_{(m,m+1)}$. We observe that $F_{m+1} = J^{(m,m+1)}_{m+1} - J^{(m)}_{m+1} A^{(m,m+1)}_{(m)}$ is a frame for g_{m+1} since $J^{(m,m+1)}_m - J^{(m)}_m A^{(m,m+1)}_{(m)} = 0$ (but it is not flat). We use this frame for g_{m+1} and the corresponding frame $F_m = J^{(m-1,m)}_m - J^{(m-1)}_m A^{(m-1,m)}_{(m-1)}$ for g_m, and we denote by Ψ the (local) section $F_{m+1} \cdot \Psi$ of $\Lambda^{r-1}T^* \otimes g_{m+1}$, where $\Psi = \Psi_{(m,m+1)}$ is a vector-valued differential form of degree $r-1$. Writing $A = A^{(m,m+1)}_{(m-1,m)}$, we find easily that

$$\delta\Psi = dA \wedge \Psi, \qquad -D_0(\delta\Psi) = dA \wedge d\Psi. \tag{2.2}$$

Of course, the image under δ of a section of g_{m+1} is independent of the choice of frame in g_{m+1} and D_0 operates as d on its components with respect to F_m, i.e., F_m is a flat frame for the connection restricted to $\delta(\Lambda^r T^* \otimes g_{m+1})$.

Now suppose that the operator $\Phi : E \to F$ has a (real) analytic symbol. That is to say, E and F are analytic vector bundles over the analytic manifold X and the maps $\sigma_l(\varphi) : S^{k+l}T^* \otimes E \to S^l T^* \otimes F$ are analytic for $l \geqslant 0$, where $\varphi : J_k(E) \to F$ is the morphism associated with Φ. The bundles $\Lambda^* T^* \otimes g_s$ are analytic and, since δ is an analytic mapping, the bundles $\delta(\Lambda^{r-1}T^* \otimes g_{s+1})$ are also analytic. The third row of (1.5) is then an analytic complex if and only if its operators D_0 are analytic, i.e., if and only if the flat connection is analytic. For a flat analytic connection it is well known that flat frames can be introduced by means of analytic transformations. Since flat frames are related to one another by transformations whose matrices have constant coefficients, we conclude that a section of $\delta(\Lambda^{r-1}T^* \otimes g_{s+1})$ is analytic if and only if it has the form $F_s \cdot \Psi$ where Ψ is analytic and F_s is a flat frame. The representations (2.2) are valid and, since δ maps analytic sections into analytic sections, we conclude that the matrices $dA = dA^{(s,s+1)}_{(s-1,s)}$, and hence the $A^{(s,s+1)}_{(s-1,s)}$, are analytic. It then follows from the structure equation by recursion that all the structure coefficients are analytic. However, given analytic coefficients satisfying the structure equation, we can construct an isomorphic analytic operator ("third fundamental theorem" for flat operators) with a flat analytic connection having these structure coefficients. Therefore we say that a flat operator is *analytic* if it has an analytic symbol and flat connection with analytic structure coefficients. An elliptic analytic flat operator is locally solvable (i.e., (1.1) is exact) since the third row of (1.5) is exact by a well known theorem.

3. Guillemin decomposition.

For simplicity we write $\Gamma^r = \Gamma^r_m = \delta(\Lambda^{r-1} T^* \otimes g_{m+1})$ where $\Gamma^0 = 0$; then the third row of diagram (1.5) becomes

$$0 \to \Theta'' \to \Gamma^1 \xrightarrow{-D_0} \Gamma^2 \xrightarrow{-D_0} \dots \xrightarrow{-D_0} \Gamma^n \to 0. \tag{3.1}$$

In order to display some formal properties of (3.1), we decompose the operator $D_0 : \Gamma^1 \to \Gamma^2$ according to Guillemin's prescription [2].

Let U be a non-characteristic sub-bundle of T^* defined over a neighborhood of a point of X, i.e., the map $\sigma(-D_0) : U \otimes \Gamma^1 \to \Gamma^2$ is injective. Let U be spanned by the differentials $dx^1, \ldots, dx^k$, and let W be the complementary sub-bundle spanned by $dx^{k+1}, \ldots, dx^n$. Writing $g = T^* \otimes g_{m+1}, g' = U \otimes g_{m+1}, g'' = W \otimes g_{m+1}$ (m a large fixed integer), we have the following exact commutative diagram :

$$
\text{(3.2)} \qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma^1_{m+1} \cap g' & \to & g' & \xrightarrow{dA} & dA \wedge g' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma^1_{m+1} & \to & g & \xrightarrow{dA} & \Gamma^2 & \to & 0 \\
 & & \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi''} & & \downarrow & & \\
0 & \to & h'' & \to & g'' & \xrightarrow{\alpha} & dA \wedge g''/dA \wedge h'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here π'' is the projection onto g'', and $h'' = \pi''(\Gamma^1_{m+1})$. Let $\lambda : dA \wedge g''/dA \wedge h'' \to g''$ be a splitting of the third row of (3.2) ; we then have the decomposition

$$g = g' \oplus h'' \oplus \lambda\alpha(g'')$$

and we let $\chi' : g \to h'', \tilde{\omega}'' : g \to \lambda\alpha(g'')$ be the projections where $\chi'', \tilde{\omega}''$ vanish on g'. Since $dA \wedge \chi''g \subset dA \wedge \pi'g$, we have the decomposition

$$\Gamma^2 = (dA \wedge \pi' g) \oplus (dA \wedge \tilde{\omega}'' g).$$

Next, writing $E_i = dx^i \wedge \Gamma^1$ we note that the map $\sigma_{dx^i}(-D_0) : \Gamma^1 \to E_i$, where $\sigma_{dx^i}(-D_0) : dA \,.\, \Psi \to dA \wedge dx^i \,.\, \Psi$, is an isomorphism for $i = 1, 2, \ldots, k$ and that we have the decomposition $dA \wedge \pi' g = E_1 \oplus E_2 \oplus \ldots \oplus E_k$ (since the dx^i are non-characteristic) ; therefore $\Gamma^2 = E_1 \oplus E_2 \oplus \ldots \oplus E_k \oplus C$ where $C = dA \wedge \tilde{\omega}''g$. Let $p : \Gamma^2 \to dA \wedge \pi'g$, $p_i : \Gamma^2 \to E_i$, $q : \Gamma^2 \to C$ be the projections, where $p : dA \wedge \Psi \to dA \wedge (\pi' + \chi'')\Psi$, $q : dA \wedge \Psi \to dA \wedge \tilde{\omega}''\Psi$, and define $D_i : \Gamma^1 \to \Gamma^1$, $D_s : \Gamma^1 \to C$ by setting $D_i = \sigma_{dx^i}(-D_0)^{-1} \circ p_i \circ (-D_0)$, $D_s = q \circ (-D_0)$. Then

$$-D_0 = p(-D_0) + q(-D_0) = \Sigma_1^k \ \sigma_{dx^i}(-D_i)D_i + D_s,$$

where $p(-D_0) : dA \cdot \Psi \to dA \wedge (\pi' + \chi'')d\Psi$, $q(-D_0) : dA \cdot \Psi \to dA \wedge \tilde{\omega}''d\Psi$, and Guillemin's argument yields the same commutation relations as were originally obtained for the decomposition of the operator $D : C^0 \to C^1$, namely : there exist operators $D'_{ij} : \Gamma^2 \to \Gamma^1$, $D'_i : \Gamma^1 \to \Gamma^1$ such that $[D_i, D_j] = D'_{ij}D_0$, $D_0 D_i = D'_i D_0$, $1 \leqslant i, j \leqslant k$.

Let $\zeta = \sum_{i=1}^{k} \xi_i dx^i + \sum_{i=k+1}^{n} \eta_i dx^i = \xi + \eta$; then we have the symbol maps

$$\sigma(p(-D_0))(\zeta) : dA \,.\, \Psi \mapsto dA \wedge (\xi + \chi''\eta)\Psi \,,$$

$$\sigma(q(-D_0))(\zeta) : dA \,.\, \Psi \mapsto dA \wedge \tilde{\omega}''\eta \,.\, \Psi$$

and $E_\eta = \ker \sigma(q(-D_0))(\zeta) = dA \,.\, \eta^{-1}(h'')$ where

$$\eta^{-1}(h'') = \{\Psi \in g_{m+1} \mid \eta\Psi \in h''\} \quad .$$

On E_η we have $\sigma(p(-D_0))(\zeta) : dA \,.\, \Psi \mapsto dA \wedge (\xi + \eta) \,.\, \Psi$ where

$$dA \wedge \eta\Psi = dA \wedge s(\eta\Psi) \quad \text{and} \quad s(\eta\Psi) = \sum_{i=1}^{k} s_i(\eta\Psi) \,.\, dx^i \in \pi'\Gamma^1_{m+1} \quad .$$

Let $-a_i(x, \eta) = dA \,.\, s_i \circ \eta \,.\, dA^{-1}|_{E_\eta}$ where $dA^{-1} : \Gamma^1 \to g_{m+1}$; then for

$$\Psi \in \eta^{-1}(h'')$$

$$\sigma(p(-D_0))(\zeta) : dA \,.\, \Psi \mapsto \sum_{i=1}^{k} \{\xi_i - a_i(x, \eta)\}(dA \,.\, \Psi) \wedge dx^i$$

where the matrices $a_i(x, \eta)$ commute, $a_i(x, \eta) = \sigma_i(x, \eta) \,.\, \text{id-}b_i(x, \eta)$, $b_i(x, \eta)$ nilpotent. If the differential operator Φ satisfies the δ-estimate then, by a theorem of MacKichan [5], the $a_i(x, \eta)$ are normal, therefore simultaneously diagonalizable and the $b_i(x, \eta)$ all vanish. In this case the method of parameterization of the characteristic variety adopted by Guillemin, Quillen and Sternberg [3] shows that the characteristics (which are the characteristics of the given operator Φ) are generically simple (simple at all points of a Zariski open subset of the maximal dimensional component of the complex characteristic variety).

REFERENCES

[1] Goldschmidt H. — Existence theorems for analytic linear partial differential equations, *Annals of Math.*, 86, 1967, p. 246-270.

[2] Guillemin V. — Some algebraic results concerning the characteristics of overdetermined partial differential equations, *Amer. Journ. Math.*, 90, 1968, p. 270-274.

[3] Guillemin V.W., Quillen D. and Sternberg S. — The integrability of characteristics, *Comm. Pure and Applied Math.*, 23, 1970, p. 39-77.

[4] Lewy H. — On the local character of the solutions of an atypical linear differential equation in 3 variables and a related theorem for functions of 2 complex variables, *Annals of Matfh.*, 64, 1956, p. 514-522.

[5] MacKichan B. — A generalization to overdetermined systems of the notion of diagonal operators. I. Elliptic Operators, *Acta Math.*, 126, 1971, p. 83-119.

[6] Spencer D.C. — Overdetermined systems of linear partial differential equations, *Bull. Amer. Math. Soc.*, 75, 1969, p. 179-239.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 257 à 265.

STRUCTURE LOCALE DES MORPHISMES ANALYTIQUES

by R. THOM

1. Les modèles locaux : Croix normale.

DEFINITION 1. – On appelle *croix normale* de codimension k dans l'espace euclidien $\mathbf{R}^m$ la figure constituée par k hyperplans se coupant transversalement. La variété linéaire W^{m-k} intersection des k hyperplans sera dite l'*axe* de la croix normale.

Lorsque le nombre k des hyperplans est égal à la dimension k de l'espace, la croix normale sera dite *canonique.* Par exemple, dans l'espace $\mathbf{R}^k$ de coordonnées $t_1, t_2, \ldots, t_k$, les k hyperplans d'équations $t_j = 1$, $j = 1, 2, \ldots k$, forment une croix normale canonique ayant pour axe le point $(1, 1, 1, \ldots 1)$. Evidemment, une croix normale de codimension k dans $\mathbf{R}^m$ est le produit topologique d'une croix normale canonique de $\mathbf{R}^k$ par l'axe W^{m-k}. Il suffit donc d'étudier les croix canoniques.

Croix normale canonique.

La croix normale canonique H_k de $\mathbf{R}^k$ divise l'espace en 2^k composantes connexes. Désignons par $(\underset{\sim}{k})$ l'ensemble $1, 2, \ldots k$ des k premiers entiers. A tout sous-ensemble A de $(\underset{\sim}{k})$ on associe l'ensemble $M(A)$ du complémentaire $\mathbf{R}^k - H_k$ défini par :

$$t_i < 1 \quad \text{pour} \quad i \in A \quad ; \quad t_j > 1 \quad \text{pour} \quad j \in (\underset{\sim}{k}) - A$$

Si A et B sont deux sous-ensembles de $(\underset{\sim}{k})$, les adhérences $\overline{M(A)}$, $\overline{M(B)}$ se rencontrent selon la variété linéaire définie par

$$t_r = 1 \quad \text{pour} \quad r \in (A;B) = A \cup B - A \cap B$$

Une telle sous-variété est l'axe d'une croix normale de dimension s-cardinal $(A\ ;\ B)$. Une telle croix normale sera dite *subordonnée* à la croix normale canonique H_k.

Si, en particulier, on prend pour A l'ensemble vide $\emptyset$, on posera

$$N(B) = \overline{M(\emptyset)} \cap \overline{M(B)}$$

$N(B)$ est l'axe d'une croix normale de codimension cardinal de B ; on l'appelle la *trace* de l'ensemble B sur la région $M(\emptyset)$. Si les deux sous-ensembles B et C de $(\underset{\sim}{k})$ sont disjoints, les variétés traces $N(B)$, $N(C)$ s'intersectent transversalement selon la trace $N(B \cup C)$.

DEFINITION 2. – *Application linéaire graduée.*

Soit $n = i_1 + i_2 + \ldots + i_k$ une partition de l'entier naturel n en k entiers non nuls. On considère la suite des applications linéaires surjectives :

(1) $$E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_j \xrightarrow{f_j} E_{j+1} \to \ldots \to E_k \to 0$$

où les E_j sont des espaces vectoriels de dimension respective : $i_j + i_{j+1} + \ldots + i_k$.

Désignons par F_j le noyau de l'application f_j : c'est un espace vectoriel de dimension i_j. On définira une décomposition de E_j en somme directe (splitting) à l'aide d'une application surjective $g_j : E_j \to F_j$. En usant de ces "scissions" g_j, on pourra dans E_1 former un drapeau-conoyau de la forme :

$$0 \subset \mathbf{R}_{i_k} \subset \mathbf{R}_{i_k + i_{k-1}} \subset \ldots \subset \mathbf{R}_{i_k + \ldots + i_j} \subset \ldots \subset \mathbf{R}_n = E_1$$

Désignons par r_j une norme dans le noyau F_j ; par abus de notation, on désignera par la même lettre r_j les fonctions induites $r_j \circ g_j$ sur E_j, et $r_j \circ g_j \circ \ldots \circ f_{j-1} \circ \ldots \circ f_i$ sur E_i, $i < j$.

Ces fonctions nous serviront à définir localement un voisinage tubulaire du drapeau-conoyau.

2. Voisinage tubulaire d'un drapeau.

Considérons dans l'espace $\mathbf{R}^k$ de coordonnées $t_i, i = 1, 2, \ldots, k$, l'application linéaire graduée

$$\mathbf{R}^k \xrightarrow{\hat{t}_k} \mathbf{R}^{k-1} \ldots \mathbf{R}^j \xrightarrow{\hat{t}_j} \mathbf{R}^{j-1} \to \ldots \to \mathbf{R}^1 \xrightarrow{\hat{t}_1} 0$$

où l'application $\hat{t}_j$ s'obtient en oubliant la coordonnée t_j.

La fonction t_j définit alors une scission de $\hat{t}_j$: projection de R^j sur l'axe Ot_j, noyau F_j de $\hat{t}_j$. Ces scissions permettent de définir un drapeau-conoyau :

$$0 \subset \mathbf{R}^1 \subset \mathbf{R}^2 \subset \ldots \subset \mathbf{R}^j \subset \ldots \subset \mathbf{R}^k ,$$

où 0 est définie par $t_1, t_2, \ldots, t_k = 0$
$\mathbf{R}^1$ $\quad t_2, t_3, \ldots, t_k = 0$
$\mathbf{R}^i$ $\quad t_{i+1}, \ldots, t_k = 0$
$\mathbf{R}^k$

On se restreindra à l'"octant" positif $\mathbf{R}^k_+$ de $\mathbf{R}^k$ ($t_j \geqslant 0$). En posant $R^i_+ - R^{i-1}_+ = V^i$, on définira l'espace R^k_+ entier comme réunion disjointe des V^i ; on se propose de définir les V^i comme des intérieurs de variétés à coins M^i, puis de reconstituer l'espace R^k_+ par identification le long des bords des M_i. Dans ce but, désignons par $s_i = [(t_{i+1})^2 + (t_{i+2})^2 + \ldots (t_k)^2]^{1/2}$ la distance euclidienne à la variété R^i du drapeau.

Considérons alors les variétés tubes :

$$s_0 = \epsilon , \quad s_1 = \epsilon^2 , \quad s_2 = \epsilon^3 , \ldots , s_{k-1} = \epsilon^k$$

Pour ϵ assez petit, ces variétés-tubes se coupent transversalement ; à un infiniment petit d'ordre supérieur près, les équations locales en un point de l'intersection s'écrivent :

$$t_1 = \epsilon, \quad t_2 = \epsilon^2, \ \ldots \quad t_i = \epsilon^i, \quad t_k = \epsilon^k$$

Dans chaque $\mathbf{R}^i$, ces équations définissent une croix normale canonique H_i ; lorsque ϵ tend vers zéro, les axes de H_i et des H_s, $s < i$ qui lui sont subordonnées, décrivent des variétés tendant vers zéro, qui avec les positions initiales sont homéomorphes à des simplexes de sommet 0 ; on obtiendra une image homéomorphe de l'ensemble de ces variétés en considérant l'application d'éclatement Q qui transforme le cube unité $0 \leqslant u_i \leqslant 1$, dans le simplexe standard de sommets $(0, \ldots, 0)$, $(1, 0, \ldots 0) \ldots, (1, 1, \ldots, 1, 0, \ldots 0)$ $(1, 1, 1, \ldots, 1)$, définie par

$$Q_k \qquad \begin{aligned} U_1 &= u_1 \\ U_2 &= u_1\, u_2 \\ U_i &= u_1\, u_2 \ldots u_i \\ U_k &= u_1\, u_2 \ldots u_k \end{aligned}$$

On associera à chaque $\mathbf{R}^i$ une variété à coins M^i, définie par

$$s_r \leqslant \epsilon^r \text{ pour } r \leqslant i \ , \quad s_j \geqslant \epsilon^j \text{ pour } j > i.$$

On pourra reconstituer l'espace R^k_+ comme réunion disjointe des M^i, identifiés le long de leur bord par la convention définie au § 1 précédent pour les régions $M(\omega)$ définies par une croix canonique. La remarque essentielle est la suivante : si l'on veut identifier la croix normale H_i située dans R^i avec un voisinage tubulaire d'une strate du bord $\mathbf{R}^s$, $s < i$, on introduira les trajectoires d'attachement obtenues en faisant tendre ϵ vers zéro, alors la croix normale H_i, pour être construite dans un voisinage normale de $\mathbf{R}^s$, devra être soumise à un homéomorphisme $\hat{Q}$ ainsi défini : sur le cube $t_i \leqslant 1$, on effectue l'application Q ; sur les régions extérieures au cube $t_i \leqslant 1$; un homéomorphisme en principe arbitraire. Une telle transformation, en général, n'est pas compatible avec la structure différentiable ambiante de V_i : un demi-hyperplan de la croix normale $0 < t_1 < 1$, $t_2 = 1$ pour H_2 par exemple est rabattu vers l'axe t_1, pour former la parabole $t_2 = {t_1}^2$. cf. Fig. 1.

La construction faite ici dans le cas de R^k avec l'application linéaire graduée définie par les $\hat{t}_j$ peut se généraliser à une application linéaire graduée arbitraire, telle que celle définie par (1). Les équations $r_j = \epsilon^j$ définiront dans E^1 une croix normale H_k, et l'application f_1 projette cette croix normale sur une croix normale subordonnée dans $E_2, \ldots$ etc.

DEFINITION 3. – *Application linéaire subordonnée à une application linéaire graduée.*

Si dans l'application linéaire graduée (1) : $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{k-1}} E_k \xrightarrow{f_k} 0$, on supprime un E_i, on obtient encore une application linéaire graduée en composant les flèches $E_{i-1} \xrightarrow{f_{i-1}} E_i \xrightarrow{f_i} E_{i+1}$ en $E_{i-1} \xrightarrow{f_i \circ f_{i-1}} E_{i+1}$ de part et d'autre de E_i. En itérant ce procédé pour tous les E_i dont les indices appartiennent à un

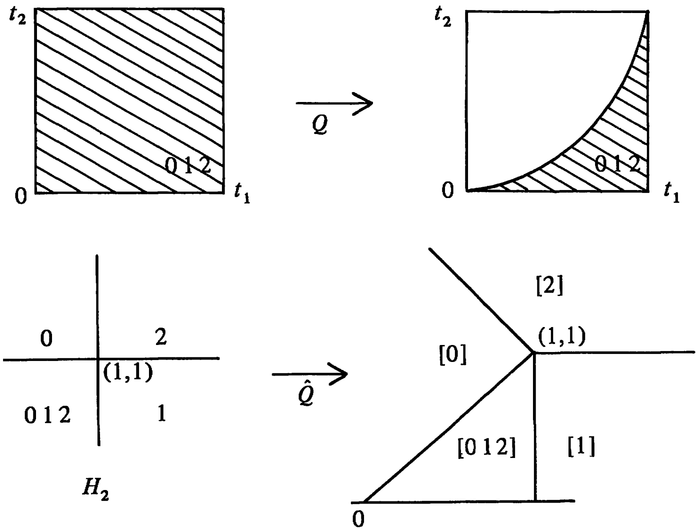

Fig. 1

sous-ensemble A de k, on définit l'application linéaire graduée F_A subordonnée à F pour le sous-ensemble A.

Le drapeau conoyau de l'application F_A (pour les scissions g_j relevées) s'obtient à partir du drapeau conoyau de F en supprimant les variétés linéaires X_j dont les indices j appartiennent à A. Si l'on construit pour F_A la croix normale définie par les fonctions r_j, cette croix normale est localement subordonnée à la croix normale définie pour F.

DEFINITION 4. – *Secteur.*

Soient $X_1, X_2, \dots X_k$ k espaces topologiques ; le joint J de ces espaces est le quotient du produit des espaces $X_1 \times \dots \times X_k$ et du simplexe standard Δ^{k-1} par la relation d'équivalence suivante :

Soient s_j, $j = 1, 2, \dots, k$, des coordonnées barycentriques sur Δ^{k-1}, alors deux points $(x_1, x_j \dots x_k, s_1 \dots, s_k)$, $(x'_1, x'_j \dots x'_k, s'_1, \dots s'_j \dots s'_k)$ sont identifiés si $s_j = s'_j$ pour tout j, et $x_i = x'_i$ pour tout indice i tel que $s_i \neq 0$. (Au contraire x_i et x'_i peuvent être arbitraires, si $s_i = 0$). Soit $q : X_1 \times \dots X_k \times \Delta^{k-1} \to J$ ce passage au quotient. On considère alors une suite d'applications surjectives :

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_i \xrightarrow{f_i} \dots \xrightarrow{f_k} X_k \tag{2}$$

Elle définit dans le produit $X_1 \times X_2 \times X_i \times \dots \times X_k$ un fermé F, constitué des suites $x_1 \in X_1$, $x_2 \in X_2$, $\dots$, $x_j \in X_j$, $\dots$, $x_k \in X_k$, telles que $x_2 = f_1(x_1), \dots$, $x_{i+1} = f_i(x_i)$, $\dots$, $x_k = f_k(x_{k-1})$. On appellera *secteur* associé à la suite (2), l'image par q du fermé F dans le joint J des espaces X_i. Pour une suite composée d'une seule flèche, $X_1 \to X_2$, le secteur associé n'est autre que le "mapping cylinder" de l'application.

DEFINITION 5. – *Secteur associé à une application linéaire graduée.*

Revenons à l'application linéaire graduée $F^{(1)}$ de § 1, avec ses scissions $g_j : E_j \to F_j$. Dans E_j, un voisinage tubulaire du drapeau-conoyau est défini par

$$r_j = \epsilon^j, \ r_{j+1} = \epsilon^{j+1}, \dots r_k = \epsilon^k.$$

Pour ϵ petit, ces équations définissent dans E_j une variété W_j. Chaque f_j applique W_j surjectivement sur W_{j+1}, par une fibration dont la fibre est définie par $r_j =$ cste dans F_j. Ces fibrations sont triviales, et par suite chaque W_i est un produit de sphères.

On appelle *secteur associé* à l'application linéaire graduée le secteur défini par la suite des applications surjectives :

$$W_1 \xrightarrow[f_1]{} W_2 \xrightarrow[f_2]{} \dots \longrightarrow W_i \xrightarrow[f_i]{} \dots \xrightarrow[f_{k-1}]{} W_k \xrightarrow[f_k]{} 0$$

Si on considère une application linéaire graduée F_A subordonnée à (F), alors la suite correspondante des W_j permet de définir un secteur qui s'injecte canoniquement dans le secteur associé à F. Pour toute suite de points x_i, $x_{i+1} = f_i(x_i)$, le simplexe sous-tendu par ces points x_i est l'image, par l'application Q, de la trace $N(A)$ dans $M(\underset{\sim}{k})$ dans la croix normale associée à F au point $(x_i, x_{i+1} \dots x_k)$.

II. **Ensembles stratifiés.**

(1) *Variétés à coins.* On appelle *variété différentiable à coins* une variété ouverte paracompacte, ouvert M dans une variété ambiante U, telle que tout point m du bord $\bar{M} - M$ admette dans M un voisinage difféomorphe à une région $M(A)$ d'une croix normale dans U. L'axe d'une telle croix normale de codimension r sera dit un *coin de codimension r* de la variété. Ainsi en tout point intérieur de M on a un coin de codimension zéro, en tout point régulier du bord, un coin de codimension un, en un coin ordinaire, un coin de codimension deux ... etc.

(2) Pour définir un ensemble stratifié E, on se donne tout d'abord son *schéma d'incidence.* Il s'agit d'un graphe fini orienté, avec une fonction définie positive de l'ensemble des sommets dans N, la dimension d. Si une arête va de a en b, alors $d(a) > d(b)$. Si $a \to b$ et $b \to c$, on a aussi une flèche $a \to c$.

(3) A tout sommet a est associé une variété à coins $M(a)$.

(4) A toute chaine de flèches $a_0 \leftarrow a_1 \leftarrow \dots \leftarrow a_k \leftarrow b$ est associée dans le bord $\partial M(b)$ un coin dont la codimension est égale à la *longueur* de la chaine c, i.e. le nombre total $|c|$ des flèches qu'elle contient. L'axe de cette croix normale $H(c)$ est une sous-variété de codimension $|c|$.

Si une chaine c' partant de b est une sous-chaine de c alors le coin $N(c')$ associé est l'axe d'une croix normale $H|c')$ subordonnée à la croix normale $N(c)$ associée à c : il suffit pour cela d'*oublier* dans la carte locale en tout point de $N(c)$, les coordonnées relatives aux sommets de c qui ne figurent pas dans c'.

(5) *Applications d'attachement.*

Soit une chaine de la forme $A\,X\,B\,Y$, où A et B sont des blocs de flèches. A la chaine $A\,X\,B\,Y$ est associée dans le bord $\partial M(Y)$ un coin de codimension

$|AXB|$, de croix normale $H(AXB)$; de même dans $\partial M(X)$, on a un coin de croix normale $H(A)$. On postule alors qu'il existe une application différentiable surjective k_{XY} définie sur un voisinage de $\partial M(Y)$ à valeurs dans un voisinage de $\partial M(X)$, telle que k_{XY} s'obtient, dans la carte locale autour de tout point de $H(ABXY)$ en oubliant les coordonnées relatives aux sommets de BY ; l'image est alors la croix normale $H(A) \subset \partial M(X)$.

En vertu même de cette définition, si l'on a $X \leftarrow Y \leftarrow Z$, alors $k_{XZ} = k_{XY} \circ k_{YZ}$ sur toutes les croix normales de $M(Z)$ et $M(Y)$ associées aux chaines de la forme $AXBYCZ$, A, B, C blocs de flèches.

Ceci étant posé, l'ensemble stratifié E se définit ainsi : sur la réunion disjointe $M(X)$ des "strates" des sommets X, on effectue les identifications obtenues en identifiant tout point $y \in \partial M(Y)$ avec son image $x = k_{XY}(y)$ dans $\partial M(X)$.

Un ensemble stratifié s'obtient donc en recollant un certain nombre de variétés à coins (les "strates") selon les modèles locaux décrits plus haut.

Au lieu d'identifier brutalement sur les bords par les applications k_{XY}, on pourrait aussi ajouter aux bords les "mapping cylinder (s)" (généralisés) des applications d'attachement. Ceci conduit à la construction suivante : A toute strate $M(X)$ on associe son *étoile* $St(X)$ ainsi définie : pour toute chaine de la forme $X \leftarrow A$, A bloc de flèches, on construit le secteur associé aux applications linéaire graduée définie par les k_{XA} sur les voisinages des croix normales des strates de A. Si XB est une sous-chaine de XA, alors une partie du bord des strates de B s'injectent dans le bord des strates de A : on prolonge alors cette injection aux secteurs construits sur ces chaines. L'ensemble stratifié E s'obtient également en prenant la réunion disjointe de toutes ces étoiles $St(X)$, et en les identifiant selon les injections canoniques : $Y \to St(X)$ si $X \leftarrow Y$.

Dans un article antérieur (EMS), j'ai proposé une autre définition d'un ensemble stratifié, usant des strates, de lambeaux d'incidence, d'applications d'attachement, et de "fonctions tapissantes". La procédure de "normalisation" décrite dans cet article revient à montrer que tout ensemble stratifié admet une présentation comme quotient de variétés à coins selon les modèles locaux décrits plus haut.

Morphismes stratifiés.

J'ai donné dans (EMS) une définition des morphismes faiblement stratifiés. Un tel morphisme $p : E \to E'$ a la propriété que l'image par p d'une strate X de E est une strate X' de E', par une application surjective de rang maximum, et que p commute aux applications d'attachement. Pour un tel morphisme, la contre-image $p^{-1}(X')$ de toute strate X' de E' est un espace fibré (Premier théorème d'isotopie).

Cette notion soulève le problème suivant :

PROBLEME. — Tout morphisme analytique (réel, resp. complexe) admet-il localement un morphisme polynomial (réel, resp. complexe) qui a localement même schéma d'incidence et lui est isotope ? Une réponse positive impliquerait :

COROLLAIRE. — Tout ensemble analytique (réel, resp. complexe) est localement homéomorphe à un ensemble algébrique (réel, resp. complexe).

La notion de morphisme faiblement stratifié n'exclut pas une grande pathologie, car, dans un diagramme de la forme

$$\begin{array}{ccc} X & \xrightarrow{k_{XY}} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow[k_{X'Y'}]{} & Y' \end{array}$$

rien n'est postulé sur le rang de l'application k_{XY} : Ker $p \mid Y \to$ Ker $p \mid X$.

On rappelle que si ces applications sont surjectives, on définit ainsi les morphismes "doux", sans éclatement.

Pour un morphisme doux, il est possible d'adapter la présentation de E à la présentation de E', de manière que toute croix normale de E' soit subordonnée à une croix normale de E par oubli de coordonnées correspondant aux flèches verticales (une flèche $X \to Y$ est dite verticale, si $p(X) = p(Y)$). Mais, dans la carte associée, la propriété que p s'exprime en oubliant certaines coordonnées verticales n'est vraie que sur une partie du complémentaire de la croix. Après application d'une transformation de type ($\hat{Q}$) à la croix normale, cette propriété peut être supposée vraie partout sur l'image de la croix normale par Q.

Exemple.

Prenons pour application stratifiée la projection parallèle à l'axe Oy du rectangle $ABCD$ du plan (Oxy) défini par $A(0, -1)$, $B(a, -1)$, $C(a, +1)$, $D(0, +1)$. L'image est le segment $[0, 1]$ dans un espace O_1X_1 quotient de (Oxy). Ajoutons à l'arête AD l'origine O comme O-strate (figure 2). Soit $\mathcal{R}$ la strate de dimension deux définie par l'intérieur du rectangle, dont l'image par p est l'intérieur de $]0\,1[$ de sur O_1X_1. Autour de O, dans la 2 strate $\mathcal{R}$, on a une croix normale définie par la chaine $O \leftarrow (OA) \leftarrow \mathcal{R}$ d'axe le point $(+1/2, -1/2)$ par exemple.

Dans cette croix normale, les courbes de la forme $p^{-1}\,p(a)$ ont l'allure ci-dessous (courbes pointillées). Elles ne sont linéaires que sur les régions $y \leqslant -1/2$ de la croix normale d'axe $(-1/2, 1/2)$. Après un homéomorphisme de type $\hat{Q}$, on transforme cette croix normale en la figure F (cf. Fig. 3) qui, elle, contient les classes $p^{-1}\,p(a)$ comme verticales.

De manière générale, dans un morphisme doux, sans éclatement une croix normale associée à une chaine (c), contient comme croix subordonnée l'image réciproque par p de la chaine image $p(c)$. C'est seulement après une transformation de type $\hat{Q}$, associée aux flèches "verticales" de c (une flèche $x_1 \leftarrow x_2$ est verticale si $p(x_1) = p(x_2)$) qu'on peut imposer la condition que p est une projection linéaire globale de la carte.

Morphismes analytiques généraux.

Dans un morphisme analytique général, d'application

$$k_{XY} : \text{Ker } p/Y \to \text{Ker } p/X$$

n'est en général pas de rang constant. Mais, très vraisemblablement, on peut

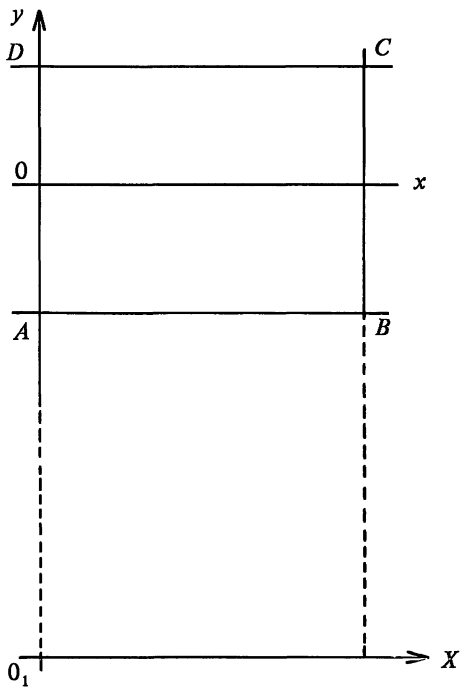

Fig. 2

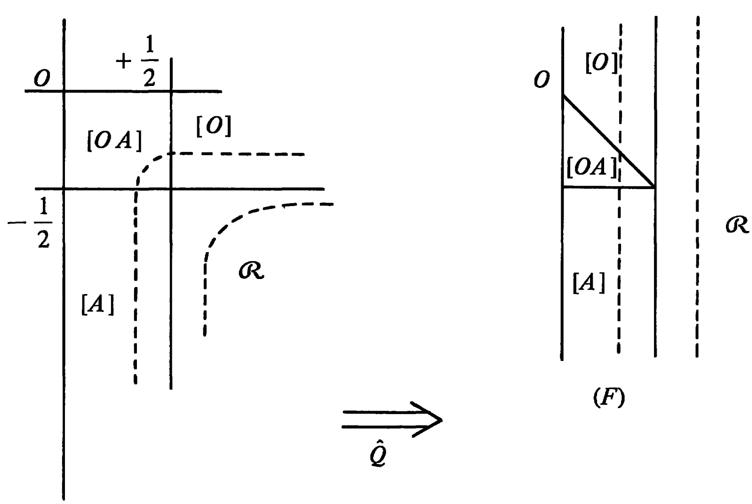

Fig. 3

substratifier X et Y de manière à ce que ces applications soient de rang constant. Ceci oblige à introduire pour toute application d'attachement k_{XY}, leurs espaces image et noyau, ainsi que leurs images par p dans le but. On définira ainsi, dans le bord de toute strate X, des germes de feuilletage (en nombre fini) auxquels on pourra imposer des conditions de transversalité ou d'inclusion. Ainsi dans une théorie généralisée des morphismes analytiques, les applications d'attachement k_{XY} ont une structure linéaire graduée, dont les noyaux et les images par p définissent les germes de feuilletage introduits plus haut.

Alors un morphisme général "rude" avec éclatement, apparait comme contenant implicitement un nombre fini de morphismes "doux" dont les sources et but sont les feuilles de ces feuilletages auxiliaires. Par suite ces morphismes doux implicitement contenus dans un morphisme rude s'organisent en familles continues. Il n'est pas interdit de penser qu'une telle description d'un morphisme permet de donner une interprétation géométrique des procédures de désingularisation utilisés en géométrie algébrique. Ces procédés d'éclatement, en effet, permettent d'expliciter ces familles continues sous jacentes en espaces topologiques éclatés au-dessus des strates du but.

On pourra se demander, en conclusion, si toute l'évolution récente de la géométrie analytique n'a pas détourné l'attention du problème central qui est l'étude topologique des *ensembles* définis par des équations (et inéquations !) algébriques et analytiques. En substituant aux ensembles les idéaux qui (parfois) les définissent, on a cru faire un grand progrès conceptuel. En fait, ce progrès n'est le plus souvent qu'illusoire. Car reconnaitre que deux algèbres locales définies par générateurs et relations sont isomorphes oblige à résoudre un problème linéaire d'une telle dimension qu'il en est en général impraticable. Il est souvent préférable (et toujours utile) d'inspecter topologiquement les germes correspondants et de rechercher s'ils sont ou non homéomorphes. De plus, la notion fondamentale d'"équisingularité" de deux algèbres locales attend toujours sa définition algébrique. Enfin les définitions algébriques sont impuissantes devant les ensembles constructibles (semi-algébriques, semi-analytiques, etc.) introduits par projection propre. Pour toutes ces raisons, je ne peux que croire qu'il n'est pas vain d'essayer de construire une théorie purement topologique des morphismes analytiques, théorie dont on a esquissé ici les premiers rudiments.

REFERENCES

[EMS] THOM R. — Ensembles et morphismes stratifiés. *Bull. Amer. Math. Soc.*, March 69, t. 75, p. 240-284.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 267 à 271.

IDEAUX DE FONCTIONS DIFFERENTIABLES

par Jean-Claude TOUGERON

Soit $\mathcal{E}_n$ (resp. $\mathcal{O}_p$) l'anneau des germes à l'origine des fonctions numériques, définies et indéfiniment dérivables (resp. analytiques) au voisinage de l'origine de $\mathbf{R}^n$ (resp. $\mathbf{R}^p$). Notons $C^\infty(n, p)$ l'ensemble des germes à l'origine de $\mathbf{R}^n$ des applications Φ, indéfiniment dérivables de $\mathbf{R}^n$ dans $\mathbf{R}^p$, et telles que $\Phi(0) = 0$.

Il est bien connu que l'anneau $\mathcal{E}_n$ n'est pas noethérien ; en outre, tout germe de fermé à l'origine de $\mathbf{R}^n$ est le germe des zéros $V(\mathcal{J})$ d'un idéal de type fini $\mathcal{J}$ de $\mathcal{E}_n$. Il est donc sans intérêt d'étudier tous les idéaux de $\mathcal{E}_n$. Aussi, avons nous considéré une famille très particulière d'idéaux :

Soit I un idéal de $\mathcal{O}_p$ engendré par des germes $f_1, \ldots, f_q$; nous étudions l'idéal de $\mathcal{E}_n$, noté $\Phi^* I$, engendré par les $f_i \circ \Phi$, $1 \leqslant i \leqslant q$. Bien entendu, si Φ est quelconque, on ne peut rien dire ; mais nous montrons que, sous des hypothèses de transversalité sur Φ, vérifiées "en général" (nous préciserons ultérieurement cette expression), l'image réciproque $\Phi^* I$ possède des propriétés analogues à celles de I.

Ce travail s'inspire d'une part, de la théorie des singularités des applications différentiables, développée d'abord par H. Whitney et R. Thom (R. Thom étudie les germes d'ensembles $\Phi^{-1}(V(I))$, où $V(I)$ désigne le germe des zéros de I, d'un point de vue topologique, en particulier leur stabilité topologique) ; d'autre part, de résultats de L. Hörmander, S. Łojasiewicz et B. Malgrange, concernant les idéaux de l'anneau $\mathcal{E}_n$, engendrés par des fonctions analytiques (c'est le cas particulier : $n = p$; Φ = identité, de notre problème).

1. Propriétés généralement vraies (J. Cl. Tougeron, [6]).

Soit $\mathcal{F}(n, p)$ l'ensemble des séries de Taylor à l'origine de $\mathbf{R}^n$ de tous les $\Phi \in C^\infty(n, p)$; de même, si $q \in \mathbf{N}^*$, soit $\mathcal{F}^q(n, p)$ l'ensemble des polynômes de Taylor d'ordre q à l'origine de $\mathbf{R}^n$, de tous les $\Phi \in C^\infty(n, p)$. On a des projections évidentes : $T : C^\infty(n, p) \to \mathcal{F}(n, p)$;

$$\pi_q : \mathcal{F}(n, p) \to \mathcal{F}^q(n, p) \quad ; \quad \pi_{q,q'} : \mathcal{F}^{q'}(n, p) \to \mathcal{F}^q(n, p), \text{ si } q' \geqslant q.$$

Pour tout $q \in \mathbf{N}^*$, soit V_q une sous-variété algébrique réelle de $\mathcal{F}^q(n, p)$ et supposons que : $\ldots \supset \pi_q^{-1}(V_q) \supset \pi_{q+1}^{-1}(V_{q+1}) \supset \ldots$

Nous dirons que $V = \bigcap_{q \in \mathbf{N}^*} \pi_q^{-1}(V_q)$ est une sous-variété algébrique de $\mathcal{F}(n, p)$ et par définition, sa codimension sera égale à $\lim_{q \to \infty} \operatorname{codim}_{\mathcal{F}^q(n,p)} V_q$. La variété V

est de codimension infinie, si et seulement si la condition suivante est satisfaite : $\forall q \in \mathbf{N}^*$ et $\forall f_q \in V_q$, il existe un entier $q' \geqslant q$ et $f_{q'} \in \pi_{q,q'}^{-1}$ (f_q), tels que $V \cap \pi_{q'}^{-1}(f_{q'}) = \emptyset$.

Definition. – Une propriété (P) relative aux éléments de $C^\infty(n, p)$ est vraie *en général* si pour tout $\xi \in \mathcal{F}^1(n, p)$ il existe une sous-variété algébrique de codimension infinie V_ξ de $\mathcal{F}(n, p)$ telle que tout Φ appartenant à $T^{-1}(\pi_1^{-1}(\xi) - V_\xi)$ satisfasse à (P).

On ne doit pas confondre cette notion avec celle de "propriété générique" au sens où l'entend Thom. Par exemple, dans le cas $p = 1$, on sait que génériquement Φ est une "fonction de Morse", i.e. l'idéal engendré dans $\mathcal{E}_n$ par les dérivées partielles $\partial\Phi/\partial x_1, \ldots, \partial\Phi/\partial x_n$ contient l'idéal maximal m_n de $\mathcal{E}_n$. Mais cette propriété n'est pas *générale* : on peut simplement affirmer qu'*en général* l'idéal engendré par les $\partial\Phi/\partial x_i$ contient une puissance de m_n.

Soit I un idéal propre de $\mathcal{O}_p$: dans les paragraphes suivants, nous énonçons quelques propriétés vérifiées *en général* par $\Phi^* I$, lorsque Φ décrit $C^\infty(n, p)$.

2. **Le théorème de quasi-transversalité (J. Cl. Tougeron, [6])**

Si $\mathcal{I}$ est un idéal de $\mathcal{E}_n$, on note $\hat{\mathcal{I}}$ l'idéal de l'anneau des séries formelles $\mathcal{F}_n = \mathbf{R}[[x_1, \ldots, x_n]]$, formé par les séries de Taylor, à l'origine de $\mathbf{R}^n$, des éléments de $\mathcal{I}$. On a d'abord le résultat suivant (conservation de la hauteur) :

Theoreme 1. – *En général :* $ht\, \widehat{\Phi^* I} = \inf\,(n, ht I)$.

En particulier, si $ht\ I \geqslant n$, *en général* $\Phi^*\ I$ est un idéal de définition de $\mathcal{E}_n$, i.e. $\Phi^*\ I$ contient une puissance de m_n. Par exemple, si $n \leqslant p$, l'idéal (Φ) engendré dans $\mathcal{E}_n$ par les composantes $\Phi_1, \ldots, \Phi_p$ de Φ est, *en général* un idéal de définition de $\mathcal{E}_n$.

Venons en au théorème de quasi-transversalité. Si $\mathcal{I}$ est un idéal de $\mathcal{E}_n$ et si $k \in \mathbf{N}^*$, notons $J_k(\mathcal{I})$ l'idéal engendré dans $\mathcal{E}_n$ par $\mathcal{I}$ et tous les jacobiens $\dfrac{D(\varphi_1, \ldots, \varphi_k)}{D(x_{i_1}, \ldots, x_{i_k})}$ où $\varphi_1, \ldots, \varphi_k$ appartiennent à $\mathcal{I}$ et $1 \leqslant i_1, \ldots, i_k \leqslant n$. Cet idéal ne dépend pas du système de coordonnées locales choisi et $J_k(\mathcal{I}) = \mathcal{I}$ si $k > n$. Désignons par $\sigma_k(\mathcal{I})$ l'idéal de $\mathcal{E}_n$ engendré par les ξ tels que $\xi \,.\, \mathcal{I}$ soit contenu dans un sous-idéal de $\mathcal{I}$ engendré par k éléments, et posons :

$$R_k(\mathcal{I}) = \sqrt{J_k(\mathcal{I})} \cap \sqrt{\sigma_k(\mathcal{I})}.$$

Si I est un idéal de $\mathcal{O}_p$ (ou de l'anneau $\mathcal{O}(U)$ des fonctions analytiques sur un ouvert U de $\mathbf{R}^p$), on définit pareillement des idéaux $J_k(I)$, $\sigma_k(I)$, $R_k(I)$.

L'interprétation de l'idéal $R_k(I)$ est facile. Par exemple, si I est un idéal de $\mathcal{O}(U)$, l'ensemble $V(I) - V(R_k(I))$ est exactement l'ensemble des points x de $V(I)$ tels que $\mathcal{O}_x \mid I_x$ soit un anneau local régulier de dimension $p - k$. En particulier, $V(I) - V(R_k(I))$ est une sous-variété analytique de codimension k de l'ouvert U.

Definition. – Une k-strate de $\mathcal{E}_n$ est un couple $(\mathcal{I}, \mathcal{I}')$ de deux idéaux de type fini de $\mathcal{E}_n$ tels que $\mathcal{I} \subset \sqrt{\mathcal{I}'} \subset R_k(\mathcal{I})$. Visiblement, si $(\mathcal{I}, \mathcal{I}')$ est une k-strate, le germe d'ensemble $V(\mathcal{I}) - V(\mathcal{I}')$ est un germe de variété C^∞, de codimension k, à l'origine de $\mathbf{R}^n$.

On définirait de même une k-strate de $\mathcal{O}_p$. Ceci dit, on a le résultat suivant :

THEOREME 2. — *Soit (I, I') une k-strate de $\mathcal{O}_p$:*

(1) *si $I' \neq \mathcal{O}_p$, en général $(\Phi^* I, \Phi^* I')$ est une k-strate de $\mathcal{E}_n$*

(2) *si $I' = \mathcal{O}_p$, en général $(\Phi^* I, m_n)$ est une k-strate de $\mathcal{E}_n$*

Le théorème précédent est une version algébrique et locale du théorème de transversalité de Thom [5] : si (I, I') est une k-strate de $\mathcal{O}_p$, on en déduit qu'*en général* Φ est transverse sur le germe de variété analytique $V(I) - V(I')$, sauf peut-être à l'origine de $\mathbf{R}^n$ (mais en fait le théorème 2 est beaucoup plus précis que cette conséquence).

3. **Idéaux fermés (J. Cl. Tougeron et J. Merrien, [7]).**

Soient Ω un ouvert de $\mathbf{R}^n$; $\mathcal{E}(\Omega)$ la $\mathbf{R}$-algèbre des fonctions numériques définies et de classe C^∞ sur Ω. Munissons $\mathcal{E}(\Omega)$ de sa structure habituelle d'espace de Fréchet (convergence uniforme des fonctions et de leurs dérivées sur tout compact). Si $a \in \Omega$, l'application $T_a : \mathcal{E}(\Omega) \to \mathbf{R}[[x_1, \ldots, x_n]]$ qui à toute fonction f associe sa série de Taylor en a est surjective (théorème de Borel généralisé). Si $\mathcal{I}(\Omega)$ est un idéal de $\mathcal{E}(\Omega)$, on pose $\widehat{\mathcal{I}(\Omega)} = \{f \in \mathcal{E}(\Omega) \mid \forall a \in \Omega, T_a f \in T_a \mathcal{I}(\Omega)\}$: un théorème de Whitney affirme que l'adhérence de $\mathcal{I}(\Omega)$ dans $\mathcal{E}(\Omega)$ est égale à $\widehat{\mathcal{I}(\Omega)}$. La notion d'idéal fermé se localise : un idéal $\mathcal{I}$ de $\mathcal{E}_n$ sera dit "fermé" s'il existe un voisinage ouvert Ω de l'origine de $\mathbf{R}^n$ est un idéal fermé $\mathcal{I}(\Omega)$ de $\mathcal{E}(\Omega)$ tels que $\mathcal{I}(\Omega)$ engendre $\mathcal{I}$ sur $\mathcal{E}_n$.

On a le résultat suivant, dû à B. Malgrange [3], et démontré d'abord dans le cas d'un polynôme par L. Hörmander [1], et dans le cas d'une fonction analytique par S. Łojasiewicz [2] :

Un idéal de $\mathcal{E}_n$ engendré par un nombre fini de germes de fonctions analytiques est fermé.

Le théorème suivant généralise le théorème précédent (la démonstration utilise les théorèmes 1 et 2 et aussi les techniques développées pas S. Łojasiewicz et B. Malgrange) :

THEOREME 3. — *Soit I un idéal de $\mathcal{O}_p$. Si $\Phi \in C^\infty(n, p)$, en général l'idéal $\Phi^* I$ est fermé.*

Enfin, il résulte facilement du théorème de B. Malgrange que l'anneau $\mathcal{E}_n$ *est plat sur l'anneau* $\mathcal{O}_n$ des germes des fonctions numériques, analytiques à l'origine de $\mathbf{R}^n$. L'analogue de ce résultat dans le présent contexte est le suivant :

THEOREME 4. — *Soit I un idéal de $\mathcal{O}_p$. Si $\Phi \in C^\infty(n, p)$, en général $TOR_1^\Phi(\mathcal{O}_p/I, \mathcal{E}_n)$ est un* $\mathbf{R}$-*espace vectoriel de dimension finie (et même, en général*

$$TOR_1^\Phi(\mathcal{O}_p/I, \mathcal{E}_n) = 0,$$

si la dimension homologique $dh(\mathcal{O}_p/I)$ de $\mathcal{O}_p/I$ sur $\mathcal{O}_p$ est $\leqslant n$).

[Un germe d'application $\Phi \in C^\infty(n, p)$ munit $\mathcal{E}_n$ d'une structure de $\mathcal{O}_p$-module : le module $TOR_1^{\mathcal{O}_p}(\mathcal{O}_p/I, \mathcal{E}_n)$ est alors noté $TOR_1^{\Phi}(\mathcal{O}_p/I, \mathcal{E}_n)$; la condition $TOR_1^{\Phi}(\mathcal{O}_p/I, \mathcal{E}_n) = 0$ signifie simplement ceci : si I est engendré sur $\mathcal{O}_p$ par $f_1, \ldots, f_q$, le module des relations entre les $f_i \circ \Phi$ à coefficients dans $\mathcal{E}_n$ est engendré sur $\mathcal{E}_n$ par les relations entre les f_i à coefficients dans $\mathcal{O}_p$].

On déduit des théorèmes précédents de nombreux renseignements sur l'idéal $\Phi^* I$: par exemple, si $\mathcal{O}_p/I$ est réduit (i.e. sans nilpotents) et si $dh(\mathcal{O}_p/I) < n$, *en général* $\mathcal{E}_n/\Phi^* I$ est réduit ; si $\mathcal{O}_p/I$ est normal et si $dh(\mathcal{O}_p/I) < n - 1$, *en général* $\mathcal{F}_n/\widehat{\Phi^* I}$ est normal.

4. Stabilité locale des idéaux (J. Cl. Tougeron, [6]).

Désignons par Dif(n) le groupe des germes $\mathcal{C}$ (à l'origine de $\mathbf{R}^n$) des difféomorphismes C^∞ d'un voisinage de l'origine de $\mathbf{R}^n$ sur un voisinage de l'origine de $\mathbf{R}^n$, tels que $\mathcal{C}(0) = 0$.

DEFINITION. — Un germe d'application $\Phi \in C^\infty(n, p)$ est *I-déterminant* s'il existe un entier q tel que la condition suivante soit satisfaite :

Pour tout $\Phi' \in C^\infty(n, p)$ tel que $\Phi - \Phi'$ soit q-plat à l'origine

(i.e. $$\pi_q \circ T(\Phi) = \pi_q \circ T(\Phi')),$$

il existe un élément de Dif(n) qui transforme l'idéal $\Phi^* I$ en l'idéal $\Phi'^* I$.

En particulier, sous cette dernière hypothèse, et si Φ_q désigne le polynôme de Taylor de degré q de Φ à l'origine, il existe un élément de Dif(n) qui transforme l'idéal $\Phi^* I$ en l'idéal $\Phi_q^* I$: donc, à difféomorphisme C^∞ près, l'idéal $\Phi^* I$ est engendré par des germes de fonctions analytiques.

Nous dirons que l'idéal I est *rigide*, si *en général* un élément de $C^\infty(n, p)$ est I-déterminant. On a le résultat suivant :

THEOREME 5. — *Soit I un idéal de hauteur k de $\mathcal{O}_p$, tel que*

$$I = \sqrt{I} \qquad \text{et} \qquad ht(R_k(I)) \geq \inf(n, p-1).$$

L'idéal I est rigide.

Soit I un idéal de $\mathcal{O}_p$ tel que $I = \sqrt{I}$ et $ht(I) \geq \inf(n-1, p-2)$: l'hypothèse du théorème précédent est alors satisfaite (l'hypothèse $I = \sqrt{I}$ entraine en effet : $ht(R_k(I)) > k = ht(I)$) et donc I est rigide. En particulier, si $n \leq 2$ ou si $p \leq 3$, tout idéal premier de $\mathcal{O}_p$ est rigide (par contre si $n \geq 3$ et si $p \geq 4$, il existe dans $\mathcal{O}_p$ des idéaux non rigides).

Signalons enfin la conséquence suivante. Soit $y_1, \ldots, y_p$ un système de coordonnées locales à l'origine de $\mathbf{R}^p$. L'idéal I engendré par $y_1, \ldots, y_p$ dans $\mathcal{O}_p$ est évidemment rigide et $\Phi^* I$ est égal à l'idéal (Φ) engendré dans $\mathcal{E}_n$ par les composantes de Φ. Ainsi, *en général* il existe un entier $q > 0$ et un élément de Dif(n) qui transforme l'idéal (Φ) en l'idéal (Φ_q) engendré dans $\mathcal{E}_n$ par les polynômes de Taylor d'ordre q des composantes de Φ (en fait, on a des résultats beaucoup plus précis : nous renvoyons le lecteur à [6], ch. II).

BIBLIOGRAPHIE

[1] HÖRMANDER L. — On the division of distributions by polynomials, *Arkiv för Mathematik*, 3, 1958, p. 555-568.

[2] ŁOJASIEWICZ S. — Sur le problème de la division, *Studia Math*, 18, 1959, p. 87-136.

[3] MALGRANGE B. — *Ideals of differentiable functions*. Oxford Uni-Press, 1966.

[4] SERRE J.P. — Algèbre locale, Multiplicités, lecture notes, in *Mathematics*, 11, 1965.

[5] THOM R. — Un lemme sur les applications différentiables, *Bol. Soc. Mat. Mexicana*, 1956, p. 59-71.

[6] TOUGERON J. Cl. — Idéaux de fonctions différentiables I, *Annales de l'institut Fourier*, Tome XVIII.

[7] TOUGERON J. Cl. et MERRIEN J. — Idéaux de fonctions différentiables II, *Annales de l'institut Fourier*, Tome XX.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 273 à 277.

C 5 - GROUPES ALGÉBRIQUES FONCTIONS AUTOMORPHES ET GROUPES SEMI-SIMPLES

ON THE ZETA-FUNCTIONS OF THE GENERAL LINEAR GROUP

by A. N. ANDRIANOV

1. Spherical functions.

Let G be any unimodular locally compact topological group, and $\mathfrak{U}$ is a compact sub-group of G. Let $L(G, \mathfrak{U})$ be the C-algebra of all complex-valued continuous functions on G with compact support which are constant on each double coset $\mathfrak{U}x\mathfrak{U}$ of $\mathfrak{U}$ in G. Multiplication in $L(G, \mathfrak{U})$ is defined as the convolution : $(f * \varphi)(x) = \int_G f(xy^{-1})\varphi(y)\,dy$, where dy is a Haar measure on G. Assume now that $L(G, \mathfrak{U})$ is commutative. Then a (zonal) spherical function on G relative to $\mathfrak{U}$ is defined to be a complex-valued continuous function ω on G which satisfies the following three conditions : $(\hat{I})$ ω is bi-invariant with respect to $\mathfrak{U}$; (2) $\omega(1) = 1$; (3) $f * \omega = \lambda_f \omega$ for all $f \in L(G, \mathfrak{U})$, where λ_f is a complex number depending on f. Denote the set of all spherical functions on G relative to $\mathfrak{U}$ by $\Omega(G, \mathfrak{U})$. For the theory of spherical functions see [4], [8], [10].

2. Spherical functions on GL_n.

Let $\mathcal{D}$ be a division algebra over a complete discrete valuation field ; let $\mathfrak{O}$ be a maximal order in $\mathcal{D}$; let $\mathfrak{p} = (\pi)$ be the maximal ideal in $\mathfrak{O}$. Suppose that the residue field of $\mathfrak{O}$ is finite and has q elements and let

$$G = GL_n(\mathcal{D}), \quad \mathfrak{U} = GL_n(\mathfrak{O}) \quad (n \geqslant 2) .$$

In this case the set $\Omega(G, \mathfrak{U})$ can be parametrized as follows [8], [II]. We set $H = \{\mathrm{diag}(\pi^\lambda) = \mathrm{diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}); (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n\}$, $N = \{(x_{ij}) \in G ;\ x_{ij} = 0$ if $i > j,\ x_{ii} = 1\ (i = 1, \dots, n)\}$.

Then $G = \mathfrak{U}HN = \mathfrak{U}H\mathfrak{U}$. For $s = (s_1, \dots, s_n) \in \mathbf{C}^n$ and $x \in \mathfrak{U}\,\mathrm{diag}\,\pi^\lambda N$ define $\Phi_s(x) = \Pi_1^n\, q^{\lambda_i(s_i - i + 1)}$. Now let

$$\omega_s(x) = \int_{\mathfrak{U}} \Phi_s(x^{-1}u)\,du \qquad (x \in G)$$

where du is the Haar measure on the compact group $\mathfrak{U}$, normalized so that the measure of $\mathfrak{U}$ is I. Then $\omega_s \in \Omega(G, \mathfrak{U})$, and all $\omega \in \Omega(G, \mathfrak{U})$ are obtained in this way.

Since ω_s is bi-invariant with respect to $\mathfrak{U}$ it follows that ω_s is uniquely determined by its values on H. Then we have the following formula.

THEOREM 1 [3]. – *Let* $\lambda \in \mathbf{Z}^n, \lambda_1 \geqslant \cdots \geqslant \lambda_n$ *and let* $s \in \mathbf{C}^n$. *Then*

$$\omega_s(\operatorname{diag} \pi^\lambda) = \frac{q^{\Sigma_1^n \lambda_i (i-1)}}{P(q^{-1})} \sum_{\sigma \in S_n} q^{-\Sigma_1^{ns} \sigma(i) \lambda_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1 - q^{-1+s_{\sigma(i)} - s_{\sigma(j)}}}{1 - q^{s_{\sigma(i)} - s_{\sigma(j)}}} \tag{1}$$

where S_n *is the symmetric group of degree n, and* $P(t) = (t-1)^{-n} \Pi_1^n (t^i - 1)$.

Remark. – It is easy to see that the expression (1) is in fact a polynomial in q^{s_i} $(i = 1, \ldots, n)$.

Theorem 1 can be reformulated as a purely algebraic statement about the explicit structure of the ring $L(G, \mathfrak{U})$ [3].

Macdonald had anounced the analogous formulas for the $\mathfrak{p}$-adic Chevalley groups [7]. Langlands conjectures that the analogous formulas take place for each quasi-split reductive group over a local field [6].

3. Local zeta-functions of GL_n.

Keep the notations and the assumptions as in § 2. For $i = 1, \ldots, n$ and $x \in G$ we set $V_i(x) = q^{-\lambda_i}$ where

$$\mathfrak{U} x \mathfrak{U} = \mathfrak{U} \operatorname{diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_i}, \ldots, \pi^{\lambda_1}) \mathfrak{U}, \ \lambda_1 \leqslant \cdots \leqslant \lambda_n .$$

The integral

$$\zeta_G(z_1, \ldots, z_n ; \omega) = \int_{G \cap M_n(\mathfrak{o})} \omega(x^{-1}) \{\Pi_1^n V_i(x)^{z_i}\} \, dx , \tag{2}$$

where $z_i (i = 1, \ldots, n)$ are complex variables, $\omega \in \Omega(G, \mathfrak{U})$, will be called the multiple zeta-function of the group G with "character" ω. The following Theorem is the consequence of Theorem I.

THEOREM 2 [3]. – *Given an arbitrary* $\omega = \omega_s \in \Omega(G, \mathfrak{U})$, *the integral* (2) *converges absolutely in the domain* $\operatorname{Re} z_j > \max_i \operatorname{Re} s_i$ $(j = 1, \ldots, n)$, *and in and domain* $\zeta_G(z_1, \ldots, z_n ; \omega)$ *has the form*

$$\zeta_G(z_1, \ldots, z_n ; \omega) = \frac{P(s ; z_2, \ldots, z_n)}{Q(s ; z_1, \ldots, z_n)} ,$$

where

$$Q(s ; z_1, \ldots, z_n) = \prod_{r=1}^{n} \prod_{1 \leqslant i_1 < \cdots < i_r \leqslant n} \left(1 - q^{-\frac{r(r-1)}{2} + \sum_{a=1}^{r} s_{i_a} - \sum_{j=n-r+1}^{n} z_j}\right) ,$$

$P(s ; z_2, \ldots, z_n)$ *is a polynomial in* $q^{s_1}, \ldots, q^{s_n} ; q^{-(z_2 + \cdots + z_n)}, \ldots, q^{-z_n}$ *with the coefficients in* $\mathbf{Q}$.

It is easy to see that the fraction $Q(s ; z_1, \ldots, z_n)^{-1}$ can be expressed as a product of the zeta-functions of Langlands [5], [6] which correspond to the exterior degrees of the standard representation of $G = GL_n(\mathfrak{O})$. The nature of $P(s ; z_2, \ldots, z_n)$ is not clear. The analogous statements can be infered for the $\mathfrak{p}$-adic Chevalley groups from the Macdonald's formulas [7].

4. Applications to Sp_n [3].

Let the notations be as in § 2 and suppose that $\mathcal{O} = k$ is a field. Let

$$S = Sp_n(k) = \left\{ g \in GL_{2n}(k) \; ; \; {}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = r(g) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \; , \; r(g) \in k \right\}$$

be the symplectic group of genus n over k, $V = S \cap GL_{2n}(\mathcal{O})$. The integral

$$\zeta_s(z \; ; \omega) = \int_{S \cap M_{2n}(\mathcal{O})} \omega(g^{-1}) \, |r(g)|_{\mathfrak{p}}^z \, dg \, ,$$

where $\omega \in \Omega(S, V)$ (see § 1), dg is a Haar measure on S and z is complex variable, is called the zeta-function of the group S with the "character" ω

Given an arbitrary $\omega \in \Omega(S, V)$, it exists $\widetilde{\omega} \in \Omega(GL_n(k), GL_n(\mathcal{O}))$ such that

$$(1 - q^{-(\gamma+z)}) \, \zeta_s(z \; ; \omega) = \zeta_{GL_n(k)}(-n, -(n-1), \ldots, -2, z + \gamma - 1 \; ; \widetilde{\omega})$$

for suitable $\gamma \in \mathbf{C}$ ($\mathrm{Re}\, z$ is large enough) [2]. It follows from this relation and Theorem 2 that $\zeta_s(z \; ; \omega)$ is the rational function in q^{-z} and its denominator has the form which was conjectured by Satake [8]. It gives also the proof of the Shimura's conjectures [9] about rationality of the Hecke series for $Sp_n(k)$ and the degrees of their nominators and denominators.

5. Global zeta-functions of GL_n.

Let A be a central simple algebra over an algebraic number field k of finite degree ; let G be the multiplicative group of A ; let $\mathcal{O}$ be a maximal order in A. For each valuation $\mathfrak{p}$ on k we shall denote by $k_{\mathfrak{p}}$ the completion of k at $\mathfrak{p}$; $A_{\mathfrak{p}} = A \otimes_k k_{\mathfrak{p}}$; $G_{\mathfrak{p}}$ is the multiplicative group of $A_{\mathfrak{p}}$. For a nonarchimedean valuation $\mathfrak{p}$ we denote by $\mathcal{O}_{\mathfrak{p}}$ the closure of $\mathcal{O}$ in $A_{\mathfrak{p}}$ and set

$$\mathfrak{U}_{\mathfrak{p}} = \{ u_{\mathfrak{p}} \in G_{\mathfrak{p}} \; ; \; u_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} \} \, .$$

For an archimedean valuation $\mathfrak{p}$ we set $\mathfrak{U}_{\mathfrak{p}} = \{ u_{\mathfrak{p}} \in G_{\mathfrak{p}} \; ; \; u_{\mathfrak{p}} u_{\mathfrak{p}}^* = 1 \}$ where $x_{\mathfrak{p}} \to x_{\mathfrak{p}}^*$ is a positive involution of $A_{\mathfrak{p}}$. For $x_{\mathfrak{p}} \in A_{\mathfrak{p}}$ we set $V_{\mathfrak{p}}(x_{\mathfrak{p}}) = |\mathfrak{N} x_{\mathfrak{p}}|_{\mathfrak{p}}$, where $\mathfrak{N}$ is the norm of the regular representation of $A_{\mathfrak{p}}$ over $k_{\mathfrak{p}}$; $|\;|$ is the usual norm in $k_{\mathfrak{p}}$. For $x_{\mathfrak{p}} \in A_{\mathfrak{p}}$ we let

$$\Phi_{\mathfrak{p}}(x_{\mathfrak{p}}) = \begin{cases} \exp(-\pi \, Tr(x_{\mathfrak{p}} x_{\mathfrak{p}}^*)) \, , & \text{if } \mathfrak{p} \text{ is archimedean} \\ \left.\begin{array}{ll} 1 \, , & \text{if } x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}} \\ 0 \, , & \text{if } x_{\mathfrak{p}} \notin \mathcal{O}_{\mathfrak{p}} \end{array}\right\} & \text{if } \mathfrak{p} \text{ is nonarchimedean} \end{cases}$$

where Tr is the reduced trace over the field of real numbers.

Let G_A be the group of adeles of the group G. For $x = (\ldots, x_{\mathfrak{p}}, \ldots) \in G_A$ we set $V(x) = \Pi_{\mathfrak{p}} V_{\mathfrak{p}}(x_{\mathfrak{p}})$, $\Phi(x) = \Pi_{\mathfrak{p}} \Phi_{\mathfrak{p}}(x_{\mathfrak{p}})$. We denote by $\Gamma \cong G$ the subgroup of principal adles, and set $\mathfrak{U} = \Pi_{\mathfrak{p}} \mathfrak{U}_{\mathfrak{p}}$.

Continuous function f on G_A are said to be Γ-automorphic if :

(1) $f(ux\gamma) = f(x)$ for all $u \in \mathfrak{U}$, $x \in G_A$, $\gamma \in \Gamma$;

(2) for any $\Phi \in L(G_A, \mathfrak{U})$ (see § 1) there exists a complex number λ_Φ such that $\Phi * f = \lambda_\Phi f$.

To every nontrivial Γ-automorphic function f there corresponds uniquely a zonal spherical function $\omega \in \Omega(G_A, \mathfrak{U})$ which satisfies the condition

$$\int_{\mathfrak{u}} f(xuy)\, du = \omega(x) f(y) \ [10] .$$

We shall say that ω belongs to f. By the spectrum $s(\Gamma)$ of Γ we shall mean the set of all $\omega \in \Omega(G_A, \mathfrak{U})$ which belong to some nontrivial Γ-automorphic function and which are positive definite functions [10] and satisfy the relation $\omega(\xi x) = \omega(x)$ for every ξ from the center of G_A.

By the zeta-function of the group G with "character"

$$\omega = \Pi_{\mathfrak{p}}\, \omega_{\mathfrak{p}} \in s(\Gamma) \ \ (\omega_{\mathfrak{p}} \in \Omega(G_{\mathfrak{p}}, \mathfrak{U}_{\mathfrak{p}}))$$

we mean the function

$$\zeta_G(z\ ;\omega) = \int_{G_A} \Phi(x)\, \omega(x^{-1})\, V(x)^z\, dx = \Pi_{\mathfrak{p}} \int_{G_{\mathfrak{p}}} \Phi_{\mathfrak{p}}(x_{\mathfrak{p}})\, \omega_{\mathfrak{p}}(x^{-1})\, V_{\mathfrak{p}}(x_{\mathfrak{p}})^z\, dx_{\mathfrak{p}} = \Pi_{\mathfrak{p}}\, \zeta_{G_{\mathfrak{p}}}(z\ ;\omega_{\mathfrak{p}})_{\mathfrak{p}},$$

where $dx = \Pi_{\mathfrak{p}} dx_{\mathfrak{p}}$ is a Haar measure on G_A. The function $\zeta_G(z\ ;\omega)$ is regular in the region $\operatorname{Re} z > 1$. The $\mathfrak{p}$-factors $\zeta_{G_{\mathfrak{p}}}(z\ ;\omega_{\mathfrak{p}})$ are computed explicitly by Tamagawa in [11]. Then we have the theorem.

THEOREM 3 [I]. – *With the notations and assumptions mentioned above the function $\zeta_G(z\ ;\omega)$ extends meromorphically over the entire z-plane with only a finite number of poles and satisfies the functional equation*

$$\zeta_G(z\ ;\omega) = W(\omega)\, \Delta^{1/2-z}\, \zeta_G(1-z\ ;\bar{\omega}) ,$$

where $W(\omega)$ is a constant depending only on ω ; $|W(\omega)| = 1$; Δ is the absolute discriminant of the algebra A.

The analogous theorem for the ground functional fields k are proved by Maloletkin (Мат. заметки 5,5 (1969)). Note, that $\zeta_G(z\ ;\omega)$ coincides with the Langlands's zeta-function of G corresponding to the standard representation of G [5], [6].

REFERENCES

[1] Андрианов А. Н. — Дзета-функции простых алгебр с неабелевыми характерам, Успехи матем. наук, 23, 4, 1968, p. 3-66.

[2] Андрианов А. Н. — Теоремы рациональности для рядов Гекке и дзета-функций групп GL_n и Sp_n над локальными полями, Известия АН СССР (сер. матем.) 33, 3, 1969, p. 466-505.

[3] Андрианов А. Н. — Сферические функции для GL_n над локальными полями и суммирование рядов Гекке, Матем. сборник, 83, 3, 1970.

[4] GODEMENT R. — Introduction aux travaux de A. Selberg, *Sem. Bourbaki*, 9, 1956-1957.
[5] LANGLANDS R.P. — Euler Products, *Lecture notes*, Yale University, 1967.
[6] LANGLANDS R.P. — Problems in the Theory of Automorphic Forms, *Lecture Notes*, Yale University, 1969.
[7] MACDONALD I.G. — Spherical functions on a $\mathfrak{p}$-adic Chevalley group. *Bull. Amer. Math. Soc.*, 74, 3, 1968, p. 520-525.
[8] SATAKE I. — Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields, *Publ. Math. I.H.E.S.*, 18, 1963, p. 5-69.
[9] SHIMURA G. — On modular correspondences for $Sp(n, \mathbf{Z})$ and their congruence relations, *Proc. Nat. Acad. Sci. U.S.A.*, 49, 6, 1963, p. 824.
[10] TAMAGAWA T. — On Selberg's trace formula, *Journ. Fac. Sci. Univ. Tokyo*, Sect. I, 8, 1960, p. 363-386.
[11] TAMAGAWA T. — On the ζ-functions of a division algebra, *Ann. of Math.*, 77, 1963, p. 387-405.

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SINGULAR ELEMENTS OF SEMI-SIMPLE ALGEBRAIC GROUPS

by E. BRIESKORN

Four years ago Steinberg lectured in Moscow on classes of elements of semi-simple algebraic groups [12]. In a very modest sense my talk may be viewed as a continuation of Steinbergs lecture. However, I shall not try to give a report on all the problems posed by Steinberg. Instead of that I shall concentrate on some recent results concerning one of Steinbergs problems. The problem is : Study the variety of unipotent elements thoroughly.

Let G be a semisimple algebraic group over an algebraically closed field $K = \overline{K}$. We want to study the conjugacy classes of elements $x \in G$. This problem can be decomposed into two parts, corresponding to the Jordan decomposition $x = x_s \cdot x_u$ of x into its semisimple and unipotent parts. The conjugacy classes of semisimple elements are obviously classified by T/W, where T is a maximal torus and W is the Weyl group. Thus, associating to x the conjugacy class $\overline{x}_s$ of x_s, one obtains a morphism $G \to T/W$, the fibres of which are unions of conjugacy classes of G.

Steinberg, Springer and Kostant studied the classes of elements of the most general type, that is, the regular elements ([11]. [9], [7]).

DEFINITION. — x is regular if and only if the dimension of its centralizer $Z_G(x)$ is minimal, i.e. equals the rank r of G.

Steinberg and Kostant have obtained various characterizations of regular elements, for instance the following ones.

THEOREM. —

(i) *x is regular if and only if x is contained only in finitely many Borel groups.*

(ii) *x is regular if and only if $G \to T/W$ is regular,* i.e. *smooth, at x.*

A good deal is known about the regular elements. For instance, they form an open dense subset in G whose complement is an algebraic set of codimension 3, and each fibre of $G \to T/W$ contains exactly one regular class. For a singular x one has $\dim Z_G(x) \geqslant r + 2$.

DEFINITION. — x is subregular if and only if $\dim Z_G(x) = r + 2$.

It was Grothendieck who recommended to study the subregular elements. In fact, he conjectured most of what is going to follow after reading a paper of mine on a mysterious connection between Weyl groups and rational singularities [2].

I am going to explain two characterizations of subregular elements, which correspond to the two characterizations of regular elements given by Steinberg. First I shall talk about the one using Borel groups.

Let D be the projective variety of all Borel groups of G, and let Y be the subvariety of $G \times D$ consisting of all pairs (x, B) such that $x \in B$. One has natural morphisms $Y \to G$ and $Y \to T$ such that the diagram formed by them, $G \to T/W$ and $T \to T/W$ commutes.

The following theorem was proved by Grothendieck and, as far as the unipotent fibre is concerned, already by Springer [10].

THEOREM. – *The following diagram is a simultaneous resolution of the singularities of the fibres of $G \to T/W$:*

$$\begin{array}{ccc} Y & \to & G \\ \downarrow & & \downarrow \\ T & \to & T/W \end{array}$$

The term "resolution" is explained by the following definition.

DEFINITION. – A resolution of $X \to S$ is a commutative diagram

$$\begin{array}{ccc} Y & \to & X \\ \downarrow & & \downarrow \\ T & \to & S \end{array}$$

where $Y \to X$ is proper and surjective,
$T \to S$ is finite and surjective,
$Y \to T$ is regular,
$Y_t \to X_{s(t)}$ is a resolution of singularities for all fibres Y_t, $t \in T$.

In general, a morphism does not admit a resolution. It is a very particular property of $G \to T/W$ and the singularities of its fibres that it has a resolution.

In order to study the singularity of a fibre X_s at a point $x \in X_s$, we consider the reduced exceptional fibre F_x over x in the resolution $Y_t \to X_s$. For unipotent x by construction $F_x = \{B \in D \mid x \in B\}$. The regular x are those with F_x a point, by Steinbergs theorem. We shall now describe the F_x for $x \in G$ subregular unipotent and G simple. – It is easy to reduce the consideration of the general situation to this case.

Choose a Borel group B_0, let Δ be the corresponding system of simple positive roots, and P_a for $a \in \Delta$ the parabolic group generated by B_0 and U_{-a}. The fibres of $G/B_0 \to G/P_a$ are projective lines in D called lines of type a. Let (n_{ab}) be the Cartan matrix, and $n'_{ab} = -n_{ab}$ if $-n_{ab} \neq \text{char } K$, and $n'_{ab} = 1$ otherwise.

DEFINITION. – A Dynkin curve is a connected curve in D, the components of which are lines of type a, $a \in \Delta$, such that any component of type a intersects n'_{ab} components of type b.

Tits and Steinberg proved the following.

THEOREM. – *Let G be simple.*

(i) *There is exactly one conjugacy class of subregular unipotent elements.*

(ii) *A unipotent $x \in G$ is subregular if and only if its exceptional fibre F_x is a Dynkin curve. All Dynkin curves occur as exceptional fibres.*

The first statement follows of course from Dynkins classification of all unipotent classes [4], if $K = \mathbf{C}$, and this proof carries over to the more general case of good characteristic. The second statement is exactly the analogue of Steinbergs first characterization of regular elements.

Exceptional curves of the type mentioned above are well known in algebraic geometry. They occur in the theory of rational singularities. This notion was introduced by M. Artin [1].

DEFINITION. – Let V be an algebraic surface, $v \in V$ a normal point and $f : V' \to V$ the minimal resolution of singularities. (V, v) is a rational singularity if for the higher direct images of the structure sheaf $(R^i f_* \mathcal{O}_{V'})_v = 0$, $i > 0$.

THEOREM. – *(V, v) is a rational singularity with* emb. $\dim_v V \leqslant 3$ *if and only if the reduced exceptional curve over v is isomorphic to a Dynkin curve of type A_r, D_r or E_r with self-intersection* -2 *for all components.*

Hence the theorem of Tits and Steinberg means that the unipotent variety has a rational singularity "along" its subregular orbit.

For $K = \mathbf{C}$, in the category of complex analytic spaces – to which I shall switch from now on – the rational singularities with embedding – dimension 3 admit the following beautiful description.

PROPOSITION. – The rational singularities with emb. dim $\leqslant 3$ are exactly the singularities of $\mathbf{C}^2/\Gamma$, Γ a finite subgroup of $SL(2, \mathbf{C})$.

The finite subgroups of $SL(2, \mathbf{C})$ are well known, they are the cyclic groups, and the binary dihedral, tetrahedral, octahedral and icosahedral groups. For example, if Γ is the binary icosahedral group, the corresponding Dynkin curve is that of E_8, and $\mathbf{C}^2/\Gamma \subset \mathbf{C}^3$ is the set of zeros of the equation

$$x^2 + y^3 + z^5 = 0 .$$

This equation is now almost one hundred years old – it first occurs in a paper of H.A. Schwarz [8] in 1872. Note that the equation is weighted homogeneous, this notion being defined as follows :

DEFINITION. – $\Sigma a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$ with $w_1 i_1 + \cdots + w_n i_n = v$ is weighted homogeneous of weight $(w_1, \dots, w_n)$ and degree v.

The equations of all $\mathbf{C}^2/\Gamma$ were determined by F. Klein in 1874 (see e.g. [6]).

PROPOSITION. – The equations of $\mathbf{C}^2/\Gamma$ are weighted homogeneous. Their weights and degrees are given in the following table.

type	weight	degree
A_r	$\left(1, \frac{r+1}{2}, \frac{r+1}{2}\right)$	$r + 1$
D_r	$(2, r-2, r-1)$	$2r - 2$
E_6	$(3, 4, 6)$	12
E_7	$(4, 6, 9)$	18
E_8	$(6, 10, 15)$	30

Up to analytic isomorphism the equations, which have to describe isolated singularities, are uniquely determined by these weights and degrees.

In order to give our second description of subregular unipotent elements, we need one more notion, that of "universal deformation" ("universal unfolding" in Thoms theory of singularities).

DEFINITION. — Let X_0 be a complex space, $x \in X_0$. A deformation of the germ (X_0, x) is a flat morphism $(X, x) \to (T, t)$ together with an isomorphism of (X_0, x) with the germ (X_t, x) of the fibre over t.

$(X, x) \to (T, t)$ is semi-universal if for all

$(X', x) \to (T', t')$ there exists a $g : (T', t') \to (T, t)$

with uniquely determined $dg|_{t'}$, such that X' is isomorphic to $X \times_T T'$.

Tjurina [13] and Schlesinger-Kas proved independently :

THEOREM. — *For isolated complete intersections semi-universal deformations exist and are unique.*

It is an easy consequence of this theorem that one can give a very explicit description of the universal deformation. For the sake of simplicity, I shall explain this only for the case of hypersurfaces.

COROLLARY. — Let X_0 be the hypersurface in $\mathbf{C}^n$ given by $f(z) = 0$, and $0 \in X_0$. The universal deformation of $(X_0, 0)$ is the germ at the origin of $\mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^1 \times \mathbf{C}^k$, where (z, t) maps to $(F(z, t), t)$ and $F(z, t) = f(z) + \sum_{i=1}^{k} g_i(z)\, t_i$, where the polynomials $1, g_1, \ldots, g_k$ represent a basis of $\mathbf{C}\{z_1, \ldots, z_n\}/(f, \partial f/\partial z_i)$.

COROLLARY. — The universal deformation of $(\mathbf{C}^2/\Gamma, o)$ is defined by a weighted homogeneous polynomial F of degree ν_r and weight $(w_1, w_2, w_3, \nu_1, \ldots, \nu_{r-1})$, where $\nu_1 \leqslant \cdots \leqslant \nu_r$ are the degrees of a minimal set of generating W-invariant polynomials.

The following result was conjectured by Grothendieck.

THEOREM. — *Let G be a simple complex Lie group of type A_r, D_r or E_r. Then a unipotent $x \in G$ is subregular if and only if there exists a factorization of map-germs*

$$\begin{array}{ccc} (G, x) & \xrightarrow{\ \pi\ } & (X, x) \\ \downarrow & \swarrow_{\varphi} & \\ (T/W, e) & & \end{array}$$

where π is regular and φ is the universal deformation of the corresponding Kleinian singularity $\mathbf{C}^2/\Gamma$.

The idea of the proof is very simple. It suffices to prove the corresponding statement for x a subregular nilpotent element in the Lie algebra. $\mathfrak{g} \to \mathfrak{t}/W$ given by a set $\varphi_1, \ldots, \varphi_r$ of G-invariant polynominals. X is a transversal subspace of dimension $r + 2$, intersecting the orbit of x in x. As pointed out by Varadarajan [15], it follows from the Jacobson-Morosov-Lemma, that $\varphi_i | X$ is weighted homogeneous of degree ν_i and weight $(w_1, w_2, w_3, \nu_1, \ldots, \nu_{r-1})$. From this one deduces $\mathrm{rank}_x(d\varphi_1, \ldots, d\varphi_{r-1}) = r - 1$ and subsequently $\varphi_r = F$.

COROLLARY. – The set of subregular $x \in G$ forms a nonsingular submanifold of codimension 3.

COROLLARY. – Any deformation of a Kleinian singularity admits a resolution.

This has independently been proved by Tjurina [14] and for A_r by Kas [5], using methods developed in [2].

The universal deformation is a nonsingular fibration over the set of regular semisimple classes. In order to analyze this fibration, one needs the fundamental group of its base space. The following result was conjectured by Tits and is proved in [3].

PROPOSITION. – Let H_{reg} be the space of regular elements in a complex Cartan algebra. $\pi_1(H_{\mathrm{reg}}/W)$ has a presentation with Δ as set of generators and with relations

$$\underbrace{\alpha\beta\alpha\ldots}_{m_{\alpha\beta}\text{ factors}} = \underbrace{\beta\alpha\beta\ldots}_{m_{\alpha\beta}\text{ factors}},$$

where $(m_{\alpha\beta})$ is the Coxetermatrix.

Applying this proposition for E_8 and Picard-Lefschetz-theory, one obtains :

COROLLARY. – $\{z \in \mathbf{C}^5 \mid \|z\| = 1,\ z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^5 = 0\}$ is an exotic 7-sphere representing Milnors standard generator of Θ_7.

Thus we see that there is a relation between exotic spheres, the icosahedron and E_8. But I still do not understand why the regular polyhedra come in. It is perhaps interesting to note that Klein in his lectures on the icosahedron emphasizes his indebtedness to Lie dating back to the years 1869-70, when Lie and Klein studied together at Berlin and Paris.

Klein writes : "At that time we jointly conceived the scheme of investigating geometric or analytic forms susceptible of transformation by means of groups of changes. This purpose has been of directing influence in our subsequent labours, though these may have appeared to lie far asunder. Whilst I primarily directed my attention to groups of discrete operations, and was thus led to the investigation of regular solids and their relations to the theory of equations, Professor Lie attacked the more recondite theory of continued groups of transformations, and therewith of differential equations".

Maybe the two theories do not lie so far asunder.

REFERENCES

[1] ARTIN M. — On isolated rational singularities of surfaces. *Amer. J. Math.*, 88, 1966, p. 129-136.

[2] BRIESKORN E. — Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, *Math. Ann.*, 178, 1968, p. 255-270.

[3] BRIESKORN E. — Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Inventiones math., 12, 1971, p. 57-61.
[4] DYNKIN E.B. — Semisimple subalgebras of semisimple Lie algebras, *Mat. Sbornik N.S.*, 30, 72, 1952, p. 349-462.
[5] KAS A. — On the resolution of certain holomorphic mappings, *Amer. J. Math.*, 90, 1968, p. 789-804.
[6] KLEIN F. — *Vorlesungen über das Ikosaeder und die Auflösung der Gleichung vom fünften Grade*, Leipzig 1884.
[7] KOSTANT B. — The principal three-dimensional subgroup and the Betti numbers of a complex simple group. *Amer. J. Math.*, 81, 1959, p. 973-1032.
[8] SCHWARZ H.A. — Uber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elements darstellt. *J. Reine Angew. Math.*, 75, 1872, p. 292-335.
[9] SPRINGER T.A. — Some arithmetical results on semisimple Lie algebras. *Publ. I.H.E.S.*, No. 30, 1966.
[10] SPRINGER T.A. — The unipotent variety of a semisimple group. *Proc. Bombay Colloq. on Algebraic Geometry*, 1968, p. 373-391.
[11] STEINBERG R. — Regular elements of semisimple algebraic groups. *Publ. I.H.E.S.*, No. 25, 1965.
[12] STEINBERG R. — Classes of elements of semisimple algebraic groups. *Proceedings of the International Congress of Mathematicians*, Moscow, 1966, p. 277-289.
[13] TJURINA G.N. — Flat locally semi-universal deformations of isolated singularities of complex spaces, *Izv. Akad. Nauk S.S.S.R. Ser. Mat.*, 33, 1969, p. 1026-1058.
[14] TJURINA G.N. — Resolution of singularities for flat deformations of rational double points, *Funkcional. Anal. i. Priložen.*, 4, 1970, p. 77-83.
[15] VARALDARAJAN V.S. — On the ring of invariant polynomicals on a semisimple Lie algebra, *Amer. J. Math.*, 90, 1968, p. 308-317.

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GROUPES ALGÉBRIQUES SEMI-SIMPLES SUR UN CORPS LOCAL

par François BRUHAT

Soit K un *corps local*, i.e. un corps *complet* pour une valuation *discrète* non impropre, à corps résiduel k *parfait* ; on sait que le groupe additif des entiers de K, ou le groupe multiplicatif des entiers inversibles, peut être muni d'une structure de *limite projective de groupes algébriques sur* k. La théorie que nous allons résumer ci-dessous fournit une construction analogue pour un groupe *semi-simple* sur K, le faisant ainsi apparaître comme un "objet algébrique de dimension infinie" sur k. D'autre part, notre théorie fournit des analogues (mais "de rang infini") des systèmes de racines, sous-groupes de Borel, sous-groupes paraboliques, systèmes de Tits (ou BN-paires) de la théorie maintenant classique des groupes réductifs sur un corps quelconque.

Nos recherches trouvent leur origine dans l'étude des sous-groupes compacts maximaux des groupes $\mathfrak{p}$-adiques. Un pas fondamental a été fait par Iwahori et Matsumoto lorsqu'ils ont démontré en 1964 l'existence d'un système de Tits à groupe de Weyl infini dans un groupe semi-simple déployé sur un corps local, résultat généralisé peu après à divers types de groupes par Hijikata. Les résultats qui suivent sont dus à J. Tits et à l'auteur.

1. – Soit $\mathfrak{G}$ un groupe algébrique réductif connexe défini sur un corps K (quelconque) ; soient $\mathfrak{S}$ un tore déployé maximal sur K de $\mathfrak{G}$, $\mathfrak{Z}$ le centralisateur et $\mathfrak{N}$ le normalisateur de $\mathfrak{S}$. On pose $G = \mathfrak{G}(K)$, $Z = \mathfrak{Z}(K)$, etc. La théorie classique permet d'introduire les objets suivants :

(1) un système de racines Φ dans le dual V^* d'un espace euclidien V ;

(2) un homomorphisme surjectif ν_0 de N sur le groupe de Weyl W_0 de Φ, de noyau Z ;

(3) pour chaque racine $a \in \Phi$, un sous-groupe unipotent $\mathfrak{U}_a$ défini sur K.

Ces objets possèdent les propriétés suivantes :

(4) $n U_a n^{-1} = U_{\nu_0(n)(a)}$ *(pour* $n \in N$ *et* $a \in \Phi$*)* ;

(5) *pour* $a, b \in \Phi$, *le groupe des commutateurs* (U_a, U_b) *est contenu dans le sous-groupe engendré par les* U_{pa+qb} *avec* p, q *entiers* > 0 *et* $pa + qb \in \Phi$;

(6) *si* $a, 2a \in \Phi$, *on a* $U_{2a} \subset U_a$;

(7) *pour* $a \in \Phi$, *posons* $M_a = \nu_0^{-1}(r_a)$, *où* r_a *est la réflexion par rapport à la racine a. Pour tout* $u \in U_a$, $u \neq 1$, *il existe un triple* $(u', m(u), u'') \in U_{-a} \times M_a \times U_{-a}$ *et un seul tel que* $u = u' m(u) u''$;

(8) *Soit* U^+ (resp. U^-) *le groupe engendré par les* U_a *pour* a *décrivant l'ensemble des racines positives* (*resp. négatives*) *pour un ordre total choisi sur* V^* ; *on a* $ZU^+ \cap U^- = \{1\}$;

(9) *le couple* (ZU^+, N) *est un système de Tits de groupe de Weyl* W_0 *dans* G ; *en particulier, l'application naturelle est une bijection de* W_0 *sur l'ensemble des doubles classes* $ZU^+\backslash G/ZU^+$.

Notons que $\mathfrak{Z}\mathfrak{U}^+$ est un sous-groupe parabolique défini sur K minimal.

D'une manière générale, nous appellerons *donnée radicielle* de type Φ dans un groupe G la donnée de sous-groupes N et U_a (pour $a \in \Phi$), engendrant G, et d'un homomorphisme surjectif $\nu_0 : N \to W_0$ de noyau noté Z, satisfaisant aux conditions (4) à (8) ci-dessus. Elles entraînent (9).

2. — Lorsque $\mathfrak{G}$ est déployé sur K, les sous-groupes U_a sont isomorphes au groupe additif de K et le choix d'une "base de Chevalley" dans l'algèbre de Lie de $\mathfrak{G}$ permet de choisir les isomorphismes $u_a : K \to U_a$ de manière "cohérente". Si K est muni d'une valuation ω non impropre, on peut alors transporter ω à chacun des U_a en posant $\varphi_a(u_a(t)) = \omega(t)$. La famille $\varphi = (\varphi_a)$ possède les propriétés suivantes :

(10) *pour tout* $k \in \mathbf{R}$, *l'image réciproque* $U_{a,k} = \varphi_a^{-1}([k, +\infty])$ *est un sous-groupe de* U_a *non réduit à* $\{1\}$ *et l'intersection des* $U_{a,k}$ *est égale à* $\{1\}$;

(11) *soient* $a, b \in \Phi$, *avec* $b \notin -\mathbf{R}_+ a$, *et soient* $h, k \in \mathbf{R}$; *le groupe des commutateurs* $(U_{a,h}, U_{b,k})$ *est contenu dans le groupe engendré par les* $U_{pa+qb,\, ph+qk}$ *pour* p, q *entiers* > 0 *et* $pa + qb \in \Phi$.

Pour énoncer commodément les autres propriétés, introduisons un langage géométrique. Pour $a \in \Phi$ et $k \in \mathbf{R}$, soit $\alpha_{a,k}$ le demi-espace fermé de V défini par l'inéquation $a(x) + k \geqslant 0$; pour $\alpha = \alpha_{a,k}$, on pose encore $U_\alpha = U_{a,k}$. Les $\alpha_{a,k}$ pour $a \in \Phi$ et $k \in \varphi_a(U_a) \cap \mathbf{R}$ seront appelés les *racines affines* de V et leurs bords les *murs* de V.

(12) *il existe un homomorphisme* ν *de* N *dans le groupe des automorphismes affines de* V *tel que* $\nu(N)$ *permute les racines affines et que* $nU_\alpha n^{-1} = U_{\nu(n)(\alpha)}$ *pour tout* $n \in N$ *et toute racine affine* α.

Posons $W = \nu(N)$: c'est une extension du groupe de Weyl W_0 par son sous-groupe des translations. On pose $H = \operatorname{Ker} \nu$.

(13) *soient* $a \in \Phi$ *et* $u \in U_a$, $u \neq 1$; *posons* $k = \varphi_a(u)$ *et écrivons* $u = u'm(u)u''$ *comme en* (7). *Alors* $\varphi_{-a}(u') = \varphi_{-a}(u'') = -k$ *et* $\nu(m(u))$ *est la réflexion orthogonale par rapport à l'hyperplan d'équation* $a(x) + k = 0$.

D'une manière générale, nous appellerons *valuation* d'une donnée radicielle $(N, (U_a))$ une famille φ de fonctions $\varphi_a : U_a \to \mathbf{R} \cup \{\infty\}$ satisfaisant aux conditions (10) à (13) et aussi à

(14) *si* $a, 2a \in \Phi$, φ_{2a} *est la restriction à* U_{2a} *de* $2\varphi_a$.

Pour simplifier l'exposé, nous supposerons désormais que Φ est *irréductible* et que G *est engendré par* H *et les* U_a (ce qui est le cas pour les groupes algébriques simples simplement connexes) ; on passe de là au cas général comme l'on passe, dans la théorie classique, du cas des groupes connexes au cas général.

Il est immédiat que, pour tout $x \in V$ et tout $\lambda \in \mathbf{R}_+^*$, la famille des $\psi_a = \lambda\varphi_a + a(x)$ est encore une valuation de la donnée radicielle, notée $\lambda\varphi + x$. On dit que φ et $\lambda\varphi + x$ sont *équivalentes.*

3. -- A la valuation φ est associée une *bornologie* sur G, à savoir la plus petite bornologie compatible avec la loi de groupe et pour laquelle sont bornés d'une part les $U_{a,k}$, d'autre part les parties M de N telles que $\nu(N)$ soit borné (c'est-à-dire relativement compact) dans le groupe des automorphismes affines de V. *Cette bornologie détermine la valuation à équivalence près.*

Revenons alors au cas des groupes algébriques simples sur un corps valué. La valuation de K détermine une bornologie naturelle sur G : une partie M de G est bornée si chaque fonction régulière reste bornée sur M. *Nous conjecturons que, si K est complet, il existe une valuation de la donnée radicielle de G (et une seule à équivalence près) dont la bornologie associée est la bornologie naturelle de G.* Sans pouvoir actuellement démontrer ce théorème dans toute sa généralité, nous savons le faire dans des cas fort larges : lorsque $\mathfrak{G}$ est *déployé* ou *quasi-déployé* sur K (et alors même si K n'est pas complet), lorsque G est un "*groupe classique*" (et même pour des groupes classiques non algébriques, associés à des corps gauches valués de rang infini sur leur centre), et, ce qui est le plus important pour les applications, lorsque K est un *corps local* au sens rappelé ci-dessus.

Notons que ce théorème entraîne que *$\mathfrak{G}$ est anisotrope sur K* (i.e. que $\Phi = \emptyset$) *si et seulement si le groupe G tout entier est borné.*

4. – Nous allons maintenant introduire des sous-groupes de G qui vont être l'analogue des sous-groupes paraboliques minimaux du cas classique. Soit $x \in V$ et soit D une chambre de Weyl de Φ dans V. Considérons le cône $x + D$ et soit $B = B_{x,D}$ le sous-groupe de G engendré par H et les U_α , où α décrit l'ensemble des racines affines contenant un voisinage de x dans $x + D$. Alors :

(15) *B est borné. Plus précisément, on a*

$$B = \prod_{a \in \Phi_+^{\text{red}}} U_{a,-a(x)} \times H \times \prod_{a \in \Phi_+^{\text{red}}} U_{-a,-a(x)+}$$

(où Φ_+^{red} désigne l'ensemble des racines indivisibles positives sur D et où $U_{a,k+}$ désigne la réunion des $U_{a,h}$ pour $h > k$).

(16) *On a une* "décomposition d'Iwasawa" $G = BNU^+ = BW_0ZU^+$. *Si de plus x est un point* spécial (i.e. *si* $-a(x) \in \varphi_a(U_a)$ *pour tout* $a \in \Phi^{\text{red}}$), *il existe un sous-groupe $P \supset B$ qui est un sous-groupe borné maximal tel que $G = PZU^+$.*

(Rappelons que ZU^+ est un sous-groupe parabolique minimal).

(17) *On a une* "décomposition de Bruhat" $G = BNB$; *plus précisément, l'application naturelle est une bijection de W sur $B\backslash G/B$.*

On peut compléter ce qui précède lorsque φ est *discrète* (i.e. $\varphi_a(U_a - \{1\})$ discret dans **R** pour tout a). Il existe alors un système de racines réduit Ψ dans V^*, de même groupe de Weyl W_0 que Φ (c'est-à-dire dont les directions des racines sont les mêmes que celles de Φ, mais Φ peut par exemple être de type B_n et Ψ de type C_n ou inversement) tel que W soit le *groupe de Weyl affine* de Ψ : quitte à remplacer φ par une valuation équivalente, les murs de V sont les hyperplans d'équation $b(x) + k = 0$ pour $b \in \Psi$ et $k \in \mathbf{Z}$. Les murs déterminent alors sur V une structure de *complexe simplicial*, les *chambres* de V, c'est-à-dire les simplexes de dimension maximale, étant ce que N. Bourbaki appelle les *alcôves* de Ψ. Le groupe $B_{x,D}$ ne dépend alors que de la chambre de V qui contient un voisinage de x dans $x + D$ et les divers groupes B sont tous conjugués (ce qui n'est pas nécessairement vrai lorsque φ n'est pas discrète). De plus, *le couple* (B, N) *est un système de Tits de groupe de Weyl* W, ce qui explique la décomposition $G = BNB$ (17).

5. – Pour $x \in V$, soit N_x le stabilisateur de x dans N et soit P_x le sous-groupe de G engendré par N_x et les U_α pour $x \in \alpha$ (lorsque x est un point spécial, c'est le sous-groupe P de (16)). Disons que deux points (g, x) et (h, y) de $G \times V$ sont équivalents s'il existe $n \in N$ tel que $y = \nu(n) . x$ et $g^{-1} hn \in P_x$: l'*immeuble* I de G est par définition le quotient de $G \times V$ par cette relation d'équivalence. Le groupe G opère sur I, l'espace affine V se plonge canoniquement dans I, l'action de G sur I prolonge celle de N sur V et le stabilisateur d'un point $x \in V$ n'est autre que P_x. Il existe sur I une *distance* et une seule invariante par G et induisant sur V la distance euclidienne. Lorsque φ est discrète, I est aussi muni d'une structure de complexe simplicial invariante par G et prolongeant celle de V.

L'espace métrique I ainsi défini joue alors un rôle analogue à celui de l'espace riemannien symétrique d'un groupe de Lie semi-simple réel. C'est un espace *contractile* ; deux points quelconques sont joints par une *géodésique unique* ; I est "*à courbure négative*" en ce sens que si $x, y, z \in I$ et si m est le milieu de la géodésique $[xy]$, on a

$$d(x, z)^2 + d(y, z)^2 \geqslant 2d(m, z)^2 + \frac{1}{2} d(x, y)^2$$

Mais I n'est pas toujours *complet*, sauf toutefois lorsque φ est discrète ou, dans le cas des groupes algébriques, lorsque le corps de base K est maximalement complet. Cependant, la propriété de courbure négative entraîne un "théorème de point fixe" : *le stabilisateur dans* G *d'une partie bornée de* I *possède au moins un point fixe dans le complété* $\hat{I}$ *de* I. Ceci permet de déterminer les *sous-groupes bornés maximaux* de G : lorsque φ est "dense", ce sont les stabilisateurs des points du complété $\hat{I}$; lorsque φ est discrète, ce sont les stabilisateurs des *sommets* du complexe simplicial I et ils se répartissent en $r + 1$ classes de conjugaison, où r est le rang de G, c'est-à-dire la dimension de V. Remarquons ici que si l'on ne fait pas les hypothèses simplificatrices de la fin du n° 2 et si l'on étudie le cas "non simplement connexe", la classification des sous-groupes bornés maximaux est un peu plus compliquée à décrire : ce sont les stabilisateurs des sommets et des centres de certaines facettes de I.

6. – Bornons-nous désormais au cas d'un groupe algébrique simple et simplement connexe sur un corps local. Les *sous-groupes parahoriques* de G sont par définition les sous-groupes contenant un conjugué de B et différents de G lui-même lorsque G n'est pas anisotrope sur K. A tout sous-groupe parahorique P de $G = \mathfrak{G}(K)$, est alors canoniquement associé un "groupe proalgébrique connexe $\mathfrak{P}$ défini sur le corps résiduel k" tel que $P = \mathfrak{P}(k)$: autrement dit, à P est associé un système projectif $(\mathfrak{P}_n)_{n \geqslant 0}$ de groupes algébriques connexes définis sur k, tel que P s'identifie à $\lim \mathfrak{P}_n(k)$. Les homomorphismes $\mathfrak{P}_m \to \mathfrak{P}_n$ sont surjectifs, de noyaux unipotents connexes, et $\mathfrak{P}_0$ est réductif. Les sous-groupes *parahoriques* contenus dans P sont les images réciproques des sous-groupes *paraboliques* de $\mathfrak{P}_0(k)$.

On a aussi des résultats relatifs à la "descente non-ramifiée" du corps de base analogues à ceux de la théorie classique : si $\widetilde{K}$ est une extension galoisienne non-ramifiée de K, un sous-groupe parahorique de $\mathfrak{G}(K)$ est le groupe des points rationnels sur K d'un sous-groupe parahorique $\widetilde{P}$ de $\mathfrak{G}(\widetilde{K})$ invariant par le groupe de Galois Γ de $\widetilde{K}$ sur K, et d'un seul, et ceci de manière cohérente avec les structures de groupe proalgébrique : l'action de Γ sur $\widetilde{P}$ définit sur chaque $\widetilde{\mathfrak{P}}_n$ une structure de groupe algébrique défini sur k et $\widetilde{\mathfrak{P}}_n$ s'identifie alors à $\mathfrak{P}_n$.

7. – Pour terminer, donnons une application : la démonstration de la nullité du H^1 d'un groupe simple simplement connexe sur un corps local dont le corps résiduel est de dimension cohomologique $\leqslant 1$, théorème dû à M. Kneser dans le cas des corps $\mathfrak{p}$-adiques. Pour cela, on peut supposer $\mathfrak{G}$ déployé sur K. Soit $\widetilde{K}$ l'extension non-ramifiée maximale et soit Γ le groupe de Galois de $\widetilde{K}$ sur K. On sait que $H^1(\mathfrak{G}) = H^1(\Gamma, \mathfrak{G}(\widetilde{K}))$. Soit donc $a = (a_\sigma)$ un 1-cocycle de Γ à valeurs dans $\mathfrak{G}(\widetilde{K})$ et soit $\mathfrak{G}_a$ la forme de $\mathfrak{G}$ sur K obtenue en tordant $\mathfrak{G}$ par a. Puisque les sous-groupes parahoriques de $\mathfrak{G}_a(K)$ correspondent aux sous-groupes parahoriques de $\mathfrak{G}_a(\widetilde{K}) = \mathfrak{G}(\widetilde{K})$ invariants par la nouvelle action de Γ, il existe au moins un tel sous-groupe parahorique Q de $\mathfrak{G}(\widetilde{K})$. Comme $\mathfrak{G}$ est déployé sur K, on voit facilement que Q est conjugué dans $\mathfrak{G}(\widetilde{K})$ d'un sous-groupe parahorique P invariant par l'ancienne action de Γ. On en déduit que a est cohomologue à un cocycle a' tel que $a'_\sigma P a'^{-1}_\sigma = P$ pour tout $\sigma \in \Gamma$. Mais P est son propre normalisateur et ceci entraîne $a'_\sigma \in P$. Ecrivons alors P comme limite projective des $\mathfrak{P}_n$: comme $\dim k \leqslant 1$, on a $H^1(\mathfrak{P}_n) = 0$, d'où l'on tire $H^1(\Gamma, P) = 0$ et a' est cohomologue à 0.

BIBLIOGRAPHIE

BRUHAT F. et TITS J. — Groupes algébriques simples sur un corps local, *Proc. Conf. Local Fields* (Driebergen 1966), Springer, Berlin, 1967, p. 23-36.

BRUHAT F. et TITS J. — *Groupes réductifs sur un corps local,* chapitre I, *Données radicielles valuées,* à paraître aux Publ. I.H.E.S.

IWAHORI N. et MATSUMOTO H. — *On some Bruhat decomposition and the structure of the Hecke ring of p-adic Chevalley groups, Publ.* I.H.E.S., 25, 1965, p. 5-48.

Université Paris 7ème
2, Place Jussieu,
Paris 5ème

Actes, Congrès intern. Math., 1970. Tome 2, p. 291 à 299.

GROUPES FORMELS, FONCTIONS AUTOMORPHES ET FONCTIONS ZETA DES COURBES ELLIPTIQUES

par P. CARTIER

A ANDRÉ WEIL et JEAN DIEUDONNÉ,
dont les travaux ont été notre source d'inspiration constante et féconde

1. Congruences pour les coefficients des fonctions automorphes.

Nous allons rappeler quelques-unes des remarquables congruences satisfaites par les coefficients des formes modulaires, et qui ont été découvertes par Ramanujan, Newman, Atkin, O'Brien et Swinnerton-Dyer (voir Atkin [1] pour les détails). Considérons d'abord la forme modulaire Δ de poids 12 (discriminant) :

$$\Delta(\tau) = e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})^{24} = \sum_{n=1}^{\infty} \tau(n) \,.\, e^{2\pi i n\tau} \tag{1}$$

Soit p un nombre premier ; les coefficients $\tau(n)$ de Δ satisfont à la relation de Ramanujan-Mordell(1) :

$$\tau(np) - \tau(p) \,.\, \tau(n) + p^{11} \,.\, \tau(n/p) = 0 \tag{2}$$

pour tout entier $n \geqslant 1$. Sous l'hypothèse $\tau(p) \not\equiv 0$ mod. p, on déduit de cette égalité des congruences comme suit : définissons par récurrence les nombres rationnels p-entiers B_α par $B_1 = \tau(p)$ et $B_{\alpha+1} = \tau(p) - p^{11}/B_\alpha$; on a alors

$$\tau(np^\alpha) \equiv B_\alpha \,.\, \tau(np^{\alpha-1}) \quad \text{mod. } p^{11\alpha} \tag{3}$$

pour $n \geqslant 1$ et $\alpha \geqslant 1$. On notera qu'il existe une unique unité p-adique B satisfaisant à l'équation $B^2 - \tau(p) \,.\, B + p^{11} = 0$ et qu'on a $B_\alpha \equiv B$ mod. $p^{11\alpha}$ pour tout $\alpha \geqslant 1$; on peut donc remplacer B_α par B dans (3), à condition de se placer dans le domaine des nombres p-adiques.

Considérons par ailleurs les coefficients $c(n)$ définis par $j(\tau) = \sum_{n=-1}^{\infty} c(n)\, e^{2\pi i n\tau}$, où j est l'invariant modulaire elliptique de poids 0 bien connu. En 1968, Atkin a obtenu le résultat suivant, qui généralise et résume une longue suite de résultats partiels : étant donné un entier $\alpha \geqslant 1$, on pose $t(n) = c(\ell^\alpha n)/c(\ell^\alpha)$; on a alors les relations

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(1) Nous faisons la convention que $\tau(a)$ est nul si a n'est pas entier ; on fera des conventions analogues pour $t(a)$ dans (4), pour $\beta(a)$ dans (9), etc . . .

$$(4) \qquad t(np) - t(p) . t(n) + p^{-1} . t(n/p) \equiv 0 \quad \text{mod. } \ell^{\alpha}$$

$$(5) \qquad t(n\ell) = t(n) . t(\ell)$$

($n \geqslant 1$, p premier $\neq \ell$) lorsque $\ell = 13$ et α quelconque ou lorsque $\ell = 17, 19, 23$ et α assez petit. Atkin a formulé une conjecture précise pour le cas des nombres premiers ℓ quelconques [1].

Le troisième exemple que nous considérerons se réfère à des formes modulaires de poids 2, c'est-à-dire à des formes différentielles de première espèce sur des courbes modulaires. D'une manière plus générale (cf. n° 5 pour le rapport entre ces deux points de vue), considérons une cubique plane C d'équation non homogène $Y^2 = X^3 - aX - b$ avec a et b entiers. Choisissons au voisinage du point à l'infini de C un paramètre local ξ tel que l'on ait $X = \xi^{-2} + \sum_{n=-1}^{\infty} \alpha(n) . \xi^n$ avec des coefficients $\alpha(n)$ entiers ; la forme différentielle de première espèce $\omega = - dX/2Y$ sur C se développe sous la forme $\omega = \sum_{n=1}^{\infty} \beta(n) . \xi^{n-1} d\xi$ avec des coefficients $\beta(n)$ entiers, et $\beta(1) = 1$. Soit p un nombre premier différent de 2 et 3 ; Atkin et Swinnerton-Dyer(1) ont établi les congruences suivantes :

$$(6) \qquad \beta(np) \equiv \beta(n) . \beta(p) \quad \text{mod. } p$$

$$(7) \qquad \beta(p) \equiv \sum_{t \text{ mod. } p} - \left(\frac{t^3\text{-}at\text{-}b}{p}\right) \text{ mod. } p \ ,$$

où $\left(\frac{a}{p}\right)$ est le symbole de Legendre. Supposons qu'on ait $\beta(p) \not\equiv 0$ mod. p, c'est-à-dire que la réduction de C modulo p soit d'invariant de Hasse-Witt non nul ; il existe alors une suite $(k_\alpha)_{\alpha \geqslant 1}$ de nombres entiers tels que

$$(8) \qquad \beta(np^\alpha) \equiv k_\alpha \beta(np^{\alpha-1}) \quad \text{mod. } p^\alpha \text{ pour tout } n \geqslant 1 \quad .$$

L'analogie avec la démonstration de (3) à partir de (2) a conduit Atkin et Swinnerton-Dyer à postuler une congruence de la forme

$$(9) \qquad \beta(np) - \beta(p) . \beta(n) + p . \beta(n/p) \equiv 0 \quad \text{mod. } p^\alpha$$

pour tout entier $n \equiv 0$ mod. $p^{\alpha-1}$, y compris lorsque $\beta(p) \equiv 0$ mod. p.

Il semble prématuré de faire des conjectures précises contenant tous ces cas particuliers (et d'autres analogues). Le schéma général semble être le suivant : on considère une certaine forme modulaire de poids $2g$, soit $h(\tau) = \sum_{n=1}^{\infty} r(n) . e^{2\pi i n \tau}$, avec des coefficients $r(n)$ entiers, normalisée par $r(1) = 1$; on est en droit d'attendre des congruences de la forme

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(1) A notre connaissance, les résultats d'Atkin et Swinnerton-Dyer n'ont pas encore été publiés et sont contenus dans la correspondance échangée entre ces auteurs et Serre. Nous remercions Serre qui, en nous communiquant cette correspondance et en nous obligeant à répondre à ses questions pertinentes, a été à l'origine des résultats exposés ici.

(10) $\quad r(np) - r(p) \, . \, r(n) + p^{2g-1} \, . \, r(n/p) \equiv 0 \quad \text{mod.}\, p^{(2g-1)\alpha}$

lorsque p est premier et $n \equiv 0$ mod. $p^{\alpha-1}$. Rappelons que la relation (2) de Ramanujan-Mordell signifie que Δ est fonction propre de l'opérateur de Hecke T_p. Par analogie, les résultats sur l'invariant modulaire elliptique j suggèrent la possibilité suivante : soit ℓ premier ; à l'aide des coefficients de Fourier de certaines formes modulaires de poids 0, on pourrait définir une "cohomologie étale ℓ-adique" qui serait un module libre H_ℓ de rang $[\ell/12]$ sur l'anneau $\mathbf{Z}_\ell$ des entiers ℓ-adiques et un opérateur de Hecke $T_{p,\ell}$ dans H_ℓ pour tout nombre premier $p \neq \ell$. Par contre, les congruences sur les courbes elliptiques suggèrent la possibilité dans certains cas de définir un opérateur de Hecke $T_{p,p}$ dans un module de cohomologie p-adique H'_p analogue à la cohomologie de Washnitzer-Monsky.

2. **Groupes *p*-adiques rigides.**

La suite de cet exposé est motivée par les congruences d'Atkin et Swinnerton-Dyer pour les différentielles de première espèce sur les courbes elliptiques. Le cadre naturel semble celui des groupes p-adiques rigides, dont nous empruntons la définition (en la simplifiant pour notre usage) à Tate [6]. Notons p un nombre premier, $\mathfrak{o}$ ou $\mathbf{Z}_p$ l'anneau des entiers p-adiques et K ou $\mathbf{Q}_p$ le corps des fractions de $\mathfrak{o}$. Pour tout entier $n \geqslant 0$, on note D^n l'ensemble des vecteurs à n composantes dans $\mathfrak{o}$ divisibles par p, et $\mathfrak{A}_n$ la $\mathfrak{o}$-algèbre des fonctions sur D^n de la forme

$$f(x) = \sum_{i_1, \ldots, i_n} a(i_1, \ldots, i_n) \, . \, x_1^{i_1} \ldots x_n^{i_n}$$

(les coefficients $a(i_1, \ldots, i_n)$ étant pris dans $\mathfrak{o}$). Une *variété rigide de dimension n* est un couple $(X, \mathfrak{A}(X))$ isomorphe à $(D^n, \mathfrak{A}_n)$; un système de coordonnées rigide sur X est une suite $(\xi_1, \ldots, \xi_n)$ d'éléments de $\mathfrak{A}(X)$ telle que l'application $x \mapsto (\xi_1(x), \ldots, \xi_n(x))$ soit un isomorphisme de X sur D^n. Une variété rigide X porte une structure de variété analytique sur le corps K pour laquelle tout système de coordonnées rigide est un système de coordonnées analytique ; les éléments de $\mathfrak{A}(X)$ sont *certaines* fonctions analytiques sur X, qualifiées de *rigides*(1). A partir des fonctions analytiques rigides sur X, on pourra définir les champs de vecteurs (ou les formes différentielles) rigides.

Les variétés rigides forment une catégorie avec produit, et l'on peut par suite définir la notion de groupe p-adique rigide. Deux exemples de tels groupes sont le groupe additif $\mathbf{G}_a$, ayant D^1 pour variété sous-jacente, et l'addition pour opération, et le groupe multiplicatif $\mathbf{G}_m$ qui se compose du groupe multiplicatif des $x \equiv 1$ mod. p dans $\mathfrak{o}$, avec la coordonnée rigide ξ donnée par $\xi(x) = x - 1$.

Dans la suite, nous désignerons par G un groupe p-adique rigide de dimension 1 (nécessairement commutatif) ; les formes différentielles rigides de degré 1 sur G

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(1) Rappelons qu'une fonction qui est localement égale à une fonction analytique est analytique. Par contre, une fonction qui appartient localement à $\mathfrak{A}(X)$ n'appartient pas nécessairement à $\mathfrak{A}(X)$, d'où la terminologie : "rigide".

invariantes par translation forment un $\mathfrak{o}$-module libre de rang 1, dont nous choisirons une base ω_0. Alors ω_0 est la différentielle $d\mathfrak{L}$ d'une fonction analytique $\mathfrak{L}$ sur G, appelée le *logarithme de G*. Ce logarithme est un isomorphisme de groupes de Lie p-adiques de G sur G_a, mais n'est pas en général une fonction analytique rigide. Pour préciser ce point, introduisons les opérateurs de Lazard $\Psi_n (n \geqslant 1)$ dans $\mathfrak{A}(G)$ par

$$(11) \qquad \Psi_n f(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x^i) \quad ;$$

si ξ est une coordonnée rigide dans G, normalisée par $\omega_0 = d\xi$ à l'origine, on a

$$(12) \qquad \mathfrak{L}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \Psi_n \, \xi(x)/n .$$

Cette formule de Lazard permet le contrôle des dénominateurs dans $\mathfrak{L}$; lorsque $G = \mathbf{G}_m$, $\xi(x) = x - 1$ et $\omega_0 = dx/x$, on a $\Psi_n \xi = \xi^n$ et (12) redonne le développement en série classique du logarithme usuel.

Le lien avec les groupes formels est le suivant. Choisissons une coordonnée rigide ξ sur G ; il existe alors une série formelle $F \in \mathfrak{o}[[X, X']]$ caractérisée par $\xi(xx') = F(\xi(x) ; \xi(x'))$ pour x, x' dans G ("Théorème d'addition"). Cette série satisfait aux identités

$$(13) \qquad F(X;0) = F(0;X) = X \quad , \quad F(X;Y) = F(Y;X) \quad ,$$
$$F(F(X;Y);Z) = F(X;F(Y;Z)) \quad ;$$

autrement dit, c'est une loi de groupe formel commutatif à coefficients dans $\mathfrak{o}$.

3. **Classification des groupes *p*-adiques rigides.**

Le théorème de classification repose sur deux notions essentielles : la hauteur et le module différentiel. Soit G un groupe p-adique rigide de dimension 1. L'anneau $\mathfrak{A} = \mathfrak{A}(G)$ est local, et son idéal maximal $\mathfrak{m}$ se compose des fonctions analytiques rigides dont les valeurs sont divisibles par p en tout point de G. La *hauteur* de G est la borne supérieure (finie ou non) $ht(G)$ des entiers $h \geqslant 1$ tels que

$$\Psi_p(\mathfrak{A}) \subset p \, . \, \mathfrak{A} + \mathfrak{m}^{p^h} .$$

On a $ht(\mathbf{G}_m) = 1$ et $ht(\mathbf{G}_a) = \infty$; la formule (12) montre facilement que tout groupe de hauteur infinie est isomorphe, comme groupe p-adique rigide, à $\mathbf{G}_a$.

Une *courbe* dans G est un morphisme de variétés rigides $\gamma : D^1 \to G$, normalisé par $\gamma(0) = e$ (élément neutre de G). Les courbes forment un groupe commutatif $C(G)$ pour l'addition définie par $(\gamma + \gamma')(t) = \gamma(t) \, . \, \gamma'(t)$. Pour tout nombre premier ℓ, l'opérateur de décalage dans $C(G)$ est défini par $V_\ell \gamma(t) = \gamma(t^\ell)$, et l'opérateur de Frobenius par $F_\ell \gamma(t) = \prod_{i=1}^{\ell} \gamma(\zeta^i t^{1/\ell})$. Dans cette dernière formule, ζ est une racine ℓ-ième de l'unité, distincte de 1, que l'on adjoint à $\mathfrak{o}$ ainsi que la racine $t^{1/\ell}$ de t, mais le résultat de la multiplication se trouve définit sur 0.

Notons maintenant t la coordonnée naturelle sur D^1 et Ω le $\mathfrak{o}$-module des formes différentielles rigides sur D^1 ; nous représenterons toujours celles-ci sous la forme

$$(14) \qquad \omega = \sum_{n=1}^{\infty} a(n) \,.\, t^{n-1} dt \qquad (a(n) \in \mathfrak{o} \text{ pour tout } n \geqslant 1) \quad ;$$

enfin, soit $d\mathfrak{A}_1$ l'ensemble des différentielles des fonctions $f \in \mathfrak{A}_1$. L'application $\gamma \mapsto \gamma^*(\omega_0)$ définit un isomorphisme u du groupe $C(G)$ des courbes de G sur un sous-groupe $\mathfrak{D}(G)$ de Ω. On dit que $\mathfrak{D}(G)$ est le *module différentiel* de G ; il caractérise G à un isomorphisme rigide près. De plus, u transforme V_ℓ et F_ℓ en les opérateurs suivants sur $\mathfrak{D}(G)$:

$$(15) \qquad V_\ell\, \omega = \sum_{n=1}^{\infty} \ell \,.\, a(n/\ell) \,.\, t^{n-1} dt \quad , \quad F_\ell\, \omega = \sum_{n=1}^{\infty} a(n\ell) \,.\, t^{n-1} dt$$

(pour ω de la forme (13)).

Soit F la loi de groupe formel définie à la fin du n° 2, et soit $F_{(p)}$ la loi de groupe formel à coefficients dans le corps $\mathbf{F}_p = \mathfrak{o}/p.\mathfrak{o}$ déduite de F par réduction modulo p. Sa hauteur au sens de Lazard et Dieudonné est égale à la hauteur h de G ; *nous la supposons désormais finie*(1). Le module de Dieudonné de $F_{(p)}$ est un $\mathfrak{o}$-module libre $\mathfrak{D}_p(G)$ de rang h muni d'un opérateur linéaire V, donc un module sur l'anneau de polynômes $\mathfrak{o}[V]$. On démontre qu'il existe un unique polynôme d'Eisenstein $P = V^h + b_1 V^{h-1} + \cdots + b_{h-1} V + b_h$ dans $\mathfrak{o}[V]$ tel que $\mathfrak{D}_p(G)$ soit isomorphe au $\mathfrak{o}[V]$-module $\mathfrak{o}[V]/(P)$. De plus, la théorie résumée dans [2] permet d'identifier $\mathfrak{D}_p(G)$ au quotient de $\mathfrak{D}(G)$ par le sous-groupe formé des différentielles de la forme $p \,.\, df + \Sigma_\ell V_\ell\, \omega_\ell$ avec $f \in \mathfrak{A}_1$ et $\omega_\ell \in \mathfrak{D}(G)$ pour tout nombre premier ℓ, et V provient de V_p par passage au quotient.

Le polynôme d'Eisenstein P, ou ce qui revient au même, les coefficients $b_1, \ldots, b_h$ déterminent entièrement le module différentiel $\mathfrak{D}(G)$ qui se compose des formes différentielles ω telles que

$$(16) \qquad V_p^h \omega + b_1 \,.\, V_p^{h-1} \omega + \cdots + b_{h-1} \,.\, V_p\, \omega + b_h \,.\, \omega \equiv 0 \quad \text{mod. } p \,.\, d\mathfrak{A}_1 \,.$$

De manière plus explicite, soient $a(1), a(2), \ldots, a(n), \ldots$ des éléments de $\mathfrak{o}$; posons

$$(17) \qquad t(n) = a(n) + \frac{p b_{h-1}}{b_h} \,.\, a(n/p) + \cdots + \frac{p^{h-1} b_1}{b_h} \,.\, a(n/p^{h-1}) + \frac{p^h}{b_h} \,.\, a(n/p^h) \,.$$

La forme différentielle $\omega = \sum_{n=1}^{\infty} a(n) \,.\, t^{n-1} dt$ *appartient à* $\mathfrak{D}(G)$ *si et seulement si l'on a les congruences* $t(n) \equiv 0$ mod. p^α *pour tout* $\alpha \geqslant 1$ *et tout entier* $n \equiv 0$ mod. p^α. De plus, tout polynôme d'Eisenstein de degré h provient d'un groupe p-adique rigide de dimension 1 et de hauteur h.

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(1) Lorsque G est de hauteur infinie, il est isomorphe (de manière rigide) à $\mathbf{G}_a$, et l'on a $\mathfrak{D}(G) = d\mathfrak{A}_1$.

En résumé, *on peut répartir les groupes p-adiques rigides de hauteur h en familles non vides* $\mathrm{F}(b_1, \dots, b_h)$ (avec $b_1, \dots, b_h$ dans $p.\mathfrak{o}$ et b_h non divisible par p^2). *Supposons que G soit de type* $\mathrm{F}(b_1, \dots, b_h)$ *et soient* ξ *une coordonnée rigide dans* G, $\omega = \sum_{n=1}^{\infty} a(n) . \xi^{n-1} d\xi$ *une forme différentielle rigide invariante par translations sur G. Alors les coefficients* $a(n) \in \mathfrak{o}$ *satisfont aux congruences* $t(np^\alpha) \equiv 0$ mod. p^α *pour* $\alpha \geqslant 1$ et $n \geqslant 1$, *en définissant* $t(n)$ *comme plus haut*(1).

4. Courbes elliptiques.

On note $\mathbf{Z}$ l'anneau des entiers rationnels, $\mathbf{Q}$ le corps des nombres rationnels et $\mathbf{F}_p$ le corps fini à p éléments. Soit $H \in \mathbf{Z}[X, Y, Z]$ un polynôme non nul, homogène de degré 3, irréductible et de discriminant non nul. On suppose que la courbe elliptique d'équation homogène $H = 0$ a un point d'inflexion à coordonnées rationnelles. Quitte à faire un changement linéaire de variables à coefficients entiers, on peut ramener H à la forme

$$(18) \quad H(X, Y, Z) = Y^2 Z + (aX + bZ) YZ + (X^3 + uX^2 Z + vXZ^2 + wZ^3)$$

et supposer que la réduction $H_{(p)}$ de H modulo p est irréductible dans $\mathbf{F}_p[X, Y, Z]$ pour tout nombre premier p. Soit Γ le schéma projectif sur $\mathbf{Z}$ associé à l'algèbre graduée $\mathbf{Z}[X, Y, Z]/(H)$; on pose $C = \mathbf{Q} \otimes_{\mathbf{Z}} \Gamma$ et $C_{(p)} = F_p \otimes_{\mathbf{Z}} \Gamma$, de sorte que C est la courbe elliptique sur $\mathbf{Q}$ d'équation $H = 0$, et que $C_{(p)}$ est la réduction modulo p de C, d'équation $H_{(p)} = 0$. On dit que Γ est le *modèle de Néron* de C (cf. [4]). On considère C (resp. $C_{(p)}$) comme un groupe algébrique sur $\mathbf{Q}$ (resp. $\mathbf{F}_p$), d'élément neutre le point à l'infini e (resp. e_p).

Soit p un nombre premier. Nous associons comme suit un groupe p-adique rigide G_p à Γ : les points de G_p sont les points de Γ dans $\mathbf{Z}_p$ qui se réduisent modulo p en e_p, et les fonctions analytiques rigides sur G_p sont les éléments du complété de l'anneau local du schéma Γ au point $e_p \in \Gamma(\mathbf{F}_p)$. De manière plus concrète, G_p se compose des points $g = (x, y, z)$ de C dans $\mathbf{Q}_p$ tels que $x/py \in \mathbf{Z}_p$, et l'on définit une coordonnée rigide ξ par $\xi(g) = x/y$. On note $\omega = \sum_{n=1}^{\infty} \beta(n) . \xi^{n-1} d\xi$ la forme différentielle de première espèce sur C normalisée par $\beta(1) = 1$; c'est une forme différentielle rigide invariante par translations sur G_p.

Supposons d'abord que $C_{(p)}$ soit une courbe elliptique sur $\mathbf{F}_p$, ce qui exclut un nombre fini de valeurs de p. Le nombre des points rationnels de $C_{(p)}$ est de la forme $1 - f_p + p$ avec $|f_p| < 2p^{1/2}$ (inégalité de Hasse-Weil). De plus, la réduction modulo p de ω est une forme de première espèce sur $C_{(p)}$ et "l'opération de Cartier" la multiplie par f_p ; comme cette opération transforme $h^{p^i-1}dh$ en $h^{p^{i-1}-1}dh$, on en déduit les

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(1) En particulier, le groupe p-adique rigide G est défini à isomorphisme près par sa réduction modulo p, qui est un groupe formel sur $\mathrm{F}_p = \mathfrak{o}/p.\mathfrak{o}$, et il n'y a donc pas de "modules". Cette situation est particulière au cas envisagé $\mathfrak{o} = \mathbf{Z}_p$ (cf. [2]).

congruences $\beta(np) \equiv f_p \,.\, \beta(n)$ et en particulier $\beta(p) \equiv f_p$ mod. p. Lorsque H est de la forme $Y^2Z - (X^3 - aXZ^2 - bZ^3)$, on a $f_p = \sum_{t \bmod . p} -\left(\frac{t^3-at-b}{p}\right)$ et l'on retrouve ainsi les congruences (6) et (7) du n° 1 (cette démonstration est due à Serre). Enfin, f_p détermine la structure du groupe p-adique rigide G_p comme suit(1) :

(a) si $f_p \neq 0$, le groupe G_p est de hauteur 1, associé au polynôme d'Eisenstein $V - pu^{-1}$ où l'unité p-adique u satisfait à $u^2 - f_p u + p = 0$;

(b) si $f_p = 0$, le groupe G_p est de hauteur 2, associé au polynôme d'Eisenstein $V^2 + p$.

La congruence (9) du n° 1 se déduit immédiatement de là et des résultats du n° 3.

La *fonction zéta* de la courbe elliptique C a été définie par A. Weil comme le produit eulérien $\zeta_C(s) = \prod_p \zeta_p(s)$; lorsque $C_{(p)}$ est une courbe elliptique, on a $\zeta_p(s) = (1 - f_p p^{-s} + p^{1-2s})^{-1}$, et l'on a une recette bien définie [7] lorsque p est un nombre premier exceptionnel pour C. On peut aussi définir le *schéma formel* $\hat{\Gamma}$ *complété de* Γ le long de la section neutre ; c'est un groupe formel sur **Z**. Le choix du paramètre local ξ permet de représenter $\hat{\Gamma}$ par une loi de groupe formel F à coefficients dans **Z**, telle que $\xi(xx') = F(\xi(x) ; \xi(x'))$ pour tout nombre premier p et x, x' dans G_p.

Un de nos résultats fondamentaux (démontré aussi partiellement par Honda [3]) est le suivant : *il existe un paramètre local bien déterminé t dans Γ au voisinage de la section neutre tel que la forme différentielle de première espèce ω s'écrive*

$$\omega = \sum_{n=1}^{\infty} b(n) \,.\, t^{n-1}\, dt$$

et que la fonction zéta de C s'écrive $\zeta_C(s) = \sum_{n=1}^{\infty} b(n) \,.\, n^{-s}$ avec les mêmes coefficients entiers $b(n)$. Le choix usuel des facteurs exceptionnels de ζ_C est le seul pour lequel ce résultat soit vrai, et l'on peut donc dire que la fonction zéta de C ne dépend que du groupe formel associé à C.

5. Relation avec les fonctions automorphes.

Les résultats précédents nous semblent jeter une lumière supplémentaire sur les conjectures de Weil [7], [8] (mais non sur leur démonstration !). Notons **C** l'ensemble des nombres complexes, **P** le demi-plan de Poincaré et, pour tout entier $N > 0$, soit $\Gamma_0(N)$ le groupe des transformations conformes de **P** de la forme $z \mapsto \frac{az+b}{cz+d}$ avec

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(1) Lorsque $C_{(p)}$ n'est pas une courbe elliptique, elle est isomorphe comme groupe algébrique sur $\mathbf{F}_p$, soit à $\mathbf{G}_a$, soit à $\mathbf{G}_m$, soit à la forme non-déployée de $\mathbf{G}_m$ qui se déploye sur l'extension quadratique de $\mathbf{F}_p$.

a , b , c , d entiers, $ad - bc = 1$ et $c \equiv 0$ mod. N. Les coefficients entiers $b(n)$ étant définis comme précédemment, on note φ la forme différentielle holomorphe $\sum_{n=1}^{\infty} b(n) . e^{2\pi i n\tau} d\tau$ sur **P**. Enfin, soit N le *conducteur* de C ; c'est un entier > 0 dont les diviseurs premiers sont les nombres premiers exceptionnels pour C. La conjecture de Weil est que φ *est toujours invariante par* $\Gamma_0(N)$.

Soient D le disque unité ouvert dans **C**, et C_c le tore complexe de dimension 1 formé des points complexes de C. Le groupe commutatif Λ formé des applications holomorphes γ de D dans C_c telles que $\gamma(0) = e$ est l'analogue du groupe $C(G)$ défini au n° 3. On définit pour chaque nombre premier p des opérateurs V_p et F_p par

$$(19) \qquad V_p \gamma(q) = \gamma(q^p) \quad , \quad F_p \gamma(q^p) = \sum_{j=1}^{p} \gamma(\zeta^j q)$$

(avec $\zeta^p = 1$, $\zeta \neq 1$) ; l'opérateur de Hecke associé à p est $T_p = V_p + F_p$. Le paramètre local t auquel il est fait allusion à la fin du n° 4 définit en fait une coordonnée locale holomorphe au voisinage de e dans C_c et il existe un élément δ de Λ caractérisé par $t(\delta(q)) = q$ pour q assez petit dans D.

Posons $H(\tau) = \delta(e^{2\pi i\tau})$; alors H est une application holomorphe de **P** dans C_c, caractérisée par la propriété suivante : *l'image réciproque par H de la forme de première espèce ω sur C_c est la forme différentielle holomorphe φ sur* **P**. Soit p un nombre premier tel que $C_{(p)}$ soit une courbe elliptique ; on peut montrer qu'on a $T_p \delta = f_p \delta$, c'est-à-dire la relation

$$(20) \qquad H(p\tau) + \sum_{j \bmod. p} H\left(\frac{\tau + j}{p}\right) = f_p . H(\tau) \quad (\tau \text{ dans } \mathbf{P}) \quad .$$

La conjecture de Weil signifie que H se factorise en $\mathbf{P} \to \mathbf{P}/\Gamma_0(N) \xrightarrow{H'} C_c$. De plus, par adjonction à $\mathbf{P}/\Gamma_0(N)$ des points à l'infini correspondant aux "pointes", on obtient une courbe algébrique complète S_N sur **C**. Or Shimura a construit dans [5] un modèle de S_N sur le corps **Q** des nombres rationnels, et l'on peut raffiner sa méthode(1) de manière à obtenir un schéma Σ_N sur **Z** tel que $S_N = \mathbf{C} \otimes_{\mathbf{Z}} \Sigma_N$. Nos résultats entraînent que, *si C satisfait à la conjecture de Weil, H' est un morphisme de schémas de Σ_N dans* Γ *au-dessus de Spec* (**Z**).

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(1) Pour tout nombre premier p, l'anneau local de Σ_N au point de Σ_N générique au-dessus de p se compose des fonctions méromorphes sur **P**, invariantes par $\Gamma_0(N)$ et qui se développent en série de Fourier $\sum_{n=-\infty}^{\infty} c_n . e^{2\pi i n\tau}$ avec des coefficients p-entiers c_n.

BIBLIOGRAPHIE

[1] ATKIN A.O.L. — Congruences for modular forms, in *Computers in Mathematical Research,* North Holland Publ., 1968.

[2] CARTIER P. — Relèvement des groupes formels commutatifs, *Séminaire Bourbaki,* 21e année, 1968-69, exposé n° 359.

[3] HONDA T. — Formal groups and zeta functions, *Osaka J. Math.*, 5, 1968, p. 199-213.

[4] NÉRON A. — Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, *Publ. Math. I.H.E.S.*, 21, 1964, p. 1-128.

[5] SHIMURA G. — Correspondances modulaires et les fonctions ζ des courbes algébriques, *Journ. Math. Soc. Japan,* 10, 1958, p. 1-28.

[6] TATE J. — Rigid analytic spaces, à paraître dans *Inventiones Mathematicae.*

[7] WEIL A. — Ueber die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Annalen,* 168, 1967, p. 149-156.

[8] WEIL A. — Dirichlet series and automorphic functions, Lecture Notes in Math. *189,* Springer 1971.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 301 à 306.

BOUNDARIES OF LIE GROUPS AND DISCRETE SUBGROUPS

by Harry FURSTENBERG

1. Introduction.

It G is a non-compact semi-simple Lie group, there is a compact homogeneous space $B(G)$ attached to it which plays an important role in the theory of harmonic functions on the symmetric space associated with the group G. $B(G)$ is a boundary component of one of the Satake compactifications of the symmetric space ([8]), but it can be characterized directly in terms of its behavior as a G-space (see § 3). In [4] the space $B(G)$ is shown to play an important role in the theory of spherical functions on G which means that it also has significance for the theory of irreducible unitary representations of G. More recently this space has appeared as a tool in proving "rigidity" theorems. If G is a locally compact topological group, a subgroup Γ is called a *lattice subgroup* if Γ is discrete and G/Γ has finite left-invariant (Haar) measure. Suppose Γ_1 is a lattice in G_1 and Γ_2 is a lattice in G_2, where G_1 and G_2 are semi-simple Lie groups. One wants to know to what extent an assertion of the following type is valid : an isomorphism of Γ_1 with Γ_2 is induced by an isomorphism of G_1 with G_2. The case where $G_1 = G_2$ and Γ_2 is obtained from Γ_1 by a continuous deformation had been treated in work by Calabi, Vesentini, Weil, Garland and Ragunathan. In some recent work the space $B(G)$ has played a major role in the arguments. For example, in [5] we considered the question of whether $\Gamma_1 \cong \Gamma_2$ implied $G_1 \cong G_2$, and we treated a fairly special case making use of the notion of Poisson boundary which is closely related to that of the space $B(G)$. Mostow has given a rather conclusive treatment for the case that the Γ_i are uniform (co-compact) subgroups showing that, in general, an isomorphism of Γ_1 with Γ_2 induces a homeomorphism of $B(G_1)$ with $B(G_2)$, and this in turn implies the isomorphism of G_1 with G_2 ([7]).

The latter results suggest the possibility that when Γ is a lattice subgroup of the semi-simple group G, the space $B(G)$ may be attached directly to the group Γ as an abstract group. More precisely, one might expect to be able to define a functor Π on the category of locally compact groups to the category of compact spaces satisfying the following conditions ;

(i) $\Pi(G)$ is a G-space.

(ii) If $h : G_1 \to G_2$ is a epimorphism so that every G_2-space can be viewed as a G_1-space, then there exists a G_1-equivariant map $h^* : \Pi(G_2) \to \Pi(G_2)$.

(iii) If Γ is a lattice subgroup of G, then there exists a Γ-equivariant map $\Pi(\Gamma) \to \Pi(G)$ which is an isomorphism.

(iv) If G is a semi-simple Lie group, then $\Pi(G) = B(G)$.

Without further restriction one cannot expect to attain all of these conditions. One can see this by considering the case of the free group F_n on n generators which is a lattice subgroup of $SL(2, R)$. Since $B(SL(2, R))$ is P^1, the one-dimensional projective space, and since free groups map readily into any group one would find an abundance of maps of P^1 into every $B(G)$. However there are even automorphisms of F_n which are not compatible with continuous maps of P^1 onto itself.

In what follows we shall discuss several candidates for the functor Π and show to what extent the conditions above are met. While in none of these do we achieve the identity of $\Pi(\Gamma)$ with $\Pi(G)$ for Γ a lattice subgroup, we nonetheless find in one case that these two spaces are sufficiently close to have some implications for rigidity type theorems. In particular we obtain in this way an alternative proof of the result announced in [5] to the effect that it G_1 has R-rank 1 and G_2 is one of the groups $SL(m, R), m \geqslant 3$, then G_1 and G_2 have no isomorphic lattice subgroups. The same method also seems to show that for $m \neq n$ $SL(m, R)$ and $SL(n, R)$ cannot have isomorphic lattice subgroups. Because of the more precise results of Mostow we haven't pursued this matter to its conclusion. Our interest in the functor Π stems from our expectation that the spaces $\Pi(G)$ will play a role in other problems. Detailed proofs will appear else where.

2. Proximal Minimal G-spaces.

This notion is borrowed from topological dynamics ([2]). Let M be a compact G-space. We say M is *minimal* if M does not contain a non-trivial, closed, G-invariant subset. Equivalently, M is minimal it every G-orbit in M is dense. These spaces are plentiful since every compact G-space must contain a minimal G-space. M is called *proximal* if for every pair $x, y \in M$ there exists a net $\{g_\alpha\}$ in G with

$$\lim g_\alpha x = \lim g_\alpha y \quad .$$

When G is abelian, or more generally, when G is nilpotent, every proximal G-space contains a fixed point for the group, so that the only proximal, minimal G-space is the trivial space. However, when G is semi-simple there exist interesting proximal minimal spaces. Namely one has

THEOREM 2.1. — *If G is semi-simple then $B(G)$ is a proximal G-space.*

Since G is transitive on $B(G)$, the latter is obviously a minimal G-space.

Now products of proximal spaces are proximal, and if M_1 and M_1 are proximal minimal, then a minimal subspace of $M_1 \times M_2$ will be both proximal and minimal and will have M_1 and M_2 as equivariant images. In this way one may prove

THEOREM 2.2. — *For an arbitrary group G, there exists a universal proximal minimal G-space $\Pi_p(G)$ such that if M is any proximal minimal G-space, then exists an equivariant map $\rho : \Pi_p(G) \to M$.*

Note that ρ is onto inasmuch as M is minimal. Moreover, ρ is unique. For suppose that ρ and σ were two such maps. Let $x \in \Pi_p(G)$ and choose a net with $\lim g_\alpha \rho(x) = \lim g_\alpha \sigma(x)$. If y is a limit of a subnet of $g_\alpha x$, then $\rho(y) = \sigma(y)$, whence $\rho(gy) = \sigma(gy)$, and since the orbit of y is dense, $\rho = \sigma$. This implies in particular that $\Pi_p(G)$ is unique.

It is easily shown that conditions (i) and (ii) are met for Π_p. The following condition related to (iii) can also be established :

THEOREM 2.3. — *If Γ_1 is a subgroup of finite index in Γ_2, then $\Pi_p(\Gamma_1) \cong \Pi_p(\Gamma_2)$*

It is an open question whether (iv) is valid in this case. (iv) would be valid if it were true that $\Pi_p(S)$ is trivial for all solvable S. This is also open. In any case, if (iv) is true so that $\Pi_p(S)$ is a manifold when G is semi-simple, then (iii) will certanly not be true since one can prove that for Γ a lattice subgroup of a semi-simple Lie group, $\Pi_p(\Gamma)$ will not be a manifold.

3. Strong proximality.

We will denote by $\mathfrak{P}(M)$ the space of regular probability measures on the compact space M. If M is a G-space, then $\mathfrak{P}(M)$ is a G-space. Moreover we endow $\mathfrak{P}(M)$ with the usual weak topology so that it becomes a compact convex set. The extremals of this set are the point measures and these are in correspondance with the points of M. Now suppose that M is a proximal G-space. One sees easily that for any finite set of points $x_1, x_2, \ldots, x_n \in M$, there exists a net g_α in G with $\lim g_\alpha x_1 = \lim g_\alpha x_2 = \cdots = \lim g_\alpha x_n$. From this it follows that if ν is any discrete measure in $\mathfrak{P}(M)$ we can find a net with $g_\alpha \nu \to$ point measure. We shall say that a G-space M is *strongly proximal* if this holds for an arbitrary $\nu \in \mathfrak{P}(M)$. Again one shows :

THEOREM 3.1. — *For an arbitrary group G there exists a universal strongly proximal minimal G-space $\Pi_{SP}(G)$ with the property that it M is any strongly proximal minimal G-space there exists a unique equivariant map ρ of $\Pi_{SP}(G)$ onto M.*

Moreover one has

THEOREM 3.2. — *If G is a connected Lie group and R its (not necessarily connected) radical, then $\Pi_{SP}(G) = B(G/R)$.*

This is an easy consequence of [3].

We will see in the next section that if Γ is a lattice subgroup of the semi-simple group G, then the space $\Pi_{SP}(G)$ is a strongly proximal minimal Γ-space. Hence there is a map of $\Pi_{SP}(\Gamma)$ onto $\Pi_{SP}(G)$. However, in general it will not be one-one. For we can construct a strongly proximal Γ-space M which is a non-trivial extension of $\Pi_{SP}(G)$. Namely we form a Γ-space M which is minimal and of which $\Pi_{SP}(G)$ is an equivariant image such that for some point $x \in \Pi_{SP}(G)$, the inverse image in M consists of a single point. It is easy to see that this implies that M is strongly proximal.

As with Π_p one has

THEOREM 3.3. — *If Γ_1 is of finite index in Γ_2 then*

$$\Pi_{SP}(\Gamma_1) \cong \Pi_{SP}(\Gamma_2) \quad ,$$

Finally we mention another characterization of the space $\Pi_{SP}(G)$.

THEOREM 3.4. — *If G acts by affine transformations on a compact convex set Q leaving no proper compact convex subset invariant, then there exists a unique affine equivariant map of* $\mathcal{P}(\Pi_{SP}(G))$ *onto Q.*

4. Mean Proximal G-spaces.

We now introduce a further strengthening of proximality which enables us to say something more about the relation between $\Pi(\Gamma)$ and $\Pi(G)$. Let M be a G-space and let μ be a probability measure on G. Form the sequence of measures

$$\mu_n = \frac{\mu + \mu * \mu + \cdots + \overbrace{\mu * \cdots * \mu}^{n}}{n} .$$

We shall say that M is *μ-proximal* if for every neighborhood $\mathcal{U}$ of the diagonal $\Delta(M) \subset M \times M$ we have

$$\mu_n \{g | (gx , gy) \notin \mathcal{U}\} \to 0$$

as $n \to \infty$ for each $x , y \in M$. Finally we say that M is *mean proximal* if it is μ-proximal for any μ whose support generates G.

Once again we have

THEOREM 4.1. — *There exists a universal mean proximal minimal G-space* $\Pi_{MP}(G)$ *such that if M is any mean proximal minimal G-space then there exists a unique equivariant map of* $\Pi_{MP}(G)$ *onto M.*

The following theorem implies that mean proximality is in fact stronger than strong proximality.

THEOREM 4.2. — *The following conditions on a metric G-space M are equivalent :*

(a) *M is μ-proximal.*

(b) *A measure ν on $M \times M$ satisfying $\mu * \nu = \nu$ is supported by the diagonal.*

(c) *If $X_1 , X_2 , \ldots , X_n , \ldots$ is a sequence of G-valued independent random variables each having distribution μ, and if ν is a measure on M with $\mu * \nu = \nu$, then with probability one, $X_1 , X_2 , \ldots , X_n \nu$ converges to a point measure.*

(d) *If $\theta \in \mathcal{P}(M)$ and u is an open subset of $\mathcal{P}(M)$ containing all point measures then $\mu_n \{g | g\theta \notin u\} \to 0$ as $n \to \infty$.*

The relationship between the various notions of proximality imply

$$\Pi_P(G) \to \Pi_{SP}(G) \to \Pi_{MP}(G)$$

As we saw in Theorem 2.2, all equivariant maps between proximal spaces are unique.

In the remainder of the paper we shall sketch a proof of the following result :

THEOREM 4.3. — *If G is a semi-simple Lie group then $\Pi_{MP}(G) = B(G)$. If Γ is a lattice subgroup of G then there is an equivariant map $\rho : \Pi_{MP}(\Gamma) \to \Pi_{MP}(G)$. The map ρ is a measurable isomorphism in the sense that there exists a map*

$$\sigma : \Pi_{MP}(G) \to \Pi_{MP}(\Gamma)$$

which is measurable as a map from the manifold $B(G)$ to $\Pi_{MP}(\Gamma)$, and such that $\rho \circ \sigma =$ identity.

A map on a C^∞-manifold is said to be *measurable* if when composed with a real-valued function, the resulting function is a lebesgue measurable function of the local coordinates.

For the first assertion of the theorem, inasmuch as $B(G)$ is strongly proximal so that there exists a map $B(G)$ onto $\Pi_{SP}(G)$, it would suffice to show that $B(G)$ is mean proximal. For the second assertion we must show that $B(G)$ is mean proximal as a Γ-space. Both of these follow from the next theorem.

THEOREM 4.4. — *Let G be a subgroup of $\mathrm{GL}(n, R)$ which together with all its subgroups of finite index acts irreducibly on R^n. Assume moreover that G acts proximally on the projective space P^{n-1}. Then P^{n-1} is a mean proximal space.*

The proof of the theorem depends upon an analysis of the behavior of random products of matrices. The type of argument is similar to that of [6, § 8].

To apply the foregoing theorem to give a proof of Theorem 4.3, we use the fact that the space $B(G)$ occurs as a component of one of the Satake compactifications of the symmetric space G/K. In these compactifications the symmetric space is identified with a subset of the projective space associated with the space of symmetric matrices of a certain dimension. Moreover, the group G acts linearly on this space. It is not hard to show that the action is irreducible and proximal. According to Theorem 4.4, it will therefore be mean proximal.

Now let Γ be a lattice subgroup of G. By [1], if G acts irreducibly on a space so does Γ and so does every subgroup of finite index. Moreover, it is easily seen that if G acts mean proximally on a space, then Γ at least acts proximally. Therefore Theorem 4.4 applies to Γ as well, and this yields the second statement of the theorem.

We now turn to the last assertion of the theorem. Let $\Pi = \Pi_{MP}(\Gamma)$ so that Π is a Γ-space. We construct a G-space by dividing the product $G \times \Pi$ by the relation $(g, x) \sim (g\gamma^{-1}, \gamma x)$ and setting $g_1(g, x) = (g_1 g, x)$. We denote this space $G \times_\Gamma \Pi$. Clearly $G \times_\Gamma \Pi$ has G/Γ as equivariant image. Both of these spaces are locally compact and it follows that the set of probability measures on $G \times_\Gamma \Pi$ which map onto the Haar measure on G/Γ form a compact convex space. Now recall that $B(G)$ can be expressed as G/H where H is a subgroup of G with the fixed point property ([3]). So there exists a measure θ on $G \times_\Gamma \Pi$ which maps onto Haar measure on G/Γ and which is invariant under H. If we lift θ to $G \times \Pi$, we obtain a measure λ on $G \times \Pi$ invariant under the action of both Γ and H. Here Γ acts by sending

(g, x) into $(g\gamma^{-1}, \gamma x)$, and H acts by sending (g, x) into (hg, x). Since θ maps into Haar measure on G/Γ, λ maps into Haar measure on G. We may therefore decompose λ :

$$\lambda = \int_G \delta_g \times \lambda_g \, dg$$

with δ_g the Dirac measure and λ_g a Haar-measurable function on G with values in $\mathfrak{L}(\Pi)$.

We now have

$$h\lambda = \int_G \delta_{hg} \times \lambda_g \, dg = \int_G \delta_g \times \lambda_{h^{-1}g} \, dg$$

whence $\lambda_{h^{-1}g} = \lambda_g$, so we can define ω_ξ for $\xi \in G/H$ by $\omega_{gH} = \lambda_{g^{-1}}$. Moreover

$$\gamma\lambda = \int_G \delta_{g\gamma^{-1}} \times \gamma\lambda_g \, dg = \int_G \delta_g \times \gamma\lambda_{g\gamma} \, dg$$

whence $\lambda_{g\gamma} = \gamma^{-1}\lambda_g$. Therefore $\gamma\omega_{gH} = \gamma\lambda_{g^{-1}} = \lambda_{g^{-1}\gamma^{-1}} = \omega_{\gamma gH}$ and $\gamma\omega_\xi = \omega_{\gamma\xi}$. We thus find a Γ-equivariant map ω of $B(G)$ into $\mathfrak{L}(\Pi)$. ω need not be continuous, but it is measurable.

We now make use of (the easy part of) Theorem 3 in [5]. According to this theorem, there exists a measure μ on Γ and an absolutely continuous measure ν on $B(G)$ with $\mu * \nu = \nu$. The map ω takes ν into a measure $\tilde{\nu}$ on $\mathfrak{L}(\Pi)$ and by the equivariance of the map Γ it follows that $\mu * \tilde{\nu} = \tilde{\nu}$. By Theorem 4.2, $\tilde{\nu}$ must be concentrated on point measures of Π. In other words, ω defines a measurable Γ-equivariant map from $B(G)$ into Π. This is the σ of our theorem.

BIBLIOGRAPHY

[1] Borel A. — Density properties for certain subgroups of semi-simple groups without compact component, *Ann. of Math.*, 72, 1960, p. 179-188.

[2] Ellis R. — *Lectures in Topological Dynamics*, W.A. Benjamin Inc., 1969.

[3] Furstenberg H. — A Poisson formula for semi-simple Lie groups, *Ann. of Math.*, 77, 1963, p. 335-386.

[4] Furstenberg H. — Translation invariant cones of functions on semi-simple Lie groups, *Bull. Amer. Math. Soc.*, 71, 1965, p. 271-326.

[5] Furstenberg H. — Poisson boundaries and envelopes of dicrete groups, *Bull. Amer. Math. Soc.*, 73, 1967, p. 350-356.

[6] Furstenberg H. — Noncommuting random products, *Trans. Amer. Math. Soc.*, 108, 1963, p. 377-428.

[7] Mostow G.D. — The rigidity of locally symmetric spaces, *Proceedings of this congress.*

[8] Satake I. — On representations and compactifications of symmetric spaces, *Ann. of Math.*, 71, 1960, p. 77-110.

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SEMISIMPLE GROUP SCHEMES OVER CURVES AND AUTOMORPHIC FUNCTIONS

by G. HARDER

Let k denote a field and let K/k denote a function field of one variable over k. We assume K/k is a regular extension, i.e. $K \otimes_k \bar{k}$ is a field ($\bar{k}$ = algebraic closure of k). Let Y/k denote a projective, smooth model of K/k.

I want to study semisimple affine groupschemes G/Y ; a satisfactory theory of such groupschemes over Y has implications for the arithmetic of semisimple algebraic groups which are defined over the function field K/k. A semisimple groupscheme G/Y is called rationally trivial if its generic fiber $G \underset{Y}{\times} K = G_K$ is a Chevalley group ; then G/Y is locally split for the Zariski topology on Y. By X/Y I denote the scheme of Borel subgroups of G/Y, this is a smooth projective scheme over Y (Compare [2], Exp. XXII). From the projectivity of this scheme follows that a Borel subgroup $B_K \subset G_K$ can be extended in a unique way to a Borel subgroup of G/Y :

$$\Gamma(X/Y) = \mathrm{Hom}_Y(Y, X) = \Gamma(X_K/\mathrm{Spec}(K)) .$$

If $B \subset G$ is a Borel subgroup of G/Y we denote its unipotent radical by B_u. The quotient $B/B_u = T$ is a split torus.

Let Δ(resp. Δ^+) be the set of roots (res. positive roots) in the charactermodule $X(T) = \mathrm{Hom}(T, G_m)$. By $\pi = \{\alpha_1 \ldots \alpha_r\}$ I denote the set of simple roots in Δ^+. There is a natural filtration of the unipotent radical

$$B_u = U_0 \supset U_1 \ldots U_{\nu+1} \ldots \supset U_n = \{e\}$$

by smooth subschemes which are normal in B such that the quotients $U_\nu/U_{\nu+1}$ are line bundles, i.e. they are locally isomorphic to G_a/Y. The action of T on $U_\nu/U_{\nu+1}$ is given by multiplication with a root $\alpha \in \Delta^+$. This yields a one-to-one correspondence between the roots $\alpha \in \Delta^+$ and the quotients $U_\nu/U_{\nu+1}$. If α corresponds to $U_\nu/U_{\nu+1}$ we put $W_\alpha = U_\nu/U_{\nu+1}$, and call W_α the line bundle associated to the root $\alpha \in \Delta^+$. If $W_{\alpha_1} \ldots W_{\alpha_r}$ are the line bundles associated to the simple roots $\{\alpha_1 \ldots \alpha_r\} = \pi$ we put

$$n_i(B) = \mathrm{degree}(W_{\alpha_i}) = c(W_{\alpha_i})$$

Thus we assigned to any Borel subgroup B of G/Y a vector

$$n(B) = (n_1(B) \ldots n_r(B)) \in \mathbf{Z}^r .$$

This makes sense because we can canonically identify the set of simple roots of two different Borel subgroups. If $\alpha_{i_0} \in \pi$ is a simple root and $B \subset G/Y$ a Borel subgroup then $P^{(i_0)} \supset B$ is the maximal parabolic subgroup of type $\pi - \{\alpha_{i_0}\}$ containing B ([1], § 4). The root system of the semisimple part of $P^{(i_0)}$. is of type $\pi - \{\alpha_{i_0}\}$. The unipotent radical $R_u(P^{(i_0)})$ is contained in B_u. The intersection of the filtration above with $R_u(P^{(i_0)})$ yields a filtration of $R_u(P^{(i_0)})$

$$R_u(P^{(i_0)}) \supset U_1' \supset U_2' \supset \ldots \supset U'_{d_{i_0}} = \{e\} .$$

The quotients are line bundles which correspond to the roots in

$$\Delta^+_{i_0} = \left\{ \alpha \in \Delta^+ \mid \alpha = \sum_{i=1}^{r} m_i \alpha_i \ ; \ m_{i_0} > 0 \right\}$$

Now we assign a second vector $\mathbf{p}(B) = (p_1(B) \ldots p_r(B))$ to our Borel subgroup $B \subset G/Y$ by putting

$$p_i(B) = \sum_{\alpha \in \Delta_i^+} c(W_\alpha)$$

The elements $\widetilde{\chi}_i = \sum_{\alpha \in \Delta_i^+} \alpha$ form a basis of $X(T) \otimes \mathbf{Q}$, in fact the $\widetilde{\chi}_i$ are multiples of the fundamental weights χ_i, so we get $\widetilde{\chi}_i = f_i \chi_i$ where the f_i are positive integers. We express the characters $\widetilde{\chi}_i$ in terms of the simple roots :

$$\widetilde{\chi}_i = \Sigma a_{ij} \, \alpha_j \qquad a_{ij} \in \mathbf{N}$$

and vice versa

$$\alpha_i = \Sigma b_{ij} \, \widetilde{\chi}_j \qquad b_{ij} \in \mathbf{Q} \ .$$

From this we get the following relations for the vectors $\mathbf{n}(B)$ and $\mathbf{p}(B)$:

$$p_i(B) = \sum_{j=1}^{r} a_{ij} \, n_j(B)$$
$$(*)$$
$$n_i(B) = \sum_{j=1} b_{ij} \, p_j(B)$$

It is an easy but important observation that for a given group scheme G/Y the numbers $p_i(B)$ are bounded from above as B is running over the set of Borel subgroups of G/Y. We call a vector $\mathbf{p}(B) = (p_1(B) \ldots p_r(B))$ maximal for a given G/Y, if there is no Borel subgroup B' of G/Y such that $p_i(B) \leqslant p_i(B')$ for all $\alpha_i \in \pi$ and $p_{i_0}(B) < p_{i_0}(B')$ for some $\alpha_{i_0} \in \pi$.

Let g denote the genus of Y/k, let $h > 0$ denote the g.c.d. of all degrees of positive divisors on Y. Then we have ([5], Satz 2.2.6 und Kor. 2.2.14).

THEOREM 1. — *If G/Y is a semisimple rationally trivial groupscheme and if $B \subset G/Y$ is a Borel subgroup such that* $\mathbf{p}(B)$ *is maximal, then*

$$n_i(B) \geqslant -2g - 2(h-1) \quad \textit{for all} \quad \alpha_i \in \pi$$

We call a Borel subgroup $B \subset G$ *reduced* if $n_i(B) \geqslant -2g - 2(h-1)$ for all α_i.

THEOREM 2. — *There exists a constant M which only depends on g, h and on the Dynkin diagram of G/Y such that the following statement holds : If $B \subset G$*

is reduced and if for α_{i_0} we have $n_{i_0}(B) > M$ then the maximal parabolic subgroup $P^{(i_0)} \supset B$ of type $\pi - \{\alpha_{i_0}\}$ contains all reduced Borel subgroups of G/Y.

To any vector $\mathbf{n} = (n_1 \dots n_r)$ we may associate a quasiprojective scheme

$$\Gamma_{\mathbf{n}}(X/Y) \to \mathrm{Spec}(k)$$

the points of which are the Borel subgroups of G satisfying $\mathbf{n}(B) = \mathbf{n}$. To be more precise we put for any scheme $S \to \mathrm{Spec}(k)$

$$\Gamma_{\mathbf{n}}(X/Y)(S) = \{B \subset G \times_Y (Y \times_k S) \mid n_i(B \times_Y k(s)) = n_i \quad \text{for any point} \quad s \in S\} .$$

The functor $S \to \Gamma_{\mathbf{n}}(X/Y)(S)$ is representable by a quasiprojective scheme over k (Comp. [3]). This functor can be defined for all groupschemes of inner type. Analagously we define for any vector $\mathbf{p} = (p_1 \dots p_r)$ the scheme $\Gamma^{\mathbf{p}}(X/Y)/k$ of Borel subgroups B satisfying $p_i(B) = p_i$. Of course we have $\Gamma_{\mathbf{n}}(X/Y) = \Gamma^{\mathbf{p}}(X/Y)$ if the relation (*) holds between $\mathbf{n}$ and $\mathbf{p}$. If $\mathbf{n} = (n_1 \dots n_r)$ is a vector whose components satisfy $n_i \leqslant -2g + 1$ and if $\Gamma_{\mathbf{n}}(X/Y)$ is not empty then the scheme $\Gamma_{\mathbf{n}}(X/Y)$ is smooth over k. Moreover we can calculate the dimension of this scheme. For this purpose we consider the corresponding vector $\mathbf{p} \leftrightarrow \mathbf{n}$. Then the dimension of $\Gamma_{\mathbf{n}}(X/Y) = \Gamma^{\mathbf{p}}(X/Y)$ is given by

$$\dim \Gamma^{\mathbf{p}}(X/Y) = -2 \sum_{i=1}^{r} \frac{p_i}{f_i} + (1-g) . \# \Delta^+$$

The following theorem 3 seems to be deeper than the preceding ones. It is only formulated in the case of a finite ground field $k = \mathbf{F}_q$, but I believe it can derived from this special case by general theorems in algebraic geometry.

THEOREM 3. — *Let $k = \mathbf{F}_q$ be a finite field and Y/k a smooth projective curve. Let G/Y be a semisimple group scheme of inner type. If the components of the vector $\mathbf{p} = (p_1 \dots p_r)$ are sufficiently small and if $\Gamma^{\mathbf{p}}(X/Y)$ is not empty then we have*

$$\dim \Gamma^{\mathbf{p}}(X/Y) = - \sum_{i=1}^{r} \frac{2p_i}{f_i} + (1-g) \# \Delta^+$$

and there is exactly one irreducible component of this dimension.

I want to give an idea of the proof. Before doing this I introduce some notation. I put

$$l_i(B) = \frac{p_i(B)}{f_i}$$

These numbers are not necessarily integers. This is due to the fact that in general the roots do not generate the lattice spanned by the fundamental characters. But under a certain assumption on the isomorphism type of G/Y they are integers, and I will explain the idea only in this special case. For any vector $\mathbf{l} = (l_1 \dots l_r)$ I put

$$n(G, l_1 \dots l_r) = \# \Gamma^{\mathbf{p}}(X/Y)(\mathbf{F}_q)$$

where $p_i = -f_i l_i$. Then I consider the Laurent series

$$E(G, t) = \sum_{l} n(G, l_1 \ldots l_r) \cdot q^{-\Sigma l_i} \cdot t_1^{l_1} \ldots t_r^{l_r}$$

It will be shown in [6] that $E(G, t)$ is a rational function, and can be written in the following form

$$E(G, t) = \frac{P(G, t)}{Q(t) \cdot \prod_{\nu=1}^{r} (1 - qt_\nu)}$$

Here $P(G, t)$ is a polynomial in the variables t_i, t_i^{-1} and $Q(t)$ is a polynomial in the variables t_i depending only on the Dynkin diagram of G/Y. Moreover the polynomial $Q(t)$ has no zeroes in the disc $D\left(\frac{1}{\sqrt{q}}\right) = \left\{(t_1 \ldots t_r) \mid |t_i| < \frac{1}{\sqrt{q}}\right\}$. We also know the residue of $E(G, t)$ at the point $(q^{-1} \ldots q^{-1})$. Here the residue is defined by

$$\operatorname*{Res}_{(t_1 \ldots t_r) = (q^{-1} \ldots q^{-1})} E(G, t) = \prod_{\nu=1}^{r} (1 - qt_\nu)\, E(G, t)|_{(t_1 \ldots t_r) = (q^{-1} \ldots q^{-1})}$$

It can be expressed in terms of values the ζ-function of our field K/F_q and we obtain

$$\operatorname*{Res}_{(t_1 \ldots t_r) = (q^{-1} \ldots q^{-1})} E(G, t) = q^{-(g-1)\,\#\,\Delta^+} + O(q^{-(g-1)\#\Delta^+ - 1/2})$$

(Here the Riemannian hypothesis comes in!). This yields an estimate

$$|\# \Gamma^{\underline{p}}(X/Y)(\mathbf{F}_q) - q^{2\sum_{i=1}^{r} l_i + (1-g).\#\Delta^+}| \leqslant C\, q^{2\sum_{i=1}^{r} l_i + (1-g).\#\Delta^+ - 1/2}$$

if the vector $\mathbf{l} = (l_1 \ldots l_r)$ has sufficiently large components, say $l_i > n_0(G) = n_0$. It can be shown that this estimate also holds with the same constant C and under the same conditions $l_i > n_0$ on $\mathbf{l}$ if we extend our ground field $\mathbf{F_q}$ to $\mathbf{F}_{q^n}$. Of course we have to substitute q^n for q. Then our theorem 3 is a consequence of a theorem of Lang and Weil [10].

Of course the properties of $E(G, t)$ I need are not at all abvious. This function depends on the isomorphism type $[G]$ of G/Y and for the investigation of $E(G, t_1 \ldots t_r)$ one has to consider this function as a function of $[G]$. Let $G_0/\mathbf{F}_q$ be a Chevalley group of the same type as G/Y, let T_0(resp. $B_0 \supset T_0$) be a maximal torus (resp. a Borel subgroup containing T_0). Then $G_1 = G_0 \underset{\mathbf{F}}{\times} Y$ is a Chevalley scheme over Y. Let $G_0(A)$ be the adele group of $G_{1,K}$ and $\mathcal{R} = \prod_{\mathfrak{y}} G_0(\sigma_{\mathfrak{y}})$ the canonical maximal compact subgroup. Then we may identify

$$H^1(Y_{\text{zar}}, G_1) \xrightarrow{\sim} \mathcal{R} \backslash G_0(A)/G_0(K)$$

From the Iwasawa decomposition we get $G_0(A) = \mathcal{R}.B_0(A)$ so for $\mathbf{x} \in G_0(A)$ we can write $\mathbf{x} = \mathbf{k_x} . \mathbf{b_x}$ (not unique).

If $(s_1, \ldots, s_r) = s$ is a vector whose components are complex numbers we put $\eta_s(\mathbf{x}) = \prod_{i=1} |\chi_i(\mathbf{b_x})|^{-1-s_i}$. Then the following series

$$E(\mathrm{x}\,,s) = \sum_{\gamma \in G_0(K)/B_0(K)} \eta_s(\mathrm{x}\gamma)$$

converges for $\mathrm{Re}(s_i) > 1$. It is analogous to the Eisenstein series considered by Langlands in the number field case (Compare [7], [8]), and I will show in [6] that Langland's theory can be carried over to the function field case.

To $\mathrm{x} \in G_0(A)$ corresponds a cohomology class in $H^1(Y\,,G_1)$ and this cohomology class defines an isomorphism class of twisted semisimple group schemes over Y and I assume that my given group scheme G/Y is in this class. Then it is clear that

$$E(\mathrm{x}\,,s) = E(G\,,q^{-s_1}\,\ldots\,q^{-s_r})$$

and all desired properties of $E(G\,,t_1\,\ldots\,t_r)$ can be derived from the theory of Eisenstein series. For the estimations of the Eisenstein series which are needed in the proof the theorem 3 the theorems 1 and 2 are important. The theorem 3 has nice consequences :

THEOREM 4. – *Let G/K be a simply connected semisimple algebraic group over the function field K/F_q. Then $H^1(K\,,G) = 0$.*

This theorem follows from theorem 3 in the case where G/K is a Chevalley group with out any case by case discussion (Comp. [6]). For the general case one has to use the methods in [4].

Finally I want to mention that the calculation of the residue of $E(x\,,s)$ at $(1\,,\ldots\,1)$ yields.

THEOREM 5. – *The Tamagawa number of a semisimple simply connected Chevalley group G/K is one.*

This is proved by the same method as Langland's in the numberfield case (Comp. [9].

REFERENCES

[1] BOREL A. et TITS J. — Groupes réductifs, *Publ. Math. I.H.E.S.*, 27, 1965, p. 55-151.
[2] DEMAZURE M. et GROTHENDIECK A. — Schémas en groupes, *Séminaire I.H.E.S.*, Bures-sur-Yvette, 1963-64.
[3] GROTHENDIECK A. — Les schémas de Hilbert, *Séminaire Bourbaki*, Exp. 221, Paris, 1960-61.
[4] HARDER G. — Uber die Galoiskohomologie halbeinfacher Matrizengruppen, *Math. Zeitschrift*, 90, 1968, p. 404-428.
[5] HARDER G. — Halbeinfache Gruppenschemata über vollständigen Kurven, *Inv. math.*, 6, 1968, p. 107-149.
[6] HARDER G. — *Chevalleygruppen über Funktionenkörpern und automorphe Funktionen* (in Preparation).
[7] HARISH-CHANDRA. — Automorphic forms on semisimple Lie groups, *Springer Lecture notes*, 62, 1968.

[8] LANGLANDS R. — *On the functional equation satisfied by Eisenstein series,* (Preprint).
[9] LANGLANDS R. — The volume of the Fundamental Domain for Some Arithmetical Subgroups of Chevalley groups, *Proceedings of Symposia in Pure Mathematics,* AMS, Providence, Vol. IX, 1966.
[10] LANG S., WEIL A. — Number of points of varieties over finite fields, *Am. Journ. of Math.,* 76, 1954, p. 819-827.

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GROUP REPRESENTATIONS AND SYMMETRIC SPACES

by Sigurdur HELGASON

1. Introduction.

In this lecture I shall discuss some special instances of the following three general problems concerning a homogeneous space G/H, H being a closed subgroup of a Lie group G.

(A) Determine the algebra $\mathbf{D}(G/H)$ of all differential operators on G/H which are invariant under G.

(B) Determine the functions on G/H which are eigenfunctions of each $D \in \mathbf{D}(G/H)$.

(C) For each joint eigenspace for the operators in $\mathbf{D}(G/H)$ study the natural representation of G on this eigenspace ; in particular, when is it irreducible and what representations of G are so obtained ?

Here we shall deal with the case of a symmetric space X of the noncompact type and with the case of the space Ξ of horocycles in X. We refer to [6] for proofs of most of the results reported here.

2. The eigenfunctions of the Laplacian on the non-Euclidean disk.

Let X denote the open unit disk in the plane equipped with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{[1 - (x^2 + y^2)]^2} \ .$$

The corresponding Laplace-Beltrami operator is given by

$$\Delta = [1 - (x^2 + y^2)]^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

We shall begin by stating some recent results about the eigenfunctions of Δ. Let B denote the boundary of X and $P(z, b)$ the Poisson kernel

$$P(z, b) = \frac{1 - |z|^2}{|z - b|^2} \qquad z \in X, \ \ b \in B \ .$$

It is then easily verified that if $\mu \in \mathbf{C}$ then $\Delta_z (P(z, b)^\mu) = 4\mu(\mu - 1) P(z, b)^\mu$ so for any measure m on B the function $z \to \int_B P(z, b)^\mu \, dm(b)$ is an eigenfunction of Δ. If $\mu \in \mathbf{R}$ and $m \geqslant 0$ this gives all the positive eigenfunctions of Δ (cf. [1],

[7]). More generally one can take m to be a distribution on B and even more generally, an *analytic functional* on B, that is a continuous linear functional on the space of analytic functions on the boundary B with the customary topology.

THEOREM 1. — *The eigenfunctions of the Laplace-Beltrami operator on the non-Euclidean disk are precisely the functions*

$$f(z) = \int_B P(z\,,b)^\mu \, dT(b) \tag{1}$$

where $\mu \in \mathbf{C}$ and T is an analytic functional on B.

The functional T is related to the boundary behaviour of f. Assuming, as we can, that μ in (1) satisfies $\operatorname{Re}\mu \geqslant 1/2$ we have as $|z| \to 1$

$$c_\mu (1 - |z|^2)^{\mu-1} f(z) \to T \qquad c_\mu = \Gamma(\mu)^2/\Gamma(2\mu - 1) \tag{2}$$

in the sense that the Fourier series of the left hand side converge formally for $|z| \to 1$ to the Fourier series of T. (For Re $\mu = 1/2$ a minor modification of (2) is necessary).

The case $\mu = 1$ in Theorem 1 is closely related to Köthe's Cauchy kernel representation of holomorphic functions by analytic functionals, [9]. For Eisenstein series a result analogous to (2) was proved by John Lewis in his thesis.

It is well known that the eigenspaces of the Laplacian on a sphere are irreducible under the action of the rotation group. The analogous statement for X is in general false : The largest connected group G of isometries of X does not act irreducibly on the space of harmonic functions ($\mu = 1$). In fact, the constants form an invariant subspace. However we have the following result.

THEOREM 2. — *For $\mu \in \mathbf{C}$ let V_μ denote the space of eigenfunctions of Δ for the eigenvalue $4\mu(\mu - 1)$ with the topology induced by that of $C^\infty(X)$. Then G acts irreducibly on V_μ if and only if μ is not an integer.*

3. The Fourier transform on a symmetric space X. Spherical functions.

In order to motivate the definition I restate the Fourier inversion formula for $\mathbf{R}^n$ in a suggestive form. If $f \in L^1(\mathbf{R}^n)$ and (,) denotes the inner product on $\mathbf{R}^n$ the Fourier transform $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda\omega) = \int_{\mathbf{R}^n} f(x)\, e^{-i\lambda(x,\omega)}\, dx \qquad \lambda \geqslant 0\,,\ |\omega| = 1\ ,$$

and if for example $f \in C_c^\infty(\mathbf{R}^n)$ we have

$$f(x) = (2\pi)^{-n} \iint_{\mathbf{R}^+ \times \mathbf{S}^{n-1}} \widetilde{f}(\lambda\omega)\, e^{i\lambda(x,\,\omega)}\, \lambda^{n-1}\, d\lambda\, d\omega \tag{3}$$

where $\mathbf{R}^+$ denotes the set of nonnegative reals and $d\omega$ is the surface element on $\mathbf{S}^{n-1}$.

Now consider a symmetric space X of the noncompact type, that is a coset space $X = G/K$ where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup. We fix an Iwasawa decomposition $G = KAN$

of G, A and N being abelian and nilpotent, respectively. The horocycles in X are the orbits in X of the subgroups of G conjugate to N ; the group G permutes the horocycles transitively and the set Ξ of all horocycles is naturally identified with the coset space G/MN where M is the centralizer of A in K. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{m}$ denote the respective Lie algebras of the groups introduced and log the inverse of the map exp : $\mathfrak{a} \to A$. It is clear from the above that each $\xi \in \Xi$ can be written $\xi = kaMN$, where $kM \in K/M$ and $a \in A$ are unique. Here the coset kM is called the *normal* to ξ and a the *complex distance* from the origin o in X to ξ. If $x \in X$, $b \in B (= K/M)$ there exists exactly one horocycle, denoted $\xi(x, b)$, through x with normal b. Let $a(x, b) \in A$ denote the complex distance from o to $\xi(x, b)$ and put $A(x, b) = \log a(x, b)$. This element of $\mathfrak{a}$ is the symmetric space analog of the inner product (x, ω) in $\mathbf{R}^n$. Denoting by $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$ and defining $\rho \in \mathfrak{a}^*$ by $\rho(H) = \frac{1}{2} \operatorname{Tr}(\operatorname{ad} H \mid \mathfrak{n})$, where ad is adjoint representation and | restriction, we can define the *Fourier transform* $\tilde{f}$ of a function $f \in C_c^\infty(X)$ by

$$(4) \qquad \tilde{f}(\lambda, b) = \int_X f(x)\, e^{(-i\lambda+\rho)(A(x,b))}\, dx, \qquad \lambda \in \mathfrak{a}^*, \ b \in B,$$

dx denoting the volume element on X, suitably normalized. The inversion formula for this Fourier transtorm is

$$(5) \qquad f(x) = w^{-1} \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda, b)\, e^{(i\lambda+\rho)(A(x,b))}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, db,$$

where w is the order of the Weyl group W of X, db the normalized K-invariant measure on B and $\mathbf{c}(\lambda)$ Harish-Chandra's function which can be expressed explicitly in terms of Γ-functions as we shall explain later in more detail.

A *spherical function* on X is by definition a K-invariant eigenfunction φ of each G-invariant differential operator on X, normalized by $\varphi(o) = 1$. By a simple reformulation of a theorem of Harish-Chandra the spherical functions are just the functions

$$(6) \qquad \varphi_\lambda(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}\, db$$

λ being arbitrary in the complex dual $\mathfrak{a}_c^*$; also $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = s\mu$ for some $s \in W$. The c-function arises in Harish-Chandra's work from a study of the behaviour of $\varphi_\lambda(x)$ for large x ; roughly speaking, $\varphi_\lambda(a)$ behaves for large a in the Weyl chamber A^+ as $\Sigma_{s\in W}\, \mathbf{c}(s\lambda)\, e^{(is\lambda-\rho)(\log a)}$ if $\lambda \in \mathfrak{a}^*$.

If f in (4) is K-invariant, then $\tilde{f}$ is independent of b and by use of (6) formula (5) reduces to Harish-Chandra's inversion formula for the spherical Fourier transform. On the other hand the general formula (5) can be derived quite easily from this special case, [5].

It is of course of interest to characterize the images of various function spaces on X under the Fourier transform $f \to \tilde{f}$. In this regard we have the following result (where $\mathfrak{a}_+^*$ denotes the positive Weyl chamber in $\mathfrak{a}^*$).

THEOREM 3. – *The Fourier transform $f \to \tilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times B)$ (with the measure $|\mathbf{c}(\lambda)|^{-2}\, d\lambda\, db$).*

A point $x \in X$ is called *regular* if the geodesic (ox) has stabilizer of minimum dimension. Since $(K/M) \times A^+$ is by the "polar coordinate representation" identified with the set X' of all regular points in X, Theorem 3 shows that "*X is self-dual under the Fourier transform*". The c-function is given by

$$\mathrm{c}(\lambda) = c_0 \prod_{\alpha \in P^+} \frac{\Gamma(\langle i\lambda, \alpha_0 \rangle)\, 2^{-i \langle \lambda, \alpha_0 \rangle}}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2} m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{1}{2} m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle\right)\right)}$$

where c_0 is a constant. Here P^+ is the set of positive roots which are not integral multiples of other positive roots, m_α and $m_{2\alpha}$ are the multiplicities, $\langle\ ,\ \rangle$ the inner product on $\mathfrak{a}_c^*$ and $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$. This formula was proved by Harish-Chandra and Bhanu-Murthy in special cases and by Gindikin and Karpelevič [2] in general. Every detail in this formula has turned out to be conceptually significant : the location of the singularities for the Paley-Wiener theorem, the asymptotic behaviour for the Fourier transform of rapidly decreasing functions, the Radon transform on X is inverted by a differential operator (not just a pseudo-differential operator) if and only if c^{-1} is a polynomial ; using the formula for c one shows [6] that this happens exactly when all Cartan subgroups of G are conjugate. Finally, we shall now see that the numerator and the denominator have their individual importance. We have in fact the following generalization of Theorem 2.

Let X have rank 1, i.e. $\dim A = 1$, let Δ denote the Laplace-Beltrami operator on X and for $\lambda \in \mathfrak{a}_c^*$ $c_\lambda \in \mathbf{C}$ the eigenvalue given by $\Delta\varphi_\lambda = c_\lambda \varphi_\lambda$. Let $\mathcal{E}_\lambda$ be the eigenspace of Δ for the eigenvalue c_λ, this space taken with the topology induced by the usual topology of $C^\infty(X)$.

THEOREM 4. – *Let* $\mathrm{e}(\lambda)^{-1}$ *denote the denominator in the expression for* $\mathrm{c}(\lambda)$. *Then the natural representation of* G *on* $\mathcal{E}_\lambda$ *is irreducible if and only if*

$$\mathrm{e}(\lambda)\, \mathrm{e}(-\lambda) \neq 0 .$$

4. The conical distribution on Ξ.

The spaces

$$X = G/K,\ \Xi = G/MN$$

have many analogies reminiscent of the duality between points and hyperplanes in $\mathbf{R}^n$. For example, we have the following natural identifications for the orbit spaces of K on X, MN on Ξ,

$$K\backslash G/K = A/W,\quad MN\backslash G/MN = A \times W . \tag{7}$$

In the spirit of this analogy we define the counterparts to the spherical functions.

DEFINITION. – A distribution on Ξ is called a *conical distribution* if it is an MN-invariant eigendistribution of each G-invariant differential operator on Ξ.

Since by (6) the set of spherical functions is parametrized by $\mathfrak{a}_c^*/W$, the identifications (7) suggest that the set of conical distributions should somehow be parametrized by $\mathfrak{a}_c^* \times W$. We shall now explain how this turns out to be essentially so. The Bruhat decomposition for G implies that Ξ decomposes into finitely many disjoint orbits under MNA

$$\Xi = \bigcup_{s \in W} \Xi_s , \quad \Xi_s = MNA \cdot \xi_s .$$

There is a natural measure $d\nu$ on the orbit Ξ_s and if $\lambda \in \mathfrak{a}_c^*$ we consider the functional

$$\Phi_{\lambda,s} : \varphi \to \int_{\Xi_s} \varphi(\xi)\, e^{(is\lambda + s\rho)\,(\log a(\xi))}\, d\nu(\xi) , \quad \varphi \in C_c^\infty(\Xi) ,$$

where $a(\xi)$ denotes the A-component of $\xi \in \Xi_s$. Since Ξ_s is not in general closed there is no guarantee of convergence. However one does have absolute convergence for all $\varphi \in C_c^\infty(\Xi)$ if and only if $\mathrm{Re}(\langle i\lambda , \alpha \rangle) > 0$ for all

$$\alpha \in P^+ \cap s^{-1} P^- \qquad (P^- = - P^+) .$$

If this is the case, $\Phi_{\lambda,s}$ is a conical distribution. One would now like to obtain a meromorphic continuation of the distribution-valued function $\lambda \to \Phi_{\lambda,s}$ because then all the values and "residues" of this extension would still be conical distributions. Remarkably enough it turns out that the singularities are the same as those in the numerator for the c-function except that one restricts the product to $P^+ \cap s^{-1} P^-$. Thus we have

THEOREM 5. – *Let* $s \in W$, $\alpha_0 = \alpha / \langle \alpha , \alpha \rangle$, $\mathrm{d}_s(\lambda) = \prod_{\alpha \in P^+ \cap s^{-1} P^-} \Gamma(\langle i\lambda , \alpha_0 \rangle)$. *Then the mapping*

$$\lambda \to \Psi_{\lambda,s} = \frac{1}{\mathrm{d}_s(\lambda)}\, \Phi_{\lambda,s}$$

extends to an entire function on $\mathfrak{a}_c^*$.

The residues of $\Phi_{\lambda,s}$, that is the values of $\Psi_{\lambda,s}$ at the removable singularities λ_0, have the following geometric interpretation. The closure of Ξ_s in Ξ is a union of Ξ_s and some other orbits $\Xi_{s'}$. Then the residue $\mathrm{Res}_{\lambda=\lambda_0} \Phi_{\lambda,s}$ is a linear combination of certain transversal derivatives of the various $\Psi_{\lambda,s'}$ constructed from the other orbits in the closure.

We have now to each $(\lambda , s) \in \mathfrak{a}_c^* \times W$ associated a conical distribution $\Psi_{\lambda,s}$. One can now prove ([6], Ch. III) that essentially all the conical distributions arise in this manner. This is done by transferring the differential equations for the conical distributions to differential equations on X by means of the dual to the Radon transform. Under this transform a conical distribution is sent into a C^∞ function so the differential equations become easier to handle. Thus our preliminary guess that the set of conical distributions can be parametrized by $\mathfrak{a}_c^* \times W$ is essentially verified.

5. Eigenspaces of invariant differential operators on Ξ.

Let $\mathrm{D}(\Xi)$ denote the algebra of all G-invariant differential operators on Ξ. For $\lambda \in \mathfrak{a}_c^*$ let the eigenvalue $\gamma_\lambda(D)$ be determined by $D\Psi_{\lambda,s} = \gamma_\lambda(D)\, \Psi_{\lambda,s}$ for all $D \in \mathrm{D}(\Xi)$. As indicated by the notation, the eigenvalue is independent of $s \in W$. Let $\mathcal{D}'(\Xi)$ denote the set of all distributions on Ξ and put

$$\mathcal{D}'_\lambda = \{ \Psi \in \mathcal{D}'(\Xi) \,|\, D\Psi = \gamma_\lambda(D)\, \Psi \quad \text{for} \quad D \in \mathrm{D}(\Xi) \} .$$

Each eigendistribution of the operators in $D(\Xi)$ lies in one of the spaces $\mathcal{D}'_\lambda$. Let τ_λ denote the natural representation of G on the distribution space $\mathcal{D}'_\lambda$ (strong distribution topology).

THEOREM 6. — *The representation* τ_λ *is irreducible if and only if* $\mathbf{e}(\lambda)\,\mathbf{e}(-\lambda) \neq 0$.

This is proved by relating the representation τ_λ to the Hilbert space representation π_λ of G induced by the one-dimensional representation $man \to e^{i\lambda(\log a)}$ of MAN. According to Harish-Chandra [3], Theorem 5, irreducibility of π_λ is equivalent to the (algebraic) irreducibility of the representation $d\pi_\lambda$ of $\mathfrak{g}$ on the space of K-finite vectors in the Hilbert space and a criterion for the algebraic irreducibility of $d\pi_\lambda$ is given by Kostant [8], p. 63. For X of rank one an entirely different proof of Theorem 6 is given in [6].

The distributions $\Psi_{\lambda,s}$ all belong to $\mathcal{D}'_\lambda$ and play the role of *extreme weight vectors* for the infinite-dimensional representation τ_λ.

If the irreducibility condition for τ_λ is satisfied the representations τ_λ and $\tau_{s\lambda}$ are equivalent for each $s \in W$. The intertwining operator realizing this equivalence is a kind of a convolution operator by means of the conical distribution $\Psi_{s\lambda,s^{-1}}$. More precisely, the map $S \to \Psi$ given by

$$\Psi(\varphi) = \int_{K/M} \left(\int_A e^{(i\lambda+\rho)(\log a)} \varphi(kaMN)\, da \right) dS(kM)$$

is a bijection of $\mathcal{D}'(B)$ onto $\mathcal{D}'_\lambda$. Thus τ_λ can be regarded as a representation of G on $\mathcal{D}'(B)$. If $S_{s\lambda,s^{-1}} \in \mathcal{D}'(B)$ corresponds to $\Psi_{s\lambda,s^{-1}}$ then the convolution operator $S \to S \times S_{s\lambda,s^{-1}}$ sets up the equivalence between τ_λ and τ_{s_λ}. The relationship of these results with the work of Knapp, Kunze, Schiffmann, Stein and Zhelobenko on intertwining operators is explained in [6], Ch. III, § 6.

For $\lambda \in \mathfrak{a}^*$ the intertwining operator is very simply described in terms of the Fourier transform $\tilde{f}(\lambda, b)$. In fact, the space $\{\tilde{f}(\lambda, .)\,|\, f \in C_c^\infty(X)\}$ is dense in $L^2(B)$ as well as in $\mathcal{D}'(B)$ and the operator $\tilde{f}(\lambda, .) \to \tilde{f}(s\lambda, .)$ extends to an isometry of $L^2(B)$ onto itself and induces the operator which intertwines τ_λ and $\tau_{s\lambda}$.

REFERENCES

[1] FURSTENBERG. — Translation-invarient cones of functions on semisimple Lie groups, *Bull. Amer. Math. Soc.*, 71, 1965, p. 271-326.

[2] GINDIKIN S.G. and KARPELEVIČ F.I. — Plancherel measure of Riemannian symmetric spaces of nonpositive curvature, *Soviet Math.*, 3, 1962, p. 962-965.

[3] HARISH-CHANDRA. — Representations of a semisimple Lie group on a Banach space I, *Trans. Amer. Math. Soc.*, 75, 1953, p. 185-243.

[4] HARISH-CHANDRA. — Spherical functions on a semisimple Lie group I, II, *Amer. J. Math.*, 80, 1958, p. 241-310, p. 553-613.

[5] HELGASON S. — Lie groups and symmetric spaces, *Battelle Rencontres*, 1967, 1-71. W.A. Benjamin, New York, 1968.

[6] HELGASON S. — A duality for symmetric spaces with applications to group representations, *Advances in Math.*, 5, 1970, p. 1-154.
[7] KARPELEVIČ. — The geometry of geodesics and the eigenfunctions of the Beltrami-Laplace operator on symmetric spaces, *Trans. Moscow Math. Soc.*, 14, 1965, p. 48-185.
[8] KOSTANT B. — On the existence and irreducibility of certain series of representations, *Bull. Amer. Math. Soc.*, 75, 1969, p. 627-642.
[9] KÖTHE G. — Die Randverteilungen analytischer Funktionen, *Math. Zeitschr.*, 57, 1952, p. 13-33.

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ARITHMETIC VARIETIES AND THEIRS FIELDS OF QUASI-DEFINITION

by D. KAJDAN

Introduction

Let G be an algebraic semi-simple Q-group and $G_{\mathbf{R}}$ be its group of real points. Let us suppose that the factor space of $G_{\mathbf{R}}$ by its maximal compact subgroup K has an invariant complex structure. We shall call this space D. Such a group as G defines a class of algebraic varieties over the field of complex numbers **C**. This comes about in the following way. For each arithmetic subgroup $\Gamma \subset G_{\mathbf{R}}$ (that is a discrete subgroup $\Gamma \subset G_{\mathbf{Q}}$ whose intersection with the group $G_{\mathbf{Z}}$ is a subgroup with finite index both in Γ and $G_{\mathbf{Z}}$) let us associate the complex space $X_\Gamma = D/\Gamma$. This space is a complex manifold if Γ has no elements $\gamma \neq 1$ of finite order. It is well known ([1]) that one can provide X_Γ with exactly one structure of an algebraic non necessarily complete variety compatible with its complex structure. We shall call such algebraic variety X_Γ an arithmetic variety. Different questions of Analysis and Number theory are connected with this algebraic structure of X_Γ. In some cases we can study X_Γ as a moduli space for a suitable problem [2]. For example the factor space of Siegel's generalized upper halfplane by the modular group is the moduli space for polarized abelian varieties. It seems natural to begin our study of the arithmetical varieties by trying to define them in algebraic terms.

In this report I want to discuss the following

THEOREM 1. – *Let X be an arithmetic variety and $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$. Then the algebraic* **C***-variety X^σ obtained from X by "base change" $\sigma : \mathbf{C} \to \mathbf{C}$ is also an arithmetic variety.*

Let us give some idea of the proof of this theorem 1. Let us choose an arithmetic subgroup $\Gamma \subset G_{\mathbf{Q}}$ which has no (non-trivial) elements of finite order. We shall drop the index Γ in the symbol for an arithmetic variety X_Γ.

For the proof of our theorem we have to know that for each $\sigma \in G(\mathbf{C}/\mathbf{Q})$, X^σ is isomorphic to some X. Our arguments are divided into two parts – analytic and group – theoretic. In the analytic part we consider the universal covering manifold $\widetilde{X^\sigma}$ for X^σ. We shall prove that there exists an isomorphism between the complex manifolds $\widetilde{X}(= D)$ and $\widetilde{X^\sigma}$. In the group theoretic part we shall consider the fundamental group Γ^σ of the variety X^σ and its representation (homomorphism) Γ^σ in $\mathrm{Aut}(\widetilde{X^\sigma}) = G_{\mathbf{R}}$. We shall prove that it is an arithmetic subgroup in $G_{\mathbf{R}}$.

1. Analytic part.

The main tool for the proof is the structure of unramified correspondences on X which was introduced by Pjateckii-Šapiro and Šafarevič [3]. To each element

$g \in G_Q$ we can associate the unramified correspondence which is the subvariety $X_g \subset X \times X$ image of the manifold $\widetilde{X}_g = \{\widetilde{x}, \widetilde{x}g\} \subset \widetilde{X} \times \widetilde{X}$ under the natural projection $\pi : \widetilde{X} \times \widetilde{X} \to X \times X$. X_g is an algebraic subvariety in $X \times X$ and its projections on X induced by projections $\pi_{1,a} : X \times X \to X$ are unramified coverings. Pjateckii-Šapiro and Šafarevič have proved that the converse is also true ; that is every irreducible algebraic subvariety $Y \subset X \times X$ whose projections on both factors are unramified coincides with X_g for some $g \in G_Q$ (if G is the adjoint group).

The structure of unramified correspondences is an algebraic notion ; in particular it is the same on X^σ. Each irreducible unramified correspondence $Y \subset X^\sigma \times X^\sigma$ defines the two-sided coset class g_Y in the group $\mathrm{Aut}(\widetilde{X}^\sigma)$ (here $\widetilde{X}^\sigma$ is regarded as a complex manifold) by its subgroup Γ^σ. We shall denote the group $\mathrm{Aut}(\widetilde{X}^\sigma)$ by ${}_{\mathbf{R}}G^\sigma$ and its subgroup generated by Γ^σ and g_Y by ${}_{\mathbf{Q}}G^\sigma$.

The group Γ^σ is a quasi-normal subgroup of ${}_{\mathbf{Q}}G^\sigma$ that is, for every $g_Y \in {}_{\mathbf{Q}}G^\sigma$ the subgroup $g\Gamma^\sigma g^{-1} \cap \Gamma^\sigma \subset \Gamma^\sigma$ has finite index in Γ^σ.

We know that the completion $\hat{\pi}_1(N)$ of the fundamental group $\pi_1(N)$ of an algebraic variety N in the Krull topology is an algebraic notion.

Therefore we have the following

THEOREM 2. – *The completion G of the group ${}_{\mathbf{Q}}G^\sigma$ in the topology defined by subgroups of finite index in Γ^σ (this is correct since Γ^σ is quasi-normal in ${}_{\mathbf{Q}}G^\sigma$) is independent from σ.*

We shall prove that the group ${}_{\mathbf{Q}}G^\sigma$ is "sufficiently large". More exactly we shall prove the following

THEOREM 3. – *The action of the closure ${}_{\mathbf{R}}G^\sigma$ of the group ${}_{\mathbf{Q}}G^\sigma$ in X^σ is transitive.*

Proof of theorem 3 is based on

THEOREM 4. – *The Bergman metric $\rho_{\widetilde{X}^\sigma}$ on $\widetilde{X}^\sigma$ is nondegenerate.*

The method of proof would be as follows. Let $\{\Gamma_n\}$ be a sequence of normal subgroups in Γ so that $\cap \Gamma_n = \{e\}$ and $X_n = X_{\Gamma_t}$.

At first we would prove

LEMMA 1. –

$$\lim \dim \frac{H^0(X_n, K)}{[\Gamma : \Gamma_n]} = C, \quad 0 < C < \infty \qquad (*)$$

K is the canonical class.

The proof of lemma 1 is based on arguments from functional analysis and representation theory. Let us clarify the method of proof. Let T be the representation of the group $G_{\mathbf{R}}$ in the space $\Omega(X)$ of holomorphic differential square-integrable forms of the highest degree on $\widetilde{X}$. T is a square-integrable (that is, it lies in the discrete series). If the representation T were absolutely integrable and X were a complete space then, using the Selberg trace formula and the duality theorem ([4]), we could find the exact dimension of the space $H^0(X_n, K)$. It turns out that for the receiving asymptotic (as regards n) formula it is enough for T to be a square-integrable representation. If Q-rang $G > 0$ we have to use Selberg-Langlands's works ([5]).

Using lemma 1 (if we note that the left side in (*) does not depend on σ) it is not difficult to prove

LEMMA 2. – There exist a natural number n_0 such that for every $n > n_0$ the Bergman metric $\rho_{X_n^\sigma}$ on X_n^σ is nondegenerate.

To complete the proof of theorem 4 it is enough to verify that for $n \to \infty$ the metrics $\rho_{X_n^\sigma}$ on X_n^σ converges to the non degenerate metric on X^σ.

Now we pass to theorem 3. Let us choose a maximal compact subgroup $G_{\mathbf{R}}$ defined over Q. The subgroup $K_{\mathbf{Q}} = K \cap G_{\mathbf{Q}}$ generates the set of correspondences on X which have a common fixed point. Therefore there exists a point $x_0^\sigma \in X^\sigma$ which is preserved by the isomorphic set of correspondences. Consequently there is a subgroup $K_{\mathbf{Q}}^\sigma \subset_{\mathbf{Q}} G^\sigma$ isomorphic to $K_{\mathbf{Q}}$ which preserves a point $\widetilde{x}_0 \in \widetilde{X}^\sigma$.

If the group $G_{\mathbf{R}}$ is simple then the group K (consequently $K_{\mathbf{Q}}$ and $K_{\mathbf{Q}}^\sigma$) act irreducibly on the tangent space at the point $\widetilde{x}_0 (\widetilde{x}_0^\sigma)$. Consider the orbit $\{_{\mathbf{R}}G^\sigma \widetilde{x}_0^\sigma\}$. To begin with, it is closed. It is easy to prove that $\dim_{\mathbf{R}} \{_{\mathbf{R}}G^\sigma \widetilde{x}_0^\sigma\} > 0$. Since the action of $K_{\mathbf{Q}}^\sigma$ is irreducible we find that the tangent space to the orbit $\{_{\mathbf{R}}G^\sigma \widetilde{x}_0^\sigma\}$ at x_0^σ is equal to the tangent space to $\widetilde{X}^\sigma$. Therefore this orbit is open and, consequently, coincides with $\widetilde{X}^\sigma$. The general case is also not difficult examine.

We can make theorem 3 more precise.

THEOREM 5. – $_{\mathbf{R}}G^\sigma \simeq G_{\mathbf{R}}$. $\widetilde{X}^\sigma \simeq \widetilde{X}$.

2. Group-theoretic part.

The remaining part of the proof for theorem 1 is based on the study of natural representation of the group $_{\mathbf{Q}}G^\sigma$ into group $G_{\mathbf{R}} (= {_{\mathbf{R}}G^\sigma})$ and $\hat{G}$. For every prime number p there is the natural homomorphism of the group $\hat{G}$ into the group $G_p (= G_{\mathbf{Q}_p})$. So for every prime p we have the representation $F_p^\tau : {_{\mathbf{Q}}G^\sigma} \to G_p$. We shall prove that F_p is quasi-defined over Q that is, for every $\tau \in G(\mathbf{Q}_p/\mathbf{Q})$ representation differs from F_p by an algebraic automorphism of the group G_p. Consequently, for every prime p there exists an algebraic Q-group H^p so that $(H^p)_p \simeq G_p$ and $\operatorname{Im} F_p \subset (H^p)_{\mathbf{Q}}$. Then we shall prove that the Q-group (H^p) does not depend on p. This will prove theorem 1.

Now we shall give a more detailed account. Let H be an algebraic group over the local field L and T' be a representation (homomorphism) $_{\mathbf{Q}}G^\sigma$ into H_L. Since the group Γ^σ is finitely generated, we may "deform" ([6]) T' in a representation T of the group $_{\mathbf{Q}}G^\sigma$ so that $T(_{\mathbf{Q}}G^\sigma) \subset H_k$ where $k \subset L$ is a number field. Furthermore there exists a finite set V of prime ideals in the field k so that $T(\Gamma^\sigma) \subset H_{\mathfrak{O}_V}$ where $\mathfrak{O}$ is the ring of integers in k.

We shall study the representation T in the case when G is the simply connected simple Q-group. From Kneser's strong approximation theorem it follows that the sequence $O \to C \to \hat{G} \to G_{\hat{\mathbf{Q}}} \to O$ is exact. (**)

Here $\hat{\mathbf{Q}}$ is the ring of finite adeles and C is "the kernel of the congruence subgroup problem". For each prime ideal $\mathfrak{p} \notin V$ the representation T may be extended to the representation $T_{\mathfrak{p}} : \hat{G} \to H_{\mathfrak{p}}$. It is convenient to study $T_{\mathfrak{p}}$ for prime ideals $\mathfrak{p} \notin V$ for which (the group G $(= (G \otimes_{\mathbf{Q}} k)_{\mathfrak{p}})$ has no compact factors. Let us call such an ideal $\mathfrak{p}$ "good".

Before formulating the main result about representations $T_{\mathfrak{p}}$ we must make two new definitions.

DEFINITION. – Let T be a representation ${}_{\mathbf{Q}}G^{\sigma}$ into H_k and $\mathfrak{p}$ be a "good" ideal. We call representation $T_{\mathfrak{p}}$ nonsingular, if $T_{\mathfrak{p}}(C)$ is a finite set, and call $\mathfrak{p}$-degenerate if $T_{\mathfrak{p}}(C)$ is a compact set.

DEFINITION. – Let M and N be two groups, N_1, N_2 be subgroups in N so that $(N_1, N_2) = 1$. If $T_{1,2}$ are representations M in $N_{1,2}$ then we shall call the representation M in N which has the form $T(g) = T_1(g)\,T_2(g)$ the (tensor) product of T_1 and T_2.

THEOREM 6. – *Each representation $T : {}_{\mathbf{Q}}G^{\sigma} \to H_k$ is a product of $\mathfrak{p}$ nonsingular and $\mathfrak{p}$-degenerate representations.*

Proof. – The image $C^{\mathfrak{p}} = T_{\mathfrak{p}}(C)$ is a compact normal subgroup in $G^{\mathfrak{p}} = T_{\mathfrak{p}}(\hat{G})$. As $G/C \simeq G_{\mathbf{Q}_p}$ is a product of semi-simple $\mathfrak{p}$-adic groups, then $G^{\mathfrak{p}}/C^{\mathfrak{p}}$ is a semi-simple p-adic groups $S(p = Nm_{k:Q}(\mathfrak{p}))$. From the Levi theorem follows that $C^{\mathfrak{p}}$ is a semidirect product S and $C^{\mathfrak{p}}$. As $C^{\mathfrak{p}}$ is a compact group, $G^{\mathfrak{p}}$ is an almost direct product. Theorem 6 is proved.

Remark. – If $\mathbf{Q}$-rang $G > 0$ then each representation of the group $G_{\mathbf{Q}}$ in H_k is nonsingular.

It may be proved that the notion of a nonsingular and a degenerate representation does not denend on a "good" prime ideal $\mathfrak{p}$.

THEOREM 7. – *The representation F_p of the group ${}_{\mathbf{Q}}G^{\sigma}$ into G_p are nonsingular for all prime numbers p.*

From theorem 7 it immediately follows that for all $\tau \in G(\mathbf{Q}_p/\mathbf{Q})$ the representations F_p^{τ} are also nonsingular. Consequently F_p^{τ} differs from F_p by an algebraic automorphism of group G_p. So as we have seen the representations $F_p : {}_{\mathbf{Q}}G^{\sigma} \to G_p$ can be pushed through representations $F_p^0 : {}_{\mathbf{Q}}G^{\sigma} \to (H^p)_{\mathbf{Q}}$. Using nonsingularity of the representations F_p^0 we can prove.

THEOREM 8. – *The $\mathbf{Q}$-group H^{σ} does not depend on p.*

We shall denote this $\mathbf{Q}$-group G^{σ}. It is not hard to prove that the fundamental group Γ^{σ} of X^{σ} is an arithmetic subgroup in $G^{\sigma}_{\mathbf{Q}}$.

THEOREM 1. – *Follows directly from theorems 8.*

3. Application to the determination to the field of quasi-definition.

Let N be an algebraic variety over the universal domain K and let k be the simple subfield of K. To each element $\sigma \subset \mathrm{Gal}(K/k)$ one can associate the variety N^{σ} which is defined over the same field K. Let us consider the subgroup $H \subset G(K/k)$ consisting of the elements σ in $G(K/k)$ so that the K-variety N^{σ} is isomorphic to N.

DEFINITION. – The subfield $k(N) \subset K$ consisting of H-invariant elements in K - we shall call the field of quasi-definition of the $\mathbf{C}$-variety N.

In order to find the field of quasi-definition of arithmetic variety $X = X_{\Gamma}$ we have to answer on the following question : for which $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ the $\mathbf{C}$

variety X^σ is isomorphic to X. This question is divided into two parts. First we have to know for which $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ the $\mathbf{Q}$-group G^σ connected (look theorem 8) with X^σ is isomorphic to G. It is evident that the answer to this question depends only on G (and does not depend on Γ). I want to formulate the following.

Conjecture 1. – If G is an absolutely simple $\mathbf{Q}$-group, then $G^\sigma \simeq G$ for every $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$.

In any case from Borel-Serre theorem [7] follows that exist a number field K_G such that for every $\sigma \in \mathrm{Gal}(\mathbf{C}/K_G)$ $G^\sigma \simeq G$.

For the description of the acting of the group $\mathrm{Gal}(\mathbf{C}/K)$ by "base change" on the set of conjugate classes of arithmetic subgroup $\Gamma \subset G_{\mathbf{Q}}$ we consider the quotient group $A(G)$ of the group $\mathrm{Aut}(G)_{\hat{\mathbf{Q}}}$ by the closure of its subgroup $\mathrm{Aut}(G)_{\mathbf{Q}}$. For every $a \in A(G)$ and a conjugate classe $\{\Gamma\}$ of arithmetic subgroups in $G_{\mathbf{Q}}$ we can define a conjugate class $\{\Gamma\}^a$. In order to do this we fix a group Γ in the classe $\{\Gamma\}$ and representation $a \in \mathrm{Aut}(G)_{\hat{\mathbf{Q}}}$ of the element a. Now we can define the group $\Gamma^a = G_{\mathbf{Q}} \subset \Gamma^a$ where Γ^a is the closure of Γ in $G_{\hat{\mathbf{Q}}}$. It is clear that the conjugate class $\{\Gamma\}^a$ containing Γ^a is correctly defined.

THEOREM 9. – *It exists a continuous homomorphism* φ: $\mathrm{Gal}(\mathbf{C}/K_G) \to A(G)$ *such that for every arithmetic subgroup* $\Gamma \subset G_{\mathbf{Q}}$ *and* $\sigma \in \mathrm{Gal}(\mathbf{C}/K_\sigma)$ *the conjugate class containing* Γ^σ *is equal to* $\{\Gamma\}^{\varphi(\sigma)}$.

COROLLARY. – Let X_p be a proalgebraic variety (look [3]) which is a projective limit of varieties $X_n = D/\Gamma(p^n)$ where $\Gamma(p^n)$ is a congruence subgroup $\mathrm{mod}(p^n)$. Then X_p is defined over some number field.

I think that the methods described in my report have a non group-theoretic foundation. For example, I think that the following conjecture is true:

Conjecture. – Let X be a nonsingular algebraic $\mathbf{C}$-variety so that its Bergman metric ρ_X is nondegenerate metric with negative curvature. Then for each element $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$, $\widetilde{X^\sigma} \simeq \widetilde{X}$ as complex manifolds.

REFERENCES

[1] PJA ECKII-SAPIRO. — Automorphic functions and geometry of classical domains. *Mathematics and its applications,* Volume 8.

[2] SHIMURA. — *Ann. of Math.,* 91, No. 1, 1970, p. 144-222.

[3] PJATECKII-SAPIRO and ŠAFAREVIČ. — *Izv. Akad. Nauk S.S.S.R.,* 30, 1966, No. 3, p. 671-705.

[4] GELFAND, GRAEV and PJATECKII-SAPIRO. — *Generalized functions,* Vol. 6.

[5] HARISH-CHANDRA. — *Automorphic forms on semisimple Lie Groups,* Springer-Verlag, 1968.

[6] SELBERG. — *Ind. Math. Soc.,* 20, 1956, p. 47-87.

[7] BOREL et SERRE. — *Comm. Math. Helv.,* 39, 1964, p. 111-164.

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AUTOMORPHIC FORMS ON *GL* (2)

by R. LANGLANDS

In [3] Jacquet and I investigated the standard theory of automorphic forms from the point of view of group representations. I would like on this occasion not only to indicate the results we obtained but also to justify our point of view.

For us it is imperative not to consider functions on the upper half plane but functions on $GL(2, \mathbf{Q})\backslash GL(2, \mathbf{A}(\mathbf{Q}))$ where $\mathbf{A}(\mathbf{Q})$ is the adèle ring of $\mathbf{Q}$. We also replace $\mathbf{Q}$ by an arbitrary number field or function field (in one variable over a finite field) F. One can introduce [3] a space of functions, called automorphic forms, and the notion that an irreducible representation π of $GL(2, \mathbf{A}(F))$ is a constituent of the space of automorphic forms on $GL(2, F)\backslash GL(2, \mathbf{A}(F))$.

Such a representation can in a certain sense be written as a tensor product

$$\pi = \otimes_v \pi_v$$

where the product is taken over all valuations of the field and π_v is an irreducible representation of $GL(2, F_v)$. F_v is the completion of F at v. Such a π_v has associated to it a local zeta-function $L(s, \pi_v)$ which can be expressed in terms of Γ-functions if v is archimedean and otherwise is of the form

$$\frac{1}{(1-\alpha\,|\widetilde{\omega}_v|^s)\,(1-\beta\,|\widetilde{\omega}_v|^s)}$$

if $\widetilde{\omega}_v$ is a uniformizing parameter for F_v. If ψ is a non-trivial character of $F\backslash\mathbf{A}(F)$ and ψ_v is its restriction to F_v there are also factors $\epsilon(s, \pi_v, \psi_v)$ which for the given collection of π_v are almost all 1. The functions

$$L(s, \pi) = \prod\nolimits_v L(s, \pi_v)$$

and, if $\widetilde{\pi}_v$ is the representation contragredient to π_v,

$$L(s, \widetilde{\pi}) = \prod\nolimits_v L(s, \widetilde{\pi}_v)$$

are defined by the products on the right for s in a half-plane and can be analytically continued as meromorphic functions to all of the complex plane. If

$$\epsilon(s, \pi) = \prod\nolimits_v \epsilon(s, \pi_v, \psi_v)$$

then

$$L(s, \pi) = \epsilon(s, \pi)\, L(1-s, \widetilde{\pi})$$

This is of course nothing but the functional equation of Hecke and of Maass except that the ground field is now more general. The converse theorem can take a number of forms. I do not give them here (cf [3]) but do mention that they involve the basic idea of [6].

The converse theorems can be used to justify some suggestions of mine [4] as well as some of Weil's [6] to which they are closely related. Assume that Weil's generalizations of the Artin L-functions are for irreducible two-dimensional representations of the Weil group entire functions. Then locally one has a map

$$\sigma_v \to \pi(\sigma_v)$$

from the two dimensional representations of the Weil group of F_v to the irreducible representations of $GL(2, F_v)$ and if σ is a representation of the Weil group of the global field F and σ_v its restriction to the Weil group of F_v the representation

$$\pi(\sigma) = \otimes_v \pi(\sigma_v)$$

is a constituent of the representation of $GL(2, \mathbf{A}(F))$ on the space of automorphic forms. Moreover the Artin L-function $L(s, \sigma)$ is equal to $L(s, \pi(\sigma))$. Only for function fields does this assertion lead to a real theorem.

Weil's suggestions have, I understand, also been verified for function fields [7]. It is possible, by stressing their local aspects, to refine these suggestions slightly. Although these refined suggestions have, so far as I know, not been verified I would like to mention them. If C is an elliptic curve over F, say of characteristic 0, then, as in [4], we can associate to each F_v a representation $\pi(C/F_v)$ of $GL(2, F_v)$ which depends only on C as a curve over F_v. The refined form of Weil's suggestion is that

$$\pi(C) = \otimes_v \pi(C/F_v)$$

is a constituent of the space of automorphic forms. $L(s, \pi(C))$ would then be, apart perhaps from a translation, the zeta-function $L(s, C)$ of the curve. Since $\pi(C)$ is its own contragredient we would have

$$L(s, C) = \epsilon(s, \pi(C))\, L(1 - s, C)$$

and the factor

$$\epsilon(s, \pi(C) = \prod_v \epsilon(s, \pi_v, \psi_v)$$

could be computed in terms of local properties of the curve without reference to the theory of automorphic forms. For those elliptic curves C over $\mathbf{Q}$ which sit in the Jacobians of the curves associated to elliptic modular functions one knows that a π occurs in the space of automorphic forms whose local factors are equal to those of $\pi(C)$ at almost all places. The assertion that they are equal everywhere is an interesting assertion about, among other things, the j-invariants of such curves. Some examples can be found in [2]. I understand that Deligne has obtained a reasonably general result along these lines. Casselman [1] has established results of a similar nature.

If G' is the multiplicative group of a quaternion algebra over F and π' is a representation of $G'_{\mathbf{A}(F)}$ occurring in the space of automorphic forms on $G'_F \backslash G'_{\mathbf{A}(F)}$ one can form a zeta-function

$$L(s, \pi') = \prod_v L(s, \pi'_v)$$

with the usual properties. Here again π' can be written in a certain sense as

$$\pi' = \otimes_v \pi'_v .$$

It is possible to define locally maps

$$\pi'_v \to \pi(\pi'_v)$$

from the representations of G'_{F_v} to those of $GL(2, F_v)$ so that globally

$$\pi(\pi') = \otimes_v \pi(\pi'_v)$$

is a constituent of the space of automorphic forms on $GL(2, F)\backslash GL(2\backslash \mathrm{A}(F))$ if π' is a constituent of the space of automorphic forms on $G'_F\backslash G'_{\mathrm{A}(F)}$. Again

$$L(s, \pi') = L(s, \pi(\pi')) .$$

If the field F has characteristic 0 the relations, mostly hypothetical, between the functions $L(s, \sigma)$, $L(s, C)$, $L(s, \pi')$ and the functions $L(s, \pi)$ do not, unlike the usual identities between L-functions, imply elementary number-theoretical statements because the local factors in the Euler products defining $L(s, \pi)$ are determined transcendentally. The corresponding factors for $L(s, \sigma)$, $L(s, C)$ and, at least when the quaternion algebra does not split at any archimedean prime, $L(s, \pi')$ are however determined in an elementary way. Thus relations between the functions $L(s, \sigma)$, $L(s, C)$ on one hand and $L(s, \pi')$ on the other are of some interest. Scattered instances of such relations can be found in the literature. To obtain a general theorem along these lines one needs a criterion for deciding when a representation π of $GL(2, \mathrm{A}(F))$ occurring in the space of automorphic forms is the representation corresponding to a representation π' occurring in the space of automorphic forms on $G'_F\backslash G'_{\mathrm{A}(F)}$. Jacquet and I sketched a proof that this is so precisely when the local factors π_v of π belong to the discrete series for all valuations at which the quaternion algebra does not split. This criterion is easily applied to the representations $\pi(\sigma)$ and $\pi(C)$.

REFERENCES

[1] Casselman W. — *Some new abelian varieties with good reduction,* preprint, Princeton University.

[2] Igusa J.I. — Kroneckerian Model of Fields of Elliptic Modular Functions, *Amer. Jour. Math.,* Vol. 81, 1959.

[3] Jacquet H. and Langlands R.P. — *Automorphic Forms on GL(2),* Springer-Verlag, 1970.

[4] Langlands R.P. — *Problems in the Theory of Automorphic Forms,* preprint, Yale University.

[5] Weil A. — Sur la Théorie du Corps de Classes, *Jour. Math. Soc. Japan.,* Vol. 3, 1951.

[6] Weil A. — Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.,* Vol. 168, 1967.

[7] Weil A. — *Lecture* notes, Institute for Advanced Study, 1970.

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HARMONIC ANALYSIS ON SEMI-SIMPLE GROUPS

by I. G. MACDONALD

By harmonic analysis I mean zonal spherical functions and the associated Plancherel measure, and my purpose in this lecture is to compare the theory of these things on a semi-simple real Lie group with the corresponding theory in the p-adic case.

1. Zonal spherical functions.

Let G be a unimodular locally compact group, K a compact subgroup of G. Fix a Haar measure on G. Let $L(G, K)$ (resp. $L^1(G, K)$, resp. $L^2(G, K)$) denote the space of compactly supported continuous (resp. integrable, resp. square-integrable) functions $f : G \to \mathbf{C}$ which are bi-invariant with respect to K (i.e., $f(gk) = f(kg) = f(g)$ for all $k \in K$ and $g \in G$). With convolution as product, $L(G, K)$ is a $\mathbf{C}$-algebra. The basic assumption is that $L(G, K)$ is *commutative.*

A *zonal spherical function* (z.s.f.) on G relative to K is a continuous function $\omega : G \to C$ such that (1) ω is bi-invariant with respect to K, (2) $\omega(1) = 1$, (3) $f * \omega$ is a scalar multiple of ω, for all $f \in L(G, K)$. Let Ω^+ be the set of all z.s.f. on G relative to K which are of positive type. If $f \in L^1(G, K)$, the *Fourier transform* of f is the function $\hat{f}$ on Ω^+ defined by

$$\hat{f}(\omega) = \int_G f(g)\, \omega(g^{-1})\, dg.$$

We give Ω^+ the coarsest topology for which all these Fourier transforms are continuous. Then Ω^+ is Hausdorff and locally compact.

The basic result, due to Godement [3], is that there exists a unique positive measure μ on Ω^+ (called the *Plancherel measure*) such that

(i) for all $f \in L(G, K)$, we have $\hat{f} \in L^2(\Omega^+, \mu)$,

(ii) the L^2 norm of f is equal to the L^2 norm of $\hat{f}$ (for the measure μ),

(iii) the mapping $f \to \hat{f}$ extends to an isometric isomorphism of $L^2(G, K)$ onto $L^2(\Omega^+, \mu)$.

In the special case where G is abelian and $K = \{1\}$, Ω^+ is the character group $\hat{G}$ of G, and μ is the Haar measure on $\hat{G}$ dual to the chosen Haar measure on G.

2. Real Lie groups.

Let G be a connected semi-simple real Lie group with finite centre, K a maximal compact subgroup in G (they are all conjugate in G). Then $L(G, K)$ is commutative. Let $G = KAN$ be an Iwasawa decomposition, and let $\mathfrak{a}$ be the Lie algebra of A. Define $\varphi : G \to \mathfrak{a}$ by the rule $g \in K.\ \exp \varphi(g)\,.\, N$, and let $\lambda : \mathfrak{a} \to \mathbf{C}$ be an $\mathbf{R}$-linear mapping. Then the main theorems are as follows.

(i) The function ω_λ on G defined by

$$\omega_\lambda(g) = \int_K e^{(i\lambda-\rho)\varphi(g^{-1}k)}\,dk\,, \tag{2.1}$$

where dk is normalized Haar measure on K and ρ is half the sum of the positive roots (each counted with its multiplicity) is a z.s.f. on G relative to K. Moreover, all z.s.f. arise in this way, and $\omega_\lambda = \omega'_\lambda$ if and only if $\lambda' = w\lambda$ for some element w in the Weyl group.

(ii) The Plancherel measure μ is supported on the set of real-valued λ and is of the form

$$d\mu(\omega_\lambda) = \text{const. } d\lambda/|\underset{\sim}{c}(\lambda)|^2\,, \tag{2.2}$$

where $d\lambda$ is Lebesgue measure on the dual of the real vector space $\mathfrak{a}$, and the function $\underset{\sim}{c}$ is given explicitly as a product of beta-functions by the following formula :

$$\underset{\sim}{c}(\lambda) = \prod_{a>0} B\left(\frac{1}{2}\,m_a\,,\,\frac{1}{4}\,m_{a/2} + \frac{1}{2}\,i\,\lambda(a^\vee)\right), \tag{2.3}$$

where the product is over the positive roots relative to some ordering, m_a is the multiplicity of the root a, and $a^\vee \in \mathfrak{a}$ is the "dual root" corresponding to a (i.e., $\lambda(a^\vee) = 2<\lambda\,,a>/<a\,,a>$).

These results are due to Harish-Chandra [4*a*, 4*b*], except for the explicit calculation (2.3) of $\underset{\sim}{c}(\lambda)$, which is due to Bhanu-Murthy for the split groups and to Gindikin and Karpelevič [2] in the general case.

3. Structure of p-adic groups.

Let $\mathfrak{K}$ be a p-adic field, $\mathfrak{G}$ a simply-connected semi-simple algebraic group defined over $\mathfrak{K}$, and G the group of $\mathfrak{K}$-rational points of $\mathfrak{G}$. Then G inherits a topology from $\mathfrak{K}$ with respect to which it is a locally compact topological group. We need to say a little about the structure of G and its root subgroups. By contrast with the real case, the maximal compact subgroups of G are not all conjugate ; they form a finite number of conjugacy classes, some of which are "better" than others. In what follows we have set things up so that the subgroup K to be defined presently is one of the "good" maximal compact subgroups.

One can associate with G a Euclidean space V and a reduced root system Σ_0 in the dual space V^*. Let W_0 be the Weyl group of Σ_0, and choose a set of positive roots for Σ_0, or equivalently a positive Weyl chamber. For each $a \in \Sigma_0$ and each integer r let $a + r$ denote the affine-linear function $x \to a(x) + r$ on V. These functions we call *affine roots.* Let Σ be the set of them. For each $\alpha \in \Sigma$ let w_α be the reflection in the hyperplane $\alpha^{-1}(0)$, and let W be the group of displacements of V generated by these reflections. If T is the subgroup of translations belonging to W, then W is the semi-direct product of W_0 and T. The "affine Weyl group" W acts on Σ by transposition.

There exist subgroups $U_\alpha (\alpha \in \Sigma)$ and N in G, and a surjective homomorphism $\nu : N \to W$ with the properties (3.1) - (3.7) listed below. In these, H is the kernel of ν, and $Z = \nu^{-1}(T)$. Elements of Σ are denoted by α, β, and integers by r, s. If X, Y are subgroups of G, then $\langle X, Y\rangle$ denotes the subgroup of G generated by X and Y.

(3.1) $nU_\alpha n^{-1} = U_{\nu(n)\alpha}$ $(n \in N, \alpha \in \Sigma)$.

(3.2) $U_{\alpha+1}$ is strictly contained in U_α, and $\bigcap_{r \geqslant 0} U_{\alpha+r} = \{1\}$.

(3.3) If $\beta = -\alpha + r$ with $r > 0$, then $\langle U_\alpha, U_\beta, H\rangle = U_\alpha H U_\beta$.

(3.4) $\langle U_\alpha, U_{-\alpha}, H\rangle = (U_\alpha \cdot \nu^{-1}(w_\alpha) \cdot U_\alpha) \cup (U_\alpha H U_{-\alpha+1})$.

(3.5) If $\alpha^{-1}(0)$ and $\beta^{-1}(0)$ are not parallel, then the group of commutators of U_α and U_β is contained in the group generated by the $U_{r\alpha+s\beta}$ such that $r, s > 0$ and $r\alpha + s\beta \in \Sigma$.

(3.6) If $a \in \Sigma_0$ let $U_{(a)} = \bigcup_{r \in \mathbf{Z}} U_{a+r}$. Let U^+ (resp. U^-) be the subgroup of G generated by the $U_{(a)}$ with a positive (resp. a negative). Then $U^+ \cap ZU^- = \{1\}$.

(3.7) G is generated by N and the U_α.

The group K generated by H and the U_α for which $\alpha(0) \geqslant 0$ is a maximal compact subgroup of G, and is open in G. We have Cartan and Iwasawa decompositions

$$G = KZ^+K = KZU^- \tag{3.8}$$

where Z^+ is the semigroup of all $z \in Z$ such that the translation $\nu(Z)$ sends the origin 0 into the positive Weyl chamber. Also the algebra $L(G, K)$ is commutative ; this assertion and (3.8) are consequences of (3.1) - (3.7).

The results in this section are all due to Bruhat and Tits [1*a*, 1*b*].

4. Spherical functions and Plancherel measure in the p-adic case.

For G as in § 3 it turns out that the theory of zonal spherical functions on G relative to K is a purely formal consequence of (3.1) - (3.7). We shall therefore change our point of view and take these properties as axioms. Let V be a Euclidean space, Σ_0 a reduced irreducible root system in V^*, Σ the associated affine root system and W the affine Weyl group. Let G be a Hausdorff topological group in which there exist closed subgroups N and $U_\alpha (\alpha \in \Sigma)$ and a surjective homomorphism $\nu : N \to W$, such that (3.1) - (3.7) are satisfied and the subgroup K defined at the end of § 3 is open and compact.

Then the subgroups U_α are compact, and U_α is open in $U_{\alpha-1}$, hence of finite index, say q_α. We have $q_\alpha = q_{\alpha+2}$ in any case, but it can happen that $q_\alpha \neq q_{\alpha+1}$. Consequently we enlarge the root system Σ_0, by adjoining to Σ_0 all half-roots $a/2$ such that $q_a \neq q_{a+1}$ $(a \in \Sigma_0)$. The resulting set of vectors is a root system in V^*, and we denote it by Σ_1. If $a \in \Sigma_0$ we put $q_{a/2} = q_{a+1}/q_a$.

The group Z normalizes U^+, and we define

$$\Delta(z) = d(zu^+z^{-1})/du^+$$

for $z \in Z$, where du^+ is a Haar measure on U^+.

Let s be a homomorphism of the translation group T into the multiplicative group $\mathbf{C}^*$. Define a function Φ_s on G by the rule $\Phi_s(g) = s(\nu(z))\,\Delta(z)^{1/2}$ if $g \in KzU^-$. Then we have

THEOREM 1 (Satake [8]). – *The function* ω_s *on* G *defined by*

$$\omega_s(g) = \int_K \Phi_s(g^{-1}k)\, dk \tag{4.1}$$

where dk is normalized Haar measure on K, is a z.s.f. on G relative to K. Conversely, all z.s.f. arise in this way, and $\omega_s = \omega_{s'}$ *if and only if* $s = ws'$ *for some* $w \in W_0$. *[*W_0 *acts on T by inner automorphisms, hence on Hom* $(T, \mathbf{C}^*)$*].*

Next, there is an analogue of Harish-Chandra's $\underset{\sim}{c}$-function (2.3), given by

$$\underset{\sim}{c}(s) = \prod_{b>0} \frac{1 - q_{b/2}^{-1/2}\, q_b^{-1}\, s(t_b)^{-1}}{1 - q_{b/2}^{-1/2}\, s(t_b)^{-1}} \tag{4.2}$$

where the product is over the positive $b \in \Sigma_1$, and t_b is the translation such that $t_b(0) = b^\vee$, the dual root associated with b. By contrast with the real case, there is a simple explicit formula for the values of the spherical function ω_s. By (3.8) and the bi-invariance of ω relative to K, it is enough to know the values of ω_s on Z^+. These are given by

THEOREM 2. – *If* $z \in Z^+$ *then*

$$\omega_s(z) = \frac{\Delta(z)^{-1/2}}{Q(q^{-1})} \sum_{w \in W_0} \underset{\sim}{c}(ws^{-1})\, ws^{-1}(\nu(z)). \tag{4.3}$$

Here $Q(q^{-1})$ is a certain polynomial in the $q_b^{-1}(b \in \Sigma_1)$ whose explicit definition we have no space to give here.

By analogy with the real case (§ 2) one might expect that the support of the Plancherel measure μ would be the set of spherical functions ω_s parametrized by *characters* s of T. In fact this is usually the case, but not always (1). To be precise :

THEOREM 3. – *Assume that* $q_{a/2} \geqslant 1$ for all $a \in \Sigma_0$. *Then the Plancherel measure* μ *is supported on the set* $\{\omega_s : s \in \hat{T}\}$ *and we have*

$$d\mu(\omega_s) = \text{const. } ds/|\underset{\sim}{c}(s)|^2 \tag{4.4}$$

where ds is Haar measure on the torus $\hat{T}$.

(If we normalize the Haar measures so that $\int_K dg = \int_{\hat{T}} ds = 1$, then the constant in (4.5) is $Q(q^{-1})/|W_0|$).

However, it can happen that $q_{a/2} < 1$ for some roots $a \in \Sigma_0$. In this case the support of the Plancherel measure is a set of tori of dimensions $l, l-1, \ldots, l-r$, for some r. On each of these tori the measure μ can be calculated expli-

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(1) This was first pointed out by Matsumoto [6] ; I had originally overlooked this possibility.

citly, and the general effect is that the factor $|\underset{\sim}{c}(s)|^{-2}$ in (4.4) is replaced by a certain residue of it. For the details, and the proofs of Theorems 1, 2 and 3 we refer to [5]. All these results are formal consequences of the axioms (3.1) to (3.7) of Bruhat and Tits.

Finally, we can unify the results for the Plancherel measure in the real and p-adic cases, at least when G is the group of rational points of a Chevalley group. First of all, if $\mathfrak{K}$ is a p-adic field then all the indices q_a defined at the beginning of this section are equal to q, the number of elements in the residue field of $\mathfrak{K}$. Hence $\Sigma_0 = \Sigma_1$, and we are in the situation of Theorem 3. If s is a character of T, then there exists $\lambda \in V$ such that $s(t_a) = q^{i\lambda(a^\vee)}$ for all $a \in \Sigma_0$, where as above t_a is the translation in W defined by $t_a(0) = a^\vee$. We write $\omega_\lambda = \omega_s$.

THEOREM 4. — *Let $\mathfrak{K}$ be any locally compact non-discrete field (real, complex or p-adic), G the group of rational points of a simply-connected simple Chevalley group defined over $\mathfrak{K}$. Then the Plancherel measure μ is given in all cases by*

$$d\mu(\omega_\lambda) = \text{const. } d\lambda / \prod_a \gamma_{\mathfrak{K}}(i\lambda(a^\vee))$$

where $\gamma_{\mathfrak{K}}$ is the gamma function [7] *associated with the field $\mathfrak{K}$, and the product is over all the roots, positive and negative.*

REFERENCES

[1a] BRUHAT F. and TITS J. — *4 notes in Comptes Rendus* 263, 1966, p. 598-601; p. 766-768; p. 822-825; p. 867-869.

[1b] BRUHAT F. and TITS J. — Groupes algèbriques simples sur un corps local, in *Proceedings of a Conference on Local Fields,* Springer, 1967.

[2] GINDIKIN S.G. and KARPELEVIC F.I. — Plancherel measure of Riemannian symmetric spaces of non-positive curvature, *Soviet Math.,* 3, 1962, p. 962-965.

[3] GODEMENT R. — Introduction aux travaux de A. Selberg, *Sém. Bourbaki,* 1957, Exp. 144.

[4a] HARISH-CHANDRA. — Spherical functions on a semi-simple Lie group I, *Amer. J. of Math.,* 80, 1958, p. 241-310.

[4b] HARISH-CHANDRA. — Discrete series for semi-simple Lie groups II, *Acta Math.,* 116, 1966, p. 1-112.

[5] MACDONALD I.G. — Spherical functions on groups of p-adic type, *Publications of the Ramanujan Institute of Advanced Study in Mathematics,* Madras, (to appear).

[6] MATSUMOTO H. — Functions sphériques sur un groupe semi-simple p-adique, *Comptes Rendus* 269, 1969, p. 829-832.

[7] SALLY P.J. and TAIBLESON M.H. — Special functions on locally compact fields, *Acta Math.,* 116, 1966, p. 279-309.

[8] SATAKE I. — Theory of spherical functions on reductive algebraic groups over p-adic fields, *Publ. Math. I.H.E.S.* no. 18, 1963, p. 1-69.

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LATTICES IN SEMISIMPLE LIE GROUPS

by M. S. RAGHUNATHAN

This talk is based for the most part on some joint work done by the speaker with H. Garland [4].

We outline here the description of a fundamental domain for lattices in real semisimple Lie groups of rank 1. The technique of the proof also yields new proofs of some theorems of Borel-Harishchandra [2] and Borel [1].

A lattice in a Lie group G is a discrete subgroup Γ such that G/Γ has finite Haar measure. Γ is non-uniform if G/Γ is not compact. A theorem of Borel-Harishchandra asserts that arithmetic subgroups of semisimple groups are lattices. A result due to Borel gives a description of a fundamental domain for these arithmetic lattices which is very useful for the study of these groups. Our results show that the description given by Borel is in fact valid for *any* non-uniform lattice in a rank 1 semisimple Lie group. The methods of proof lead us to new proofs of the theorems of Borel-Harishchandra and Borel, quoted above (different also from the adèle techniques of [5]).

To deal with non-uniform lattices of **R**-rank 1 groups as well as certain arithmetic groups we introduce the notion of an *admissible discrete* subgroup of a semisimple Lie group. Towards this end we fix some notation and definitions first.

Let G be a real connected linear semisimple Lie group. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition for $\mathfrak{g}$. Let K be the (maximal compact) subgroup corresponding to $\mathfrak{k}$. The Killing form restricted to $\mathfrak{p}$ is positive definite and defines a G-invariant symmetric Riemannian metric on $X = K\backslash G$. In the sequel we denote by $B(\ ,\ )$ the Killing form on $\mathfrak{g}$ and for $Y \in \mathfrak{p}$, $\|Y\| = B(Y, Y)$. Let e be the identity of G and $\dot{e}$ the corresponding point on $K\backslash G$. Let $\pi : G \to K\backslash G$ be the natural map (then $\pi(e) = \dot{e}$). It is then known that the geodesics through $\dot{e}$ parametrised by arc-length are precisely the curves $t \to \pi(\exp t(Y))$, $Y \in \mathfrak{p}$, $\|Y\| = 1$, $t \in \mathbf{R}$, where $\exp : \mathfrak{g} \to G$ is the exponential map. Now let $\Gamma \subset G$ be a discrete subgroup of G. Then Γ acts (on the right) as a properly discontinuous group of isometries on X. Let $d(\ ,\ )$ denote the distance function on X. Then the Poincaré fundamental domain for Γ acting on X is the (closed) set

$$E_\Gamma = \{x/x \in X,\ d(x, e) \leqslant d(x\gamma, e) \quad \text{for all} \quad \gamma \in \Gamma\}.$$

It is well known and easily proved that $E_\Gamma \Gamma = X$.

DEFINITION 1. – A *ray* for Γ is an element $Y \in \mathfrak{p}$ such that $\|Y\| = 1$ and $\pi(\exp - t\, Y) \in E_\Gamma$ for all $t \geqslant 0$.

Suppose now $Y \in \mathfrak{p}$ is any element. Then the endomorphism ad $Y : \mathfrak{g} \to \mathfrak{g}$, it is well known is semisimple with all eigen values real. For $\lambda \in \mathbf{R}$, let

$$\mathfrak{g}^{\lambda} = \{v \mid v \in \mathfrak{g},\ [Y, v] = \lambda v\}$$

Then $\mathfrak{g} = \coprod_{\lambda > 0} \mathfrak{g}^{\lambda} \oplus \mathfrak{g}^{0} \oplus \coprod_{\lambda > 0} \mathfrak{g}^{-\lambda}$ and the subspace

$$\mathfrak{n}_Y^+ = \coprod_{\lambda > 0} \mathfrak{g}^{\lambda} \Big(\text{resp. } \mathfrak{n}_Y^- = \coprod_{\lambda > 0} \mathfrak{g}^{-\lambda}\Big)$$

is a nilpotent subalgebra of $\mathfrak{g}$. Let N_Y^+(resp. N_Y^-) be the (closed) connected subgroup of G corresponding to $\mathfrak{n}_Y^-$(resp. $\mathfrak{n}_Y^+$). Let $\mathfrak{g}^0 = \mathfrak{z}(Y)$; it is clearly the centraliser of Y in $\mathfrak{g}$. Let $Z(Y)$ be the (closed) connected subgroup of G corresponding to $\mathfrak{z}(Y)$. Then $Z(Y)$ normalses N_Y^+ and N_Y^-. Let $P_Y^+ = Z(Y).\ N_Y^+$ and $P_Y^- = Z(Y)N_Y^-$. let $A_Y = \{\exp tY\}_{-\infty < t < \infty}$ and for $c \in \mathbf{R}$, let $A_{y,c} = \{\exp tY\}_{-\infty < t < c}$.

DEFINITION 2. — A closed normal subgroup Q_Y^+ of P_Y^+ is a *parabolic supplement* to Y in G if $P_Y^+ = A_Y Q_Y^+$ and $A_Y \cap Q_Y^+ = e$.

One sees easily that $Q_Y^+ \supset N_Y^+$ and that there always exists a parabolic supplement to any $Y(\neq 0)$ in p.

DEFINITION 3. — A discrete subgroup $\Gamma \subset G$ is *admissible* if every ray Y for Γ admits a parabolic supplement Q_Y^+ such that $Q_Y^+/Q_Y^+ \cap \Gamma$ is compact.

With these definitions, the main result is

THEOREM 1. — *Let $\Gamma \subset G$ be an admissible discrete subgroup. Then the set Ξ^* of rays for Γ is finite. Moreover if G/Γ is not compact we can find a subset $\Xi \subset \Xi^*$ and for every $Y \in \Xi$ a relatively compact open subset $\eta_Y \subset Q_Y^+$ and a real constant $c > 0$ such that the following hold.*

(i) $\eta_Y(\Gamma \cap Q_Y^+) = Q_Y^+$.

(ii) if $\Omega = \bigcup_{Y \in \Xi} KA_{Y,c}\, \eta_Y$, then $\Omega\Gamma = G$

(iii) the set $\{\gamma \in \Gamma\ ;\ \Omega\gamma \cap \Omega \neq \emptyset\}$ is finite.

Finally given $c_1 \in \mathbf{R}$ we can find $c_2 \in \mathbf{R}$ such that for $Y, Y' \in \Xi$

$$KA_{Yc_1}\eta_Y\gamma \cap KA_{Y'}\eta_{Y'} = \emptyset \text{ for } \quad \gamma \in \Gamma\ \cdot$$

only if $Y = Y'$ and $\gamma \in Q_Y^+ \cap \Gamma$.

COROLLARY. — An admissible discrete subgroup is a lattice.

The notion of an admissible group is fruitful in view of the following two results (Theorems 2 and 3).

THEOREM 2. — *If G is a semisimple group of rank 1, then any lattice in G is an admissible discrete subgroup.*

Theorem 2 is a partial converse to Corollary to Theorem 1. The proof uses the results of Kazdan-Margolies [6]. The following lemma is very useful both for Theorem 2 as well as Theorem 3 stated below.

LEMMA 1. — Let G^* be a semisimple algebraic group defined over a field k of characterestic 0. Let G_k be the set of k-rational points of G and assume that

G has k-rank 1 (see Borel-Tits [3] for definitions). Then any unipotent element of G_k is contained in a *unique* maximal unipotent algebraic subgroup of G defined over k.

Theorem 3 below is the first step in obtaining a fundamental domain for arithmetic groups.

THEOREM 3. – *Let G^* be an algebraic semisimple Lie group defined and of rank 1 over* Q. *Let G denote the identity component of $G_{\mathbf{R}}$ the group of* R-*rational points of G. Let $\Gamma \subset G$ be an arithmetic subgroup. Then Γ is admissible. In particular Γ is a (non-uniform) lattice.*

This result is proved by the techniques of Mostow-Tamagawa [8] in their proof of the Godement criterion ; the Godement criterion itself is also used.

One decuce from Theorems 3 and 1 the following result due to Borel.

THEOREM 4. – *Let G^* be a semisimple algebraic group defined over* Q *and P^* a minimal parabolic subgroup defined over* Q. *Let $\Gamma \subset G_{\mathbf{R}}$ be an arithmetic subgroup. Then the set $P_{\mathbf{Q}} \backslash G_{\mathbf{Q}} / \Gamma$ is finite where $G_{\mathbf{Q}}$*(resp $P_{\mathbf{Q}}$) *is the set of* Q-*rational points of G^** (resp. P^*).

One proves Theorem 4 first for Q-rank 1 groups by applying theorem 1 and 3. Then using the Bruhat decompositions one obtains the theorem in the general case.

Borel [1] shows how to obtain a good description of a fundamental domain for arithmetic groups starting from Theorem 4. Thus the notion of admissible discrete subgroups achieves a certain amount of unification in dealing with lattices and arithmetic subgroups (which à priori are not known to be lattices).

Theorem 2 in combination with Theorem 1 leads to some interesting applications. We assume from now on that G is a semisimple Lie group of *rank 1 without compact factors* (hence simple) and $\Gamma \subset G$ is a lattice. Then we have

(1) G/Γ is the interion of a compact differentiable manifold with boundary.

(2) Γ in finitely presentable ; (hence) Γ has a subgroup of finite index which is torsion free (see Selberg [11]).

(3) if Γ is torsion free, X/Γ is the interior of a compact differentiable manifold with boundary

(4) if M is a finitely generated abelian group on which Γ acts (as group automorphisms) then $H^p(\Gamma, M)$ are finitely generated.

These results are proved by constructing a suitable C^∞ function with no critical points outside a compact set on G/Γ.

A more delicate use of Theorems 1 and 2 leads to the following results.

(5) (rigidity) if G is not locally isomorphic to $SL(2, \mathbf{R})$ or $SL(2, \mathbf{C})$ then any homomorphism of Γ in G close to the inclusion $i : \Gamma \to G$ is obtained from i by an inner conjugation.

(6) if G is the identity component of the R-rational points of an algebraic group G^* defined over a number field $k \subset \mathbf{R}$ then we can find a number-field l, $k \subset l \subset \mathbf{R}$ and $g \in G$ such that $g \Gamma g^{-1} \subset G_l$ the set of l-rational points of G^*.

These results are proved by appealing to the de Rham theorem for sheavs and certain vanishing theorems for cohomology ; the techniques are those of [10].

A result in a different direction is the following.

(7) Let G and G' be two simple Lie groups of rank 1. Let Γ(resp. Γ') be a lattice in G(resp. G'). If Γ is isomorphic to Γ' and one of G/Γ and G'/Γ' is *not* compact, then G and G' are locally isomorphic.

We also have the following

(8) Let Γ(resp. Γ') be a lattice in a rank 1 simple Lie group G(resp. G'). Let $\varphi : \Gamma \to \Gamma'$ be an isomorphism. Then there exists a subgroup Γ_1 of finite index in Γ such that for $\gamma \in \Gamma_1, \varphi(\gamma)$ is unipotent (resp. semisimple) if and only if γ is unipotent (resp. semisimple).

A final application deduced using the S-cobordism theorem is the following

THEOREM 5. – *Let* Γ, Γ' *be two lattices in a rank 1 connected simple Lie group G. Let X be the symetric space associated to G and a maximal compact subgroup K. Assume that* Γ *and* Γ' *are torsion free. Then if* X/Γ *and* X/Γ' *are diffeomorphic and* $\dim X \geqslant 6$ *they are quasiconformally equivalent (and hence by a theorem of Mostow* [7] *isometric).*

Detailed proofs are to appear in [4] and [9].

Higher rank groups. – The techniques used for rank 1 groups above rely heavily on Lemma 1. We do not know a suitable generalisation to obtain results for groups of higher rank. However considerable progress can be made with the use of the following *conjecture* : Let $X \in M(n, \mathbf{R})$ $(X \neq 0)$ be a nilpotent matrix. Let $\pi : SL(n, \mathbf{R}) \to SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ be the natural map. Then the composite map $t \to \pi(\exp tX)$ of $\mathbf{R}$ in $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ is not proper.

This conjecture can be proved very simply when $n = 2$. The general case has been claimed by Margolis ; the present writer has so far not seen a proof. (I am however informed that a proof for $n = 2$ by Margolies has appeared). We give below our proof for $n = 2$ (obtained in collaboration with H. Garland) below.

Using Mahler's criterion (see for instance [1]) we need only show that there cannot be a sequence φ_n of non-zero Lattice points in $\mathbf{R}^2$ such that $q^n \varphi_n$ tends to zero as n tends to ∞ where $q = \exp X$. If in fact there were such a sequence, we find that $q^n \varphi_{n+1}$ tends to zero as well so that $\varphi_n \wedge \varphi_{n+1} = q^n \varphi_n \wedge q^n \varphi_{n+1}$ tends to zero as well. Since the φ_n are lattice points $\varphi_n \wedge \varphi_{n+1} = 0$. But then $\varphi_{n+1} = \pm \varphi_n$ (if we assume as we may that the φ_n are primitive). But then we have a non-zero vector $\varphi \in \mathbf{R}^2$ such that $q^n \varphi$ tends to zero, a contradiction.

REFERENCES

[1] BOREL A. — *Introduction aux groupes arithmétiques,* Hermann, Paris 1969.

[2] BOREL A. and CHANDRA Harish. — Arithmetic subgroups of algebraic groups, *Ann of Math,* 75, 1962, p. 485-535.

[3] BOREL A. and TITS J. — Groupes Reductifs, Inst. Hautes Etudes Sci., *Publ. Math.*, 27, 1965, p. 55-151.
[4] GARLAND H. and RAGHUNATHAN M.S. — Fundamental domains for lattices in (R-) rank 1 semisimple Lie groups, to appear in the *Ann of Math.*
[5] GODEMENT R. — Domaines fondamentaux des groupes arithmétiques, *Séminaire Bourbaki,* Vol. 15, Exposé 257, Secrétariat Mathématiques, Paris.
[6] KAZDAN D.A. and MARGOLIES G.A. — A proof of Selberg's hypothesis, *Matsb* (N.S.), 75 (117), 1968, p. 163-168 (Russian).
[7] MOSTOW G.D. — Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, Inst. Hautes Etudes Sci., *Publ. Math.*, 34, 1968, p. 53-104.
[8] MOSTOW G.D. and TAMAGAWA T. — On the compactness of arithmetically defined homogeneous spaces, *Ann. of Math.*, 76, 1962, p. 446-463.
[9] RAGHUNATHAN M.S. — *Discrete subgroups of Lie groups,* to appear.
[10] RAGHUNATHAN M.S. — Cohomology of Arithmetic Subgroups of algebraic groups II, *Ann of Math.*, 87, 1968, p. 279-304.
[11] SELBERG A. — *On discontinuous groups in higher dimensional symmetric spaces, Contributions to Function Theory,* TIFR, Bombay 1960, p. 147-164.

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ON ARITHMETIC AUTOMORPHIC FUNCTIONS

by Goro SHIMURA

1. – To explain our problems, we start with a semi-simple algebraic group G^1 defined over Q satisfying the following condition :

(*) *The quotient of* $G^1_{\mathbf{R}}$ (see Notation below) *by a maximal compact subgroup defines a bounded symmetric domain* $\mathfrak{H}$.

If Γ is an arithmetic subgroup of G^1, there is a Γ-invariant holomorphic map φ of $\mathfrak{H}$ into a projective space which induces a biregular isomorphism of $\mathfrak{H}/\Gamma$ onto a Zariski open subset V of a projective variety (see Baily and Borel [1]). We call such a couple (V, φ) a *model of* $\mathfrak{H}/\Gamma$. Then our first problem, in a naive form, is as follows.

(P) *Associate with each* Γ *of congruence type a model* $(V_\Gamma, \varphi_\Gamma)$ *and an algebraic number field* k_Γ *of finite degree so that the following conditions are satisfied :*

(P.1) V_Γ *is defined over* k_Γ.

(P.2) *If* $\alpha \in G^1_{\mathbf{Q}}$ *and* $\alpha\Gamma\alpha^{-1}$ *is contained in another arithmetic group* Δ, *then* $k_\Delta \subset k_\Gamma$, *and the morphism* $J_{\Delta\Gamma}(\alpha)$ *of* V_Γ *onto* V_Δ, *defined by* $J_{\Delta\Gamma}(\alpha) \circ \varphi_\Gamma = \varphi_\Delta \circ \alpha$, *is rational over* k_Γ.

(P.3) *If* w *is an isolated fixed point on* $\mathfrak{H}$ *of an element of* $G^1_{\mathbf{Q}}$, *the field* $k_\Gamma(\varphi_\Gamma(w))$ *generated over* k_Γ *by the coordinates of* $\varphi_\Gamma(w)$ *has a certain class field theoretical property similar to that of singular values of elliptic modular functions.*

My purpose of this lecture is to give a brief survey of the recent development in this and other related problems.

As Weil [9] showed, any Q-simple algebraic group G^1 can be represented, up to isogeny and with few exceptions, as the group of automorphisms of an involutorial algebra. Under the condition (*), such a group belongs to one of the following four types (modulo the center in each case).

(I) The unitary group of a hermitian form over a quaternion algebra B whose center is a totally real algebraic number field F. (B can be the matrix algebra $M_2(F)$.)

(II) The special unitary group of a hermitian form over a division algebra, with respect to an involution of the second kind, whose center is a totally imaginary quadratic extension of a totally real algebraic number field.

(III) The orthogonal group of a quadratic form over a totally real algebraic number field.

(IV) The unitary group of an anti-hermitian form over a quaternion algebra whose center is a totally real algebraic number field.

(In each case, certain conditions on the signature of the quadratic or hermitian form are necessary).

At present, the above problem has been settled for the groups of type I (by the author [8]), and of type II (by K. Miyake [3]). Actually, instead of assigning

$$(V_\Gamma, \varphi_\Gamma, k_\Gamma)$$

to a discontinuous group Γ, we consider the adelization $G_{\mathbf{A}}$ of a certain reductive group G whose semi-simple part is G^1, and assign a model (V_S, φ_S) and a number field k_S to a subgroup S of $G_{\mathbf{A}}$ of a certain type. This formulation is almost inevitable for practical, aesthetic, and philosophical reasons, and suggests an analogy with class field theory, as will be seen later.

Notation. – For an algebraic group G over $\mathbf{Q}$, we denote by $G_{\mathbf{A}}$ its adelization. The archimedean and non-archimedean parts of $G_{\mathbf{A}}$ are denoted by G_∞ and G_0, respectively, so that $G_{\mathbf{A}} = G_0 G_\infty$. We identify G_∞ with $G_{\mathbf{R}}$, the group of all $\mathbf{R}$-rational points of G. Then $G_{\infty+}$ denotes the identity component of G_∞. We put $G_{\mathbf{A}+} = G_0 G_{\infty+}$, $G_{\mathbf{Q}+} = G_{\mathbf{A}+} \cap G_{\mathbf{Q}}$. A $\mathbf{Q}$-rational homomorphism λ of G to another $\mathbf{Q}$-rational algebraic group G' defines naturally a continuous homomorphism of $G_{\mathbf{A}}$ to $G'_{\mathbf{A}}$, which we denote again by λ. For any algebraic number field F of finite degree, we view $F^\times = F - \{0\}$ as (the $\mathbf{Q}$-rational points of) an algebraic group over $\mathbf{Q}$. Then $F^\times_{\mathbf{A}}$ denotes the idele group, and $F^\times_{\infty+}$ the identity component of $F^\times_\infty = (F \otimes_{\mathbf{Q}} \mathbf{R})^\times$. Further F_{ab} denotes the maximal abelian extension of F, and F_c the closure of $F^\times F^\times_{\infty+}$ in $F^\times_{\mathbf{A}}$.

2. – First let us make preliminay considerations about an object (F, Θ) consisting of an algebraic number field F of finite degree, and an absolute equivalence class Θ of $\mathbf{Q}$-linear representations of F by complex matrices. If $g = [F : \mathbf{Q}]$ and $\tau_1, \ldots, \tau_g$ denote all the isomorphisms of F into $\mathbf{C}$, then Θ is determined by the multiplicity, say r_i, of each τ_i in Θ. We write then $\Theta \sim \sum_{i=1}^{g} r_i \tau_i$, and put $\det \Theta(x) = \prod_{i=1}^{g} (x^{\tau_i})^{r_i}$, $tr\ \Theta(x) = \sum_{i=1}^{g} r_i x^{\tau_i}$ for $x \in F$. Given (F, Θ), let F' denote the field generated over $\mathbf{Q}$ by $tr\,\Theta(x)$ for all $x \in F$. Then there is a unique $(F \otimes_{\mathbf{Q}} F')$-module V such that Θ is equivalent to the representation of F into the ring of F'-linear endomorphisms of V. Mapping F' into the ring of F-linear endomorphisms of V, we obtain an equivalence class Θ' of representations of F. The couple (F', Θ') is called *the reflex of* (F, Θ). We see easily that $\det \Theta'(y)$ and $tr\,\Theta'(y)$ belong to F for every $y \in F'$.

3. – To state our results for the group of type I, let F be a totally real algebraic number field of degree g, and B a quaternion algebra over F, which may or may not be a division algebra. Define an algebraic group G over $\mathbf{Q}$ so that

$$G_{\mathbf{Q}} = \{\alpha \in GL_n(B) \mid {}^t\alpha^\iota \cdot \alpha = \nu(\alpha) \cdot 1_n \quad \text{with} \quad \nu(\alpha) \in F^\times\},$$

where ι is the main involution of B. The semi-simple part of G is

$$G^1 = \{\alpha \in G \mid \nu(\alpha) = 1\}.$$

Now $B \otimes_{\mathbf{Q}} \mathbf{R} = M_2(\mathbf{R})^r \oplus \mathbf{H}^{g-r}$ with an integer $r \geqq 0$, where $\mathbf{H}$ denotes the Hamilton quaternions. Accordingly we have $G_{\mathbf{R}}^1 = Sp(n, \mathbf{R})^r \times Sp(n)^{g-r}$, so that $G_{\mathbf{R}}^1$ modulo a maximal compact subgroup can be identified with the product $\mathfrak{H}_n^r$ of r copies of the Siegel upper half space $\mathfrak{H}_n$ of degree n. Let $\tau_1, \ldots, \tau_r$ be the isomorphisms of F into $\mathbf{R}$ corresponding to the first r factors $M_2(\mathbf{R})$ of $B \otimes_{\mathbf{Q}} \mathbf{R}$. Consider (F, Θ) with $\Theta \sim \sum_{i=1}^{r} \tau_i$, and let (F', Θ') be the reflex of (F, Θ). Put $\lambda = \det \Theta'$, and define two subgroups $\mathfrak{G}_+$ and $\overline{\mathfrak{G}}_+$ of G_{A+} by

$$\mathfrak{G}_+ = G_{\mathbf{Q}+} G_{\infty+} \cdot \{x \in G_{\mathbf{A}+} \mid \nu(x) \in \lambda(F_{\mathbf{A}}'^{\times})\},$$

$$\overline{\mathfrak{G}}_+ = \{x \in G_{\mathbf{A}+} \mid \nu(x) \in \lambda(F_{\mathbf{A}}'^{\times}) F_c\}.$$

It can be shown that $\overline{\mathfrak{G}}_+$ is the closure of $\mathfrak{G}_+$.

Let $\mathfrak{Z}$ denote the set of all the subgroups S of $\overline{\mathfrak{G}}_+$ satisfying the following two conditions : (i) $F_c G_{\infty+} \subset S$; (ii) *$S/F_c G_{\infty+}$ is open and compact in $\overline{\mathfrak{G}}_+/F_c G_{\infty+}$.* (If especially $r = 1$, we have $\overline{\mathfrak{G}}_+ = G_{\mathbf{A}+}$, so that $\mathfrak{Z}$ is in one-to-one correspondence with the set of all open compact subgroups of $G_0/(G_0 \cap F_c)$.) For every $S \in \mathfrak{Z}$, put $\Gamma_S = S \cap G_{\mathbf{Q}}$. Then $F^\times \subset \Gamma_S$, and Γ_S, or rather $\Gamma_S/F^\times$, is a discontinuous group of transformations on $\mathfrak{H}_n^r$. We are going to assign, to each $S \in \mathfrak{Z}$, a model (V_S, φ_S) of $\mathfrak{H}_n^r/\Gamma_S$ and a number field k_S. To define k_S, we consider the canonical isomorphism of $F_{\mathbf{A}}'^{\times}/F_c'$ onto $\mathrm{Gal}(F_{ab}'/F')$. Let $\mathfrak{k}$ denote the subfield of F_{ab}' corresponding to the kernel of the map $\lambda^* : F_{\mathbf{A}}'^{\times} \to \lambda(F_{\mathbf{A}}'^{\times}) F_c/F_c$ which is naturally obtained from λ. Then $\mathrm{Gal}(\mathfrak{k}/F')$ is canonically isomorphic to $\lambda(F_{\mathbf{A}}'^{\times})F_c/F_c$. Since $\nu(\overline{\mathfrak{G}}_+) \subset \lambda(F_{\mathbf{A}}'^{\times}) F_c$, composing this isomorphism with ν, we obtain a homomorphism $\sigma : \overline{\mathfrak{G}}_+ \to \mathrm{Gal}(\mathfrak{k}/F')$. Now k_S is defined to be the subfield of $\mathfrak{k}$ corresponding to $\sigma(S)$. After these preparations, our first main theorem can be stated as follows.

THEOREM 1. – *There exists a system*

$$\{V_S, \varphi_S, J_{TS}(x), (S, T \in \mathfrak{Z}; x \in \overline{\mathfrak{G}}_+)\}$$

formed by the objects satisfying the following conditions.

(1) *For each $S \in \mathfrak{Z}$, (V_S, φ_S) is a model of $\mathfrak{H}_n^r/\Gamma_S$.*

(2) *V_S is rational over k_S.*

(3) *$J_{TS}(x)$, defined if and only if $xSx^{-1} \subset T$, is a morphism of V_S onto $V_T^{\sigma(x)}$, rational over k_S, and has the following properties :*

(3a) *$J_{SS}(x)$ is the identity map if $x \in S$;*

(3b) $J_{TS}(x)^{\sigma(y)} \circ J_{SR}(y) = J_{TR}(xy)$;

(3c) *$J_{TS}(\alpha) \circ \varphi_S = \varphi_T \circ \alpha$ for every $\alpha \in G_{\mathbf{Q}+}$ if $\alpha S \alpha^{-1} \subset T$.*

(4) *A certain reciprocity law holds at every isolated fixed point of $G_{\mathbf{Q}+}$.*

Thus the symbol $J_{TS}(x)$ includes $J_{\Delta\Gamma}(\alpha)$ of (P.2) as a special case. For an element x of $\overline{\mathfrak{G}}_+$ not contained in $G_{\mathbf{Q}+}$, the existence of the morphism $J_{TS}(x)$ of V_S to $V_T^{\sigma(x)}$ is quite a non-trivial fact. This is especially so when the class number of F is greater than one.

For simplicity, we give the precise statement of (4) in the special case where $r = 1$, τ_1 is the identity map, and the isotropy group at the fixed point spans a number field. (See [6], [8] for the general case.) Let P_0 be a totally real extension of F of degree n, and P a totally imaginary quadratic extension of P_0 that splits B. Then there exists an F-linear isomorphism f of P into $M_n(B)$ such that $f(a^\rho) = {}^t f(a)^\iota$ for all $a \in P$, where ρ denotes the complex conjugation. If $a \in P^\times$ and $aa^\rho \in F$, we see that $f(a) \in G_{\mathbf{Q}+}$, and these elements $f(a)$ have a unique common fixed point z on $\mathfrak{H}_n$. We can then find a $\mathbf{Q}$-linear representation $\Psi : P \to M_n(\mathbf{C})$ and a holomorphic isomorphism π of $\mathfrak{H}_n$ onto the unit ball $\mathfrak{D}_n$ of complex symmetric matrices of size n so that

$$(\pi \cdot f(a) \cdot \pi^{-1})(w) = \overline{\Psi(a)} \cdot w \cdot \Psi(a)^{-1} \qquad (a \in P^\times, aa^\rho \in F ; w \in \mathfrak{D}_n)$$

Let (P', Ψ') be the reflex of (P, Ψ). Put $\eta(x) = f(\det \Psi'(x))$ for $x \in P'$. It can be shown that $F' \subset P'$, η maps $P'^\times$ into G, and $\nu \circ \eta = \lambda \circ N_{P'/F'}$, hence $\eta(P'^\times_{\mathbf{A}}) \subset \mathfrak{G}_+$. Now the property (4) in this special case is as follows : *For each $S \in \mathfrak{Z}$, the point $\varphi_S(z)$ is rational over P'_{ab}, and for every $\nu \in P'^\times_{\mathbf{A}}$ and $T = \eta(\nu) S \eta(\nu)^{-1}$ one has*

$$\text{(4a)} \qquad \varphi_S(z)^{[\nu]} = J_{ST}(\eta(\nu)^{-1})(\varphi_T(z)),$$

where $[\nu]$ denotes the element of $\mathrm{Gal}(P'_{ab}/P')$ corresponding to ν.

4. – Let K_S denote the field of all functions (in the sense of algebraic geometry) on V_S rational over k_S. Define a field $\mathfrak{K}$ of meromorphic functions on $\mathfrak{H}_n^r$ by

$$\mathfrak{K} = \bigcup_{S \in \mathfrak{Z}} \mathfrak{K}_S, \qquad \mathfrak{K}_S = \{f \circ \varphi_S \mid f \in K_S\}.$$

The elements of $\mathfrak{K}$ may be called *arithmetic automorphic functions with respect to G.* Denote by $\mathrm{Aut}(\mathfrak{K}/\mathfrak{F})$ the group of all automorphisms of $\mathfrak{K}$ over any subfield $\mathfrak{F}$, and make it a topological group by taking as a basis of neighborhoods of the identity the subgroups $\mathrm{Aut}(\mathfrak{K}/\mathfrak{L})$ for all subfields $\mathfrak{L}$ finitely generated over $\mathfrak{F}$. For each $x \in \overline{\mathfrak{G}}_+$, we can define an element $\tau(x)$ of $\mathrm{Aut}(\mathfrak{K}/F')$ by the rule

$$(f \circ \varphi_S)^{\tau(x)} = f^{\sigma(x)} \circ J_{ST}(x) \circ \varphi_T \qquad (T = x^{-1} S x, f \in K_S).$$

THEOREM 2. – *The map $x \to \tau(x)$ is a continuous homomorphism of $\overline{\mathfrak{G}}_+$ into* $\mathrm{Aut}(\mathfrak{K}/F')$, *and has the following properties :*

(1) *The kernel of τ is $F_c \cdot G_{\infty+}$.*

(2) *$\tau(x) = \sigma(x)$ on $\mathfrak{k}$.*

(3) *$h^{\tau(\alpha)} = h \circ \alpha$ for every $\alpha \in G_{\mathbf{Q}+}$ and $h \in \mathfrak{K}$*

(4) *Let P, f, P', z, and η be as above (assuming $r = 1$). Then, for every $h \in \mathfrak{K}$ defined at z, $h(z)$ is rational over P'_{ab}. Moreover, if $\nu \in P'^\times_{\mathbf{A}}$ and $u = \eta(\nu)^{-1}$, then $h^{\tau(u)}$ is also defined at z, and $h(z)^{[\nu]} = h^{\tau(u)}(z)$.*

The last property (or (4a) of Th.1) is a generalization of the wellknown behavior of the singular values of the classical modular function j.

THEOREM 3. – *The map τ induces a topological isomorphism of $\overline{\mathfrak{G}}_+/F_c G_{\infty+}$ onto an open subgroup of* $\mathrm{Aut}(\mathfrak{K}/F')$. *For every $S \in \mathfrak{Z}$, $\mathfrak{K}$ is an infinite Galois* *extension of* $\mathfrak{K}_S$, *and* $\tau(S) = \mathrm{Gal}(\mathfrak{K}/\mathfrak{K}_S)$. *Moreover, if* $r = 1$ *and* $\mathfrak{H}_n/\Gamma_S$ *for any* $S \in \mathfrak{Z}$ *is compact, or* $G_{\mathbf{Q}} = GL_2(\mathbf{Q})$, *then* τ *is surjective.*

One can conjecture that τ is surjective so long as $r = 1$. If $r > 1$, $\mathrm{Aut}(\mathfrak{K}/F')$ may be larger than $\tau(\mathfrak{G}_+)$, since the "permutations" of the factors of $\mathfrak{H}_n^r$ may give rise to automorphisms of $\mathfrak{K}$, see [8, II].

We notice that the relation (4) of Th. 2 explains the deep arithmetic meaning of the map τ, exactly similar to the fact that the canonical isomorphism of $F_{\mathbf{A}}^{\times}/F_c$ onto $\mathrm{Gal}(F_{ab}/F)$, for any number field F, is defined locally by the Frobenius automorphisms. Thus the above two theorems provide an analogue of class field theory for the field $\mathfrak{K}$ which is of Kroneckerian dimension > 1.

The construction of V_S is based on the existence of families of abelian varieties $\Sigma = \{A_z \mid z \in \mathfrak{H}_n^r\}$ with the property that A_z and A_w (endowed with structures of polarization and endomorphisms) are isomorphic if and only if $z = \gamma(w)$ for some $\gamma \in \Gamma_S$. We shall not discuss further the ideas of the proofs beyond this remark, since they are explained rather in detail in [4] and the Introduction of [5].

5. – As is already mentioned, Miyake [3] has obtained the results completely parallel to Theorems 1, 2, 3 for the groups of type II by the same method. One can naturally propose the generalization to a wider class of groups. Since there are families of abelian varieties, similar to the above Σ, for each of the groups belonging to the remaining types III and IV, it is quite certain that one will be able to construct, without much difficulties, a system like that of Th. 1 for a suitably chosen reductive group G' containing the given semi-simple group of any type, excluding exceptional ones. We call such a system a *canonical system for* G', and can define *arithmetic automorphic functions with respect to* G' in the same manner as above. Now there is an important aspect of this framework : *canonical systems for two groups are not only defined and characterized individually, but also consistent with each other.* To be more specific, let

$$\{V_S', \varphi_S', J_{TS}'(x), k_S'\} \quad \text{and} \quad \{V_L'', \varphi_L'', J_{ML}''(x), k_L''\}$$

be canonical systems for G' and G'', respectively, and f a $\mathbf{Q}$-rational homomorphism of G' into G''. Suppose that f defines a holomorphic embedding ϵ of the corresponding bounded domain $\mathfrak{H}'$ into $\mathfrak{H}''$. If $f(S) \subset L$, we have $f(\Gamma_S) \subset \Gamma_L$, so that there is a morphism E_{LS} of V_S' into V_L'' such that $E_{LS} \circ \varphi_S' = \varphi_L'' \circ \epsilon$. In some cases we can prove : (i) $k_L'' \subset k_S'$; (ii) E_{LS} *is rational over* k_S' ; (iii) *If further* $f(T) \subset M$, $x \in G_{\mathbf{A}}'$, $xSx^{-1} \subset T$, $f(x)Lf(x)^{-1} \subset M$, *then*

$$E_{MT}^{\sigma'(x)} \circ J_{TS}'(x) = J_{ML}''(f(x)) \circ E_{LS} \quad ,$$

provided that $J_{TS}'(x)$ *and* $J_{ML}''(f(x))$ *are defined, where* σ' *is the symbol defined for* G' *corresponding to* σ. If $G' = G''$, and f is the identity map, this means the unicity of the canonical system. We may consider (4, 4a) of Th. 1 as an example of such a relation. ($\mathfrak{H}'$ is a single point in this case). More examples, in which G' and G'' are of type I, are given in [8, I, § 8]. The consistency of this kind can also be formulated in terms of $\mathrm{Aut}(\mathfrak{K}/F')$ and τ . Let $\mathfrak{K}'$ τ', $\mathfrak{K}''$, τ'' be the corresponding symbols for G' and G'', respectively. If an element h of $\mathfrak{K}''$ is holomorphic at some point of $\epsilon(\mathfrak{H}')$, then $h \circ \epsilon \in \mathfrak{K}'$, and $h^{\tau''(f(x))} \circ \epsilon = (h \circ \epsilon)^{\tau'(x)}$ for every $x \in G'_{\mathbf{A}}$ such that $\tau'(x)$ and $\tau''(f(x))$ are meaningful. Again (4) of Th. 2 is a special case.

Because of the restriction on the length, I have to give up the discussion of two important topics related to the above theory, namely :

(A) *Construction of a system of varieties over finite fields,* as Ihara [2] has given in the one-dimensional case. The reader is referred to his lecture and [8, I, § 2.8].

(B) *Local representations of the Galois group of an infinite extension of a number field obtained from the points of the coverings lying on an algebraic point of V_S.* The characteristic roots of the Frobenius automorphisms, defined through these representations, have absolute values of the Riemann-Ramanujan-Weil type. Thus they provide generalizations of l-adic representations of abelian varieties. See [7] and [8, I, §§7, 8].

REFERENCES

[1] Baily W.L. and Borel A. — Compactification of arithmetic quotients of bounded symmetric domains, *Ann. of Math.*, 84, 1966, p. 442-528.

[2] Ihara Y. — On Congruence Monodromy Problems, *lecture notes,* University of Tokyo, I, 1968; II, 1969.

[3] Miyake K. — *On models of certain automorphic function fields,* to appear.

[4] Shimura G. — Number fields and zeta functions associated with discontinuous groups and algebraic varieties, *Proc. Int. Congress of Math. Moscow,* 1966, p. 290-298.

[5] Shimura G. — Construction of class fields and zeta functions of algebraic curves, *Ann. of Math.*, 85, 1967, p. 58-159.

[6] Shimura G. — Algebraic number fields and symplectic discontinuous groups, *Ann. of Math.*, 86, 1967, p. 503-592.

[7] Shimura G. — Local representations of Galois groups, *Ann. of Math.*, 89, 1969, p. 99-124.

[8] Shimura G. — On canonical models of arithmetic quotients of bounded symmetric domains, I, *Ann. of Math.*, 91, 1970, p. 144-222; II, 92, 1970, p. 528-549.

[9] Weil A. — Algebras with involutions and the classical groups, *J. Ind. Math. Soc.*, 24, 1960, p. 589-623.

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HOMOMORPHISMES ET AUTOMORPHISMES "ABSTRAITS" DE GROUPES ALGÉBRIQUES ET ARITHMÉTIQUES

par J. TITS

1. – En 1928, O. Schreier et B.L. van der Waerden ont déterminé tous les automorphismes du groupe projectif spécial (ou "unimodulaire") $PSL_n(K)$ sur un corps commutatif K, et ont montré qu'à deux exceptions près ($PSL_2(\mathrm{F}_4) \cong PSL_2(\mathrm{F}_5)$ et $PSL_2(\mathrm{F}_7) \cong PSL_3(\mathrm{F}_2)$) l'isomorphisme de $PSL_n(K)$ et $PSL_{n'}(K')$ entraîne $n = n'$ et $K = K'$. Par la suite, de nombreux travaux ont été consacrés, par J. Dieudonné et d'autres, à l'étude des automorphismes de groupes classiques et des isomorphismes entre groupes classiques de "provenance" différente. Le chapitre IV de [8] donnait l'état de la question en 1963. Depuis lors, de nouveaux progrès ont été réalisés et le champ des recherches a été élargi, notamment par l'introduction d'anneaux de base plus généraux (au lieu des corps), le passage des groupes classiques aux groupes algébriques simples (et, corrélativement, la recherche de démonstrations "générales", ne faisant pas intervenir la classification) et la prise en considération d'homomorphismes non nécessairement surjectifs. Le présent exposé vise à donner une vue d'ensemble des principaux résultats obtenus dans ces divers domaines.

2. – Les problèmes envisagés ici peuvent aussi être vus dans la perspective d'un résultat de H. Freudenthal qui a montré, en 1941 [11], que la topologie d'un groupe de Lie absolument simple (c'est-à-dire simple et dont l'algèbre de Lie reste simple lorsqu'on la tensorise par C) est entièrement déterminée par sa structure de groupe "abstrait" (pour le cas compact, voir aussi [5], [29]). La question analogue qui se pose naturellement dans le cas d'un groupe algébrique $\mathcal{G}$, disons sur un corps K, est de savoir si le groupe $\mathcal{G}(K)$ de ses points rationnels sur K "porte en lui la donnée du corps K et de la structure d'ensemble algébrique de $\mathcal{G}$", c'est-à-dire si tout isomorphisme (éventuellement soumis à certaines restrictions) $\alpha : \mathcal{G}(K) \to \mathcal{G}'(K')$ de $\mathcal{G}(K)$ sur un groupe de même nature est "induit par un isomorphisme $\sigma : K \to K'$ et par un K-isomorphisme de groupes algébriques $\mathcal{G} \to \mathcal{G}'$, une fois K' identifié à K par σ".

3. – Précisons et généralisons ce dernier énoncé. Soient K, K' deux corps commutatifs, ${}_K\mathcal{G}$ (resp. ${}_{K'}\mathcal{G}'$) un schéma en groupes sur K (resp. K') et H (resp. H') un sous-groupe de ${}_K\mathcal{G}(K)$ (resp. ${}_{K'}\mathcal{G}'(K')$). Pour tout homomorphisme de corps $\sigma : K \to K'$, notons ${}^{\sigma}{}_K\mathcal{G}$ le schéma en groupes sur K' déduit de ${}_K\mathcal{G}$ par le changement de base σ et σ_* l'homomorphisme canonique ${}_K\mathcal{G}(K) \to {}^{\sigma}{}_K\mathcal{G}(K')$. Nous

dirons, par abus de langage, qu'un homomorphisme $\alpha : H \to H'$ est *semi-algébrique* s'il existe σ et un K'-homomorphisme $\varphi : {}^{\sigma}{}_{K}\mathcal{G} \to {}_{K'}\mathcal{G}'$ tel que

$$\alpha = \varphi(K') \circ \sigma_{*}|_{H} \tag{1}$$

(cette notion est évidemment relative à la donnée de K, K', ${}_{K}\mathcal{G}$, ${}_{K'}\mathcal{G}'$ et des inclusions $H \subset {}_{K}\mathcal{G}(K)$ et $H' \subset {}_{K'}\mathcal{G}(K')$). Le contenu principal de presque tous les résultats qui font l'objet de cet exposé répond au schéma suivant : on considère certains types de schémas en groupes semi-simples (v. à ce sujet le n° 11) et certains types de sous-groupes H, H' et on montre que, pour tout homomorphisme $\beta : H \to H'$ soumis éventuellement à certaines conditions,

(*) *il existe un unique homomorphisme semi-algébrique $\alpha : H \to H'$ et un unique homomorphisme χ de H dans le centre de H' tels que $\beta(h) = \alpha(h) . \chi(h)$ pour tout $h \in H$.*

Cette assertion, lorsqu'elle est vraie, ne résoud pas entièrement le problème de la détermination des homomorphismes "abstraits" β, puisqu'il faut encore, notamment, déterminer pour tout $\sigma : K \to K'$ les homomorphismes algébriques $\varphi : {}^{\sigma}{}_{K}\mathcal{G} \to {}_{K'}\mathcal{G}'$ et rechercher parmi eux ceux pour lesquels l'homomorphisme α défini par (1) envoie H dans H'. Ces questions, qui ne sont pas toujours faciles, ont été résolues de façon assez satisfaisante dans la plupart des cas où on a pu établir (*). Toutefois, nous laisserons de côté cet aspect des choses, de même que certains résultats annexes, concernant par exemple les précisions que l'on peut apporter sur α et χ lorsqu'on suppose que β est un isomorphisme, ou encore la question de savoir dans quelle mesure α détermine σ et φ. A propos de cette dernière, remarquons seulement que si K' est un corps parfait de caractéristique $p \neq 0$, α ne change pas lorsqu'on remplace σ et φ respectivement par l'homomorphisme $\sigma' : x \to \sigma(x)^{1/p}$ et par $\varphi \circ \mathrm{Fr}$ où $\mathrm{Fr} : {}^{\sigma'}{}_{K}\mathcal{G} \to {}^{\sigma}{}_{K}\mathcal{G}$ désigne le morphisme de Frobenius.

4. – Les travaux dont il est rendu compte dans les n^{os} 5 à 7 ont trait à des *anneaux de base* : on se donne au départ des anneaux intègres R, R', ayant K, K' pour corps des quotients, et des schémas en groupes $\mathcal{G}$, $\mathcal{G}'$ sur R, R' qui deviennent ${}_{K}\mathcal{G}$, ${}_{K'}\mathcal{G}'$ par extension des scalaires, et on suppose que $H \subset \mathcal{G}(R)$, $H' \subset \mathcal{G}'(R')$ (sauf toutefois dans la situation envisagée en 7.1). Disons alors qu'un homomorphisme $\alpha : H \to H'$ est *semi-entier* s'il existe $\sigma : R \to R'$ et $\varphi : {}^{\sigma}\mathcal{G} \to \mathcal{G}'$ tels qu'on ait (1). Dans certains cas, par exemple ceux considérés ci-dessous en 5 (cf. [13], [14]) et en 7.2 (cf. [2], théorème 4.3), on peut préciser l'assertion (*) en y remplaçant "semi-algébrique" par "semi-entier".

Anneaux intègres quelconques.

5. – Soit $R = R'$ un anneau intègre quelconque. Soit $\mathcal{G} = \mathcal{G}'$ le (schéma en) groupe (s) linéaire général $\mathcal{GL}$ ou spécial $\mathcal{SL}$ d'un R-module libre de rang fini $\geqslant 3$, ou le groupe symplectique $\mathcal{S}p$ d'un module libre de rang pair fini $\geqslant 4$ muni d'une forme alternée *standard*, et soit $H = H'$ le groupe $\mathcal{G}(R)$ lui-même ou le sous-groupe $T\mathcal{G}(R)$ engendré par les transvections. Alors ([13], [14], [17]), *tout automorphisme β de H possède la propriété* (*).

Le rapporteur a été informé de ce que O.T. O'Meara et son école ont généralisé ce résultat dans diverses directions : substitution de modules "bornés" (sous-modules de modules libres de type fini) quelconques aux modules libres (pour les groupes $\mathfrak{GL}$ et $\mathfrak{SL}$ d'un tel module sur un anneau de Dedekind, cf. aussi [13]) ; extension à d'autres schémas en groupes $\mathfrak{G}$ ($\mathfrak{PGL}, \mathfrak{PSL}, \mathfrak{PSp}$) et à des sous-groupes H de congruence (pour le cas d'un anneau de Dedekind, cf. aussi [18]) ; pour certains types de groupes, généralisation à des isomorphismes (au lieu d'automorphismes).

Anneaux locaux et anneaux arithmétiques.

6. – Soient R un anneau intègre de caractéristique $\neq 2$ et K son corps des quotients. Supposons remplie l'une des conditions suivantes :

(G) K est un corps global (extension finie de $\mathbf{Q}$ ou d'un corps $\mathbf{F}_p(t)$), S un ensemble fini de places de K contenant toutes les places à l'infini (places archimédiennes) et R l'intersection des anneaux de valuation des places n'appartenant pas à S ;

(L) R est un anneau local.

Soit M un R-module "borné" (cf. n° 5) de rang $n \geqslant 7$, $n \neq 8$, muni d'une forme quadratique Q non-dégénérée (i.e. restriction d'une forme quadratique non dégénérée sur l'espace vectoriel $M \otimes K$). Soient $\mathfrak{G}$ le "groupe orthogonal" $\mathfrak{O}(M, Q)$ (1) et H un sous-groupe de $\mathfrak{G}(R)$ contenant -1 et un groupe de congruence (groupe des éléments de $\mathfrak{G}(R)$ congrus à 1 modulo un idéal donné non nul de R). Alors, le principal résultat de [16] est que, *sous certaines hypothèses supplémentaires, toujours remplies par exemple dans le cas (L), ou dans le cas (G) si K est un corps de nombres algébriques qui n'est pas totalement réel, tout automorphisme β de H possède la propriété* (*).

7. – Dans ce numéro, où sont décrits quelques-uns des résultats de [2], $K = K'$ est une extension algébrique finie de $\mathbf{Q}$, S, R ont la même signification qu'en 6 (G), $R' = R, \mathfrak{G}$, $\mathfrak{G}'$ sont des R-schémas en groupes semi-simples connexes, presque simples sur K, et r désigne le K-rang de $\mathfrak{G}$. En 7.1 et 7.2, on suppose $\mathfrak{G}$ simplement connexe ou $\mathfrak{G}'$ adjoint.

7.1. Soient $K = \mathbf{Q}$, $R = \mathbf{Z}$ et $\mathfrak{G}(\mathbf{Z}) \subset H \subset \mathfrak{G}(\mathbf{Q})$. Alors, *si $r \geqslant 2$, tout homomorphisme $\beta : H \to H' = \mathfrak{G}'(\mathbf{Q})$ tel que $\beta(\mathfrak{G}(\mathbf{Z}))$ soit Zariski-dense dans $\mathfrak{G}'$ possède la propriété* (*).

7.2. Soit $\mathfrak{G} = \mathfrak{G}'$ un schéma de Chevalley. Supposons $r \geqslant 2$ ou Card $S \geqslant 2$. Alors, *tout automorphisme β de $H = \mathfrak{G}(R)$ possède la propriété* (*). (Le théorème 4.3 de [2] donne une description plus précise de ces automorphismes).

7.3. Soient $R = \mathbf{Z}$, $K = \mathbf{Q}$. *Si l'algèbre de Lie de $\mathfrak{G}(\mathbf{R})$ n'a pas de facteur isomorphe à* $\mathfrak{sl}(2, \mathbf{R})$, *le groupe* Aut $\mathfrak{G}(\mathbf{Z})/$Int $\mathfrak{G}(\mathbf{Z})$ *est fini.* (Cf. le théorème 1.5 (ii) de [2], qui est d'ailleurs un peu plus général que ceci).

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(1) Dans le cas (L), $\mathfrak{G}$ n'est pas nécessairement un schéma en groupe, mais il est toujours un R-foncteur-groupe au sens de [9].

Corps

8. – Soient K un corps de caractéristique $\neq 2$, non isomorphe à $\mathbf{F}_3$, V un espace vectoriel sur K, de dimension $n \geqslant 7$, $n \neq 8$, muni d'une forme quadratique Q non dégénérée, $\mathcal{G}$ le "groupe orthogonal" $\mathcal{O}(Q)$ et $H \subset \mathcal{G}(K)$ un sous-groupe qui est soit le noyau de la norme spinorielle (dans $\mathcal{O}^+(Q, K)$) soit le groupe dérivé de $\mathcal{G}(K)$. Alors [15], *tout automorphisme β de H possède la propriété* (*). (Pour d'autres résultats, plus anciens, concernant les groupes classiques et, en particulier, les groupes orthogonaux $\mathcal{G}(K)$ eux-mêmes, cf. [8]).

9. – Dans ce numéro, K et K' désignent des corps, $\mathcal{G}$ (resp. $\mathcal{G}'$) un groupe algébrique absolument presque simple défini sur K (resp. K'), G^+ le sous-groupe de $\mathcal{G}(K)$ engendré par les sous-groupes de la forme $\mathcal{U}(K)$, où $\mathcal{U} \subset \mathcal{G}$ est K-isomorphe au groupe additif, et H un sous-groupe de $\mathcal{G}(K)$ contenant G^+. On suppose $\mathcal{G}$ simplement connexe ou $\mathcal{G}'$ adjoint.

9.1. *Tout homomorphisme $\beta : H \to \mathcal{G}'(K')$ tel que $\beta(G^+)$ soit Zariski-dense dans $\mathcal{G}'$ possède la propriété* (*) ([3], [4]). Remarquons que l'existence de β implique que K soit infini et que le groupe $\mathcal{G}$ soit isotrope sur K ($G^+ \neq \{1\}$) ; en particulier, ce résultat ne redonne pas celui de E. Cartan [5] et B.L. van der Waerden [29] sur les groupes compacts (cf. n° 2), ni ceux qui concernent les groupes orthogonaux et unitaires (cf. par exemple le n° 8) lorsque les formes quadratiques et hermitiennes considérées sont anisotropes.

9.2. Supposons K et K' finis, non isomorphes à $\mathbf{F}_2$ et $\mathbf{F}_3$, et soit $\beta : H \to \mathcal{G}(K')$ un homomorphisme tel que $\beta(H)$ contienne le groupe dérivé de $\mathcal{G}'(K')$. Alors, *β possède la propriété* (*) *ou bien K et K' sont respectivement isomorphes à* $\mathbf{F}_4$ *et* $\mathbf{F}_5$ *et* $\mathcal{G}$, $\mathcal{G}'$ *sont de type A_1 (isomorphes à $\mathcal{SL}_2$ ou $\mathcal{PSL}_2$)*. C'est une conséquence immédiate des résultats de [1] (dûment complétés : cf. par exemple [27], 4.5) et [24]. Lorsqu'on n'exclut pas $\mathbf{F}_2$ et $\mathbf{F}_3$, quelques autres exceptions, bien connues, viennent s'ajouter (cf. par exemple [1], ou [27], tableau 4).

Application : une généralisation du "théorème fondamental de la géométrie projective".

10. – Pour $i = 1, 2$, soient K_i un corps commutatif, $\mathcal{G}_i$ un groupe algébrique absolument simple adjoint défini sur K_i et de K_i-rang $\geqslant 2$ et P_i l'ensemble des K_i-sous-groupes paraboliques de $\mathcal{G}_i$ ordonnés par inclusion. Alors ([28], théorème 5.8), *pour tout isomorphisme d'ensembles ordonnés $\pi : P_1 \to P_2$, il existe un isomorphisme $\sigma : K_1 \to K_2$ et une isogénie $\varphi : {}^\sigma\mathcal{G}_1 \to \mathcal{G}_2$ induisant π ; l'isogénie φ est un isomorphisme sauf peut-être si K est parfait de caractéristique* 2 *et $\mathcal{G}_i$ est déployé de type B_n, C_n ou F_4, ou si K est parfait de caractéristique* 3 *et $\mathcal{G}_i$ est déployé de type G_2*. Ce résultat généralise aussi un théorème bien connu de W.L. Chow (cf. [8], chap. III, § 4).

Groupes non semi-simples : exemples.

11. – Tous les groupes algébriques considérés plus haut étaient supposés semi-simples. Les exemples suivants illustrent quelques uns des phénomènes qui se présentent lorsqu'on abandonne cette hypothèse. Comme au n° 9, $\mathfrak{G}$ (resp. $\mathfrak{G}'$) désigne un groupe algébrique défini sur le corps commutatif K (resp. K').

11.1. Soit $K = K' = \mathbf{R}$ et soit $\mathfrak{G} = \mathfrak{G}'$ le groupe additif. Le groupe $\mathfrak{G}(\mathbf{R})$ (qui est un espace vectoriel de dimension $2^{\aleph_0}$ sur $\mathbf{Q}$) possède $2^{2^{\aleph_0}}$ endomorphismes. Parmi eux, seuls les endomorphismes de la forme $x \to ax$ $(a \in \mathbf{R})$ sont semi-algébriques.

11.2. Soit $\mathfrak{G}$ (resp. $\mathfrak{G}'$) le groupe sur K (resp. K') produit semi-direct du groupe multiplicatif par le groupe additif pour l'opération usuelle du premier sur le second ("groupe des transformations affines $x \mapsto ax + b$"). Il est facile de voir que tout homomorphisme injectif de $\mathfrak{G}(K)$ dans $\mathfrak{G}'(K')$ est semi-algébrique.

11.3. Soit $K = K'$ et soit $\mathfrak{G} = \mathfrak{G}' = \mathfrak{GL}_n \cdot \mathfrak{M}_n$ le produit semi-direct du groupe $\mathfrak{GL}_n$ par le groupe (algébrique) additif des matrices carrées d'ordre n sur lequel $\mathfrak{GL}_n$ opère par conjugaison. On a donc, avec les notations usuelles, $\mathfrak{G}(K) = GL_n(K) . M_n(K)$. Soit d une dérivation de K. Alors, il est facile de vérifier que l'application $\beta : \mathfrak{G}(K) \to \mathfrak{G}(K)$ définie par

$$\beta(x , y) = (x , x^{-1} . \ dx + y)$$

est un homomorphisme qui ne possède la propriété (*) que si d est nul. Remarquons que si $K = \mathbf{R}$ et si $d \neq 0$, la restriction de β au sous-groupe $GL_n(\mathbf{R})$ est une "section de Levi non continue" du groupe de Lie $\mathfrak{G}(\mathbf{R})$.

BIBLIOGRAPHIE

Cette bibliographie, établie à l'aide d'une liste de références très complète obligeamment communiquée à l'auteur par O.T. O'Meara, porte essentiellement sur la période 1966-1970. Pour les travaux antérieurs nous renvoyons aux bibliographies de [8] (cf. essentiellement la littérature citée au chapitre IV) et [13], nous contentant ici de mentionner quelques articles qui n'y figurent pas.

[1] Artin E. — The orders of the classical simple groups, *Comm. Pure Appl. Math.*, 8, 1955, p. 455-472.
[2] Borel A. — On the automorphisms of certain subgroups of semi-simple Lie groups, *Proc. Bombay Colloquium on Algebraic Geometry*, 1968, p. 43-73.
[3] Borel A. and Tits J. — On « abstract » homomorphisms of simple algebraic groups, *Proc. Bombay Colloquium on Algebraic Geometry*, 1968, p. 75-82.
[4] Borel A. et Tits J. — Homomorphismes « abstraits » de groupes algébriques semi-simples, *Ann. of Math.*, à paraître.

[5] CARTAN E. — Sur les représentations linéaires des groupes clos, *Comm. Math. Helv.*, 2, 1930, p. 269-283
[6] COHN P.M. — On the structure of the GL_2 of a ring, *Inst. Hautes Etudes Sci. Publ. Math.* 30, 1966, p. 5-53.
[7] COHN P.M. — Automorphisms of two-dimensional linear groups over Euclidean domains, *J. London Math. Soc.* (2), 1, 1969, p. 279-292.
[8] DIEUDONNÉ J. — *La géométrie des groupes classiques,* Second Edition, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
[9] DEMAZURE M. et GABRIEL P. — *Groupes algébriques I,* Masson, Paris, 1970.
[10] DULL M.H. — On the automorphisms of the two-dimensional linear groups over integral domains, *University of Notre Dame thesis,* 1969.
[11] FREUDENTHAL H. — Die Topologie der Lieschen Gruppen als algebraisches Phänomen I, *Ann. of Math* (2) 42, 1941, p. 1051-1074. Erratum ibid. 47, 1946, p. 829-830.
[12] HUMPHREYS J.E. — On the automorphisms of infinite Chevalley groups, *Canadian J. Math.,* 21, 1969, p. 908-911.
[13] O.MEARA O.T. — The automorphisms of the linear groups over any integral domain, *J. reine angew. Math.* 223, 1966, p. 56-100.
[14] O'MEARA O.T. — The automorphisms of the standard symplectic group over any integral domain, *J. reine angew. Math.,* 230, 1968, p. 104-138.
[15] O'MEARA O.T. — The automorphisms of the orthogonal groups $\Omega_n(V)$ over fields, *Amer. J. Math* 90, 1968, p. 1260-1306.
[16] O'MEARA O.T. — The automorphisms of the orthogonal groups and their congruence subgroups over arithmetic domains, *J. reine angew. Math.* 238, 1969, p. 169-206.
[17] O'MEARA O.T. — Group-theoretic characterization of transvections using CDC, *Math. Zeit* 110, 1969, p. 385-394.
[18] O'MEARA O.T. and ZASSENHAUS H. — The automorphisms of the linear congruence groups over Dedekind domains, *J. Number Theory* 1, 1969, p. 211-221.
[19] REINER I. — Automorphisms of the symplectic modular group, *Trans. Amer. Math. Soc.,* 80, 1955, p. 35-50.
[20] SOLAZZI R.E. — The automorphisms of certain subgroups of $PGL_n(V)$, *University of Notre Dame thesis,* 1969.
[21] SOLAZZI R.E. — The automorphisms of the symplectic congruence groups, preprint.
[22] SPIEGEL E. — On the automorphisms of the unitary group over a field of characteristic 2, *Amer. J. Math.,* 89, 1967, p. 43-50.
[23] SPIEGEL E. — On the automorphisms of the projective unitary group over a field of characteristic 2, *Amer. J. Math.,* 89, 1967, p. 51-55.
[24] STEINBERG R. — Automorphisms of finite linear groups, *Canadian J. Math.*, 12, 1960, p. 606-615.
[25] STEINBERG R. — Representations of algebraic groups, *Nagoya Math. J.,* 22, 1963, p. 33-56.
[26] STEINBERG R. — Lectures on Chevalley groups, *Yale University lecture notes,* 1967.
[27] TITS J. — Groupes simples et géométries associées, *Proc. Intern. Congress Math.,* Stockholm, 1962, p. 197-221.
[28] TITS J. — Buildings of spherical types and finite BN-pairs of rank $\geqslant 3$, *Springer Lecture Notes,* à paraître.
[29] WAERDEN B.L. v. d. — Stetigkeitssätze für halb-einfache Liesche Gruppen, *Math. Zeit,* 36, 1933, p. 780-786.

[30] WONENBURGER M.J. — The automorphisms of the group of rotations and its projective group corresponding to quadratic forms of any index, *Canadian J. Math.*, 15, 1963, p. 302-303.
[31] WONENBURGER M.J. — The automorphisms of $U_n^+(k, f)$ and $PU_n^+(k, f)$, *Rev. Mat. Hisp.-Amer.*, (4), 24, 1964, p. 52-65.
[32] XU C.-H. — On the automorphisms of orthogonal groups over perfect fields of characteristic 2, *Chinese Math.*, 8, 1966, p. 475-523.
[33] ZASSENHAUS H. — Characterization of unipotent matrices, *J. Number Theory*, 1, 1969, p. 222-230.

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D

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D 1 - ESPACES VECTORIELS TOPOLOGIQUES

COMPACT CONVEX SETS

by D. A. EDWARDS

Although Choquet boundary theory was put into definitive shape in the early years of the past decade, the pace of development in convexity theory has since then never slackened. The past five years have seen particularly vigorous activity. I attempt here to survey a few of the main lines of thought of that period, though I am conscious that I scarcely have time to do justice to any of them.

1. Choquet simplexes

We consider a non-empty compact convex subset X of a Hausdorff locally convex real topological vector space, together with the space $A = A(X)$ of all real affine continuous functions on X. Under the natural partial ordering and the uniform norm, A is a partially ordered Banach space with closed positive cone. The evaluations at points of X are precisely the positive linear functionals on A that are of unit norm ; this remark embeds X affinely, and homeomorphically for the vague topology, into A^*.

A partially ordered vector space V has the *Riesz separation property* if whenever $u_1, u_2, v_1, v_2 \in V$ with

$$u_r \leqslant v_s \qquad (r = 1, 2 \; ; s = 1, 2)$$

there exists a $w \in V$ with

$$u_r \leqslant w \leqslant v_s \qquad (r = 1, 2 \; ; \; s = 1, 2).$$

It is well known that A has this property if and only if A^* is a vector lattice, and that A^* is in that case an AL-space in the sense of Kakutani. In these circumstances X is called a *Choquet simplex*. If $A(X)$ is a vector lattice then the Riesz condition is satisfied and $A(X)$ is an AM-space ; the simplexes which arise in this way are precisely [7] those for which the set X_e of extreme points of X is closed. Many characterizations of Choquet simplexes are known. A useful one, which requires the space $S(X)$ of all upper semicontinous convex maps of X into $[-\infty, \infty)$, is as follows :

THEOREM 1. – *The compact convex set X is a Choquet simplex if and only if, whenever $f, -g \in S(X)$ and $f \leqslant g$, there exists an $h \in A(X)$ such that $f \leqslant h \leqslant g$.*

(For this and some generalizations see [14, 8]). Theorem 1 has a large number of consequences, of which only some can be considered here :

(1) If Q is a G_δ closed face of a Choquet simplex X then Q is a peak set for $A(X)$ [11, 28]. In particular (Bauer's conjecture) if $x \in X_e$ and X is metrizable then x is a peak point (for a slightly weaker result and an application to potential theory see [25]).

(2) Suppose that E is a non-empty closed subset of the Choquet simplex X and is a union of closed faces of X. Let $A(E)$ denote the set of real continuous functions on E that are affine on convex subsets of E. Suppose that $h \in A(E)$, that $f, -g \in S(X)$ with $f \leqslant g$, and that $f|E \leqslant h \leqslant g|E$. Then there is an $\overline{h} \in A(X)$ that extends h and satisfies $f \leqslant \overline{h} \leqslant g$.

Some comments on this extension theorem should be made : (i) A somewhat weaker extension theorem, obtained by modifying the proof of Theorem 1, was used by Effros [18] in his structure theory for simplex spaces (see § 2). (ii) In the special case when E is a closed non-empty subset of X_e we have $A(E) = C(E)$. Choquet [10] has shown that for compact convex metrizable X this special case of the extension theorem holds only for Choquet simplexes ; Alfsen [3] has shown that this statement becomes false if the metrizability condition is dropped. (iii) If we take $E = Q_0 \cup Q_1$, where Q_0 , Q_1 are disjoint closed faces of X then it is clear that we can choose $h \in A(X)$ so that $0 \leqslant h \leqslant 1$ and $h|Q_0 = 0$, $h|Q_1 = 1$ [28, 29]. (iv) A *Lindenstrauss space* is a Banach space whose first dual is isometric to some AL-space. A partial generalization of the extension theorem to Lindenstrauss spaces has been obtained by Lazar and Lindenstrauss [31].

(3) Let X be a Choquet simplex and let $f : X_e \to \underset{\sim}{R}$ be a bounded continuous function. Define $f^* : X \to \underset{\sim}{R}$ by

$$f^*(x) = \inf \{g(x) : g \in A(X), g|X_e \geqslant f\}$$

and let $f_* = -(-f)^*$. Then there exists an $\overline{f} \in A(X)$ that extends f if and only if the restrictions of f^*, f_* to $\overline{X}_e$ are continuous. (This is the simplest case of Alfsen's "Dirichlet problem" for a compact convex set, on which see [7, 1, 30, 19, 32].

For basic simplex theory and a full discussion of Theorem 1 see [35, 37].

2. Ideal theory for simplexes

With the aid of a special case of the extension theorem of § 1 Effros [18, 19, 21] has developed a substantial structure theory for a Choquet simplex X which relates, by duality, certain order ideals of $A(X)$ with the closed faces of X. Only the most basic results of his theory can be described here.

If $F \subseteq X$ then, by definition,

$$F_\perp = \{f \in A(X) : f(x) = 0, \forall\, x \in F\} ;$$

if $J \subseteq A(X)$ then, by definition,

$$J^\perp = \{x \in X : f(x) = 0, \forall\, x \in J\}.$$

An *ideal* of $A = A(X)$ is a positively generated order ideal. The set of closed ideals is denoted by $\mathfrak{J}$, the set of closed faces of X by $\mathfrak{F}$. The basic duality theorem is :

THEOREM 2. – *If X is a Choquet simplex then the two maps $J \to J^{\perp}$ ($J \in \mathfrak{J}$ and $F \to F_{\perp}$ ($F \in \mathfrak{F}$) are mutually inverse bijections between $\mathfrak{J}$ and $\mathfrak{F}$.*

It is easy to prove [18, 28] that the convex hull of two closed faces of a Choquet simplex is again a face. From this it is obvious that the family

$$\{Q \cap X_e : Q \in \mathfrak{F}\}$$

satisfies the axioms for the closed sets of a topology on X_e. This is the *structure topology*, and X_e with this topology is denoted by max A. For part (i) of the following result see [18] ; for part (ii) see [23], where applications of both parts to potential theory can be found. We first recall that a topological space is *anti-Hausdorff* if no two disjoint non-empty open sets exist, and that a partially ordered vector space V is an *antilattice* if the existence of $u \vee v$ implies that u , v are comparable ($u \leqslant v$ or $u \geqslant v$).

THEOREM 3. – *Let X be a Choquet simplex. Then* (i) max $A(X)$ *is Hausdorff if and only if $A(X)$ is a vector lattice* ; (ii) max $A(X)$ *is anti-Hausdorff if and only if $A(X)$ is an antilattice.*

Effros coined the term *simplex space* for any partially ordered Banach space V having closed positive cone and such that V^* is an AL-space. Under the conditions of Theorem 2, A, J and A/J are simplex spaces ; the structure theory extends to them, and max J and max A/J have been computed [18].

Davies and Azimow (independently) gave the first intrinsic characterization of simplex spaces. Davies [12] later proved more :

THEOREM 4. – *Let V be a partially ordered Banach space with closed positive cone. Then V^* is a Banach lattice if and only if*

(i) *for all $x \in V$, $||x|| = \inf \{||y|| : -y \leqslant x \leqslant y\}$,*

(ii) *V satisfies the Riesz separation condition.*

V is a simplex space if and only if it satisfies (i), (ii), and

(iii) *for all $x , y \geqslant 0$ in V we can find $z \in V$ such that $z \geqslant x , y$ and*

$$||z|| \leqslant \max (||x|| , ||y||).$$

Davies [12] extended a substantial part of the Effros structure theory to spaces V satisfying the conditions of the first sentence of Theorem 4. In another direction Effros [20] has very recently extended much of the structure theory to Lindenstrauss spaces. Further theorems on the structure topology of a simplex have been obtained by Gleit [26] and Taylor [40].

3. Uniform approximation

The following theorem [17, 16] extends to $A(X)$ the well-known Kakutani-Stone approximation theorem for function lattices.

THEOREM 5. – *Let X be a compact non-empty convex set and let L be a linear subspace of $A(X)$ such that for each $k \in A(X)$ the set $\{g \in L : g > k\}$ is downward filtering. Then for each $f \in A(X)$ we have $f \in L$ if and only if whenever $\epsilon > 0$ and $x_1, x_2 \in X_e$ we can find $g \in L$ such that*

$$|g(x_r) - f(x_r)| < \epsilon \ (r = 1,2).$$

We have adopted here the convention that the empty set is downward filtering.

If, under the conditions of Theorem 5, L contains the constant functions and if $x \in X_e$ then

$$Q(x) \equiv \{y \in X : f(y) = f(x), \forall f \in L\}$$

is a face of X ; such we call *L-faces.*

Corollary 1. — *Let L satisfy the conditions of Theorem 5 and contain the constant functions. Then* $\overline{L}$ *consists of those functions in* $A(X)$ *that are constant on the L-faces of X.*

Corollary 2. — *Let* Ω *be a compact Hausdorff space and let L, M be linear subspaces of* $C(\Omega)$ *that contain the constant functions and are such that* $L \subseteq M$ *and M separates the points of* Ω. *Then L is dense in M if and only if*

(i) $\{g \in L : g > f\}$ *is downward filtering for each* $f \in M$ *;*

(ii) *L separates the points of the Choquet boundary of* Ω *relative to M.*

From Theorem 5 one can deduce approximation theorems concerning simplex spaces [17, 16]. One such is as follows.

Theorem 6. — *Let X be a compact convex set and let L be a linear subspace of* $A(X)$ *that contains the constant functions, satisfies the Riesz separation condition and is such that for each* $x \in X_e$ *the family*

$$\{g \in L : g(x) = 0\}$$

is upward filtering. Then L satisfies the conditions of Corollary 1 and so $\overline{L}$ *consists of the functions of* $A(X)$ *that are constant on the L-faces.*

Vincent-Smith [41] has shown that Theorem 6 implies a density theorem of Keldych in potential theory. By sharpening Theorem 6 he has been able to extend the density theorem to certain axiomatic potential theories (see also [34]).

Theorems 1 and 5 together provide a quick proof that the closure of an ideal in a simplex space is again an ideal [18]. Theorem 6 has also been used by Fakhoury to characterize the elements of the largest linear subspace (called the ideal centre) of a simplex space that is a vector lattice (see [24, 36]).

4. Other developments

It has not been possible to treat here Archimedean ideals, ideal centres, split faces, nor the applications to C^*-algebras [22, 2, 39, 4, 43]. Other important omissions include Lazar's work on selection theorems [29], the recent applications of split face theory to function algebras [6, 9, 5], and the theory of tensor products and projective limits [13, 27, 30, 33]. Finally, mention ought to be made of Vincent-Smith's [42] generalization to infinite dimensions of Carathéodory's theorem, and of subsequent work on this problem [38, 16].

REFERENCES

[1] Alfsen E.M. — On the Dirichlet problem of the Choquet boundary, *Acta Math.*, 120, 1968, p. 149-159.
[2] Alfsen E.M. — Facial structure of compact convex sets, *Proc. Lond. Math. Soc.*, 18, 1968, p. 385-404.
[3] Alfsen E.M. — *Convex compact sets and boundary integrals*, Springer-Verlag, Berlin (to appear).
[4] Alfsen E.M. and Andersen T.B. — Split faces of compact convex sets, *Aarhus Preprints*, No. 32, 1968-69.
[5] Alfsen E.M. and Hirsberg B. — On dominated extensions in linear subspaces $\mathcal{C}_C(X)$, *Oslo Preprints*, No. 1, 1970.
[6] Anderson T.B. — *On dominated extensions of continuous affine functions on split faces* (to appear).
[7] Bauer H. — Šilovscher Rand und Dirichletsches Problem, *Ann. Inst. Fourier* (Grenoble), 11, 1968, p. 89-136.
[8] Boboc N. and Cornea A. — Convex cones of lower semicontinuous functions on compact spaces, *Rev. Roum. Math. Pures et Appl.*, 12, 1967, p. 471-525.
[9] Briem E. — Restrictions of subspaces of C(X), dittoed notes, *Aarhus*, 1970.
[10] Choquet G. — *Lectures on Analysis I-III*, New York, 1969.
[11] Davies E.B. — A generalized theory of convexity, *Proc. Lond. Math. Soc.*, 17, 1967, p. 644-652.
[12] Davies E.B. — The structure and ideal theory of the pre-dual of a Banach lattice, *Trans. Amer. Math. Soc.*, 131, 1968, p. 544-555.
[13] Davies E.B. and Vincent-Smith G.F. — Tensor products, infinite products, and projective limits of Choquet simplexes, *Math. Scand.* 22, 1968, p. 145-164.
[14] Edwards D.A. — Minimum-stable wedges of semicontinuous functions, *Math. Scand.*, 19, 1966, p. 15-26.
[15] Edwards D.A. — On uniform approximation of affine functions on a compact convex set, *Oxford Quart. J.*, 20, 1969, p. 139-42.
[16] Edwards D.A. — An extension of Choquet boundary theory to certain partially ordered compact convex sets, *Studia Math.*, 36, 1970, p. 177-193.
[17] Edwards D.A. and Vincet-Smith G.F. — A Weierstrass-Stone theorem for Choquet simplexes, *Ann. Inst. Fourier* (Grenoble), 18, 1968, p. 261-282.
[18] Effros E.G. — Structure in simplexes, *Acta Mathematica*, 117, 1967, p. 103-121.
[19] Effros E.G. — Structure in simplexes II, *J. Functional Analysis*, 1, 1967, p. 379-391.
[20] Effros E.G. — *On a class of real Banach spaces* (to appear).
[21] Effros E.G. and Gleit A. — *Structure in simplexes III* (to appear).
[22] Effros E.G. and Hahn F. — Locally compact transformation groups and C*-algebras, *American Mathematical Society Memoirs*, 75, Providence 1967.
[23] Effros E.G. and Kazdan J.L. — *Application of Choquet simplexes to elliptic and parabolic boundary value problems* (to appear).
[24] Fakhoury H. — Espaces fortement réticulés de fonctions affines. Propriétés des simplexes dont l'ensemble des points extremaux est $\mathcal{K}$-analytique. *Thèse*, Paris 1970.
[25] Gamelin T.W. and Rossi H. — Jensen measures and algebras of analytic functions, *Proc. Tulane Symposium on Function Algebras* 1965, p. 15-35, Chicago 1966.
[26] Gleit A.S. — Topics in simplex space theory, *Thesis*, Stanford 1969.

[27] JELLETT F. — Homomorphisms and inverse limits of Choquet simplexes, *Math. Zeitschr.*, 103, 1968, p. 219-226.

[28] LAZAR A.J. — On affine continuous functions on simplexes, *Thesis,* Jerusalem 1965.

[29] LAZAR A.J. — Spaces of affine continuous functions on simplexes, *Trans. Amer. Math. Soc.*, 134, 1968, p. 503-525.

[30] LAZAR A.J. — Affine products of simplexes, *Math. Scand.*, 22, 1968, p. 165-172.

[31] LAZAR A.J. and LINDENSTRAUSS J. — *Banach spaces whose duals are L_1 spaces and their representing matrices* (to appear).

[32] MCDONALD J.N. — *Maximal measures and abstract Dirichlet problems* (to appear).

[33] NAMIOKA I. and PHELPS R.R. — Tensor products of compact convex sets, *Pacific J. Math.*, 31, 1969, p. 469-480.

[34] PRADELLE A. DE LA — A propos du mémoire de Vincent-Smith sur l'approximation des fonctions harmoniques, *Ann. Inst. Fourier* (Grenoble), 19, 1970, p. 355-370.

[35] ROGALSKI M. — Espaces de Banach ordonnés, simplexes, frontières de Silov et problème de Dirichlet, *Séminaire Choquet* (Paris) 5e année, 1965-66, No. 12.

[36] ROGALSKI M. — Etude du quotient d'un simplexe par une face fermée et application à la théorème de Alfsen; quotient par une relation d'équivalence, *Séminaire Brelot-Choquet-Deny,* 12e année, 1967-68, No. 2.

[37] RUGY A., GOULLET DE. — *Géométrie des simplexes,* Centre de Documentation Universitaire, Paris, 1968.

[38] SKAU C. Fr. — Existence of simplicial boundary measures on compact convex sets, dittoed notes, *Aarhus,* 1969.

[39] STØRMER E. — On partially ordered vector spaces and their duals, with applications to simplexes and C^*-algebras, *Proc. Lond. Math. Soc.*, 18, 1968, p. 245-65.

[40] TAYLOR P.D. — The structure space of a Choquet simplex, *Thesis,* Harvard 1969.

[41] VINCENT-SMITH G.F. — Uniform approximation of harmonic functions, *Ann. Inst. Fourier* (Grenoble), 19, 1970, p. 339-353.

[42] VINCENT-SMITH G.F. — *An extension of Carathéodory's theorem to infinite dimensions using Choquet boundary theory* (to appear).

[43] WILS W. — *The ideal centre of partially ordered vector spaces* (to appear).

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Actes, Congrès intern. Math., 1970. Tome 2, p. 365 à 372.

THE GEOMETRIC THEORY OF THE CLASSICAL BANACH SPACES

by Joram LINDENSTRAUSS

In the theory of Banach spaces a rather small class of spaces has always played a central role (actually even before the formulation of the general theory). This class –the class of classical Banach spaces– contains the $L_p(\mu)$ spaces (μ a measure, $1 < p < \infty$) and the $C(K)$ spaces (K compact Hausdorff) and some related spaces. These spaces are very important in various applications of Banach spaces in mathematical analysis. They are, however, also of major importance in the abstract theory of Banach spaces. Among the questions studied in the theory of classical Banach spaces are (i) classification of the classical spaces, (ii) special properties of the classical spaces, in particular properties which characterize these spaces, and to a lesser extent (iii) the relation between the classical spaces and general Banach spaces. The purpose of this talk is to give a condensed survey of recent developments in some aspects of this theory. For the sake of simplicity we shall concentrate our attention on the separable classical spaces.

Before continuing, we emphasize the fact that there are two Banach space theories. The *isometric theory* –in which two Banach spaces are identified if there is an isometry (i.e. a norm preserving linear map) from one space onto the other, and the *isomorphic theory*– in which we identify two Banach spaces if there is an isomorphism (i.e. a bounded linear map with a bounded inverse) from one space onto the other. Though there are of course close connections between those theories, it should be clear that the difference between them is quite large and very often problems which are relatively easy in one theory become very difficult in the other. It turns out in particular that the natural concept of a classical Banach space is not the same in both theories.

Our notation for the common Banach spaces is standard : $L_p(\mu)$, $L_p(0,1)$ (if μ is the Lebesgue measure on $[0,1]$), $C(K)$, l_p, c_0, l_p^n (the n dimensional L_p space). p and q will always denote conjugate exponents i.e. $p^{-1} + q^{-1} = 1$ (in particular $p = 1$, resp. ∞ iff $q = \infty$, resp. 1). The distance coefficient $d(X, Y)$ between two isomorphic Banach spaces is defined as $\inf \|T\| \, \|T^{-1}\|$ where the inf is taken over all isomorphisms T from X onto Y. A complemented subspace of a Banach space is a subspace on which there is a bounded linear projection. Isometry between spaces is denoted by = while isomorphism is denoted by $\approx$. The bibliography contains only papers to which we refer in the text. Further references may be found in those papers. The bibliography does not contain references to older papers which appear already in the bibliography of Day's book [3].

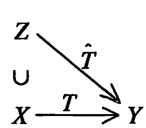
The natural choice for the class of classical Banach spaces in the framework of the isometric theory turns out to be the class of spaces X such that $X^* = L_q(\mu)$

for some $1 \leqslant q \leqslant \infty$ and some measure μ. For $1 < p < \infty$ it is well known that $X = L_p(\mu)$ iff $X^* = L_q(\mu)$. The situation is similar for $p = 1$ [7]. The class of spaces X such $X^* = L_1(\mu)$ includes the $C(K)$ spaces but also other Banach spaces. These are thus the only classical spaces in the isometric theory which are not "classical" in the usual meaning of this word. The preduals of $L_1(\mu)$ have most of the special properties of $C(K)$ spaces as Banach spaces (what distinguishes mostly the $C(K)$ spaces among the other spaces of this class is the additional structure of $C(K)$ as an algebra or a vector lattice). The preduals of $L_1(\mu)$ have a non-trivial and interesting structure (cf. [5], [6], [11], and [18]). We cannot go here into details and mention only that if $X = A(S)$, the space of affine continuous functions on a Choquet simplex S, then $X^* = L_1(\mu)$ and that the theory of Choquet simplexes can be carried over in a natural way to the setting of general preduals of $L_1(\mu)$. *Every separable predual of $L_1(\mu)$ can be represented as a subspace of an $A(S)$ space Y on which there is a projection of norm 1 from Y* [11]. The class of all classical Banach spaces in the isometric theory is closed under taking duals, preduals and projections of norm 1.

The classification of separable infinite-dimensional $L_p(\mu)$ spaces, $1 < p < \infty$, is very easy (modulo standard results in measure theory). For $p = 2$ there is of course only one isometry type of such spaces. For $p \neq 2$ there is a countably infinite number of isometry types ; the isometry type is determined by the number of atoms of μ and the existence or nonexistence of a continuous part in μ. There are two isomorphism types namely those represented by l_p and $L_p(0,1)$. Let us underline the fact that the exponent p is the basic isomorphism invariant of a classical space and thus all nontrivial problems on isometric or isomorphic classification, involve spaces with a given exponent p. The class of spaces with exponent ∞ i.e. the preduals of $L_1(\mu)$ is much richer than the other classes. There are, for example, an uncountable number of isomorphism types of separable preduals of $L_1(\mu)$. The isometric classification of $C(K)$ spaces is easy and well known. The isometry type of $C(K)$ determines, and is determined by, the topological type of K. A result of a similar nature (but of doubtful use) is easy to prove for $A(S)$ spaces and even general preduals of $L_1(\mu)$. The isomorphic classification of the separable preduals of $L_1(\mu)$ spaces is much harder and at present a complete solution of this classification problem is known only for $C(K)$ spaces : *If K is compact metric uncountable then $C(K) \approx C(0,1)$* [21], *while if K is countable the isomorphic type of $C(K)$ is determined by the first ordinal α for which the α-th derived set of K is empty. The isomorphism invariant is the ordinal α^ω* [1] (and thus there are uncountably many such types).

From the isometric properties which characterize the classical spaces we mention here only those connected with the basic extension and lifting diagrams for operators. (All the operators which we consider are bounded and linear).

Consider first the extension diagram



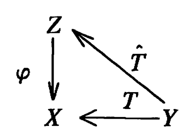
(i) X^* *is an* $L_1(\mu)$ *space iff for every* Y, Z *and compact* T *there is a compact* $\hat{T}$ *with* $\|\hat{T}\| = \|T\|$ ([8], [12]).

(ii) Y^* *is an* $L_1(\mu)$ *space iff for every* X, Z *compact* T *and* $\epsilon > 0$ *there is a compact* $\hat{T}$ *with* $\|\hat{T}\| \leqslant (1 + \epsilon)\|T\|$ ([8], [12]).

(iii) Z *is an* $L_2(\mu)$ *space iff for every* X, Y *and compact* T *there is a compact* $\hat{T}$ *with* $\|\hat{T}\| = \|T\|$ (Kakutani, it is assumed that dim $Z \geqslant 3$).

Actually much weaker forms of the extension properties above already characterize preduals of $L_1(\mu)$ (resp. Hilbert spaces). It is for example enough to consider in (i), (ii) or (iii) only operators T with dim $TX \leqslant 3$ and even $\leqslant 2$ in some cases. (This is a minimal assumption since if dim $TX = 1$ then $\hat{T}$ always exists, with $\|\hat{T}\| = \|T\|$, by the Hahn Banach theorem.) On the other hand, if stronger extension properties than those of (i) and (ii) are considered then usually only a small subclass of the preduals of $L_1(\mu)$ spaces have these properties and thus one gets into the isometric classification problem of preduals of $L_1(\mu)$ spaces. For exemple : Y *has the extension property expressed in* (ii) *with* $\epsilon = 0$ *iff* Y^* *is an* $L_1(\mu)$ *space and the unit cell of every finite dimensional subspace of* Y *is a polytope* (c_0 is such a space) [10]. Another example : X (resp. Y) *has the extension property obtained from* (i) (resp. (ii)) *by replacing "compact" with "bounded" iff* X (resp. Y) $= C(K)$ *with* K *extremally disconnected* (cf. [3, p. 95], [12, p. 82]).

A similar situation arises if we consider the lifting diagram



where φ is a quotient map. If we keep Y (resp. X) fixed and require that for all compact T there is a compact lifting $\hat{T}$ with $\|\hat{T}\| = \|T\|$ (resp. $\|\hat{T}\| \leqslant (1 + \epsilon)\|T\|$) we get properties which characterize $L_1(\mu)$ spaces [7] [13], while if we keep Z fixed we get a characterization of Hilbert spaces. Here again it would suffice to consider weaker lifting properties, while stronger lifting properties are shared only by a subclass of the $L_1(\mu)$ spaces. Since, however, the classification of $L_1(\mu)$ spaces is essentially trivial it is easy to determine the subclass of $L_1(\mu)$ which has the desired lifting property. The only thing which usually matters is whether or not the measure μ is purely atomic.

Before leaving the discussion of the classical spaces in the isometric theory I want to mention one further result which serves also as motivation to our approach to the isomorphic theory : (*) *Let* X *be an infinite-dimensional separable Banach space. Then* $X^* = L_q(\mu)$ *for some* μ, $1 \leqslant q \leqslant \infty$, *iff* $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ *with* $E_n \subset E_{n+1}$ *and* $E_n = l_p^n$ *for every* n. In other words, the separable classical spaces are exactly all the direct limits of l_p^n spaces in which the mappings $l_p^n \to l_p^{n+1}$ are isometries. The asymptotic behaviour of these isometries determines of course the isometry type of the direct limit X. The most interesting case is again the case $p = \infty$ (i.e. $q = 1$). If follows for example from (*) with $p = \infty$ that to every métrizable Choquet simplex there corresponds in a natural way a class of Markoff chains (see [11] for details).

We turn now to the isomorphic theory. For a long time the objects considered in the isomorphic theory of the classical Banach spaces were simply those spaces which are isomorphic to the classical spaces in the isometric theory. It turned out however that from many points of view this was not a natural approach. If one takes, for example, an isometric property which characterizes $L_1(\mu)$ spaces then, in general, its isomorphic version will not characterize spaces isomorphic to $L_1(\mu)$ but rather the larger class of $\mathcal{L}_1$ spaces defined below.

DEFINITION – *Let $1 \leqslant p \leqslant \infty$ and $\lambda \geqslant 1$. A Banach space X is called an $\mathcal{L}_{p,\lambda}$ space if for every finite dimensional subspace B of X there is a subspace C of X containing B with $d(C, l_p^n) \leqslant \lambda$ where $n = \dim C < \infty$. A Banach space is called an $\mathcal{L}_p$ space if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda < \infty$.*

The $\mathcal{L}_p$ spaces seem to be the right choice for the term "classical spaces" in the isomorphic theory of Banach spaces. The fact that these spaces are defined in terms of their "local" behaviour is a part of a general phenomenon concerning Banach spaces. Many properties of a Banach space (in particular those involving inequalities) are determined already be the metric structure of the class of its finite-dimensional subspaces. The way in which those finite-dimensional subspaces are embedded in each other is often of a secondary importance only. For $1 \leqslant p < \infty$, $p \neq 2$, an $\mathcal{L}_p$ space is not in general isomorphic to an $L_p(\mu)$ space. (For $p = 2$ there always exists such an isomorphism, the question whether every $\mathcal{L}_\infty$ space is isomorphic to a predual of $L_1(\mu)$ is open.).

We bring now several of the known facts concerning $\mathcal{L}_p$ spaces. *A Banach space X is an $\mathcal{L}_{p,1+\epsilon}$ space for every $\epsilon > 0$ iff $X^* = L_q(\mu)$ for some μ* [15]. For separable X this result is of course related (though not identical with) the result (*) mentioned above.

X is an $\mathcal{L}_p$ space iff X^* is an $\mathcal{L}_q$ space. [17].

This is not obvious, since by passing to the dual in the definition, we pass from a direct limit to an inverse limit. It follows therefore from the result above that the $\mathcal{L}_p$ spaces are also exactly those spaces which can be obtained as inverse limits of l_p^n spaces (with a suitable uniform bound on the operators involved in the inverse limit). The cases $p = 1, \infty$ of the result deserve a special mention since $\mathcal{L}_1$ or $\mathcal{L}_\infty$ spaces are not reflexive. The validity of the result for those values of p is connected to the fact that for every Banach space X the finite dimensional subspaces of X^{**} are arbitrarily close to those of X (cf. the "local reflexivity principle" in [17]).

A complemented subspace of an $\mathcal{L}_p$ space is either an $\mathcal{L}_p$ space or an $\mathcal{L}_2$ space. If $p = 1, \infty$ it must be an $\mathcal{L}_p$ space. A separable Banach space X with $X \not\approx l_2$ is an $\mathcal{L}_p$ space for $1 < p < \infty, p \neq 2$ iff it is isomorphic to a complemented subspace of $L_p(0,1)$. [15], [17].

The question of the structure of the complemented subspaces of $L_p(0,1)$ has been considered for quite a time. The similar question for l_p has been answered in [22] : *Every infinite-dimensional complemented subspace of c_0 or l_p $1 \leqslant p < \infty$ is isomorphic to the space itself.* In view of what is known at present on the classification of the $\mathcal{L}_p$ spaces (see below) it is very unlikely that a functional representation can be given (in reasonable terms) to all types of complemented subspaces of $L_p(0,1)$, $1 < p < \infty, p \neq 2$. There are many open questions concerning direct decompositions of $L_p(0,1)$. Here are two : (i) Assume $L_p(0, 1) \approx X \oplus Y$. Is either X or Y isomorphic to $L_p(0,1)$? (in this connection cf. [16]) (ii) Is every complemented subspace of $L_1(0,1)$ isomorphic either to l_1 or $L_1(0,1)$? In various areas of analysis there appear naturally complemented subspaces of $L_p(\mu)$. Some of them (e.g. several Besov spaces, according to J. Peetre) are not isomorphic to $L_p(\mu)$ spaces and thus provide examples of nontrivial $\mathcal{L}_p$ spaces which appear naturally in analysis. The study of projections in $L_p(\mu)$ spaces enables also to prove that several Banach spaces which look quite different are in fact isomorphic. For example, the space of analytic functions on the unit disc whose *p-th* power is integrable with respect to the planar Lebesgue measure is (for $1 \leqslant p < \infty$ and the obvious norm) isomorphic to l_p [16]. Though basically the proof is constructive and not really hard there is no known practical way to exhibit this isomorphism explicitly (unless $p = 2$). This lack of an explicit map is a common situation with isomorphisms whose existence is proved in the geometric theory of Banach spaces.

The study of general (i.e. not necessarily complemented) subspaces of $L_p(0,1)$ is of course much harder than the study of the $\mathcal{L}_p$ spaces. Naturally less is known on these spaces. A recent paper [2], closely connected to probability theory, contains some interesting results in this direction. It characterizes all subspaces of $L_p(0,1)$, $1 \leqslant p < 2$ which have a certain symmetric structure (e.g. a symmetric basis). We return to the $\mathcal{L}_p$ spaces.

A separable $\mathcal{L}_1$ space is isomorphic to a subspace of $L_1(0,1)$ and its dual is isomorphic to l_∞ [15].

Every separable $\mathcal{L}_p$ space $1 \leqslant p \leqslant \infty$ has a Schauder basis [9].

This is one of the few cases in Banach space theory in which the existence of a Schauder basis has been proved for a class of spaces which are not given by an explicit functional representation. All the known examples of separable $\mathcal{L}_p$ spaces, $1 < p < \infty$, have even an unconditional basis. For $L_p[0,1]$ the existence of such a basis is rather deep and due to Paley (cf. [25] for a modern treatment). It is not known whether this is true for all separable $\mathcal{L}_p$ spaces, $1 < p < \infty$. A separable $\mathcal{L}_1$ (resp. $\mathcal{L}_\infty$) space has an unconditional basis only if it is isomorphic to l_1 (resp. c_0) [15].

As we already mentioned before, the $\mathcal{L}_2$ spaces are exactly those spaces which are isomorphic to Hilbert spaces and thus there is only one isomorphism type of separable infinite-dimensional $\mathcal{L}_2$ spaces. *There are infinitely many isomorphism types of separable infinite dimensional $\mathcal{L}_1$ spaces* [14]. For $1 < p < \infty$, $p \neq 2$ there are at present seven known isomorphism types of separable infinite-dimensional $\mathcal{L}_p$ spaces. There are a few more "candidates" and it is very likely that there exist infinitely many such isomorphism types. Of the known isomorphism types we mention here (besides $L_p(0,1)$ and l_p) also the spaces $l_p \oplus l_2$, $(l_2 \oplus l_2 \oplus \ldots)_p$ and X_p which is (for $p > 2$) the space of all sequences of scalars $\lambda = \{\lambda_n\}$ such that

$$\|\lambda\| = \sup \Big(\big(\sum_n |\lambda_n|^p\big)^{1/p} , \big(\sum_n |\omega_n \lambda_n|^2\big)^{1/2} \Big) < \infty$$

where $\{\omega_n\}$ is a sequence of positive numbers tending to 0 and satisfying $\sum_n \omega_n^{2p/(p-2)} = \infty$. The isomorphism type of X_p does not depend on the specific choice of the sequence $\{\omega_n\}$. It is interesting to note that (again for $p > 2$) l_2, l_p, $l_2 \oplus l_p$ and X_p *represent isomorphism types (and the only such types) of subspaces of* $L_p(0,1)$ *spanned by a sequence of independent random variables* (cf. [24] for the study of the space X_p). On the classification of $\mathcal{L}_\infty$ spaces very little is known beyond the facts mentioned already above concerning preduals of $L_1(\mu)$. It is not even known whether every $\mathcal{L}_\infty$ space is isomorphic to a $C(K)$ space.

We turn to some geometric properties which characterize the $\mathcal{L}_p$ spaces. Let us first discuss the isomorphic versions of the basic extension and lifting properties. As in the isometric theory the classical spaces with exponent ∞ (i.e. the $\mathcal{L}_\infty$ spaces) are characterized by the possibility of extending compact operators having these spaces as domain or range spaces [12], [17]. Of course we require here not norm preserving extensions but simply that for every compact T there exists a compact extension $\hat{T}$. Whether the analogue of the isometric characterization of Hilbert spaces by the extension diagram, is true in the isomorphic theory is a famous open problem. It is equivalent to the question whether a Banach space which is not isomorphic to a Hilbert space must have a noncomplemented subspace(*). By considering the lifting diagram for compact operators we get, in analogy to the isometric situation, properties which characterize $\mathcal{L}_1$ spaces [17].

The $\mathcal{L}_p$ spaces in general and in particular the spaces with exponents 1, 2 and ∞ are also characterized by the fact that the Boolean algebras of projections on these spaces are in a sense well behaved [17], [20]. This is of course of relevance to the spectral theory of operators on these spaces. We cannot go here into the technical details and restrict ourselves to stating results which involve Schauder bases instead of Boolean algebras of projections (the proofs of the basis versions involve the same basic facts and inequalities as the Boolean algebra versions). Two bases $\{x_n\}$ of X and $\{y_n\}$ of Y are said to be equivalent if there is an isomorphism T from X onto Y such that $Tx_n = y_n$ for all n. A basis $\{x_n\}$ is said to be normalized if $\|x_n\| = 1$ for all n. A sequence $\{z_i\}$ is said to be a block basic sequence with respect to a basis x_n if $z_i = \sum_{n=n_i+1}^{n_{i+1}} \lambda_n x_n$ for suitable sequence of scalars $\{\lambda_n\}$ and an increasing sequence of integers $\{n_i\}$. We can now state the results.

A basis of a Banach space X, is equivalent to the unit vector basis of c_0 *or* l_p *for some* $1 \leqslant p < \infty$ *iff it is equivalent to every normalized block basic sequence with respect to itself.* [27].

The only Banach spaces which have, up to equivalence, a unique normalized unconditional basis are c_0, l_1 *and* l_2 [19].

We conclude with a few words on the relation of classical spaces to general Banach spaces. A well known and easy fact is that every Banach space is isometric to a subspace of a $C(K)$ (and hence $\mathcal{L}_\infty$) space, in particular every separable Banach space is isometric to a subspace of $C(0,1)$. Every Banach space is a quotient space of an $L_1(\mu)$ (and hence $\mathcal{L}_1$) space, in particular every separable

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(*) Added in proof : This problem has been solved recently. If in a Banach space X every closed subspace is complemented then X is an $\mathcal{L}_2$ space [28].

Banach space is a quotient space of l_1. The $\mathcal{L}_\infty$ and $\mathcal{L}_1$ spaces are thus maximal in mutually dual senses. A deep result of Dvoretzky [4] shows that the $\mathcal{L}_2$ spaces (and obviously only the $\mathcal{L}_2$ spaces) are locally minimal in both senses : *Every infinite-dimensional Banach space has for every* $\epsilon > 0$ *and every integer* n *a subspace (resp. a quotient space)* B *with* $d(B, l_2^n) \leqslant 1 + \epsilon$. Two other results in the same circle of ideas deserve mention. *A Banach space* X *is a subspace of an* $\mathcal{L}_1$ *space and a quotient space of an* $\mathcal{L}_\infty$ *space iff* X *is an* $\mathcal{L}_2$ *space* [8], [15]. *A separable Banach space contains a subspace isomorphic to* l_1 *iff it has a quotient space isomorphic to* $C(0,1)$ [23].

It is not known whether every Banach space has a subspace isomorphic to c_0 or to some l_p $1 \leqslant p < \infty$. A partial positive answer, due to James, is known for spaces having an unconditional basis (cf. [26] for a recent generalization). Another open question : Is it true that for every infinite-dimensional Banach space X there exists a λ so that for every integer n there is a subspace B of X such that $d(B, l_p^n) \leqslant \lambda$ for some p and there is a projection of norm $\leqslant \lambda$ from X onto B ? (we may clearly restrict p to the set 1, 2, ∞).

BIBLIOGRAPHY

[1] BESSAGA C. and PELCZYNSKI A. — Spaces of continuous functions (IV), *Studia Math.*, 19, 1960, p. 53-62.

[2] BRETAGNOLLE J. et DACUNHA-CASTELLE D. — Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces L^p, *Ann. Scien. Ecole Normale Sup.*, 2, 1969.

[3] DAY M.M. — *Normed linear spaces*, Springer Verlag, 1958.

[4] DVORETZKY A. — Some results on convex bodies and Banach spaces, *Proc. Symp. on Linear Spaces*, Jerusalem 1961, p. 123-160.

[5] EFFROS E. — Structure in simplexes, *Acta Math.*, 117, 1967, p. 103-121.

[6] EFFROS E. and GLEIT A. — Structure in simplexes III, *Trans. Amer. Math. Soc.*, 142, 1969, p. 355-379.

[7] GROTHENDIECK A. — Une caractérisation vectorielle métrique des espaces L^1 *Canadian J. Math.*, 7, 1955, p. 552-561.

[8] GROTHENDIECK A. — Résumé de la théorie métrique des produits tensoriels topologiques, *Bol. Soc. Math. São Paulo*, 8, 1956, p. 1-79.

[9] JOHNSON W., ROSENTHAL H.P. and ZIPPIN M. — (in preparation).

[10] LAZAR A. — Polyhedral Banach spaces and extensions of compact operators, *Israel J. Math.*, 7, 1969, p. 357-364.

[11] LAZAR A. and LINDENSTRAUSS J. — Banach spaces whose duals are L_1 spaces and their representing matrices, *Acta Math.* (to appear).

[12] LINDENSTRAUSS J. — Extension of compact operators, *Mem. Amer. Math. Soc.*, 48, 1964.

[13] LINDENSTRAUSS J. — On the extension of operators with a finite dimensional range, *Illinois J. Math.*, 8, 1964, p. 488-499.

[14] LINDENSTRAUSS J. — A remark on $\mathcal{L}_1$ spaces, *Israel J. Math.*, 8, 1970, p. 80-82.

[15] LINDENSTRAUSS J. and PELCZYNSKI A. — Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications, *Studia Math.*, 29, 1968, p. 275-326.

[16] LINDENSTRAUSS J. and PELCZYNSKI A. — Contributions to the theory of the classical Banach spaces, to appear in *J. Funct. Anal.*

[17] LINDENSTRAUSS J. and ROSENTHAL H.P. — The $\mathcal{L}_p$ spaces, *Israel J. Math.*, 7, 1969, p. 325-349.

[18] LINDENSTRAUSS J. and WULBERT D. — On the classification of the Banach spaces whose duals are L_1 spaces, *J. Funct. Anal.*, 4, 1969, p. 332-349.

[19] LINDENSTRAUSS J. and ZIPPIN M. — Banach spaces with a unique unconditional basis, *J. Funct. Anal.*, 3, 1969, p. 115-125.

[20] LINDENSTRAUSS J. and ZIPPIN M. — Banach spaces with sufficiently many Boolean algebras of projections, *J. Math. Anal. Appl.*, 25, 1969, p. 309-320.

[21] MILUTIN A.A. — Isomorphism of spaces of continuous functions on compacta of power continuum, *Tieoria Funct., Funct. Anal. i Pril* (Kharkov), 2, 1966, p. 150-156.

[22] PELCZYNSKI A. — Projections in certain Banach spaces, *Studia Math.*, 19, 1960, p. 209-228.

[23] PELCZYNSKI A. — On Banach spaces containing L_1 (μ), *Studia Math.*, 30, 1968, p. 231-246.

[24] ROSENTHAL H.P. — On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables, *Israel J. Math.*, 8, 1970, p. 273-303.

[25] STEIN E.M. — *Topics in Harmonic Analysis related to the Littlewood-Paley Theory*, Ann. Math. Studies, 63, 1970.

[26] TZAFRIRI L. — Reflexivity in Banach lattices and their subspaces, to appear.

[27] ZIPPIN M. — On perfectly homogeneous bases in Banach, *Israel J. Math.*, 4, 1966, p. 265-272.

[27] LINDENSTRAUSS J. and TZAFRIRI L. — On the complemented subspaces problem, *Israel J. Math.*, 9, 1971.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 373 à 378.

INTERPOLATION FUNCTORS AND BANACH COUPLES

by Jaak PEETRE

0. Introduction.

The *theory of interpolation spaces* originally arose from an attempt to generalize the classical interpolation theorems of M. Riesz and Marcinkiewicz to a more abstract setting. However it should more correctly be described as a theory of "families" of abstract spaces : Given a number of (usually two) spaces contained in a common "large" space, we try to find as many "families" of new such spaces as possible. The primary goal is to gain a deeper insight into such classical cases as L_p spaces, Lip_α spaces etc. Thus ultimately the whole theory should be judged from the point of view to what extent it provides tools useful in other domains of analysis.

1. Interpolation functors.

To make precise the vague ideas expressed above let us introduce some terminology. By a *Banach couple* $\vec{A} = \{A_0, A_1\}$ we mean an entity consisting of two Banach spaces A_0 and A_1 both linearly and continuously embedded in some Hausdorff topological vector space, say $\mathcal{Q}$. Since linear subspaces of $\mathcal{Q}$ form a lattice, we can consider the sum $\Sigma(\vec{A}) = A_0 + A_1$ and the *intersection* $\Delta(\vec{A}) = A_0 \cap A_1$ of $\vec{A}$. Given any two Banach couples $\vec{A}$ and $\vec{B}$ we next define a continuous linear mapping T from $\vec{A}$ into $\vec{B}$ (in brief : $T : \vec{A} \to \vec{B}$) to be a linear mapping from $\Sigma(\vec{A})$ into $\Sigma(\vec{B})$ such that its restriction to A_i maps A_i continuously into B_i $(i = 0, 1)$. We define the norm of T by $\|T\| = \max(\|T\|_0, \|T\|_1)$ where $\|T\|_i$ denotes the norm of the restriction to A_i $(i = 0, 1)$. The class of all couples and all mappings obviously forms a category, indeed what might be called a *normed category* $\mathcal{C}_1$ (the category of Banach couples). Finally we define an *interpolation functor* to be a functor F from $\mathcal{C}_1$ into $\mathcal{C}_0$ (the usual category of Banach spaces) which to each couple $\vec{A}$ in $\mathcal{C}_1$ assigns a space $F(\vec{A})$ in $\mathcal{C}_0$ such that $\Delta(\vec{A}) \subset F(\vec{A}) \subset \Sigma(\vec{A})$, the inclusions being *natural*. From this definition follows that if $\vec{A}$ and $\vec{B}$ are any two couples and $T : \vec{A} \to \vec{B}$ a mapping then we have $T : A \to B$ where A and B are any two spaces such that

$$A \subset F(\vec{A}),\, B \supset F(\vec{B}), \tag{1}$$

F being any interpolation functor (*interpolation property*). It is important that the following converse holds true (Aronszajn-Gagliardo [1] ; cf. Deutsch [2] for related ideas and further Gagliardo [3]) : If A and B are any spaces with $\Delta(\vec{A}) \subset A \subset \Sigma(\vec{A})$; $\Delta(\vec{B}) \subset B \subset \Sigma(\vec{B})$ such that from $T : \vec{A} \to \vec{B}$, follows $T : A \to B$, then there exists an interpolation functor F such that (1) is fulfilled. Therefore all concrete interpolation theorems can in principle be obtained in this manner using interpolation functors. Accordingly we also say ("par abus de langage") that $F(\vec{A})$ is an *interpolation space* (or, as some authors prefer, *intermediate space*).

Ex. 1. – If we can show that there exists an interpolation functor F such that $L_p = F(L_{p_0}, L_{p_1})$, we have essentially a proof of the M. Riesz theorem. Similarly, if we can find another one F^* such that $L_p = F^*(L^*_{p_0}, L^*_{p_1})$ we have a proof of the Marcinkiewicz theorem.

We pause to indicate generalizations of the above ideas in two different directions :

a) Instead of Banach couples one might consider Banach $(n+1)$-tuples $\vec{A} = \{A_0, ..., A_n\}$, leading to a category $\mathcal{C}_n$. Not much has been done along these lines. Also really significant applications seem to be lacking.

b) Instead of parting from the concept of Banach space we could use other types of topological vector spaces. We are then lead to consider *Hilbert couples, quasi-Banach couples* (Krée, Holmstedt), *locally convex couples* (Deutsch, Goullaouic) etc. For some type of applications it seems however that the setting of topological vector space (= this Section of the Congress !) is not at all appropriate. Here one should, the rule says, conserve the metric structure and drop (or modify) partly or entirely the algebraic one. We are thus lead to study *interpolation of metric spaces* (Peetre [4]) and *interpolation of normed Abelian groups* (Peetre-Sparr [5]).

Let us now give some examples of interpolation spaces. The most important interpolation spaces are perhaps the "complex" spaces $[\vec{A}]_\theta$ $(0 < \theta < 1)$, introduced by Calderón and Lions (around 1960), which are closely related to Thorin's method of proof in the M. Riesz theorem. Here let me concentrate upon the "real" so-called K- and J-spaces, which go back to, and replace the "real" spaces introduced by Gagliardo on one hand and Lions ("espaces de traces", "espaces de moyennes") on the other hand (around 1959). We introduce a family of (equivalent) norms in $\Sigma(\vec{A})$ as follows

$$K(t, a ; A) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

Then we obtain the K-spaces $(\vec{A})^K_{\theta q}$ $(0 < \theta < 1,\ 1 \leqslant q \leqslant \infty)$ as the subspace of $\Sigma(\vec{A})$ determined by

$$\|a\|_{(\vec{A})^K_{\theta q}} = \Phi_{\theta q}(K(t, a ; \vec{A})) = \left(\int_0^\infty (t^{-\theta} K(t, a ; \vec{A}))^q \frac{dt}{t}\right)^{1/q} < \infty. \qquad (2)$$

J-spaces are introduced in a dual way ; we start with the family of norms in $\Delta(\vec{A})$ defined by

$$J(t, a ; \vec{A}) = \max(\|a\|_{A_0}, t\,\|a\|_{A_1})$$

and end up with the J-spaces $(\vec{A})^J_{\theta q}$. We obtain still more general spaces $(\vec{A})^K_\Phi$ and $(\vec{A})^J_\Phi$ if we use a general function norm Φ instead of just $\Phi_{\theta q}$ (cf. (2)).

An important question is the following : For which couples $\vec{A}$ and $\vec{B}$ are all interpolation spaces K-spaces (i.e. : we can take $F(\vec{A}) = (\vec{A})^K_\Phi, F(\vec{B}) = (\vec{B})^K_\Phi$ in (1))? Let us say that $\vec{A}$ and $\vec{B}$ are *K-adequate* if this is the case.

The above definition suggests at once a method of "explicitizing" K-spaces in concrete cases. To "explicitize" first $K(t,a;\vec{A})$ and then apply (2). We cite two simple examples, each of which is the prototype of a whole series of more complicated results in the same sense.

Ex. 2. – $\{L_1, L_\infty\}$. Then

$$K(t,a) = \int_0^t a^*(x)\,dx$$

where $*$ stands for "decreasing absolute rearrangement of". It follows that

$$(L_1, L_\infty)^K_{\theta q} = L_{pq} \left(\frac{1}{q} = 1 - \theta\right)$$

where L_{pq} are the so-called Lorentz spaces. In particular $(q = p)$ $L_{pp} = L_p$ and $(q = \infty)$ $L_{p\infty} = L_p^*$ (cf. ex. 1). Similar results hold e.g. for the couple $\{L_{p_0}, L_{p_1}\}$.

Ex. 3. – $\{C^0, C^1\}$. Then.

$$K(t,a) = \frac{1}{2}\,\omega^*(2t,a)$$

where $*$ now means "least concave majorant of", $\omega(t,a)$ being the modulus of continuity of a. It follows that

$$(C^0, C^1)^K_{\theta\infty} = \mathrm{Lip}_\theta$$

Similar results hold e.g. for the Sobolev couple $\{L_p, W_p^N\}$. We then get the Besov spaces B_p^{sq}.

2. Applications of the theory of interpolation spaces.

We now give a review of in which areas the theory of interpolation spaces has been applied and –what is mote important !– in which areas further applications might be made. Grossly speaking, we may suspect that there are such applications in every branch of analysis where there intervene such classical "families" of spaces as L_p spaces, Lip α spaces etc. (If I could possibly attract more people to look at such matters, I would have achieved the primary goal of my talk !) Waht we gain in this way is : new proofs of known results, various extensions, simplifications, unification.

One of the original motivations (Aronzajn, Lions) for the study of interpolation spaces was given by certain specific applications to *partial differential equations*. In a long series of works Lions and Magenes have since then made a systematic use of interpolation spaces in connection with "higher" order elliptic and parabolic problems with "smooth" coefficients (cf. e.g. Magenes [6] or Lions-Magenes [7] for references). I think however that the interpolation spaces should be of utility also for second order problems with "discontinuous" coefficients (à la de Giorgi-Nash), including also the non-linear case. (The interpolation theory of metric space might be of interest here). They are further useful in connection with the study of Green's function and other kernels associated with differential problems (Spanne).

The most natural domain of application of the theory lies however in the direction of *approximation theory* and, related to that, *numerical analysis.* In particular in the latter area more work could be done ; until now interpolation spaces have been used in connection with finite difference approximations to differential problems (cf. e.g. Thomée [8]) and in connection with spline approximations of functions (cf. e.g. Varga [9]). Recently I noted (Peetre [10]) also a link with the work of Stečkin and his school on the problem of best numerical differentiation (cf. e.g. Stečkin [11]). (Here and in other non-linear problems (approximation by spline functions with variable nodes, rational functions etc.) the interpolation theory of normed Abelian groups seems to be very useful). More related to classical approximation theory and allied branches of analysis, there is however a series of applications involving direct and inverse theorems (à la Jackson and Bernstein), multipliers, Sobolov spaces ect. In fact there is a nice abstract setting for these questions (cf. Peetre [12]). The point of departure is a class of (unbounded) operators P in a Banach space X for which there is possible to construct a spectral calculus such that the "Riesz means" $\left(1 - \frac{P}{t}\right)_+^{\alpha}$ of order α are uniformly bounded (in t) for some α. The classical case corresponds to the case when P is an elliptic partial differential operator, say $P = -\Delta$ (Laplacian), acting on $X = L_p(\mathbf{R}^d)$ or $L_p(\mathbf{T}^d)$.

Finally, let me remark that interpolation spaces might be of interest also in connection with *norm ideals* of operators in Banach space (cf. e.g. Pietsch [13]), and furthermore in connection with *Banach algebras.*

3. The category of Banach couples.

I have a strong feeling that to pursue any further one should temporarily leave aside the interpolation functors and instead concentrate upon the study of the couples per se. I shall now sketch some fragmentary results which might give an indication of what could be achieved in this direction (cf. Peetre [14]). The sources of inspiration are obvious : on one hand general ideas from the theory of categories, on the other hand the parallel theory of Banach spaces.

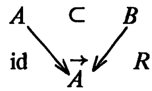
First let me suggest a suitable definition fo subcouple. We say that $\vec{A} = \{A_0, A_1\}$ is a (K-) subcouple of $\vec{B} = \{B_0, B_1\}$ if all the four spaces involved are embedded in the same space $\mathfrak{A}$ such that

$$A_i \subset B_i,\ \|a\|_{A_i} = \|a\|_{B_i} \text{ if } a \in A_i\ (i = 0,1)$$

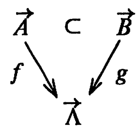
and such that moreover

$$K(t,a\,;\vec{A}) = K(t,a\,;\vec{B}) \text{ if } a \in \Sigma(\vec{A}).$$

A particularly important instance of subcouples is obtained when $\vec{A}$ is a *retract* of $\vec{B}$, i.e. there exists a mapping R (retraction) of norm 1 such that we have the commutative diagram



(We remark that using the idea of retract the interpolation theory of Sobolev and Besov spaces (Grisvard) can easily be reduced to the interpolation theory of vector valued L_p spaces. Cf. ex 3). We can now prove an analogue of the Hahn-Banach theorem. Let $\vec{\Lambda} = \{\mathbf{R}, \mathbf{R}\}$ or $\{\mathbf{C}, \mathbf{C}\}$ (scalar couple), the field of scalars being $\Lambda = \mathbf{R}$ or $\mathbf{C}$. Then every mapping $f : \vec{A} \to \vec{\Lambda}$ can be extended to a mapping $g : \vec{B} \to \vec{\Lambda}$, $\vec{A}$ being any subcouple of $\vec{B}$.



Thus we are on the right track ! We can now go on and define *injective* couples. E.g. Lipschitz couples $\{\mathrm{Lip}_{a_0}, \mathrm{Lip}_{a_1}\}$ are injective. There are indeed plenty of injective couples : Every couple is isomorphic to a subcouple of some injective couple (analogue of the Banach-Mazur theorem). But we have also direct applications to questions pertaining to interpolation spaces ! We quote the following result which gives a partial answer to the problem of K-adequacy (cf. Section 1) : If $\vec{B}$ is injective then for every $\vec{A}$, the couples $\vec{A}$ and $\vec{B}$ are K-adequate. It is also possible to characterize all couples $\vec{B}$ with this property. Such couples we might term *partially injective*. Thus the couple $\{L^*_{p_0}, L^*_{p_1}\}$ is partially injective, which result obviously is of interest in connection with the Marcinkiewicz theorem.

Guide to the literature.

The book by Butzer-Berens [15] contains a chapter entirely devoted to interpolation spaces and has also an excellent bibliography. Here are some other works of survey character, where additional references can be found : Magenes [6], Krein-Petunin [16] (devoted to the important work of Krein and his school, which we have completely neglected here !), Peetre [17].

REFERENCES

[1] Aronszajn N. and Gagliardo E. — Interpolation spaces and interpolation methods. *Ann. Mat. Pura Appl.*, 68, 1965, p. 51-118.

[2] Deutsch N. — Interpolation dans les espaces vectoriels topologiques localement convexes. *Bull. Soc. Math. France, Suppl. Mém.*, 13, 1968.

[3] Gagliardo E. — Caratterizzazione construtiva di tutti gli spazi di interpolazione tra spazi di Banach. *Symposia Matematica,* 2, 1968, p. 95-106.

[4] Peetre J. — Interpolation of Lipschitz operators and metric spaces. *Mathematica* (Cluj).

[5] Peetre J. and Sparr G. — Interpolation of normed Abelian groups. In preparation.

[6] MAGENES E. — Spazi di interpolazione ed equazioni a dérivate parziali. *Atti del VII Congresso dell' U.M.I.*, Genova, 1963, Edizioni Gremonese, Roma, 1964.
[7] LIONS J.L. and MAGENES E. — Problèmes aux limites non homogènes et applications. Dunod, Paris, 1968.
[8] THOMÉE V. — Stability theory for partial difference operators. *S.I.A.M. Rev.*, 11, 1969, p. 152-195.
[9] VARGA R. — Error bounds for spline interpolation. Approximations with special emphasis on spline functions. Academic Press, New York, 1969, p. 367-488.
[10] PEETRE J. — Approximation of linear operators. Varna meeting, May 1970.
[11] STEČKIN S.B. — Best approximation of differentiation operators. Budapest meeting, Aug. 1969 (Russian).
[12] PEETRE J. — Some classes of operators in Banach space. Berlin meeting, Aug. 1969.
[13] PIETSCH A. — Ideale von S_p-Operatoren in Banachräumen. *Studia Math.*, 28, 1970, p. 59-69.
[14] PEETRE J. — Banach couples, I. In preparation.
[15] BUTZER P.L. and BERENS H. — Semi-groups of operators and approximation. Springer, Berlin, 1967.
[16] KREIN S.G. and PETUNIN J.J. — Scales of Banach Spaces. *Usp. Mat. Nauk.*, 21, 1966, p. 89-168. (Russian).
[17] PEETRE J. — On the connection between the theory of interpolation spaces and approximation theory. Budapest meeting, Aug. 1969.

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D 2 - ALGÈBRES D'OPÉRATEURS REPRÉSENTATIONS DES GROUPES LOCALEMENT COMPACTS

SOME TOPICS IN THE THEORY OF OPERATOR ALGEBRAS

by Huzihiro ARAKI

1. Asymptotic ratio set

Powers [14] has shown that a family of factors R_x, $0 \leqq x \leqq 1$ are mutually non $*$ isomorphic. Araki and Woods [1] have introduced an asymptotic ratio set $r_\infty(R)$ for a W^* algebra R as the set of all $x \in [0, \infty)$ such that $R \sim R \otimes R_x$ ($*$ isomorphism) where $R_{(x^{-1})} = R_x$ for $x \neq 0$. $r_\infty(R) \cap (0, \infty)$ is a group. For R on a separable space, $r_\infty(R)$ is closed and is one of the following sets : S_ϕ = empty, $S_0 = \{0\}$, $S_1 = \{1\}$, $S_{01} = \{01\}$, $S_x = \{x^n ; n \in Z\} \cup \{0\}$ for $0 < x \neq 1$ and $S_\infty = [0, \infty)$. [2, 5, 14].

Among the countable infinite tensor products of type I factors on separable spaces (abreviated as ITPFI's), there exists a unique R_∞ with $r_\infty(R_\infty) = S_\infty$, R_x satisfies $r_\infty(R_x) = S_x$ and $r_\infty(R_0 \otimes R_1) = S_{01}$. If an ITPFI R satisfies $r_\infty(R) = S_x$, $0 \leqq x \leqq \infty$, then $R \sim R_x$. If it is type II_∞ and satisfies $r_\infty(R) = S_{01}$, then

$$R \sim R_0 \otimes R_1 .$$

Let (a, b) denote ∞ if a/b is irrational and the largest c if a and b are integer multiplies of c. For $x, y \in (0, \infty)$, $R_x \otimes R_y \sim R_z$ with $\log z = (\log x, \log y)$. $R_\infty \otimes R \sim R_\infty$ for any ITPFI R.

According to Powers, R has the property L_λ if for any $\epsilon > 0$ and any normal state ω of R, there exists $N \in R$ such that $N^2 = 0$, $N^*N + NN^* = 1$ and $|(1-\lambda)\,\omega(QN) - \lambda\omega(NQ)| \leqq \epsilon\,\|Q\|$ for all $Q \in R$. We say that R has the property L'_λ if the statement holds for any finite collection of normal states $\omega_1 \ldots \omega_n$ of R. [5] R on a separable space has the property L'_λ if and only if $\lambda/(1-\lambda) \in r_\infty(R)$ $(0 \leqq \lambda \leqq 1/2)$. Any finite continuous von Neumann algebra has the property $L_{1/2}$. Property L_0, Property L'_0, $1 \in r_\infty(R)$ and R being properly infinite are equivalent. The property L_λ for R implies that R is purely infinite if $0 < \lambda < 1/2$, R is continuous if $\lambda = 1/2$.

If R does not have the property L, $r_\infty(R) = S_\phi$ or S_0 according as R is finite or infinite. (cf. [17]).

For ITPFI, there are no R with $r_\infty(R) = S_\phi$ and no purely infinite R with $r_\infty(R) = S_0$. All ITPFI except R_0 (type I_∞) has 1 in $r_\infty(R)$ and hence the property L. Question : do all hyperfinite factors share these properties. (There are non-hyperfinite counter-examples).

There exist uncountably many $ITPFI_2$ with $r_\infty(R) = S_{01}$. Araki and Woods have introduced another invariant $\rho(R)$ which is the set of $x \in [0, \infty)$ such that $R \otimes R_x \sim R_x$. This invariant separates some ITPFI in the class S_{01}. Woods has shown that $\rho(R)$ has Lebesgue measure 0 for any type III $ITPFI_2$ (countable infinite tensor product of type I_2 factors).

Krieger [10, 11, 12] has constructed for every $x \in (0,1)$, an ITPFI such that $r_\infty(R) = S_{01}$ and $R \otimes R \sim R_x$, hyperfinite factors $A_{x,p}$, $1/2 \leqq p < 1$ such that $r_\infty(A_{x,p}) = S_{01}$, $A_{x,p} \otimes A_{x,p} \sim R_x$ and $A_{x,p} \not\sim A_{x,q}$ for $p \neq q$, and a hyperfinite factor such that $r_\infty(R) = S_{01}$ and $R \otimes R \sim R_\infty$.

Williams [18] has shown that $A \otimes R_x$ does not have the property L_λ except for $\lambda/(1-\lambda) \in S_x$ if A has a restricted semifinite part. Using the free group with two generators, he has non hyperfinite $A \otimes R_x$ with $r_\infty(A \otimes R_x) = S_x$, $0 \leqq x \leqq 1$. (See also [8], [16].) He has also shown that a countable ITP of finite factors is $*$ isomorphic to $F \otimes I$ where F is finite, I is an ITPFI and if $I \neq R_0$, F is an ITP of given finite factors with respect to cyclic trace vectors.

Nielsen [13] has shown that any W^*-algebra R on a separable space has a unique decomposition $R = R_{(\phi)} \otimes R_{(01)} \otimes R_{(\infty)} \otimes \int_0^1 d\mu(x) R_{(x)}$, coarser than the central decomposition, where μ is a Borel measure on [0,1] and $R_{(a)}$ is of pure type S_a in the sense that almost all factors in its central decomposition have the asymptotic ratio set S_a.

2. Representations of the CCR (canonical commutation relations).

For an isomorphism ϕ of a group G into a topological group H, there exists the weakest group topology on G which makes ϕ continuous. It can serve as an invariant in the classification of representations. For a unitary representation U of a real vector space V_ϕ, such that $U(\lambda f)$ is continuous in $\lambda \in R$ for $f \in V_\phi$, the weakest vector topology, τ_ϕ, making U continuous, is given by a collection of distances $d_\phi(f) = \sup_{0 \leqslant \lambda \leqslant 1} \|\{U(\lambda f) - 1\} \Phi\|$. It is the weakest group topology on V_ϕ making the representation of $f \in V_\phi$ by the infinitesimal generator $\phi(f)$ of $U(\lambda f) = \exp i\lambda\phi(f)$ continuous relative to the topology of resolvent convergence. If Ω is separating for $\{U(f)\}''$, then τ_ϕ is metrizable by d_Ω. [7, 9, 19].

If a unitary operator $V(g)$ for some $g \in V_\phi^*$ (algebraic dual) satisfies

$$V(g)\,U(f)\,V(g)^*U(f)^* = \exp ig(f) \quad \text{and} \quad |(\Phi, V(g)\Phi)| > 1/2,$$

then $d_\phi(f)^{-1} g(f)$ is uniformly bounded for $d_\phi(f) < 1/2$. Hence $g(f)$ is τ_ϕ continuous in f. If a subspace $V_\pi \subset V_\phi^*$ has a unitary representation $V(g)$, $g \in V_\pi$ having this commutation property, the pair U, V is called a representation of CCR over V_ϕ, V_π. The above boundedness implies the non-existence of a representation of CCR over V_ϕ, V_ϕ^*. Hence V_ϕ^* does not have any V^*-quasi-invariant measures – a special case of a known result.

A representation U of V_ϕ can be extended uniquely to the topological completion $\overline{(V_\phi, \tau_\phi)}$. If there exists a separating vector in the common domain of $\phi(f)$, $f \in \overline{V}_\phi$, then τ_ϕ is a Hilbert space topology. If a pair V_ϕ, $V_\pi (\subset V_\phi^*)$ can be imbedded in a real Hilbert space algebraically ($g(f) = (g, f)$), then it has a representation of CCR. The converses of both statements hold for a countable infinite tensor product of Schrödinger representations of one dimensional CCR.

One usually requires that $U(\lambda f)$ and $V(\lambda g)$ are continuous in λ. If B is a σ-field generated by cylinder sets over V_ϕ in V_ϕ^*, μ is a V_π-quasi-invariant measure on (V_ϕ^*, B), $H_\mu = L_2(V_\phi^*, B, \mu)$, $U_\mu(f)$ is multiplication by $\exp i\xi(f)$, $\xi \in V_\phi^*$ and $[V_\mu(g)\psi](\xi) = [d\mu(\xi + g)/d\mu(\xi)]^{1/2}\psi(\xi + g)$, then $U_\mu(\lambda f)$ and $V_\mu(\lambda g)$ are continuous in λ. [4] If V_ϕ and V_π are finite linear spans of countable dual bases, then all multipliers (first order cocycles) can be explicitly given and hence concrete structure of all representations are known.

One usually requires in addition that the bilinear form $(g, f) = g(f)$ on $V_\pi \times V_\phi$ be nondegenerate (V_π and V_ϕ separate each other). It can be uniquely extended to $\overline{V}_\pi \times \overline{V}_\phi$ (the closure relative to $\tau_\pi \times \tau_\phi$) but may fail to be nondegenerate. We call a representation of CCR closable or non-closable according as (g, f) is nondegenerate or not on $\overline{V}_\pi \times \overline{V}_\phi$.

3. Quasiequivalence criterion for quasifree states.

Let K be a complex linear space, Γ an involution of K, and γ a nondegenerate hermitian form on K satisfying $\gamma(\Gamma h, \Gamma h') = \sigma\gamma(h', h)$, $\sigma = +$ or $-$ (CAR or CCR). For $\sigma = +$, $\gamma(h, h) > 0$ is assumed for $h \neq 0$. $\mathfrak{A}(K, \Gamma, \gamma)$ denotes a free $*$ algebra over the symbols $B(f), f \in K$ adjoined by an identity 1 and divided by the two-sided $*$ ideal generated by

$$B(cf + dg) - cB(f) - dB(g), \qquad B(f)^* - B(\Gamma f),$$
$$B(f)^* B(g) + \sigma B(g) B(f)^* - \gamma(f, g) 1.$$

Any state φ defines a hermitian form $S(f, g) = \varphi(B(f)^* B(g))$, satisfying $S(f, g) + \sigma S(\Gamma g, \Gamma f) = \gamma(f, g)$ and $S(f, f) \geqq 0$. Conversely, there exists a unique quasifree state φ_S giving rise to any such S. In the associated representation π_S, $\pi_S(B(f))$ for $f \in \mathrm{Re}K = \{h\,;\, \Gamma h = h\}$ is essentially selfadjoint (bounded for $\sigma = +$) and defines an induced vector topology τ_S on ReK and hence on K. It is given by a positive definite form $(f, g)_S \equiv S(f, g) + S(\Gamma g, \Gamma f)$. Let $\overline{K}_S = \overline{(K, \tau_S)}$, $S(f, g) = (f, Sg)_S$, $f, g \in \overline{K}_S$. Then $1 \geqq S \geqq 0$. For $\sigma = -$, the representation is closable if and only if $1/2$ is not a discrete eigenvalue of S, which we shall assume.

If S is a projection, φ_S is called a Fock state. Any φ_S is a restriction of a Fock state $\varphi_{P(S)}$ of $\mathfrak{A}(\hat{K}, \hat{\Gamma}, \hat{\gamma})$, $\hat{K} = K \oplus K$, $\hat{\Gamma} = \Gamma \oplus -\sigma\Gamma$, $\hat{\gamma} = \gamma \oplus \sigma\gamma$. φ_S and $\varphi_{S'}$, are quasiequivalent [3,6] if and only if

(1) $\tau_S \sim \tau_{S'}$, which implies $\tau_{P(S)} \sim \tau_{P(S')}$ on $\hat{K}$, and

(2) $P(S) - P(S')$ is in the Hilbert Schmidt class relative to any Hilbert space norm equivalent to $\tau_{P(S)}$. For $\sigma = +$, (1) always holds and (2) is equivalent to $S^{1/2} - (S')^{1/2}$ being in the HS class. If φ_S and $\varphi_{S'}$ are gauge invariant (relative to $K = L \oplus \Gamma L$), then $S = S_1 \oplus (1 - S_1)$, $S' = S_1' \oplus (1 - S_1')$ and the result agrees with [15].

REFERENCES

[1] Araki H. and Woods E.J. — *Publ. Res. Inst. Math. Sci. Kyoto Univ.* A, 4, 1968, p. 51-130.
[2] Araki H. — *Publ. Res. Inst. Math. Sci. Kyoto Univ.*, A 4, 1968, p. 585-593.
[3] Araki H. — *Publ. Res. Inst. Math. Sci. Kyoto Univ.*, 6, 1970, No. 3.
[4] Araki H. — *Commun. Math. Phys.*, 20, 1971, p. 9-25.
[5] Araki H. — *Publ. Res. Inst. Math. Sci. Kyoto Univ.*, 6, 1970, No. 3.
[6] Araki H. and Shiraishi M. — *Quasieiquivalence of quasifree states of C.C.R.*, *RIMS* 76, 77.
[7] Araki H. and Woods E.J. — *Topologies induced by representations of the canoninal commutation relations.*
[8] Ching W.M. — *A continuum of non-isomorphic non-hyperfinite factors.*
[9] Hegerfeldt G.C. and Klauder J.R. — *Commun. Math. Phys.*, 16, 1970, p. 329-346.
[10] Krieger W. — *J. Functional Analysis,* 6, 1970, p. 1-13.
[11] Krieger W. — *On a class of hyperfinite factors that arise from null-recurrent Markov chains.*
[12] Krieger W. — *On hyperfinite factors and non-singular transformations of a measure space.*
[13] Nielsen O.A. — *The asymptotic ratio set and direct integral decompositions of a von Neumann algebra.*
[14] Powers R.T. — *Annals of Math,* 86, 1967, p. 138-171.
[15] Powers R.T. and Størmer E. — *Commun. Math. Phys.*, 16, 1970, p. 1-33.
[16] Sakai S. — *An uncountable family of non-hyperfinite type III-factors.*
[17] Tomiyama J. — *Pacific J. Math.*, 30, 1969, p. 263-270.
[18] Williams J.J. — *Non-isomorphic tensor products of von Neumann algebras.*
[19] Woods E.J. — *Commun. Math. Phys.*, 17, 1970, p. 1-20.

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BANACH *-ALGEBRAIC BUNDLES AND INDUCED REPRESENTATIONS

by J. M. G. FELL

The notion of an induced representation first arose in the work of Frobenius on finite groups at the end of the nineteenth century. It was generalized to the functional-analytic context of locally compact groups in the now classical work of Mackey [12]. In [11] Mackey also obtained the fundamental abstract characterization of induced representations of locally compact groups, the so-called Imprimitivity Theorem. The object of this talk is to present a more general context in which induced representations can be defined and the Imprimitivity Theorem proved, namely, the context of Banach *-algebraic bundles. Using this generalization we shall obtain a strengthened form of the Imprimitivity Theorem even in the purely group-theoretic situation.

Special cases of Banach *-algebraic bundles and their representation theory, or closely related concepts, have been studied by several mathematicians. Thus, special sorts of "semidirect product bundles" occur in the work of Glimm [7], Zeller-Meier [14], and Effros and Hahn [4]. Dopplicher, Kastler, and Robinson [3] defined what amounts to the general "semidirect product Banach *-algebraic bundle". Induced representations and the Imprimitivity Theorem were studied by Takesaki [13] for this case ; and Ernest [5] has raised questions of duality for semidirect product bundles. Meanwhile Leptin [8], [9], has developed a generalization of the semidirect product context, which he refers to as generalized L_1 algebras. Roughly speaking these generalize the semidirect product situation to the same extent that arbitrary group extensions generalize semidirect product group extensions. In [6], using a quite different approach from Leptin, Fell defined a still more general notion called a Banach *-algebraic bundle. Leptin's structures correspond roughly to what are called in [6] *homogeneous* Banach *-algebraic bundles. In Part II of [6] induced representations are defined, and an Imprimitivity Theorem proved, for homogeneous Banach *-algebraic bundles. A novel feature of the present talk is the removal of the requirement of homogeneity. Indeed, we are now able to obtain an Imprimitivity Theorem valid for arbitrary Banach *-algebraic bundles.

Here are the definitions. A *bundle* over a Hausdorff space X is a Hausdorff space B together with a continuous open surjection $\pi : B \to X$. A *Banach bundle* over X is a bundle in which each fiber $B_x = \pi^{-1}(x)$ $(x \in X)$ is a complex Banach space satisfying the following conditions ($+$, . , $\| \ \|$ being as usual the operations and norm in each fiber, and 0_x denoting the zero of B_x) : (i) Addition is continuous on $\{(s, t) \in B \times B \mid \pi(s) = \pi(t)\}$ to B ; (ii) scalar multiplication is continuous on B ; (iii) the norm is continuous on B ; (iv) if $\{s_i\}$ is a net of elements of B such that $\|s_i\| \to 0$ and $\pi(s_i) \to x \in X$, then $s_i \to 0_x$ in B.

A Banach bundle each of whose fibers is a Hilbert space is a *Hilbert bundle.*

For any Banach bundle B, $L(B)$ will denote the space of all continuous cross-sections of B which have compact support.

By a *Banach *-algebraic bundle* over a locally compact group G (with unit e) we mean a Banach bundle B, π over G together with two continuous operations $\cdot$ (binary) and $*$ (unary) on B, satisfying : (v) $\pi(s.\,t) = \pi(s)\,\pi(t)$ and $\pi(s^*) = (\pi(s))^{-1}$ for $s,\ t \in B$; (vi) for each $x,\ y \in G$, the operation $\cdot$ is bilinear on $B_x \times B_y$, and $*$ is conjugate-linear on B_x ; (vii) the operation $\cdot$ is associative, $s^{**} = s$, and $(s.\,t)^* = t^*.\,s^*$ $(s, t \in B)$; (viii) $\|s.\,t\| \leqslant \|s\|\ \|t\|$ and $\|s^*\| = \|s\|$.

These postulates are very suggestive of those of a Banach *-algebra. The fiber B_e is closed under all the operations (by (v)) and is a Banach *-algebra ; but the other fibers are not closed under $\cdot$ and $*$.

Two further conditions are imposed on all the Banach *-algebraic bundles to be considered here – first, that there are enough continuous cross-sections of B to pass through any point of B, and secondly, that B has an *approximate unit,* that is, a norm-bounded net $\{w_i\}$ of elements of B_e such that $\|w_i\, s - s\| \to 0$ and $\|sw_i - s\| \to 0$ uniformly for s running over any compact subset of B.

As an example, the "semidirect product bundles" are obtained as follows : Take any Banach *-algebra A with an approximate unit ; and let τ be a strongly continuous homomorphism of G into the group of all isometric *-automorphisms of A. Define $B = G \times A$; put $\pi(x, a) = x$ (so that each fiber B_x has the Banach space structure of A) ; and set $(x, a) \cdot (x', a') = (xx', a\ \tau_x(a'))$ and

$$(x, a)^* = (x^{-1}, \tau_{x^{-1}}(a^*)) .$$

Then $B, \pi, \cdot, *$ turns out to be a Banach *-algebraic bundle, which we call the *τ-semidirect product of A and G.*

These semidirect products, and many other Banach *-algebraic bundles too, have an important property called homogeneity. If we assume (to avoid technicalities) that our Banach *-algebraic bundle has a unit 1 (necessarily in B_e), then B is *homogeneous* if every fiber B_x contains an element u which is unitary (i.e., $uu^* = u^*u = 1$) and if certain additional topological conditions on the set of unitary elements hold. Homogeneity has important implications for representation theory.

A *representation* of a Banach *-algebraic bundle B, π over G is a mapping T of B into the space of bounded linear operators on a Hilbert space $X(T)$ such that (i) T is linear on each fiber, (ii) T carries $\cdot$ and $*$ into product and adjoint, and (iii) T is continuous with respect to the strong operator topology. We shall always assume that our representations are non-degenerate, that is, that

$$\{ T_s\, \xi \mid s \in B,\ \xi \in X(T)\}$$

is total in $X(T)$.

We are now ready to prepare the ground for defining induced representations. Fix a Banach *-algebraic bundle B, π over a locally compact group G ; and let H be a closed subgroup of G. To avoid technicalities, we shall make the inessential assumption that there is a G-invariant measure μ on the left coset space G/H. The part $B_H = \pi^{-1}(H)$ of B which lies over H is clearly a Banach *-algebraic bundle over the subgroup H.

Now fix a representation S of B_H. We are going to define a representation T of B *induced* by S. To do this, we shall first construct a Hilbert bundle $\mathfrak{X}$ over G/H. Fix a coset $\alpha \in G/H$. Let Y_α be the direct sum $\sum_{x \in \alpha}^{\oplus} (B_x \otimes X(S))$; and define a conjugate-bilinear form $(\ ,\ )_\alpha$ on Y_α as follows :

$$(b \otimes \xi, c \otimes \eta)_\alpha = (S_{c^*b}\, \xi, \eta) \quad (b, c \in B_\alpha = \bigcup_{x \in \alpha} B_x\ ;\ \xi, \eta \in X(S))\,.$$

Now it may well happen that $(\ ,\ )_\alpha$ is not positive. If it does happen to be positive for every α in G/H, we shall say that S is *inducible up to* B. Suppose this is the case. Then for each coset α we can construct a Hilbert space $\mathfrak{X}_\alpha$ by factoring out from Y_α the null space of $(\ ,\ )_\alpha$ and completing. Let $\mathfrak{X}$ be the union of the Hilbert spaces $\mathfrak{X}_\alpha$. By adopting a somewhat different approach one can assign a natural topology to $\mathfrak{X}$, making the latter a Hilbert bundle over G/H (having the $\mathfrak{X}_\alpha$ as its fibers). Let κ_α be the quotient map $Y_\alpha \to \mathfrak{X}_\alpha$. Notice that $\mathfrak{X}_{eH}$ can be identified in a natural way with $X(S)$; indeed, we have only to identify $\kappa_{eH}(b \otimes \xi)$ with $S_b \xi$ $(b \in B_H\ ;\ \xi \in X(S))$.

One can now define a natural "action" of B on $\mathfrak{X}$. Let $b \in B_x$ $(x \in G)$. It turns out that for each α in G/H there is a (unique) continuous linear map $\tau_b^{(\alpha)} : \mathfrak{X}_\alpha \to \mathfrak{X}_{x\alpha}$ satisfying the condition that

$$\tau_b^{(\alpha)}[\kappa_\alpha(c \otimes \xi)] = \kappa_{x\alpha}((bc) \otimes \xi) \quad (c \in B_\alpha\ ;\ \xi \in X(S))\,.$$

Taking the union of the $\tau_b^{(\alpha)}$ (b fixed, α varying), we obtain a continuous map $\tau_b : \mathfrak{X} \to \mathfrak{X}$ which satisfies the following conditions : (i) For each coset α, τ_b sends $\mathfrak{X}_\alpha$ linearly into $\mathfrak{X}_{\pi(b)\alpha}$; (ii) we have $\tau_b \tau_c = \tau_{bc}$ for all b, c in B ; and (iii) $(\tau_b \xi, \eta)_{x\alpha} = (\xi, \tau_{b^*} \eta)_\alpha$ whenever $x \in G$, $b \in B_x$, $\alpha \in G/H$, $\xi \in \mathfrak{X}_\alpha$, and $\eta \in \mathfrak{X}_{x\alpha}$. We are now ready to define the induced representation T. Its space will be the Hilbert space $L_2(\mathfrak{X}, \mu)$ of all measurable cross-sections of $\mathfrak{X}$ which are square-integrable with respect to the G-invariant measure μ on G/H. Its action follows the "action" τ of B on $\mathfrak{X}$:

$$(T_b f)(\alpha) = \tau_b(f(\pi(b)^{-1}\alpha)) \quad (b \in B\ ; f \in X(T)\ ;\ \alpha \in G/H)\,.$$

Conditions (i) – (iii) above guarantee that T is indeed a representation of B ; we denote it by $\operatorname{Ind}_{B_H \uparrow B}(S)$ or simply Ind (S).

If B is homogeneous (in particular if B is a semidirect product) every representation of B_H is inducible up to B.

This definition generalizes the classical definition of Mackey. Indeed, suppose that B is the trivial semidirect product (the direct product !) of the complex field with G. We shall call this the "group case", or describe it by saying that "B is essentially just the group G". In this case, representations of B or B_H are just unitary representations of G or H respectively ; and Ind (S) is the same as Mackey's induced representation of G.

We now take up the Imprimitivity Theorem. Fix a locally compact Hausdorff topological G-space M. A *system of imprimitivity* for B and M is defined as a representation R of B together with a regular projection-valued measure P on the Borel σ-field of M which satisfies

$$R_b P(W) = P(\pi(b)\, W)\, R_b \quad (b \in B \ ; W \text{ a Borel subset of } M)\,.$$

Every induced representation is automatically accompanied by a system of imprimitivity. Indeed, in the above description of $T = \operatorname{Ind}(S)$, let us define $P(W)$, for each Borel subset W of G/H, to be multiplication (of cross-sections of $\mathfrak{X}$) by the characteristic function of W. Then T, P is a system of imprimitivity for B and G/H, which (like T itself) we shall refer to as being *induced* by S.

One would now conjecture the following Imprimitivity Theorem :

IMPRIMITIVITY THEOREM. – *If H is a closed subgroup of G and T, P is a system of imprimitivity for B and G/H, then there is an inducible representation S of B_H, unique to within unitary equivalence, such that T, P is unitarily equivalent to the system of imprimitivity induced by S.*

Rather trivial examples show that this is not quite true as it stands. To make it true we have to impose one further condition on T, P, which we will call the Spanning Condition. This is a little too technical to formulate here ; in case G is discrete it says that the images of range (P_{eH}) under the T_b $(b \in B)$ span a dense subspace of $X(T)$. If B is homogeneous the Spanning Condition always holds, so that the above Theorem is true as it stands. In particular, in the group case we recover Mackey's classical result.

To conclude this report we would like to sketch a development leading to a strengthened version of the above Imprimitivity Theorem.

The above inducing construction would be more satisfying from the point of view of symmetry if we could start, not just with a representation of B_H, but with a more or less arbitrary system of imprimitivity for B_H, and then induce up to a system of imprimitivity for B. Let us sketch how such a construction would proceed.

As before H is a closed subgroup of G. Let N be any locally compact topological H-space, and S, Q a system of imprimitivity for B_H and N. To construct an induced system of imprimitivity for B, our first step must be to construct from N a locally compact G-space M. This is easy. We let H act (to the right) on the Cartesian product $G \times N$ as follows : $(x\,,n)\,h = (xh\,,h^{-1}\,n)$; and define M to be the (locally compact Hausdorff) quotient space of all H-orbits in $G \times N$. Then G has a natural continuous action on M given by $y\,(x\,,n)^{\sim} = (yx\,,n)^{\sim}$ ($(x\,,n)^{\sim}$ denoting the H-orbit of $(x\,,n)$). Notice that, when we identify n with $(e\,,n)^{\sim}$, N becomes a closed H-stable subset of M. We can therefore regard Q as a projection-valued measure on M which is carried by the closed subset N. Now we shall define a system of imprimitivity T, P for B and M. Put $T = \operatorname{Ind}(S)$. Thus

$$X(T) = L_2(\mathfrak{X}\,,\mu)\,,$$

$\mathfrak{X}$ being the Hilbert bundle over G/H constructed earlier. It turns out that for each α in G/H and each Borel subset W of M, there is a unique projection $P^{(\alpha)}(W)$ on $\mathfrak{X}_\alpha$ satisfying :

$$P^{(\alpha)}(W)(\tau_b \xi) = \tau_b(Q(\pi(b)^{-1} W)\xi) \quad \text{for all} \quad \xi \text{ in } X(S)\ (\cong \mathfrak{X}_{eH})$$

$$\text{and all} \quad b \text{ in } B_\alpha\,(= \bigcup_{x \in \alpha} B_x)\,.$$

The equation $(P(W)f)(\alpha) = P^{(\alpha)}(W)(f(\alpha))$ $(f \in X(T)\,; \alpha \in G/H)$ then defines $P(W)$ as a projection on $X(T)$; $P : W \to P(W)$ is a projection-valued measure on M ; and T, P is the required induced system of imprimitivity for B and M. We shall use the notation Ind (S, Q) to refer to T, P.

In terms of this construction we can prove a strengthened form of the Imprimitivity Theorem :

STRENGTHENED IMPRIMITIVITY THEOREM. — *Let M be a locally compact Hausdorff topological G-space, H a closed subgroup of G, and $F : M \to G/H$ a continuous surjection satisfying $F(xm) = xF(m)$ $(x \in G\,; m \in M)$. Let T, P be a system of imprimitivity for B and M satisfying an analogue of the Spanning Condition (which we shall not formulate here, and which is always satisfied if B is homogeneous). Let N be the H-stable closed subset $F^{-1}(eH)$ of M, considered as an H-space in its own right. Then* : (I) *M coincides in all essentials with the G-space constructed from N in the preceding paragraph* ; *and* (II) *there* is *a system of imprimitivity S, Q for B_H and N, unique to within unitary equivalence, such that T, P is unitarily equivalent to* Ind (S, Q).

A theorem very closely related to this, in the context of separable locally compact groups, is to be found in [1] (Theorem 2, p. 77).

In proving the results described above we have followed the topological route laid out by Loomis [10] and Blattner [2], which is adapted to the non-separable situation, rather than the (in some ways more enlightening) measure-theoretic approach of [11]. The proof of the strengthened Imprimitivity Theorem consists in reducing it to the "ordinary" form of the Theorem by means of a construction which was suggested by Blattner.

BIBLIOGRAPHY

[1] AUSLANDER L. and MOORE C.C. — Unitary Representations of Solvable Lie Groups, *Memoirs of the A.M.S.*, No. 62, 1966.

[2] BLATTNER R.J. — Positive Definite Measures, *Proc. A.M.S.*, 14, 1963, p. 423-428.

[3] DOPPLICHER S., KASTLER D. and ROBINSON D.W. — Covariance Algebras in Field Theory and Statistical Mechanics, *Commun. Math. Phys.*, 3, 1966, p. 1-28.

[4] EFFROS E.G. and HAHN F. — Locally Compact Transformation Groups and C*-Algebras, *Memoirs of the A.M.S.*, No. 75, 1967.

[5] ERNEST J. — A Duality Theorem for the Automorphism Group of a Covariant System, to appear in *Commun. Math. Phys.*

[6] FELL J.M.G. — An Extension of Mackey's Method to Banach *-Algebraic Bundles, *Memoirs of the A.M.S.*, No. 90, 1969.

[7] Glimm J. — Families of Induced Representations, *Pac. J. Math.*, 12, 1962, p. 885-911.

[8] Leptin H. — Verallgemeinerte L^1-Algebren und Projective Darstellungen Lokal Kompakter Gruppen, I, II, *Inventiones Math.*, 3, 1967, p. 257-281; 4, 1967, p. 68-86.

[9] Leptin H. — Darstellungen Verallgemeinerten L^1-Algebren, *Inventiones Math.*, 5, 1968, p. 192-215.

[10] Loomis L.H. — Positive Definite Functions and Induced Representations, *Duke Math. J.*, 27, 1960, p. 569-580.

[11] Mackey G.W. — Imprimitivity for Representations of Locally Compact Groups, *Proc. Nat. Acad. Sci. U.S.A.*, 35, 1949, p. 537-545.

[12] Mackey G.W. — Induced Representations of Locally Compact Groups, I, *Ann. of Math.*, 55, 1952, p. 101-139.

[13] Takesaki M. — Covariant Representations of C*-Algebras and their Locally Compact Automorphism Groups, *Acta Math.*, 119, 1967, p. 273-303.

[14] Zeller-Meier G. — Produits Croisés d'une C*-Algèbre par un Groupe d'Automorphismes, *C. R. Acad. Sci. Paris*, 263, 1966, p. 20-23.

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MAPPINGS OF OPERATOR ALGEBRAS

by Richard V. KADISON *

Let $\mathcal{H}$ be a Hilbert space over the complex numbers and $\mathcal{B}(\mathcal{H})$ the family of bounded (continuous) linear operators on $\mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$ is an algebra under the usual operations of addition and multiplication of transformations ; and the adjoint (*) operation $A \to A^*$ is an involutory anti-automorphism of $\mathcal{B}(\mathcal{H})$. With the norm of an operator A in $\mathcal{B}(\mathcal{H})$ defined as its bound $\|A\|$, $\mathcal{B}(\mathcal{H})$ becomes a Banach space and the * operation is an isometry. The weak-operator topology, defined as the weakest topology on $\mathcal{B}(\mathcal{H})$ in which the functionals $A \to (Ax, y)$ are continuous, will be needed along with the norm topology associated with the operator-bound norm.

The subalgebras of $\mathcal{B}(\mathcal{H})$ stable under the * operation and closed in the norm topology – the C^*-algebras, as well as their special subclass consisting of those closed in the weak-operator topology, the von Neumann algebras, are the principal objects of attention in this report. The main purpose of this exposition is to describe the developments which have occurred over the past five years in the study of special classes of mappings of such algebras. The primary concern is with the (*) automorphisms and derivations ; but, as an outgrowth of these considerations, the recent work on cohomology of these algebras will be discussed.

A (*) automorphism α of a C^*-algebra $\mathfrak{A}$ is an algebraic automorphism of $\mathfrak{A}$ such that $\alpha(A^*) = \alpha(A)^*$. If U is a unitary operator in $\mathfrak{A}$, $A \to UAU^*$ is an automorphism of $\mathfrak{A}$. Such automorphisms are said to be *inner*. Automorphisms tend to be *outer* (i.e., not inner). If $\mathcal{C}$ is the compact operators on $\mathcal{H}$ and $\mathfrak{A}$ is the C^*-algebra generated by $\mathcal{C}$ and I, each U in $\mathcal{B}(\mathcal{H})$ induces an automorphism of $\mathfrak{A}$, though many unitary operators are not the sum of a scalar and a compact operator. These last automorphisms are *spatial* – induced by a unitary operator in $\mathcal{B}(\mathcal{H})$.

In general automorphisms of C^*-algebras will not be spatial. Homeomorphisms of locally compact measure spaces which don't preserve null sets of the measure produce automorphisms of the C^*-algebra of multiplication operators by continuous functions which are not spatial. Automorphisms of von Neumann algebras, on the other hand, tend to be spatial – provided that their action on the center respects certain elementary numerical invariants. In the case of the *factors*-the von Neumann algebras with center consisting of scalars – automorphisms will be spatial (with a possible exception [14] in the case of a factor of type II_∞ with II_1 commutant).

Though spatial, in general, automorphisms of von Neumann algebras tend not to be inner. If G is a countable (discrete) group with conjugate classes infinite

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and $\mathcal{H}$ is the Hilbert space of complex-valued, square-integrable functions on G, then the weak-operator closed algebra generated by the unitary operators U_a defined by $(U_a f)(g) = f(a^{-1} g)$ is a factor $\mathcal{M}$ (of type II_1). Each automorphism of G induces a spatial automorphism of $\mathcal{M}$. If G is the free group on two generators a and b, the automorphism interchanging a and b will be outer [2 : Ex. 15, p. 288]. If G is the group of those permutations of the integers which move at most a finite set then each locally compact group with a countable base has a (faithful, strong-operator-continuous) representation on $\mathcal{H}$ by unitary operators which (with the exception of I) induce outer automorphisms of $\mathcal{M}$ [1].

An automorphism α of a C^*-algebra is an isometry ; for A and $\alpha(A)$ have the same spectrum. Thus $\|\alpha(A)\| = \|A\|$ when A is self-adjoint. For arbitrary T in the algebra, $\|T\|^2 = \|T^*T\| = \|\alpha(T^*T)\| = \|\alpha(T)\|^2$. Hence α is, in particular, a bounded operator on $\mathfrak{A}$ (as a Banach subspace of $\mathcal{B}(\mathcal{H})$) ; and $\|\alpha\| = 1$. If ι denotes the identity automorphism of the von Neumann algebra $\mathfrak{R}$, $\|\alpha - \iota\| \leqslant 2$. While outer automorphisms of von Neumann algebras abound, if $\|\alpha - \iota\| < 2$ then α is inner [11 : Theorem 7]. This theorem is established by C^*-and von Neumann algebra techniques combined with analytic methods. The proof is directed toward showing that α lies on a oneparameter group of automorphisms of the form $\exp(t\delta)$, where δ is a bounded linear operator on $\mathfrak{R}$. Because the mappings $\exp(t\delta)$ are automorphisms, δ is a derivation of $\mathfrak{R}$

$$\text{(i.e. } \delta(AB) = \delta(A)B + A\delta(A)\text{)}.$$

The theorem that derivations of von Neumann algebras are inner [7, 10, 15, 19] applies ; and there is an iH in $\mathfrak{R}$ such that $\delta(A) = i(HA - AH)$ for each A in $\mathfrak{R}$. Since α preserves adjoints, the same is true for δ ; and H may be chosen self-adjoint. The automorphism α, with which we started, is induced by the unitary operator $\exp(iH)$ (in $\mathfrak{R}$).

The development leading up to the theorem that derivations of von Neumann algebras are inner began with the observation that this is true for type I von Neumann algebras [16]. The prototype of these algebras is $\mathcal{B}(\mathcal{H})$. There is a group $\mathcal{U}$ of unitary operators in $\mathcal{B}(\mathcal{H})$ whose linear span has norm closure a C^*-algebra $\mathfrak{A}$ with weak-operator closure $\mathcal{B}(\mathcal{H})$; and $\mathcal{U}$ is the ascending union of finite groups. Choosing an orthonormal basis for $\mathcal{H}$, $\mathcal{U}$ can be taken as the group generated by those unitary operators which either permute or reflect through 0 a finite number of basis elements and fix the others. Since $\mathcal{U}$ is an ascending union of finite groups, $\mathcal{U}$ has a (two-sided, invariant) mean μ. If φ is a bounded function from $\mathcal{U}$ into $\mathcal{B}(\mathcal{H})$, meaning $U \to (\varphi(U)x, y)$, for each pair of vectors x, y in $\mathcal{H}$, leads to a bounded bilinear functional on $\mathcal{H}$ and, thence, to an operator $\mu(\varphi)$ in $\mathcal{B}(\mathcal{H})$. If $\varphi(U) = U^*\delta(U)$, then $\delta(V) = VT - TV$ for V in $\mathcal{U}$, where $T = \mu(\varphi)$. This follows from meaning φ_V, where

$$\varphi_V(U) = \varphi(UV) = (UV)^*\delta(UV) = V^*U^*[U\delta(V) + \delta(U)V] =$$
$$= V^*\delta(V) + V^*U^*\delta(U)V.$$

From the properties of the mean, $T = V^*\delta(V) + V^*TV$. By linearity and norm continuity $\delta(A) = AT - TA$ for each A in $\mathfrak{A}$. At this point, we can make use of special (automatic) continuity properties of derivations [15 : Lemma 3], to conclude that $\delta(A) = AT - TA$ for all A in $\mathcal{B}(\mathcal{H})$. This continuity results from the observation that, if $I \geqslant A \geqslant 0$, then $\delta(A) = \delta(A^{1/2})A^{1/2} + A^{1/2}\delta(A^{1/2})$; so that, if (Ax, x) $(= \|A^{1/2}x\|^2)$ is small, $(\delta(A)x, x)$ is small.

The same argument, slightly embellished, will prove that derivations of type I von Neumann algebras are inner. More general results can also be proved by this method. If $\mathfrak{R}$ is a von Neumann algebra, $\mathfrak{M}$ is a two-sided (unital) $\mathfrak{R}$-module which is the dual of a Banach space $\mathfrak{M}_*$, and if the bilinear mappings $(A, m) \to Am$ and $(A, m) \to mA$ are bounded and w^* continuous in m, $\mathfrak{M}$ is said to be a *dual (Banach) $\mathfrak{R}$-module*. If these mappings are ultraweakly continuous in A (i.e. weak-operator continuous in A on bounded subsets of $\mathfrak{R}$), $\mathfrak{M}$ is said to be *normal*. The argument just sketched will show that a derivation of a type I von Neumann algebra $\mathfrak{R}$ into a normal dual $\mathfrak{R}$-module $\mathfrak{M}$ (i.e. a linear mapping δ of $\mathfrak{R}$ into $\mathfrak{M}$ such that $\delta(AB) = \delta(A)B + A\delta(B)$), has the form $A \to Am - mA$, for some m in $\mathfrak{M}$ [9 : Cor. 5.4]. In particular, if δ is a derivation of $\mathfrak{R}$ into $\mathcal{B}(\mathcal{H})$ there is a T in $\mathcal{B}(\mathcal{H})$ such that $\delta(A) = AT - TA$.

This module formulation of derivation results lends itself, at once, to considerations of cohomology of C^*-algebras with coefficients in a module [4,5]. With $\mathfrak{U}$ a C^*-algebra and $\mathfrak{M}$ a Banach $\mathfrak{U}$-module, let $C_c^n(\mathfrak{U}, \mathfrak{M})$ be the linear space of bounded n-linear mappings of $\mathfrak{U}$ into $\mathfrak{M}$. The coboundary operator Δ is defined [4] by :

$$(\Delta\rho)(A_1, \ldots, A_{n+1}) = A_1\rho(A_2, \ldots, A_{n+1}) - \rho(A_1A_2, A_3, \ldots, A_{n+1}) + \\ + \ldots \pm \rho(A_1, \ldots, A_{n-1}, A_nA_{n+1}) \mp \rho(A_1, \ldots, A_n)A_{n+1},$$

for ρ in $C_c^n(\mathfrak{U}, \mathfrak{M})$. Such mappings ρ are the (bounded) *n-cochains*. Those ρ for which $\Delta\rho = 0$ are the (bounded) *n-cocycles*. They form a subspace $Z_c^n(\mathfrak{U}, \mathfrak{M})$ of $C_c^n(\mathfrak{U}, \mathfrak{M})$. Since $\Delta\Delta = 0$; the n-cochains of the form $\Delta\xi$ with ξ an n-1-cochain are cocycles. They are the *n-coboundaries*. The factor space of $Z_c^n(\mathfrak{U}, \mathfrak{M})$ by the space $B_c^n(\mathfrak{U}, \mathfrak{M})$ of (bounded) n-coboundaries is the n-th *cohomology group* $H_c^n(\mathfrak{U}, \mathfrak{M})$ of $\mathfrak{U}$ with coefficients in $\mathfrak{M}$. Note that the 1-cocycles are those linear mappings δ of $\mathfrak{U}$ into $\mathfrak{M}$ such that

$$(\Delta\delta)(A, B) = A\delta(B) - \delta(AB) + \delta(A)B = 0$$

– that is to say, the derivations of $\mathfrak{U}$ into $\mathfrak{M}$. When $\mathfrak{M}$ is $\mathfrak{U}$ (with action given by the multiplication on $\mathfrak{U}$) the 1-cocycles become the standard derivations of $\mathfrak{U}$ into $\mathfrak{U}$. The 0-cochains are the constant mappings – the elements of $\mathfrak{M}$; and the coboundary of m is $Am - mA$ (at A). To say that a derivation δ of $\mathfrak{U}$ into $\mathfrak{M}$ cobounds is to say, then, that, for some m in $\mathfrak{M}$, $\delta(A) = Am - mA$, for each A in $\mathfrak{U}$. The theorem that the derivations of a von Neumann algebra $\mathfrak{R}$ (into itself) are inner is the assertion that $H_c^1(\mathfrak{R}, \mathfrak{R}) = 0$. In this framework, it is known that $H_c^n(\mathfrak{R}, \mathfrak{M}) = 0$ when $\mathfrak{R}$ is a type I von Neumann algebra and $\mathfrak{M}$ is a normal dual $\mathfrak{R}$-module. The same is true for all hyperfinite von Neumann algebras $\mathfrak{R}$ [6, 9, 12, 13].

If $\mathfrak{U}$ is a C^*-algebra and $\mathfrak{K}$ is a (norm-closed) two-sided ideal in $\mathfrak{U}$, then $\mathfrak{U}/\mathfrak{K}$ is, again, a C^*-algebra [3 : Prop. 1.8.2, p. 17]. The problem of "lifting" a derivation δ of $\mathfrak{U}/\mathfrak{K}$ to $\mathfrak{U}$ leads to considerations of 2-cohomology of $\mathfrak{U}$ with coefficients in $\mathfrak{C}$. If ξ is a (norm-continuous) linear mapping of $\mathfrak{U}$ into $\mathfrak{U}$ which lifts δ then $A\xi(B) - \xi(AB) + \xi(A)B (= \rho(A, B))$ is in $\mathfrak{C}$ for all A and B in $\mathfrak{U}$, Moreover $\Delta\rho = 0$; so that ρ is in $Z_c^2(\mathfrak{U}, \mathfrak{C})$. If $\rho = \Delta\eta$ with η in $C_c^1(\mathfrak{U}, \mathfrak{C})$, then $\xi - \eta$ lifts δ (as does ξ). As $\Delta(\xi - \eta) = \rho - \rho = 0$, $\xi - \eta$ is a derivation of $\mathfrak{U}$ into $\mathfrak{U}$.

A new element of difficulty enters the higher cohomology arguments by virtue of the fact that higher order cocycles do not enjoy the automatic continuity properties of derivations. Derivations of a C^*-algebra are norm continuous [8, 18] and ultraweakly continuous [15]. If ξ is a (norm) discontinuous linear mapping of $\mathfrak{R}$ into $\mathfrak{R}$, $\Delta\xi$ is a 2-coboundary (hence, 2-cocycle) which is not norm continuous (in general). Similarly, starting with ξ norm but not ultraweakly continuous, $\Delta\xi$ *must* fail to be ultraweakly continuous. A Tauberian result to the effect that if the 2-coboundary $\Delta\xi$ is ultraweakly continuous (in its first argument), then ξ is ultraweakly continuous [9 : Lemma 4.7], governs this situation. A sketch of the proof follows.

It suffices [22] to show that $\xi(\Sigma E_j) = \xi(I) = \Sigma\xi(E_j)$ (ultraweak convergence), where $\{E_j\}$ is a family of orthogonal projections in the von Neumann algebra $\mathfrak{R}$. By ultraweak continuity of $A \to (\Delta\xi)(A, B) = A\xi(B) - \xi(AB) + \xi(A)B$, with E_j for A and E_k for B, summing over j, we have

$$\xi(I)E_k = (\Delta\xi)(I, E_k) = (\Delta\xi)(\Sigma E_j, E_k) = (\Sigma E_j)\xi(E_k) - \xi(E_k) + \\ + (\Sigma\xi(E_j))E_k = (\Sigma\xi(E_j))E_k .$$

As this holds for each E_k and $\Sigma E_k = I$, $\Sigma\xi(E_j) = \xi(I)$. The same is not true for 3-cocycles ; for a (discontinuous) 2-coboundary can always be added to a 2-cochain without changing its coboundary.

The evidence is every strong that $H_c^n(\mathfrak{R}, \mathfrak{R}) = 0$ for a general von Neumann algebra $\mathfrak{R}$, but this remains to be completed. Although derivations of C^*-algebras are not inner, in general (the algebra generated by the compact operators and I illustrates this), there are special instances in which they are. The most striking of these is the case of simple C^*-algebras with a unit. For such algebras, all derivations are inner [17, 20, 21]. It may well be the case that all cohomology groups vanish for such algebras ; but this, too, awaits further study.

REFERENCES

[1] Blattner R. — Automorphic group representations. *Pacific J. Math.*, 8, 1958, p. 665-677.

[2] Dixmier J. — *Les algèbres d'opérateurs dans l'espace Hilbertien* (*algèbres de von Neumann*), 2nd edition. Gauthier-Villars, Paris, 1969.

[3] Dixmier J. — *Les C*-algèbres et leurs représentations*. Gauthier-Villars, Paris, 1964.

[4] Hochschild G. — On the cohomology groups of associative algebras. *Ann. of Math.*, 46, 1945, p. 58-67.

[5] HOCHSCHILD G. — On the cohomology theory for associative algebras. *Ann. of Math.*, 47, 1946, p. 568-579.
[6] JOHNSON B. — *Cohomology in Banach algebras.*
[7] JOHNSON B. and RINGROSE J. — Derivations of operator algebras and discrete group algebras, *Bull. London Math. Soc.*, 1, 1969, p. 70-74.
[8] JOHNSON B. and SINCLAIR A. — Continuity of derivations and a problem of Kaplansky. *Amer. J. Math.*, 90, 1968, p. 1067-1073.
[9] JOHNSON B., KADISON R., RINGROSE J. — Cohomology of operators algebras III. Reduction to normal cohomology.
[10] KADISON R. and RINGROSE J. — Derivations of operator group algebras. *Amer. J. Math.*, 88, 1966, p. 562-576.
[11] KADISON R. and RINGROSE J. — Derivations and automorphisms of operator algebras. *Comm. Math. Phys.*, 4, 1967, p. 32-63.
[12] KADISON R. and RINGROSE J. — Cohomology of operator algebras I. Type I von Neumann algebras. *Acta Math.*, 126, 1971.
[13] KADISON R. and RINGROSE J. — *Cohomology of operator algebras II. Extended cobounding and the hyperfinite case.* Arkiv für Mat., to appear.
[14] KADISON R. — Isomorphisms of factors of infinite type. *Canadian J. Math.*, 7, 1955, p. 322-327.
[15] KADISON R. — Derivations of operator algebras. *Ann. of Math.*, 83, 1966, p. 280-293.
[16] KAPLANSKY I. — Modules over operator algebras. *Amer. J. Math.*, 75, 1953, p. 839-859.
[17] LANCE E. — Inner automorphisms of UHF algebras. *J. London Math. Soc.*, 43, 1968, p. 681-688.
[18] SAKAI S. — On a conjecture of Kaplansky. *Tôhoku Math. J.*, 12, 1960, p. 31-33.
[19] SAKAI S. — Derivations of W*-algebras. *Ann. of Math.*, 83, 1966, p. 273-279.
[20] SAKAI S. — Derivations of uniformly hyperfinite C^*-algebras. *Pub. Res. Inst. Math. Sci.* (Kyoto), 3, 1967, p. 167-175.
[21] SAKAI S. — Derivations of simple C^*-algebras. *J. Functional Analysis,* 2, 1968 p. 202-206.
[22] TAKESAKI M. — On the conjugate space of an operator algebra. *Toohku Math. J.*, 10, 1958, p. 194-203.

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ORBITS AND QUANTIZATION THEORY

by Bertram KOSTANT

1. The representation $\operatorname{ind}(\eta_g, \mathfrak{h})$

Let G be a simply-connected Lie group and let $\mathfrak{g}$ be its Lie algebra. The (real) dual space $\mathfrak{g}'$ to $\mathfrak{g}$ is a module for G with respect to the coadjoint representation.

If $g \in \mathfrak{g}'$ let $G_g \subseteq G$ be the isotropy subgroup at g. The Lie algebra $\mathfrak{g}_g$ of G_g is the set of all $x \in \mathfrak{g}$ such that g vanishes on Im ad x. Thus

$$< g , [\mathfrak{g}_g, \mathfrak{g}] > = 0 \quad .$$

In particular g vanishes on the commutator $[\mathfrak{g}_g, \mathfrak{g}_g]$ so that

$$2\pi i g : \mathfrak{g}_g \to i\mathbf{R} \tag{1.1}$$

is a homomorphism of Lie algebras where $i\mathbf{R}$ is regarded (via ordinary exponentiation) as the Lie algebra of the circle group $\mathbf{T}$.

The linear function g on $\mathfrak{g}$ will be extendable if there exists a character

$$\eta_g : G_g \to \mathbf{T}$$

whose differential is the restriction $2\pi i g | \mathfrak{g}$.

Remark 1. — Since G_g is not in general connected, note that extendability may be more than just saying that (1.1) induces a character on the subgroup G_g^0 of G corresponding to $\mathfrak{g}_g$.

In fact, given $g \in \mathfrak{g}'$, let $\mathcal{L}_g$ be the set of all characters η on G_g whose differential is given by (1.1).

Let $0 = G \cdot g$ be the orbit of G in $\mathfrak{g}'$ through g so that as a homogeneous space 0 is isomorphic to G/G_g. But the fundamental group $\Pi = \Pi_1(0)$ of 0 is naturally isomorphic with the quotient group G_g/G_g^0. Any character μ on Π may then be regarded as a character on G_g which is trivial on G_g^0. But then if $\eta \in \mathcal{L}_g$ and $\mu \in \Pi^*$ (the character group of Π) one defines a new element $\eta' \in \mathcal{L}_g$ by the relation

$$\eta'(a) = \mu(a)\, \eta(a) \tag{1.2}$$

for any $a \in G_g$. This is clear since η' and η have the same differential. But clearly any character on G_g with this differential can be obtained this way. Thus we have

PROPOSITION 1. — *If $\mathcal{L}_g$ is not empty (that is g is extendable) then $\mathcal{L}_g$, using* (1.2), *has the structure of a principal homogeneous space for Π^*, the character group of the fundamental group of the G orbit 0 through g.*

In [*I*] Auslander and I studied the unitary representation theory of G in the case where G is solvable. One knows that unlike the semi-simple or nilpotent case there is the question of type. With regard to this we have proved

THEOREM 1. — *If G is solvable then it is type I if and only if*

(1) *every linear function $g \in \mathfrak{g}'$ is extendable and*

(2) *all G orbits $O \subseteq \mathfrak{g}'$ are locally closed (the intersection of an open and closed set).*

Remark 2. — Examples exist of a non-type *I* solvable group G which satisfy (1) or (2), but of course, not both of the conditions.

1.2. Now return to the general case and let $g \in \mathfrak{g}'$. One knows that $\dim \mathfrak{g}/\mathfrak{g}_g$ is even, say $2n$. Indeed as observed first by Kirillov the alternating bilinear form $\{x, y\} = \langle g, [y, x] \rangle$ on $\mathfrak{g}$ induces a non-singular alternating bilinear form B_g on $\mathfrak{g}/\mathfrak{g}_g$. A polarization at g is a complex subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_{\mathbf{C}}$ (the complexification of $\mathfrak{g}$) which satisfies the following conditions (1) $\mathfrak{h}$ is stable under Ad G_g, (2) $\dim_{\mathbf{C}} \mathfrak{g}_{\mathbf{C}}/\mathfrak{h} = n$, (3) $\langle g, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$ and (4) $\mathfrak{h} + \overline{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_{\mathbf{C}}$. (Conjugation is over the real form $\mathfrak{g}$).

The polarization $\mathfrak{h}$ defines 2 subalgebras $\mathfrak{d}$ and $\mathfrak{e}$ of $\mathfrak{g}$ stable under Ad G_g where $\mathfrak{d} = \mathfrak{g} \cap \mathfrak{h}$ and $\mathfrak{e} = \mathfrak{g} \cap (\mathfrak{h} + \overline{\mathfrak{h}})$. If $\dim_{\mathbf{R}} \mathfrak{g}/\mathfrak{e} = k$ then one has

$$\underbrace{\mathfrak{g}_g \subseteq \mathfrak{d}}_{k} \underbrace{\subseteq \mathfrak{e}}_{2(n-k)} \underbrace{\subseteq \mathfrak{g}}_{k}$$

where the integer below each consecutive pair is the codimension of the first in the second of the pair. Furthermore there is a natural non-singular symmetric bilinear form S_g on $\mathfrak{e}/\mathfrak{d}$ defined by

$$S_g(u, v) = \overline{B}_g(ju, v)$$

where $\overline{B}_g$ is the non-singular alternating form on $\mathfrak{e}/\mathfrak{d}$ defined by B_g and j is the operator on $\mathfrak{e}/\mathfrak{d}$ satisfying $j^2 = -1$ given so that $j = -i$ on $\mathfrak{h}/\mathfrak{h} \cap \overline{\mathfrak{h}}$ and $j = i$ on $\overline{\mathfrak{h}}/\mathfrak{h} \cap \overline{\mathfrak{h}}$. The polarization $\mathfrak{h}$ is called positive if S_g is positive definite. The polarization is said to satisfy the Pukansky condition in case the orbit $E_0 \cdot g \subseteq \mathfrak{g}'$ is closed where E_0 is the subgroup of G corresponding to $\mathfrak{e}$.

Now assume $g \in \mathfrak{g}'$ is integral and let $\mathfrak{h}$ be a polarization at g satisfying the Pukansky condition. Let $\eta_g \in \mathcal{L}_g$. Then one defines a unitary representation $\operatorname{ind}(\eta_g, \mathfrak{h})$ of G as follows. First of all if D_0 is the subgroup of G corresponding to $\mathfrak{d}$ then $D = G_g D_0$ is a closed subgroup of G and the character η_g extends in a natural way to a character χ_g on D. The representation $\operatorname{ind}(\eta_g, \mathfrak{h})$ is a subrepresentation of the induced representation $\operatorname{ind}_G \chi_g$. To describe this subrepresentation, let $\mathcal{H}$ be the module for $\operatorname{ind}_G \chi_g$. Now one has $E = G_g E_0$ is also a closed subgroup of G. Also we may regard $e_{\mathbf{C}} = \mathfrak{h} + \overline{\mathfrak{h}}$ as complex left invariant vector fields on E. Then if $\mathcal{H}(\eta_g, \mathfrak{h})$ is the set of all $\varphi \in \mathcal{H}$ which are smooth functions on E and satisfy on E the differential equations

$$x \cdot \varphi = 2\pi i g(x) \varphi$$

for all $x \in \mathfrak{h}$ it follows that $\mathcal{H}(\eta_g, \mathfrak{h})$ is a closed subspace of $\mathcal{H}$ stable under G. The restriction to $\mathcal{H}(\eta_g, \mathfrak{h})$ defines $\operatorname{ind}(\eta_g, \mathfrak{h})$.

1.3. Now assume again that G is solvable. Let $\mathfrak{n} \subseteq \mathfrak{g}$ be the nilradical. Let $g \in \mathfrak{g}'$ and $f = g|\mathfrak{n}$. The group G naturally operates on $\mathfrak{n}'$ and we let G_f be the isotropy group at f. A polarization $\mathfrak{h}$ at g is called admissible if $\mathfrak{h}$ is stable under G_f, positive and such that $\mathfrak{h} \cap n_{\mathbf{C}}$ is a polarization at f. Any admissible polarization $\mathfrak{h}$ automatically satisfies the Pukansky condition so that if g is integral we can form the unitary representation $\operatorname{ind}(\eta_g, \mathfrak{h})$ for any $\eta_g \in \mathcal{L}_g$. The second main result in [1], generalizing Kirillov's result for nilpotent groups, is

THEOREM 2. — *If G is solvable and $g \in \mathfrak{g}'$ is arbitrary then there exists an admissible polarization $\mathfrak{h}$ at g. Moreover if g is integrable and $\eta_g \in \mathcal{L}_g$ then* $\operatorname{ind}(\eta_g, \mathfrak{h})$ *is (1) irreducible and (2) is independent of the choice of admissible polarization. Furthermore if G is of type I then any irreducible unitary representation is equivalent to a representation of this form.*

The independence of polarization in Theorem 2 enables us to assign for any solvable G and integral $g \in \mathfrak{g}'$ an irreducible unitary representation $\pi(\chi_g)$ for each $\chi_g \in \mathcal{L}_g$, But now if $f, g \in \mathfrak{g}'$ lie on the same orbit 0 then $\mathcal{L}_f$ and $\mathcal{L}_g$ are canonically isomorphic and if χ_f corresponds to χ_g then $\pi(\chi_f)$ and $\pi(\chi_g)$ are equivalent.Thus for each integral orbit 0 (an orbit through integral points) we may introduce a set $\mathcal{L}(0)$ which is naturally isomorphic to $\mathcal{L}_f$ for each $f \in 0$. Also to each $c \in \mathcal{L}(0)$ one associates an equivalence class $\pi(c)$ of irreducible unitary representations. Our final main result in [1] also generalizes a result of Kirillov since in the nilpotent, case $\mathcal{L}(0)$ reduces to one element.

THEOREM 3. — *Assume G is solvable simply connected of type I and let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G. Then the correspondence $c \to \pi(c)$ sets up a bijection*

$$\cup \mathcal{L}(0) \to \hat{G}$$

where the union is over all orbits 0 (see Theorem 1).

2. Quantization 1.

The construction of the representation $\operatorname{ind}(\eta_g, \mathfrak{h})$ seemingly technical and without very much motivation does in fact arise quite naturally from a quantization theory which we now describe. The notion of quantization presently to be made precise incorporates the idea associated by physicists to that term. Roughly speaking, it is an operation which associates operators to certain functions on a symplectic manifold.

Let (X, ω) be a symplectic manifold. That is X is an even dimensional $(2n)$ manifold and ω is a real closed 2-form on X which is everywhere non-singular. Now one may associate to (X, ω) an exact sequence of Lie algebras

$$0 \to \mathbf{C} \to C \to \mathfrak{a} \to 0 \ .$$

Here α is the Lie algebra of all smooth hamiltonian vector fields on X and C is the space of all smooth functions on X made into a Lie algebra by Poisson bracket. The classical map $C \to \alpha$ is given by $\varphi \mapsto \xi_\varphi$ where if $i(\xi_\varphi)$ denotes the interior product, then ξ_φ is the vector field defined by $i(\xi_\varphi)\,\omega = d\varphi$. The exact sequence makes C into a (central) extension of α by the constants $\mathbf{C}$ and in general the extension is not split.

Now the symplectic manifold (X, ω) will be called integral if the de Rham class $[\omega] \in \mathcal{H}^2(X, \mathbf{R})$ of ω lies in the image of $\mathcal{H}^2(X, \mathbf{Z})$ in $\mathcal{H}^2(X, \mathbf{R})$. The first key point in the notion of *prequantization* is that if (X, ω) is integral then C admits natural modules S which are not modules for α (that is, they are non-trivial, for the constants $\mathbf{C}$). To obtain these modules we first recall that if

$$\begin{array}{ccc} \mathbf{C} \to L \\ \downarrow \\ X \end{array}$$

is a line bundle with connection α (α is a suitable 1-form in the associated principal bundle L^*) over X then the curvature, written $\text{curv}(L, \alpha)$, is a closed 2-form on X. In fact $\text{curv}(L, \alpha)$ is the 2-form which pulls up to $d\alpha$. But now, starting with a closed 2-form ω on X, as one does if (X, ω) is a symplectic manifold, then one shows that a line bundle with connection (L, α) exists over X such that

$$\omega = \text{curv}(L, \alpha) \tag{2.1}$$

if and only if (X, ω) is integral. Assume this to be the case and let (L, α) satisfy (2.1). A module S for C is obtained as follows : Let S be the space of all smooth sections $s : X \to L$ of the line bundle L. Then if

$$\nu : C \to \text{End}\, S$$

is the map obtained by letting $\nu(\varphi)$, for all $\varphi \in C$, be the operator given by

$$\nu(\varphi)\, s = (\nabla_{\xi_\varphi} + 2\pi i \varphi)\, s$$

(Here $s \in S$ and ∇_{ξ_φ} is covariant differentiation by the hamiltonian field ξ_φ) then ν is in fact a representation of C. That is $\nu[\varphi, \psi] = [\nu(\varphi), \nu(\psi)]$ for any $\varphi, \psi \in C$. Also note that $\nu(1) \neq 0$ so that S is not an α-module. The assignment $\varphi \to \nu(\varphi)$ we call prequantization. Motivation for ν may be given by the fact that the operators on S of the form $\nu(\varphi)$ exactly correspond to all vector fields on L^* whose Lie derivative annihilates α and which commute with the action of $\mathbf{C}^*$.

Now the module S depends upon the choice of (L, α). In a natural way one defines an equivalence relation on the set of all pairs (L, α) such that $\omega = \text{curv}(L, \alpha)$. In fact given (L, α) one defines a $\mathbf{C}^*$-valued function Q on the set Γ of all closed piece-wise smooth curves γ on X. For each $\gamma \in \Gamma$, $Q(\gamma) \in \mathbf{C}^*$ is the multiple in the line at a point $p \in \gamma$ obtained by parallel transport of the line around γ. One shows easily that 2 line bundles with connection are equivalent if and only if they define the same Q function. Also the Q function takes its values on the unit circle $\mathbf{T}$ if and only if there is an invariant (under parallelism) Hermitian structure on L. Thus if $\mathcal{L}_c(X, \omega)$ is the set of all equivalence classes of (L, α) admitting an invariant

Hermitian structure and such that $\omega = \mathrm{curv}(L, \alpha)$ then for each $\ell \in \mathcal{L}_c(X, \omega)$ one defines a T-valued function Q^{ℓ} on Γ. Also prequantization ν and the module S depend (up to equivalence) only on ℓ so that one can write ν^{ℓ} and S^{ℓ}. Now if $\Pi^* = \Pi^*(X)$ is the character group of the fundamental group of X then Π^* operates on $\mathcal{L}_c(X, \omega)$ by the relation

$$Q^{\mu\ell}(\gamma) = \mu(\gamma)\, Q^{\ell}(\gamma)$$

for all $\mu \in \Pi^*$, $\ell \in \mathcal{L}_c(X, \omega)$ and $\gamma \in \Gamma$. The set (parametrized by $\mathcal{L}_c(X, \omega)$) of all possible (Hermitian) prequantizations for C is given by

PROPOSITION 2. — *If the symplectic manifold* (X, ω) *is integral then* $\mathcal{L}_c(X, \omega)$ *is a principal homogeneous space for the fundamental group* $\Pi^*(X)$.

If X is simply connected, Proposition 2 says that $\mathcal{L}_c(X, \omega)$ has exactly 1-element so that prequantization is unique. In fact in that case the unique Q function is given by

$$Q(\gamma) = e^{-2\pi i \int_\sigma \omega}$$

where σ is any surface of deformation of the curve γ.

2.2. Now assume that (X, ω) is a homogeneous space of a simply connected Lie group G with Lie algebra $\mathfrak{g}$ in such a way that the differential induces a homomorphism $\sigma : \mathfrak{g} \to \mathfrak{a}$ (strongly symplectic homogeneous space). If (X, ω) is integral the modules S are not available to $\mathfrak{g}$ and hence to G unless one has a homomorphism $\lambda : \mathfrak{g} \to C$ lifting σ and hence giving rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{C} & \to & C & \to & \mathfrak{a} & \to & 0 \\ & & & & \nwarrow & & \uparrow{\scriptstyle \sigma} & & \\ & & & & & & \mathfrak{g} & & \end{array}$$

The obstruction to λ existing is evidently an element $[\sigma] \in \mathcal{H}^2(\mathfrak{g}, \mathbf{C})$.

Now if $0 \subseteq \mathfrak{g}$ is an orbit for G then the bilinear forms $B_g, g \in 0$, (see § 1.2) define a closed 2-form ω_0 on 0 giving $(0, \omega_0)$ the structure of a strongly symplectic homogeneous space for G. In this case a lifting λ exists and in fact for $x \in \mathfrak{g}$ one has $\lambda(x)(g) = \langle g, x \rangle$ for any $g \in 0$. Thus if $(0, \omega_0)$ is integral, prequantization will give rise to modules $(\nu^{\ell} \circ \lambda)$ for $\mathfrak{g}$ and hence for G. But it is only for orbits $0 \subseteq \mathfrak{g}'$ and their covering spaces that one can do this.

THEOREM 4. — *If* (X, ω) *is a strongly symplectic homogeneous space for G then a lifting* λ *exists if and only if* (X, ω) *covers an orbit* $(0, \omega_0)$, *in* $\mathfrak{g}'$.

Remark 3. — If $\mathcal{H}^1(\mathfrak{g}, \mathbf{C}) = 0$, i.e. if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ then any symplectic homogeneous space is strongly symplectic and if, in addition, $\mathcal{H}^2(\mathfrak{g}, \mathbf{C}) = 0$, e.g. if $\mathfrak{g}$ is semi-simple, then the most general symplectic homogeneous space is a covering of an orbit 0 in $\mathfrak{g}'$.

Now let $0 \subseteq \mathfrak{g}'$ be an orbit and let $g \in 0$. We now find that the question of integrality of $(0, \omega_0)$ is the same as the extendability of g (see § 1.1).

THEOREM 5. — $(\mathcal{O}, \omega_0)$ *is integral if and only if* $g \in \mathcal{O}$ *is extendable. Moreover there exists a natural bijection* $\mathcal{L}_c(\mathcal{O}, \omega_0) \to \mathcal{L}_g$, $\ell \to \eta^\ell$ *and the bijection commutes with the action of the character group* $\Pi^*(\mathcal{O})$ *on the fundamental group of* $\mathcal{O}$. *Furthermore the unitary representations of G defined by exponentiating* $\nu^\ell \circ \lambda$ *is just* $\mathrm{ind}_G\, \eta^\ell$.

One consequence is

COROLLARY 6. — *The condition of extendability in Theorem* 1 *for a solvable G to be type I may be replaced by the condition that the de Rham class* $[\omega_0]$ *should be zero for all orbits* $\mathcal{O} \subseteq \mathfrak{g}'$.

A polarization of a symplectic manifold (X^{2n}, ω) is a complex involutory distribution F of dim n such that (1) $\omega(F_p, F_p) = 0$ at all $p \in X$ and (2) $F + \overline{F}$ is an involutory distribution of constant dimension on X. The polarization F distinguishes a Lie algebra C_F^1 of C by defining C_F^1 to be the set of all functions $\varphi \in C$ such that the corresponding hamiltonian fields ξ_φ preserve (by Lie differentiation) F. Furthermore if (X, ω) is integral the polarization also distinguishes a subspace S_F^ℓ of S^ℓ, $\ell \in \mathcal{L}_c(X, \omega)$. One defines S_F^ℓ to be the space of all sections $s \in S^\ell$ such that under (covariant differentiation) $\nabla_v s = 0$ for all $v \in F_p$ and all $p \in X$. The subspace S_F^ℓ is stable under $\nu^\ell(C_F^1)$ inducing a representation

$$\nu_F^\ell : C_F^1 \to \mathrm{End}\, S_F^\ell \quad .$$

The assignment $\varphi \mapsto \nu_F^\ell(\varphi)$ for $\varphi \in C_F^1$ is called quantization.

If an integral (X, ω) is a strongly symplectic homogeneous space for G with a lifting λ then a polarization F of (X, ω) yields a module $\nu_F^\ell \circ \lambda$ for $\mathfrak{g}$ provided the image of λ lies in C_F^1, that is, if one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{C} & \to & C & \to & \mathfrak{g} & \to & 0 \\ & & & & \uparrow & \nwarrow & \uparrow & & \\ & & & & C_F^1 & \leftarrow & \mathfrak{g} & & \end{array}$$

In case of an orbit $\mathcal{O} = G \cdot g$ all such polarizations exactly correspond to all polarizations $\mathfrak{h}$ at g defined in § 1.2 setting up a bijection $\mathfrak{h} \to F(\mathfrak{h})$. The representation $\mathrm{ind}(\eta^\ell, \mathfrak{h})$ of G defined in § 1.2 is just the unitary representation obtained by exponentiating $\nu_F^\ell(\mathfrak{h}) \circ \lambda$.

REFERENCES

[1] KOSTANT B. — *Quantization and Unitary Representations,* Lectures in Modern Analysis and Applications III, *Lecture Notes in Mathematics,* Vol. 170, Springer-Verlag.

[2] AUSLANDER L. and KOSTANT B. — Polarization and Unitary Representations of Solvable Lie Groups, to appear *Inventiones Mathematicae.*

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ERGODICITY IN THE THEORY OF GROUP REPRESENTATION *

by George W. MACKEY

Let G be a separable locally compact group and let S be a standard Borel space on which G acts as a group of Borel automorphisms. More precisely let sx be defined for all $s \in S$, $x \in G$ so that for

$$s \in S\,;\quad x\,, x_1, x_2 \in G\,;\quad (sx_1)x_2 = sx_1x_2\,,\quad se = s$$

and $s, x \to sx$ is a Borel function. Given any σ finite measure μ defined on all Borel sets and given any $x \in G$ we obtain a new measure μ_x by setting $\mu_x(E) = \mu(Ex)$ for all Borel sets $E \subseteq S$. If μ is invariant in the sense that $\mu_x = \mu$ for all x then we obtain a unitary representation V of G with Hilbert space $\mathcal{L}^2(S, \mu)$ by defining $V_x(f)(s) = f(sx)$. Suppose now that μ is not invariant but is *quasi invariant* in the sense that μ and μ_x have the same sets of measure zero for all x. Then there exists a Borel function ρ from $S \times G$ to the positive real numbers such that if we define $V_x(f)(s) = \rho(s, x) f(sx)$ then $x \to V_x$ is a unitary representation of G and ρ is unique modulo changes on sets of measure zero. For each fixed x, ρ^2 is a Radon Nikodym derivative of μ_x with respect to μ. Finally let A be a Borel function from $S \times G$ to the unitary operators in some separable Hilbert space $\mathcal{H}$. Then for each $x \in G$ we may define a unitary operator V_x^A in the Hilbert space $\mathcal{L}^2(S, \mu, \mathcal{H})$ by letting $(V_x^A f)(s) = \rho(s, x) A(s, x) f(sx)$, and an easy calculation shows that $x \to V_x^A$ is a unitary representation of G if and only if A is a normalized Borel cocycle in the sense that for all $x_1, x_2 \in G$ we have $A(s, e) = I\; A(s, x_1x_2) = A(s, x_1)\, A(sx_1, x_2)$ for almost all $s \in S$.

Suppose that S admits a decomposition $S = S_1 \cup S_2$ where $S_1 \cap S_2 = 0$, $\mu(S_1)\,\mu(S_2) \neq 0$, $S_1x = S_1$, $S_2x = S_2$ for all $x \in G$. Then S_1 and S_2 are Borel G-spaces in their own right and we obtain normalized Borel cocycles A^1 and A^2 by restricting A to $S_1 \times G$ and $S_2 \times G$ respectively. Evidently V^A is equivalent to the direct sum of V^{A_1} and V^{A_2}. Thus the most interesting case to consider is that in which μ is *ergodic* in the sense that no such decomposition is possible and in what follows we shall restrict attention to that case.

If $\mathcal{H}'$ is a Hilbert space of the same dimension as $\mathcal{H}$ and B is Borel function from S to the set of all unitary maps of $\mathcal{H}$ on $\mathcal{H}'$ then we obtain a unitary map $\widetilde{B}$ from $\mathcal{L}^2(S, \mathcal{H}, \mu)$ to $\mathcal{L}^2(S, \mathcal{H}', \mu)$ by writing $(\widetilde{B}f)(s) = B(s) f(s)$ and one computes at once that $x \to \widetilde{B}\, V_x^A\, \widetilde{B}^{-1}$ is identical with V^{A^1} where

$$A^1(s, x) = B(s)\, A(s, x)\, B^{-1}(sx)\,.$$

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(1) This report was written while the author held a John Simon Guggenheim memorial fellowship.

Thus in particular A^1 is a normalized Borel cocycle. Two normalized Borel cocycles related by a B as are A^1 and A will be said to be *cohomologous.* Also we shall not distinguish between normalized Borel cocycles which are almost everywhere equal. For brevity we shall refer to a cohomology class of normalized Borel cocycles as a *cocycle class.* Given an ergodic quasi-invariant measure μ in S the mapping $A \to V^A$ defines a mapping from cocycle classes to equivalence classes of unitary representations of G.

Ergodic quasi-invariant measures fall sharply into two classes ; those that are *properly ergodic* in the sense that every G orbit is of measure zero and those that are *essentially transitive* in the sense that some G orbit is of positive measure. In the latter case the complement of the orbit of positive measure must (by ergodicity) be of measure zero. Hence ignoring an invariant null set we are reduced to thecase in which the action is transitive in the sense that there is only one orbit. In the transitive case S may be identified with the coset space G/H where H is the closed subgroup leaving an origin fixed in S. Choosing different origin merely replaces H by one of its conjugates.

The transitive case is interesting in that we can in a sense determine all cocycle classes. More precisely we may set up a natural one-to-one correspondence between the cocycle classes and the equivalence classes of unitary representations of the closed subgroup H. Modulo resolvable difficulties about sets of measure zero one obtains this correspondence by observing that the identity

$$A(s, x_1, x_2) = A(s, x_1)\, A(sx_1, x_2)$$

becomes $A(s, x_1, x_2) = A(s, x_1)\, A(s, x_2)$ whenever $x_1 \in H$, $x_2 \in H$ and $sH = s$. Combining this correspondence with $A \to V^A$ we obtain a mapping from equivalence classes of unitary representations of H to equivalence classes of unitary representations of G which turns out to be identical with that defined by the now well-known "inducing" construction $L \to U^L$.

The natural one-to-one correspondence between cocycle classes and equivalence classes of unitary group representations of H suggests that one should be able to express the main concepts of the unitary representation theory of H in terms of the cocycle classes for the system G/H, G, μ. This turns out to be the case. For example given normalized Borel cocycles $A^1 \dots A^r$ one can define their direct sum A by the equation

$$A(s, x) = A^1(s, x) \oplus A^2(s, x) + \dots A^r(s, x)$$

and obtain a normalized Borel cocycle whose cohomology class depends only upon that of the A^j. One shows without difficulty that this notion of direct sum coincides with that which one obtains by passing to the corresponding unitary representations of H and one generalises easily from direct sums to direct integrals. In a similar spirit one defines the tensor product of the normalized Borel cocycles A^1 and A^2 to be the normalized Borel cocycle

$$s, x \to A^1(s, x) \otimes A^2(s, x)$$

and one defines an "intertwining operator" for the Borel cocycles A^1 and A^2 with Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$ to be a Borel function T from S to the bounded linear operators from $\mathcal{H}^1$ to $\mathcal{H}^2$ such that for all x

$$A^2(s, x)\, T(sx) = T(s)\, A^1(s, x)$$

for almost all s and one identifies functions T which are almost everywhere equal.

The fact that this translation from properties of unitary representations to properties of cocycle classes is possible is interesting because once the translation is made the definition in no way depends upon the transitivity. In other words the definitions of direct sums, direct integrals, intertwining operators, tensor products etc. make sense for the cocycle classes attached to a properly ergodic action. Moreover to a large extent the theorems which one can prove about unitary irreducible representations of H continue to be true for the cocycle classes attached to a properly ergodic action. In other words for every ergodic action one has a theory of the associated cocycle classes which is very closely analogous to the unitary representation theory of a group and in fact reduces to such a theory when the action is essentially transitive. In particular every finite dimensional cocycle class is uniquely a direct sum of irreducible cocycle classes and one has close analogues to the more elaborate decomposition theorems which one can prove for infinite dimensional unitary representations of a group.

Now let φ be a Borel mapping from S onto second standard Borel space $\widetilde{S}$ and suppose that for each $z \in \widetilde{S}$ we have $\varphi^{-1}(z)x = \varphi^{-1}(z')$ for some $z' \in \widetilde{S}$. Then setting $zx = \varphi(\varphi^{-1}(z)x)$ we convert $\widetilde{S}$ into a Borel G space. Moreover defining $\widetilde{\mu}(E) = \mu(\varphi^{-1}(E)$ we obtain a quasi invariant measure $\widetilde{\mu}$ in $\widetilde{S}$. $\widetilde{\mu}$ is clearly ergodic whenever μ is ergodic and transitive whenever μ is transitive. However $\widetilde{\mu}$ can be transitive even when μ is properly ergodic. For example S can always be taken to be a one point space. In the special case in which the actions on S and $\widetilde{S}$ are both transitive so that $S = G/H$, $\widetilde{S} = G/\widetilde{H}$ one can always choose H so that $H \subseteq \widetilde{H}$. Conversely of course if $\widetilde{H} \supseteq H$, $\widetilde{S}$ can be obtained from H as indicated above. Now an important part of the theory of group representation has to do wtih the relationship between the representations of a group and those of its subgroups. This suggests that one seek relationships between cocycle classes on $S \times G$ and $\widetilde{S} \times G$ which in the transitive case correspond to the familiar relationship between the representations of H and $\widetilde{H}$.

For example if A is a normalized Borel cocycle on $\widetilde{S} \times G$ we obtain a normalized Borel cocycle A^1 on $S \times G$ by setting $A^1(s, x) = A(\varphi(s), x)$ and one shows easily that the cocycle class of A^1 depends only on that of A. The resulting mapping of cocycle classes for $\widetilde{S} \times G$ into those for $S \times G$ translates in the transitive case to the operation of restricting a unitary representation from $\widetilde{H}$ to H. The dual operation of inducing a representation from H to $\widetilde{H}$ can also be expressed in terms of cocycle classes, and so extended to the properly ergodic case. We shall give details only in the relatively simple special case in which each $\varphi^{-1}(z)$ is a finite set. In that case it follows from ergodicity that the cardinal number of $\varphi^{-1}(z)$ is almost everywhere the same and hence without loss of generality can be assumed to be a constant n. Given a normalized Borel cocycle A in $S \times G$ we define $A^1(s, x)$ to be $\sum_{s \in \varphi^{-1}(z)} \oplus A(s, x)$. The resulting map from cocycle classes for $S \times G$ to cocycle classes for $\widetilde{S} \times G$ translates into the inducing map for unitary representations. More generally, one must use the theory of fiber decomposition of measures to put a measure on each $\varphi^{-1}(z)$ and replace $\Sigma \oplus A(s, x)$ by a corresponding direct integral. There is a problem here about identifying the Hilbert spaces for the different $\varphi^{-1}(z)$ but it is easily overcome. In the extreme case in which $\widetilde{S}$ has only one point the cocycle classes for $\widetilde{S} \times G$ correspond one-to-one to the equivalence classes of unitary representations of G and our generalization of the inducing construction is (modulo equivalences) just the mapping $A \to V^A$ with which we started our discussion.

Once these definitions have been made the question presents itself of finding the extent to which known theorems in the theory of induced representations have generalizations valid for induced cocycle classes. Little detailed work has been done on this question but preliminary investigations suggest that one will have only routine difficulties to face in proving suitable generalizations of the Frobenius reciprocity theorem, the imprimitivity theorem, the stages theorem, the subgroup theorem etc.

Now let us look more closely at the concept of a properly ergodic action of a group on a measure space. It is in a way a rather bizarre notion whose very existence is at first a little surprising. If every orbit is so small as to have measure zero how can there fail to be non trivial measurable invariant subsets. On the other hand the existence of this notion is at the very heart of the modern theory of probability – although this fact is not as widely appreciated at it might be. The point here is that it is the existence of properly ergodic actions of the integers which makes it possible for an infinite sequence to have terms which vary in a "random" way. Indeed it is no doubt this connection between "randomness" and proper ergodicity which gives to the latter notion its rather bizarre and apparently pathological character. To be just a bit more specific, let me remark that one can give a short argument justifying the following statement. If the integers did not admit a properly ergodic action then for every discrete stationary stochastic process with probability one the results of an infinite sequence of trials would be periodic.

Accordingly it is especially interesting to find that every separable locally compact group (which is not compact) admits a great many properly ergodic actions and (as we have seen in detail above) that they behave individually and in their relationship to one another in very much the same ways as the closed subgroups of the group behave. Indeed for many purposes one cannot study closed subgroups and properly ergodic actions in isolation from one another but must consider them together as different kinds of instances of one concept. From this point of view ergodic theory is not only an essential component of the theory of locally compact groups and their representations but cannot be properly understood without looking at it as such. Elsewhere [1] I have given several examples in which concepts which seemed rather special to ergodic theory can be interpreted in an illuminating way, by regarding a properly ergodic action as an analogue of a closed subgroup.

In view of the significance of properly ergodic actions for probability theory it is interesting to note that one has a natural probabilistic interpretation for normalized Borel cocycles. This is easiest to explain in the case in which the quasi-invariant measure μ in S is finite and invariant and the group G is countable and discrete. Let K be any group with a Borel structure e.g. the group of all unitary operators in an Hilbert space $\mathcal{H}$. Then the group K^G of all functions from G to K has a natural Borel structure and G acts on this group by translation. When G is the additive group of all the integers and K is the additive group of all the real numbers one obtains the conventional probability theory model for a discrete real valued stochastic process by putting a probability measure μ in the group K^G and this process is said to be *stationary* if μ is invariant under translation. More generally one can consider random variables parameterized by other groups than the integers. In statistical mechanics for example one is interested in the case in which G is a discrete subgroup of the Euclidean group and one can have random variables which are K valued rather than real valued. A model for such would be obtained by taking a G invariant probability measure in the appropriate K^G. More generally still one can consider the coset space K^G modulo the subgroup K of all constant functions and choose a G invariant probability measure in the coset space. If this measure is ergodic under G it defines an ergodic action G and a normalized Borel cocycle with values in K. Conversely every pair consisting of an ergodic action of G and a normalized Borel cocycle can be obtained in this way from an invariant probability measure in the coset space, modulo passage to an equivalence class space S.

From a probabilistic point of view choosing an invariant probability measure in K^G/K amounts to choosing a K valued, "G stationary" stochastic process in which only increments are defined. In other words we replace the notion of a random function by that of a function with random increments.

BIBLIOGRAPHY

[1] Mackey George W. — Ergodic theory and virtual groups. *Math. Annalen*, 166, 1966, p. 187-207.

[2] Mackey George W. — *Induced representation of groups and quantum mechanics*, W.A. Benjamin, 1968, Chap. V.

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SOME PROBLEMS AND RESULTS IN REPRESENTATION THEORY OF COMPLEX SEMISIMPLE LIE GROUPS

by M. A. NAIMARK

1. Introduction.

In what follows G denotes a connected complex semisimple Lie group, $\mathfrak{U}$ its maximal compact subgroup, X its group algebra (the convolution algebra of all infinitely differentiable function on G with compact support supplied with the Schwartz topology).

The main purpose of this lecture is to formulate some problems. Results will be exposed in so far as they are necessary for formulation of the problems.

2. Fundamental notions.

We consider representations $T : g \to T_g$ of G in a complete locally convex space E which are continuous in the following sense : the correspondence $\{g, \xi\} \to T_g \xi$ is a continuous mapping of $G \times E$ into E.

Let $C(E)$ be the algebra of all continuous linear operators $A : E \to E$ supplied with the weak topology (the locally convex topology defined by the seminorms $p_{\xi\eta}(A) = |(A \xi, \eta)|$, $\xi \in E$, $\eta \in E'$, where E' is the dual to E space and (ξ, η) is the canonical bilinear form between E and E'). T is called completely irreducible (Zhelobenko [1]), if the closed linear hull of all T_g contains $C(E)$.

This is a generalization of a definition given by Godement [2] for representations in Banach spaces. We recall that :

2.1. Complete irreducibility implies topological irreducibility ; for unitary representations the converse is also true.

A representation T in E is called a chain of representations $T^1, T^2, \dots$, if in E closed subspaces $E^1 \subset E^2 \subset \cdots$ exist with the following properties :

(1) the closed linear hull of all E^j oincides with E ;

(2) every E^j is invariant with respect to T ;

(3) T generates in E^1, E^2/E^2, $E^3/E^2, \dots$ the representations T^1, T^2, $T^3, \dots$ If moreover T^1, $T^2, \dots$ are completely irreducible they are called the completely irreducible components of T.

3. Elementary representations.

The following construction gives a class of representations of G which are called elementary.

Let $\mathfrak{G}$ be the Lie algebra of G, $\mathfrak{H}$ its Cartan subalgebra, ω a root, e_ω the corresponding root vector, Z the simply connected subgroup with tangential vectors e_ω, $\omega < 0$. We put $D = \exp \mathfrak{H}$ and $\mathcal{K} = ZD$. The pair $\alpha = \{p, q\}$, p, $q \in \mathfrak{H}$ is called a signature if $\nu = p - q$ is a weight of $\mathfrak{U}$. For every signature we put $f(h) = (p, h) + (q, \bar{h}) - (d, h + \bar{h})$, where (,) is the Killing-Cartan form in $\mathfrak{H}$, $h \to \bar{h}$ is a conjugation, for which all roots are real, d is the halfsum of all positive roots. Then the function $\alpha(\kappa) = \alpha(\delta) = \exp f(h)$ for $\kappa = \delta z$, $\delta = \exp \text{h}$ is a character of $\mathcal{K}$.

Let $e(\alpha)$ be the representation of G induced by the character $\kappa \to \alpha(\kappa)$ of $\mathcal{K}$ in the space of all infinitely differentiable functions on G. In virtue of the Ivasava decomposition the representation space $\mathcal{D}(\alpha)$ of $e(\alpha)$ can be considered as a subspace of the Montel space $C^\infty(\mathfrak{U})$. The operators of $e(\alpha)$ can be extended to continuous operators in the Hilberts space $H(\alpha)$ obtained from $\mathcal{D}(\alpha)$ by completion in the metric of $L^2(\mathfrak{U})$.

$e(\alpha)$ is called the elementary representation with signature α. For complex classical groups and for some values of α the $e(\alpha)$ where first constructed by Gelfand and Naimark (see [3] ; for further references see [1]). The following most complete result about elementary representations was obtained by Zhelobenko [1] :

3.1. $e(\alpha)$ is completely irreducible if and only if for no one of the roots $\omega > 0$ the numbers $p_\omega = 2(p, \omega)/(\omega, \omega)$, $q_\omega = 2(q, \omega)/(\omega, \omega)$ are both positive integers or both negative integers.

In this case $e(\alpha)$ is called non degenerate ; in the contrary case it is called degenerate. The following problems arise :

I. To represent the degenerate $e(\alpha)$ as a chain of its completely irreducible components. Which of these components is induced by a finite dimensional representation of a group $\tilde{\mathcal{K}}$ satisfying $\mathcal{K} \subset \tilde{\mathcal{K}} \subset G$?

II. Under what conditions the non degenerate $e(\alpha)$ and the completely irreducible components of the degenerate $e(\alpha)$ can form a chain which is not equivalent to their direct sum ?

III. Under what conditions the non degenerate $e(\alpha)$ and the completely irreducible components of the degenerate $e(\alpha)$ are unitary with respect to some positive definite inner product ?

Till now only partial solutions of these problems are known.

4. Homogeneous representations.

Let $c^\lambda_{je}(u)$ be the matrix elements in some canonical basis of the irreducible representation c^λ of weight λ of $\mathfrak{U}$. For a given representation T of G in E we put (Naimark [4]) $T_j^\lambda = \dim c^\lambda \int c^\lambda_{jj}(u)\, T_u\, du$, where du denotes the differential of the Haar measure on $\mathfrak{U}$ such, that $\int du = 1$; T_j^λ is a continuous projection in E. Let $j(\lambda)$ be the index j, for which the corresponding basis vector is the vector of heighest weight of c^λ. We put $T^\lambda = T^\lambda_{j(\lambda)}$ and $\lambda_0(T) = \min \{\lambda : T^\lambda \neq 0\}$.

A representation T will be called homogeneous of weight λ_0 (the unitary case see in Naimark [4]), if for T and for the conjugate representation $\hat{T}$ we have $\lambda_0(T) = \lambda_0(\hat{T}) = \lambda_0$ and the closed linear hulls of all $\{T_g T^{\lambda_0}E, g \in G\}$ and $\{\hat{T}_g \hat{T}^{\lambda_0}E', g \in G\}$ coincide with E and E'.

The following problems are of interest :

IV. Under what conditions is a homogeneous representation

(a) completely irreducible ?

(b) a direct integral (if E is a Hilbert space) of completely irreducible representations ?

Conjecture : IV a) takes place if and only if

$$\dim T^{\lambda_0} E = \dim \hat{T}^{\lambda_0} E' = 1 \ . \tag{4.1}$$

V. Under what conditions T is a direct sum of homogeneous representations ?

We mark that a unitary T is a direct orthogonal sum of homogeneous representations (see [4]).

5. Representations in spaces with indefinite metric.

Let E be a Hilbert space, P an orthogonal projection in E, $Q = 1 - P$. We put $\kappa = \min(\dim PE, \dim QE)$ and $[x, y] = -(Px, Py) + (Qx, Qy)$, where (x, y) is the inner product in E. The space E with the so defined indefinite inner product $[x, y]$ is called a Π_κ space ; if moreover $\kappa < \infty$ it is called a Pontryagin space. In all this section orthogonality and unitarity are meant with respect to $[x, y]$; in case of a Pontryagin space Π_κ we suppose, that $\kappa = \dim PE$.

The unitary representations in Π_κ, unlike the usual have a more complicated structure, as they do not in general decompose into a direct integral of irreducible representations.

Unitary representations T in a separable Pontryagin spaces Π_κ of an arbitrary locally compact group G with a countable neighborhoods basis were discussed by Naimark [6]. For such representation T in [6] two closed subspaces $M, N \subset \Pi_\kappa$ are in a special manner constructed, which are invariant with respect to T and such, that :

(1) $N \subset M$, $N \perp M$ and on $M^\perp$ the operators of T are comparatively simple described ;

(2) N is finite dimensional ;

(3) $[x, y]$ and T go under the canonical mapping $f : M \to M/N$ respectively into a positive definite inner product $[\xi, \eta]$ and a unitary with respect to $[\xi, \eta]$ representation T^1. Moreover the decomposition of T^1 into the direct integral of irreducible components can be obtained by diagonalization of the immage under f of a maximal commutative algebra in $(T_g)'$. From this explicit formulas for the T_g are obtained in terms of the irreducible components of T^1 and the values of T_g on $M^\perp$.

To obtain these results the following proposition was used :

5.1. (Naimark [7]) For every commutative family $\{\mathfrak{U}\}$ of unitary operators in a Pontryagin space Π_κ there exists in Π_κ a κ-dimensional subspace M, which is invariant with respect to every $\mathfrak{U} \in \{\mathfrak{U}\}$ and on which $[x, x] \leqslant 0$.

For the case, when $\mathfrak{U}$ consists of one operator 5.1 coincides with the known Pontryagin-Krein-Iohvidov theorem (see e.g. Iohvidov and Krein [8]).

It is naturaly to expect, that for complex semisimple Lie groups G further detalization is possible. Till now only unitary representations T in Pontryagin spaces Π_κ of $G = SL(2, C)$ were studied by Ismahilov [9]. He proved, that if we remove the trivial cases, then in Π_κ a κ-dimensional subspace $\mathfrak{N}$ exists, which is invariant with respect to T and on which $[x, y] = 0$. Then $\mathfrak{N}^\perp = \mathfrak{N} \oplus H$, where H is a Hilbert space and the restriction T^M of T on $M = \mathfrak{N}^\perp$ is a chain of a finite dimensional representation on $\mathfrak{N}$ with a unitary representation in H. It turns out that a decomposition of T^M into a direct orthogonal sum can be extended to a decomposition of the initial T. On the other hand T^M is a finite orthogonal sum of representations which can be only of the following types :

(1) finite dimensional ;

(2) unitary in the usual sense ;

(3) indecomposable chain of a finite dimensional representation which is a multiple of a non identity irreducible representation with a multiple of an irreducible unitary representation ;

(4) a chain of an identity representation with a unitary representation containing in its decomposition the representations of the complementary series and those representations of the principal series, which are relatives to the identity representation.

The problem of extension Ismahilovs results to other groups remains open.

Till now also unitary representations in Π_∞ were not studied ; one of the main difficulties is, that it is unknown whether the assertion of Proposition 5.1 remains valid for Π_∞. Only Langer [10] succeded to prove an analog to Proposition 5.1 for bounded Hermitian with respect $[x, y]$ operators A in Π_∞ under strong restriction on A, which are fulfilled for $\kappa < \infty$ but seem to be unnecessary for $\kappa = \infty$.

We conclude with the following three problems, which seem to be of interest :

VI. To find the conditions of unitary equivalence (in the Π_κ sense) of two unitary in Π_κ representations.

VII. If two unitary (in Π_κ) representations are equivalent in Naimark's sence (see [11] and [12]), are they also unitary equivalent in Π_κ sense ?

VIII. Does topological irreducibility of a unitary in Π_κ representation imply its complete irreducibility ? For Pontryagin spaces the answer is positive (see Ismahilov [13]).

A survey of the subjects considered in this section and of allied topics see in Naimark and Ismahilov [14].

Added in proof. The author succeded to give partial solutions of Problems IVa) and IVb). Particularly (4.1) is sufficient for a homogeneous representation to be irreducible and under some additional conditions to be completely irreducible.

REFERENCES

[1] ZHELOBENKO D.P. — *Izvestia Akad. Nauk S.S.S.R.*, 32, 1968, p. 108-133.
[2] GODEMENT R. — *Trans. Amer. Math. Soc.*, 73, 1952, p. 496-556.
[3] GELFAND I.M. and NAIMARK M.A. — *Trudy Matem. Inst. Steklova*, 36, 1950.
[4] NAIMARK M.A. — *Matem. Sbornik*, I, 35, 1954, p .317-356: II, 37 1955, p. 121-140.
[5] NAIMARK M.A. — *Studia Math.*, 31, 1968, p. 383-398.
[6] NAIMARK M.A. — *Izvestia Akad. Nauk S.S.S.R.*, 30, 1966, p. 1111-1132.
[7] NAIMARK M.A. — *Acta Sci. Szeged*, 24, 1963, p. 177-189.
[8] IOHVIDOV I.S. and KREIM M.G. — *Trudy Mosk. Maten.*, ob-va 5, 1956, p. 367-432; 8, 1959, p. 413-496.
[9] ISMAHILOV R.S. — *Izvestia Akad. Nauk S.S.S.R.*, 30, 1966, p. 497-522.
[10] LANGER H.K. — *Doklady Akad. Nauk S.S.S.R.*, 169, 1966, p. 12-15.
[11] NAIMARK M.A. — *Doklady Akad. Nauk S.S.S.R.*, 97, 1954, p. 969-972.
[12] ZHELOBENKO D.P. and NAIMARK M.A. — *Izvestia Akad. Nauk S.S.S.R.*, 34, 1970, p. 57-82.
[13] ISMAHILOV R.S. — *Doklady Akad. Nauk S.S.S.R.*, 171, 1966, p. 269-277.
[14] NAIMARK M.A. and ISMAHILOV R.S. — *Itogi Nauki, Math. Analysis Moscow*, 1969, p. 73-105.

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NEW RESULTS IN THE REPRESENTATION THEORY OF SOLVABLE GROUPS

by L. PUKANSZKY

Not long ago L. Auslander and B. Kostant, following the line of research started by A.A. Kirillov ([6]) and continued by P. Bernat ([3]), gave a necessary and sufficient condition in order that a connected and simply connected solvable Lie group be of type I, and provided in this case a complete description of its dual ([1] [2]). The purpose of the present communication is to outline the construction of a class of factor representations for an arbitrary connected and simply connected Lie group G. If G is of type I, our representation are multiples of the irreducible representations described by Auslander and Kostant. In the general case, as a substitute for the completeness statement we show, that our family is ample enough to permit a central decomposition, into a continuous direct integral, of the regular representation. Finally we apply these results to derive a striking property of the left ring of our group.

1. – Let $\mathfrak{g}$ be the Lie algebra of G and let us put $\mathfrak{b} = [\mathfrak{g}, \mathfrak{g}]$. If f is some element of $\mathfrak{b}'$ (= dual of the underlying space of $\mathfrak{b}$), we denote by B_f the skew symmetric bilinear form on $\mathfrak{b}_{\mathbf{C}} \times \mathfrak{b}_{\mathbf{C}}$ defined by $B_f(x, y) = ([x, y], f)$ $(x, y \in \mathfrak{b}_{\mathbf{C}})$. A subalgebra $\mathfrak{f} \subset \mathfrak{b}_{\mathbf{C}}$ will be called an admissible polarization with respect to f, in symbols $\mathfrak{f} = \mathrm{pol}(f)$, if the following conditions are satisfied. 1) $\mathfrak{f}$ is maximal selforthogonal with respect to B_f, 2) α) $\bar{\mathfrak{f}} + \mathfrak{f}$ is a subalgebra of $\mathfrak{b}_{\mathbf{C}}$, If

$$x + iy \in \mathfrak{b}_{\mathbf{C}}\ (x, y \in \mathfrak{b})$$

we have $B_f(x, y) \geqq 0$, and $B_f(x, y) = 0$ implies $x, y \in \mathfrak{f} \cap \mathfrak{b}$. 3) Denoting by G_f the stabilizer of f in G (acting on $\mathfrak{b}'$ via the representation, which is contragredient to the restriction of the adjoint representation of G to $\mathfrak{b}$), we have $G_f\, \mathfrak{f} \subseteqq \mathfrak{f}$. – The existence, for any $f \in \mathfrak{b}'$, of admissible polarizations follows from ([2], Lemma II.3.1). – Let g be some element of $\mathfrak{g}'$, G_g its stabilizer, with respect to the coadjoint representation, in G, and $\mathfrak{g}_g \subset \mathfrak{g}$ the Lie algebra of G_g. There is a character χ_g on $(G_g)_0$ (= connected component of the unity in G_g), such that $d\chi\,(l) = i(l, g)$ $(l \in \mathfrak{g}_g)$. Let us put $\overset{0}{G}_g = \mathrm{Ker}(\chi_g)$ and observe, that $\overset{0}{G}_g$ is an invariant subgroup in G_g.

DEFINITION 1. – *The reduced stabilizer of* g, *denoted by* $\bar{G}_g$, *is the complete inverse image, in* G_g, *of the center of* $G_g / \overset{0}{G}_g$.

We write $\overset{A}{G}_g$ for the set of all characters of $\bar{G}_g$, which, when restricted to $(G_g)_0$, coincide with χ_g. – Given $g \in \mathfrak{g}'$, $\chi \in \overset{A}{G}_g$ and $\mathfrak{f} = \mathrm{pol}(g \mid \mathfrak{b})$ we can construct, through a procedure inspired by [2], a unitary representation in $(\mathfrak{f}, \chi, g)$ of G as follows. Let us put $d = \mathfrak{f} \cap \mathfrak{g}$, and let us denote by D the connected subgroup of G corresponding to d. There is a character χ' of the closed subgroup $A = G_g D$ of G, such that $\chi'|\overline{G} \equiv \chi$, and $\chi'|D \equiv \chi$, where $d\chi_f(l) = i(l, f)$ $(l \in d)$. Let us put $T' = \operatorname{ind}_{A\uparrow G} \chi'$, and denote by $H(T')$ the space of T'. In the present case T' can be described as follows. Let da and dg be elements of right invariant measures on A and G resp., such that $d(a_0 a) = \Delta_A(a_0)\, da$ and

$$d(g_0 g) = \Delta_G(g_0)\, dg \quad (a_0 \in A \,,\, g_0 \in G).$$

One can show, that there is a continuous homomorphism h of G into the multiplicative group of positive numbers, such that $h(a) \equiv \Delta_A(a)/\Delta_G(a)$ $(a \in A)$, and hence there is a Borel measure $d\nu(p)$ on G/A, such that

$$d\nu(pg) = h(g) \,.\, d\nu(p) \,.$$

Then $H(T')$ is the Hilbert space associated with the prehilbert space of all measurable functions on G, satisfying $f(ax) = h(a)\, \chi'(a)\, f(x)$ for all a in A and x in G, and

$$\int_A |f(x)|^2/h(x)\, d\nu < +\infty$$

The operator $T'(g)$ $(g \in G)$ corresponds to translation on the right by g. – Let H and E be the connected subgroups in $G_{\mathbf{C}}$ corresponding to $\mathfrak{f}$ and $e = \mathfrak{f} + \overline{\mathfrak{f}} \cap \mathfrak{g}$ resp. The set HE is open in $E_{\mathbf{C}} \subset G_{\mathbf{C}}$. One can show, that the linear variety H_0 in $H(T')$, corresponding to all those functions, for which the map $hk \to \chi_f(h) f(ka_0)$ $(h \in H, k \in E$; a_0 arbitrarily fixed in G ; χ_f holomorphic character of H, such that $d\chi_f(l) = i(l, f)$ for $l \in \mathfrak{f}$) is holomorphic on $HE \subset E_{\mathbf{C}}$, is closed in $H(T')$ and invariant by T'. We define $\operatorname{ind}(\mathfrak{f}, \chi, g)$ as $T'|H_0$. – Given $a \in G$ and $\chi \in \overset{A}{G}_g$, we write $a\chi$ for the element of $\overset{A}{G}_{ag}$ defined by $(a\chi)(b) = \chi(a^{-1}ba)$ $(b \in G_{ag})$.

THEOREM 1. – *Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$. For $g \in \mathfrak{g}'$, $\chi \in \overset{A}{G}_g$ and $\mathfrak{f} = \operatorname{pol}(g|\mathfrak{h})$ the representation* $\operatorname{ind}(\mathfrak{f}, \chi, g)$ *is a semifinite factor representation. It is of type I if and only if the index of the reduced stabilizer of g in its stabilizer is finite. We have*

$$\operatorname{ind}(\mathfrak{f}, \chi, g) = \operatorname{ind}(\mathfrak{f}_1, \chi_1, g_1)$$

in the sense of quasi-equivalence if and only if there is an element a in G such that $ag = g$, $a\chi = \chi_1$. In this case these representations are unitarily equivalent.

Remark 1. – Let us recall (cf. [2]), that a necessary condition, in order that G be of type I, is that $\overline{G}_g = G_g$.

2. – The factor representations provided by Theorem 1 lead to a central decomposition of the regular representation only under a regularity assumption on the action of the coadjoint representation on $\mathfrak{g}'$. In the general case a more involved construction is needed, the principal steps of which are as follows.

LEMMA 1. – *There is an equivalence relation R on $\mathfrak{g}'$, uniquely determined by the following properties* 1) *Any R orbit $\mathfrak{O}$ is locally closed and G invariant,* 2) *For any $p \in \mathfrak{O}$, Gp is dense in $\mathfrak{O}$. – We have $Gp = \mathfrak{O}$ if and only if Gp is locally closed.*

Let $\mathfrak{O}$ be an orbit of R. One can show, that the closed subgroup $L\bar{G}_g (L = [G, G])$ is independent of the choice et $g \in \mathfrak{O}$; we denote it by $\bar{K}$. We put

$$\overset{\theta}{K} = \{\varphi \,;\, \varphi \in \hat{\bar{K}}\,,\ \varphi \mid K_0 \equiv 1\}\,.$$

$\overset{\theta}{K}$ operates on the set $\mathfrak{L}(\mathfrak{O}) = U_{g \in \mathfrak{O}} \hat{G}_g$ by the rule $\varphi(g, \chi) = (g, (\varphi \mid \bar{G}_g)\,\chi)$. – Let us write $J = \bar{K}/\bar{K}_0$; for any $g \in \mathfrak{O}$, J is isomorphic to $\bar{G}_g/(G_g)_0$, which is a free abelian group of rank α, say. Given $g_0 \in \mathfrak{O}$ and $a \in J$, one can readily show the existence of a neighborhood U of g_0 in $\mathfrak{O}$, and of a continuous map $f : U \to G$, such that 1) $f(g) \in \bar{G}_g$, 2) $\Phi(f(g)) \equiv a$ $(g \in U)$, where Φ is the canonical homomorphism form K onto J. Let $\{a_j \,;\, 1 \leqq j \leqq \alpha\}$ be a basis of J, and let $f_j (1 \leqq j \leqq \alpha)$ correspond to $a_j (1 \leqq j \leqq \alpha)$ as f did above to a ; suppose, the maps f_j are all defined on the neighborhood U of g_0. Une can prove, that there is a topology on $\mathfrak{L}(\mathfrak{O})$ well determined by the condition, that the map β : $U \times \hat{J} \to \mathfrak{L}(\mathfrak{O})$ defined by $\beta(g, \varphi) = (g, \chi) \in \mathfrak{L}(\mathfrak{O})$, where χ is determined by $\chi(f_j(g)) = \varphi(a_j)$ $(1 \leqq j \leqq \alpha)$, be a homeomorphism with its image for all choice of U and $\{f_j\}$ as above. This turns $\mathfrak{L}(\mathfrak{O})$ into a principal bundle over $\mathfrak{O}$ with the structure group $\overset{\theta}{K}$ $(\sim T^m)$. Let us define, if $p = (g, \chi) \in \mathfrak{L}(\mathfrak{O})$ and $a \in G$,

$$ap = (ag, a\chi)\,.$$

We have $a\varphi p = \varphi a p$ $(\varphi \in \overset{\theta}{K})$, and G acts on $\mathfrak{L}(\mathfrak{O})$ as a group of homeomorphisms.

LEMMA 2. – *The bundle* $\mathfrak{L}(\mathfrak{O})$ *is trivial.*

Let p_0 $\mathfrak{O} \to \mathfrak{L}(\mathfrak{O})$ be a continuous crossection. Let us put $ap_0(g) \equiv \mu(a, g)\, p_0(ag)$; the map $\mu : G \times \mathfrak{O} \to \overset{\theta}{K}$ is continuous.

LEMMA 3. – *There is a crossection, such that* $\mu(a, g)$ *is independent of* $g \in \mathfrak{O}$.

By aid of Lemma 3 one proves

LEMMA 4. – *There is an equivalence relation* S *on* $\mathfrak{L}(\mathfrak{O})$, *uniquely determined by the property, that any* S *orbit* 0 *be locally closed and* G *invariant, and for* $p \in 0$, Gp *be always dense in* 0.

Remark 2. – Let τ be the canonical projection from $\mathfrak{L}(\mathfrak{O})$ onto $\mathfrak{O}$, that is $\tau(g, \chi) = g$ $(g \in \mathfrak{O})$. If $\mathfrak{O}$ is acted upon transitively by G, which is the case, in particular, if G is of type I, then $(0, \tau)$ is a simple covering of $\mathfrak{O}$. In the general case, however, for $g \in \mathfrak{O}$, $\tau^{-1}(g) \cap 0$ does not even need to be countable.

Notation. – We shall write $\mathfrak{S}$ for the collection of all S orbits (for all possible choice of the R orbit $\mathfrak{O}$).

LEMMA 5. – *There is a nontrivial, positive and* G *invariant Borel measure* μ *on* 0, *uniquely determined up to a multiplicative constant.*

DEFINITION 2. – *For* $0 \in \mathfrak{S}$ *a field of polarizations* $\{\mathfrak{f}_p \,;\, p \in 0\}$ *is a map from* 0 *into the set of subspaces of* $\mathfrak{h}_{\mathbf{C}}$, *such that for any* $p \in 0$ *we have*

$$\mathfrak{f}_p = \mathrm{pol}\,(\tau(p) \mid \mathfrak{h})$$

(cf. 1 above).

For $p = (g, \chi) \in \mathfrak{L}(\mathfrak{O})$ and $\mathfrak{f} = \mathrm{pol}\,(\tau(p) \mid \mathfrak{h})$ let us put ind $(\mathfrak{f}, p) =$ ind $(\mathfrak{f}, \chi, g)$. – One can show, that if $\{\mathfrak{f}_p\ ; p \in 0\}$ is a field of polarizations, the field of concrete representations $\{\mathrm{ind}\,(\mathfrak{f}_p, p)\ ; p \in 0\}$ can be endowed with a measurable structure (cf. [5], 18.7.2, p. 324), and hence we can form the representation

$$\int_0 \oplus \,\mathrm{ind}\,(\mathfrak{f}_p, p)\, d\mu \tag{1}$$

where μ is as in Lemma 5.

THEOREM 2. – *The direct integral* (1) *defines a factor representation which, up to unitary equivalence, is uniquely determined by* $0 \in \mathfrak{S}$. *It is of type I if and only if* 1) 0 *is a G orbit,* 2) *If* $p = (g, \chi) \in 0$, $\overline{G}_g$ (cf. Definition 1) *is of a finite index in* G_g. *In this case* (1) *is a multiple of ind* $(\mathfrak{f}_p, p)$ (p arbitrary in 0).

We recall, that the unitary representations U and V of G are called quasi-equivalent, if there is a $*$ isomorphism φ from the v. Neumann algebra generated by U onto that generated by V, such that $V(a) = \varphi(U(a))$ $(a \in G)$.

THEOREM 3. – *The left regular representation of G is a central continuous direct sum of factor representations belonging to the quasi-equivalence classes of the representations of Theorem 2.*

Given a v. Neumann algebra M, we denote by M_{I}, M_{II} and M_{III} the type I, II, and III components resp. of M (cf. [4]). Let $L(G)$ be the left ring of G. We showed earlier (cf. [8], Corollaire 1), that $L(G)_{\mathrm{III}}$ is always trivial.

THEOREM 4. – *Let G be a connected and simply connected solvable Lie group. Then its left ring coincides with its type* I *or type* II *components ; in other words* $L(G) = L(G)_{\mathrm{I}}$ *or* $L(G) = L(G)_{\mathrm{II}}$.

Remark 3. – *That* $L(G) = L(G)_{\mathrm{I}}$ *does not imply, that* G is of type I was shown by G.W. Mackey (cf. [7], p. 324).

Remark 4. – Although $L(G)_{\mathrm{III}} = 0$ is valid even if G is not simply connected, this assumption is necessary in order that Theorem 4 be true.

BIBLIOGRAPHY

[1] AUSLANDER L. and KOSTANT B. — Quantization and representations of solvable Lie groups, *Bull. A.M.S.*, 73, 1967, p. 692-695.

[2] AUSLANDER L. and KOSTANT B. — *Polarization and unitary representations of solvable Lie groups* (to appear).

[3] BERNAT P. — Sur les représentations unitaires des groupes de Lie résolubles, *Ann. Ec. Norm. Sup.*, Vol. 82, 1965, p. 37-99.

[4] DIXMIER J. — Sur la réduction des anneaux d'opérateurs, *Ann. Ec. Norm. Sup.*, Vol. 68, 1951, p. 185-202.

[5] DIXMIER J. — *Les C* algèbres et leurs représentations,* Paris, Gauthier-Villars 1964.

[6] KIRILLOV A.A. — Unitary representations of nilpotent Lie groups, *Usp. Mat. Nauk*, 17, 1962, p. 57-110.
[7] MACKEY G.W. — Induced representations and normal subgroups, *Proc. of Int. Symp. on Linear Spaces*, Jerusalem, 1960.
[8] PUKANSZKY L. — Sur la représentation régulière des groupes de Lie résolubles connexes, *C. R. Acad. Sci. Paris*, Vol. 268, 1969, p. 1172-1173.

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NON-LINEAR QUANTUM PROCESSES AND AUTOMORPHISM GROUPS OF C^*-ALGEBRAS

by I. E. SEGAL

Introduction.

The cogent words of Hilbert on the role of external ideas in mathematics are no less applicable today than when enunciated in his Paris address of 1900. I propose to discuss "quantum field theory" in relation to aspects of functional analysis in their spirit.

The mathematical sense in which "solution" is to be understood for a non-linear "quantized" partial differential equation involves new ideal constructs, whose mathematical formulation and treatment are naturally essential preliminaries to the subsumption of the theory under pure mathematics. There is now a repertoire of such constructs which may be adequate for the foundations of the theory. As yet only limited results have been based on these notions, the roots of some of which go back several decades ; but a direct and mutually stimulating rapport with frontiers of pure functional analysis has become established.

Varieties of quantum field theory.

The different formal approaches to the crucial problem of the nature of the temporal evolution in quantum field theory may reasonably be classified under three main headings, associated roughly with different periods in the development of the subject :

(1) quantized non-linear partial differential equations (Dirac, Heisenberg, Pauli et al., 1925-40) ;

(2) the scattering ($S-$) operator as a function of "free fields", via the method of variation of constants (Feynman, Schwinger, Tomonaga et al., 1940-55) ;

(3) in terms of automorphisms and states of C^*-algebras (1955-70).

Approach (1) begins with a given partial differential equation ; a typical one is the equations $\square\phi = c\phi + p(\phi)$, where c is a given non-negative constant, p is a given real polynomial such that $p(0) = p'(0) = 0$, and $p = q'$ for some non-negative polynomial q ; and $\square = \Delta - \partial^2/\partial t^2$. When ϕ is a strict function $\phi(x, t)$, this equation is essentially clear-cut ; its global theory has made much progress in the past decade, through the work of Jörgens, Strauss, and others. In question however is an operator-valued solution satisfying "canonical commutation relations" (or "CCRs") :

$$[\phi(x, t), \phi(x', t)] = 0 = [\phi\ (x, t), \phi\ (x', t)], [\phi(x, t), \phi\ (x', t)] = i\delta(x - x')$$

for arbitrary points x and x' in space and all times t ; these require that ϕ be a generalized function of x, for fixed t. For such a function, the term $p(\phi)$ has no *a priori* meaning.

A variant of this approach is to attempt to determine a quantum hamiltonian from the classical one by substitution of hermitian operators satisfying the CCRs, for the classical canonical variables. For the cited equation the classical hamiltonian is $H_p(\phi) = H_0(\phi) + V_p(\phi)$, where $H_0(\phi) = \int [(\text{grad } \phi)^2 + c\phi^2 + \phi_t^2]\, dx$ and $V_p(\phi) = \int q(\phi)\, dx$. This is meaningful for strict functions ϕ, but not for generalized ones. In addition, the formal unicity of this procedure derived from the belief that any two irreducible representations of the CCRs are, if reasonably regular, unitarily equivalent ; this is now known to be quite false in the infinite-dimensional case relevant here, although valid in the finite-dimensional case by the Stone-von Neumann theorem.

In Approach (2), the temporal evolution from time t to time t' is expressed, modulo the temporal evolution defined by the linear equation $\Box\phi = c\phi$, by an operator $S(t', t)$ whose putative limit as t' and $-t$ tend to ∞ is the "scattering operator" S. Formally, S is expressible in terms of the solution ϕ_0 of the quantized equation $\Box\phi_0 = c\phi_0$ as the product integral over the t-interval $(-\infty, \infty)$ of the exp $(i \int V_p(\phi_0(x, t))\, dx)$. This approach involves the explicit concept ϕ_0 in place of the highly implicit one ϕ, but the indicated integral lacks clear meaning in several basic respects.

Approach (3) developed from the C^*-algebra quantum phenomenology of 1947-51. In this the concepts : "observable", "state", and "temporal transition" are no longer represented by : self-adjoint operator in a given Hilbert space, vector (possibly generalized) in this space, unitary operator on this space, as in classical quantum mechanics. Instead they correspond to : self-adjoint element of a given (abstract) C^*-algebra, normalized positive linear functional on the algebra, and automorphism of the algebra. The special applicability of this phenomenology for quantum field theory was indicated by the result (1959) that any two irreducible representations of the CCRs are equivalent in the sense of the existence of a natural isomorphism between associated C^*-algebras ; in the event (in practice, rare) of a unitary equivalence, the latter could be recovered from the C^*-isomorphism ; and explicit criteria for the "unitary implementability" of a given C^*-isomorphism were obtained.

Linear fields and functional integration.

The CCRs are appropriately treated in terms of the more succinct and mathematically viable notion of "Weyl system", where "Weyl system over a given complex pre-Hilbert space H" is defined as a pair (K , W) consisting of a Hilbert space K and a continuous map W from H to the unitary operators on K, satisfying the relations : $W(z)\, W(z') = \exp [i\text{Im}\,(\langle z, z'\rangle)]\, W(z + z')$, for arbitrary z and z' in H. When H is finite-dimensional, there is a well-known Weyl system with $K = L_2(H')$, where H$'$ is any real subspace of H such that H = H$'$ + iH$'$, and with simple explicit expressions for the $W(z)$. When H$'$ is infinite-dimensional, the concept $L_2(H')$ is *a priori* undefined, and it is not obvious that a Weyl system over H exists. An explicit heuristic representation of the infinite CCRs was first given by V. Fock (1932). This was given a rigorous and invariant mathematical expression, now classical, in terms of tensor products of Hilbert spaces by Cook (1952). The development of abstract functional integration ideas served to give a meaning to $L_2(\mathbf{H}')$ in the infinite case, and these ideas have been crucial in the treatment of non-linear processes. In particular, a canonical unitary equivalence was set up between the structures associated with the Fock-Cook space and $L_2(\mathbf{H}')$, which structures respectively diagonalize certain self-adjoint operators having non-negative integral proper values ("particle numbers") and operators representing the $\phi(x, t)$ for a fixed value of t (the "field variables" throughout space), expressing explicitly the so-called "particle-wave duality" ; and the indicated Weyl relations were a by-product.

These functional integration ideas were in part closely related to those of Wiener on the homogeneous chaos and Brownian motion, especially in the vein developed by Cameron and Martin, and treated by Kac, Kakutani, and others. But basing the theory, in a formally somewhat different way, on an abstract Hilbert space rather than a space of continuous functions on the reals, as in the cited work, not only facilitated its application in quantum field theory but furnished improvements and a natural setting for relevant results on absolute continuity of transformations in function space. For example, the transformation $x \to Tx + a$ is absolutely continuous with respect to the generalized invariant normal probability measure in $\mathbf{H}'$ for arbitrary a in $\mathbf{H}'$, and for precisely those invertible linear transformations T such that $T^*T - I$ is of Hilbert-Schmidt class. There is a corollary (and similar) condition for the unitary implementability of an important class of automorphisms of the Weyl (C^*-) algebra, due to Shale. During the period 1955-70 there has been a considerable further development of the more abstract functional integration ideas, for both classical and quantum purposes, by, among many others, Bochner (in part, earlier), Dudley, Gross, Feldman, Shale, and Stinespring in the U.S.A. and Gelfand, Minlos, Sazonov, and Sudakov in the U.S.S.R.

The vacuum and C^*-algebras.

The Fock-Cook representation also included concrete versions of the additional elements of structure postulated in quantum field theory, – the "vacuum vector" ν and the "temporal evolution group" $\Gamma(t)$, $t \in R^1$. "Weyl process" over $\mathbf{H}$ may be defined as a triple $(\mathbf{K}, W, \nu)$ such that $(\mathbf{K}, W)$ is a Weyl system and ν is a unit vector in $\mathbf{K}$. The "expectation functional" $A \to \langle A\nu, \nu \rangle$ determined by ν for the linear operators A on $\mathbf{K}$ plays a role analogous to the expectation on a probability measure space. "Covariant Weyl process" over $\mathbf{H}$, relative to a given topological group G and continuous symplectic representation V of G on $\mathbf{H}$, is a quadruple $(\mathbf{K}, W, \nu, \Gamma)$ such that $(\mathbf{K}, W, \nu)$ is a Weyl process, and Γ is a continuous unitary representation of G on $\mathbf{K}$ having the properties : $\Gamma(a)\,W(z)\,\Gamma(a)^{-1} = W(V(a)z)$, $\Gamma(a)\,\nu = \nu$, for all $a \in G$ and $z \in \mathbf{H}$. The Fock-Cook system is covariant with respect to the full unitary group on $\mathbf{H}$, in its given representation $U \to U$; *a fortiori*, with respect to any given one-parameter unitary group in $\mathbf{H}$. In particular, the Fock-Cook system for the Hilbert space constituted by the solution manifold of a typical linear relativistic partial differential equation, in its essentially unique Lorentz-invariant Hilbert norm, defines the "free quantum field" for this equation.

On the other hand, the crucial end product of quantum field theory is a treatment of "interaction", usually represented by a perturbation of a given invariant linear equation ; but even the simplest non-trivial perturbations of a typical such equation give rise to formal temporal evolutions which are not representable by one-parameter unitary groups in the Fock-Cook space. E.g. this is true for the perturbed equation $\Box\phi = c\phi + V\phi$in relation to the equation $\Box\phi = c\phi$, B being a general bounded regular non-negative given function on space. In addition, in the absence of an appropriate intrinsic characterization for the Fock-Cook system, its privileged position could be questioned.

Approach (3) provided ideal constructs with which these problems could be resolved, and in particular dealt with the vacuum as a linear functional, rather than as a vector. The vacuum relative to a given one-parameter automorphism group of a C^*-algebra **C** could appropriately be defined, in connection with problems such as those indicated, as a state on **C** which is invariant, and such that the corresponding unitary group $U(t)$ in the associated canonical Hilbert space structure has the form e^{itH} for some non-negative self-adjoint operator H. This formulation substantially enlarged the scope of the linear theory, as indicated e.g. in recent work of Weinless ; a simple example of the results which is readily quoted at this point is that no vacuum exists for the quantized version of the equation $\Box\phi = c\phi$ if $c < 0$. More significantly, for present purposes, it was potentially applicable to nonlinear quantum fields.

Nonlinear local functions of Weyl processes.

The formal hamiltonian for an equation such as $\Box\phi = c\phi + V\phi$ is a quadratic perturbation of that for the equation $\Box\phi = c\phi$; through this connection the *a priori* meaningless square of a quantum field could be defined effectively ; from a formal standpoint these squares differ by an "infinite constant" from the "true" (non-existent) squares, and so are called "renormalized". Higher powers could not be resolved in this fashion (the at most quadratic expressions are formally special in that they form a closed set under bracketing, as a result of which they may be dealt with in a variety of ways) ; but related formal considerations, among others, have led to a definition for higher renormalized powers and products which appears to satisfy all the basic desiderata. Unlike the theory up to this point, which essentially depends only on the linear topological and group-theoretic structure in **H**, the nonlinear theory depends materially on the multiplicative structure in **H**, which is determined in practice by the specific fashion in which the vectors of **H** are represented by functions on spacetime.

The essential idea of the definition is that the formal relations

$$[\phi(x, t)^n, \phi(x', t)] = in\,\phi(x, t)^{n-1}\,\delta(x - x')$$

should remain valid to the maximal extent consistent with the desideratum that the vacuum expectation values of the $\phi(x)^n$ should all vanish ($n = 1, 2, \ldots$). The main conclusion is that under fairly general conditions, generalized operators which are formally identifiable with the renormalized $\phi(x, t)^n$ exist, are unique, are local in their dependence on x, enjoy certain regularity properties, etc. In these considerations, space may be taken as an arbitrary locally compact abelian group, and the laplacian replaced by a member of a general class of group-invariant operators. In the special case of the Fock-Cook representation and vacuum, the corresponding renormalized powers for linear relativistic quantum fields turn out to be formally identifiable with those defined and treated heuristically by G.-C. Wick (1950), in which ambiguities in the transition from a classical to quantized canonical variables in a given polynomial are resolved by a standardized rearrangement procedure.

Solution of quantized partial differential equations.

As a preliminary to the treatment of $H_p(\phi)$ in the hamiltonian version of Approach 1, one might consider the apparently simpler although time-dependent object $H_p(\phi_0(.\,,t))$. These two formal operators are *a priori* very different, but one might conceive that ultimately the difference might be immaterial in a C^*-treatment, as in the case of a quadratic perturbation. In any event, a study of the semiboundedness of a certain version of $H_p(\phi_0(.\,,0))$ in the case in which space is taken as T^1, $q(x)$ as const. x^4, was sketched by Nelson (1966). It is now known that this $H_p(\phi_0)$ is, when suitably formulated, for arbitrary, p of the type earlier indicated, essentially self-adjoint, has a unique proper lowest vector ν', enjoys certain regularity properties, etc. Employing the canonical representation structure for the state corresponding to ν', it follows that there exists a temporally covariant Weyl process for which the corresponding ϕ satisfies the equation $\Box\phi = c\phi + \widetilde{p}(\phi)$, where $\widetilde{p}$ is a certain polynomial whose highest-order term is the same as that of p, but which includes lower-order "counter terms" distinct from those of p ; the coefficients of the latter are simple functions of the expectation values with respect to ν' of the renormalized powers of ϕ with respect to ν. Thus, modulo the hypothesis that the mapping $p \to \widetilde{p}$ is surjective, any equation of the earlier indicated type has a solution (except in the case $c = 0$, which requires separate treatment, and has not yet been studied).

In a relativistic treatment, space must be taken as R^1 rather than T^1, in which case $V_p(\phi_0(.\,,0))$ no longer appears as a generalized operator in a usual sense ; however a new ideal notion, describable as a quantized variant of the notion of one-dimensional differential form in function space, in sufficient for its description. In particular, $V_p(\phi_0(.\,,0))$ determines a derivation of the relevant C^*-algebra, which generates a one-parameter automorphism group, which is not unitarily implementable for non-trivial p. At this point hyperbolicity considerations dependent on special properties of Δ in euclidean space (up to this point the theory would apply to a quite general class of group-invariant operators) and on locality properties of $V(\phi\,(.\,,0))$, intervene. The abstract idea is briefly that the sum of generators of automorphism groups of "finite speed" of a "graduated" C^*-algebra again generates a group of finite speed, if it generates a group at all. Convergence questions are thereby reduced to the study of densely defined hermitian operators in Hilbert space corresponding to the formal expressions $H_0(\phi_0) + \int q(\phi(x\,,0)f(x)\,dx$, f being a given smooth non-negative function of compact support. It follows that there exists a one-parameter group of automorphisms corresponding to $H_p(\phi_0(.\,,0)$. The polynomial $\widetilde{p}$ is however f-dependent, and the question of a differential equation is more complicated than in the case of T^1.

Work in a related formal direction, and based on the same C^*-algebra framework, but involving major differences in method and spirit, has recently been published or announced by Cannon, Jaffe, and Glimm, primarily in the case in which $q(x) = \text{const. } x^4$, and by Rosen.

Moderated perturbation theory.

This is the main analytic basis of the preceding section, and is interesting as an example of general mathematical fallout from a concrete investigation of partly external origin. The moderated sum $H \tilde{+} V$ of given self-adjoint operators H and V in a Hilbert space $\mathbf{K}$ is defined as the putative self-adjoint limit H', in the sense of convergence of the one-parameter unitary groups generated, of all sequences of the form $H + f_n(V)$, the f_n being real bounded Baire functions such that $f_n(\lambda) \to \lambda$ and $|f_n(\lambda)| \leqslant |\lambda|$ $(n = 1\ 2, \ldots)$. A typical result is : *Let M be a given probability measure space, $\mathbf{K} = L_2(M)$, V the operator of multiplication by the given real measurable function F on M. Suppose that e^{-tH} is bounded from L_2 to L_p for some $t > 0$ and $p > 2$; that, for all sufficiently small t, e^{-tH} is bounded in norm from L_q to L_q by e^{ct} for some constant c and $q > 2$; and that F and e^{-F} are in all L_p with $p < \infty$. Then $H \tilde{+} V$ exists, is the closure of $H + V$, is bounded from below, etc.* Many further conclusions can be drawn, e.g. the associativity of the operation $\tilde{+}$ within the context of this theorem. Under simple and applicable conditions, information is available concerning the spectrum of $H \tilde{+} V$; e.g. if a proper lowest vector exists, it is essentially unique and may be taken as positive a.e., if the semigroup e^{-tH}, $t > 0$, is positivity-preserving and has global support ; under further conditions, the existence of such a vector is assured. On the whole, substantially more singular perturbations than are admitted by the well-known theory in Hilbert space due to Rellich, Kato, Birman, and others, can be effectively treated, when the additional structure involved here is present.

Adiabatic perturbations and distribution propagators.

For a variety of reasons, some suggested by the foregoing, but basically the greater singularity of the perturbations involved in the higher-dimensional theory, the combination of Approaches (2) and (3) appears more promising and important than that of Approaches (1) and (3). To describe ideal notions designed to deal with this greater singularity, note that if H is a given self-adjoint operator in a Hilbert space $\mathbf{K}$, and V is a bounded self-adjoint operator on $\mathbf{K}$, there is for any given real bounded measurable function g of compact support on R^1, a unitary operator $S(g)$ giving the transition from a very early to a very late time for a solution of the differential equation

$$u'(t) = ig(t)\, V(t)\, u(t) \ ; \ V(t) = e^{-itH}\, V e^{itH} .$$

In case g is the characteristic function of the interval $[t, t']$, $S(g)$ is identical with the operator $S(t', t)$ earlier indicated. On the other hand, when $g \in C_c^\infty(R^1)$, $S(g)$ may exist in an appropriate sense even for extremely singular perturbations V ; this is probably the case for the perturbation of an arbitrary self-adjoint elliptic operator on a compact manifold by (the generalized operator of multiplication by) an aribtrary real distribution. Let then "distribution propagator" (relative to the given H) be defined as a mapping S from the non-negative real functions in $C_c^\infty(R^1)$ to the unitary operators on $\mathbf{K}$ which is in the minimal class of such mappings containing all mappings arising as indicated from a bounded operator V, and closed under sequential convergence. If now V is a given continuous bilinear form on the domain $\mathbf{D}_\infty(H)$ of all infinitely differentiable vectors with respect to H, the "adiabatic perturbation of H by V" is said to exist if there exists a distribution propagator S such that for all g, $S(g/n)^n \to e^{iH(g)}$ (as $n \to \infty$), $H(g)$ being a self-adjoint operator whose associated bilinear form extends $\int V(t)\, g(t)\, dt$. The "adiabatic sum" of H and V may be said to exist in the special case in which $S(\cdot)$ can be extended by continuity to characteristic functions of intervals ; the formal sum $H + V$ is then representable by a self-adjoint operator. This is the case e.g. if $H = -\Delta$ and V is a member of a general class of non-absolutely continuous measures on R^1 (cf. also explicit work of Berezin and Faddeev for the case $V = \delta$).

In the theory of non-linear quantum processes in higher dimensions, the relevant bilinear forms $\int V(t)\, g(t)\, dt$, which involve a function f of compact support on space which ultimately $\to 1$, are indeed representable by densely defined operators admitting self-adjoint extensions. It is plausible that a unique adiabatic perturbation (but no adiabatic sum, except in trivial cases as of a quadratic perturbation) exists, at least in the form of automorphisms of a Weyl algebra, and has the natural locality and covariance properties. Hyperbolicity considerations would then yield as earlier an explicit passage to the limit $f \to 1$, resulting in an automorphism $\Sigma(g)$ of the Weyl C^*-algebra. The putative $\lim_{g \to 1} \Sigma(g)$ is an automorphism which is necessarily unitarily implementable by an essentially unique unitary operator, then appropriately definable as the S-operator. The controlled existence of the indicated adiabatic perturbations is probably sufficient to provide a satisfactory mathematical foundation for basic nonlinear quantum field theory ; and it may well be that those formulations which involve temporal transitions between precise times cannot, in the case of a four-dimensional space-time, be effectively correlated with rigorous mathematics.

Concluding remarks.

Limitations of time and space have made it necessary to omit discussion of some interesting related questions. I mention only :

(i) the treatment of quantum fields satisfying anti-commutation relations ;

(ii) the theory of symplectic manifolds, particularly solution manifolds of evolutionary differential equations, and their "quantization".

REFERENCES

(These include references to relevant earlier work cited above.)

[1] Nonlinear functions of weak processes, I and II. *Jour. Funct. Analysis,* 4, 1969, p. 404-457 and 6, 1970, p. 29-75.
[2] Transformations in Wiener space and squares of quantum fields, *Adv. in Math.*, 4, 1970, p. 91-108.
[3] Construction of nonlinear local quantum processes, I and II. *Annals of Math.*, 92, 1970, p. 462-481; and to appear.

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ONE PARAMETER AUTOMORPHISM GROUPS AND STATES OF OPERATOR ALGEBRAS

by Masamichi TAKESAKI

In order to solve an outstanding problem in the tensor products of operator algebras :

$$(\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2)' = \mathfrak{M}_2' \bar{\otimes} \mathfrak{M}_2',$$

Tomita developed in 1967 the theory of modular Hilbert algebras (which I will call Tomita algebras). His theory enables us not only to solve the above problem but also to develop a new theory of non-commutative integration together with applications to mathematical physics.

To describe a thermodynamical equilibrium state for a quantum system, physists introduced the so-called Kubo-Martin-Schwinger (KMS) boundary conditions as follows : when a C^*-algebra $\mathfrak{A}$ and a one parameter automorphism group σ_t are given, a σ_t-invariant positive linear functional(1) φ of $\mathfrak{A}$ is said to satisfy the KMS-boundary condition for $\beta > 0$ if for every $x, y \in \mathfrak{A}$ there exists a bounded function $F(z)$ continuous on and holomorphic in the strip, $0 \leqslant \operatorname{Im} z \leqslant \beta$, with boundary values :

$$F(t) = \varphi(\sigma_t(x)\, y) \quad \text{and} \quad F(t + i\beta) = \varphi(y\sigma_t(x)) .$$

Here β means $1/kT$, k the Boltzman constant and T the absolute temperature of the system consideration. Considering $\sigma_{\beta t}$, we may assume $\beta = 1$ without loss of generality.

Although the KMS condition looks strange apparently, it plays a vital role in non-commutative integration theory. To explain the situation concerning the KMS-condition, let $\mathfrak{M}$ be the von Neumann algebra of all bounded operators on a separable Hilbert space $\mathfrak{H}$. Let h be a positive nuclear operator on $\mathfrak{H}$ such that $h\xi \neq 0$ for every non-zero $\xi \in \mathfrak{H}$. Let $\varphi(x) = Tr(xh)$ for $x \in \mathfrak{M}$. Then φ is a faithful normal positive linear functional on $\mathfrak{M}$ and h^{it}, $t \in R$, is a one parameter strongly continuous unitary group which induces a one parameter automorphism group $\sigma_t(x) = h^{it}\, x\, h^{-it}$ of $\mathfrak{M}$. For each $x, y \in \mathfrak{M}$ we have

$$F(t) = \varphi(\sigma_t(x)\, y) = Tr(h^{it+1}\, xh^{-it}\, y) .$$

If $\operatorname{Re} z \geqslant 0$, then h^z is bounded. If $0 \leqslant \operatorname{Re}^z \leqslant 1$, then $h^{1-z}\, xh^z\, y$ is nuclear and the function : $z \to Tr(h^{1+iz}\, xh^{-iz}\, y)$ is holomorphic in and continuous on the strip : $0 \leqslant \operatorname{Im} z \leqslant 1$; and we have

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(1) If $\mathfrak{A}$ has a unit, then the KMS-condition implies the σ_t-invariance of φ by Strum's theorem for bounded holomorphic functions.

$$F(t+i) = Tr(h^{it}xh^{-it+1}y) = Tr(hyh^{it}xh^{-it}) = \varphi(y\sigma_t(x)).$$

Therefore, φ satisfies the KMS-condition with respect to σ_t.

Suppose a C^*-algebra $\mathfrak{A}$, its one parameter automorphism group σ_t, and a KMS-state φ are given. Then the left kernel $N_\varphi = \{x \in \mathfrak{A} : \varphi(x^*x) = 0\}$ of φ is actually a two sided ideal and coincides with the kernel $\pi_\varphi^{-1}(0)$ of the cyclic representation π_φ induced by φ. Hence the quotient space $\mathfrak{A} = \mathfrak{A}/N_\varphi$ has an involutive algebra structure and an inner product induced by φ. Then we can see easily in this algebra $\mathfrak{A}$ that the involution : $\pi_\varphi(x)\xi_\varphi \to \pi_\varphi(x_\varphi^*)\xi_\varphi$ is preclosed, where ξ_φ is the cyclic vector of the cyclic representation $\{\pi_\varphi, \mathfrak{H}_\varphi\}$.

In general, an involutive algebra $\mathfrak{A}$ with involution : $\xi \in \mathfrak{A} \to \xi^\# \in \mathfrak{A}$ is called a *left Hilbert algebra* if it has an inner product such that

(I) the involution : $\xi \to \xi^\#$ is preclosed ;

(II) $(\xi\eta | \zeta) = (\eta | \xi^\# \zeta)$;

(III) for each $\xi \in \mathfrak{A}$, the map : $\eta \to \xi\eta$ is continuous ;

(IV) the subalgebra $\mathfrak{A}^2$, spanned by $\xi\eta$, $\xi, \eta \in \mathfrak{A}$, is dense in $\mathfrak{A}$.

Namely, the algebra $\mathfrak{A}$, obtained by a KMS-state, is precisely a left Hilbert algebra. There is another very important example of a left Hilbert algebra. Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra on a Hilbert space with separating and cyclic vector ξ_0.

Let $\mathfrak{A} = \mathcal{M}\xi_0$. Then $\mathfrak{A}$ has naturally an involutive algebra structure as follows :

$$(x\xi_0)(y\xi_0) = xy\xi_0,\ xy \in \mathcal{M} ;$$

$$(x\xi_0)^\# = x^*\xi_0.$$

Since we have, for every $x \in \mathcal{M}$ and $y \in \mathcal{M}'$,

$$(x^*\xi_0 | y\xi_0) = (y^*\xi_0 | x\xi_0)$$

and $\mathcal{M}'\xi_0 = \mathfrak{A}'$ is dense in $\mathfrak{H}$, the involution : $x\xi_0 \to x^*\xi_0$ in $\mathfrak{A}$ is pre-closed. Therefore, then algebra $\mathfrak{A}$ is also a left Hilbert algebra.

Now, suppose $\mathfrak{A}$ is a left Hilbert algebra and $\mathfrak{H}$ its completion. To each $\xi \in \mathfrak{A}$, there corresponds a unique operator $\pi(\xi)$ on $\mathfrak{H}$ defined by $\pi(\xi)\eta = \xi\eta$, $\eta \in \mathfrak{A}$. The von Neumann algebra generated by $\pi(\xi)$, $\xi \in \mathfrak{A}$. is called the *left von Neumann* algebra of $\mathfrak{A}$ and denoted by $\mathcal{L}(\mathfrak{A})$. Let S be the closure of the involution : $\xi \in \mathfrak{A} \to \xi^\# \in \mathfrak{A}$ as a closed operator and F the adjoint of S. Let $D^\#$ and D^b be the definition domains of S and F respectively. Notice that S and F are both conjugate linear ; hence

$$(S\xi | \eta) = (F\eta | \xi),\ \xi \in D^\#,\ \eta \in D^b .$$

Let $\Delta = FS$. Then Δ is a self-adjoint positive operator on $\mathfrak{H}$ and we can conclude that

$$D^\# = D(\Delta^{1/2}) \quad \text{and} \quad D^b = D(\Delta^{-1/2})$$

and the polar decompositions of S and F :

$$S = J\Delta^{1/2} = \Delta^{-1/2}J ;$$

$$F = J\Delta^{-1/2} = \Delta^{1/2} J .$$

Here J is a conjugate linear isometry and $J^2 = 1$.

A vector $\eta \in D^b$ is called π'-bounded if $\xi \in \mathfrak{A} \to \pi(\xi)\,\eta$ is bounded. If $\eta \in D^b$ is π'-bounded, the map : $\xi \in \mathfrak{A} \to \pi(\xi)\,\eta$ is extended to a bounded operator $\pi'(\eta)$. Denote by $\mathfrak{A}'$ the set of all π'-bounded elements in D^b and define an involution and a product in $\mathfrak{A}'$ by :

$$\eta^b = F\eta \ ; \ \eta_1\eta_2 = \pi'(\eta_2)\,\eta_1 \, .$$

Then $\mathfrak{A}'$ turns out to be a *right* Hilbert algebra as the dual notion of the *left* Hilbert algebra. The map $\pi' : \eta \in \mathfrak{A}' \to \pi'(\eta) \in B(\mathfrak{H})$ is an anti-representation of $\mathfrak{A}'$. We can show, without particular difficulties that the commutant $\mathcal{L}(\mathfrak{A})'$ of $\mathcal{L}(\mathfrak{A})$ is generated by $\pi'(\mathfrak{A}')$. Starting from the right Hilbert algebra $\mathfrak{A}'$, we get a *left* Hilbert algebra $\mathfrak{A}''$ which contains $\mathfrak{A}$ as a subalgebra. If we proceed these arguments, then we get

$$\mathfrak{A}' = \mathfrak{A}''' = \mathfrak{A}^{(v)} = \ldots\ldots \ ; \ \mathfrak{A}'' = \mathfrak{A}^{(iv)} = \mathfrak{A}^{(vi)} = \ldots$$

If $\mathfrak{A} = \mathfrak{A}''$, then it is said to be *achieved.*

After a deep spectral analysis of Δ by making use of Cauchy integral, Tomita proved the following :

THEOREM. – *If* $\mathfrak{A}$ *is an achieved left Hilbert algebra, then the one-parameter unitary group* Δ^{it}, $t \in R$, *is a group of automorphisms of* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *Furthermore, there exists a self-adjoint subalgebra* $\mathfrak{A}_0$ *of* $\mathfrak{A}$ *such that*

(i) $\mathfrak{A}_0'' = \mathfrak{A}$; *hence* $\mathcal{L}(\mathfrak{A}_0) = \mathcal{L}(\mathfrak{A})$;

(ii) $\mathfrak{A}_0$ *is contained in* $\bigcap_{a \in C} D(\Delta^a)$ *and* Δ^a *satisfies the following properties in addition to* (I) – (IV) :

(V) $(\Delta^a \xi)^\# = \Delta^{-\bar{a}}\,\xi^\#$;

(VI) $(\Delta^a \xi \,|\, \eta) = (\xi \,|\, \Delta^{\bar{a}}\eta)$;

(VII) $(\Delta\xi^\# | \eta^\#) = (\eta \,|\, \xi)$;

(VIII) $(1 + \Delta^t)\mathfrak{A}_0$ *is dense in* $\mathfrak{A}_0$ *for every* $t \in R$;

(IX) $(\Delta^\alpha \xi \,|\, \eta)$ *is a holomorphic function of* α *on the whole plane* C.

In general, an involutive algebra $\mathfrak{A}$ equipped with inner product and complex parameter automorphism group $\Delta(\alpha)$ is called a *Tomita* algebra if conditions (I) – (IX) are satisfied. Notice that condition (I) follows from other conditions (II) – (IX). A Tomita algebra $\mathfrak{A}$ is acturely a left and right Hilbert algebra, if we put $\xi^b = \Delta\xi^\#$.

THEOREM. – *If* $\mathfrak{A}$ *is a Tomita algebra, then the commutant* (*the right von Neumann algebra of* $\mathfrak{A}$) *of* $\mathcal{L}(\mathfrak{A})$ *is generated by* $\pi'(\mathfrak{A})$; *and the unitary involution J maps* $\mathcal{L}(\mathfrak{A})$ *onto* $\mathcal{L}(\mathfrak{A})'$, *namely*

$$J\mathcal{L}(\mathfrak{A})\,J = \mathcal{L}(\mathfrak{A})' \quad \text{and} \quad J\mathcal{L}(\mathfrak{A})'J = \mathcal{L}(\mathfrak{A}) ,$$

where the unitary involution J in the extension of the conjugate linear isometry : $\xi \epsilon \mathfrak{D} \to \Delta^{1/2}\xi^{\#} = \xi^*$.

Therefore, a von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$ with separating and generating vector is anti-isomorphic to its commutant $\mathcal{M}'$. These results yield the following solution for Tensor products :

$$(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \bar{\otimes} \mathcal{M}_2'$$

Now, we can show by making use of the above machinery, that if φ_0 is the normal positive linear functional on $\mathcal{M}$ defined by $\varphi_0(x) = (x\xi_0 \mid \xi_0)$, $x \in \mathcal{M}$, where ξ_0 is a fixed separating and cyclic vector, then φ_0 satisfies the KMS-boundary condition with respect to the one-parameter automorphism σ_t of $\mathcal{M}$ induced by Δ^{it}, where Δ is the modular operator of the left Hilbert algebra $\mathfrak{A} = \mathcal{M}\xi_0$. This result is stated more precisely as follows :

THEOREM. — *To every faithful normal positive linear functional φ on a von Neumann algebra $\mathcal{M}$, there corresponds a unique one-parameter automorphism group σ_t^φ of $\mathcal{M}$ for which φ satisfies the KMS-boundary condition at $\beta = 1$.*

This σ_t^φ is called the *modular automorphism group* of $\mathcal{M}$ associated with φ.

THEOREM. — *The automorphism group σ_t^φ is inner in the sense that there exists a strongly continuous one parameter unitary group $\Gamma(t)$ in $\mathcal{M}$ which implements σ_t^φ, if and only if $\mathcal{M}$ is semi-finite.*

If $\mathcal{M}$ has a faithful normal semi-finite trace τ, then σ_t^φ is implemented by h^{it}, where h is defined by $\varphi(x) = \tau(xh)$, $x \in \mathcal{M}$. Therefore, the modular automorphism group σ_t^φ can be expected to carry some information of φ as h^{it} does. Actually, we have the following.

THEOREM. — *If $\mathcal{M}$ is a factor, then any positive linear functional ψ satisfying the KMS-boundary condition for σ_t^φ at $\beta = 1$ is a scalar multiple of φ. In general, ψ is a center multiple of φ in the following sense : there exists a positive self-adjoint operator h affiliated with the center $\mathfrak{Z}$ of $\mathcal{M}$ such that*

$$\psi(x) = (xh\xi\varphi \mid h\xi\varphi),\ x \in \mathcal{M},$$

where $\varphi(x) = (x\xi\varphi \mid \xi\varphi)$.

Let $\mathcal{M}_\varphi$ be the subalgebra of $\mathcal{M}$ consisting of all those x such that

$$\varphi(xy) = \varphi(yx),\ y \in \mathcal{M}.$$

The subalgebra $\mathcal{M}_\varphi$ is called the *centralizer* of φ. Then $\mathcal{M}_\varphi$ is precisely the fixed point algebra of σ_t^φ. Now, we define the commutativity of positive functionals φ and ψ by $|\varphi + i\psi| = |\varphi - i\psi|$, where $|\cdot|$ means the absolute value of $\cdot$ in the sense of the polar decomposition in the predual $\mathcal{M}_*$.

THEOREM. — *Let φ be a faithful normal positive linear functional of $\mathcal{M}$. For any positive normal linear functional ψ of $\mathcal{M}$, the following conditions are all equivalent :*

(i) *φ and ψ commute, that is,*

$$|\varphi + i\psi| = |\varphi - i\psi| ;$$

(ii) ψ *is* σ_t^φ*-invariant* ;

(iii) *there exists a positive self-adjoint operator h affiliated with the centralizer* $\mathfrak{M}_\varphi$ *of* φ *such that*

$$\psi(x) = (xh\xi_\varphi \mid h\xi_\varphi),\ x \in \mathfrak{M},$$

where

$$\varphi(x) = (x\xi_\varphi \mid \xi_\varphi).$$

If ψ is faithful, then the following conditions are equivalent to the above conditions :

(iv) σ_t^φ and σ_s^ψ commute ;

(v) φ is σ_t^ψ-invariant.

If we have a normal faithful finite trace τ on a von Neumann algebra $\mathfrak{M}$ and its von Neumann subalgebra $\mathfrak{N}$ then we can define the conditional expectation ϵ of $\mathfrak{M}$ onto $\mathfrak{N}$ by :

$$\tau(xy) = \tau(\epsilon(x)\,y),\ x \in \mathfrak{M},\ y \in \mathfrak{N}$$

and this conditional expectation was studied by Umegaki in great detail [24]. Using our theory, we can show the following :

THEOREM. – *Let* φ *be a faithful normal state on a von Neumann algebra* $\mathfrak{M}$ *with associated modular automorphism group* σ_t. *Suppose* $\mathfrak{M}$ *is a von Neumann subalgebra of* $\mathfrak{M}$. *Then* $\mathfrak{N}$ *is invariant under* σ_t, *if and only if there exists a unique conditional expectation* ϵ *of* $\mathfrak{M}$ *onto* $\mathfrak{N}$ *determined by :*

$$\varphi(xy) = \varphi(\epsilon(x)\,y)\ x \in \mathfrak{M}\ ,\ y \in \mathfrak{N}\ .$$

As well as the conditional expectation is defined for semi-finite trace, this new conditional expectation is defined for a semi-finite weight, an unbounded positive linear functional under a suitable consideration. If we have done this, then the projection of G-finite von Neumann algebra $\mathfrak{M}$ to the fixed points algebra $\mathfrak{M}^G$ is understood as the conditional expectation of $\mathfrak{M}$ onto $\mathfrak{M}^G$ determined by a suitable weight.

REFERENCES

[1] ARAKI H. — Multiple time analyticity of a quantum statistical state satisfying the KMS-boundary condition, *Publ. R.I.M.S.*, Kyoto Univ., Ser. A, 4, 1969.

[2] ARAKI H. and MIYATA H. — On the KMS-boundary condition, *Publ. R.I.M.S.*, Kyoto Univ., Ser. A, 4, 1968, p. 361-371.

[3] DIXMIER J. — *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, 2nd edition, 1969.

[4] DIXMIER J. — Algèbres quasi-unitaires, *Comm. Math. Helv.*, 26, 1952, p. 275-322.

[5] DIXMIER J. — Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. Fr.*, 81, 1953, p. 9-39.

[6] DYE H.A. — The Radon-Nikodym theorem for finite rings of operators, *Trans. Amer. Math. Soc.*, 72, 1952, p. 243-280.

[7] HAAG R., HUGENHOLTZ N.M. and WINNINK M. — On the equilibrium states in quantum statistical mechanics, *Comm. Math. Phys.*, 5, 1967, p. 215-236.

[8] HERMAN R. and TAKESAKI M. — States and automorphism groups of operator algebras, to appear in *Comm. Math. Phys.*, 1970.

[9] HUGENHOLTZ N.M. and WIERINGA J.D. — On locally normal states in quantum statistical mechanics, *Comm. Math. Phys.*, 11, 1969, p. 183-197.

[10] KASTLER D., POOL J.C. and THUE POULSEN E. — Quasi-unitary algebras attached to temprature states in statistical mechanics — a comment on the work of Haag, Hugenholtz and Winnink, *Comm. Math. Phys.*, 12 1969, p. 175-192.

[11] MURRAY F.J. and VON NEUMANN J. — On rings of operators, *Ann. Math.*, 37, 1936, p. 116-229.

[12] VON NEUMANN J. — On rings of operators III, *Ann. Math.*, 41, 1940, p. 94-161.

[13] PUKANSZKY L. — On the theory of quasi-unitary algebras, *Acta Sci. Math.*, 17, 1955, p. 103-121.

[14] SAKAI S. — A Radon-Nikodym theorem in W^*-algebras, *Bull. Amer. Math. Soc.*, 71, 1965, p. 149-151.

[16] SEGAL I.E. — A non-commutative extension of abstract integration, *Ann. Math.*, 57, 1953, p. 401-457.

[17] TAKESAKI M. — Tomita's theory of modular Hilbert algebras and its applications, *Lecture Notes*, Springer 128, 1970.

[18] TAKESAKI M. — Disjointness of the *KMS*-states of different temperatures, *Comm. Math. Phys.*, 17, 1970, p. 33-41.

[19] TOMITA M. — Standard forms of von Neumann algebras, *The Vth functional analysis symposium of the Math. Soc. of Japan*, Sendai, 1967.

[20] WINNINK M. — Algebraic aspects of the Kubo-Martin-Schwinger condition, *Cargèse Lecture Notes*, 1969.

[21] TAKESAKI M. — The theory of operator algebras, *Lecture Notes*, U.C.L.A., 1969.

[22] COMBES F. — *Poids associé a une algèbre hilbertienne à gauche*, to appear.

[23] PERDRIZET F. — *Elements positifs relativement à une algèbre hilbertienne à gauche*, to appear.

[24] UMEGAKI H. — Conditional expectation in operator algebra, I, II, III and IV, *Tôhoku Math J.*, 6, 1954, p. 177-181; *Tôhoku Math. J.*, 8, 1956, p. 86-100; *Kodai Math. Sem. Rep.*, 11, 1959, p. 51-64; *Kodai Math. Sem. Rep.*, 14, 1962, p. 59-85.

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D 3 - THÉORIE SPECTRALE

SOME APPLICATIONS OF STRUCTURAL MODELS FOR OPERATORS ON HILBERT SPACE

by Ciprian FOIAŞ

1. – Among objects in mathematics which are easily defined but are far from being well understood, one must rank operators on Hilbert spaces(1) ; indeed, many natural problems concerning the structure of these operators are not yet solved (see for instance [9]). One of the reasons for this situation may lie in the structural poverty of the definition of an operator on a Hilbert space and, perhaps, in the future, people working in mathematical logic will illuminate some deep questions concerning the structure of such an operator. One way to forestall this eventuality is to build models involving more elaborate structures (if not more concrete) for general operators, up to suitable equivalence relations. The aim of this address is to present some models of this kind together with some achievements based on them.

2. – We shall begin by defining a rather complicated category(2) involving sophisticated operator structures, which, however, implicitly appear in the study of an operator on Hilbert space ; moreover this category is connected with abstract Prediction Theory (compare the subcategory Σ_I defined in the sequel with [1]) and, in case the group Z of all integers is replaced by that of the real line R, with Scattering Theory [10](3). The objects of this category are systems

$$\sigma = \{U, J; \mathfrak{K}_-, \mathfrak{K}_+\}$$

formed by operators U and J acting on a Hilbert space $\mathfrak{K}$ spanned by its subspaces $\mathfrak{K}_-$ and $\mathfrak{K}_+$, and satisfying the following conditions : (i) J is a symmetry (i.e. $J^* = J = J^{-1}$) and U is J-unitary (i.e. $U^*JU = J = UJU^*$) ; moreover

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(1) All operators are supposed linear, bounded and everywhere defined ; the Hilbert spaces are supposed complex. The terminology and notations are those of the forthcoming English version of the monograph [17] to which, for the sake of brevity, constant reference will be made when possible.

(2) We shall use in part the language of the theory of categories with the hope that it might suggest some natural questions in Operator Theory. For the terminology and notations concerning this language we refer to [4].

(3) If Z is replaced by the Poincaré group (i.e. the inhomogeneous Lorentz group) a similar category may play an interesting role in the theory of the analytic Lorentz invariant S-matrix theory (as sketched in [5]).

$U^{\pm 1}\mathfrak{K}_{\pm} \subset \mathfrak{K}_{\pm} = J\mathfrak{K}_{\pm}$. (ii) If $\mathfrak{H} = \mathfrak{K}_{-} \cap \mathfrak{K}_{+}$ then $J/\mathfrak{H} = I_{\mathfrak{H}}$ (1) and

$$\mathfrak{K}_{-} \ominus \mathfrak{H} \perp \mathfrak{K}_{+} \ominus \mathfrak{H}\,, \quad \mathfrak{K}_{\pm} = \bigvee_{n \geqslant 0} U^{\pm 1}\,\mathfrak{H}.$$

If $\sigma' = \{U', J'\,;\, \mathfrak{K}'_{-}, \mathfrak{K}'_{+}\} \in Ob\,\Sigma$ (i.e. is an object of Σ), then, by definition,

$$X \in \mathrm{Hom}_{\Sigma}(\sigma', \sigma)$$

(i.e. is a morphism, in Σ, from σ' to σ) is an operator from $\mathfrak{K}'$ ($= \mathfrak{K}'_{-} \vee \mathfrak{K}'_{+}$) into $\mathfrak{K}$ satisfying the following condition : (iii) X intertwins U' with U (i.e. $XU' = UX$),

$$X^*JX \leqslant J' \quad , \quad XJ'X^* \leqslant J \quad \text{and} \quad X\mathfrak{K}'_{+} \subset \mathfrak{K}_{+} \quad , \quad X^*\mathfrak{K}_{-} \subset \mathfrak{K}'_{-}\,.$$

For σ', $\sigma \in Ob\Sigma$, $X \in \mathrm{Hom}_{\Sigma}(\sigma', \sigma)$ let us define the operator $\mathrm{C}(\sigma)$ on $\mathfrak{H}$ (see (ii) above) by $\mathrm{C}(\sigma) = P_{\mathfrak{H}}\,U/\mathfrak{H}$ and the operator $\mathrm{C}(X)$ from $\mathfrak{H}'(= \mathfrak{K}'_{-} \cap \mathfrak{K}'_{+})$ to $\mathfrak{H}$ by $\mathrm{C}(X) = P_{\mathfrak{H}}X/\mathfrak{H}'$. We obtain thus a (covariant) functor C from Σ into the category Ω of all operators on Hilbert spaces, whose morphisms are the intertwining operators. We shall call C the *compression functor* from Σ into Ω. This functor, which occurs in most of the structure theories for normal operators developed in the last 25 years, has the following basic property :

(I) *Any operator* T (i.e. $T \in Ob\,\Omega$) *is unitarily equivalent*(2) *to some* $\mathrm{C}(\sigma)$ *with* $\sigma \in Ob\,\Sigma$.

This theorem (in an equivalent form), as well as the following construction of such a σ, is due to C. Davis [6] : Let $D_T = |I_{\mathfrak{H}} - T^*T|^{1/2}$, $J_T = sgn(I_{\mathfrak{H}} - T^*T)$ and let $\mathfrak{K}$ be the following orthogonal sum

$$\begin{matrix} & (-2) & (-1) & (0) & (1) & (2) & \\ \cdots \oplus & \mathfrak{D}_{T^*} \oplus & \mathfrak{D}_{T^*} \oplus & \mathfrak{H} \oplus & \mathfrak{D}_T \oplus & \mathfrak{D}_T & \oplus \cdots \end{matrix}$$

where $\mathfrak{H}$ is the Hilbert space where T operates, $\mathfrak{D}_{T^*} = \overline{D_{T^*}\mathfrak{H}}$, $\mathfrak{D}_T = \overline{D_T\,\mathfrak{H}}$, and the upper indices stand for the coordinates. Let $\mathfrak{K}_{-}$(resp. $\mathfrak{K}_{+}$) be the subspace of those elements of $\mathfrak{K}$ whose positive (resp. negative) components vanish. Finally, for $h = (h_i)_{-\infty}^{\infty} \in \mathfrak{K}$ put $Uh = (k_i)_{-\infty}^{\infty}$ with

$$k_i = h_{i-1} \text{ for } i \neq 0, 1,\; k_0 = D_{T^*}h_{-1} + Th_0\,,\;\; k_1 = -T^*J_{T^*}h_{-1} + D_T h_0\,,$$

and $Jh = (l_i)_{-\infty}^{\infty}$ with $l_i = J_T h_i$ for $i < 0$, $l_0 = h_0$, $l_i = J_T h_i$ for $i > 0$. For the system $\sigma = \{U, J\,; \mathfrak{K}_{-}, \mathfrak{K}_{+}\}$ obtained in this way, $\mathrm{C}(\sigma)$ is unitarily equivalent to T. In case T is a contraction, (I) is essentially the existence theorem, of the unitary dilation U for T, of B. Sz.-Nagy (see [17], Secs. I.4 and II.1), while the above construction coincides with that of J. J. Schäffer (as improved by B. Sz.-Nagy ; see [17], Sec. I.5). The basic role played by the compression functor can be clearly seen in this particular case. Indeed, if we denote by Σ_I the full subcategory (see(2) page 235) of Σ whose objects are those σ for which $J = I_{\mathfrak{K}}$ and by Ω_1 the

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(1) We denote by I the identity operator on any Hilbert space $\mathfrak{K}$ and by $P_{\mathfrak{H}}$ the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$, whenever $\mathfrak{H}$ is a subspace of $\mathfrak{K}$.

(2) Note that unitary equivalence is stronger than isomorphism in Ω, which coincides with affine equivalence (i.e. similarity) ; however remark that in the category Ω_1, defined in the sequel, isomorphism means unitary equivalence.

subcategory of Ω whose objects are the contractions and whose morphisms are the intertwining contractions, then $\mathbf{C}(\sigma) \in Ob\,\Omega_1$ *if and only if* $\sigma \in Ob\,\Sigma_I$ and the restriction $\mathbf{C}_I$ of $\mathbf{C}$ to Σ_I is a functor from Σ_I into Ω_1, in terms of which the basic theorems of the theory of the unitary dilations of contractions can be stated as follows :

(II) *Any* $T \in Ob\,\Omega_1$ *is isomorphic to some* $\mathbf{C}_I(\sigma)$ *with* $\sigma \in \Sigma_I$(1).

(III) $\mathbf{C}_I(\sigma)$ *and* $\mathbf{C}_I(\sigma')$ *are isomorphic if and only if* σ *and* σ' *are isomorphic.*

(IV) *Any* $A \in \mathrm{Hom}_{\Omega_1}(\mathbf{C}_I(\sigma'), \mathbf{C}_I(\sigma))$ *is of the form* $\mathbf{C}_I(X)$ *with some*

$$X \in \mathrm{Hom}_{\Sigma_I}(\sigma', \sigma) .$$

Statement (II) is another formulation of the above quoted theorem of B. Sz.-Nagy, (III) is his uniqueness theorem for the minimal unitary dilation (see [17], Sec. I.4), while (IV) is precisely the dilation theorem for intertwining operators between contractions (see [17], Sec. II.2 and [8]). Many of the known properties of contractions are actually implied by the above properties of $\mathbf{C}_I$. For instance, J. von Neumann's inequality (see [17], Sec. I.8), D. Sarason's theorem on the commutant of the restrictions of the backward shift of multiplicity one (see [17], Sec. VI.3.8) and Z. Nehari's extrapolation theorem (as well as its generalizations ; see [11]) are direct consequences of the properties (II) – (IV) of $\mathbf{C}_I$. Moreover, (II) – (IV) easily imply T. Ando's theorem asserting the existence of a unitary dilation of a pair of commuting contractions (see [17], Sec. I.6.1). Since this theorem fails if the pair T_1, T_2 is replaced by a triplet T_1, T_2, T_3 of commuting contractions (as shown by an unexpectedly simple counter example of S. Parrott ; see [17], Sec. I.6.3), one can easily conclude that *the construction of* σ *(in* (II)*) and* X *(in* (IV)*) cannot be given a functorial character.*

3. – In virtue of the properties (II) – (IV) we can consider $\mathbf{C}_I(\sigma)$ together with all its structure inherited from $\sigma \in Ob\,\Sigma_I$ as a structural model for any $T \in Ob\,\Omega_1$ isomorphic to $\mathbf{C}_I(\sigma)$, obtaining thus a structural model for any contraction. A basic feature of these models is that, up to an isomorphism in Σ_I, any $\sigma \in Ob\,\Sigma_I$ is of the following form :

$$U = U_*^{\times} \oplus (U^{\times} / \overline{\Delta L^2(\mathfrak{E})}) \oplus V \tag{1}$$

$$K_- = [L^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})} \oplus \mathfrak{F}] \ominus \{\Theta u \oplus \Delta u \oplus O : u \in H^2(\mathfrak{E})\} \tag{2}$$

$$K = H^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})} \oplus \mathfrak{F} , \tag{3}$$

where :

(i) $\mathfrak{E}$, $\mathfrak{E}_*$ and $\mathfrak{F}$ are Hilbert spaces (in many interesting concrete cases $\mathfrak{E}$ and $\mathfrak{E}_*$ are finite dimensional and $\mathfrak{F} = \{0\}$), $L^2(\mathfrak{E})$ (resp. $L^2(\mathfrak{E}_*)$) is the L^2-space of $\mathfrak{E}$(resp. $\mathfrak{E}_*$)-valued functions defined on $[0, 2\pi)$, while $H^2(\mathfrak{E})$ (resp. $H^2(\mathfrak{E}_*)$) is the corresponding Hardy subspace of $L^2(\mathfrak{E})$ (resp. $L^2(\mathfrak{E}_*)$) (see [17], Sec. V.1) ;

(ii) $U^{\times}$(resp. $U_*^{\times}$) denotes the multiplication by the exponential function e^{it} on $L^2(\mathfrak{E})$ (resp. $L^2(\mathfrak{E}_*)$), while V is a unitary operator on $\mathfrak{F}$;

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(1) See footnote (2), previous page.

(iii) $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ is a pure contractive analytic function, i.e. an analytic function defined on the unit disc $D = \{\lambda : |\lambda| < 1\}$ whose values are contractions from $\mathfrak{E}$ into $\mathfrak{E}_*$ and which satisfies the condition $\|\Theta(0)\, e\| < \|e\|$ whenever $e \neq 0$;

(iv) $\Delta(t) = (I_{\mathfrak{E}} - \Theta(e^{it})^* \Theta(e^{it}))^{1/2}$, which makes sense almost everywhere on $[0,2\pi)$. Clearly the space $\mathfrak{F}$ in (2) – (3) is $\{0\}$ if and only if $\mathbf{C}_I(\sigma)$ is not reduced to a unitary operator by any non trivial subspace, i.e. if $\mathbf{C}_I(\sigma)$ is completely non unitary. In this case the system σ given above depends only on Θ and will be denoted by σ_Θ. Thus *every completely non unitary contraction on a Hilbert space is unitarily equivalent to some* $\mathbf{C}_I(\sigma_\Theta)$. This is the functional model introduced and studied in a series of common papers by B. Sz.-Nagy and the author beginning in early 1963 (for a detailed exposition, see [17], Chs. VI and VII). Another way of attaching to any analytic contractive function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ a "canonical" contraction $\mathbf{R}_\Theta$ was proposed in 1964 by L. de Branges and J. Rovnyak (see [2], Appendix) ; namely $\mathbf{R}_\Theta$ is the restriction of $(U_*^x/H^2(\mathfrak{E}_*))^*$ to the Hilbert space formed by those functions $u_* \in H^2(\mathfrak{E}_*)$ for which

$$\|u_*\|^2 = \sup\{\|u_* + \Theta u\|^2 - \|u\|^2 : u \in H^2(\mathfrak{E})\} < \infty$$

and endowed with the norm : $u_* \to \|u_*\|$(1). It is easy to see that $\mathbf{R}_\Theta$ does not change if Θ is replaced by its pure part (in the sense of [17], Sec. V.2), so that we can suppose that $\Theta(\lambda)$ is pure. To exhibit the connection (suggested by R.G. Douglas) between the two operators $\mathbf{C}_I(\sigma_\Theta)$ and $\mathbf{R}_\Theta$, let us point out (see (1) – (3) and Sec. 3) that $\mathbf{C}_I(\sigma)^*$ *is the restriction of*

$$(4) \qquad (U_*^x/H^2(\mathfrak{E}_*))^* \oplus (U^x/\Delta L^2(\mathfrak{E}))^*$$

to
$$\mathfrak{H}_\Theta = [H^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})}] \ominus \{\Theta u \oplus \Delta u : u \in H^2(\mathfrak{E})\}$$

and moreover that

$$(5) \qquad \mathfrak{H}_\Theta^0 = [H^2(\mathfrak{E}_*) \oplus \overline{\Delta H^2(\mathfrak{E})}] \ominus \{\Theta u \oplus \Delta u : u \in H^2(\mathfrak{E})\}$$

is the smallest subspace of $\mathfrak{H}_\Theta$ *such that* $\mathfrak{H}_\Theta \ominus \mathfrak{H}_\Theta^0$ *is invariant under* $\mathbf{C}_I(\sigma_\Theta)^*$ *and* $\mathbf{C}_I(\sigma_\Theta)^*/\mathfrak{H}_\Theta \ominus \mathfrak{H}_\Theta^0$ *is isometric.* The connection between $\mathbf{C}_I(\sigma_\Theta)$ and $\mathbf{R}_\Theta$ can now be stated as follows :

(V) *The mapping* $A(u_* \oplus v) = u_*$ *from* $\mathfrak{H}_\Theta^0$ *into* $H^2(\mathfrak{E}_*)$ *realizes a unitary equivalence between* $P_{\mathfrak{H}_\Theta^0}\mathbf{C}_I(\sigma_\Theta)^*/\mathfrak{H}_\Theta^0$ *and* $\mathbf{R}_\Theta$.

In particular, *every contraction T having no isometric restriction to any non trivial invariant subspace is, up to a unitary equivalence, of the form* $\mathbf{R}_\Theta$. However this functional model of L. de Branges and J. Rovnyak has lost some of the superstructure of σ_Θ which explains why the model $\mathbf{C}_I(\sigma_\Theta)$ is more powerful as the following sections will also illustrate.

4. – In the study of $\mathbf{C}_I(\sigma_\Theta)$ one of the significant facts is that if T is isomorphic (in Ω_1) to $\mathbf{C}_I(\sigma_\Theta)$ then $\Theta(\lambda)$ coincides, up to constant (as functions of λ) unitary factors, with the *characteristic function* of T (see [17], Sec. VI. 1-3). Let

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(1) That this is indeed a Hilbert space follows also from the next Proposition (V).

us recall that for any operator T on a Hilbert space $\mathfrak{H}$ the characteristic function $\Theta_T(\lambda)$ is defined by

$$\Theta_T(\lambda) = [-TJ_T + \lambda D_{T^*}(I_{\mathfrak{H}} - \lambda T^*)^{-1} D_T] / \mathfrak{D}_T \quad (6)$$ (1)

whenever $(I_{\mathfrak{H}} - \lambda T^*)^{-1}$ makes sense ; its values are operators from $\mathfrak{D}_T$ into $\mathfrak{D}_{T^*}$. This function was introduced a long time ago (see for instance [13.]) as a Möbius transform of T, since

$$\Theta_T(\lambda)\, D_T J_T = D_{T^*}(I_{\mathfrak{H}} - \lambda T^*)^{-1}(\lambda I_{\mathfrak{H}} - T) .$$

The connection between the characteristic function and the functional model for a completely non unitary contraction has permitted among other things the characterization of the invariant subspaces for such contractions in terms of their characteristic functions (see [17], Ch. VII and [16]), to obtain a spectral decomposition for any weak contraction (see [17], Ch. VIII), to build a Jordan model theory for C_0-operators (see [17], Secs. III.4-7, VI.5, IX.2-3 and the Notes to Ch. IX), and to characterize in terms of the characteristic function those contractions which are similar to unitary operators (see [17], Sec. IX.1). This last result goes as follows.

(VI) *A contraction T is similar to a unitary operator if and only if $\Theta_T(\lambda)$ is defined for $|\lambda| \neq 1$ and* $\sup\{\|\Theta_T(\lambda)\| : |\lambda| \neq 1\} < \infty$ (2).

This condition on the characteristic function $\Theta_T(\lambda)$ turned out to be sufficient to assure the similarity to a unitary operator of an arbitrary operator T, as proved by L.A. Sahnovič [12].

It is noteworthy that from a geometrical point of view the characteristic function, considered as a multiplication operator from $H^2(\mathfrak{E})$ into $H^2(\mathfrak{E}_*)$, appears in σ_Θ as the restriction to $\mathfrak{K}_+ \ominus \mathfrak{H}$ of the orthogonal projection of $\mathfrak{K}$ onto $\bigvee_{n \geq 0} U^n \mathfrak{K}_-$. The construction given in Sec. 2, permits one to carry on a part of this geometrical interpretation of Θ_T for any operator T (see [7]) ; this structural property plays a basic role in the proof of the following similarity theorem of [7] :

(VII) *If $\Theta_T(\lambda)$ is defined on $D(=\{\lambda : |\lambda| < 1\})$ and*

$$\sup\{\|\Theta_T(\lambda)\| : |\lambda| < 1\} < \infty .$$

then T is similar to a contraction.

Since if T is invertible, $\Theta_{T^{-1}}(\lambda)$ coincides, up to constant affine factors, with $\Theta_T(1/\lambda)$, Theorem (VII) along with the classical similarity theorem of B. Sz.-Nagy (concerning power-bounded operators [15](3)) plainly implies the theorem of L.A. Sahnovič refered above. However, (VII) does not completely contain (VI), since the boundedness of $\Theta_T(\lambda)$ on D is not implied by the similarity of T to a contraction ; indeed C. Davis has remarked that the unilateral weighted shift with

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(1) If T is a contraction, since $J_T / \mathfrak{D}_T = I_{\mathfrak{D}_T}$, J_T can be omitted in (6).

(2) Note that $\|\Theta_T(\lambda)\| \leq 1$ for $|\lambda| < 1$ whenever T is a contraction.

(3) This theorem is used also in the proof of (VII).

the weights $\exp(-1)^n$, $n = 0,1,2,\ldots$, is similar to the unilateral shift of multiplicity one and though its characteristic function is unbounded on D. It is not known neither how the similarity of T to a contraction is reflected in $\Theta_T(\lambda)$, nor how the boundedness on D of $\Theta_T(\lambda)$ is reflected in T. Moreover it is not yet known if (VII) contains the following similarity theorem (see [17], Secs. I.11 and II.8) :

(VIII) *The operator T is similar to a contraction whenever its spectrum lies in $\overline{D}(=\{\lambda : |\lambda| \leqslant 1\})$ and*

(7) $\|(\lambda I_{\mathfrak{H}} - T)^{-1}\| \leqslant (|\lambda| - 1)^{-1}$ *for* $0 < |\lambda| - 1$ *enough small.*

J. Stampfli proved [14] that if T and T^{-1} satisfy (7) then T is unitary. This deep result and (VI) assure that (VII) is not implied by (VIII).

5. – There is still plenty of work to be done on structural models involving the functorial properties of compressions of operators. For instance, in Ω_1 the strict injections(1) corespond to closed invariant subspaces and apparently they must be connected by $\mathbf{C}_I$ with strict injections in Σ_I ; nothing is known on this question. Moreover with respect to $\mathbf{C}_I$ and Σ_I, the functor $\mathbf{C}$ and the category Σ (which perhaps is not the most adequate one) are scarcely studied. To give a sample let us recall that isomorphism in Ω means similarity and hence the facts presented in Sec. 4 may lie on some general unknown properties of $\mathbf{C}$. Also, in virtue of the recent researches [3] and [7], part of [17] can "almost surely" be extended from $\mathbf{C}_I$ and Σ_I to $\mathbf{C}$ and Σ. This would lied to a geometrical insight in the analytical approach (of V.M. Brodskiĭ, I.T. Gohberg and M.G. Kreĭn [3]) in the theory of the characteristic functions of invertible operators.

In this approach (see [3]) the basic object is a system $\kappa = \{\mathfrak{H}, \mathfrak{G} ; T, R, G\}$ formed by operators T on $\mathfrak{H}$, G on $\mathfrak{G}$ and R from $\mathfrak{G}$ into $\mathfrak{H}$ satisfying the following condition : T is invertible, G is a symmetry, $I_{\mathfrak{H}} - T^*T = RGR^*$ and $\mathfrak{H} = \bigvee_{n=-\infty}^{\infty} T^n R\mathfrak{G}$. Two systems $\kappa = \{\mathfrak{H}, \mathfrak{G}; T, R, G\}$ and $\kappa' = \{\mathfrak{H}', \mathfrak{G}'; T', R', G'\}$ are called isomorphic if there exists a unitary operator X from $\mathfrak{H}'$ onto $\mathfrak{H}$ such that $XT' = TX$ and $XR' = R$. To any κ there corresponds a characteristic function (which in case $\mathfrak{G} = \mathfrak{D}_T$, $R = D_T$ and $G = J_T$ coincides with (6)) determining κ up to an isomorphism. The connection between this approach and that of this address can be seen as follows : Let $\mathfrak{K}$ denote the orthogonal sum

$$\ldots \oplus \underset{(-2)}{\mathfrak{G}} \oplus \underset{(-1)}{\mathfrak{G}} \oplus \underset{(0)}{\mathfrak{H}} \oplus \underset{(1)}{\mathfrak{G}} \oplus \underset{(2)}{\mathfrak{G}} \oplus \ldots$$

where the indices stand for the coordinates and let $\mathfrak{K}_\pm$ be defined in the same way as in the construction following (I) ; let moreover

$$J = \ldots \oplus G \oplus G \oplus I_{\mathfrak{H}} \oplus G \oplus G \oplus \ldots$$

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(1) Let us recall that an injection j is called *strict* if for any surjection s and any morphisms q, r such that $sr = jq$ there exists a morphism x satisfying $jx = r$.

and define U on $\mathfrak{K}$ by $U(h_i)_{-\infty}^{\infty} = (k_i)_{-\infty}^{\infty}$ where $k_i = h_{i-1}$ for $i \neq 0,1$ and

$$k_0 = (T^*)^{-1} RQ^{-1} h_{-1} + Th_0 \, , \; k_1 = - GQ^{-1} h_{-1} + R^* h_0 \, ,$$

Q being an invertible operator satisfying the relation

$$G + ((T^*)^{-1} R)^* ((T^*)^{-1} R) = Q^*GQ \ (^1)$$

Then, at least if R is injective, $\sigma = \{U, J \, ; \, \mathfrak{K}_-, \mathfrak{K}_+\} \in Ob\Sigma$ and $C(\sigma)$ is unitarily equivalent to T.

6. – Concluding, one may hope that this kind of structure theory has not yet been fully exploited and that, in the future, analoguous affine (not unitary) general models can be found.

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REFERENCES

[1] ADAMJAN V.M. and AROV D.Z. — On the unitary coupling of semi-unitary operators (in Russian), *Mat. Issled. Kishinev*, 1-2, 1966, p. 3-66.

[2] DE BRANGES L. and ROVNYAK J. — Canonical models in quantum scattering theory, *Perturbation theory and its application in quantum mechanics*, New York, London, Sidney, 1966, p. 295-391.

[3] BRODSKII B.M., GOHBERG I.T. and KREIN M.G. — On characteristics functions of invertible operators (in Russian), *Acta Scient. Math.*, 32, 1971, to appear.

[4] BUCUR I. and DELEANU A. — *Introduction to the theory of categories and functors*, New York, London, Sidney, 1968.

[5] CHEW G.F. — *The analytic S-matrix; a basis for nuclear democracy* (New York, 1966).

[6] DAVIS C. — J-unitary dilation of a general operator, *Acta Scient. Math.*, 31, 1970, p. 75-86.

[7] DAVIS C. and FOIAŞ C. — Operators with bounded characteristic functions and their J-unitary dilations, *Acta Scient. Math.*, 32, 1971, to appear.

[8] DOUGLAS R.G., MUHLY P.S. and PEARCY C. — Lifting commuting operators, *Michigan Math. Journ.*, 15, 1968, p. 385-395.

[9] HALMOS P.R. — Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, 76, 1970, p. 887-933.

[10] LAX P. and PHILLIPS R.S. — *Scattering theory*, New York, 1967.

[11] PAGE L.B. — Application of the Sz.-Nagy and Foias lifting theorem, *Indiana Univ. Math. Journ.*, 20, 1970, p. 135-146.

[12] SAHNOVIČ L.A. — Non-unitary operators with absolutely continuous spectrums (in Russian), *Izvestija Akad. Nauk S.S.S.R., Ser. Mat.*, 33 1969, p. 52-64.

[13] SMUL'JAN Ju. L. — Operators with degenerate characteristic function (in Russian), *Doklady Akad. Nauk S.S.S.R.*, 93, 1953, p. 985-988.

---

(1) The existence of Q follows from [3], Lemma 1.1.

[14] Stampfli J.G. — A local spectral theory of operators, *Journ. of Funct. Analysis*, 4, 1969, p. 1-10.
[15] Sz.-Nagy B. — On uniformly bounded linear transformations in Hilbert space, *Acta Scient. Math.*, 11, 1947, 152-157.
[16] Sz.-Nagy B. — *Sous espaces invariants d'un opérateur et factorisation de sa fonction characteristique* (Adress to the present Congress).
[17] Sz.-Nagy B. and Foiaş C. — *Analyse harmonique des opérateurs de l'espace de Hilbert*, Budapest-Paris, 1967.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 441 à 445.

SPECTRAL REPRESENTATIONS AND THE SCATTERING THEORY FOR SCHRÖDINGER OPERATORS

by S. T. KURODA

1. – The main purpose of the present paper is to indicate the possibility that some problems in the scattering theory for differential operators such as generalized Schrödinger operators, exterior problems, etc., may be handled systematically by applying an abstract stationary method developed by T. Kato and the writer ([5], [6], [7]). The problems to be considered are :

(1) to prove the existence and completeness of wave operators ; and

(2) to construct perturbed eigenfunction expansions.

Problem (1) will be solved under assumptions that, roughly speaking, coefficients of perturbing differential operators decay as $0(|x|^{-(1+\epsilon)})$, $\epsilon > 0$, at infinity. Our solution to problem (2) requires a stronger condition.

Throughout the present paper a key role is played by operators of the type $\mathfrak{F}A_k^*$ (cf. Assumption 2.1) introduced in [7]. As we will see in 2, these operators can also be used to reformulate the results given in [6]. This reformulation, which is seemingly more straightforward, requires to restrict the generality of the situation. But the restriction is inessential for such applications as discussed in 3. We also need a slight improvement in order to allow perturbing terms to have the same order as the unperturbed one. This is done by introducing the products of resolvents in (ii) of Assumption 2.1. Some ideas used in 2 go back to P.A. Rejto (see, e.g., [9]).

M.Š. Birman (see, e.g., [2]) has developed another abstract stationary method and applied it extensively to various differential operators. Birman's condition, however, requires the decay rate $0(|x|^{-(n+\epsilon)})$, $\epsilon > 0$, n being the dimension of the space.

In this paper most of the arguments and expressions such as (2.1) will remain formal. A precise formulation with detailed proofs will be given in subsequent publications. The references are rather limited. For a more complete list, see the article of T. Kato in these proceedings.

2. – We consider two self-adjoint operators H_1 and H_2 in a Hilbert space $\mathfrak{H}$ related to each other as

$$H_2 = H_1 + \sum_{k=1}^{r} A_k^* C_k B_k \ . \tag{2.1}$$

To simplify the exposition, we suppose that $\mathfrak{H} = L^2(R^n)$, $n \geqslant 2$, and

$$H_1 = -\Delta = \sum_{k=1}^{n} D_k^2 \ , \ D_k = i\partial/\partial x_k \ .$$

However, the following argument will work if, for instance, H_1 admits a spectral representation on $L^2(I\,;\mathfrak{A})$, the L^2-space of $\mathfrak{A}$-valued functions over an interval I. We assume furthermore that the range of at least one of A_k^* is dense in $\mathfrak{H}$.

The following notations will be used : $\mathbf{B}(\mathfrak{X}, \mathfrak{Y})$ is the set of all bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$; $\mathfrak{D}(A)$ is the domain of A ; $R_1(z) = (H_1 - z)^{-1}$, $R_2(z) = (H_2 - z)^{-1}$; $\mathfrak{H}_{ac}(H_2)$ is the subspace of absolute continuity with respect to H_2 ; $\mathfrak{F}$ is the Fourier transform :

$$(\mathfrak{F}u)(\xi) = (2\pi)^{-n/2} \int_{R^n} u(x) \exp(-i\xi x)\, dx \ .$$

ASSUMPTION 2.1. –

(i) For every $k = 1, \ldots, r$, the expressions

$$(\mathfrak{F}A_k^* u)(\lambda\omega) \text{ and } (\mathfrak{F}B_k^* u)(\lambda\omega) \ , \ \lambda > 0 \ , \ \omega \in S^{n-1} \ , \ u \in L^2(R^n) \ ,$$

determine $\mathbf{B}(L^2(R^n), L^2(S^{n-1}))$-valued functions of λ which are locally Hölder continuous in λ with respect to the operator norm.

(ii) $B_j R_1(z) R_1(z') A_k^*$, $Im\ z \neq 0$, $Im\ z' \neq 0$, $1 \leqslant j, k \leqslant r$, is completely continuous.

This assumption ensures the existence of boundary values of various families of operators. In particular, there exists a closed null set $e_0 \subset (0, \infty)$ such that $C_j B_j R_2(\lambda \pm i\epsilon) A_k^*$, $\lambda \in (0, \infty) - e_0$, converges as $\epsilon \downarrow 0$ to $Q_{jk}^{(2)}(\lambda \pm i0)$ in the operator norm. Put $G_{jk}^{(2)}(\lambda \pm i0) = \delta_{jk} - Q_{jk}^{(2)}(\lambda \pm i0)$.

THEOREM 2.2. – *There exists* $\mathfrak{F}_\pm \in \mathbf{B}(L^2(R^n))$ *having the following properties*:

(i) $(\mathfrak{F}_\pm A_k^* u)(\xi) = \sum_{j=1}^{r} (\mathfrak{F}A_j^* G_{jk}^{(2)}(|\xi|^2 \pm i0)\, u)(\xi)$;

(ii) $\mathfrak{F}_\pm$ *is a partial isometry on to* $L^2(R^n)$ *with initial set* $\mathfrak{H}_{ac}(H_2)$;

(iii) $(\mathfrak{F}_\pm H_2 u)(\xi) = |\xi|^2 (\mathfrak{F}_\pm u)(\xi)$.

COROLLARY 2.3. – $W_\pm = \mathfrak{F}_\pm^* \mathfrak{F}$ is an isometry from $L^2(R^n)$ onto $\mathfrak{H}_{ac}(H_2)$ and satisfies the intertwining relation $H_2 W_\pm \supset W_\pm H_1$.

THEOREM 2.4. – *We have* $W_\pm = s - \lim_{t \to \pm\infty} \exp(itH_2) \exp(-itH_1)$. *More generally, the so-called invariance of wave operators holds.*

Corollary 2.3 and Theorem 2.4 solve problem (1).

It follows that the singular spectrum of H_2 in $(0, \infty)$ is confined to e_0. Further information about e_0 will not be discussed in this paper.

A rough idea for treating problem (2) is as follows. Let

$$\varphi_1(\xi) = \varphi_1(x, \xi) = (2\pi)^{-n/2} \exp(i\xi x)$$

and put $V = \Sigma A_k^* C_k B_k$. Then, a formal manipulation using $(\mathfrak{F}w)(\xi) = \langle w, \varphi_1(\xi)\rangle$ gives

$$(\mathfrak{F}_\pm A_k^* u)(\xi) = < A_k^* u, \{1 - R_2(|\xi|^2 \mp i0) V\} \varphi_1(\xi) > .$$

Thus, we get
$$(\mathfrak{F}_\pm u)(\xi) = < u, \varphi_\pm(\xi) >$$
with
$$\varphi_\pm(\xi) = \{1 - R_2(|\xi|^2 \mp i0) V\} \varphi_1(\xi) .$$

In other words, $\{\varphi_\pm(\xi) \mid \xi \in R^n, |\xi|^2 \notin e_0\}$ is a complete set of "eigenfunctions" associated with H_2 in $\mathfrak{H}_{ac}(H_2)$. $\varphi_\pm(\xi)$ is a unique solution of the Lippmann-Schwinger equation $\varphi_\pm(\xi) = \varphi_1(\xi) - R_1(|\xi|^2 \mp i0) V\varphi_\pm(\xi)$. Other properties of $\varphi_\pm(\xi)$ can be deduced from the fact that $\mathfrak{F}_\pm$ is a spectral representation.

In order to make all these arguments rigorous, one has to introduce a space in which $1 - R_2(|\xi|^2 \mp i0) V$ has a legitimate meaning. One convenient choice is the space $\mathfrak{X}$ obtained by completing $\bigcap_{k=1}^{r} \mathfrak{D}(A_k)$ with respect to the norm $\Sigma \|A_k u\|$. Thus, for solving problem (2) we need an adittional assumption that $\varphi_1(\xi) \in \mathfrak{X}$.

3. – Throughout the present section we assume that all functions $a(x) = a_{jk}(x)$, $b_k(x)$, $q(x)$ appearing in our differential operators satisfy the following condition :

$$(3.1) \quad \begin{cases} \text{there exist } \alpha > 1 \text{ and } c \geq 0 \text{ such that} \\ |a(x)| \leq c(1 + |x|)^{-\alpha} \;, \quad x \in R^n . \end{cases}$$

(1) *Schrödinger operators.*

To illustrate the method, let us first consider the Schrödinger operator for the potential scattering (cf. [5], [7]). Namely, let $H_1 = -\Delta$ and $H_2 = -\Delta + q(x) \cdot$ in $L^2(R^n)$. Here, $q(x)$ is a real valued function satisfying (3.1). Write

$$q(x) = (1 + |x|)^{-s} c(x) (1 + |x|)^{-s} \;, \quad s = \alpha/2 \;, \quad c \in L^\infty(R^n) \;,$$

and put $A = B = (1 + |x|)^{-s} \cdot$, $C = c(x) \cdot$. Then, $\mathfrak{F}A^* u = \mathfrak{F}B^* u$, $u \in L^2(R^n)$, belongs to the Sobolev space $H^s(R^n)$ (of the variable ξ). Since $s > 1/2$, (i) of Assumption 2.1 follows from the well-known fact concerning the L^2-trace on hypersurfaces of H^s-functions. Other assumptions being checked easily, problem (1) is solved.

In order to solve problem (2), assumption (3.1) need to be strengthened as $\alpha > (n + 1)/2$. Write $q(x) = (1 + |x|)^{-s} c(x) (1 + |x|)^{-t}$, $s > n/2$, $t > 1/2$, and put $A = (1 + |x|)^{-s}$ etc. Then, the space $\mathfrak{X}$ is the L^2-space over the measure $(1 + |x|)^{-2s} dx$. Since $s > n/2$, $\mathfrak{X}$ contains $\exp(i\xi x)$. The Lippmann-Schwinger equation becomes the integral equation originally considered by Povzner [8] and Ikebe [3]. Thus, their results are recaptured.

(2) *Generalized Schrödinger operators.*

Let $H_1 = -\Delta$ and

$$(3.2) \quad H_2 = \sum_{j,k=1}^{n} D_j \{\delta_{jk} + a_{jk}(x)\} D_k + \sum_{k=1}^{n} \{b_k(x) D_k + D_k \overline{b_k(x)}\} + q(x) .$$

Assume that a_{jk}, b_k, q satisfy (3.1) and that q is real. Suppose further that H_2 is (uniformly) elliptic. The last condition is required to define H_2 correctly.

The difference between this problem and the Schrödinger operator is just a matter of formal complication. To illustrate, let us write

$$a_{jk}(x) = (1 + |x|)^{-s} c_{jk}(x) (1 + |x|)^{-s}$$

and put $A_{jk} = (1 + |x|)^{-s} D_j$, $B_{jk} = (1 + |x|)^{-s} D_k$, etc. Then, (i) of Assumption 2.1 is verified quite similarly, because D_j in $A^*_{jk} = D_j(1 + |x|)^{-s}$ can be taken out in front of $\mathcal{F}$ as a multiplication operator. In this way problem (1) is solved. Problem (2) is solved if $\alpha > (n + 1)/2$.

(3) *Exterior Dirichlet problems.*

Let $\Omega \subset R^n$ be a domain exterior to a compact set K with smooth boundary. Let H_2' and H_2'' be self-adjoint operators in $L^2(\Omega)$ and $L^2(K)$, respectively, determined by $-\Delta$ with the Dirichlet boundary condition. Let $H_2 = H_2' \oplus H_2''$ in $L^2(R^n) = L^2(\Omega) \oplus L^2(K)$. The scattering problem for the pair $\{H_1, H_2'\}$ can be reduced to that for the pair $\{H_1, H_2\}$ (cf. [1], [4]). We can then apply the method given in 2 to the pair $\{R_1, R_2\}$, $R_j = (H_j + 1)^{-1}$. In fact, denoting by P the projection in $H^1(R^n)$ onto the subspace of all functions whose trace on $\partial\Omega$ vanishes, one obtains $R_2 = R_1 - (1 - P) R_1$. We factor $(1 - P) R_1 \geqslant 0$ into the product of its non-negative square root. Then, (i) of Assumption 2.1 can be verified on the basis of the fact that a function belonging to the range of $1 - P$ satisfies $-\Delta u + u = 0$ in Ω and hence decays exponentially at infinity. Thus, problem (1) is solved. Similar consideration can be performed when H_2 is defined by expression (3.2) and the Dirichlet boundary condition.

(4) *Uniformly propagative systems.*

Uniformly propagative systems in the sense of Wilcox [11] can be handled similarly. Let E_1 and $E_2(x)$ be the energy density matrices for the unperturbed and the perturbed systems, respectively. Assume that : a) the systems have no static solutions ; b) $E_2(x) - E_1$ and $D_k E_2(x)$ satisfy (3.1). Then, problem (1) can be solved. In the proof the problem is reduced to the one of comparing two symmetric hyperbolic systems in the same Hilbert space. Schulenberger and Wilcox [10] solved problem (1) without assuming a). But their assumption about the rate of decay requires $\alpha > n$.

Finally, we remark that the method can be applied to more general problems, say, higher order operators.

4. – One of the problems not discussed above is the problem concerning the type of the scattering matrix $\mathcal{S}(\lambda)$. The scattering operator $S = W_+^* W_-$ commutes with H_1 and hence is represented in the Fourier space as

$$(\mathcal{F} S u)(\lambda\omega) = \mathcal{S}(\lambda) (\mathcal{F} u)(\lambda\omega)$$

by a family of unitary operators $\{\mathcal{S}(\lambda)\}_{\lambda > 0}$ in $L^2(S^{n-1})$. It can be seen (cf. [7]) that the von Neumann-Schatten class of $\mathcal{S}(\lambda)$ and that of operators of the form $\mathcal{F} A_k^*$ are closely related. Thus, in each example considered above we can show, for instance, that $\mathcal{S}(\lambda)$ belongs to the Hilbert-Schmidt class (or the trace class) of $L^2(S^{n-1})$ if α in (3.1) can be taken as $\alpha > (n + 1)/2$ (or $\alpha > n$).

REFERENCES

[1] BIRMAN M.Š. — Perturbations of the continuous spectrum of a singular elliptic operator under the change of the boundary and boundary conditions, *Vest. Leningrad Univ., Ser. Mat., Meh., Astron.*, 1, 1962, p. 22-25 (in Russian).

[2] BIRMAN M.Š. — Scattering problems for differential operators with constant coefficients, *Funkcional. Anal. i Priložen*, 3, No. 3, 1969, p. 1-16 (in Russian).

[3] IKEBE T. — Eigenfunction expansions associated with the Schroedinger operators and their applications to scattering theory, *Arch. Rat. Mech. Anal.*, 5, 1960, p. 1-34.

[4] KATO T. — Scattering theory with two Hilbert spaces, *J. Funct. Anal.*, 1, 1967, p. 342-369.

[5] KATO T. — Some results on potential scattering, *Proc. Intern. Conf. Functional Analysis and Related Topics*, Tokyo 1969, Univ. Tokyo Press, 1970, p. 206-215.

[6] KATO T. and KURODA S. T. — Theory of simple scattering and eigenfunction expansions, *Functional Analysis and Related Fields*, Springer, 1971, p. 99-131.

[7] KURODA S.T. — Some remarks on scattering for Schrödinger operators, *J. Fac. Sci. Univ. Tokyo*, Sect. I A, 17, 1970, p. 315-329.

[8] POVZNER A. Ya. — On the expansions of arbitrary functions in terms of the eigenfunctions of the operator $-\Delta u+cu$, *Mat. Sb.*, 32 (74), 1953, p. 109-156 (in Russian).

[9] REJTO P.A. — On partly gentle perturbations, I, II, *J. Math. Anal. Appl.*, 17, 1967, p. 435-462; 20, 1967, p. 145-187.

[10] SCHULENBERGER J.R. and WILCOX C.H. — Completeness of the wave operators for perturbations of uniformly propagative systems, *Tech. Summary Rep.*, 5, Univ. of Denver, 1970.

[11] WILCSX C.H. — Wave operators and asymptotic solutions of wave propagation problems of classical physics, *Arch. Rat. Mech. Anal.*, 22, 1966, p. 37-78.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 447 à 457.

CERTAIN PROBLEMS OF SPECTRAL THEORY OF DIFFERENTIAL OPERATORS

by B. M. LEVITAN

Introduction.

The spectral theory of differential operators belongs to the branches of mathematics which seem to develop now with great intensity.

Surely there is no chance to review in this report, even briefly, all trends of this theory.

Therefore for my present report *I* have chosen three trends of the spectral theory, according to my personal interests during recent years.

They are :

(1) Asymptotic behaviour of the spectral function of a self-adjoint elliptic differential operator.

(2) Spectral analysis of differential operators with operator coefficients.

(3) Inverse problems of spectral analysis.

1. **Asymptotic behaviour of the spectral function of a self-adjoint differential elliptic operator.**

For differential operators with partial derivatives an asymptotic expansion cannot be obtained for a separate eigenfunction.

Instead of that the asymptotic of eigenfunctions is studied on the average. This means : let L be an elliptic differential operator given in a certain domain D with boundary S of n-dimensional Euclidian space R^n. In order no to overload the essence of the problem we shall assume further that the spectrum of the operator L is positive and discrete. Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ denote the eigenvalues of L and $\{\varphi_n(x)\}$ – the corresponding orthonormal eigenfunctions. Now let us introduce the the folloving functions

$$\Theta(x, y; \lambda) = \sum_{\lambda_n < \lambda} \varphi_n(x)\varphi_n(y), \quad N(\lambda) = \sum_{\lambda_n < \lambda} 1 .$$

As $N(\lambda) = \int_{(D)} \Theta(x, x; \lambda)\,dx$ then, if we know the asymptotic behaviour of the function $\Theta(x, x; \lambda)$ as $\lambda \to \infty$, up to the boundary S of the domain D, we can find out the asymptotic behaviour of the function $N(\lambda)$.

The function $\Theta(x, y; \lambda)$ is called in spectral analysis a spectral function of L. Its asymptotic behaviour as $\lambda \to \infty$ and with fixed x and y within the domain D was the subject of numerous researches, started by Carleman [1].

For this aim a number of methods have been worked out. They are summarized in the introduction to Hörmander's paper [2], where their comparative analysis is given. To this paper of Hörmander we shall offen be referring.

We shall speak on one of the methods wich is based on the studying of the singularities of the fundamental solution of the Cauchy problem for hyperbolic equation (or systems). To make clear the essence of the method, we shall explain it briefly. Suppose L is an elliptic operator of the second order :

$$L(u) = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x) u ,$$

$c(x) > 0$ with coefficients smooth enough. Let us consider an auxiliary Cauchy problem

(1) $$\frac{\partial^2 u}{\partial t^2} + Lu = 0 ,$$

(2) $$u \Big|_{t=0} = f(x) , \quad \frac{\partial u}{\partial t} \Big|_{t=0} = 0 ,$$

where $f(x)$ is an arbitrary smooth function. If a problem for eigenvalues has a boundary condition it must be added to problem (1) – (2). Suppose this condition looks like

(3) $$u/s = 0 .$$

Solving the problem (1), (2), (3) by Fourier method we shall have

$$u(x, t) = \sum_{\kappa=1}^{\infty} c_\kappa \varphi_\kappa(x) \cos \mu_\kappa t ,$$

(4) $$\mu_\kappa = \sqrt{\lambda_\kappa} \quad , \quad c_\kappa = \int_{(D)} f(y) \varphi_\kappa(y) \, dy .$$

The problem (1), (2) is a Cauchy problem for hyperbolic equation. Therefore it has a finite domain of dependence. Hence if x is an inner point of domain D and t is so small that the domain of dependence with its centre in the point x does not come out of the domain D, then the boundary condition (3) should not be taken into consideration and a pure Cauchy problem (1), (2) may be solved.

Let the solution be

(5) $$u(x, t) = \int_{(D)} \mathcal{K}(x, y ; t) f(y) \, dy ,$$

where $\mathcal{K}(x, y ; t)$ is some generalized function with finite support (the support of the generalized function $\mathcal{K}$ coincides with the domain of dependence for equation (1)). From (4) and (5) and from the uniqueness of the solution of Cauchy problem it follows

$$\sum_{\kappa=1}^{\infty} c_\kappa \varphi_\kappa(x) \cos \mu_\kappa t = \int_{(D)} \mathcal{K}(x, y ; t) f(y) \, dy .$$

Now let $g(t)$ be an even infinitely differentiable function with finite support and let

$$\psi(\mu) = \int_0^\infty g(t) \cos\mu t \, dt$$

be its cosine-Fourier transformation. If we multiply both sides of the identity (6) by $g(t)$ and then integrate with respect ot t, we shall have using the arbitrariness of the function $f(y)$

$$\int_0^\infty \psi(\mu) \, d_\mu \, \Theta(x, y; \mu^2) = \int_0^\infty \mathcal{K}(x, y; t) g(t) \, dt . \tag{7}$$

Formula (7) gives us the regularized cosine – Fourier transformation of the spectral function $\Theta(x, y; \mu^2)$. By virtue of that and certain simple special Tauberian theorems [3], [4] one can extract the main term as λ tends to infinity, of the spectral function $\Theta(x, y; \lambda)$ with non-improvable estimation of the remainder.

In order to use the Tauberian theorems mentioned above, one must know the nature of singularities of the fundamental solution $\mathcal{K}(x, y; t)$ of the Cauchy problem (1), (2). For $L = -\Delta$, where Δ is a Laplacian, there is an explicit formula for $\mathcal{K}(x, y; t)$ ([5], chapter 1V, § 12) which *I* have used in 1954 [6]. For the more complicated operator $L = -\Delta + q(x)$ in order to examine the function $\mathcal{K}(x \;\; y; t)$, *I* have used in 1955-56 Sobolev's method [7], [8]. The same method gives good results in case of the general elliptic operator of the second order [9]. Hadamard's method may also help in studying the singularities of the function $\mathcal{K}(x, y; t)$. It was done by Bureau in 1960-62 [10].

Some modifications of the method also give good results in studying the asymptotic behaviour of Riesz means of spectral function and of Riesz means of Fourier expansions [4], [6].

In comparison with other methods, this method has the following advantages :

(1) Clearly revealed localisation, which is expressed in the following :

(a) asymptotics of spectral function and its Riesz means depends on the behaviour of the coefficients of the operator only in the immediate neighbourhood of the given points ;

(b) Riesz means of the order $s > (n-1)/2$ of Fourier expansion of function $f(x)$, which belongs to the class $\mathcal{L}^2(D)$, depends on the behaviour of the operator coefficient and function $f(x)$ only in the immediate neighbourhood of the given point

(2) Another advantage is that the Tauberian theorems used in this method are simplier and clearer.

Nevertheless the method has some shortcomings.

(1) It does not allow to study the asymptotics of a spectral function up to the boundary of the domain. From the above mentioned it follows that it could be done if we knew the structure of the fundamental solution of problem (1), (2), (3), at least, for small values of *t*. Yet it has been done only near a plane piece of boundary [11].

(2) Another shortcoming is that the method, as it is described here could be applied only to elliptic equations of the second order. The method has got its further development in Hörmander's paper [2] in which in particulary the last shortcoming was removed.

Hörmander assumes that L is a self-adjoint positive pseudo-differential operator of the first order. He studies the Cauchy problem

$$\text{(8)} \qquad \frac{1}{i} \cdot \frac{\partial u}{\partial t} = Lu \ , \quad u\,|_{t=0} = f(x) \ .$$

For problem (8) there is no finite domain of dependence of the initial value but the singularities of the fundamental solution propagate with a finite speed. This means that there is a finite domain of dependence with accuracy to a amooth nucleus. It is enough to make possible the use of the Tauberian theorems said above. Hörmander did not clearly use them in his paper, but if he had, the second part of his paper could have been simplier and more transparent.

To solve problem (8) and subsequently deduce the formula of type (7) Hörmander used and developed Lax's method [12] of solving Cauchy problem for hyperbolic equations.

Because of lack of time *I* have no opportunity to discuss Lax's method in details. The more so, as it seems to be widely known at present to the specialist in differential equations.

I shall only point out that Hörmander has managed to prove that the phase function, playing an important part in Lax's method, can be chosen linear in t, when the symbol of operator L does not depend on t (which always occurs in our case). This fact turned out to be decisive when for spectral problems. It is interesting to notice that if the main part of the L-operator symbol does not depend on x, then already from Lax's theory it follows that the phase is linear with respect to t, and in this case it seems to be the only possibility.

It should be mentioned that Eskin [13], [14] and Maslov [15] have applied Lax's method to pseudo-differential operators independent of Hörmander.

Now let P be an elliptic, positive self-adjoint differential operator of order m. Let us put $L = P^{1/m}$ where the root is defined by spectral theory. In virtue of an important theorem of Seeley [16], L is a pseudo-differential operator of the first order. Having substituted it into equation (8) and applying the method said above Hörmander has got for the spectral function $\Theta(x, x; \lambda)$ (on diagonal) of the operator P the following asymptotic formula :

$$\text{(9)} \qquad \Theta(x, x; \lambda) = \frac{1}{(2\pi)^n} \int_{\mathfrak{P}(x,\xi)<\lambda} d\xi + O(\lambda^{(n-1)/m}) \ ,$$

where $\mathfrak{P}(x, \xi)$ is the main symbol of the operator P.

This method may also be applied to some systems of differential and pseudo-differential operators of the first order, for instance to systems with simple caracteristics.

In paper [2] Hörmander assumes that the operator L is bounded below. Therefore the usual systems of differential operators of the first order (for instance the classical Dirac's system) is not included formally in Hörmander's theory, as a system of differential operators of the first order can never be semi-bounded. Nevertheless the method can be easily applied to such systems. Asymptotics of the spectral function as $\lambda \to +\infty$ and $\lambda \to -\infty$ appeared to be essentially the same. For one-dimensional Dirac's systems, for which problem (8) can be solved quite elementarily, this had been found out by Sargsjan [17].

Concluding this first point we shall notice that if the fundamental solution is known for all values of t (and not only for small ones) the remainder estimation in formula (9) can be improved by substituting $O(\lambda^{(n-1)/m})$ for $o(\lambda^{(n-1)/m})$. It follows from Marchenko's Tauberian theorem [18].

2. Spectral analysis of differential operators with operator coefficients.

Studying differential operators, whose coefficients are operators in a Hilbert of Banach space, permits to consider from the same point of view both ordinary differential operators and operators with partial derivatives. Unlike other fields of the theory of differential equations, in spectral analysis this approach was developed not long ago.

Let H be an abstract, separable Hilbert space. Let us consider the set of all vector valued functions $f(x)$ $(-\infty \leqslant a \leqslant x \leqslant b \leqslant \infty)$ with values in H, mesurable by Bochner and such that $\int_a^b \|f(x)\|^2\,dx < \infty$. This set forms a new Hilbert space say H_1, if we define a scalar product in it by an equality

$$(f(x)\,, g(x))_1 = \int_a^b (f(x)\,, g(x))\,dx\,.$$

The problem is to develop in H_1 a spectral theory of differential operators. We shall be considering only the simpliest operator

$$l(y) = -y'' + Q(x)y\,, \tag{10}$$

which is analogous to a classical operator of Sturm-Liouville. As we are going to include in our theory differential operators with partial derivatives, we should assume that the operators $Q(x)$ (acting in H) are non-bounded.

Besides it is natural to suppose that there is an everywhere dense set in H (we shall denote it by $D\{Q(x)\}$), common for all x, where all the operators $Q(x)$ are defined. We shall assume that the operators $Q(x)$ in the domain $D\{Q(x)\}$ are symmetrical and bounded form below by a number not depending on x. Then the operator l which acts in H_1, is also semi-bounded from below and hence it has a classical self-adjoint extension by Friedrichs. We shall denote it by L.

Here a simple question arises : when does operator L have a discrete spectrum ?

It can be proved [19] that 1) in case of finite interval $(a\,, b)$, it is enough that the operator $Q(x)$ for each x should be inverse to a completely continuous one ; 2) in case of an infinite interval $(a\,, b)$, except condition 1) it is sufficient that for any $\omega > 0$

$$\lim_{x\to\infty} \int_x^{x+\omega} \alpha_1(x)\,dx = \infty\,,$$

where $\alpha_1(x)$ is the least eigenvalue of operator $Q(x)$.

More profound results have been received in papers [20] and [21]. The first one studies the asymptotic behaviour of function $N(\lambda)$ – the number of eigenvalues of the operator L, which are less than the given number λ. The second work studies the properties of the Green function of the operator L. $I : d$ like to speak on the later problem. For constructing a Green function of the operator L a classical "parametrix" method is used. It is evident that the operator-function

$$g(x, \eta ; \mu) = \frac{1}{2} \kappa^{-1} \exp(- |x - \eta| \kappa),$$

where $\kappa = \{Q(x) + \mu I\}^{1/2}$ (x is fixed) satisfies the equation $- g''_{\eta\eta} + \kappa^2 g = 0$ that is, it is a "parametrix" for the operator (10).

Therefore for constructing a Green function an integral equation must be composed

$$(11) \quad G(x, \eta ; \mu) = g(x, \eta ; \mu) - \int_{-\infty}^{\infty} g(x, \xi ; \mu) [Q(\xi) - Q(x)] G(\xi, \eta ; \mu) d\xi .$$

It is possible to show [21] that if the operators $Q(x)$ satisfy the conditions :

(1) for $|x - \xi| \leqslant 1$, $\|[Q(\xi) - Q(x)] Q^{-a}(x)\| \leqslant A |x - \xi|$, where $0 < a < 3/2$, $A > 0$ are constant numbers.

(2) for $|x - \xi| > 1$ there is $\|Q(\xi) \exp\left(- \frac{1}{2} |x - \xi| Q^{1/2}(x)\right)\| \leqslant B$, where B is a constant number, then for large enough μ equation (11) may be solved by the method of successive approximations. Its solution is a Green operator-function for the operator L. Then the function $G(x, \eta ; \mu)$ can be analytically extended for other values of μ (not belonging to the spectrum of the operator L). From the various results we shall point out the following [21] : suppose that the operator $Q(x)$ is for every x an inverse to a completely continuous one. Let us denote its eigenvalues in ascending order by $\alpha_1(x)$, $\alpha_2(x), \ldots, \alpha_n(x), \ldots$ and suppose that the series $\sum_{n=1}^{\infty} \alpha_n^{-3/2}(x)$ converges and its sum $F(x)$ is a function of the class $\mathfrak{L}^1(-\infty, \infty)$. Then the integral operator $Af = \int_{-\infty}^{\infty} G(x, \eta ; \mu) f(\eta) d\eta$ is an operator of Hilbert-Schmidt.

When studying the integral equation (11) we used a method which has been worked out by Titchmarsh in his paper [22].

Out of the asymptotic behaviour of Greens function with $\mu \to -\infty$ and classical Tauberian theorems we can obtain an asymptotic formula for $N(\lambda)$. Let us represent the formula :

$$N(\lambda) \sim \frac{1}{\pi} \sum_{\alpha_i(x) < \lambda} \int_{\alpha_i(x) < \lambda} \{\lambda - \alpha_i(x)\}^{1/2} dx .$$

Concluding this point *I'd* like to point out two unsolved problems. Their solving seems to be very important for the whole theory.

(1) We assumed from the very beginning that the initial Hilbert space H does not depend on x. Therefore as far as differential operators are concerned, it is possible only to consider straight cylindrical domains. For comprising the case

of non-sylindrical domains it would be necessary to suppose that the initial Hilbert space H depends on x. Very little has been done in this direction (at any rate as far as spectral problems are concerned).

(2) It is desirable to describe all the self-adjoint extensions of the operator l. For symmetrical differential operators with bounded operator coefficients in a finite interval (a, b) all the self-adjoint extensions have been described by Rofe-Beketov [23].

It is particulary interesting to find out when the extension by Friedrichs is the unique self-adjoint extension of operator l. It is true in case of $a = -\infty$, $b = +\infty$, $Q(x) = A^2 + R(x)$, where A^2 is a constant self-adjoint positive operator, $R(x)$ is a bounded operator, uniformly in respect to x bounded below. This last result can be generalized in the direction of the Sear's – Tichmarsh's theorm [24].

3. **Inverse problems of spectral analysis.**

Generally speaking it is possible to say that the inverse problems of spectral anslysis consist of the restoration of a differential operator by some of its spectral characteristics (spectra, spectral functions and so on). If we don't take into consideration an important result of Ambarzumjan [25], which he had obtained in 1929, the intensive development of inverse problems was started by Borg [26] and Levinson [27].

The first work proved that the two spectra of a Sturm-Liouville classical operator $l(y) = -y'' + q(x)y$, $0 \leqslant x \leqslant \pi$ correspondingly to the boundary conditions :

$$\text{(I)} \left.\begin{array}{l} y'(0) - hy(0) = 0 \\ y'(\pi) + Hy(\pi) = 0 \end{array}\right\} \qquad \text{(II)} \left.\begin{array}{l} y'(0) - h_1 y(0) = 0 \\ y'(\pi) + Hy(\pi) = 0 \end{array}\right\} \quad h_1 \neq h \ .$$

define uniquely the operator (that is the function $q(x)$).

Levinson's work examines Sturm-Liouville operator on a half-line $(0, \infty)$ in the case when the potential function $q(x)$ satisfies the condition

$$\int_0^\infty x\, |q(x)|\, dx < \infty$$

With this condition the solution $\varphi(x, s)$ of the equation $l(y) = s^2 y$, $y(0) = 0$, admits as $x \to +\infty$ an asymptotic representation

$$\varphi(x, s) = M(s) \sin(sx + \delta(s)) + o(1) \ .$$

The function $\delta(s)$ is called a scattering phase. Levinson's main result is that if the problem has no negative eigenvalues, the scattering phase defines uniquely the potential function $q(x)$.

Further development on inverse problems has been made in the USSR. The decisive factor was the using of analytic apparatus, which naturally appeared within Delsarte's theory of generalized translation operators. I mean the operators, which in the USSR are called transformation operators and in France they are called transmutation operators. Their general definition is the following :

Let E_1 and E_2 be two linear topological spaces, A and B – two linear continuous operator in E_1 and in E_2 correspondingly. The linear continuous operator T, acting from E_1 into E_2 is called a transformation operator if it satisfies the following two conditions :

(1) $TA = BT$,

(2) T has a continuous inverse operator T^{-1}.

Example : Let E_1, E_2 be linear subspaces of the space $C_2(0, \infty)$ of functions, which satisfy the boundary condition $f'(0) = h_1 f(0)$, respectively $f'(0) = h_2 f(0)$, where h_1 and h_2 are constants ; let

$$A = \frac{d^2}{dx^2} - q(x), \qquad B = \frac{d^2}{dx^2} - r(x).$$

The topology in E_1 and E_2 is given by uniform convergence in each finite interval of the function and its first two derivatives.

Then the T-operator does exist and can be realized as an operator of Volterra :

$$Tf = f(x) + \int_0^x \mathcal{K}(x, t) f(t)\, dt. \tag{12}$$

Formula (12) is that important result, which naturally appeared from the theory of generalized translation operators [28] ; [29], [30] and that I have mentioned above.

Formula (12) was applied to inverse problems initially by Marchenko, in 1952 [31] who obtained by its mean the most general uniqueness theorem, which is the following :

The Weyl-Titchmarsh spectral function of the Sturm-Liouville operator uniquely defines this operator.

This theorem includes Borg's result as well as Levinson's one.

Later, with the help of formula (12) Gelfand and I [32] managed to determine a differential operator by its Weyl-Titchmarsh function and to indicate necessary and sufficient conditions for a given monotonous function $\rho(\lambda)$ to be a Weyl-Titchmarsh function of the Sturm-Liouville operator.

Another method for solving inverse problems was worked out in the same time by M.G. Krein [33], [34]. He has got similar results to ours.

For the effective restoration of the Sturm-Liouville operator by scattering phase, formula (12) is not suited. Nevertheless B.Ja. Levin [35] has proved that for problems of scattering theory, there are transformation operators with some conditions at infinity. This operators have the following form :

$$Tf = f(x) + \int_x^\infty A(x, t) f(t)\, dt. \tag{13}$$

Marchenko [36] and Marchenko and Agranovich [37] successfully applied formula (13) to the inverse problem of scattering theory.

Important results, concerning the inverse problem of scattering theory had been obtained earlier by Jost and Newton [38].

Later on analogies of operators (12) and (13) were obtained for systems of the first order, the type of Dirac's system [39], [41]. Main results of the inverse problem theory were extended to such systems [39], [40], [41].

I should like to mention one of my result wich *J* obtained in 1964 [42], and wich consist in the following : for two infinite intermittent sequences $\{\lambda_n\}$ and $\{\mu_n\}$ $(\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \cdots < \lambda_n < \mu_n < \cdots)$ satisfying classical asymptotics : $\sqrt{\lambda_n} = n + \frac{a_0}{n} + \cdots$; $\sqrt{\mu_n} = n + \frac{b_0}{n} + \cdots$, $a_0 \neq b_0$ there exists an equation $-y'' + q(x)y = \lambda y$, for which these sequences are spectra, corresponding to boundary conditions (I) and (II).

Among later results on inverse problem theory for second order operators, *I'd* like to mention papers [43] and [44], which put forward and solved the problem of stability for the inverse problem by scattering phase.

The question is put as follows : suppose that at some finite interval $(0, N)$ scattering phases $\delta_1(s)$ and $\delta_2(s)$ of two problems coincide or do not differ much. What then can we say about the corresponding deviation of potentials ?

All said above refered to equations or systems of equations not higher than of the second order. It is natural to state inverse problems for ordinary equations of higher orders.

Attempts to apply formula (12) for differential operators of higher orders have failed, as it was proved by Sachnovich in 1961 [45] and Mazaev in 1960 [46], that already an operator of the fourth order $l(y) = y^{IV} + q(x)y$ with an non-infinitely differentiable coefficient $q(x)$ could not be transformed linearly with the help of *T*-operator of kind (12) into y^{IV} operator.

However recently Leibenzon [47], [48] has obtained important results on the inverse problems for differential operators of higher orders. He seems to have worked out an analytic apparatus, replacing successfully transformation operators.

LITERATUR QUOTED

[1] CARLEMAN T. — Propriétés asymptotiques des fonctions fondamentales des membranes Vibrantes. *C.R. 8e Congr. de Math. Scand.* Stockholm, 1934, p. 34-44.

[2] HÖRMANDER L. — The spectral function of an elliptic operator, *Acta Math.*, v. 121, 1968, p. 193-218.

[3] LEVITAN B.M. — On a Special Tauberian Theorem. *Izvestija Akad. Nauk S.S.S.R.*, Serie Math., 17, 1953, p. 269-284.

[4] LEVITAN B.M. — On Expansion in Eigenfunction of the Equation $\Delta u + \{\lambda - q(x_1, \ldots, x_n)\} u = O$, *Izv. Akad. Nauk S.S.S.R.*, Serie Math., 20, 1956, p. 437-468.

[5] COURANT R. — *Partial Differential Equations*, New-York, 1962.

[6] LEVITAN B.M. — On expansion in eigenfunction of the Laplacian operator, *Mat. Sb. (N.S.)*, 35 (77), 1954, p. 267-316.
[7] LEVITAN B.M. — On Solution of the Cauchy problem for equation $\partial^2u/\partial t^2 = \Delta u - q\ (x_1, \ldots, x_n)\, u$ by Sobolev's method, *Izv. Akad Nauk S.S.S.R.*, Serie Math. 20, 1956, p. 337-376.
[8] LEVITAN B.M. — Application of Sobolev's method in spectral analysis of self-adjoint differential operators of the second order, *U.S.S.R. and U.S.A., Joint Symposium on Equations with Partial Derivatives,* Novosibirsk, 1963.
[9] LEVITAN B.M. — On Expansion in eigenfunction of self-adjoint partial differential operator, *Trudy Moskov. Mat. Obsc.*, t. 5, 1956, p. 269-298.
[10] BUREAU F.J. — Asymptotic representation of the spectral function of self-adjoint elliptic operators of the second order with variable coefficients, *I. J. Math. Anal. and Applic,* 1, 1960, p. 423-483, *II J. Math. Anal. and Applic,* 4, 1962, p. 181-192.
[11] LEVITAN B.M. and GASYMOV M.G. — On the asymptotic behaviour of a spectral function of Shrödinger's operator near plane piece of boundary, *Izv. Akad. Nauk Serie Math.*, 28 : 5, 1964, p. 527-552.
[12] LAX P.D. — Asymptotic solutions of oscillatory int. problems, *Duke Math. J.*, 24, 1957, p. 627-646.
[13] ESKIN G.I. — Cauchy problem for hyperbolic equations in convolution, *Math. Sb,* 74 : 2, 1967, p. 262-297.
[14] ESKIN G.I. — Systems of pseudo-differential equations with simple real charactrestics, *Mat. Sb.*, 77 : 2, 1963, p. 174-200.
[15] MASLOV V.P. — On regularization of Cauchy problem for pseudo-differential equations, *Doklady Acad. Nauk S.S.S.R.*, t. 177 : 6, 1967, p. 1277-1280.
[16] SEELEY R.T. — Complex powers of an elliptic operator, *Amer. Math. Soc. Proc. Symp. Pure Math.*, 10, 1968, p. 288-307.
[17] SARGSJAN I.S. — Asymptotic behaviour of spectral matrix of Dirac one-dimensional system, *Doklady Acad. Nauk S.S.S.R.*, 166 : 5, 1966, p. 1058-1061.
[18] MARCHENKO V.A. — Theorems of Tauberian type in spectral analysis of differential operators, *Izv. Akad. Nauk S.S.S.R.*, Serie Mat., 19, 1955, p. 381-422.
[19] LEVITAN B.M. and SUVORCHENKOVA G.A. — Sufficient conditions for the discretnes of spectra of Sturm-Liouville equation with operator coefficient, *Functional Anal. i Prilozen,* v. 2, No. 2, 1968, p. 56-62.
[20] LEVITAN B.M. and KOSTJUCHENKO A.G. — Asymptotic behaviour of eigenvalue of the operator Sturm-Liouville problem, *Funct. Analysis i Prilozen,* v. 1, No. 1, 1967, p. 86-96.
[21] LEVITAN B.M. — Properties of the Green Function for a Sturm-Liouville equation with an operator coefficient, *Mat. Sb.*, t. 76, 118 : 2, 1968, p. 239-270.
[22] TITCHMARSH E.C. — Eigenfunction Expansions Associated with Partial Differential Equations, III, *Proc. London Math. Soc.* (3), 3, 1953, p. 153-169.
[23] ROFE-BEKETOV F.S. — Self-Adjoint extensions of diffenrential operators in space of vector functions, *Dokl. Akad. Nauk S.S.S.R.*, v. 184 : 5, 1969, p. 1034-1037.
[24] SEARS D.B. — Note on the uniqueness of Green's functions associated with certain differential equations, *Canadian J. of Math.*, 2, 1950, p. 314-325.
[25] AMBARZUMJAN V.A. — Uber eine frage der Eigenwertheorie, *Zeitschrift für Physik,* 53, 1929, p. 690-695.
[26] BORG G. — Eine Umkehrung der Sturm-Liouvillschen Eigenvertaufgabe. Bestimmung der Differetialgleichung durch die Eigenwerte, *Acta Math,* 78, No. 2, 1945, p. 1-96.
[27] LEVINSON N. — On the Uniquences of the Potential in a Schrödinger Equation for a Given Asymptotic Phase, *Danske Vid. Selsk. Mat.-Fys. Medd.*, 25, No. 9, 1949, p. 29.

[28] DELSARTE J. — Sur certaines transformations fonctionnelles relatives aux equations linéaires aux dérivées partielles du second ordre, *C.R. Acad. Sci.*, Paris, 206, 1938, p. 178-182.

[29] POVZNER A. Ja. — On differential equations of Sturm-Liouville type on a half-axis, *Mat. Sbornik*, t. 23, (65), 1948, p. 3-52.

[30] LEVITAN B.M. — Application of Generallized translation Operators to linear differential equations of second order, *Uspehi Mat. Nauk*, 4, 1 (29), 1949, p. 3-111.

[31] MARCHENKO V.A. — Some questions of the theory of one dimensional linear differential operators of the second order, *Trudy Moskov. Mat. Obsc.* 1, 1952, p. 324-420.

[32] GELFAND I.M. and LEVITAN B.M. — On the determination of a differential equation by its spectral function, *Izv. Akad. Nauk S.S.S.R.*, Ser. Mat. 15, 1951, p. 309-360.

[33] KREIN M.G. — Solution of the inverse Sturm-Liouville problem, *Doklady Akad. Nauk S.S.S.R.*, 76, 1951, p. 21-24.

[34] KREIN M.G. — On the inverse problem for non-homegenous string, *Doklady Akad. Nauk*, 85, 2, 1952, p. 669-672.

[35] LEVIN B. Ja. — Transformations of Fourier and Laplacian type with the help of solutions of differential equation of the second order, *Dokl. Akad. Nauk S.S.S.R.*, 106, 2, 1956, p. 187-190.

[36] MARCHENKO V.A. — Restoration of potential energy ty phases of scattering waves, *Doklady Akad. Nauk.*, v. 104, 5, 1955, p. 695-698.

[37] MARCHENKO V.A. and AGRANOVICH Z.S. — *Inverse problem of Scattering theory*, Charkov University Edition, 1960.

[38] JOST R. and NEWTON R.C. — The construction of potentials from the S-matrix for systems of differential equations II, *Nuovo Cimento*, v. 1, No. 4, 1955, p. 590-622.

[39] TOLL J.S. and PRATS F. — Connections of phase shift and potential for the Dirac equation, *Phys. Rev.*, v. 113, No. 1, 1959, p. 363.

[40] LEVITAN B.M. and GASYMOV M.G. — Inverse problem for Dirac system, *Dokl. Akad. Nauk S.S.S.R.*, 167 : 5, 1966, p. 967-970.

[41] LEVITAN B.M. and GASYMOV M.G. — Determination of Dirac's system by scattering phase, *Doklady Akad. Nauk. S.S.S.R.*, 167 : 6, 1966, p. 1219-1222.

[42] LEVITAN B.M. — On the determination of differential Sturm-Liouville equation by two spectra, *Izv. Akad. Nauk S.S.S.R.*, Ser. Mat. 28 : 1, 1964, p. 63-78.

[43] MARCHENKO V.A. — Stability of the irverse scatering problems, *Mat. Sb.* v. 77 : 2, 1968, p. 139-162.

[44] MARCHENKO V.A. and LUNDINA D. Sh. — Specifying of the inequalities, characterising the stability of the inverse scattering problem, *Mat. Sb.*, v. 78 : 4, 1969, p. 475-484.

[45] SACHNOVICH L.A. — Sufficient conditions for the existence of a transformation operator for an equations of the forth order, *Uspehi Mat. Nauk*, 16, No. 5, 1961, p. 199-205.

[46] MAZAEV V.I. — On existence of transformation operator for differential equations of higher orders, *Doklady Akad. Nauk S.S.S.R.*, v. 130 : 3, 1960, p. 499-502.

[47] LEIBENZON Z.L. — The uniqeness of solution of the inverse problem for ordinary differential equations of order $n > 2$ and transformations of such operators, *Doklady Akad. Nauk S.S.S.R.*, 142 : 3, 1962, p. 519-522.

[48] LEIBENZON Z.L. — Inverse problem of spectral analysis for ordinary differential operators of higher orders, *Trudy Moskov. Mat. Obsc.*, 15, 1966, p. 70-144.

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SOUS-ESPACES INVARIANTS D'UN OPÉRATEUR ET FACTORISATIONS DE SA FONCTION CARACTÉRISTIQUE

par Béla SZ.-NAGY

1. – On s'est aperçu dès les commencements de l'étude des "fonctions caractéristiques" associées aux opérateurs T de certains types dans l'espace de Hilbert (1), qu'il existe des relations entre les sous-espaces invariants pour T et les factorisations de la fonction caractéristique de T. Notamment, on a pu associer à tout sous-espace invariant pour T une factorisation de la fonction caractéristique, et à toute factorisation de la fonction caractéristique un sous-espace invariant, *sinon de T, mais d'un opérateur $T' = T \oplus V$* où le terme V est un opérateur à structure simple, par exemple unitaire ou autoadjoint, suivant le cas traité. Cf. en particulier [1], [2], [3].

Le problème de trouver les relations exactes entre les factorisations de la fonction caractéristique et les sous-espaces invariants *pour T lui-même*, n'a été résolu que vers la fin de l'année 1963, par Sz.-Nagy et Foiaş, pour les contractions complètement non-unitaires (c.n.u.) (2); cf. [4] et [5], chap. VII. Ces résultats ne sont donc pas tout récents. Mais ils semblent être peu connus, même les journaux analytiques n'en ont donné qu'une idée vague, sinon incorrecte. Il ne nous a pas semblé donc inutile de revenir à ce sujet dans cette conférence.

Notons que le cas des opérateurs à partie imaginaire positive se réduit à celui des contractions par une transformation de Cayley.

2. – Pour un espace de Hilbert $\mathfrak{E}$ on designera par $L^2_{\mathfrak{E}}$ l'espace de Hilbert des fonctions $u(\cdot)$ définies sur le cercle unité, à valeurs vecteurs dans $\mathfrak{E}$, mesurables et de norme carrée intégrable pour la mesure de Lebesgue normée. Soit $H^2_{\mathfrak{E}}$ le sous-espace formé des fonctions

$$u(e^{it}) \sim u_0 + e^{it} u_1 + e^{2it} u_2 + \cdots .$$

On pose $\chi(e^{it}) \equiv e^{it}$.

On envisagera aussi des fonctions $\Phi(\cdot)$ à valeurs opérateurs $\mathfrak{E} \to \mathfrak{E}_*$, mesurables et bornées par 1 : une telle fonction sera appelée *contractive* ; elle engendre par $(\Phi u)(\cdot) = \Phi(\cdot)\, u(\cdot)$ un opérateur $\Phi : L^2_{\mathfrak{E}} \to L^2_{\mathfrak{E}_*}$, notamment une contraction ; Φ permute à la multiplication par toute fonction scalaire mesurable bornée.

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(1) Tout espace de Hilbert envisagé est supposé complexe et séparable ; par opérateur on entend un opérateur linéaire et borné.

(2) Toute contraction est la somme orthogonale d'un opérateur unitaire et d'une contraction c.n.u.

Lorsque Φ transforme le sous-espace $H^2_{\mathfrak{E}}$ de $L^2_{\mathfrak{E}}$ dans le sous-espace $H^2_{\mathfrak{G}}$ de $L^2_{\mathfrak{G}}$, nous disons que la fonction est *analytique* ; telle fonction est notamment la limite radiale, p.p. sur le cercle unité, d'une fonction $\Phi(\lambda) = \Phi_0 + \lambda\Phi_1 + \lambda^2 \Phi_2 + \cdots$ analytique et contractive dans l'intérieur du cercle unité. (Ici on fait usage de ce que les espaces envisagés sont séparables). La fonction analytique contractive (f.a.c.) $\Phi(\cdot)$ est *pure* lorsque $\|\Phi_0 a\| < \|a\|$ pour tout $a \in \mathfrak{E}$, $a \neq 0$. Toute f.a.c. $\Phi(\cdot)$ est la somme orthogonale d'une f.a.c. pure $\Phi^0(\cdot)$ et d'une fonction constante unitaire $\Phi^1(\cdot) \equiv Z$:

$$\begin{array}{ccccc} \mathfrak{E} & = & \mathfrak{E}^0 & \oplus & \mathfrak{E}^1 \\ \Phi(\cdot)\big\downarrow & & \Phi^0(\cdot)\big\downarrow & & \Phi^1(\cdot)\big\downarrow \\ \mathfrak{G} & = & \mathfrak{G}^0 & \oplus & \mathfrak{G}^1 \end{array}$$

La f.a.c. $\Phi(\cdot)$ s'appelle ***intérieure*** si ses valeurs sont des opérateurs isométriques $\mathfrak{E} \to \mathfrak{G}$ en presque tous les points du cercle unité ; une définition équivalente est que l'opérateur Φ soit isométrique. Elle s'appelle *extérieure* si l'image de $H^2_{\mathfrak{E}}$ par Φ est dense dans $H^2_{\mathfrak{G}}$.

Si $\Phi(\cdot)$ est une f.a.c., il en est de même de la fonction $\Phi^{\sim}(\cdot)$ définie par

$$\Phi^{\sim}(e^{it}) = \Phi(e^{-it})^* .$$

La fonction $\Phi(\cdot)$ s'appelle $*$-intérieure ou $*$-extérieure si son associée est intérieure ou extérieure, selon les cas.

Pour deux fonctions à valeurs opérateurs on dit qu'elles *coïncident* lorsqu'elles ne diffèrent qu'à des facteurs constants unitaires de chaque côté.

3. – Pour une contraction quelconque T dans l'espace de Hilbert $\mathfrak{H}$ on définit les opérateurs de défaut $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$ et les espaces de défaut $\mathfrak{D}_T = \overline{D_T \mathfrak{H}}$, $\mathfrak{D}_{T^*} = \overline{D_{T^*} \mathfrak{H}}$. La fonction

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T] | \mathfrak{D}_T$$

est analytique dans l'intérieur du cercle unité et on peut montrer qu'elle est même contractive pure : ses valeurs sont des opérateurs $\mathfrak{D}_T \to \mathfrak{D}_{T^*}$. Cette fonction, ou plutôt la fonction $\Theta_T(\cdot)$ qui en dérive comme limite radiale p.p. sur le cercle unité, s'appelle la *fonction caractéristique* de la contraction T.

Les cas particuliers a) $T \in C_{.0}$, b) $T \in C_{.1}$, c) $T \in C_{0.}$, d) $T \in C_{1.}$ (1) correspondent aux cas où la fonction caractéristique $\Theta_T(\cdot)$ est a) intérieure, b) extérieure, c) $*$-intérieure, d) $*$-extérieure.

Les opérateurs $T \in C_{.0}$ admettent un modèle fonctionnel simple. On prend une fonction intérieure, à valeurs opérateurs $\mathfrak{E} \to \mathfrak{G}$; $\Theta H^2_{\mathfrak{E}}$ est alors un sous-espace de $H^2_{\mathfrak{G}}$. On pose

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(1) On écrit $T \in C_{.0}$ si $T^{*n}h \to 0$ pour tout $h \in \mathfrak{H}$ $(n \to \infty)$, et $T \in C_{.1}$ si $T^{*n} h \to 0$ pour $h = 0$ seulement. De même, $T \in C_{0.}$ si $T^n h \to 0$ pour tout h, et $T \in C_{1.}$ si $T^n h \to 0$ pour $h = 0$ seulement. Enfin, $C_{\alpha\beta} = C_{\alpha.} \cap C_{.\beta}$.

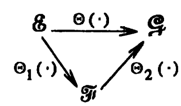
$$\mathcal{H}(\Theta) = H^2_{\mathfrak{G}} \ominus \Theta H^2_{\mathfrak{E}} \;;$$

$\mathcal{H}(\Theta) \neq \{0\}$ sauf si $\Theta(\cdot)$ est une constante unitaire. On définit dans $\mathcal{H}(\Theta)$ l'opérateur $S(\Theta)$ par

$$S(\Theta)u = P_{\mathcal{H}(\Theta)}(\chi u) \quad \text{ou} \quad S(\Theta)^* u = \chi(u - u_0) \quad (u \in \mathcal{H}(\Theta))\ (^1).$$

On obtient de cette façon toutes les contractions de type $C_{\cdot 0}$, à une équivalence unitaire près. Pour T donné il n'y a qu'à prendre pour $\Theta(\cdot)$ la fonction caractéristique $\Theta_T(\cdot)$, ou n'importe quelle autre fonction intérieure dont la partie pure $\Theta^0(\cdot)$ coïncide avec $\Theta_T(\cdot)$.

Dans ce cas à toute factorisation

(1) 

de $\Theta(\cdot)$ en produit $\Theta_2(\cdot)\Theta_1(\cdot)$ de deux fonctions intérieures il correspond une décomposition de l'espace $\mathcal{H}(\Theta)$, notamment

$$\mathcal{H}(\Theta) = H^2_{\mathfrak{G}} \ominus \Theta_2 \Theta_1 H^2_{\mathfrak{E}} = \Theta_2(H^2_{\mathfrak{F}} \ominus \Theta_1 H^2_{\mathfrak{F}}) \oplus (H^2_{\mathfrak{G}} \ominus \Theta_2 H^2_{\mathfrak{F}}) =$$
$$= \Theta_2 \cdot \mathcal{H}(\Theta_1) \oplus \mathcal{H}(\Theta_2) \equiv \mathcal{H}_1 \oplus \mathcal{H}_2 \;.$$

Il est facile de montrer que $\mathcal{H}_1$ est invariant pour $S(\Theta)$. La matrice de $S(\Theta)$ par rapport à cette décomposition est donc de la forme

$$\begin{bmatrix} S_1 & X \\ 0 & S_2 \end{bmatrix}$$

où $S_1 = S(\Theta)|\mathcal{H}_1$ est unitairement équivalent à $S(\Theta_1)$, et de plus $S_2 = S(\Theta_2)$.

Ainsi, les fonctions caractéristiques de S_1 et S_2 coïncident avec les parties pures de $\Theta_1(\cdot)$ et $\Theta_2(\cdot)$. Il s'ensuit en particulier que si la factorisation (1) est non banale (c'est-à-dire aucun des facteurs n'est une constante unitaire), $\mathcal{H}_1$ est un sous-espace invariant propre pour $S(\Theta)$.

Or on obtient de cette façon *tous* les sous-espaces $\mathcal{H}_1$ invariants pour $S(\Theta)$, ou, ce qui revient au même, tous les sous-espaces $\mathcal{H}_2$ invariants pour $S(\Theta)$. Cela résulte d'une manière simple du théorème bien connu de Beurling – Lax – Halmos sur la relation entre les sous-espaces invariants d'une translation unilatérale et les fonctions intérieures.

Ainsi, pour $T \in C_{\cdot 0}$ il y a une correspondance complète entre les sous-espaces invariants de T et les factorisations en fonctions intérieures de $\Theta_T(\cdot)$. Malheureusement, le problème de factoriser une fonction intérieure $\Theta(\cdot)$ en produit de deux fonctions intérieures ne s'avère maniable (à nos connaissances actuelles) que si $\Theta(\cdot)$ admet un multiple scalaire. Des cas importants où cette condition est vérifiée sont traités dans les chapitres V et VIII de [5].

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(1) P désigne la projection orthogonale sur l'espace indiqué comme indice.

4. – Cela impose le problème d'étudier les contractions qui n'appartiennent pas nécessairement à la classe $C_{.0}$.

Une telle étude exige des moyens plus fins et a été rendue possible notamment par l'utilisation de la théorie générale des *dilatations isométriques* des contractions c.n.u. et du modèle fonctionnel général de ces contractions qui en dérive par une analyse de Fourier.

Ce modèle est le suivant : On considère une fonction analytique contractive quelconque $\Theta(\cdot)$, à valeurs opérateurs $\mathfrak{E} \to \mathfrak{F}$, et on y associe l'espace de Hilbert

$$\mathfrak{H}(\Theta) = [H^2_{\mathfrak{F}} \oplus \overline{\Delta L^2_{\mathfrak{E}}}] \ominus \{\Theta u \oplus \Delta u : u \in H^2_{\mathfrak{E}}\}$$

où $\Delta(\cdot) = [I - \Theta(\cdot)^* \Theta(\cdot)]^{1/2}$ (1) ; on a $\mathfrak{H}(\Theta) \neq \{0\}$ sauf si $\Theta(\cdot)$ est une constante unitaire. Dans $\mathfrak{H}(\Theta)$ on définit l'opérateur $S(\Theta)$ par

$$S(\Theta)\,(u \oplus v) = P_{\mathfrak{H}(\Theta)}(\chi u \oplus \chi v) \qquad \text{ou} \qquad S(\Theta)^*(u \oplus v) = \bar{\chi}(u - u_0) \oplus \bar{\chi} v$$

$(u \oplus v \in \mathfrak{H}(\Theta))$. Ces définitions se réduisent à celles envisagées plus haut dans le cas où $\Theta(\cdot)$ est intérieure, car alors $\Delta(\cdot) \equiv 0$.

L'opérateur $S(\Theta)$ est une contraction c.n.u. et sa fonction caractéristique coïncide avec la partie pure de $\Theta(\cdot)$. De plus, on obtient de cette façon, à une équivalence unitaire près, *toutes* les contractions c.n.u. T. En effet, pour T donnée il n'y a qu'à prendre pour $\Theta(\cdot)$ n'importe quelle f.a.c. dont la partie pure coïncide avec $\Theta_T(\cdot)$.

Observons alors que si $\Theta(\cdot) = \Theta_2(\cdot)\,\Theta_1(\cdot)$ est une factorisation de $\Theta(\cdot)$ en produit de deux fonctions de même type suivant le diagramme (1), on a

$$\Delta(\cdot)^2 = \Theta_1(\cdot)^* \Delta_2(\cdot)^2\, \Theta_1(\cdot) + \Delta_1(\cdot)^2 \,, \tag{2}$$

Il s'ensuit que l'application

$$\Delta(e^{it})\, g \to \Delta_2(e^{it})\, \Theta_1(e^{it})\, g \oplus \Delta_1(e^{it})\, g \quad (g \in \mathfrak{E})$$

est isométrique pour presque tout point fixé e^{it}, donc se prolonge par continuité en une isométrie

$$Z(e^{it}) : \overline{\Delta(e^{it})\, \mathfrak{E}} \to \overline{\Delta_2(e^{it})\, \mathfrak{F}} \oplus \overline{\Delta_1(e^{it})\, \mathfrak{E}} \,.$$

De (2) il dérive aussi que l'application

$$\Delta u \to \Delta_2\, \Theta_1 u \oplus \Delta_1 u \qquad (u \in L^2_{\mathfrak{E}})$$

est isométrique et se prolonge donc par continuité en une isométrie

$$Z : \overline{\Delta L^2_{\mathfrak{E}}} \to \overline{\Delta_2 L^2_{\mathfrak{F}}} \oplus \overline{\Delta_1 L^2_{\mathfrak{E}}} \,.$$

---

(1) L'ensemble linéaire $\{\Theta u \oplus \Delta u \,:\, u \in H^2_{\mathfrak{E}}\}$ est fermé : conséquence de ce qu'il est l'image de $H^2_{\mathfrak{E}}$ par l'application isométrique $u \to \Theta u \oplus \Delta u$.

On montre sans difficulté que les conditions suivantes sont équivalentes :

condition locale : $Z(e^{it})$ est unitaire p.p.,

condition globale : Z est unitaire.

Lorsque ces conditions (équivalentes) sont vérifiées, on appelle la factorisation $\Theta(\cdot) = \Theta_2(\cdot)\,\Theta_1(\cdot)$ *régulière.*

Si c'est le cas, on peut identifier les éléments des espaces

$$\overline{\Delta L^2_{\mathcal{E}}} \quad \text{et} \quad \overline{\Delta_2 L^2_{\mathcal{F}}} \oplus \overline{\Delta_1 L^2_{\mathcal{E}}}$$

qui se correspondent par l'opérateur unitaire Z et on obtient alors pour $\mathcal{H}(\Theta)$ la forme à trois composantes

$$\mathcal{H}(\Theta) = [H^2_{\mathcal{G}} \oplus \overline{\Delta_2 L^2_{\mathcal{F}}} \oplus \overline{\Delta_1 L^2_{\mathcal{E}}}] \ominus \{\Theta_2\Theta_1 u \oplus \Delta_2\Theta_1 u \oplus \Delta_1 u : u \in H^2_{\mathcal{E}}\}.$$

Il en dérive la décomposition $\mathcal{H}(\Theta) = \mathcal{H}_1 \oplus \mathcal{H}_2$ où

$$\mathcal{H}_1 = \omega_2 \cdot \mathcal{H}(\Theta_1) \quad \text{et} \quad \mathcal{H}_2 = \mathcal{H}(\Theta_2) \oplus \{0\}, \tag{3}$$

ω_2 étant la restriction à $\mathcal{H}(\Theta_1)$ de l'isométrie

$$u \oplus v \to \Theta_2 u \oplus \Delta_2 u \oplus v \qquad (u \in H^2_{\mathcal{F}},\ v \in L^2_{\mathcal{E}}) .$$

Il s'ensuit aussi que $\mathcal{H}_1$ est invariant pour $S(\Theta)$ et que dans la matrice correspondante

$$S(\Theta) = \begin{bmatrix} S_1 & X \\ 0 & S_2 \end{bmatrix}, \tag{4}$$

les opérateurs S_1 et S_2 sont unitairement équivalents à $S(\Theta_1)$ et $S(\Theta_2)$, selon les cas ; en particulier $\mathcal{H}_1$ est un sous-espace invariant *propre* si la factorisation n'est pas banale.

Ces calculs sont directs et relativement simples. Le point difficile (et c'est ici où une étude approfondie de la structure des dilatations isométriques est utilisée) est de montrer qu'on obtient de cette façon *tous* les sous-espaces invariants $\mathcal{H}_1$ pour $S(\Theta)$, et d'arriver ainsi au théorème suivant :

THEOREME. – *Lorsque* $\Theta(\cdot) = \Theta_2(\cdot)\,\Theta_1(\cdot)$ *parcourt la totalité des factorisations régulières de* $\Theta(\cdot)$, *le sous-espace correspondant* $\mathcal{H}_1$ (*par* (3)) *parcourt la totalité des sous-espaces invariants pour* $S(\Theta)$. (Cf. [4], [5]).

5. – En vertu de ce théorème il importe de trouver des critères maniables pour qu'une factorisation

$$\Theta(\cdot) = \Theta_2(\cdot)\,\Theta_1(\cdot) \tag{F}$$

d'une f.a.c. en facteurs de même type soit régulière. En voici quelques uns (cf. [5] prop. VII. 3.3).

(a) *(F) est régulière si la factorisation duale $\Theta^{\sim}(\cdot) = \Theta_1^{\sim}(\cdot)\,\Theta_2^{\sim}(\cdot)$ est régulière.*

(b) *Pour $\Theta(\cdot)$ intérieure (*-intérieure), (F) est régulière si et seulement si les facteurs sont aussi des fonctions intérieures (*-intérieures).*

(c) *(F) est régulière si, en presque tous les points du cercle unité où $\Theta_2(e^{it})$ n'est pas isométrique, $\Theta_1(e^{it})$ est co-isométrique.*

Un cas particulier où le dernier critère s'applique est celui où $\Theta(e^{it})$ admet un inverse (non nécessairement borné) en presque tous les points du cercle unité (cas qui se présente par exemple si $\Theta(\cdot)$ est *-extérieure). Notamment on peut montrer (faisant usage d'un raisonnement de Lowdenslager [6]) qu'il existe alors pour tout sous-ensemble borélien α du cercle unité une factorisation

$$\Theta(\cdot) = \Theta_{2\alpha}(\cdot)\,\Theta_{1\alpha}(\cdot)$$

telle que

(i) $\Theta_{1\alpha}(e^{it})$ est *unitaire* ($\mathfrak{E} \to \mathfrak{F}_\alpha$) en presque tout point du complémentaire α' de α par rapport au cercle unité,

(ii) $\Theta_2(e^{it})$ est *isométrique* ($\mathfrak{F}_\alpha \to \mathfrak{G}$) en presque tout point de α.

Comme l'unitarité entraîne la co-isométrie, cette factorisation est régulière, donc il y correspond par (3) un sous-espace invariant $\mathfrak{H}_\alpha$ de $\mathfrak{H}(\Theta)$. Pour que $\mathfrak{H}_\alpha$ soit un sous-espace *propre*, il faut choisir α de façon que $\Theta_{1\alpha}(\cdot)$ et $\Theta_{2\alpha}(\cdot)$ ne soient pas des constantes unitaires. Cela est certainement possible, et même d'une infinité de manières, si la fonction n'est pas intérieure, c'est-à-dire que si sa valeur n'est pas isométrique aux points d'un sous-ensemble ϵ de mesure positive du cercle unité : il n'y a qu'à choisir α tel que $\alpha \cap \epsilon$ et $\alpha' \cap \epsilon$ soient de mesure positive.

Si T est une contraction c.n.u. telle que $T \in C_{1.}$ et $T \notin C_{.0}$ (en particulier si $T \in C_{11}$), sa fonction caractéristique est *-extérieure et non-intérieure, donc les résultats ci-dessus s'appliquent : il existe pour chaque sous-ensemble borélien α du cercle unité un sous-espace invariant $\mathfrak{H}_\alpha$ pour T, et ce sous-espace est propre pour une infinité de choix de α. Pour $T \in C_{11}$ une analyse plus approfondie montre que $\mathfrak{H}_\alpha$ est même ultrainvariant pour T et que de plus, si $\Theta_T(\cdot)$ admet un multiple scalaire, le spectre de $T \mid \mathfrak{H}_\alpha$ est compris dans $\overline{\alpha}$.

Remarquons que l'existence de sous-espaces invariants propres pour une contraction c.n.u. $T \in C_{11}$ peut être démontrée aussi à partir du fait que T est quasi-similaire à un opérateur unitaire, mais la méthode originelle était celle utilisant les factorisations de la fonction caractéristique. Cette méthode, complétée par une étude approfondie de l'arithmétique des factorisations régulières, conduit à établir une sorte de décomposition spectrale pour toutes les contractions "faibles" ; cf. [5], chap. VIII.

6. – La construction (3) a un sens même si la factorisation envisagée n'est pas régulière, mais fournit alors un sous-espace invariant non pour $S(\Theta)$, mais pour une somme orthogonale de $S(\Theta)$ et d'un opérateur unitaire d'un espace non-banal.

La question se pose s'il existe quelque autre moyen d'associer à une factorisation (F) non régulière une décomposition $\mathfrak{H}(\Theta) = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ de l'espace, de sorte que la matrice correspondante de $S(\Theta)$ soit de la forme $\begin{bmatrix} S_1 & X \\ 0 & S_2 \end{bmatrix}$, avec (i) S_1 unitairement équivalent à $S(\Theta_1)$, (ii) S_2 unitairement équivalent à $S(\Theta_2)$.

Lorsque $S(\Theta) \in C_{.0}$ on devra avoir aussi $S_1, S_2 \in C_{.0}$. On en déduit que si $\Theta(\cdot)$ est intérieure, les conditions (i) et (ii) entraînent que $\Theta_1(\cdot)$ et $\Theta_2(\cdot)$ soient aussi intérieures, selon les cas. Ainsi, la réponse à la question posée est certainement négative pour les factorisations non-banales, construites dans [5] n° V.5, d'une fonction intérieure $\Theta(\cdot)$ en produit d'une fonction intérieure $\Theta_1(\cdot)$ et d'une fonction extérieure $\Theta_2(\cdot)$.

*

Dans cette conférence nous avons considéré seulement des contractions c.n.u. et leurs modèles fonctionnels se rattachant à la théorie des dilatations isométriques. Pour d'autres aspects de notre sujet nous citons par exemple les Notes [7] — [10].

BIBLIOGRAPHIE

[1] Livšic M.S. — *Mat. sbornik,* 19 (61), 1946, p. 239-260.
[2] Brodsky M.S. and Livšic M.S. — *Uspehi mat. nauk,* 19 : 1 (115), 1958, p. 3-85.
[3] Smulyan Yu. L. — *Mat. sbornik,* 57 (99), 1962, p. 105-136.
[4] Sz.-Nagy B. and C. Foias C. — *Comptes rendus,* 256, 1963, p. 3236-3239 et p. 3413-3415; 258, 1964, p. 3426-3429; *Acta scient. math.,* 25, 1964, p. 293-316; 26, 1965, p. 193-196.
[5] Sz.-Nagy B. and Foiaş C. — *Analyse harmonique des opérateurs de l'espace de Hilbert* (Budapest-Paris, 1967).
[6] Lowdenslager D.B. — *Annals of math.,* 78, 1963, p. 450-454.
[7] Sz.-Nagy B. and Foiaş C. — *Acta scient. math.,* 28, 1967, p. 201-212.
[8] Kužel' A.V. — *ibidem,* 30, 1969, p. 225-233.
[9] Brodsky V.M. — *ibidem,* 32, 1971, à paraître.
[10] De Branges L. — *J. math. analysis and applications,* 29, 1970, p. 163-200.

Bolyai Intézet,
Aradi vértanúk tere 1
Szeged
Hongrie

Actes, Congrès intern. Math., 1970. Tome 2, p. 467 à 473.

D 4 - ALGÈBRES DE FONCTIONS : ANALYSE DE FOURIER

THÉORÈMES DE PLONGEMENT DES ESPACES FONCTIONNELS

par O.V. BESOV

La théorie du plongement des espaces fonctionnels de fonctions différentiables à plusieurs variables réelles a pris naissance dans les travaux de S.L. Sobolev en rapport avec la recherche de la solution d'une série de problèmes de la physique mathématique. Son développement ultérieur a été déterminé aussi bien par la théorie des problèmes aux limites, que par sa propre problématique intérieure. Le problème principal de cette théorie est le suivant :

On introduit, sur un ensemble de fonctions suffisamment large, une famille de normes dépendant d'un ou de plusieurs paramètres qui caractérisent la différentiabilité et la sommabilité de ces fonctions. Ces normes définissent les espaces fonctionnels correspondants. En partant de l'appartenance d'une fonction à un de ces espaces, il faut déduire son appartenance à un autre espace fonctionnel. Ainsi, la fonction est considérée comme un élément de deux espaces au moins, et on se propose d'étudier l'opérateur de plongement ainsi obtenu d'un espace fonctionnel dans un autre espace. Généralement, on pose la question de savoir si cet opérateur est borné ou s'il est complètement continu. On appelle les propositions correspondantes *théorèmes de plongement* d'un espace normé dans un autre et, respectivement, *théorèmes de compacité*.

Nous disons que un espace fonctionnel normé E est plongé dans un espace fonctionnel F, et nous désignons ce fait par $E \to F$ si $E \subset F$ et s'il existe une constante C, indépendante de f, telle que $\|f\|_F \subset \|f\|_E, f \in E$.

S.L. Sobolev a étudié les espaces isotropes $W_p^{(l)}(\Omega)$ des fonctions $f(x)$, définies sur le domaine $\Omega \subset E^n$, qui vérifie la condition du cône et dont la norme est :

$$\sum_{|\alpha| \leqslant l} \|\mathcal{D}^\alpha f\|_{p,\Omega} ,$$

où l est un entier naturel et $p \geqslant 1$, avec

$$\|f\|_{p,\Omega} = \{\int_\Omega |f(x)|^p \, dx\}^{1/p} .$$

Il a obtenu, grâce aux représentations intégrales des fonctions, les premiers théorèmes de plongement, qui affirment la q-sommabilité d'une fonction $f(x)$ sur le domaine Ω ou sur les sections de ce domaine de dimensions inférieures.

Par la suite, S.M. Nikolski a obtenu les théorèmes de plongement des espaces de fonctions de différentiabilité différente suivant les directions et avec des degrés de différentiabilité non-entiers (conditions de Hölder dans la métrique L_p). S.M. Nikolski a également démontré, le premier, les inversions exactes des théorèmes de plongement. Sa méthode consiste en approximation par des fonctions entières.

Arrêtons-nous sur certains résultats obtenus au cours des dernières années en Union Soviétique.

A présent, on a trouvé la surjectivité des traces de fonctions de $W_p^l(\Omega)$, sur la frontière $\Gamma = \partial\Omega$, qui vérifie localement la condition de Lipchitz. Nous la donnons pour le cas typique suivant :

Soient

$$x = (x', x'') \in E^n, \; x' \in E^m, \; x'' \in E^{n-m},$$

$$\Gamma = \{x : x'' = \mathcal{S}(x'), \; |\mathcal{S}(x') - \mathcal{S}(y')| \leqslant M_0 \, |x' - y'|, \; x', y' \in E^m\},$$

$$V = \{x : |x''| > M_1 |x'|\}, \; M_1 > M_0, \quad 1 < p < \infty,$$

$$0 \leqslant S < l - \frac{n-m}{p} < S + 1,$$

S, l des nombres entiers.

Alors, pour

$$f \in W_p^{(l)}(E^n \setminus \Gamma)$$

on a

$$(1) \quad \sum_{|\beta| \leqslant S} \left\{ \int_{E^n \setminus \Gamma} \int_{\Gamma(x)} \int_{\Gamma(x)} \left| \frac{P_S^{(\beta)}(x - y; y) - P_S^{(\beta)}(x - z; z)}{\rho(x, \Gamma)^{l + 2\frac{m}{p} - |\beta|}} \right|^p dy' dz' dx \right\} \leqslant C \|f\|_{W_p^{(l)}(E \setminus \Gamma)}$$

où $\Gamma(x) = \Gamma \cap (x - V)$, C ne dépend pas de f, $P_S(x ; y)$ est un polynôme de Taylor en la variable x au point y.

$$(2) \qquad P_S(x ; y) = \sum_{|\alpha| \leqslant S} \frac{f^{(\alpha)}(y)}{\alpha!} x^\alpha$$

Inversement, soit donné sur Γ un système de fonctions $\{f_\alpha(x)\}$,

$$(3) \qquad |\alpha| \leqslant S, \; x \in \Gamma \quad \text{et} \quad P_S(x ; y) = \sum_{|\alpha| \leqslant S} \frac{f_\alpha(y)}{\alpha!} x^\alpha$$

telles que, avec ces notations, la partie gauche de (1) soit finie. Alors il existe une fonction $f(x)$, définie sur $E^n \setminus \Gamma$, telle que

$$f^{(\alpha)}(x) = f_\alpha(x)|_{x \in \Gamma} \quad \text{et} \quad \|f\|_{W_p^{(l)}(E^n \setminus \Gamma)}$$

soit majorée par la partie gauche de (1) multipliée par une certaine constante.

En plus de la norme habituelle L_p, nous allons considérer la norme mixte L_p des fonctions $f(x)$ pour $x \in E^n$. Dans ce cas $p = (p_1, \dots, p_n)$

$$\|f\|_p = \|f\|_{p,E^n} = (\int(\int|f|^{p_1}dx_1)^{\frac{1}{p_1}} \dots)^{p_n}dx_n)^{\frac{1}{p_n}} .$$

Pour le cas $p_i = \infty$ au lieu de $(\int|g|^{p_i}dx_i)^{\frac{1}{p_i}}$ il faut lire $\operatorname{ess\,sup}_{-\infty<x_i<\infty} |g|$.

On a pour $\Omega \subset E^n$,

$$\|f\|_{p,\Omega} = \|f_\Omega\|_{p,E^n} ,$$

où
$$f_\Omega(x) = \begin{cases} f(x) , x \in \Omega \\ 0 \quad ,x \notin \Omega \end{cases} .$$

Pour $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$ nous écrirons $p < q$ $(q \leqslant q)$ si pour tous $i = 1, \dots, n$ nous avons $p_i < q_i$ $(p_i \leqslant q_i)$.

Supposons encore que $\vec{1} = (1, \dots, 1)$, $\vec{\infty} = (\infty, \dots, \infty)$

$$p : q = \left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}\right) \quad ; \quad \frac{1}{p} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right) .$$

Lors de l'étude des espaces anisotropes de fonctions, il est naturel de considérer les domaines $\Omega \subset E^n$, qui vérifient la condition de la corne.

On appelle r-corne le domaine $V(r)$, $r = (r_1, \dots, r_n)$, $r_i > 0$ $(i = 1, \dots, n)$ tel que

$$(4) \quad V(r) = \bigcup_{0<h<H} \left\{ x : \frac{x_i}{a_i} > 0 , h < \left(\frac{x_i}{a_i}\right)^{r_i} < (1 + \epsilon) h \ (i = 1, \dots, n) \right\}$$

Nous disons que le domaine $\Omega \subset E^n$ vérifie la condition de la r-corne s'il existe un recouvrement fini $\{\Omega_k\}_1^K$ par des ensembles ouverts Ω_k et une suite de cornes $V_k(r)$ pour lesquelles les conditions suivantes se vérifient :

1/ $\Omega = \bigcup_{k=1}^{K} \Omega_k$;

2/ $x \in \Omega_k \Rightarrow x \in V_k(r) \subset \Omega$ $(k = 1, \dots, K)$;

3/ $\exists\, \epsilon > 0$ tel que $\Omega = \bigcup_{k=1}^{K} \Omega_k^{(\epsilon)}$

où
$$\Omega_k^{(\epsilon)} = \{x : x \in \Omega_k , \rho(x , \partial\Omega_k \setminus \partial\Omega) > \epsilon\} .$$

Nous disons, dans ce cas que $\Omega \subset E^n$ vérifie la condition forte de r-corne, si dans 3) on peut remplacer $\Omega_k^{(\epsilon)}$ par

$$\Omega_k^{[\epsilon]} = \{x : x \in \Omega_k , \rho(x , \Omega \setminus \Omega_k) > \epsilon\} .$$

Soient $l = (l_1, \ldots, l_n)$, l_i des nombres naturels. On a

$$(5) \qquad \|f\|_{W_p^l(\Omega)} = \|f\|_{p,\Omega} + \sum_{i=1}^{n} \|\mathcal{D}_i^{l_i} f\|_{p,\Omega}$$

Pour un domaine $\Omega \subset E^n$ vérifiant la condition de r-corne, on a le théorème suivant de plongement de l'espace $W_p^l(\Omega)$:

$$(6) \qquad \|\mathcal{D}^\alpha f\|_{q,\Omega} \leqslant C \|f\|_{W_p^l(\Omega)}$$

pour $\vec{1} \leqslant p \leqslant q \leqslant \vec{\infty}$

$$(7) \qquad \left| \left(\frac{1}{p} - \frac{1}{q} + \alpha\right) : r \right| < \min_{1 \leqslant i \leqslant n} \frac{l_i}{r_i}$$

L'inégalité (7) peut être remplacée par l'égalité si

$$1 < p_n < q_n < \infty \quad \text{ou bien} \quad \vec{1} < p = q < \vec{\infty}.$$

Considérons que dans (6) $q = (p_1, \ldots, p_m, \infty, \ldots, \infty)$. Alors l'estimation (6) indique en particulier, le degré de différentiabilité de la trace (restriction) de la fonction f sur la section de dimension m du domaine Ω définie par

$$x_{m+1} = x_{m+1}^0, \ldots, x_n = x_n^0 .$$

Ces propriétés des traces des fonctions sont optimales en termes de fonctions généralisées (d'ordre entier), cependant elle peut être améliorés et devenir caractéristiques si l'on utilise les termes de comportement des différences finies des fonctions. Nous désignons par $\Delta_i^m(t\,;\Omega) f(x)$ la différence d'ordre m de la fonction $f(x)$ avec un pas t en direction du vecteur des coordonnées e_i si $[x, x + mte_i] \subset \Omega$, et par $\Delta_i^m(t\,;\Omega) f(x) = 0$ dans le cas contraire.

Supposons que le domaine $\Omega \subset E^n$ vérifie la condition de r-corne.

Alors

$$\|\Delta_j^M(\tau^{\frac{1}{r_j}}\,;\Omega)\mathcal{D}^\alpha f\|_q \leqslant$$

$$(8) \qquad \begin{aligned} &\leqslant C T^{M:r_j} H^{-M:r_j - |(\frac{1}{p^0} - \frac{1}{q} + \alpha):r|} \|f\|_{p^0,\Omega} \\ &+ C \sum_{i=1}^{n} \left\{ \int_0^T \|\Delta_i^{m_i}(t^{\frac{1}{r_i}}\,;\Omega) f\|_{p^i}^{\theta_i} \frac{dt}{t^{1+\theta_i l_i : r_i}} \right\}^{\frac{1}{\theta_i}} \\ &+ C \sum_{i=1}^{n} T^{M:r_j} \int_T^H \|\Delta^{m_i}(t^{\frac{1}{r_i}}\,;\Omega) f\|_{p^i} \frac{dt}{t^{1+l_i:r_i+M:r_j}} \end{aligned}$$

où

$$\vec{1} \leqslant p^i \leqslant q \leqslant \vec{\infty},\ l_i : r_i \geqslant \left| \left(\frac{1}{p^i} - \frac{1}{q} + \alpha\right) : r \right| \geqslant 0$$

et en outre

$$\theta_i = 1 \quad \text{ou bien} \quad 1 \leqslant \theta_i \leqslant q_n \ , \ 1 \leqslant p_n^i < q_n < \infty \ ; \tag{9}$$

la constante C ne dépend pas des f, τ, T, H

$$(0 < \tau \leqslant T \leqslant H \leqslant H_0).$$

A partir de l'estimation (8), on déduit les estimations des normes (théorèmes de plongement) pour des espaces fonctionnels de fonctions vérifiant la condition de Hölder et ses généralisations. Pour la formulation de ces résultats, donnons la définition suivante :

On appelle $\mathcal{H}$*-fonctionnelles* une fonctionnelle non-négative

$$\mathcal{H}[\psi] = \mathcal{H}[\psi(\cdot)] = \mathcal{H}[\psi(T)] \geqslant 0$$

définie sur les fonctions non-négatives mesurables $\psi(T) \geqslant 0$, $0 < T < H < \infty$ et vérifiant, pour tous les $a \geqslant 0$, $b \geqslant 0$, $\delta > 0$, les conditions suivantes :

1/ $\mathcal{H}[a\psi_1 + b\psi_2] \leqslant Ca\,\mathcal{H}[\psi_1] + Cb\,\mathcal{H}[\psi_2]$;

2/ $\mathcal{H}[\psi_1] \leqslant C\,\mathcal{H}[\psi_2]$ pour $0 \leqslant \psi_1(T) \leqslant \psi_2(T)$;

3/ $\mathcal{H}\left[\int_0^1 \psi(Tt)\, t^{-1+\delta}\, dt\right] + \mathcal{H}\left[\int_1^{\frac{H}{T}} \psi(Tt)\, t^{-1-\delta}\, dt\right] \leqslant C(\delta)\,\mathcal{H}[\psi(T)]$;

4/ $\mathcal{H}[T^\delta] < \infty$

5/ $\int_0^H \psi(t)\, t^{-1+\delta}\, dt \leqslant C(\delta)\,\mathcal{H}[\psi]$.

La caractérisation des classes de fonctions différentiables par des $\mathcal{H}$-fonctionnelles et les théorèmes de plongement des espaces correspondants ont été établis par S.M. Nikolsky pour $\mathcal{H}[\psi] = \sup\limits_{0<T<H} \psi(T)$ puis par l'auteur pour

$$\mathcal{H}[\psi] = \left\{ \int_0^H \psi^\theta(T) \frac{dT}{T} \right\}^{1/\theta} \quad , \quad 1 \leqslant \theta < \infty$$

et ensuite par K.K. Golovkin pour les fonctionnelles du type de maximisation pour des semi-normes convenables et même plus générales.

En appliquant une $\mathcal{H}$-fonctionnelle à l'inégalité (8), on obtient

$$\mathcal{H}\left[\frac{\sup\limits_{0<\tau\leqslant T} \| \Delta_j^M(\tau^{\frac{1}{r_j}} ; \Omega)\, \mathcal{D}^\alpha f \|_q}{T^{\frac{S}{r_j}}} \right] \leqslant CH^{-S:r_j - |(\frac{1}{p^0} - \frac{1}{q} + \alpha):r| - \delta} \|f\|_{p^0;\Omega} + C \sum_{i=1}^n \mathcal{H}\left[\frac{\| \Delta_i^{m_i}(T^{\frac{1}{r_i}} ; \Omega) f \|_{p^i}}{T^{\frac{l_i}{r_i} + \frac{S_i}{r_i}}} \right] \tag{10}$$

si $0 < S < M$, $0 < \delta r_j < M - S$ et si les conditions (9) se vérifient.

Si, pour la fonction $f(x)$, la partie droite de (10) est finie, il est naturel de considérer que la fonction $f(x)$ est différentiable avec l'ordre de différentiabilité $l_i + S\frac{r_i}{r_j}$ pour la variable x_i. Le théorème de plongement (10) démontre que l'ordre de différentiabilité pour x_i diminue de l_i lors du passage de la métrique L_{p^i} à la métrique L_q.

La méthode principale d'étude des propriétés (des classes) de fonctions différentiables dans un domaine est la méthode des représentations intégrales, proposée par S.L. Sobolev et développée ensuite par V.P. Ilin. L'essentiel de cette méthode consiste en la représentation de la fonction par la somme des intégrales, du type potentiel, de ses dérivées (ou des différences) d'un ordre déterminé avec ensuite une estimation de ces opérateurs intégraux dans les normes correspondantes.

Supposons que le domaine Ω vérifie la condition de l-corne. Alors pour l'espace $W_p^l(\Omega)$ ont lieu les mêmes théorèmes de plongement que pour $W_p^l(E^n)$. En plus, si le domaine Ω vérifie la condition forte de l-corne, alors les fonctions de $W_p^l(\Omega)$ peuvent être prolongées sur E^n, et, dans ce cas, l'opérateur de prolongement est linéaire et borné. Ceci est aussi vrai pour des espaces, construits à l'aide de $\mathcal{H}$-fonctionnelles, de fonctions ayant la différentiabilité d'ordre l_i pour x_i et la condition (forte) de l-corne.

Cette extension peut s'effectuer par la méthode de prolongement de Calderon.

Indiquons le lien entre les espaces $W_p^l(\Omega)$ et les espaces des fonctions construits à l'aide de $\mathcal{H}$-fonctionnelles. Notons pour $\Omega \subset E^n$

(11) $$\|f\|_{B_p^l(\Omega)} = \|f\|_{p,\Omega} + \sum_{i=1}^{n} \mathcal{H}\left[\frac{\|\Delta_i^{m_i}(T^{\frac{1}{l_i}}\,;\Omega)f\|_p}{T}\right]$$

où $$m_i > l_i\ ,\ \mathcal{H}[\psi] = \left\{\int_0^H \psi^p(T)\,\frac{dT}{T}\right\}^{\frac{1}{p}}$$

Alors pour les l_i entiers naturels on a

(12) $$W_p^l(E^n) \rightleftarrows B_p^S(E^m)\quad ,\quad 1 < p < \infty,$$

où $$S_j = l_i\left(1 - \frac{1}{p}\sum_{m+1}^{n}\frac{1}{l_i}\right) > 0\,.$$

Les espaces de S.L. Sobolev $W_p^l(E^n)$ peuvent être de façon naturelle, définis pour des l non-entiers.

Soient $F[f](\xi)$ la transformée de Fourier d'une fonction $f(x), l = (l_1,\ldots,l_n) \geqslant \vec{0}$.

Posons

$$\mathcal{D}^l f = F^{-1}[(i\xi)^l\ F(f)(\xi)]\,,$$

où $$(i\xi)^l = (i\xi_1)^{l_1}\ldots(i\xi_n)^{l_n}\,,(i\xi_j)^{l_j} = e^{i\frac{\pi}{2}\operatorname{sign}\xi_j}\,|\xi_j|^{l_j}$$

D'après la définition de la fonction $f \in L_p^l(E^n)$ si la norme pour f est finie, on a

$$(13) \qquad \|f\|_{L_p^l(E_n)} = \|f\|_{p,E^n} + \sum_{j=1}^{n} \|\mathscr{D}_j^{l_j} f\|_{p,E^n} .$$

Pour des entiers l_j on a $L_p^l(E_n) = W_p^l(E^n)$. Pour les espaces $L_p^l(E^n)$ on a le théorème de plongement du type (6), (10).

Une telle extension de la classification de S.L. Sobolev de $W_p^l(E^n)$ aux l non-entiers a été proposé pour $p = 2$ par N. Aronszajn et L.N. Slobodetski, et ensuite a été développée dans une série de travaux d'autres auteurs.

Dans sa forme définitive, elle a été proposée par P.I. Lizorkin. C'est lui aussi qui a démontré les théorèmes de plongement des espaces $L_p^l(E^n)$.

Les fonctions de ces classes sont aussi caractérisées en termes de propriétés de différences finies dans certaines métriques, ce qui permet de considérer ces classes dans les domaines bornées (P.I. Lizorkin).

BIBLIOGRAPHIE

[1] Соболев С.Л. — Некоторые применения функционального анализа в математической физике. ЛГУ, 1950; Из-во СО АН СССР, 1962.

[2] Никольский С.М. — Приближение функций многих переменных и теоремы вложения. Наука, Москва, 1969.

[3] Бесов О.В., Ильин В.П., Кудрявцев Л.Д., Лизоркин П.И., Никольский С.М. — Теория вложений классов дифференцируемых функций. Дифференциальные уравнения с частными производными. Труды симпозиума, посвященного 60-летию академика С.Л. Соболева. СОАН СССР, Наука, Москва, 38-63.

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FOURIER THEORY AND FLOWS

by Frank FORELLI

Let S be a locally compact, Hausdorff space and suppose R, the real line, acts on S as a topological transformation group. I will denote by T_t the homeomorphism of S that goes with t in R. Then T_{s+t} is $T_s T_t$ and (t, x) to $T_t x$ is continuous from $R \times S$ to S. Let $C_0(S)$ denote the space of continuous, complex functions f on S that vanish at infinity and let $M(S)$ denote the space of bounded, complex, Baire measures λ on S. Then R acts on these spaces by putting

$$(T_t f)(x) = f(T_{-t} x)$$

for each x in S and $(T_t \lambda) E = \lambda(T_{-t} E)$ for each Baire set E. Let $J(f)$ denote the set of F in $L^1(R)$ such that

$$\int T_t f F(t)\, dt = 0$$

and $J(\lambda)$ the set of F in $L^1(R)$ such that

$$\int T_t \lambda F(t)\, dt = 0 .$$

Then $J(f)$ and $J(\lambda)$ are closed ideals of $L^1(R)$.

DEFINITION [1]. – The spectrum of f in $C_0(S)$ is the hull of $J(f)$ and the spectrum of λ in $M(S)$ is the hull of $J(\lambda)$.

A spectrum is a closed subset of R. Suppose S is R and $T_t x$ is $x + t$. Then the spectrum of λ is the closure of the set where the Fourier transform of λ does not vanish.

I will call λ in $M(S)$ analytic if the half line $[0, \infty)$ contains the spectrum of λ. An equivalent definition is that λ is analytic if $\lambda(T_t E)$ is in $H^\infty(R)$ for each Baire set E. λ in $M(S)$ is called quasi-invariant if whenever E is a null set of λ so is $T_t E$ for each t in R.

Suppose S is R and $T_t x$ is $x + t$. Then a nonzero analytic measure and linear measure on R have the same null sets. This is the classic F. and M. Riesz theorem. It is true for any flow (R, S) when it is restated in terms of quasi-invariance [2, 3, 6].

THEOREM. – *Analytic measures are quasi-invariant.*

Let A denote the set of functions f in $C_0(S)$ such that the half line $[0, \infty)$ contains the spectrum of f. Then A is a closed subalgebra of $C_0(S)$. An equivalent definition of A is that f in $C_0(S)$ is in A if $f(T_t x)$ is in $H^\infty(R)$ for each x in S. Suppose S is the circle and T_t is the rotation of S through an angle t. Then A is the disc algebra.

To say that A is a Dirichlet algebra means that $A + \bar{A}$ is dense in $C_0(S)$. The disc algebra is a Dirichlet algebra. However, it is not hard to see that in general A is not a Dirichlet algebra.

PROBLEM. – Find necessary and sufficient conditions on the flow (R, S) for A to be a Dirichlet algebra.

The disc algebra is a maximal closed subalgebra of the algebra of continuous functions on the circle [7]. There is a generalization of this. First, the flow (R, S) is called minimal if each orbit is dense in S.

THEOREM. – *Suppose the flow (R, S) is minimal. Then A is a maximal closed subalgebra of $C_0(S)$.*

I do not know if A is a Dirichlet algebra when the flow is minimal.

Let μ be a positive measure in $M(S)$. When is μ the total variation measure of an analytic measure ? The distant future of $L^2(\mu)$ bears on this question. Let M_t be the closure in $L^2(\mu)$ of the space of functions f in $C_0(S)$ such that the half line (t, ∞) contains the spectrum of f. The closed subspaces M_t decrease with increasing t. The distant future of $L^2(\mu)$ is the intersection of the M_t taken over all t in R. It is not hard to see that if μ is the total variation measure of an analytic measure, then just the function 0 belongs to the distant future of $L^2(\mu)$. I do not know if this goes the other way. However, there is the following [4, 5].

THEOREM. – *Suppose that just 0 belongs to the distant future of $L^2(\mu)$. Then there is a Baire function g on S such that $g\bar{g} \leqslant 1$, μ is absolutely continuous with respect to $g\mu$, and $g\mu$ is analytic.*

What is not known then is whether g can always be got with $g\bar{g} = 1$.

REFERENCES

[1] DOMAR Y. — Harmonic analysis based on certain commutative Banach algebras, *Acta Math.*, 96, 1956, p. 1-66.

[2] DE LEEUW K. and GLICKSBERG I. — Quasi-invariance and analyticity of measures on compact groups, *Acta Math.*, 109, 1963, p. 179-205.

[3] FORELLI F. — Analytic and quasi-invariant measures, *Acta Math.*, 118, 1967, p. 33-59.

[4] FORELLI F. — What makes a positive measure the total variation measure of an analytic measure ?, *J. London math. Soc.* (2) (to appear).

[5] HELSON H. and LOWDENSLAGER D. — Prediction theory and Fourier series in several variables II, *Acta Math.*, 106, 1961, p. 175-213.

[6] MANDREKAR V. and NADKARNI N. — Quasi-invariance of analytic measures on compact groups, *Bull. Amer. Math. Soc.*, 73, 1967, p. 915-920.

[7] WERMER J. — On algebras of continuous functions, *Proc. Amer. Math. Soc.*, 4, 1953, p. 866-869.

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COCYCLES IN HARMONIC ANALYSIS

by Henry HELSON

The generalizations of ordinary Fourier series on the circle T

$$f(e^{ix}) \sim \sum_{-\infty}^{\infty} a_n e^{nix} \qquad (1)$$

extend in many directions. One direction is determined by the question : How much of the theory of Hardy spaces H^p is harmonic analysis, meaningful in a more general setting ? Many classical results have been shown to hold in the context of compact abelian groups whose duals are linearly ordered [13]. Some of the methods invented for these groups have gone beyond harmonic analysis to the study of function algebras [5].

It is remarkable how far one can extend results about the circle. Nevertheless one also meets new problems on this class of compact groups. The purpose of this paper is to describe some results and techniques in harmonic analysis that are trivial on the circle.

K will always denote a compact abelian group with elements $x, y, \ldots$, which is dual to a discrete group Γ with elements $\lambda, \tau, \ldots$ assumed to be a subgroup of R_d, the discrete real line. Normalized Haar measure on K is $d\sigma$. When λ in Γ is considered as a character on K it is written χ_λ. Thus Fourier coefficients and series on K are written

$$a_\lambda(f) = \int f(x)\, \overline{\chi_\lambda(x)}\, d\sigma(x), \qquad f(x) \sim \Sigma\, a_\lambda \chi_\lambda(x) \qquad (2)$$

The Lebesgue spaces based on σ are written $L^p(K)$. $H^p(K)$, $1 \leqslant p \leqslant \infty$, is the set of all f in $L^p(K)$ such that $a_\lambda(f) = 0$ for all $\lambda < 0$ in Γ. A function is called *analytic* if it belongs to some $H^p(K)$.

The parallel between T and general K is very close for analytic functions such that $a_0 \neq 0$, or more generally having a first non-vanishing Fourier coefficient. For example Szegö's theorem is true : *a non-negative summable function w has the form $|f|^2$ for some f in $H^2(K)$ with $a_0(f) \neq 0$ if and only if*

$$\int \log w \, d\sigma > -\infty. \qquad (3)$$

However (3) is false in general for the modulus of functions in $H^2(K)$. This is not entirely surprising, for (3) is invariant under automorphisms of K, which however change the class of analytic functions.

Denote by K_0 the set of elements e_t of K defined by $e_t(\lambda) = e^{it\lambda}$ ($\lambda \in \Gamma$), where t is a real number. If K is not a circle (an assumption we make from this point), the mapping from t to e_t embeds the line R continuously, homomorphically and one-one into K. The image is dense and has well-known ergodic properties.

K_0 in K carries information about the order relation in Γ. Two theorems illustrate this point. The first [9] is the best result known in the direction of the theorem of Szegö just quoted. *A function in H^1 is almost everywhere different from 0 unless it vanishes identically. A positive function w satisfies $0 < |f| \leqslant w$ a.e. for some f in H^1 if and only if*

$$\int_{-\infty}^{\infty} \log w(x - e_t) \frac{dt}{1 + t^2} > -\infty \ a.e. \tag{4}$$

The second theorem states [7] : *a function f in $L^2(K)$ belongs to $H^2(K)$ if and only if for almost every fixed x, $f(x + e_t)(1 - it)^{-1}$ belongs to the Paley-Wiener class.* (The function of t obviously belongs to $L^2(-\infty, \infty)$ for almost every x ; the requirement is that its Fourier transform should vanish on the negative real axis).

After Beurling's paper [1] it has been fruitful to study a given analytic function f, for simplicity say in $H^2(K)$, by describing the structure of the smallest closed subspace containing f and invariant under multiplications by all χ_λ, $\lambda \geqslant 0$ in Γ. More generally we investigate arbitrary closed subspaces $\mathfrak{M}$ of $L^2(K)$ admitting these multiplications. Such a subspace is called *invariant.* Unless $\mathfrak{M}$ consists simply of all functions supported on some fixed measurable subset of K we have

$$\cap \chi_\lambda \mathfrak{M} = \{0\}, \ \cup \chi_\lambda \mathfrak{M} \text{ dense in } L^2(K) \tag{5}$$

Beurling proved on the circle that every invariant subspace of H^2 has the form $q \,.\, H^2$, where q is an *inner function* : an element of H^2 such that $|q(e^{ix})| = 1$ a. e. For invariant subspaces of L^2 satisfying (5) the same statement is true, except that q is merely a *unitary function*, of modulus one but not necessarily analytic.

To prove such a result in L^2 [6, 10] one must use methods of Hilbert space rather than of function theory. The proof is surprisingly easy. As a corollary we obtain this function-theoretic result : *each f in H^2 has a factoring qg, where q is inner and g is outer* : the smallest invariant subspace containing g is H^2 itself. Function-theoretic properties of f derive mainly from its inner factor, which can be described simply.

Thus we are led to ask for a description of the invariant subspaces of $L^2(K)$ in general. The main result of [10] is that the natural analogue of Beurling's theorem is false for K the Bohr group (dual to R_d). Here there are two obvious types of invariant subspace : $q \,.\, H^2(K)$ (q a unitary function on K) and $q \,.\, H_0^2(K)$, where $H_0^2(K)$ is the subspace of all f in $H^2(K)$ for which $a_0(f) = 0$. Other invariant subspaces satisfying (5) exist, and the structure of analytic functions must be correspondingly complicated (but see Problem A below).

Here is an outline of the proof. Let $\mathfrak{M}$ be can invariant subspace satisfying (5). Denote by P_λ the orthogonal projection of $L^2(K)$ on $\chi_\lambda \mathfrak{M}$. Then $\{I-P_\lambda\}$ is a decomposition of the identity, and

$$V_t = -\int_{-\infty}^{\infty} e^{it\lambda} \, dP_\lambda \tag{6}$$

is a continuous unitary group in $L^2(K)$. If $\mathfrak{M}$ is $H^2(K)$ or $H_0^2(K)$ then V_t is a translation along K_0 : $V_t f(x) = f(x + e_t)$. Call this translation operator T_t. The form of V_t in general is

$$V_t = A_t T_t \text{ meaning } V_t f(x) = A_t(x) f(x + e_t), \tag{7}$$

where A_t is a measurable function on K for each t. These functions satisfy

$$\begin{aligned} &|A_t(x)| = 1 \ a.e. \\ &A_t \text{ varies continuously in } L^2(K) \text{ as a function of } t \\ &A_{t+u} = A_t \, . \, T_t A_u \ \text{ for each real } t, u. \end{aligned} \tag{8}$$

Such a family is called a *cocycle* on K. (It is a cocycle of dimension one in a certain cohomology group that does not interest us further). The cocycle is an object determined by $\mathfrak{M}$; conversely, any cocycle leads backwards by Stone's theorem to an invariant subspace, or possibly to two subspaces one dimension apart like $H^2(K), H_0^2(K)$.

The cocycle of $q \, . \, H^2(K)$, q a unitary function, is $q(x)\, q(x + e_t)^{-1}$. Such cocycles are called *coboundaries*. The analogue of Beurling's theorem would state that every cocycle is a coboundary. This was disproved by constructing a cocycle that is not a coboundary.

If K is not the Bohr group the situation is more complicated. First, there are cocycles that are not coboundaries for trivial reasons, so the problem has to be re-stated. Second, the construction in [10] needed the existence of an infinite linearly independent set in Γ ; thus it could not be carried out if K is, for example, a torus T^2. Kahane gave a proof in 1964 that non-trivial cocycles exist on every K. He used a Diophantine argument to circumvent the need for independent elements in Γ. More recently Gamelin has found a way to represent the elements of the cohomology group, from which he deduces the existence of *real* non-trivial cocycles on a large class of groups [5]. Keith Yale has related these cocycles to projective representations of the line, providing a connection with known results about such representations [15], and another way to construct non-trivial cocycles.

There is another relation besides the definition between a cocycle A_t and its invariant subspace $\mathfrak{M}$. After modifying $A_t(x)$ on a null-set of x for each t, we obtain a function $A(t, x)$ measurable on the product space. For almost every x this is a measurable function on the line of modulus one almost everywhere. *A function f in $L^2(K)$ belongs to $\mathfrak{M}$ if and only if $A(t, x)\, f(x + e_t)\ (1 - it)^{-1}$ belongs to the Paley-Wiener space as a function of t, for almost every x.* (If two invariant subspaces are attached to the cocycle, $\mathfrak{M}$ should be the larger one). This theorem exhibits the cocycle as an analogue of the unitary function of Beurling's theorem.

Say that a cocycle is *analytic* if $A(t, x)\ (1 - it)^{-1}$ belongs to the Paley-Wiener space in t, for almost every x. Cocycles whose complex-conjugates are analytic correspond to invariant subspaces of $H^2(K)$. Thus the analytic cocycles take the place of inner functions on the circle. Inclusion of subspaces is related to divisibility of their cocycles, just as for inner functions. These facts [7] incorporate function-theoretic information into statements about cocycles. They are not all obvious, and they can replace complicated reference to K_0 and its cosets in further development.

The structure of analytic cocycles is not yet fully known. Say that $A(t, x)$ is a Blaschke or a singular cocycle if it is, respectively, a Blaschke or a singular inner function in the upper half-plane for almost every x. An arbitrary analytic cocycle $A(t, x)$ can be factored into analytic cocycles, one of which which contains exactly the zeros of $A(t, x)$ above a given line in the upper half-plane [8]. As a corollary one can factor $A(t, x)$ into Blaschke and singular cocycles ; but actually it is easy to see that the singular part is always trivial. The consequences of this fact for function theory on K have not been explored.

The Titchmarsh convolution theorem is closely related to the structure of invariant subspaces [14]. Once the convolution theorem is formulated as a density theorem it is a consequence of the analytic fact that every inner function in the upper half-plane without singularities in the finite plane has the form e^{iuz} for some positive u. The same argument, fortified with some information about cocycles, gives this discrete result : *if α and β are square-summable sequences defined on the positive ray of R_d, neither vanishing identically on any interval $[0, \epsilon)$, then $\alpha * \beta$ does not vanish on any such interval either.*

These simple questions are unanswered.

(A) Which cohomology classes contain analytic cocycles ? Is every analytic cocycle a coboundary ? Does every class contain an analytic cocycle ?

(B) Can every analytic cocycle be properly factored ? In other words, is $H_0^2(K)$ the only maximal invariant subspace of $H^2(K)$?

(C) Does every invariant subspace have a single generator ? Has $H_0^2(K)$?

(D) Do the answers to these questions depend on K ?

All this account has referred to cocycles whose values are elements of $L^2(K)$. The projections that led to cocycles can be defined in other function spaces with good effect. This has been the case in generalizing the important theorem of F. and M. Riesz : *if μ is a complex measure on the circle such that*

$$d\mu(x) \sim \sum_0^\infty a_n e^{nix} \qquad (9)$$

then $|\mu|$ and Lebesgue measure are mutually absolutely continuous.

This elegant extension was proved by de Leeuw and Glicksberg [2] : *if μ is a complex measure on K with one-sided Fourier-Stieltjes series, then $|\mu|$ is absolutely continuous with respect to each of its translates by an element of K_0.* Their proof uses the technique of restriction to K_0 and its subsets, in order to apply known theorems on the line. Mandrekar and Nadkarni reproved the theorem [11] in the following way. We construct L^2 based on the measure $|\mu|$. The hypothesis means that the characters χ_λ with $\lambda \geqslant 0$ span a proper subspace $\mathfrak{M}$ of L^2, evidently invariant under multiplication by these characters. Then (5) holds and we can write (6). For trigonometric polynomials, at least, (7) is true with the same proof as in $L^2(K)$. (The cocycle is no longer a unitary function, of course). The theorem now follows easily.

Forelli has extended the F. and M. Riesz theorem much farther [3, 4]. K_0 is finally replaced by a flow on a topological space. Probably a good deal of function theory could be reproduced on a space with a flow and an invariant measure.

In studying discrete stationary stochastic processes, Mandrekar and Nadkarni have been led to certain remarkable measure spaces and cocycles defined in them. Their most recent work is a joint paper with Patil [12].

Cocycles have appeared before in many parts of analysis, but it may be fair to say that they were appreciated only when they were coboundaries. I have been reporting a specialized but fairly detailed body of knowledge concerning a non-trivial cohomology group.

REFERENCES

[1] Beurling A. — On two problems concerning linear transformations in Hilbert space, *Acta Math.*, 81, 1949, p. 239-255.

[2] De Leeuw K. and Glicksberg I. — Quasi-invariance and analyticity of measures on compact groups, *Acta Math.*, 109, 1963, p. 179-205.

[3] Forelli F. — Analytic and quasi-invariant measures, *Acta Math.*, 118, 1967, p. 33-59.

[4] Forelli F. — Conjugate functions and flows, *Quaterly J. Math.*, 20, 1969, p. 215-233.

[5] Gamelin T.W. — *Uniform Algebras*, Prentice-Hall, 1969.

[6] Halmos P. — Shifts on Hilbert spaces, *J. Reine Angew. Math.*, 208, 1961, p. 102-112.

[7] Helson H. — Compact groups with ordered duals, *Proc. London Math. Soc.* (3), 14 A, 1965, p. 144-156.

[8] Helson H. — Compact groups with ordered duals II, *J. London Math. Soc.*, (2), 1, 1969, p. 237-242.

[9] Helson H. and Lowdenslager D. — Prediction theory and Fourier series in several variables II, *Acta Math.*, 106, 1961, p. 175-213.

[10] Helson H. and Lowdenslager D. — Invariant subspaces, *Proc. Int. Symp. on Linear Spaces, Jerusalem*, 1960, Macmillan (Pergamon), 1961, p. 251-262.

[11] Mandrekar V. and Nadkarni M. — Quasi-invariance of analytic measures on compact groups, *Bull. Amer. Math. Soc.*, 73, 1967, p. 915-920.

[12] Mandrekar V., Nadkarni M. and Patil D. — Singular invariant measures on the line, *Studia Math.*, 35, 1970, p. 1-13.

[13] Rudin W. — Fourier Analysis on Groups, *Interscience*, 1962.

[14] Sarason D. — Invariant subspaces, Studies in Operator Theory, *Math. Assoc. Amer.* (to appear).

[15] Yale K. — *Pac. J. Math.* (to appear).

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CENTRAL IDEMPOTENT MEASURES

by Daniel RIDER [(1)]

1. Introduction.

Let G be a locally compact group. $M(G)$, the space of regular finite measures on G, forms a Banach algebra where the multiplication is given by convolution :

$$\mu * \nu(E) = \int \mu(Ex^{-1})\, d\nu(x).$$

A measure μ is *idempotent* if $\mu * \mu = \mu$.

Idempotents arise in the study of homomorphisms of group algebras [2], [3] and in the study of projections of $L^1(G)$ that commute with right translations [12 ; Chapter 3], [5].

Some ways of obtaining idempotents are :

(1) The Haar measure (normalized) of a compact subgroup is idempotent.

(2) If μ is idempotent and ψ is a continuous complex homomorphism of G then $\psi d\mu$ is idempotent.

(3) If μ and ν are commuting idempotents, then $\mu * \nu$, $\mu + \nu - \mu * \nu$ and $\delta - \mu$ are idempotents (δ is the unit of $M(G)$).

Thus starting with compact subgroups and complex homomorphisms we can obtain a family of idempotents by using 2 and 3 finitely many times.

When G is *abelian*, Cohen [1] has shown this process gives all idempotents (see [7] for a simplified proof). Several other special cases have also been studied. Wendel [15] characterized positive idempotents ; Rudin [13] and Greenleaf [3] characterized idempotents of norm 1 ; and Parthasarathy [8] characterized positive idempotents on complete separable metric (not necessarily locally compact) groups.

Two facts come from Cohen's characterization of idempotents on abelian groups (or they are easily seen directly). First, an idempotent measure on an abelian group is supported on a compact subgroup and second, an abelian group of order n has exactly 2^n idempotents. Both of these are false if "abelian" is omitted. Rudin [13] has an example of a discrete group having an idempotent with infinite support and *any* finite non-abelian group has uncountably many idempotents.

Because of these examples we will consider only central idempotents ; i.e. idempotent measures in the center of $M(G)$. The Haar measure of a compact subgroup is central if and only if the subgroup is normal. The center is much

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easier to work with than all of $M(G)$; for example, it is easily seen that if G is finite then G has exactly 2^n central idempotents where n is the number of conjugacy classes of G.

This paper is concerned with some partial solutions to the following two problems which will appear in [10] and [11].

(I) Is a central idempotent measure supported on a compact subgroup ?

(II) Characterize central idempotent measures on compact groups in terms of closed normal subgroups and characters.

2. Support groups of central idempotents.

We will say $G \in \mathrm{SCI}$ if there is a central idempotent measure on G which is not supported on a proper closed subgroup. In [9] it was shown that if G is discrete and $G \in \mathrm{SCI}$, then G is finite. This has been extended to :

THEOREM 1 [11]. – *If* $G \in \mathrm{SCI}$, *then every open normal subgroup of* G *has finite index.*

The proof uses the following lemma which solves the problem for abelian groups.

LEMMA 2. – *If* $G \in \mathrm{SCI}$, *then* G/G' *is compact where* G' *is the closure of the commutator subgroup of* G.

The proof of Lemma 2 is the same as those for abelian groups [12 ; Theorem 3.3.2] and discrete groups [14].

This is as far as I have been able to go toward solving Problem I for general groups. We will say $G \in \mathrm{SIN}$ if every neighborhood of the identity of G contains a neighborhood of the identity which is invariant under all inner automorphisms.

THEOREM 3 [11]. – *If* $G \in \mathrm{SIN} \cap \mathrm{SCI}$ *then* G *is compact.*

The proof depends in several places on the condition that $G \in \mathrm{SIN}$. For example if x is in the carrier of μ, a central idempotent supported by G, and the conjugacy class containing x does not have compact closure, then there is a sequence $\{x_n\}$ and a neighborhood U of the identity such that $\{x_n x x_n^{-1} U\}$ are disjoint. If $G \in \mathrm{SIN}$, then we can assume that $\{x_n x U x_n^{-1}\}$ are disjoint. This gives a contradiction since

$$|\mu|(x_n x U x_n^{-1}) = |\mu|(xU) > 0$$

and

$$\|\mu\| \geqslant \Sigma\ |\mu|(x_n x U x_n^{-1})$$

It follows that for $G \in \mathrm{SCI} \cap \mathrm{SIN}$, every conjugacy class has compact closure. Also, if G_1 is the connected component of G and $G \in \mathrm{SIN}$ then G/G_1 contains a compact open normal subgroup. Together with Theorem 1, this implies that G/G_1 is compact.

The proof of Theorem 3 follows from these facts and the following lemma which is a corollary to work of Grosser and Moskowitz [4].

LEMMA 4. – *If G/G', G/G_1 and the closure of every conjugacy class are all compact and $G \in$ SIN then G is compact.*

3. Central idempotents on compact groups.

Throughout G will be compact. If G is abelian then a measure μ is idempotent if and only if $\hat{\mu}$, the Fourier transform, assumes only the values 0 and 1 on Γ, the dual group of G. Cohen's result can be stated as.

THEOREM 5. – *If $E \subset \Gamma$ then the characteristic function of E is the transform of a measure if and only if E is in the coset ring of Γ.*

For (non-abelian) compact G we want to state an analogous theorem and indicate that for certain G it is true.

Let Γ denote the set of equivalence classes of irreducible unitary representations of G. For $\alpha \in \Gamma$, T_α is a member of the class α, χ_α is the character of the class and d_α the degree. Γ has a hypergroup structure (cf. [6] and [10]). For $\alpha, \beta \in \Gamma$ let $\mu_{\alpha,\beta}(\gamma)$ be the number of times T_γ appears in the decomposition of $T_\alpha \otimes T_\beta$.

If H is a closed normal subgroup of G, then $H^\perp = \{\alpha \in \Gamma : T_\alpha(x) = E$ for all $x \in H\}$, E is the identity transformation. Helgason [6] has shown that $H^\perp$ is a normal subhypergroup and every normal subhypergroup is given in this way. If $\mathcal{H} \subset \Gamma$ and $\beta \in \Gamma$ then define

$$\beta\mathcal{H} = \{\gamma : \mu_{\alpha,\beta}(\gamma) \neq 0 \text{ for some } \alpha \in \mathcal{H}\}.$$

If $\mathcal{H}$ is a normal subhypergroup, then $\beta\mathcal{H}$ is called a hypercoset.

If μ is a central measure on G, then μ has a Fourier series of the form

$$\mu \sim \Sigma\, \hat{\mu}(\alpha)\, d_\alpha \chi_\alpha(x)$$

where

$$\hat{\mu}(\alpha) = \frac{1}{d_\alpha} \int \overline{\chi_\alpha}(x)\, d\mu(x).$$

μ is idempotent provided $\hat{\mu}(\alpha)$ is always 0 or 1. If μ is idempotent, let $S(\mu) = \{\alpha \in \Gamma : \hat{\mu}(\alpha) = 1\}$. The family Ω of all sets $S(\mu)$, for central idempotent μ, is a ring of sets ; i.e., it is closed under the formations of unions, intersections and complements. The hypercoset ring is the ring generated by the hypercosets.

THEOREM 6 [10]. –

(a) *Ω contains the hypercoset ring.*

(b) *Let H be a closed normal subgroup of G with Haar measure m. Let $\beta \in \Gamma$* and

$$\frac{1}{c} = \int_H |\chi_\beta(h)|^2\, dm(h).$$

Then $d\mu(x) = c d_\beta \chi_\beta(x) dm(x)$ is a central idempotent measure on G and $S(\mu) = \beta H^\perp$.

It should be noted that in general if μ is central idempotent and $\alpha \in \Gamma$ then for no constant c is $c\chi_\alpha d\mu$ idempotent. It is not known if $\alpha S(\mu) = S(\nu)$ for some central idempotent ν.

The following conjecture is analogous to Theorem 1.

CONJECTURE 7. — *$E \subset \Gamma$ is $S(\mu)$ for some central idempotent μ if and only if E is in the hypercoset ring.*

This conjecture has been proved in [10] for certain groups including the unitary groups. The groups of unitary $n \times n$ matrices satisfy both of the following conditions.

CONDITION A. — If $x \notin Z =$ center G then $\dfrac{\chi_a(x)}{d_a} \to 0$ as $d_a \to \infty$.

CONDITION B. — Modulo multiplication by characters of degree 1, there are only finitely many representations of any given degree.

THEOREM 8. — *If G satisfies condition A and μ is a central idempotent then*

$$\mu = \nu + \lambda$$

where ν is a central idempotent supported on Z and $\hat{\lambda}(\alpha) = 0$ if d_a is large enough.

Cohen's theorem can be used to tell all about ν. The proof itself also depends on Cohen's theorem as well as Helson's translation Lemma [12 ; Lemma 3.5.1].

If G satisfies condition B then it is easily seen that the measure λ in Theorem 8 is the sum of finitely many measures of the form

$$d_\beta \chi_\beta \cdot d\theta$$

where θ is a central idempotent measure on G with $\hat{\theta}(\alpha) = 0$ for $d_a > 1$. Then θ is given by an idempotent measure on the abelian group G/G' so that Cohen's theorem may be applied again, this time to θ. This then gives.

THEOREM 9 [10]. — *If G satisfies conditions A and B, then conjecture 7 is true.*

It is a very strong condition to require that a group satisfy both conditions A and B. For example, any closed normal subgroup H of such a group must either be contained in Z or have finite index in HG'.

Recently I have been able to prove conjecture 7 for compact groups which are products of groups satisfying both conditions A and B. The proof involves the following lemma.

LEMMA 10. — *There is $\delta > 0$ such that every central idempotent polynomial P on a compact group with $\|P\|_1 > 1$ satisfies $\|P\|_1 \geqslant 1 + \delta$.*

It is known [12 ; Theorem 3.7.2] that if μ is an idempotent measure on an abelian group with $\|\mu\| > 1$ then $\|\mu\| \geqslant \sqrt{5}/2$. Lemma 10 seems to indicate that a similar statement should be true for central idempotent measures. On the other hand, if G is the non-abelian group with six elements, every number $\geqslant 1$ is the norm of some idempotent on G.

REFERENCES

[1] Cohen P.J. — On a conjecture of Littlewood and idempotent measures, *Amer. J. Math.*, 82, 1960, p. 191-212.

[2] Cohen P.J. — On homomorphisms of group algebras, *Amer. J. Math.*, 82, 1960, p. 213-226.

[3] Greenleaf F.P. — Norm decreasing homomorphisms of group algebras, *Pacific J. Math.*, 15, 1965, p. 1187-1219.

[4] Grosser S. and Moskowitz M. — On central topological groups, *Trans. Amer. Math. Soc.*, 127, 1967, p. 317-340.

[5] Helgason S. — Topologies of group algebras and a theorem of Littlewood, *Trans. Amer. Math. Soc.*, 86, 1957, p. 269-283.

[6] Helgason S. — Lacunary Fourier series on noncommutative groups, *Proc. Amer. Math. Soc.*, 9, 1958, p. 782-790.

[7] Itô T. and Amemiya J. — A simple proof of the theorem of P.J. Cohen, *Bull. Amer. Math. Soc.*, 6, 1964, p. 774-776.

[8] Parthasarathy K.R. — A note on idempotent measures in topological groups, *J. London Math. Soc.*, 42, 1967, p. 534-536.

[9] Rider D. — Central idempotents in group algebras, *Bull. Amer. Math. Soc.*, 72, 1966, p. 1000-1002.

[10] Rider D. — Central idempotent measures on unitary groups, *Canad. J. Math.* (to appear).

[11] Rider D. — Central idempotent measures on SIN groups, *Duke Math. J.* (to appear).

[12] Rudin W. — Fourier Analysis on Groups, *Interscience*, New York, 1962.

[13] Rudin W. — Idempotents in group algebras, *Bull. Amer. Math. Soc.*, 69, 1963, p. 224-227.

[14] Rudin W. and Schneider H. — Idempotents in group rings, *Duke Math. J.*, 31, 1964, p. 585-602.

[15] Wendel J.G. — Haar measure and the semigroup of measures on a compact group, *Proc. Amer. Math. Soc.*, 5, 1954, p. 923-929.

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HARMONIC ANALYSIS IN POLYDISCS

by Walter RUDIN

Introduction.

This lecture describes some features of

(harmonic analysis) $\cap$ (function theory in polydiscs).

The notation is as in [6]. Points $z \in \mathbb{C}^n$ have coordinates $z_i (1 \leqslant i \leqslant n)$. $z \in T^n$, U^n, V^n iff $|z_i| = 1$, $|z_i| < 1$, $|z_i| > 1$ for all i. $M_R(T^n)$ is the set of all real Borel measures on T^n. Each $\mu \in M_R(T^n)$ has (a) its Poisson integral $P[d\mu]$, an n-harmonic function in U^n, (b) its Fourier transform $\hat{\mu}$, a function on Z^n. $\mu \in RP(T^n)$ iff $P[d\mu] \in RP(U^n)$, the space of all real parts of holomorphic functions in U^n. This happens exactly when $\hat{\mu} = 0$ outside $Z_+^n \cup (-Z_+^n)$, where $k \in Z_+^n$ iff $k_i \geqslant 0$ for all i. *Note that* $RP(T) = M_R(T)$, *but that* $RP(T^n)$ *is a small subspace of* $M_R(T^n)$ *if* $n > 1$.

Suppose f is holomorphic in U^n. Then $f \in H^\infty(U^n)$ iff f is bounded ; $f \in N(U^n)$ [or $H^p(U^n)$] iff $\log^+ |f|$ [or $|f|^p$] has an n-harmonic majorant in U^n ; $f \in A(U^n)$ iff f is continuous on the closure of U^n and holomorphic in U^n. If $f \in N(U^n)$ ($f \not\equiv 0$) then $\log |f|$ has a *least* n-harmonic majorant $u[f]$. Those $g \in H^\infty(U^n)$ whose radial limits have absolute value 1 a.e. on T^n are called *inner functions*. An inner function is *good* iff $u[g] = 0$. (When $n = 1$, the good inner functions are exactly the Blaschke products).

Localization.

If $\mu \in M(T^n)$ and if E is an open set in T^n that does not intersect the support of μ, it may nevertheless happen that the radial limits of $P[d\mu]$ are not 0 at some points of E. This "failure of localization" occurs, for instance, when $n > 1$ and μ is a point mass [6 ; p. 25]. $RP(T^n)$ behaves better :

THEOREM 1. – *Let $E \subset T^n$ be open. There is an open set Ω in $\mathbb{C}^n$, $\Omega \supset U^n \cup E \cup V^n$, with the following property : If $\mu \in RP(T^n)$ and E does not intersect the support of μ, then $P[d\mu]$ has an n-harmonic extension in Ω which is 0 on E.*

Sketch of proof : $P[d\mu] = u = \mathrm{Re}\, f$ for some holomorphic f in U^n. If $u_r(w) = u(rw)$ ($w \in T^n$) then $u_r \to d\mu$ weak*, as $r \nearrow 1$. So f has pure imaginary distribution limit in E. The hypotheses of the "edge-of-the-wedge" theorem [2] are thus satisfied by f and the reflected function f_0 in V^n. Hence f_0 is an analytic continuation of f, across E. That Ω is independent of f is part of the conclusion of the edge-of-the-wedge theorem.

Division problems.

Two holomorphic functions f_1 and f_2 in U^n are said to *have the same zeros* if both f_1/f_2 and f_2/f_1 are holomorphic. The following results illustrate the curious relation that exists between the zeros of a function f on the one hand and the nature of $u[f]$ on the other. The proofs are in [6 ; § 5.4].

THEOREM 2. –

(a) *A holomorphic function f in U^n has the same zeros as some $h \in H^\infty(U^n)$ iff* $\log|f| \leqslant u$ *for some* $u \in RP(U^n)$. *There exists* $f \in \bigcap_{p<\infty} H^p(U^2)$ *which violates this condition.*

(b) *$f \in N(U^n)$ has the same zeros as some good inner function g iff $u[f] \in RP(U^n)$. In that case, g is unique (up to multiplication by constants). Also,* $\|f/g\|_p = \|f\|_p$ *for* $0 < p \leqslant \infty$.

(c) *Every $f \in A(U^n)$ has the same zeros in U^n as some inner function, but there exists $f \in H^\infty(U^n)$ for which this is false.*

(d) *Suppose f is a polynomial in $\mathbb{C}^n$, $f \not\equiv 0$. Then the condition* (b) *holds iff no irreducible factor of f has zeros in both U^n and V^n.*

As regards the example in (a) [6 ; Th. 4.1.1] it is not known whether every $f \in H^p(U^n)$ has the same zeros as some $h \in H^q(U^n)$ if $p < q < \infty$. A counter-example with $p = 1$, $q = 2$, $n = 2$ whose zero-set is an *irreducible* subvariety of U^2 would give an $f \in H^1(U^2)$ which is not a product of two H^2-functions. This non-factorization phenomenon has so far been exhibited only for $n \geqslant 4$. [6 ; Th. 4.2.2].

Note that (b) always holds when $n = 1$. It follows from (b) that good inner functions are determined by their zeros. Concerning their regularity on T^n, we have

THEOREM 3. – *If g is a good inner function in U^n, W is open in $\mathbb{C}^n$, and g has no zeros in $U^n \cap W$, then g extends holomorphically across $T^n \cap W$.*

To prove this one first shows that $\log|g|$ has distribution limit 0 on T^n, since g is good. The reflection theorem can then be applied to the holomorphic function $\log g$ in $U^n \cap W$, as in Theorem 1.

Invariant subspaces.

A closed subspace S of $H^2(U^n)$ is invariant if multiplication by polynomials carries S to S. Beurling's theorem about the shift operator describes these spaces completely when $n = 1$. If $n > 1$, there exist invariant subspaces that contain no bounded function (other than 0) [6 ; p. 71], there exist some that are not finitely generated [6 ; p. 72], and there exist outer functions in $H^2(U^n)$ that generate *proper* invariant subspaces [6 ; Th. 4.4.8]. $\left(\textit{Outer} \text{ means : } \log|f(0)| = \int_{T^n} \log|f|\right)$. The invariant subspaces of finite codimension can be completely described in terms of polynomial ideals [1], [6 ; Th. 4.5.2].

These results were all proved by working inside U^n. Recently, C.A. Jacewitz has constructed an invariant subspace that cannot be generated by a single function, although it is generated by two functions that have no common zero in U^2, namely by $f_1(z, w) = g(z)$, $f_2(z, w) = w$, where g is a non-constant singular inner function in U. His (unpublished) proof works entirely on T^2. It uses some of the Helson-Lowdenslager theory of H^2-spaces on compact abelian groups with ordered duals ; two orders on Z^2 are used, given by the right half plane and the upper half plane, respectively.

Interpolation sets on T^n

A compact $K \subset T^n$ is an (I)-set iff every continuous function on K extends to an $f \in A(U^n)$. (I)-sets are known to be the same as peak sets, or peak-interpolation sets, or zero sets of $A(U^n)$; they are also those that cannot be charged by any measure orthogonal to $A(U^n)$; [6 ; Th. 6.1.2]. The best general theorem in this area is Forelli's [3] [6 ; Th. 6.2.2]. Its statement is fairly complicated and is omitted, because of lack of space. See also [4].

It is still a difficult problem to decide for a given K whether it is an (I)-set or not. For arcs on T^2, however, our information is now fairly good :

THEOREM 4. – *Let* $K = \{(e^{it}, e^{ih(t)}) : a \leqslant t \leqslant b\}$, *where h is a real continuous* function on $[a, b] \subset [-\pi, \pi]$.

(a) *If h is strictly decreasing, then K is an* (I)*-set.*

(b) *If h is strictly increasing and* $h' = 0$ a.e., *then K is an* (I)*-set.*

(c) *If h is increasing and has a continuous third derivative, then K is not an* (I)*-set.*

Part (a) is Th. 6.3.5 of [6] ; its proof uses Forelli's theorem. A student of mine, C.S. Davis, proved (b) ; the relevant property of K is that K is a union of two Borel sets A, B, such that A projects vertically in a $1-1$ manner onto a set of linear measure 0, and the same holds for the horizontal projection of B.

(c) Is an unpublished result of Lennart Carleson. (See also [6 ; Th. 6.3.4]). Suppose first that $[a, b] = [-\pi, \pi]$, $h > 0$, $h(-\pi) = -\pi$, $h(\pi) = \pi$, $h'(-\pi) = h'(\pi)$, $h''(-\pi) = h''(\pi)$. If $f \in A(U^2)$ and $g(e^{it}) = f(e^{it}, e^{ih(t)})$, two integrations by part and an application of the Schwarz inequality lead to the estimate $g(n) = O(|n|^{-1})$ as $n \to -\infty$. Since this estimate fails for some $g \in C(T)$, K is not an (I)-set. (When $h(t) = t$, then $g(n) = 0$ for $n < 0$. This is the clue !) The theorem is deduced from this special case by moving the problem from T^n to R^n, as is done in the proof of Forelli's theorem.

The smoothness assumption in (c) can be somewhat weakened : it is enough to assume that h' is the integral of a function of bounded variation. But the gap between (b) and (c) is not filled. The following very recent theorem of Torbjörn Hedberg [7] is a step in this direction.

THEOREM 5. – *If ω is a positive subadditive increasing function on* $(0, \infty)$ *such that* $\omega(t)/t \to \infty$ *as* $t \to 0$, *then there exist increasing functions* h_1 *and* h_2 *on* $[a, b] \subset [-\pi, \pi]$ *such that.*

(i) $\sup\limits_{|s-t|<\delta} |h_i(s) - h_i(t)| = O(\omega(\delta))$ *as* $\delta \to 0$, *and*

(ii) *the set* $K = \{(e^{ih_1(t)}, e^{ih_2(t)}) : a \leqslant t \leqslant b\}$ *is an* (I)-*set. In fact, the restrictions to* K *of functions of the form*

$$f(z_1, z_2) = g_1(z_1) + g_2(z_2) \qquad (g_i \in A(U))$$

cover $C(K)$.

The proof (which is similar to one that Kahane had used in [8]) actually shows, with respect to a certain natural metric defined in terms of ω, that the set of pairs (h_1, h_2) that satisfy (i) but not (ii) is of the first category.

An extreme point problem.

Let K_n be the set of all positive $\mu \in RP(T^n)$ with $\mu(T^n) = 1$. K_n is convex and weak*-compact, hence has extreme points. What are they ? (A good answer would be informative ; see [5], for example). When $n = 1$, they are the unit point masses. When $n > 1$, no $\mu \in RP(T^n)$ has a discrete component. Here are some facts about K_2 :

(a) Let r and s be positive integers, let $G_{r,s}$ be the circle subgroup of T^2 consisting of $(e^{-ir\theta}, e^{is\theta})$. The Haar measure of $G_{r,s}$, multiplied by suitable trigonometric polynomials, gives extreme points of K_2. Also, translates of extreme measures are extreme. These are the simple ones.

(b) Put $g(w) = \exp\{(w + 1)/(w - 1)\}$. The real part of $[1 - zg(w)]^{-1}[1 + zg(w)]$ is $P[d\sigma]$ for some $\sigma \in K_2$. Its support S is the union of the circle on which $w = 1$ and the spiral in T^2 on which $zg(w) = 1$. The set of all $\mu \in K_2$ supported by S is convex and weak*-compact ; each of its extreme points in an extreme point of K_2. It follows that K_2 has extreme points that are supported by S and have mass on the spiral.

(c) It is not known whether every extreme point of K_2 is singular with respect to the Haar measure of T^2.

REFERENCES

[1] AHERN P.R. and CLARK D.N. — Invariant subspaces and analytic continuation in several variables, *J. Math. and Mech.*, 19, 1970, p. 963-969.

[2] BROWDER F.E. — On the "Edge of the wedke" theorem, *Candian PJ. Math.* 15, 1963, p. 125-131.

[3] FORELLI F. — Measures orthogonal to polydisc algebras, *J. Math. and Mech.*, 17, 1968, p. 1073-1086.

[4] GLICKSBERG I. — Extensions of the F. and M. Riesz theorem, *J. of Functional Analysis*, 5, 1970, p. 125-136.

[5] HEINS M. — A lemma on positive harmonic functions, *Annals of Math.*, 52, 1950, p. 568-573.

[6] RUDIN W. — *Function Theory in Polydiscs*, Benjamin, 1969.

[7] HEDBERG T. — Sur les réarrangements de fonctions de la classe A et les ensembles d'interpolation pour $A(D^2)$, *C. R. Acad. Sci. Paris*, 270, 1970, p. 1491-1494.

[8] KAHANE J.-P. — Sur les réarrangements de fonctions de la classe A, *Studia Math.*, 31, 1968, p. 287-293.

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GROUPES DE FONCTIONS CONTINUES EN ANALYSE HARMONIQUE

by Nicholas VAROPOULOS

Soit G un groupe abélien compact et soit $E \subset G$ un sous-ensemble fermé de E, on dit alors que E est un ensemble de type $H_\alpha (0 < \alpha \leqslant 1)$ (un Helson-α) si pour toute $f \in C(E)$ de norme uniforme inférieure à 1 ($\|f\|_\infty < 1$) il existe $\varphi \in A(G)$ telle que

$$\|\varphi\|_A \leqslant \alpha^{-1} \qquad \varphi|_E = f \quad .$$

On dit que E est un ensemble de Kronecker si pour toute $f \in C(E)$ de module 1 et tout $\epsilon > 0$ il existe $\chi \in \hat{G}$ tel que

$$\sup_e |\chi(e) - f(e)| \leqslant \epsilon \ .$$

On note $Gp(E)$ le sous-groupe de G engendré par E.

Remarquons qu'un ensemble de Kronecker est de type H_1. Les théorèmes que nous démontrerons sur les ensembles de Helson sont les suivants.

THEOREME 1. — *Soit G un groupe abélien compact ; soit E un ensemble métrisable de type H_α de G et soit aussi $E_1 \subset G$ un compact de G disjoint de E. Alors pour tout $\epsilon > 0$ il existe une fonction $f \in A(G)$ telle que*

(i) $\|f\|_A \leqslant C_\alpha \epsilon^{-1}$ (C_α une constante ne dépendant que de α)

(ii) $f(e) \equiv 1 \ \ \forall e \in E$

(iii) $|f(e_1)| \leqslant \epsilon \ \ \forall e_1 \in E_1$.

THEOREME 2. — *Soit G un groupe abélien compact et soit E un ensemble métrisable de type H_α de G, soit aussi $E_1 \subset G$ un ensemble tel que :*

$$E_1 \cap Gp(E) = \emptyset$$

alors pour tout $\epsilon > 0$ il existe une fonction $f \in A(G)$ telle que

(i) $\|f\|_A \leqslant \alpha^{-2}$

(ii) $|f(e) - 1| \leqslant \epsilon \ \ \forall e \in E$

(iii) $|f(e_1)| \leqslant \epsilon \ \ \forall e_1 \in E_1$.

Les théorèmes 1 et 2 ne sont démontrés dans toute leur généralité qu'à la fin.

Le travail essentiel consiste à démontrer le cas particulier du théorème 1 et 2 dans le cas où $\alpha = 1$ (ou même dans le cas où E est un ensemble de Kronecker).

C'est le théorème 2 qui est fondamental et qui nécessite pour sa démonstration des techniques de groupes de fonctions continues. A partir du théorème 2 on déduit le théorème 1 assez facilement.

Comme corollaire du théorème 2 on obtient le théorème suivant :

THEOREME 3. — *Soit G un groupe abélien compact et soit E_1, E_2 deux ensembles métrisables de type H_α, alors l'ensemble $E_1 \cup E_2$ est un ensemble de type H_β où β ne dépend que de α.*

On peut démontrer que le théorème 2 est le meilleur possible dans un certain sens. En effet, on a le théorème suivant :

THEOREME 4. — *Il existe G un groupe abélien compact métrisable et K, E deux sous-ensembles fermés de G tels que K est ensemble de Kronecker et*

$$E \cap Gp(K) = \emptyset$$

et que, pour toute fonction $f \in A(G)$ satisfaisant

$$f(e) = 0 \quad \forall e \in E \quad , \quad f(k) = 1 \quad \forall k \in K ,$$

on a $\|f\|_A \geqslant C > 1$ où C est une constante numérique.

A partir du théorème 4 on peut déduire des résultats sur les algèbres $\widetilde{A}(E)$; plus exactement pour tout sous-ensemble fermé $E \subset G$, on note

$$I(E) = \{f \in A(G) \mid f^{-1}(0) \supset E\} \subset A(G) ,$$

$$A(E) = A(G)/I(E) ;$$

munie de la norme quotient $A(E)$ est une algèbre de Banach. Notons aussi $\widetilde{A}(E)$ l'algèbre de Banach

$$\widetilde{A}(E) = \{f \in C(E) \mid \exists \{f_n \in A(E)\}_{n=1}^{\infty} \ t.q. \ \|f_n - f\|_\infty \xrightarrow[n\to\infty]{} 0, \sup_n \|f_n\|_A < +\infty$$

munie de la norme canonique. A partir des théorèmes 2 et 4 on a alors le théorème suivant.

THEOREME 5. — *Soit G un groupe infini abélien compact ; il existe alors $E \subset G$, sous-ensemble fermé, tel que l'algèbre $A(E)$ n'est pas une sous-algèbre fermée de $\widetilde{A}(E)$.*

Le théorème 5 a des applications au problème de la synthèse harmonique. Plus exactement, on dit que $E \subset G$, sous-ensemble fermé, est un ensemble de synthèse harmonique si pour toute pseudomesure $S \in PM(E)$ (i.e. $\hat{S} \in \ell^\infty(\hat{G})$; supp $S = Sp\, \hat{S} \subset E$) il existe un filtre de mesures $\{\mu_\alpha \in M(E)\}_{\alpha \in A}$ tel que

$$\mu_\alpha \xrightarrow[\alpha \in A]{} S \quad \text{pour la topologie} \quad \sigma(PM ; A) \quad .$$

On dit que E est un ensemble de synthèse bornée si pour toute pseudomesure $S \in PM(E)$ il existe un filtre de mesures $\{\mu_\alpha \in M(E)\}_{\alpha \in A}$ tel que

$$\begin{cases} \mu_\alpha \to S & \text{pour la topologie} \quad \sigma(PM ; A) \\ \sup_\alpha \|\hat{\mu}_\alpha\|_\infty < +\infty & . \end{cases}$$

A partir du théorème 5 on obtient le théorème suivant.

THEOREME 6. — *Soit G un groupe infini abélien compact, il existe alors $E \subset G$, sous-ensemble fermé, qui est de synthèse harmonique sans être de synthèse bornée.*

Les techniques des groupes de fonctions continues qu'on utilise pour démontrer le théorème 2 consistent essentiellement en deux étapes. La première étape consiste à démontrer l'analogue du théorème de Bochner pour le groupe multiplicatif

$$S(K) = \{f \in C(K) \quad ; \quad |f(k)| = 1 \quad \forall k \in K\}$$

où K est un compact totalement discontinu. La seconde étape établit un homomorphisme entre $S(E)$ et $S(E_1)$ où E et E_1 satisfont les hypothèses du thèorème 2.

Des démonstrations différentes des théorèmes 1 et 2 (ne faisant pas intervenir des techniques des groupes des fonctions continues) peuvent être obtenues dans le cadre des groupes classiques $G = \mathbf{T}^n$.

BIBLIOGRAPHIE

VAROPOULOS N. — Groups of continuous functions in harmonic analysis, *Acta Math.*, 125, 1970, p. 109-154.

VAROPOULOS N. — *C. R. Acad. Sci.*, 271, 1970, p. 251-253.

DRURY S. W. — *C. R. Acad. Sci.*, 271, 1970, p. 162-163.

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D 5 - THÉORIE DU POTENTIEL PROCESSUS DE MARKOV

BOUNDARY BEHAVIOR OF MARKOV CHAINS AND ITS CONTRIBUTIONS TO GENERAL PROCESSES

by Kai Lai CHUNG [1]

In contemporary studies of homogeneous Markov processes on a topological space, under the name of Hunt or standard process, it is assumed that the only discontinuities of (almost all) sample functions are jumps, for all time or up to the lifetime of the process, respectively. If the same assumption is made on a Markov chain, where the state space is discrete and may be labeled by the integers, this results in a rather trivial situation long since "solved". If other types of discontinuity are allowed, then the typical sample function will have infinity as a limiting value when such a discontinuity is approached, from one or both directions of time. Various ways of reaching and returning from infinity should then be distinguished, and the consequent ramification has been called a boundary in analogy with classical potential theory. The problem is then to set up a suitable boundary and investigate the behavior of the sample functions relative to it. For the proper object of study of stochastic processes is the collection of sample functions or paths – it is through the underpinning a groundwork of paths that modern probability theory makes its most original contribution to mathematics[2].

It is easy to give a formulation in a more general context. For instance, given a standard process X with its lifetime T, we may inquire after the structure of all homogeneous Markov processes X^* with X as its initial portion and hence (if some form of strong Markov property is to hold for X^*) as a germinal constituent, in the sense that the paths should behave as they do in X away from a certain boundary, or again that X^* should be decomposable into X and a boundary derived from X. In the case of Markov chains, X may be a minimal chain (see [3] for this and other standard terminology) whose paths are of the trivial type mentioned above, controlled by an initial derivative matrix Q which will be assumed to be conservative. This leads to the so-called complete construction problem : given Q, to find all transition semigroups P such that $P'(0) = Q$. This formulation is popular among those mathematicians who wish to stake out an easily stated analytic pro-

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([1]) Research supported by the Office of Scientific Research Office of the United States Air Force, under AFOSR contract F44620-67-C-0049.

([2]) In this vein it is curious to compare the works on stochastic processes by Lévy and Doob on one hand, and Feller on the other. Behind Feller's analytic doing, however, there always lurks his thinking in terms of paths.

blem devoid of probabilistic content. However, the way to all construction is of course an adequate understanding of the fundamental structure of the would-be-constructed object, as any school child who has figured out a regular hexagon should know. From my point of view, therefore, the main thing is to describe the evolution of time of a process, in other words to trace a typical path through its ups and downs at the boundary. Definitive results are known only in the case where the exit boundary is finite or discretely countable (see [2], [4], [9], [13]). It is probably inevitable that as more general cases are treated, the finer details will become blurred, and it is not clear what kind of meaningful results can be achieved in utter generality.

We have yet to define a boundary. The word brings to mind several cognate names in other contexts, and the tendency is strong nowadays to fit a ready-made blueprint onto an emerging situation. This has its obvious advantages, but one runs the risk of losing sight of a green field because of skyscrapers and superhighways, figuratively speaking. Since the specific case that can be handled is simple enough, I choose to describe it intuitively and without punctilio. Assuming then that the minimal chain is transient as we may, it is known (after Blackwell) that the path will ultimately enter and remain in an invariant set, namely a set A such that liminf $\{X_t \in A\}$ = limsup $\{X_t \in A\}$ almost surely (a.s.) as t increases to T, where T is the lifetime of the minimal chain and is also the first boundary hitting time of the whole process. Note that T is a predictable time (see [10]) as the limit of a sequence of strictly increasing jump times. Now if we assume that there is only a finite number of atomic invariant sets that can be reached in finite time, we will identify each of them with a boundary point and say that $X(T-) = b$ if b corresponds to the set A above. The path has thus crossed the boundary **B** at b and the question is what it does thereafter. A classification of boundary points into "sticky" and "nonsticky" ones will be made. The boundary point b is sticky iff after first hitting b the path must a.s. hit it infinitely many times immediately afterward. b is nonsticky iff after first hitting b the path must a.s. not hit it again for a strictly positive time. This dichotomy is a form of special zero-one law (which does not hold in general as for a standard process). The distinction is important because in the sticky case it precludes the possibility of considering successive hits after the first. To circumvent this difficulty, a simple but crucial device is used as follows. Instead of successive hitting, we consider the successive "switching" of boundary encounter ; namely, after the first hit at T_1 we define T_2 to be the first time (an infimum in the usual manner) the path hits a boundary point different from that of the first hit, T_3 to be the first time the path hits a boundary point different from that of the second hit (but may of course be the same as the first hit), and so forth. This recursive definition is complicated by the possibility that a switch may occur instantly, for instance T_1 may equal T_2 if $X(T_1-) = b_1$ but the path hits some other boundary point b_2 in $(T_1, T_1 + \epsilon)$ for every $\epsilon > 0$. This can happen only if b_1 is nonsticky and b_2 is sticky, hence an instant switch cannot happen twice in succession. Thus the sequence of switching times T_n must strictly increase at least every other time. They cannot accumulate to a finite limit, for at such a time the path would have to oscillate between distinguishable boundary points, which is a.s. impossible by a martingale argument. Thus either $T_n = \infty$ from some random value of n on, or $T_n \nearrow \infty$ a.s. We have therefore partitioned the time axis $[0, \infty)$ into disjoint abutting subintervals :

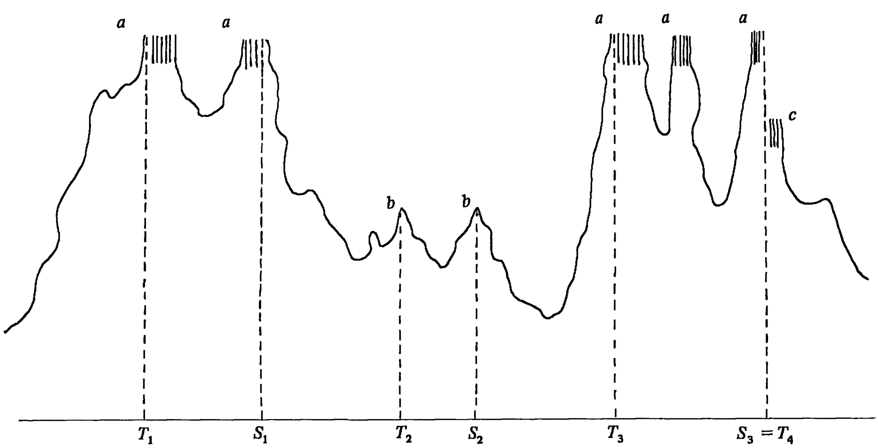
$$[0, T_1) \cup [T_1, T_2) \cup [T_2, T_3) \cup \dots$$

in each of which at most one particular boundary point can be hit. Such a reduction to one is clearly desirable.

If $X(T_1 -) = a$ we call the process in $[T_1, \infty)$ the post-a process ; and we call the process in $[T_1, T_2)$ the sub-a process. It transpires by virtue of a strengthened Markov property applicable at the boundary, to be discussed below, that whenever $X(T_n-) = a$ the process in $[T_n, \infty)$ is stochastically equivalent to the post-a process, and the process in $[T_n, T_{n+1})$ is equivalent to the sub-a process. Clearly, a post-a process is just the process starting at the boundary point a, and a sub-a process is this process killed at $\mathbf{B} - \{a\}$. Now consider a typical nondegenerate subinterval and denote it by $[T, T')$. We know by definition that

$$X(T-) = a, \quad X(T'-) = b$$

where $a \neq b$; we know also that the path does not hit any boundary point except possibly a in the interval. Let the last hit (defined as a supermum) of a be S. This may coincide with T (which can happen only if a is nonsticky) or with T' (which can happen only if b is sticky). The following picture illustrates the various possibilities :



We shall indicate how the basic quantities for the process can be derived from the preceding description of the paths in three stages. The reader will have to consult [2] or [4] for formal definitions of the symbols below. From the first hitting of the boundary, we get

$$(1) \qquad p_{ij}(t) = f_{ij}(t) + \sum_{a \in \mathbf{B}} \int_0^t l_i^a(s)\, \xi_j^a(t-s)\, ds \; ;$$

where $\Pi = (p_{ij})$ is the transition function of the whole chain, $\Phi = (f_{ij})$ that of the minimal chain, l^a is the first hitting time density at a, which is an exit law for Φ, ξ^a is the normalized entrance law for the post-a process. From the switching, we get

$$(2) \qquad \xi_j^a(t) = \rho_j^a(t) + \sum_{b \in \mathbf{B}} \int_0^t F^{ab}(ds)\, \xi_j^b(t-s)\ ;$$

where ρ^a is the entrance law of the sub-a process (which can be normalized by adjoining the usual death point), $F^{ab}(dt)$ is the switching time distribution from a to b. From the last exit in each subinterval, we get

$$(3) \qquad \rho_j^a(t) = \int_0^t E^a(ds)\, \eta_j^a(t-s)\ ;$$

where $E^a(dt)$ is the distribution of S in the sub-a process (provided a is not a recurrent trap), and η^a is the canonical Φ-entrance law linked to the exit a, to be discussed below.

Putting together these three formulas and introducing Laplace transforms for conciseness, we obtain the complete decomposition formula

$$(4) \qquad \hat{p}_{ij}(\lambda) = \hat{f}_{ij}(\lambda) + \sum_{a \in \mathbf{B}} \sum_{b \in \mathbf{B}} \hat{l}_i^a(\lambda)\, [(1 - \hat{F}(\lambda))^{-1} \hat{E}(\lambda)]_{(a,b)}\, \hat{\eta}_j^b(\lambda)\ ;$$

where $[1 - \hat{F}(\lambda)]^{-1}\, \hat{E}(\lambda)$ is a matrix on $\mathbf{B} \times \mathbf{B}$. This formula is the key to the construction problem mentioned earlier.

For a thorough analysis of the movement of the paths, certain critical combinations of the quantities above, and some new ones such as Π-exit laws, should be introduced. These become quite technical and so rather than going into them I will now discuss some of the ideas arising from this boundary study which will be found useful, indeed has already been, in the general theory of Markov processes.

The very first step in crossing the boundary involves a form of strengthened Markov property, specifically : if T is the first boundary hitting time, $\mathcal{F}_{T-}$ the strict pre-T field, $\mathcal{F}'_T$ the post-T field, then we have for every $\Lambda \in \mathcal{F}_{T-}$ and $M \in \mathcal{F}'_T$:

$$P\{\Lambda \cap M \mid X(T-)\} = P\{\Lambda \mid X(T-)\}\, P\{M \mid X(T-)\}\ ;$$

or equivalently for every $t > 0$ and A a measurable set of the state space :

$$P\{X(T+t) \in A \mid \mathcal{F}_{T-}\} = P_t(X(T-), A)\ ;$$

where $P_t(x, A)$ is the usual notation for a transition probability. Observe that this differs from the usual strong Markov property in that $T-$ replaces $T+$ everywhere. We recall that such a property is known to hold for a homogeneous Markov chain in its right-lower-semi-continuous version, whenever $X(T+)$ belongs to the state space (see [1], [3]). This is not necessarily the case at a boundary hitting time, whence the need for a new departure(1). Although much work was done

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(1) There are brief remarks about the boundary in [1], and some illustrations of the problem in [3]. In retrospect, the approaches to a boundary theory for Markov processes have progressed rapidly.

in the early days on the strong Markov property, this seems to be the first entry (see [2]) of a left-oriented version to deal with changed circumstances(1). In fact, although the right field $\mathcal{F}_{T+}$ has been in use since the beginning of Hunt's theory, its natural companion $\mathcal{F}_{T-}$ and the concomitant predictable time such as the T in question was a more recent addition (see [5] and [10]). Later it turned out that this is the form of Markov property, named "moderate Markov property", that holds in general when a strong Markov process with right continuous paths having left limits is reversed in time (see [6] and [11]). It is not a consequence of the other, right-oriented form even when both are meaningful, but it holds for a Hunt process as well, from which quasi-left continuity follows at once. There is now reason to think that the moderate Markov property, rather than the customary quasi-left continuity, is entitled to the status of a preliminary hypothesis. Details of this suggestion will be developed elsewhere. (2)

An interesting case is that of an instant switch already mentioned. On the set where the first hitting of a is also an incipient hitting of b, the usual strong Markov property also holds as if $X(T+) = b$. At least in some compactification (see [15]), nonsticky boundary points coincide with branch points and the instant switch becomes a jump from a to b. Now the existence of branch points is known from abstract considerations (see [14]), but the boundary theory furnishes genuine examples of them so that their admittance to the general theory seems imminent. Instant switch from a last exit time, rendering the possibility of $T = T'$ in the discussion above, is a related phenomenon, the difference being that such a time is inaccessible instead of predictable.

Under our hypotheses, the Φ-exit laws l^a and the Π-entrance laws ξ^a are immediately definable from their probabilistic meanings. An essential difficulty, analytically as well as stochastically, is to find the Φ-entrance laws η^a. In the approach sketched above, these are picked out, so to speak, by the paths themselves, one for each exit. (There is no need of an entrance boundary, even as to its existence, although this may be a good thing to have (see [8]).) This derivation depends on the important observation that the potential of the sub-a process is finite, namely :

$$\forall j : e_j^a = \int_0^\infty p_j^a(t)\, dt < \infty ;$$

except when a is a recurrent trap in which case the e^a below is to be replaced by a quasi-stationary measure for the post-a process (identical to the sub-a process). Now e^a is an excessive measure for Φ and the excess has a continuous derivative which is precisely η^a :

$$e^a - e^a\,\Phi(t) = \int_0^t \eta^a(s)\, ds\ .$$

As a hindsight, it can be shown that $\eta^a\ (t)$ is also the limit

$$\lim_{s \downarrow 0} \frac{\xi^a(s)\,\Phi(t-s)}{1 - \langle\, \xi^a(s)\,, L^a(\infty)\,\rangle}\ .$$

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(1) Compactifiers were of course hellbent on regularization to an old pattern, and ignored the opening to the left, but this has now been noticed (see [7]).

(2) At the *Convegno sul Calcolo delle probabilità* in Rome, March 1971.

This is intuitively more suggestive but perhaps conceals a fundamental limiting process. The method of converting a generally infinite potential for the post-a process $\int_0^\infty \xi^a(t)\,dt$ into the generally finite sub-a potential may be worth investigation. It is done by imposing a taboo set (here the boundary set $B - \{a\}$), a familiar device in Markov chain theory. The standard method of considering λ-potentials is of course just the resolvent theory, which has proved to be such a powerful tool. But calculations with resolvents tend to be purely algebraic manipulations, since it corresponds to a killing at an exponential time totally independent of and therefore alien to the process. By contrast, killing under a suitable taboo leads to simple probabilistic interpretations and more easily recognizable results(1). I do not know the scope of applicability of such an alternative for the general theory, but submit that we be on the look-out for it.

The idea of a last exit time from the boundary plays a curious role in the final step of the decomposition, expressing the entrance law of a one-boundary process in terms of that of a no-boundary process. In Markov chain theory, the last exit time from an ordinary state (particularly when it is instantaneous) is known to bear tricky consequences such as the differentiability of the transition function. A last exit being a first entrance when the sense of time is reversed, it should not be surprising that it figures prominently in the behavior of the paths, and its deeper impact is presumably due to an implicit reversal. Thus the true meaning of (3) is through a reversed viewing of the sub-a process $[T\,.\,T')$ from the terminal end T', so that the last exit time S becomes the lifetime of the minimal chain of the reverse subprocess. Indeed a final dénouement is obtained when the reverse subprocess is reversed again to retrieve the original subprocess. This doubling-back practice is by no means wasteful, as it shows up some fine features which are obscured by one-way thinking. In particular, one sees that the process from S to T' is also Markovian (although as stressed by Meyer, it is not a "subprocess" as defined by Dynkin), as well as the process from T to S, and furthermore there is stochastic independence between these two portions relative to their common epoch S. This is the reason for (3)(2). In general, a last exit time is both a death time and a birth time, and the notion has now been generalized to "co-optional" and "co-terminal" in [12], in the same sense that historically a first entrance time generalizes to "optional" and "terminal". The ensuing symmetry or duality with a concatenation should prove fertile.

It is well known that Hunt constructed the Martin boundary for a discrete parameter Markov chain by considering a sequence of last exit times from finite subsets swelling up to the state space. For a continuous parameter minimal chain this can be done at one swoop by reversing from its (finite) lifetime (see [3]).

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(1) This remark applies to reversing from a finite lifetime, see below.

(2) Other proofs of (3) have been given based on a local time $A(t)$ at the boundary point a (see [9], [16]) and going back to an analysis by Neveu. This is not surprising, since $E^a(t) = \mathbf{E}\{S \leqslant t\} = \mathbf{E}\{A(t)\}$ in our notation, where the last two $\mathbf{E}$'s are expectations for the sub-a process. But so far the intuitive meaning has not been made clearer by this method than by considering the last exit time and reversing the time.

The same ideas were used in [6] to reverse a general homogeneous Markov process to obtain a homogeneous Markov process. By insisting on following the paths faithfully and refraining from forcing them into any preconceived pattern, it is possible to set things on a natural course. We obtain thus a pair of homogeneous Markov processes *in reverse* sharing the same collection of sample functions with the arrow of time pointing in opposite directions. This entails two Markovian semi-groups in duality but enriched with the common structure of the paths. Much more needs be done to substantiate this "reversal" (vs. "dual") point of view, but I think it is a good instance where the general theory can learn from the chains.

REFERENCES

[1] Chung K.L. — Continuous parameter Markov chains, *Proc. International Congress of Mathematicians,* 1958, p. 510-517.
[2] Chung K.L. — On the boundary theory for Markov chains, *Acta. Math.,* 110. 1963, p. 19-77; 115, 1966, p. 111-163.
[3] Chung K.L. — *Markov chains with stationary transition probabilities,* Second Edition, Springer-Verlag, 1967.
[4] Chung K.L. — *Lectures on boundary theory for Markov chains, Princeton University* Press, 1970.
[5] Chung K.L. and Doob J.L. — Fields, optionality and measurability, *Amer. J. Math.,* 1965, p. 397-424.
[6] Chung K.L. and Walsh J.B. — To reverse a Markov Process, *Acta Math.,* 123, 1969, p. 225-251.
[7] Doob J.L. — Compactification of the discrete state space of a Markov process, *Z. Wahrscheinlichkeitstheorie verw. Geb.,* 10, 1968, p. 236-251.
[8] Dynkin E.B. — General boundary conditions for denumerable Markov processes, *Teor. Veroyatnost. i Primenen,* 12, 1967, p. 222-257.
[9] Lamb Charles W. — Boundary theory for Markov chains, *Doctorate Dissertation,* University of Illinois, 1969.
[10] Meyer P.A. — Guide détaillé de la théorie « générale » des processus, *Séminaire de probabilité,* II, p. 140-165, Springer-Verlag, 1968.
[11] Meyer P.A. — Le retournement du temps d'après Chung et Walsh, *Séminaire de probabilités,* V (to appear).
[12] Meyer P.A., Smythe R.T. and Walsh J.B. — *Birth and death of Markov processes,* Sixth Berkeley Symposium (to appear).
[13] Pittenger A. — Boundary decomposition of Markov processes, *Doctorate Dissertation,* Stanford University, 1967.
[14] Ray D. — Resolvants, transition functions and strongly Markovian processes, *Annals of Math.,* 70, 1959, p. 43-72.
[15] Walsh J.B. — The Martin boundary and completion of Markov chains, *Z. Wahrscheinlichkeitstheorie verw. Geb.,* 14, 1970, p. 169-188.
[16] Williams D. — Fictitious states, coupled laws and local time, *Z. Wahrscheinlichkeitstheorie verw. Geb.,* 11, 1969, p. 288-310.

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ENTRANCE AND EXIT SPACES FOR A MARKOV PROCESS

by E. B. DYNKIN

Entrance and exit spaces (closely related to Martin entrance and exit boundaries) play an important rôle in the theory of Markov processes (see [1] — [6]). We shall outline a new method of constructing these spaces based on the consideration of conditional processes. The method is applicable to Markov processes in the most wide sence. The theory becomes not only more general but also much simpler. Particularly it becomes completely invariant with respect to the time reversion.

We consider inhomogeneous processes. The relation of the stated theory to the usual one dealing with the homogeneous processes will be treated in another place.

1. Denumerable time set.

1.1. – We shall consider processes having the state space $(E, \mathcal{B})$ where E is a locally compact Hausdorff space with countable basis and $\mathcal{B}$ is the σ-field of its Borel sets.

Let T be a denumerable subset of the interval $(a, +\infty)$ where $-\infty \leqslant a < +\infty$ and let a be a limit point of T. Denote by Ω the set of all functions $\omega(t)$ defined on T with values in E. For each $T' \subseteq T$ denote by $\mathcal{N}(T')$ the σ-field generated by the sets $\{\omega : \omega(t) \in \Gamma\}$ $(t \in T\,;\, \Gamma \in \mathcal{B})$. Set

$$\mathcal{N}_t = \mathcal{N}(T \cap (a, t]), \mathcal{N}^t = \mathcal{N}(T \cap [t, \infty)) \ N = \mathcal{N}(T) .$$

Denote by $\mathfrak{N}$ the set of all measures $\mathbf{P}$ on $\mathcal{N}$ satisfying condition $\mathbf{P}(\Omega) \leqslant 1$. The set of all $\mathbf{P} \in \mathfrak{N}$ for which $\mathbf{P}(\Omega) = 1$ will be denoted by $\mathfrak{N}_0$. Let $\mathfrak{C}$ be the collection of all functions $\xi(\omega)$ of the form

$f_1[\omega(t_1)] \dots f_n[\omega(t_n)]$ where $n = 1, 2, \dots, t_1, \dots, t_n \in T$ and $f_1, \dots, f_n$ are non-negative continuous functions with compact supports. Introduce into $\mathfrak{N}$ a topology setting $\mathbf{P}_k \to \mathbf{P}$ if $\mathbf{M}_k\, \xi \to \mathbf{M}\, \xi$ (1) for all $\xi \in \mathfrak{C}$. The topological space $\mathfrak{N}$ is a compactum. We shall study stochastic processes $(x_t, \mathbf{P})$ where $x_t(\omega) = \omega(t)$ $(t \in T)$ and $\mathbf{P} \in \mathfrak{N}_0$. Set $P \in \mathfrak{M}$ if the process $(x_t, \mathbf{P})$ has the Markov property : for each $t \in T$, $\xi \in \mathcal{N}_t$, $\eta \in \mathcal{N}^t$,

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(1) **M** means the integral with respect to the measure **P**.

$$(1.1)\qquad \mathbf{M}(\xi\eta \mid x_t) = \mathbf{M}(\xi \mid x_t)\,\mathbf{M}(\eta \mid x_t) \quad (\text{a.s.}\mathbf{P})(^1)$$

This property is equivalent to the following one : for each $t \in T$, $\eta \in \mathcal{N}^t$

$$(1.2)\qquad \mathbf{M}(\eta \mid \mathcal{N}_t) = \mathbf{M}(\eta \mid x_t) \quad (\text{a.s.}\mathbf{P})$$

Let $\mathbf{P}^0 \in \mathfrak{M}$. For each $(s, x) \in T \times E$, a measure $\mathbf{P}_{s,x}$ on $\mathcal{N}$ may be constructed in such a way that, for every $\xi \in \mathcal{N}$, $\mathbf{M}_{s,x}\,\xi$ is a Borel function of x and

$$(1.3)\qquad \mathbf{M}^0\{\xi \mid x_s\} = \mathbf{M}_{s,x_s}\xi \quad (\text{a.s.}\mathbf{P}^0)$$

Let us fixe a family of measures $\{\mathbf{P}_{s,x}\}$ and let us denote by $\mathfrak{R}$ the class of all $P \in \mathfrak{N}_0$ for which

$$(1.4)\qquad \mathbf{M}\{\eta \mid \mathcal{N}_t\} = \mathbf{M}_{t,x_t}\eta \quad (\text{a.s.}\mathbf{P}) \quad \text{for each} \quad t \in T,\ \eta \in \mathcal{N}^t.$$

It is clear that $\mathfrak{R} \subseteq \mathfrak{M}$. The formulae (1.2) and (1.3) imply that $\mathbf{P}^0 \in \mathfrak{R}$.

Starting from the family $\{\mathbf{P}_{s,x}\}$, we shall construct a measurable space $(V, \mathcal{C})$ and we shall associate with any $\nu \in V$ a measure $\mathbf{P}_\nu \in \mathfrak{R}$ in such a way that, for any $\mathbf{P} \in \mathfrak{R}$,

$$(1.5)\qquad \lim_{s \downarrow a} \mathbf{P}_{s,x_s} = \mathbf{P}_\rho \quad (\text{a.s.}\mathbf{P})$$

where ρ is a measurable function on Ω with values in V. Moreover the formulae

$$(1.6)\qquad \mu(\Gamma) = \mathbf{P}\{\rho \in \Gamma\}$$

$$(1.7)\qquad \mathbf{P}(A) = \int_V \mathbf{P}_\nu(A)\,\mu(d\nu)$$

establish a one-to-one correspondence between $\mathbf{P} \in \mathfrak{R}$ and probability measures μ on $(V, \mathcal{C})$. The collection $(V, \mathcal{C}, \mathbf{P}_\nu)$ is called the entrance space for the process $(x_t, \mathbf{P})$ (and for the class $\mathfrak{R}$ too).

1.2. – Set $\mathbf{P} \in \mathfrak{N}_{a+}$ if there exists a sequence $(s_n, x_n) \in T \times E$ such that $s_n \downarrow a$ and $\mathbf{P}_{s_n,x_n} \to \mathbf{P}$. The set $\mathfrak{N}_{a+}$ is compact. Consider now an arbitrary compactum $\mathcal{E}$ homeomorphic to $\mathfrak{N}_{a+}$ and denote by $\mathbf{P}_\nu$ a measure corresponding to $\nu \in \mathcal{E}$ under a fixed homeomorphism from $\mathcal{E}$ onto $\mathfrak{N}_{a+}$.

Set $\mathcal{N}_{a+} = \bigcap_{t \in T} \mathcal{N}_t$.

Let $\mathbf{P} \in \mathfrak{R}$. By virtue of (1.4)

$$(1.8)\qquad \lim_{s \downarrow a} \mathbf{M}_{s,x_s}\eta = \lim_{s \downarrow a} \mathbf{M}\{\eta \mid \mathcal{N}_s\} = \mathbf{M}\{\eta \mid \mathcal{N}_{a+}\}.$$

The right side is an integral of η with respect to a measure depending on ω (conditional probability distribution). Evidently this measure belongs to $\mathfrak{N}_{a+}$. Denote by $\rho(\omega)$ the corresponding element of $\mathcal{E}$. Then

$$(1.9)\qquad \mathbf{M}_\rho\eta = \mathbf{M}\{\eta \mid \mathcal{N}_{a+}\} \quad (\text{a.s.}\mathbf{P})$$

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(1) For any σ-field $\mathcal{F}$ the notation $\xi \in \mathcal{F}$ means that ξ is a non-negative $\mathcal{F}$-measurable function. The notation (a.s.P) means "almost surely with respect to measure P".

The formulae (1.8) and (1.9) imply (1.5).

The equality (1.9) may be extended to all $\eta \in \mathcal{N}$. Particularly we have $\mathbf{M}_\rho \eta = \eta$ (a.s.P) for $\eta \in \mathcal{N}_{a+}$. Therefore the σ-field $\mathcal{N}_{a+}$ is generated by the sets

$$\{\omega : \rho(\omega) \in \Gamma\} \ (\Gamma \in \mathcal{C})$$

and by the sets of measure 0.

Using (1.9), we have, for any $\eta \in \mathcal{N}$ and any Borel function $\varphi \geqslant 0$ on $\mathcal{E}$

$$\mathbf{M}\,\varphi(\rho)\eta = \mathbf{M}(\varphi(\rho)\,\mathbf{M}_\rho\,\eta) = \int_{\mathcal{E}} \varphi(\nu)\,\mathbf{M}_\nu\,\eta\,\mu(d\nu) \tag{1.10}$$

where the measure μ is defined by (1.6). Setting here $\eta = \psi(\rho)$ we have

$$\int_{\mathcal{E}} \psi(\nu)\,\varphi(\nu)\,\mu(d\nu) = \int_{\mathcal{E}} \mathbf{M}_\nu\,\psi(\rho)\,\varphi(\nu)\,\mu(d\nu)\,.$$

This implies that, for almost all ν, $\mathbf{M}_\nu\,\psi(\rho) = \psi(\nu)$ and hence

$$\mathbf{P}_\nu(\rho = \nu) = 1\,. \tag{1.11}$$

Relying on (1.9) and (1.4) it is easy to deduce that $M_\rho \xi\eta = M_\rho(\xi \mathbf{M}_{t,x_t}\eta)$ (a.s.P) for all $t \in T$, $\xi \in \mathcal{N}_t$, $\eta \in \mathcal{N}^t$. It follows from here that $\mathbf{P}_\nu \in \mathfrak{K}$ for μ-almost all ν.

Denote by V the set of all $\nu \in \mathcal{E}$ for which $\mathbf{P}_\nu \in \mathfrak{K}$ and $\mathbf{P}_\nu(\rho = \nu) = 1$. Let $\mathcal{C}$ be the totality of all Borel sets of $\mathcal{E}$ contained in V. It has been proved that $\mu(\mathcal{E}\setminus V) = 0$. Therefore the formula (1.10) may be rewritten in the form

$$\mathbf{M}\,\varphi(\rho)\,\eta = \int_V \varphi(\nu)\,\mathbf{M}_\nu\,\eta\mu(d\nu) \tag{1.12}$$

By setting $\varphi = 1$, $\eta = \chi_A$ we obtain (1.7).

It is easy to show that, for any probability measure μ, a measure $\mathbf{P}$ defined by (1.7) belongs to $\mathfrak{K}$. On the other hand if μ and $\mathbf{P}$ are connected by (1.7) then by virtue of (1.11), for any $\Gamma \in A$

$$\mathbf{P}\{\rho \in \Gamma\} = \int_V \mathbf{P}_\nu\{\rho \in \Gamma\}\,\mu(d\nu) = \int_V \chi_\Gamma(V)\,\mu(d\nu) = \mu(\Gamma)\,.$$

Thus formulae (1.6) and (1.7) determine a one-to-one correspondence between $\mathbf{P} \in \mathfrak{K}$ and probability measures μ(1).

1.3. – Now let $-\infty < b \leqslant +\infty$ and let T be a denumerable subset of the interval $(-\infty, b)$ and b be a limit point of T. The exit space $(V^*, \mathcal{C}^*, \mathbf{P}^\nu)$ may be constructed in the same way as the entrance space $(V, \mathcal{C}, \mathbf{P}_\nu)$. Instead of the family $\mathbf{P}_{s,x}$, we consider measures $\mathbf{P}^{s,x}$ on $\mathcal{N}_s$ satisfying the condition

$$\mathbf{M}^0\{\xi \,|\, x_s\} = \mathbf{M}^{s,x_s}\,\xi \quad (\text{a.s.}\mathbf{P}^0) \quad (\xi \in \mathcal{N}^s) \tag{1.13}$$

Instead of the class $\mathfrak{K}$ we define the class $\mathfrak{K}^*$ of measures $\mathbf{P} \in \mathfrak{N}_0$ such that

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(1) This correspondence implies that the set of all extremal points of convex set $\mathfrak{K}$ coincides with the set of measures $\mathbf{P}_\nu (\nu \in V)$.

$\mathbf{M}(\xi \mid \mathcal{N}^t) = \mathbf{M}^{t,x_t}\,\xi$ (a.s.P) for all $t \in T$, $\xi \in \mathcal{N}_t$. Note that the time reversion transforms the exit and the entrance spaces into each other.

1.4. – The family $\mathbf{P}_{s,x}$ may be constructed starting from an arbitrary transition function $p(s, x ; t, \Gamma)$ $(s < t \in T, x \in E, \Gamma \in \mathcal{B})$. The associated class $\mathfrak{K}$ may be described as a set of all measures $\mathbf{P} \in \mathfrak{N}_0$ such that

$$\mathbf{P}(x_t \in \Gamma \mid \mathcal{N}_s) = p(s, x_s ; t, \Gamma) \quad \text{(a.s.P)}$$

for all $s < t \in T$, $\Gamma \in \mathcal{B}$. Analogously, the family $\mathbf{P}^{s,x}$ may be constructed starting from any "co-transition function" $p^*(s, x ; t, \Gamma)$ $(s > t \in T, x \in E, \Gamma \in \mathcal{B})$. The class $\mathfrak{K}^*$ consists of all measures $\mathbf{P} \in \mathfrak{N}_0$ for which

$$\mathbf{P}(x_t \in \Gamma \mid \mathcal{N}^s) = p^*(s, x_s ; t, \Gamma) \quad \text{(a.s.P)} \quad \text{for all} \quad s > t \in T, \Gamma \in \mathcal{B}.$$

2. Harmonic functions.

2.1. – A non-negative function h on the space $T \times E$ is called P-harmonic if $(h(t, x_t), \mathcal{N}_t, \mathbf{P})$ is a martingale. Functions h_1 and h_2 are called equivalent if $h_1(t, x_t) = h_2(t, x_t)$ (a.s.P) for all $t \in T$. Our aim is to describe, up to equivalence, all P-harmonic functions subject to condition $\mathbf{M}\,h(t, x_t) = 1$.

To each harmonic function h, there corresponds one and only one measure $\mathbf{P}_h$ on the σ-field $\mathcal{N}$ such that

$$\mathbf{M}_h\,\xi = \mathbf{M}\,\xi\,h(t, x_t) \quad \text{for every} \quad t \in T,\ \xi \in \mathcal{N}_t\,.$$

If $\{\mathbf{P}^{s,x}\}$ is a family of measures connected with P by relation (1.13) then $\mathbf{M}_h(\xi \mid \mathcal{N}^t) = \mathbf{M}^{t,x_t}\xi$ (a.s.$\mathbf{P}_h$) for all $\xi \in \mathcal{N}_t$, hence $\mathbf{P}_h \in \mathfrak{K}^*$.

Denote by $\mathfrak{K}_p^*$ the set of all $\mathbf{P}' \in \mathfrak{K}^*$ such that the measure $p'(t, \Gamma) = \mathbf{P}'(x_t \in \Gamma)$ is absolutely continuous relative to $p(t, \Gamma) = \mathbf{P}(x_t \in \Gamma)$ for each $t \in T$. It is clear that $p_h(t, dy) = h(t, y)\ p(t, dy)$ and therefore $\mathbf{P}_h \in \mathfrak{K}_p^*$. On the other hand, if $\mathbf{P}' \in \mathfrak{K}_p^*$ and $p'(t, dy) = h(t, y)\ p(t, dy)$, then

$$\mathbf{M}\xi h(t, x_t) = \mathbf{M}\mathbf{M}(\xi \mid \mathcal{N}^t)\,h(t, x_t) = \mathbf{M}(\mathbf{M}^{t,x_t}\xi)\,h(t, x_t) = \mathbf{M}'\mathbf{M}^{t,x_t}\xi =$$
$$= \mathbf{M}'\mathbf{M}'(\xi \mid \mathcal{N}^t) = \mathbf{M}'\xi\,, \quad \text{for} \quad s \leqslant t \in T,\ \xi \in \mathcal{N}_s\,.$$

It is obvious from here that $(h(t, x_t), \mathcal{N}_t, \mathbf{P})$ is a martingal. Thus we have a one-to-one correspondence between $\mathbf{P}' \in \mathfrak{K}_{\mathbf{P}}^*$ and classes of equivalent P-harmonic functions.

2.2. – To proceed further, we need the following assumption about the measures $\{\mathbf{P}^{s,x}\}$:

(P) For any $t < u \in T$, there exists a measure m such that $\mathbf{P}^{u,y}(x_t \in \Gamma) = 0$ for all Γ of m-measure 0 and all $y \in E$.

Let $\mathbf{P}' \in \mathfrak{K}^*$. Then

$$p'(t, \Gamma) = \mathbf{P}'(x_t \in \Gamma) = \mathbf{M}'\mathbf{P}^{u,x_u}(x_t \in \Gamma)\,,$$

so that $p'(t, \Gamma) = 0$ if $m(\Gamma) = 0$. The densities of measures $p'(t, -)$, $p(t, -)$ relative to m will be denoted by $p'(t, y)$, $p(t, y)$. It is easy to see that the measure $p'(t, -)$ is absolutely continuous relative to the measure $p(t, -)$ if and only if $p'(t, A_t) = 0$, where $A_t = (y : p(t, y) = 0)$. Thus the class $\mathfrak{K}_p^*$ may be described as the set of all measures $\mathbf{P}' \in \mathfrak{K}^*$ satisfying condition :

$$\mathbf{P}'(x_t \in A_t) = 0 \quad \text{for all} \quad t \in T . \tag{2.2}$$

2.3. – Let $(V^*, \mathfrak{A}^*, \mathbf{P}^\nu)$ be the exit space for $\mathfrak{K}^*$. According to the section 1, each measure $\mathbf{P}'$ may be uniquely represented in the form

$$\mathbf{P}' = \int_{V^*} \mathbf{P}^\nu \mu(d\nu) .$$

It is clear that condition (2.2) is fulfilled for P' if and only if $\mu(V^* \setminus V_{\mathbf{P}}^*) = 0$ where

$$V_{\mathbf{P}}^* = \{\nu : \nu \in V^*, \mathbf{P}^\nu\{x_t \in A_t = 0\} \text{ for all } t \in T\} .$$

The measure $\mathbf{P}^\nu$ belongs to $\mathfrak{K}_p^*$; therefore $p^\nu(t, dy) = k^\nu(t, y)\, p(t, dy)$ where k^ν is $\mathbf{P}$-harmonic. The formula

$$\mathbf{P}' = \int_{V_{\mathbf{P}}^*} \mathbf{P}^\nu \mu(d\nu)$$

determines a one-to-one correspondence between probability measures μ on V_p^* and $\mathbf{P}' \in \mathfrak{K}_{\mathbf{P}}$, and the formula

$$h(t, y) = \int_{V_p^*} k^\nu(t, y)\, \mu(d\nu) \tag{2.3}$$

determines a one-to-one correspondence between the same measures μ and the classes of equivalent $\mathbf{P}$-harmonic functions.

3. Continuous time parameter.

3.1. – A family of probability measures $\nu_t(\Gamma)$ $(t \in T, \Gamma \in \mathcal{B})$ is called an entrance law for the transition function $p(s, x ; t, \Gamma)$ if

$$\int_E \nu_s(dx)\, p(s, x ; t, \Gamma) = \nu_t(\Gamma) \quad (s < t \in T, x \in E, \Gamma \in \mathcal{B}) .$$

Formulae

$$\nu_+(\Gamma) = P\{x_t \in \Gamma\},$$

$$\mathbf{P}\{x_{t_1} \in dy_1, \ldots, x_{t_n} \in dy_n\} = \nu_{t_1}(dy_1)\, p(t_1, y_1 ; t_2, dy_2) \ldots p(t_{n-1}, y_{n-1} ; t_n, dy_n)$$

establish the one-to-one correspondence between the class $\mathcal{H}$ defined by the condition (1.4) and the set $\mathfrak{L}$ of all entrance laws.

Now let T be an interval (a, b) and let $\overline{T}$ be a denumerable everywhere dence subset of the set T. The restriction $\overline{\nu}$ of the entrance law $\nu \in \mathfrak{L}$ to the set $\overline{T}$ belongs to $\overline{\mathfrak{L}}$. On the other hand, to every $\overline{\nu} \in \overline{\mathfrak{L}}$ there corresponds an entrance law

$$\nu_t(\Gamma) = \int_E \nu_s(dx)\, p(s\,,x\,;\,t\,,\Gamma) \quad (s \in (a\,,t) \cap \overline{T})$$

(it is clear that the value $\nu_t(\Gamma)$ is independent of s).

We have a chain of one-to-one mappings $\overline{\mathfrak{K}} \to \overline{\mathfrak{Q}} \to \mathfrak{Q} \to \mathfrak{K}$ and therefore an one-to-one mapping $\overline{\mathfrak{K}}$ onto $\mathfrak{K}$.

Let $(\overline{V}, \overline{\mathcal{A}}, \overline{\mathbf{P}}_\nu)$ be the entrance space related to the class $\overline{\mathfrak{K}}$. Set $V = \overline{V}$, $\mathcal{A} = \overline{\mathcal{A}}$ and denote by $\mathbf{P}_\nu$ the elements of $\mathfrak{K}$ corresponding to $\overline{\mathbf{P}}_\nu \in \overline{\mathfrak{K}}$. Then the collection $(V, \mathcal{A}, \mathbf{P}_\nu)$ determines the entrance space for the class $\mathfrak{K}$.

3.2. – For every function h on $T \times E$, denote by $\overline{h}$ its restriction to $T \times E$. If h is **P**-harmonic, then $\overline{h}$ is $\overline{\mathbf{P}}$-harmonic and

$$\mathbf{M}\{h(u\,,x_u)\,|\,x_t\} = \mathbf{M}\{h(u\,,x_u)\,|\,\mathcal{N}_t\} = h(t\,,x_t) \quad \text{(a.s.P)} \quad \text{for all} \quad t < u \in T.$$

On the other hand, if q is **P**-harmonic, then $\mathbf{M}\{q(u\,,x_u)\,|\,x_t\}$ is independent of $u \in T \cap [t\,,b)$. Therefore there exists a function h on $T \times E$ such that

$$\text{(3.1)} \qquad \mathbf{M}\{q(u\,,x_u)\,|\,x_t\} = h(t\,,x_t) \quad \text{(a.s.P)} \quad \text{for all} \quad t \in (a\,,b)\,.$$

It is easy to show that h is **P**-harmonic and h is equivalent to q.

Assume the condition (P) and consider the space $\overline{V}^*_{\mathbf{P}}$ and the **P**-harmonic functions k^ν constructed in section $\underline{2}$. Set $V^*_P = \overline{V}^*_P$ and denote by k^ν the **P**-harmonic function corresponding to $\overline{k}^\nu$. Then the formula (2.3) establishes a one-to-one correspondence between probability measures on $V^*_{\mathbf{P}}$ and the classes of equivalent **P**-harmonic functions.

REFERENCES

[1] Doob J.L. — Discrete potential theory and boundaries, *J. Math. Mech.*, 8, 3, 1959, p. 433-458.

[2] Hunt G.A. — Markoff chains and Martin boundaries, *Illinois J. Math.*, 4, 1960, p. 313-340.

[3] Kunita H. and Watanabe T. — Markov processes and Martin boundaries I, *Illinois J. Math.*, 9, 3, 1965, p. 485-526.

[4] Meyer P.-A. — *Processus de Markov : la frontière de Martin*, Springer-Verlag, Berlin-Heidelberg-New York, 1968.

[5] Дынкин Е. Б. — Пространство выходов марковского процесса, Успехи матем. наук, 24, 4 (148), 1969, p. 89-152.

[6] Дынкин Е. Б. — Эксцессивные функции и пространство выходов марховского процесса, Теория вероятностей и её примен., 15, 1, 1970, p. 38-55.

Added in proof. See also :

[7] Dynkin E. B. — Classes of similar Markov processes and corresponding exit and entrance spaces, *J. Faculty of Sci., Univ. Tokyo*, Sec. 1, 17, 1-2, 1970, p. 87-100.

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FINE CONNECTIVITY AND FINELY HARMONIC FUNCTIONS

by Bent FUGLEDE

Introduction.

The theory of balayage of measures permits us, in particular, to define the harmonic measure ρ_x^V relatively to a *finely*(1) open set V and a point $x \in V$ by $\rho_x^V = \epsilon_x^{\complement V} = \epsilon_x^{\partial V}$, the swept-out of the Dirac measure ϵ_x on $\complement V$, more precisely on the fine boundary ∂V of V.

It is therefore natural to introduce a corresponding notion of harmonicity and of hyperharmonicity for functions f defined in a *finely* open set U, the crucial condition being

$$f(x) = \int f\, d\epsilon_x^{\complement V} \quad , \text{ resp. } f(x) \geqslant \int^* f\, d\epsilon_x^{\complement V},$$

for a suitable family of finely open sets V with fine closure $\widetilde{V} \subset U$.

A major difficulty in carrying out this project is the failure of compactness arguments for the fine topology, the only finely compact sets being the finite sets. It turns out, nevertheless, that a satisfactory theory of such "finely harmonic" and "finely hyperharmonic" functions can be developed, very analogous to the usual theory of harmonic and hyperharmonic functions, and even containing large parts thereof.

The framework is throughout the case (A_1) of Brelot's axiomatic theory [3], [5] including the domination axiom (D) which is indispensable. The content of this lecture is, however, new even in the newtonian case.

As it might be expected the study of fine harmonicity is tied up with the connectivity properties of the fine topology. An independent study of these latter properties and their role for the balayage of measures was initiated in [7] and will be discussed briefly in § 1 below.

Notations.

We denote by X a harmonic space satisfying the group of axioms (A_1) in Brelot's theory [3]. For any set $A \subset X$ the base $b(A)$ of A is the set of points of X at which A is not thin. Any set of the form $B = b(A)$ for some set A (hence $B = b(B)$) is called a *base*. The sets A such that $b(A) \subset A$ are precisely the closed sets in the fine topology on X. The fine closure of any set A is denoted by $\widetilde{A}$, and the *fine* boundary of A by ∂A. A finely open set V is called *regular* if $\complement V$ is a base.

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(1) The qualification fine(ly) refers to the fine topology.

An *admissible* measure means a positive Radon measure μ on X such that $\int p \, d\mu < +\infty$ for every finite and continuous potential p on X which is harmonic outside some compact set. For any set A the swept-out measure $\mu^A = \mu^{b(A)}$ is carried by $b(A)$ and is likewise admissible.

For any numerical function f, defined almost everywhere and integrable in the wide sense (allowing infinite values for the integral) with respect to ϵ_x^A, for some $x \in X$, we write

$$f^A(x) = \int f \, d\epsilon_x^A \ .$$

The function f^A thus defined in part of X coincides with f on $b(A)$. If f is hyperharmonic and $\geqq 0$ in X, then $f^A = \hat{R}_f^A$ by definition of the swept-out measure ϵ_x^A.

A potential p on X is called *semibounded* (cf. [1], [5]) if p is representable as the sum of an infinite sequence of locally bounded potentials, or equivalently if

$$\bigwedge_{t>0} R_{(p-t)^+} = 0 \ .$$

where $\wedge$ indicates the infimum in the complete lattice of all hyperharmonic functions $\geqq 0$ in X with the pointwise order. Every finite potential is semibounded.

1. Connectivity properties of the fine topology.

1.1. – By hypothesis the space X is connected and locally connected in the initially given topology. Our first result is that the same holds for the fine topology [7]. In particular the fine components of a finely open set are finely open, and their number is at most countable in view of the quasi Lindelöf principle (Doob [6]).

The local connectivity of the fine topology is a consequence of (a) and (b) of the following result applied to $B = b(\complement V)$ and $\mu = \epsilon_x$ for V finely open and $x \in V$:

1.2. – Let μ denote an admissible measure and B a base such that $\mu(B) = 0$. Then :

(a) Among all bases $E \supset B$ such that $\mu^E = \mu^B$ there exists a largest, henceforth denoted by B_μ ; and $\mu(B_\mu) = 0$.

(b) The fine components of $\complement B_\mu$ are precisely those fine components V of $\complement B$ for which $\mu(V) > 0$(1).

(c) The fine boundary ∂B_μ is the smallest finely closed set carrying μ^B (that is, the fine support of μ^B).

(d) If X is a Green space with Green kernel G, then B_μ is the base of the finely closed set $[G\mu^B = G\mu]$ and differs from it at most in the polar set $[G\mu = +\infty]$.

The proof of (a), (d), and part of (b) is given in [7]. The rest is established in [7] only in the greenian case, but follows in the general case from 3.3 below.

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(1) More generally $\mu^*(V) > 0$ for every fine component V of $\complement(B_\mu \cup e)$ when e is a polar set such that $\mu(e) = 0$.

2. Finely harmonic and finely hyperharmonic functions.

2.1. *Definitions.* – A numerical function f, defined in a finely open set $U \subset X$, is called *finely hyperharmonic* (in U) if f is finely l.s.c. (lower semicontinuous) and $> -\infty$ in U, and if the induced fine topology on U has a base consisting of finely open sets V of fine closure $\widetilde{V} \subset U$ such that $f(x) \geqslant \int^* f \, d\epsilon_x^{\complement V}$ for every $x \in V$.

A numerical function f is called *finely hypoharmonic* in U if $-f$ is finely hyperharmonic in U. Finally, f is called *finely harmonic* in U if f and $-f$ are both finely hyperharmonic in U ; that is, if f is finite valued, finely continuous, and such that the induced fine topology on U has a base consisting of finely open sets V with $\widetilde{V} \subset U$ such that $f = f^{\complement V}$ in V.

2.2. *Remarks.* – Any finely semicontinuous function in U is measurable with respect to $\epsilon_x^{\complement V}$ for any finely open V, $x \in V \subset \widetilde{V} \subset U$, because this harmonic measure does not charge any polar set.

Since the fine topology is (completely) regular, the notions of fine harmonicity or fine hyperharmonicity have the sheaf property.

In the case of a Green space X, the Green potential $G\mu$ of any admissible measure μ on X is finely harmonic in any finely open set $U \subset [G\mu < +\infty]$ such that $\mu(U) = 0$.

2.3. *The fine minimum principle.* – (1) Let f be finely hyperharmonic in a finely open set U, and suppose that there is a polar set e such that

$$\text{fine} \liminf_{x \to y,\, x \in U} f(x) \geqslant 0 \quad \text{for every} \quad y \in (\partial U) \setminus e .$$

If moreover there exists a semibounded potential(2) p on X such that $f \geqq -p$ in U, then $f \geqq 0$ (in U).

The proof is easily reduced to establishing the following lemma :

LEMMA A. – Let $f \geqq 0$ be finely u.s.c. in X and majorized q.e. (quasi everywhere, that is, except in some polar set) by a semibounded potential p. Then the family of all bases B for which $f \leqq f^B$ q.e., is stable under infimum (in the lattice of all bases).

This lemma, in turn, depends on the Choquet property for the capacity $E \mapsto R_1^E(x)$ for given $x \in X$, as established by Brelot [4]. It follows that, for f as stated in the lemma,

$$\wedge \{\hat{R}_{f-\varphi} \mid \varphi \in \mathcal{H}_0 ,\, \varphi \leqq f\} = 0,$$

where $\mathcal{H}_0$ denotes the class of all u.s.c. functions on X (in the initial topology) of compact support and with finite values $\geqq 0$. Next, such φ is represented suitably

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(1) In view of 2.6 below this result contains that obtained by Brelot [2, lemme 1] corresponding to the case U *open* and relatively compact, f bounded below.

(2) At least in the case of a Green space it can be shown that the semiboundedness of the potential p may be dropped at the expense that the boundary condition above should hold for every $y \in \partial U$.

as the pointwise limit of functions $\varphi_n \geqq 0$ of the form $\varphi_n = p_n - q_n$ of compact support, p_n and q_n being finite and continuous potentials. Here we use the approximation theorem (Hervé [8]). Finally the following further lemma is applied to p_n and q_n :

LEMMA B. – Let p be a semibounded potential, and $(B_i)_{i\in I}$ any family of bases with the infimum $B = b(\cap B_i)$. Then the swept-out potential p^B is the infimum (in the lattice of all hyperharmonic functions $\geqq 0$) of all functions obtained by repeated balayage, starting with p, on finitely many B_i, $i \in I$.

The proof of Lemma B depends on the above Choquet property and the theorem on capacity for decreasing families (Brelot [5]), together with the lattice properties of the specific order for superharmonic functions.

2.4. *The fine Dirichlet problem.* – Let U denote a *regular* finely open set (that is, $\complement U$ is a base). Let f be a finely continuous function defined on the fine boundary ∂U and such that $|f|$ is majorized there by some locally bounded(1) potential p on X. Then there exists one and only one finely continuous extension u of f to $\widetilde{U}$ such that u is finely harmonic in U and $|u|$ is majorized in U by some semibounded potential. This unique extension is $u = f^{\complement U}$ (and hence $|u| \leqslant p$ in U).

The uniqueness follows from 2.3. In proving that $u = f^{\complement U}$ has the desired properties, say for $f \geqq 0$, we may assume (on account of a certain "quasi normality" of the fine topology) that f is defined, finely continuous, and $\leqq p$ in all of X. Using the Choquet property as before it can be shown that(2)

$$\inf\{R_{|f-\varphi|}(x) \mid \varphi \in \mathscr{C}_0^+\} = 0 \ .$$

This allows us, by virtue of the approximation theorem, to reduce the matter to the simple case where f is itself a finite potential.

Likewise it is possible to extend the Perron-Wiener-Brelot method to the case of an arbitrary finely open set U with an arbitrary function f given on ∂U. If f is resolutive, or equivalently if f is integrable with respect to $\epsilon_x^{\complement U}$ for every $x \in U$, then the generalized solution obtained by this method is finely harmonic in U and coincides there with $f^{\complement U}$. The proof is based on 2.5 and 2.8 below.

2.5. *The general global character of fine (hyper) harmonicity.* – If f is finely hyperharmonic in U (finely open), then the inequality $f \geqq f^{\complement V}$ in V holds for *any* finely open set V with $\widetilde{V} \subset U$ such that $f \geqq -p$ in V for some *finite* potential p on X. Note that these sets (even the regular ones) form a base for the fine topology on U. (But unlike the situation for ordinary hyperharmonic functions, this base depends on the order of magnitude of the negative part of f). – More generally we have the following global result :

Let U be a finely open set, and f a numerical function defined and finely l.s.c. in $\widetilde{U}$ and finely hyperharmonic in U. If moreover $f \geqq -p$ in $\widetilde{U}$ for some finite potential p on X, then $f \geqq f^{\complement U}$ in U.

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(1) The local boundedness may be replaced by finiteness on account of 2.9 below.

(2) $\mathscr{C}_0^+$ denotes the class of all continuous functions on X (in the initial topology) of compact support and with finite values $\geqq 0$.

This is proved by application of 2.3 and 2.4. It follows that the finely hyperharmonic functions form a convex cone, stable under upper directed supremum.

2.6. *The case of an ordinary open set.* – Let U be open in the initial topology on X. A finely hyperharmonic (resp. finely harmonic) function u in U is hyperharmonic (resp. harmonic) in the usual sense if (and only if) u is locally bounded from below (resp. locally bounded from one side). – The question remains open whether such a local boundedness condition has to be assumed.

As an application we easily obtain a new proof that every connected open set U in the initial topology is finely connected. For let $V \subset U$ be finely open and finely closed relatively to U. Then the function which equals $+\infty$ in V and 0 in $U \setminus V$ is finely hyperharmonic in U, hence hyperharmonic, hence l.s.c., hence V is open, etc.

2.7. *The local extension theorem.* – Let f be finely hyperharmonic in a finely open set U. Every point $x \in U$ at which $f(x) < +\infty$ has a fine neighbourhood V with $\tilde{V} \subset U$ such that f is representable in V as $f = u - v$, where u and v are locally bounded potentials on X, and where v is finely harmonic in V.

The proof depends on 2.5, 2.6, and the existence of a strict potential ; and it proceeds along similar lines as the proof of the ordinary extension theorem.

An important consequence of the result obtained is that any finely hyperharmonic function is *finely continuous*. Hence there is no new "fine-fine" topology.

2.8. *Monotone families of finely harmonic functions.* – The pointwise limit f of a directed family of finely harmonic functions f_i in the same finely open set U is finely harmonic in the finely open set $U \cap [|f| < +\infty]$.

In view of the latter result of 2.7 this follows by use of 2.5 and the quasi Lindelöf principle [6].

As a first application we obtain a new proof of the local connectivity of the fine topology in a manner quite similar to the proof given by Bauer for the initial topology. – We mention two further applications :

2.9. – Let p be a semibounded potential. For any point $x \in X$ at which $p(x) < +\infty$ we have (cf. the proof of Lemma A in 2.2)

$$\inf\{R_{p-\varphi}(x) \mid \varphi \in \mathcal{H}_0 \,,\, \varphi \leqq p\} = 0 \ .$$

2.10. – For any locally bounded function $f \geqq 0$ on X and any admissible measure μ the capacity $E \mapsto \int \hat{R}_f^E \, d\mu$ has the Choquet property. (This extends a result of Brelot [4] in which it was supposed that μ does not charge any polar set).

3. Further results concerning fine harmonicity and fine connectivity.

3.1. *Removable singularities.* – Let u be finely hyperharmonic in $U \setminus e$ for some polar subset e of a finely open set U. The extension of u to U defined by putting

$$u(y) = \underset{x \to y,\, x \in U \setminus e}{\text{fine lim inf}}\ u(x) \ , \qquad y \in e \ ,$$

is finely hyperharmonic in U if (and only if) $u(y) > -\infty$ for every $y \in e$.

If u is finely harmonic in $U \setminus e$, and if u is bounded in some deleted fine neighbourhood of any point of e, then u may be extended uniquely (by fine continuity) to a finely harmonic function in U.

These results are obtained by application of 2.5 and the fine continuity result in 2.7.

3.2. – As a corollary of the latter part of 3.1 we have the following (established by another method in [7] for the case of a Green space) :

If U is a *fine domain* (that is, a finely connected, finely open set), then so is $U \setminus e$ for every polar set $e \subset U$.

As in [7] this in turn implies the identity

$$b(\partial A) = b(A) \cap b(\complement A)$$

for any set $A \subset X$. In particular, the fine boundary ∂B of a base B (or of a regular finely open set) is a base.

3.3. – Let $u \geqq 0$ be finely hyperharmonic in a fine domain U. Then either $u > 0$ or $u \equiv 0$. (In fact, the pointwise limit of the increasing sequence u, $2u$, $3u, \ldots$ is finely hyperharmonic and hence finely continuous by 2.7).

This result is the key to the proof of the theorem on the fine support of a swept-out measure (see 1.2 above), which in turn leads to the following :

A finely hyperharmonic function u in a finely open set U is either $\equiv +\infty$ in some fine component of U, or else finite q.e. in U. (In the latter case u is said to be finely *superharmonic*).

3.4. – Dually to 1.2 consider a hyperharmonic function $u \geqq 0$ on X and a base B. Then $B_u := b([u^B = u])$ is the largest base $E \supset B$ such that

$$u^E = u^B (= \hat{R}_u^B) .$$

If u is a semibounded potential, the fine components of $\complement B_u$ are precisely those fine components V of $\complement B$ for which u is *not* finely harmonic in all of $V \cap [u < +\infty]$.

3.5. – By application of 3.4 one may establish the following equivalence, communicated to the author by Ng.-Xuan-Loc, and valid for arbitrary sets $A, B \subset X$ and any $x \in X$:

$$[(\epsilon_x^A)^B = \epsilon_x^B] \Leftrightarrow [\epsilon_x^{A \cup B} = \epsilon_x^A] .$$

3.6. *Concluding remarks.* – After the theory of fine harmonicity has been brought to this point, one may use traditional methods to carry over further results from the theory of ordinary harmonic and hyperharmonic functions (in the present axiomatic case (A_1) of Brelot's theory). First one introduces notions relatively to any finely open set $U \subset X$, e.g. the finely reduced function relatively to U, and its finely l.s.c. envelope, the finely swept-out function relatively to U ; further the notion of a fine potential relatively to U, etc. One obtains for instance the unique decomposition of a *finite* valued, finely hyperharmonic function $u \geqq 0$ in U into a sum of a fine potential relatively to U and a finely harmonic function in U.

A detailed account of the present theory is in preparation.

REFERENCES

[1] Boboc N. et Cornea A. — Espaces harmoniques. Axiome D et théorème de convergence. *Revue Roum. Math.*, 13, 1968, p. 933-947.
[2] Brelot M. — Sur l'allure des fonctions harmoniques et surharmoniques à la frontière, *Math. Nachr.*, 4, 1950-51, p. 298-307.
[3] Brelot M. — *Axiomatique des fonctions harmoniques*, Montréal, 1966.
[4] Brelot M. — Recherches axiomatiques sur un théorème de Choquet concernant l'effilement, *Nagoya Math. J.*, 30, 1967, p. 39-46.
[5] Brelot M. — La topologie fine en théorie du potentiel, *Lecture Notes in Math.*, 31, 1967, p. 36-47.
[6] Doob J.L. — Applications to analysis of a topological definition of smallness of a set, *Bull. Amer. Math. Soc.*, 72, 1966, p. 579-600.
[7] Fuglede B. — Connexion en topologie fine et balayage des mesures. (To appear in *Ann. Inst. Fourier.*)
[8] Hervé R.-M. — Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, *Ann. Inst. Fourier*, 12, 1962, p. 415-471.

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SOME RECENT DEVELOPMENTS IN THE THEORY OF DUAL PROCESSES

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Throughout this paper X and $\hat{X}$ are two standard processes in duality relative to a Radon measure ξ. We refer the reader to [1] for all terminology and notation not explicitly defined here. More specifically, X and $\hat{X}$ satisfy the conditions on page 259 of [1]. However, we make no regularity assumptions on the resolvents (U^{α}) and $(\hat{U}^{\alpha})$ of X and $\hat{X}$ respectively.

Let $M = (M_t)$ be a multiplicative functional (MF) of X. (In this report all MF's are assumed to be right continuous, decreasing, and to satisfy $0 \leqslant M_t \leqslant 1$. Also equality between MF's always means equivalence). The semigroup and resolvent generated by M are defined as follows :

$$Q_t f(x) = E^x\{f(X_t) M_t\},\ t \geqslant 0$$

and

$$V^{\alpha} f(x) = E^x \int_0^{\infty} e^{-\alpha t} f(X_t) M_t \, dt \ .$$

Here f is any bounded or nonnegative measurable function. The following bounded operator associated with M is of fundamental importance. For $\alpha \geqslant 0$ define

$$(1)\quad P_M^{\alpha} f(x) = -E^x \int_0^{\infty} e^{-\alpha t} f(X_t) \, dM_t,\ x \in E_M \ ;\ P_M^{\alpha} f(x) = f(x)\,,\ x \notin E_M \ .$$

Here $E_M = \{x : P^x(M_0 = 1) = 1\}$ is the set of permanent points for M. If T_B is the hitting time of a Borel set B and $M_t = I_{[0,T_B)}(t)$, then $P_M^{\alpha} = P_B^{\alpha}$. Thus P_M^{α} extends the notion of "hitting operator" or "harmonic measure" to a general MF. If $U^{\alpha} f$ is finite, then

$$(2)\qquad U^{\alpha} f - V^{\alpha} f = P_M^{\alpha} U^{\alpha} f\,,$$

and this identity is the key to our development. Of course, P_M^{α} is given by a kernel $P_M^{\alpha}(x\,,dy)$.

If $\hat{M} = (\hat{M}_t)$ is a MF of $\hat{X}$, we write $(\hat{Q}_t)$ and $(\hat{V}^{\alpha})$ for the semigroup and resolvent generated by $\hat{M}$. In keeping with the pattern of notation established in Section VI – 1 of [1] we write the action of these operators as follows :

$$f\hat{Q}_t(x) = \int f(y)\,\hat{Q}_t(dy\,,x) = \hat{E}^x\{f(X_t) M_t\}$$

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$$f\hat{V}^\alpha(x) = \int f(y)\,\hat{V}^\alpha(dy\,,x) = \hat{E}^x \int_0^\infty e^{-\alpha t} f(X_t) M_t\, dt\ .$$

As in [1] we omit the hat "ˆ" in those places where it is redundant. For example $\hat{E}^x\{f(X_t)M_t\}$ is short for $\hat{E}^x\{f(\hat{X}_t)\hat{M}_t\}$. For notational convenience we write $\hat{E}_M$ in place of $E_{\hat{M}}$ for the set of permanent points of $\hat{M}$. Similarly we write $\hat{P}^\alpha_M$ in place of $P^\alpha_{\hat{M}}$ for the operator defined in (1) relative to $\hat{M}$, and we write the action of $\hat{P}^\alpha_M$ on a function f as $f\hat{P}^\alpha_M(x) = \int f(y)\,\hat{P}^\alpha_M(dy\,,x)$. With these conventions (2) becomes $f\hat{U}^\alpha - f\hat{V}^\alpha = f\hat{U}^\alpha\hat{P}^\alpha_M$ provided $f\hat{U}^\alpha$ is finite.

We now have the necessary notation to describe the main results of [2]. We write $dx = \xi(dx)$ and $(f\,,g) = \int f(x)\,g(x)\,dx$.

THEOREM A. – *Let M be an exact MF of X. Then there exists a unique exact MF, $\hat{M}$, of $\hat{X}$ such that for all $\alpha \geqslant 0$*

$$P^\alpha_M u^\alpha(x\,,y) = u^\alpha \hat{P}^\alpha_M(x\,,y)\ . \tag{3}$$

For each $\alpha > 0$, (3) is equivalent to $(f\hat{V}^\alpha\,,g) = (f\,,V^\alpha g)$ for all $f,\ g \in C^+_K$. In addition there exists a unique function $v^\alpha(x\,,y) \geqslant 0$ such that

$$V^\alpha(x\,,dy) = v^\alpha(x\,,y)\,dy\ :\ \hat{V}^\alpha(dy\,,x) = dy\ v^\alpha(y\,,x)\ , \tag{4}$$

$$u^\alpha(x\,,y) = v^\alpha(x\,,y) + P^\alpha_M u^\alpha(x\,,y) = v^\alpha(x\,,y) + u^\alpha\hat{P}^\alpha_M(x\,,y)\ , \tag{5}$$

and such that $x \to v^\alpha(x\,,y)$ is $\alpha - (X\,,M)$ excessive for each y and $y \to v^\alpha(x\,,y)$ is $\alpha - (\hat{X}\,,\hat{M})$ excessive for each x. Moreover, $v^\alpha(x\,,y)$ vanishes off $E_M \times \hat{E}_M$. Finally $E_M \Delta \hat{E}_M$ is semipolar, and if $F = E_M - \hat{E}_M$ then $M_{T_F} = 0$ almost surely on $\{T_F < \zeta\}$. In particular if M doesn't vanish on $[0\,,\zeta)$, then $E_M = E$ and $E - \hat{E}_M$ is polar.

The map $M \to \hat{M}$ is bijective from the class of exact *MF's* of X to the class of exact *MF's* of $\hat{X}$. We write $M \leftrightarrow \hat{M}$ for this correspondence and we say that M and $\hat{M}$ are *dual* (exact) functionals. It is immediate that if M and $\hat{M}$ are exact *MF's* of X and $\hat{X}$ respectively, then M and $\hat{M}$ are dual functionals if and only if $(f\hat{V}^\alpha\,,g) = (f\,,V^\alpha g)$ for all $\alpha > 0$ and $f, g \in C^+_K$, or, equivalently,

$$(f\hat{Q}_t\,,g) = (f\,,Q_t g) \quad \text{for all} \quad t > 0\ .$$

The next theorem is the main result of [2]. If M and N are *MF's*, then $MN = (M_t N_t)$ is again a *MF*, and if both M and N are exact so is MN. If $\lambda > 0$, $M^\lambda = (M^\lambda_t)$ – the λ - *th* power of M – is a *MF* which is exact whenever M is.

THEOREM B. – *The bijection $M \to \hat{M}$ is multiplicative, that is, $(MN)\hat{} = \hat{M}\hat{N}$. Furthermore, for each $\lambda > 0$, $(M^\lambda)\hat{} = \hat{M}^\lambda$.*

COROLLARY B_1 – Let T be an exact terminal time of X. Then there exists a unique (up to equivalence) exact terminal time $\hat{T}$ of $\hat{X}$ such that

$$P^\alpha_T u^\alpha(x\,,y) = u^\alpha \hat{P}^\alpha_T(x\,,y)\ .$$

We will say that T and T are dual terminal times and write $T \leftrightarrow \hat{T}$.

We now give some examples of the correspondence $M \leftrightarrow \hat{M}$. Let B be a Borel set. Then it follows from the familiar "switching identity" for dual processes (VI – 1.16 of [1]) that T_B and $\hat{T}_B$ are dual (exact) terminal times. Of course $\hat{T}_B$ denotes the hitting time of B by $\hat{X}$. Next let h be a bounded nonnegative Borel function. Then

$$M_t = \exp\left[-\int_0^t h(X_s)\,ds\right]$$

and

$$\hat{M}_t = \exp\left[-\int_0^t h(\hat{X}_s)\,ds\right]$$

are dual exact MF's. This can be extended to unbounded h provided one exercises a modicum of care. See (5.5) of [2]. Combining these examples with Theorem B one obtains the full strength of the duality relations proved by Hunt in [4].

The following examples lie somewhat deeper. See [3]. Let $0 < a < b < \infty$ and let d be a metric for E. Then

$$T = \inf\{t : d(X_{t-}, X_t) \in (a, b]\} \quad \text{and} \quad \hat{T} = \inf\{t : d(\hat{X}_{t-}, \hat{X}_t) \in (a, b]\}$$

are dual exact terminal times. In this statement one can replace $(a, b]$ by any Borel subset Γ of $(0, \infty)$ which is at a positive distance from the origin. Next, let A and B be Borel subsets of E such that $d(A, B) > 0$. Then

$$T_{A,B} = \inf\{t : X_{t-} \in A, X_t \in B\} \quad \text{and} \quad \hat{T}_{B,A} = \inf\{t : \hat{X}_{t-} \in B, \hat{X}_t \in A\}$$

are dual exact terminal times.

Let us now discuss the kernels $P_M^\alpha(x, dy)$ and $\hat{P}_M^\alpha(dy, x)$. A much sharper result than the next theorem is given in [2], but since it involves some technical conditions we content ourselves here with what follows. We let

$$S = \inf\{t : M_t = 0\}$$

THEOREM C. – *Let M and $\hat{M}$ be dual exact MF's and suppose that $M_{S-} = 0$ on $\{S < \zeta\}$ and that $\{\hat{M}_{\hat{S}-} = 0\}$ on $\{\hat{S} < \hat{\zeta}\}$. Suppose further that M (or $\hat{M}$) is natural. Then there exists a unique σ-finite measure ν carried by $E_M \cap \hat{E}_M$ such that $P_M^\alpha(x, dy) = v^\alpha(x, y)\,\nu(dy)$ if $x \in E_M$ and $\hat{P}_M^\alpha(dx, y) = \nu(dx)\,v^\alpha(x, y)$* if $y \in \hat{E}_M$.

Making use of Theorem C one can prove that (roughly speaking) M is natural (continuous) on $[0, S)$ if and only if $\hat{M}$ is natural (continuous) on $[0, \hat{S})$. See Theorem 8.6 of [2] for the precise result.

Finally we describe the relationship between the Lévy systems of X and $\hat{X}$. Let $\mathbf{A}_c(\hat{\mathbf{A}}_c)$ denote the collection of continuous additive functionals of $X(\hat{X})$ which are finite on $[0, \zeta)$ ($[0, \hat{\zeta})$) respectively. We say that $A \in \mathbf{A}_c$ and $\hat{A} \in \hat{\mathbf{A}}_c$ are dual CAF's provided that $M_t = \exp(-A_t)$ and $\hat{M}_t = \exp(-\hat{A}_t)$ are dual MF's. It follows from Theorem C (see also [6]) that if A and $\hat{A}$ are dual CAF's then there exists a unique σ-finite measure μ not charging semipolar sets such that

$$U_A^\alpha(x, dy) = u^\alpha(x, y)\,\mu(dy)\ ;\ \hat{U}_A^\alpha(dy, x) = \mu(dy)\,u^\alpha(y, x)\ .$$

THEOREM D. – *The Lévy systems $(N(x, dy), H_t)$ and $(\hat{N}(dy, x), \hat{H}_t)$ for X and $\hat{X}$ respectively may be chosen so that H and $\hat{H}$ are dual CAF's in $\mathbf{A}_c$ and $\hat{\mathbf{A}}_c$*

respectively and such that if μ *is the measure associated with* H *and* $\hat{H}$, *then* N *and* $\hat{N}$ *are dual kernels relative to* μ, *that is,*

$$\int f(Ng)\, d\mu = \int (f\hat{N})\, g\, d\mu \ .$$

Usually Lévy systems are discussed under the assumption that the process is special standard. However, it follows from recent results of M.J. Sharpe that at least for dual processes this assumption may be dispensed with.

In [6] Revuz has completely characterized *CAF's* and natural pure jump *AF's* of dual processes. It is now easy to characterize quasi-left-continuous pure jump *AF's*. It is well known that any such *AF*, A, has the form

$$A_t = \sum_{s \leqslant t} f(X_{s-}, X_s)$$

with f a nonnegative Borel function on $E \times E$ such that $f(x, x) = 0$. The next theorem characterizes those f's which give rise to *AF's*. For simplicity we assume that f is bounded. The extension to general f is the same as in [6].

THEOREME E. — *Let* f *be a bounded nonnegative Borel function on* $E \times E$ *which vanishes on the diagonal. Then* $A_t = \sum_{s \leqslant t} f(X_{s-}, X_s)$ *is an AF of* X *if and only if there exists an increasing sequence* $\{E_n\}$ *of Borel subsets of* E *whose union is* E *and such that :* (i) $\int u^1(x\ z) \int f(z, y) I_{E_n}(y) N(z, dy)\, \mu(dz)$ *is bounded for each* n *; and* (ii) *if* T_n *is the hitting time of* E_n^c *and* $T = \lim T_n$, *then* $T > 0$ a.s. *In this case* A *is finite a.s. on* $[0, T)$. *Moreover, if* $T \geqslant \zeta$ a.s., *then there exists a polar set* P *such that* $\hat{A}_t = \sum_{s \leqslant t} f(\hat{X}_s, \hat{X}_{s-})$ *defines a finite AF of* $\hat{X}$ *restricted to* $E - P$.

REFERENCES

[1] BLUMENTHAL R.M. and GETOOR R.K. — *Markov Processes and Potential Theory*, Academic Press, New York and London, 1968.

[2] GETOOR R.K. — Multiplicative functionals of dual processes. To appear *Ann. Inst. Fourier Grenoble.*

[3] GETOOR R.K. — *Duality of Lévy systems*, to appear.

[4] HUNT G.A. — Markoff processes and potential theory, III. *Illinois J. Math.*, 2, 1958, p. 151-213.

[5] MEYER P.A. — Quelques résultats sur les processus. *Invent. Math.*, 1, 1966, p. 101-115.

[6] REVUZ D. — Mesures associées aux fonctionelles additives de Markov, I and II: I) *Trans. Amer. Math. Soc.*, 148, 1970, p. 501-531; II) to appear.

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HITTING OF SETS BY PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

by Harry KESTEN

In this note we would like to report on some recent results concerning the probability that a set B is hit by a (right continuous) process $X = \{X_t\}_{t \geqslant 0}$ with stationary independent increments, i.e., $P\{T_B < \infty\}$, where T_B is the first hitting time of B,

$$T_B = \inf \{t > 0 : X_t \in B\} . \tag{1}$$

Most of our remarks will concern one point sets $B = \{b\}$. In this case, we write T_b for $T_{\{b\}}$ and

$$h(b) = P\{T_b < \infty\} . \tag{2}$$

We also discuss the related question of regularity of b, i.e., when is $P\{T_b = 0\} = 1$, as well as the existence of a nice local time. Unless otherwise stated we talk about processes taking values in $\mathbf{R}^d$ and starting at the origin, i.e., we take

$$P\{X_0 = 0\} = 1 . \tag{3}$$

The simplest results about $h(b)$ concern Brownian motion. In dimension one $h(b) \equiv 1$ since X is continuous and unbounded ; for a higher dimensional Brownian motion it is well known that $h(b) \equiv 0$ (see [9], 2.7.9 or [18], sect. 5). Other early results treat $h(b)$ for one dimensional stable processes with

$$Ee^{i\lambda X_t} = \exp - t|\lambda|^{\alpha}(1 + i\beta \operatorname{sgn}(\lambda)) \quad 0 < \alpha \leqslant 2 , \alpha \neq 1 , \tag{4}$$

respectively

$$Ee^{i\lambda X_t} = \exp - t|\lambda|(1 + i\beta \operatorname{sgn}(\lambda) \log|\lambda|), \quad \alpha = 1 : \tag{5}$$

(6) If $\alpha < 1$, and X_t symmetric, i.e., $\beta = 0$ in (4), then $h(b) \equiv 0$ (according to [10] this follows from results of Lévy ; the proof in the beginning of sect. 5 in [18] is also applicable).

(7) If $\alpha = 1$ and $\beta = 0$ in (5), then $h(b) \equiv 0$ (Erdös, unpublished, Port [18], sect. 5). If $\alpha = 1$, but the process is not symmetric, i.e., $\beta \neq 0$ in (5), then $h(b) > 0$ for all b (Port and Stone [19]).

(8) If $1 < \alpha \leqslant 2$ then $h(b) > 0$ for all b (Kac [10] and Port [18], sect. 2).

Finally, Hunt's capacity theory, [7] sect. 19, allows one to conclude under suitable hypotheses that $h(b) > 0$ at least for some b (see also [3], sect. VI 4.3, [18], sect. 2, [17], sect. 3, [20], sect. 6,7).

Before stating the general result we mention a problem of Chung [6], which was the motivation for studying the strict positivity of $h(b)$: Let σ be a right continuous, decreasing function on $[0, \infty)$ satisfying

$$(9) \qquad \sigma(y) \downarrow 0, y \to \infty, \quad \text{and} \quad \left| \int_0^\infty \min(1, y)\, d\sigma(y) \right| < \infty .$$

Does there exist a Borel measure W on $[0, \infty)$ for which

$$(10) \qquad \int_{0 \leqslant s \leqslant r} \sigma(r-s)\, W(ds) = 1$$

for all $r > 0$? Neveu, [16] p. 40, 41, and Chung, [6], proved with simple Laplace transform techniques that (10) holds for (Lebesgue) *almost all* r when

$$W(B) = E \int_0^\infty I_B(\widetilde{X}_t)\, dt = \int_0^\infty P\{\widetilde{X}_t \in B\}\, dt ,$$

where $\widetilde{X}$ is a process with stationary independent increments and characteristic function

$$(11) \qquad Ee^{i\lambda \tilde{X}_t} = Ee^{i\lambda(\tilde{X}_{t+s} - \tilde{X}_s)} = \exp - t \int_0^\infty [e^{i\lambda y} - 1]\, d\sigma(y) .$$

Meyer [15] showed that the left hand side of (10) equals $1 - h(r)$, where $h(\cdot)$ is given by (1) and (2) with X_t replaced by $\widetilde{X}_t$ (this fact can also be obtained from [8], theorem 1). Thus, Chung's question reduces to : Is $h(\cdot) \equiv 0$ for $\widetilde{X}$? The affirmative answer to this question is contained in theorem A, case (i), below. Another interesting interpretation of the result was given by Blumenthal and Getoor [3], p. 220, 221, and Meyer [15], sect. 4 : Under mild conditions, if Y is a standard Markov process and y a regular, instantaneous point for Y then for each fixed $r > 0$, $P^y\{Y_r = y\} = 0$.

Consider now a completely general one-dimensional process with stationary independent increments. Its characteristic function has the form

$$(12) \qquad Ee^{i\lambda X_t} = e^{-t\psi(\lambda)} ,$$

$$\psi(\lambda) = -ia\lambda + \frac{\sigma^2\lambda^2}{2} - \int_{\mathbf{R}-\{0\}} \left[e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2} \right] \nu(dy)$$

where ν is a measure on $\mathbf{R} - \{0\}$ for which

$$\int \min(1, y^2)\, \nu(dy) < \infty$$

(see [13], Ch. 7). If

$$(13) \qquad \int \min(1, |y|)\, \nu(dy) < \infty$$

we shall always write

$$(14) \qquad \psi(\lambda) = -ia'\lambda + \frac{\sigma^2\lambda^2}{2} - \int_{\mathbf{R}-\{0\}} [e^{i\lambda y} - 1]\, \nu(dy)$$

$$a' = a - \int \frac{y}{1+y^2}\, \nu(dy) .$$

For brevity *we shall assume throughout*

(15) $$\nu(\mathbf{R} - \{0\}) = \infty \, .$$

THEOREM A. – *Let* $C = \{b : h(b) > 0\}$. *If* X *is of the form* (12) *and satisfies* (15) *then* $C = \emptyset$, $\mathbf{R}$, $(0, \infty)$ *or* $(-\infty, 0)$; $C = \emptyset$ *if and only if*

(16) $$\int_{\mathbf{R}} \mathrm{Re}\ (\gamma + \psi(\lambda))^{-1}\, d\lambda = \infty \quad \textit{for any} \quad \gamma > 0 \, .$$

(i) $C = \emptyset$ *when* (13) *holds and* $a' = 0, \sigma^2 = 0$ *or when* (13) *fails,* $\sigma^2 = 0$,

(17) $$\int_{0 < y \leq 1} |y|\, \nu(dy) = \int_{-1 \leq y < 0} |y|\, \nu(dy) = \infty \, ,$$

as well as (16).

(ii) $C = (0, \infty)$ $((-\infty, 0))$ *when* (13) *holds,* $\sigma^2 = 0$, *and* $a' > 0, \nu((-\infty, 0)) = 0$ (resp. $a' < 0, \nu((0, \infty)) = 0$).

(iii) $C = \mathbf{R}$ *in the other cases, i.e., when* $\sigma^2 > 0$ *or when* (13) *holds and* $a' > 0$, $\nu((-\infty, 0)) > 0$ *or* (13) *and* $a' < 0, \nu((0, \infty)) > 0$ *or when* (13) *and* (17) *fail or when* (17) *holds, but* (16) *fails.*

Let us also mention that for "honestly higher dimensional" processes $h(\cdot) \equiv 0$ ([11], theorem 3).

Even though some incomplete proofs for case (i) of theorem A had been published, the first complete proof seems to be in [11] theorem 1 and 2. An interesting, completely analytic proof for the special case of Chung's problem was given by *L.* Carleson (unpublished). By far the most beautiful proof is a recent purely probabilistic one, due to J. Bretagnolle [5].

Theorem A also gives us information about the regularity of 0 for itself, i.e., about when $P^0\{T = 0\} = 1$. Clerarly it is necessary that $h(0) > 0$ for this. If one also takes into account a result of Shtatland ([21], theorem 1) to the effect that $t^{-1} X_t \to a'$ w.p.1 when (13) holds, one sees that $P^0\{T = 0\} = 1$ is possible only in case (iii), when in addition (13) does not hold. We conjectured ([11], p. 9) that 0 is indeed regular in those cases ; certain more special situations had already been treated by Port [18], Orey [17] and [11], p. 51, 52. Bretagnolle [5] proved the conjecture true in general. Notice the additional information contained in his

THEOREM B. – *When* X *is of the form* (12), *satisfies* (15) *and* $C \neq \emptyset$ *(i.e., in cases* (ii) *and* (iii) *of theorem A), the potential operator* U^γ *has the density* $u^\gamma(x) = k\, E \exp - \gamma T_x$ *for some* $0 < k = k^\gamma < \infty$, *i.e., for* $f \geq 0$

$$U^\gamma f \equiv E \int_0^\infty e^{-\gamma t} f(X_t)\, dt = \int_{\mathbf{R}} u^\gamma(x) f(x)\, dx \ ;$$

$u^\gamma(x)$, *and hence* $Ee^{-\gamma T_x}$, *is continuous on* $\mathbf{R} - \{0\}$. *Moreover*

(18) $$Ee^{-\gamma T_0} = \liminf_{x \to \infty} Ee^{-\gamma T_x} \leq \limsup_{x \to \infty} Ee^{-\gamma T_x} = 1 \, .$$

If in addition (13) holds and $a' \neq 0$, $\sigma^2 = 0$ *then* u^γ *has a jump discontinuity at* 0, 0 *is not regular for* $\{0\}$, *and the fine topology induced by* X *is the right (left) topology if* $a' > 0$ *(resp.* $a' < 0$*). If (13) does not hold (i.e., in the remaining cases subsumed under* (iii)) u^γ *is continuous,* 0 *is regular for* $\{0\}$, *and the fine topology is the ordinary topology of* $\mathbf{R}$.

The above questions have been generalized in two directions by Port and Stone [20]. Firstly they consider processes on an arbitrary 2^{nd} countable locally compact abelian group. Secondly, and of greater interest from our point of view, they consider the hitting time T_B of an arbitrary measurable set B, instead of one point sets only. Hunt's capacity theory, [7] sect. 19, applied to processes with *smooth* potential operators. Apart from the many results in [20] on the asymptotic behavior of hitting probabilities and potential operators for processes with stationary independent increments, Port and Stone make the major contribution of a capacity theory for such processes, *without any smoothness assumptions.* In particular, they define for any $\gamma > 0$ a (Choquet) capacity $C^\gamma(\cdot)$ such that for a Borel set B $C^\gamma(B) = 0$ if and only if $P^x\{T_B < \infty\} = P\{T_{B-x} < \infty\} = 0$ for almost all starting points x. If the state space is $\mathbf{R}$ (and (15) holds), then this result combined with theorem A shows

$$(19) \qquad C^\gamma(B) = 0 \quad \text{if and only if} \quad P^x\{T_B < \infty\} = 0 \quad \text{for all} \quad x,$$

when B is a one point set, and consequently also for countable B. This leads us to the

Problem. – For what sets B is (19) valid ?

It is not valid for all sets B. For instance one can construct a process on $\mathbf{R}$ (satisfying (15)) and a dense group $G \subset \mathbf{R}$ of zero Lebesgue measure such that $P\{X_t \in G \text{ for all } t \geq 0\} = 1$. Clearly $P^x\{T_G < \infty\} = P\{T_{G-x} < \infty\} = 0$ or 1 according as $x \notin G$ or $x \in G$. On the other hand, one easily shows that (19) holds for every measurable B when the measure U^γ is absolutely continuous

$$\left(U^\gamma(A) = \int_0^\infty e^{-\gamma t} P\{X_t \in A\}\, dt \right).$$

So far we discussed $P^x\{R_\infty \cap B \neq \emptyset\}$, where $R_t = R_t(\omega) = \{X_s(\omega) : 0 < s \leq t\}$ is the range of the process up till time t. Many other aspects of R_t have been investigated, such as its Hausdorff dimension and measure in the case where R_t has zero measure w.p.1 ([1], sect. 8, [22], sect. 7). We want to close here with some measurement of how evenly the values of the process are spread out in the cases where R_t has positive measure w.p.1. We restrict ourselves to one dimensional processes with all points regular and satisfying (15). By theorem B U^γ has a continuous bounded density in such a situation and by results of Blumenthal and Getoor, ([2], sect. 3 ; see also [12], theorem 2) there exists a local time $L_t^x(\omega)$ which is jointly measurable in (x, t, ω). For fixed x it is a continuous additive functional, and alsmost surely

$$(20) \qquad \int_0^t I_B(X_s)\, ds = \int_B L_t^x\, dx, \; t > 0, \; B \text{ Borel set in } \mathbf{R}.$$

Thus L_t^x measures in some way the amount of time spent by $\{X_s\}_{0 \leq s \leq t}$ at or near x and the continuity or possible Hölder conditions satisfied by L_t indicate how evenly the values of X are spread out in space. Trotter [23] showed that for one dimensional Brownian motion

$$|L_t^x - L_t^y| = 0(|x-y| \log |x-y|^{-1})$$

(see [9], sect. 2.8 for later results). Boylan [4] and Meyer [14] generalized Trotter's estimates, and a further refinement of their work shows that one can choose L_t^x continuous in (x, t) for all ω essentially under the condition (1)

$$(21) \qquad \Sigma n^{1/2+\epsilon}[1 - E\exp(-T_{2^{-n}}) + 1 - E\exp(-T_{-2^{-n}})]^{1/2} < \infty$$

for some $\epsilon > 0$; the precise condition is a bit more complicated than (21), see [12]. There are, however, processes which do not satisfy (21), and it is shown in [12] that for a process of the form (12) with

$$(22) \qquad \limsup_{\gamma \to \infty} \log \gamma \int_{\mathbf{R}} d\lambda \operatorname{Re}[\gamma + \psi(\lambda)]^{-1} > 0$$

there exists no continuous version of L_t. E.g., the asymmetric Cauchy process (5) with $\beta \neq 0$ does have a local time satisfying (20), but not one that is continuous in the space variable. Hence, even though $T_x \to 0$ in probability as $x \to 0$ for these processes (by theorem B), they jump around rather badly. There is a small gap between (21) and (22) (see [12]) and this suggests the following

Problem. – Find a n.a.s.c. for the existence of a local time L_t^x continuous in (x, t), for a process with stationary independent increments.

REFERENCES

[1] Blumenthal R.M. and Getoor R.K. — *J. Math. Mech.*, 10, 1961, p. 493-516.
[2] Blumenthal R.M. and Getoor R.K. — *Z. Wahrscheinlichkeitsth. verw. Geb.*, 3, 1964, p. 50-74.
[3] Blumenthal R.M. and Getoor R.K. — *Markov processes and potential theory*, Academic Press, 1968.
[4] Boylan E.S. — *Ill. J. Math.*, 8, 1964, p. 19-39.
[5] Bretagnolle J. — To appear, 1970.
[6] Chung K.L. — *C. R. Acad. Sci. Paris*, 260, 1965, p. 4665-4667 and 6794-6796.
[7] Hunt G. — *Ill. J. Math.*, **1**, 1957, p. 44-93 and p. 316-369, and **2**, 1958, p. 151-213.
[8] Ikeda N. and Watanabe S. — *J. Math. Kyoto Univ.*, 2, 1962, p. 79-95.
[9] Ito K. and Mc Kean H.P. Jr. — *Diffusion processes and their sample paths*, Springer-Verlag, 1965.
[10] Kac M. — *Publ. Inst. Stat. Univ. Paris*, 6, 1957, p. 303-306.
[11] Kesten H. — *Am. Math. Soc.*, Memoir No. 93, 1969.

(1) [12] will not appear in its original form, but in an improved version, written jointly with R.K. Getoor. It will be shown there that (21) may be replaced by

$$\sum_n \sup_{|x| \leq 2^{-n}} [1 - E\exp(-T_x) + 1 - E\exp(-T_{-x})]^{1/2} < \infty.$$

[12] Kesten H. — *Continuity of local times for Markov processes,* 1971, to appear.
[13] Lévy P. — *Théorie de l'addition des variables aléatoires,* Gauthier-Villars, 1937.
[14] Meyer P.A. — *Publ. Inst. Stat. Univ. Paris,* 15, 1966, p. 295-310.
[15] Meyer P.A. — Séminaire de probabilités III, *Lecture notes in mathematics,* Vol. 88, Springer-Verlag, 1969, p. 175-189.
[16] Neveu J. — *Abh. Math. Sem. Univ. Hamburg,* 25, 1961, p. 36-61.
[17] Orey S. — *in* :Markov processes and potential theory, *Proc. of Symposium at Univ. of Wisconsin,* 1967, p. 117-126, edited by J. Chover, John Wiley and Sons.
[18] Port S.C. — *J. d'Anal. Math.,* 20, 1967, p. 371-395.
[19] Port S.C. and Stone C.J. — *Ann. Math. Stat.,* 40, 1969, p. 137-143.
[20] Port S.C. and Stone C.J. — Infinitely divisible processes and their potential theory, to appear *Ann. Inst. Fourier Grenoble,* 1970 (see also *Bull. Amer. Math. Soc.,* 75, 1969, p. 848-851).
[21] Shtatland E.S. — *Teor. Verojatnost. i Primenen,* 10, 1965, p. 344-350.
Prnitt and S.S. Taylor, *Trans. Am. Math. Soc.,* 146, 1969, p. 299-321).
[22] Taylor S.J. — *J. Math. Mech.,* 16, 1967, p. 1229-1246 (corrected in Pruitt W.E. and Taylor S.J., *Trans. Am. Math. Soc.,* 146, 1969, p. 299-321).
[23] Trotter H.F. — *Ill. J. Math.,* 2, 1958, p. 425-433.

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DUALITÉ FORMELLE ET REPRÉSENTATION INTÉGRALE DES FONCTIONS EXCESSIVES

par Gabriel MOKOBODZKI

Hypothèses, notations :

Soit $(X, \mathcal{B})$ un espace mesurable, $(V_\lambda)_{\lambda \geq 0}$ une famille résolvante sous-markovienne de noyaux ≥ 0 sur $(X, \mathcal{B})$, satisfaisant à l'hypothèse (L) de Meyer et telle que le noyau $V_0 = V$ soit borné. Il existe alors une mesure $\mu_0 \geq 0$, bornée sur $(X, \mathcal{B})$ telle que pour toute mesure $\alpha \geq 0$ sur $(X, \mathcal{B})$, la mesure αV soit de base μ. Désignons par $\mathfrak{S}$ le cône convexe des fonctions excessives par rapport à la famille résolvante (V_λ). Dans ce qui suit, on va montrer que l'on peut définir sur l'espace vectoriel $E = \mathfrak{S} - \mathfrak{S}$ une topologie localement convexe séparée telle que $\mathfrak{S}$ soit un cône *complet* et *métrisable* pour la topologie faible sur E, ce qui permettra d'appliquer le théorème de représentation intégrale de Choquet.

Ce résultat peut être obtenu par deux méthodes, l'une formelle, qui s'appuie sur la dualité des cônes de potentiels, l'autre plus élémentaire, qui repose sur la construction d'une résolvante duale satisfaisant aussi à la condition (L) et de même base μ_0. Seule la première méthode est ici présentée.

I. Dualité des cônes de potentiels.

La notion de cône de potentiels a d'abord été définie, en collaboration avec Daniel Sibony, dans un cadre fonctionnel, dans [5] en 1968, puis affinée par l'auteur dans [3] et [4].

La présentation formelle qui suit est reprise de [4].

Récemment et indépendamment, Nicu Boboc et Aurel Cornea ([1] et [2]) ont introduit une notion voisine qui leur permet d'obtenir un analogue du théorème n° 3 ci-dessous et de résoudre certains problèmes de dualité.

Notations

Soient $(E, \leq)$ un espace vectoriel ordonné, et E^+ le cône positif de E, E^* le dual algébrique de E, E^{*+} le cône des formes linéaires positives sur E. On supposera toujours que E^{*+} sépare E.

DEFINITION 1. — Soit $C \subset E^+$ un cône convexe. On dira que le couple $(C, \leq)$ définit un *cône de potentiels* si la condition (P) est satisfaite :

$$(P) \left\{ \begin{array}{l} \text{Pour tous } u, v, C, R(u-v) = \inf\limits_{\leq}\{w \in C ; w \geq u - v\} \text{ existe ;} \\ \\ R(u-v) \in C, R(u-v) \prec u \text{ et } (u-v) \leq R(u-v) \end{array} \right.$$

(la relation "$s \prec t$" signifie $(t - s) \in C$).

Cette définition ne fait intervenir que le sous-espace $C - C$ de E et la relation $\leqslant$. On vérifie que toute face de C est un cône de potentiels.

DEFINITION 2. — On appellera cône dual de $(C, \leqslant)$ le couple $(C^*, \prec)$ défini de la manière suivante :

(a) C^* est l'ensemble des formes linéaires sur $(C - C)$, croissantes pour l'ordre défini par la relation $\leqslant$.

(b) Si $\mu, \nu \in C^*$, $(\mu \prec \nu) \Leftrightarrow (< \mu, v > \leqslant < \nu, v >)$ pour tout $v \in C$. On dira alors que μ est balayée de ν relativement à C.

Le cône des fonctions excessives, le cône des mesures excessives sont des cônes de potentiels.

THEOREME 3. —

(1) *Le couple* $(C^*, \prec)$ *est un cône de potentiels, et pour tout* $\mu, \nu \in C^*$ *et*

$$p \in C, < R(\mu - \nu), p > = \sup\{< \mu - \nu, q > \mid q \in C, 0 \leqslant q \leqslant p\}$$

(2) *Le cône* C^* *est réticulé (et même complètement réticulé) pour son ordre propre*

(3) *Le cône* C^* *est inf. stable pour la relation d'ordre du balayage*

(4) *Le cône* C^* *est faiblement complet pour la topologie* $\sigma(C^*, C)$.

Exemple. — Supposons que X soit un espace compact, que $V_\lambda(\mathcal{C}(X)) \subset \mathcal{C}(X)$ et que $V(\mathcal{C}(X))$ soit dense dans $\mathcal{C}(X)$. Le cône $C = \overline{V(\mathcal{C}^+(X))}$ est un cône de potentiels, le cône C^* s'identifie au cône $M^+(X)$ des mesures de Radon $\geqslant 0$ sur X et pour $\mu, \nu \in C^*$, la relation $\mu \prec \nu$ equivaut à $\mu V \leqslant \nu V$. Le théorème précédent affirme en particulier qu'il existe $\sigma \in M^+(X)$ telle que $\sigma V = \inf\limits_{\leqslant} (\mu V, \nu V)$.

La première méthode de représentation intégrale dans le cône des fonctions excessives, consiste à trouver un cône de potentiels $\mathcal{S}_0^* \subset \mathcal{S}^*$ tel que l'on ait $\mathcal{S} \equiv (\mathcal{S}_0^*)^*$.

Soit $(C, \leqslant)$ un cône de potentiels ; on va définir différentes classes d'éléments de C^*.

DEFINITION 4. — On dira que $\theta \in C^*$ est *régulier* si pour toute famille $(\theta_j) \subset C^*$, filtrante croissante pour l'ordre $\prec$, telle que $\theta = \sup\limits_{\prec} \theta_j$ on a $\inf\limits_{\prec}. R(\theta - \theta_j) = 0$.

DEFINITION 5. — On dira que $\theta \in C^*$ est *accessible* si l'on a

$$\theta = \sup_{\prec} \{\theta_p \mid p \in C\} \quad \text{où} \quad \theta_p = \inf_{\prec} \{\theta' \in C^* \mid \theta' \prec \theta \,; < \theta - \theta', p > \leqslant 1\}$$

DEFINITION 6. — On dira que $\sigma \in C^*$ est *dominé* par $\theta \in C^*$, ce qu'on notera $\sigma \in d(\theta)$, si pour tout $\epsilon > 0$, il existe $p \in C$, tel que pour $\sigma' \in C^*$,

$$- \qquad (0 \leqslant \sigma' \leqslant \sigma) \quad \text{et} \quad (< \sigma - \sigma', p > \leqslant 1) \Rightarrow (\sigma - \sigma' \prec \epsilon\theta)$$

On dira que $\sigma \in C^*$ est *dominable* s'il existe $\theta \in C^*$ dominant σ.

L'élément 0 de C est à la fois régulier, accessible et dominable.

Désignons par C_r^*, C_a^* respectivement l'ensemble des éléments *réguliers* et *accessibles*.

THEOREME 7. — *Les ensembles C_r^*, C_a^* sont des cônes convexes, qui sont des faces de C^*. De plus, C_r^* et C_a^* sont fermés pour la topologie de la convergence uniforme sur la famille des ensembles $A_p = \{q \in C \mid q \leqslant p\}$, p parcourant C.*

Enfin pour toute famille $(\sigma_i) \subset C_a^*$, filtrante, croissante et majorée pour l'ordre $\prec$, $\sup\limits_{\prec} \sigma_i = \sigma \in C_a^*$.

LEMME. — Si $\sigma, \theta \in C^*$ et si $\sigma \in d(\theta)$, alors $R(\sigma - \lambda\theta) \in d(\sigma)$ pour tout $\lambda > 0$.

THEOREME. — *Si $C = \bigcup\limits_n A_{p_n}$ où $A_{p_n} = \{q \in C \mid q \leqslant p_n\}$ et $(p_n)_{n\in\mathrm{N}} \subset C$, alors tout élément de C qui est à la fois régulier et accessible est dominable par un élément $\theta \in C^*$ de la forme $\theta = \sum\limits_{k=1}^{\infty} \alpha_k$ où $\alpha_k \leqslant \sigma$, $\alpha_k \in C^*$.*

Expression de la domination en termes de compacité

Soient $\theta \in C^*$, $K_\theta = \{p \in C \mid < p, \theta > \leqslant 1\}$ et $H_\theta = \{\mu \in C^* - C^* \mid \exists\, n$ tel que $-n\theta \prec \mu \prec n\theta\}$; alors θ est une unité d'ordre sur H_θ et définit une norme p_θ sur H_θ :

$$p_\theta(\mu) = \inf\{\lambda > 0 \mid -\lambda\theta \prec \mu \prec \lambda\theta\}$$

THEOREME. — *Si $\sigma \in C^*$ est régulière et si $\sigma \in d(\theta)$, alors*

(1) *$M_\sigma = \{\sigma' \in C^* \mid 0 \leqslant \sigma' \leqslant \sigma\}$ est compact pour la norme p_θ*

(2) *K_θ est précompact pour la convergence uniforme sur M_σ.*

(Si C est un cône de fonctions et σ une mesure $\geqslant 0$ (2) signifie que K_θ est précompact dans $L^1(\sigma)$).

Exemple. — Soit $(V_\lambda)_{\lambda \geqslant 0}$ une famille résolvante sous-markovienne sur $(X, \mathcal{B})$, $V_0 = V$ borné, B l'espace des fonctions mesurables bornées sur X, $\mathfrak{M}(X)$ l'espace des mesures bornées sur X et $C = \{Vf \mid f \in B^+\}$

(1) Pour toute $\nu \in \mathfrak{M}^+(X)$, νV définit un élément régulier de

$$C^* : < \nu V, Vf > = \int V^2 f\, d\nu \quad .$$

(2) Si (V_λ) satisfait à l'hypothèse (L) de Meyer, toute $\nu \in \mathfrak{M}^+(X)$ définit un élément accessible de C^*.

Nous conservons les notations de l'exemple précédent : $C = V(B^+)$, $\mathfrak{S}$ est le cône des fonctions excessives finies μ_0. p.p., V étant supposé de base μ_0 (hypothèse (L)).

Pour tout $\nu \in C^*$ définissons le prolongement de ν à $\mathcal{S}$, par

$$< \nu , v > = \sup \{< \nu , p > \mid p \in C , p \leqslant v\}$$

THEOREME. — *Soient $\alpha \in \mathcal{M}^+(X)$, ν et $\theta \in C^*$ tels que $\nu , \theta \leqslant \alpha V$ et $\nu \in d(\theta)$; pour tout $\lambda > 0$, $R(\nu - \lambda\theta)$ est fini sur $\mathcal{S}$. Posons $C_b^* = \{\nu \in C^* \mid \nu$ fini sur $\mathcal{S}\}$ et*

$$C_0^* = \{\nu \in C_a^* \cap C_b^* \cap C_r^* \mid \exists\theta \in C_b^* \quad \textit{tel que} \quad \nu \in d(\theta)\}$$

THEOREME. —

(1) *C_0^* est une face de C^*, donc un cône de potentiels.*

(2) *il existe un ensemble dénombrable $D \subset C_0^*$ qui est riche en ce sens que pour toute $\mu \in C_0^*$, $\exists \nu \in D$,* tel que *$\mu \in d(\nu)$ et pour toute $\sigma \in C^*$ on a*

$$\sigma = \sup_{\prec}\{\mu \in D \mid \mu \prec \sigma\}$$

(3) *$(C_0^*)^*$ est métrisable et complet pour la topologie de la convergence simple dans C_0^*.*

(4) $\mathcal{S} \equiv (C_0^*)^*$

THEOREME. — *Il existe une suite (μ_n) de mesures $\geqslant 0$ sur (X , B) telle que $\mu_n \leqslant \mu_0 \, \forall n$ et telle que* : (a) *chaque μ_n définit un élément de C_0^* donc μ_n est fini sur $\mathcal{S}$* ; (b) *la famille (μ_n) sépare $\mathcal{S}$ et $\mu_0 = \sup \mu_n$* ; (c) *sur $\mathcal{S}$ la topologie $\sigma(\mathcal{S}, C_0^*)$ et la topologie de la convergence forte dans tous les espaces $L^1(\mu_n)$ sont identiques* ; (d) *pour une suite $(v_n) \subset \mathcal{S}$, et $v \in \mathcal{S}$ les conditions suivantes sont équivalentes* :

(i) $\lim\limits_{p \to \infty} \int |v_p - v| \, d\mu_n = 0$ pour tout n

(ii) pour toute suite $(v'_p) \subset v_p$ $\qquad \widehat{\text{lim.inf. } v'_p} = v$

(voir aussi la topologie de la convergence en graphe [6]).

Représentation intégrale individuelle.

Soit $v_0 \in \mathcal{S}$ et soit $\nu \in \mathcal{M}^+(X)$ équivalente à μ_0 et telle que $\int v_0 d\nu \leqslant 1$. Il existe alors $\nu' \leqslant \nu$, ν' équivalente à ν, telle que $\nu' \in d(\nu)$ de sorte que

$$K_\nu = \{w \in \mathcal{S} \mid \int w \, d\nu \leqslant 1\}$$

est compact pour la topologie forte de $L^1(\nu')$. Les ensembles K_ν forment donc une famille fondamentale de chapeaux du cône $\mathcal{S}$; en particulier, pour toute suite $(v_n) \subset \mathcal{S}$ convergente vers $v_0 \in \mathcal{S}$ pour la topologie $\sigma(\mathcal{S}, C_0^*)$ il existe un chapeau K_ν qui contient la suite (v_n) et v_0 ; et pour toute $\nu' \in d(\nu)$, (v_n) converge vers v_0 dans $L^1(\nu')$.

BIBLIOGRAPHIE

[1-2] BOBOC N. et CORNEA A. — *Notes aux comptes rendus de l'Académie des Sciences,* Paris, 270, Série A, 1970, Séances du 16 fév. 1970 et du 8 juin 1970.

[3] MOKOBODZKI G. — *Cônes de potentiels et noyaux subordonnés* (Centro Internazionale matématico estivo, C.I.M.E.), Stresa 1969.

[4] MOKOBODZKI G. — Structure des cônes de potentiels, *Séminaire Bourbaki,* 22e année, 1969-70, n° 377.

[5] MOKOBODZKI G. et SIBONY D. — Sur une propriété caractéristique du cône des potentiels. *Comptes-rendus Acad. Sci., Paris,* 266, Série A, 215.

[6] MOKOBODZKI G. — Représentation intégrale des fonctions surharmoniques au moyen des réduites, *Annales de l'Institut Fourier,* T. XIV.

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SUR LES ÉQUATIONS STOCHASTIQUES AUX DÉRIVÉES PARTIELLES

par A.N. SHIRYAYEV

1.

Le problème du filtrage non linéaire des processus stochastiques amène à une recherche de la structure de l'espérance mathématique conditionnelle

$$\Pi_t(f) = M\,[f(\theta_t)|\mathcal{F}_t^X]$$

où le processus aléatoire $(\theta\,,X) = \{(\theta_t\,,X_t)\,,\,t \geqslant 0\}$ a une première composante θ qui s'estime à partir des résultats d'observations de la deuxième ; $\mathcal{F}_t^X = \sigma\{X_s\,,\,s \leqslant t\}$ (cf. [4], [12], [15], [8]). Dans le cas où la composante observée X a un caractère diffusionnel, on peut espérer que l'espérance conditionnelle $\Pi_t(f)$ est aussi un processus de type diffusionnel. Ce compte rendu est consacré à la recherche des différentielles stochastiques de $\Pi_t(f)$ et à l'étude des équations différentielles stochastiques aux dérivées partielles qui apparaissent dans ce problème.

2.

Le processus observé $X = \{x_t\,,\,0 \leqslant t \leqslant 1\}$ est supposé vérifier l'équation différentielle stochastique

$$(1) \qquad dX_t = A\,(t\,,\theta\,,X)\,dt + dW_t\quad,\quad X_0 = 0$$

où $W = \{W_t\,,\,0 \leqslant t \leqslant 1\}$ est un processus de Wiener standard et $\theta = \{\theta_t\,,\,0 \leqslant t \leqslant 1\}$ le processus qu'il nous faut estimer. Au sujet de l'équation (1) nous nous intéressons au problème de l'existence et de l'unicité de sa solution et à la continuité absolue de la mesure μ_X associée au processus X par rapport à la mesure de Wiener μ_W. La connaissance de ces mesures est importante pour trouver la différentielle $d\Pi_t(f)$.

Considérons le processus aléatoire réel

$$\tilde{\theta} = \{\tilde{\theta}_t(\tilde{\omega})\,,\,0 \leqslant t \leqslant 1\}\quad,\quad \tilde{\mathcal{F}}_t = \sigma\{\tilde{\omega} : \tilde{\theta}_s(\tilde{\omega})\,,s \leqslant t\},$$

défini sur l'espace probabilisé $(\tilde{\Omega}\,,\tilde{\mathcal{F}},\tilde{P})$. Soit C l'espace de Banach des fonctions continues $g = g_t\,,\,0 \leqslant t \leqslant 1\,,g_0 = 0\,,\,\|g\| = \sup_t |g_t|$; soit $B(C)$ la σ-Algèbre des boréliens de C ; on peut engendrer $B(C)$ par les ensembles cylindriques. Dans les § 2 et 3, nous noterons $(\Omega\,,\mathcal{F}\,,P)$ un espace probabilisé où $\Omega = C\,,\mathcal{F} = B(C)$ et P est la mesure de Wiener, $\xi = \{\xi_t(\omega)\,,\,0 \leqslant t \leqslant 1\}$ est un processus de Wiener par rapport à $(\mathcal{F}_t\,,P)$, $\mathcal{F}_t = \sigma\{g : g_s\,,\,s \leqslant t\}$.

Soit ensuite $A\,(t\,,\theta\,,X)$ des fonctionnelles mesurables telles que

$$A\,(t\,,\theta'\,,X') = A\,(t\,,\theta''\,,X'')$$

si $X_s = X''_s$, $\theta'_s = \theta''_s$, $s \leqslant t$. Sur l'espace probabilisé $(\Omega \times \widetilde{\Omega}, \mathcal{F} \times \widetilde{\mathcal{F}}, P \times \widetilde{P})$ nous définissons

$$\alpha_\xi (t , \omega , \widetilde{\omega}) = A (t , \widetilde{\theta} (\widetilde{\omega}) , \xi(\omega))$$

et

$$(2)\ \varphi_\xi (t , \omega , \widetilde{\omega}) = \exp\left\{ \int_0^t \alpha_\xi (s , \omega , \widetilde{\omega})\, d\xi_s(\omega) - \frac{1}{2} \int_0^t \alpha_\xi^2 (s , \omega , \widetilde{\omega})\, ds \right\}$$

DEFINITION 1. — *Nous dirons que* (1) *a une solution si sont définis les objets* $\mathfrak{A} = (\overline{\Omega} , \overline{\mathcal{F}}, \overline{P} , \overline{\mathcal{F}}_t , W , X , \theta)$, $0 \leqslant t \leqslant 1$, *tels que* $(\overline{\Omega} , \overline{\mathcal{F}} , \overline{P})$ *est un espace probabilisé,* $\overline{\mathcal{F}}_t$ *est une famille de sous σ-algèbres, non décroissante, de* $\overline{\mathcal{F}}$;

$$W = \{W_t (\overline{\omega}) , 0 \leqslant t \leqslant 1 , \overline{\omega} \in \overline{\Omega}\}$$

est un processus de Wiener (par rapport à $(\overline{\mathcal{F}}_t , \overline{P})$), $X = \{X_t(\overline{\omega}) , 0 \leqslant t \leqslant 1 , \overline{\omega} \in \overline{\Omega}\}$ *est une application de* $[0 , 1] \times \overline{\Omega}$ *dans* C, $\theta = \{\theta_t(\overline{\omega}) , 0 \leqslant t \leqslant 1 , \overline{\omega} \in \overline{\Omega}\}$ *est un processus aléatoire qui prend ses valeurs dans le même domaine que* θ *et, pour* $0 \leqslant t \leqslant 1$:

$$(1) \qquad \overline{P} \{\theta_t \in B\} = \widetilde{P} \{\widetilde{\theta}_t \in B\} \ , \ B \in \mathcal{B} (\mathbf{R}) ,$$

$$(2) \qquad X_t(\overline{\omega}) = \int_0^t A (s , \theta , X)\, ds + W_t(\overline{\omega}),$$

$\overline{P}$ *presque partout.*

$$(3) \qquad \mathcal{F}_t^{X,\theta} \ \text{ne dépend pas de } \ \mathcal{F}^W_{t,t+\tau} , \ \tau > 0$$

où $\mathcal{B}(\mathbf{R})$ *est un système borélien d'ensembles sur* $\mathbf{R}$;

$$\mathcal{F}_t^{X,\theta} = \sigma \{\overline{\omega} : X_s , \theta_s , s \leqslant t\} ,$$

$$\mathcal{F}^W_{t,t+\tau} = \sigma \{\overline{\omega} : W_s - W_t , t \leqslant s \leqslant t + \tau\}$$

DEFINITION 2. — *Deux solutions* $\mathfrak{A}^i = (\Omega^i , \mathcal{F}^i , P^i , \mathcal{F}^i_t , W^i , X^i , \theta^i)$, $i = 1, 2$ *sont dites faiblement équivalentes si*

$$P^1 (X^1 \in A) = P^2 (X^2 \in A) \ , \ A \in \mathcal{B}(C)$$

THEOREME 1. — *(Théorème d'existence)*

On suppose que les conditions suivantes sont vérifiées :

$$\text{I.} \qquad P \times \widetilde{P} \left\{ \int_0^1 \alpha_\xi^2 (s , \omega , \widetilde{\omega})\, ds < \infty \right\} = 1 ,$$

$$\text{II.} \qquad M \times \widetilde{M} \ \varphi_\xi (\omega , \widetilde{\omega}) = 1$$

où $\varphi_\xi (\omega , \widetilde{\omega}) = \varphi_\xi (1 , \omega , \widetilde{\omega})$, $M \times \widetilde{M}$ *est l'espérance mathématique par rapport à la mesure* $P \times \widetilde{P}$.

Alors sur $(\Omega \times \widetilde{\Omega}, \mathcal{F} \times \widetilde{\mathcal{F}})$ *nous pouvons définir la mesure de probabilité*

$$\mathcal{P}(d\omega, d\widetilde{\omega}) = \varphi_\xi(\omega, \widetilde{\omega}) P(d\omega) \widetilde{P}(d\widetilde{\omega}),$$

le processus de Wiener W *(par rapport à* $\mathcal{F}_t \times \widetilde{\mathcal{F}}_t, \mathcal{P})$*) et les processus* X, θ *tels que les objets*

$$\mathfrak{A} = (\Omega \times \widetilde{\Omega}, \mathcal{F} \times \widetilde{\mathcal{F}}, \mathcal{P}, \mathcal{F}_t \times \widetilde{\mathcal{F}}_t, W, X, \theta)$$

soient une solution de (I).

Cette solution est telle que

III. $$\mathcal{P}\left\{ \int_0^1 \alpha_X^2(s, \omega, \widetilde{\omega})\, ds < \infty \right\} = 1$$

IV. $$\mathcal{M} \frac{1}{\varphi_X(\omega, \omega)} = 1$$

V. *Les mesures* μ_X et μ_W *sont équivalentes* $(\mu_X \sim \mu_w)$.

Ici $\mathcal{M}$ *est l'espérance mathématique par rapport à* $\mathcal{P}$,

$$\alpha_X(s, \omega, \omega) = A(s, \theta(\omega, \omega), X(\omega, \omega)),$$

μ_X *et* μ_w *sont des mesures sur* $(C, \mathcal{B}(C))$ *engendrées par les processus* X *et* W *respectivement.*

THEOREME 2. – (*Unicité*)

On suppose que les conditions I et II sont vérifiées. Alors tout couple de solutions de l'équation (I) ayant les propriétés III et IV est formé de solutions faiblement équivalentes.

THEOREME 3. – *(Sur la densité de* μ_X *par rapport à* μ_ξ*)*

Sous les conditions I, II et

VI. $$P\left\{ \int_0^1 \widetilde{\alpha}_s^2(\xi)\, ds < \infty \right\} = 1,$$

où

(2) $$\widetilde{\alpha}_s(\xi) = \frac{\widetilde{M}\,[\alpha_\xi(s, \omega, \widetilde{\omega})\, \varphi_\xi(s, \omega, \widetilde{\omega})]}{\widetilde{M}\, \varphi_\xi(s, \omega, \widetilde{\omega})},$$

on a, pour $0 \leqslant t \leqslant 1$ *et P-presque partout*

(3) $$\frac{d\mu_X}{d\mu_\xi}(\mathcal{F}_t^\xi) = \exp\left\{ \int_0^1 \widetilde{\alpha}_s(\xi)\, d\xi_s - \frac{1}{2}\int_0^1 \widetilde{\alpha}_s^2(\xi)\, ds \right\}$$

Remarque 1 : La quantité $\widetilde{\alpha}_s(\xi)$ est certainement définie si $\mathcal{M}\,|A(s, \theta, X)| < \infty$. Dans ce cas $\widetilde{\alpha}_s(\xi) = \overline{\alpha}_s(\xi)$, où $\overline{\alpha}_s(X) = \mathcal{M}\,[A(s, \theta, X) | \mathcal{F}_t^X]$

Remarque 2 (M. Nisio, S. Watanabe) : Si $|A| \leqslant c < \infty$ alors l'équation (I) a une seule solution pour laquelle la représentation (3) est valable avec $\widetilde{\alpha}_s(\xi) = \overline{\alpha}_s(\xi)$.

Remarque 3. – Pour la démonstration des théorèmes 1 et 2 on utilise le résultats de Girsanov [2] qui dit que

$$W_t(\omega, \widetilde{\omega}) = X_t(\omega, \widetilde{\omega}) - \int_0^t A(s, \theta, X)\, ds$$

où

$$X_t(\omega, \widetilde{\omega}) = \xi_t(\omega) \quad , \quad \theta_t(\omega, \widetilde{\omega}) = \widetilde{\theta}_t(\widetilde{\omega})$$

est un processus de Wiener (par rapport à $(\mathcal{F}_t^{X,\theta}, \mathcal{P})$).

THEOREME 4. – *Soit la solution de l'équation (I) unique en ce sens que pour tout couple de ses solutions* $\mathcal{C}^i$, $i = 1, 2$,

$$P^1(X_t^1 \in A, \theta_t^1 \in B) = P^2(X_t^2 \in A, \theta_t^2 \in B),$$

$A, B \in \mathcal{B}(\mathbf{R})$; *alors, si les conditions I et III sont remplies (pour* $X = X^1, X^2$*) la propriété II est vérifiée et (d'après le théorème 1)* $\mu_{X^i} \sim \mu_\xi$, $i = 1, 2$.

4.

Dans la théorie du filtrage non linéaire, le modèle le plus étudié est celui où les processus (θ, X) définis sur un espace probabilisé sont tels que

$$\theta = \{\theta_t(\omega), 0 \leqslant t \leqslant 1\}$$

est un processus de Markov et le processus observé $X = \{X_t(\omega), 0 \leqslant t \leqslant 1\}$ vérifie l'équation stochastique :

$$dX_t = A(t, \theta_t, X_t)\, dt + dW_t, X_0 = 0 \tag{4}$$

où $W = \{W_t(\omega), 0 \leqslant t \leqslant 1\}$ est un processus de Wiener indépendant de θ. Nous supposerons que

$$M[\sup_s |\theta_s|] < \infty \quad , \quad \int_0^1 M\theta_s^2 .\, ds < \infty$$

et que

$$|A(t, \theta, x)| \leqslant C[1 + |x| + |\theta|],$$
$$|A(t, x', \theta) - A(t, x'', \theta)| \leqslant c\, |x' - x''|;$$

dans ces conditions l'équation (4) a une solution et une seule (module l'équivalence stochastique) (cf. [11]) pour laquelle $M \sup_t X_t^2 < \infty$ et, d'après les théorèmes 2 et 4 on a, P-presque partout,

$$\frac{d\mu_X}{d\mu_W}(\mathcal{F}_t^W) = \exp\left\{\int_0^t \overline{A}(s, W)\, dW_s - \frac{1}{2}\int_0^t \overline{A}^2(s, W)\, ds\right\} \tag{5}$$

où

$$\overline{A}(s, X) = M[A(s, \theta_s, X_s) | \mathcal{F}_t^X].$$

Nous supposerons aussi que la fonction $f = f(\theta)$ appartient au domaine de définition $D(G)$ de l'opérateur infinitésimal généralisé G (cf. [4]) c'est-à-dire que

$$M\,|f(\theta_t)| < \infty\,,\ 0 \leqslant t \leqslant 1\,,$$

et qu'il existe une fonction $(G_t f)\,(\theta)$ telle que

$$\int_0^1 M\,|(G_s f)\,(\theta_s)|\,ds < \infty$$

et

$$M\,[f(\theta_\tau)\,|\,\theta_t] - \int_t^\tau M\,[(G_s f)\,(\theta_s)\,|\,\theta_t]\,ds = f(\theta_t)\,, \tag{6}$$

$\tau \geqslant t$, P-presque partout.

THEOREME 5. – Si

$$\int_0^1 M\,[f^2(\theta_s)\,(A\,(s\,,X_s\,,\theta_s) - \bar{A}\,(s\,,X))^2]\,ds < \infty\,,$$

alors

$$\Pi_t\,(f) = M\,[f(\theta_t)\,|\,\mathcal{F}_t^X]$$

admet pour différentielle stochastique (au sens de Ito)

$$(7)\quad d\Pi_t(f) = M\,[(G_t f)\,(\theta_t)\,|\,\mathcal{F}_t^X]\,dt + M[(A\,(t\,,\theta_t\,,X_t) - \bar{A}(t\,,X))\cdot f\,(\theta_t)\,|\,\mathcal{F}_t^X]\,d\bar{X}_t$$

où $\bar{X}_t = X_t - \int_0^t \bar{A}\,(s\,,X)\,ds$ *est un processus de Wiener (par rapport à* $(\mathcal{F}_t^X\,,P)$).

Remarque : Des cas particuliers de ce théorème sont exposés dans [4] – [7], [9], [12] – [15].

Exemple : Soit $\theta = \{\theta_t\,,0 \leqslant t \leqslant 1\}$ un processus de Markov diffusionnel qui est décrit par l'équation stochastique

$$d\theta_t = a\,(t\,,\theta_t)\,dt + b\,(t\,,\theta_t)\,d\eta_t \tag{8}$$

où $\eta = \{\eta_t\,,0 \leqslant t \leqslant 1\}$ est un processus de Wiener. On suppose que θ_0, W et η sont indépendants, que

$$|a\,(t\,,\theta)| + |b\,(t\,,\theta)| \leqslant c\,[1 + |\theta|]\,,$$

$$|a\,(t\,,\theta') - a(t\,,\theta'')| + |b\,(t\,,\theta') - b\,(t\,,\theta'')| \leqslant c\,|\theta' - \theta''|.$$

Alors, pour des fonctions suffisamment "douces" $f = f(\theta)$ on trouve, d'après (7),

$$(9)\quad d\Pi_t\,(f) = M\,[(L_t f)\,(\theta_t)\,|\,\mathcal{F}_t^X]\,dt + M\,[(A\,(t\,,\theta_t\,,X_t) - \bar{A}\,(t\,,X))\,f\,(\theta_t)\,|\,\mathcal{F}_t^X]\,d\bar{X}_t$$

où

$$(L_t f)(\theta) = a\,(t\,,\theta)\,\frac{\partial f}{\partial \theta} + \frac{1}{2}\,b^2\,(t\,,\theta)\,\frac{\partial^2 f}{\partial \theta^2}.$$

Si la densité a posteriori $\Pi(t, \theta) = \dfrac{\partial P(\theta_t \leqslant \theta \,|\, \mathcal{F}_t^X)}{\partial \theta}$ est suffisamment "douce" par rapport à θ, alors de (9) on tire une équation différentielle stochastique aux dérivées partielles :

(10) $d\Pi(t, \theta) = L_t^* \Pi(t, \theta)\, dt + \Pi(t, \theta)\, [A(t, X_t, \theta) - \bar{A}(t, X)]\, d\bar{X}_t$.

où

$$L_t^*(\Pi) = -\frac{\partial(a\Pi)}{\partial\theta} + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}(b^2\,\Pi).$$

5.

Les équations qui donnent $\Pi_t(f)$ et $\Pi(t, \theta)$ que nous venons de voir dans l'exemple précédent ont été obtenues dans un cas plus général dans [12], [8], [3].

Soit $(\theta, X) = \{(\theta_t, X_t), 0 \leqslant t \leqslant 1\}$ un processus diffusionnel bidimensionnel engendré par le système

$$\begin{aligned} (11) \qquad d\theta_t &= a(t, \theta_t, X_t)\, dt + b_1(t, \theta_t, X_t)\, dW_t^1 \\ &\quad + b_2(t, \theta_t, X_t)\, dW_t^2 \\ dX_t &= A(t, \theta_t, X_t)\, dt + B_1(t, X_t)\, dW_t^1 \\ &\quad + B_2(t, X_t)\, dW_t^2 \end{aligned}$$

où les coefficients sont lipschitziens et ont une croissance au plus linéaire (par rapport aux variables de phase), θ_0 et X_0 ne dépendent pas des processus de Wiener W^1 et W^2 (qui sont eux-mêmes indépendants l'un de l'autre).

DEFINITION 3. – *Nous appellerons la fonctionnelle $\phi_t(\theta, X)$ une I-fonctionnelle si $\phi_t(\theta', X') = \phi_t(\theta'', X'')$ pour $\theta'_s = \theta''_s$, $X'_s = X''_s$, $s \leqslant t$ et*

$$\phi_t(\theta, X) = \phi_0(\theta, X) + \int_0^t [\psi(s, \theta, X)\, ds + \psi^\theta(s, \theta, X)\, d\theta_s + \psi^X(s, \theta, X)\, dX_s]$$

(les intégrales stochastiques par rapports à $d\theta_s$ et dX_s sont supposées exister).

Posons

$$\begin{aligned} \beta &= b_1 B_1 + b_2 B_2 \\ b^2 &= b_1^2 + b_2^2 \\ B^2 &= B_1^2 + B_2^2 \end{aligned}$$

THEOREME 6. – *Soit $\phi_t(\theta, X)$ une I-fonctionnelle, avec*

$$M\,|\phi_t| < \infty, \ 0 \leqslant t \leqslant 1, \ \inf_{t,\, |X| \leqslant N} B(t, X) > 0, \ N \geqslant 0.$$

Supposons que

$$\int_0^1 M\,[A^2(1 + \phi_s^2) + |\psi_s| + |\psi_s^X|^2 + |a\,\psi_s^\theta| + b^2\,|\psi_s^\theta|^2]\, ds < \infty$$

alors $\Pi_t(\phi) = M\,[\phi_t \mid \mathfrak{F}_t^X]$ *est une I-fonctionnelle,*

$$(12) \qquad d\Pi_t(\phi) = \psi_t\, dt + \psi_t^X\, dX_t\,,$$

où

$$(13) \quad \psi_t = B^{-2}\,\overline{A}\,(\overline{A}\overline{\phi} - \overline{A\phi}) + \overline{\psi} + \overline{A\psi^X} - \overline{A}\,\overline{\psi}^X + \overline{a\psi^\theta} - B^{-2}\overline{A}\;\overline{\beta\psi^\theta}$$

$$(14) \quad \psi_t^X = B^{-2}\,(\overline{A\phi} - \overline{A}\overline{\phi}) + \overline{\psi^X} + B^{-2}\,\overline{\beta\psi^\theta}$$

où
$$\overline{\Sigma} = M\,(\Sigma\,(t\,,\theta\,,X) \mid \mathfrak{F}_t^X)$$

Remarque 1 : Si la densité $\Pi(t\,,\theta) = \dfrac{\partial P(\theta_t \leqslant \theta \mid \mathfrak{F}_t^X)}{\partial\theta}$ et les coefficients dans (11) sont suffisamment "doux" alors, d'après (12)-(14), (en intégrant par parties) elle vérifie une équation différentielle stochastique aux dérivées partielles

$$(15) \quad \begin{aligned} d\Pi\,(t\,,\theta) &= L_t^*\,\Pi\,(t\,,\theta)\,dt \\ &+ \Pi\,(t\,,\theta)\,B^{-2}\,(t\,,X_t) \times [A\,(t\,,\theta\,,X_t) - \overline{A}\,(t\,,X)]\,[dX_t - \overline{A}\,dt] \\ &- \frac{\partial}{\partial\theta}\,[\beta\,B^{-2}\,(t\,,X_t)\,\Pi\,(t\,,\theta)]\,[dX_t - \overline{A}\,dt] \end{aligned}$$

où
$$\overline{A}\,(t\,,X) = M\,[A\,(t\,,\theta_t\,,X_t) \mid \mathfrak{F}_t^X]$$

$$L_t^*\,(\Pi) = -\frac{\partial}{\partial\theta}\,(a\Pi) + \frac{1}{2}\,\frac{\partial^2}{\partial\theta^2}\,(b^2\Pi)$$

Remarque 2 : *A* un degré de généralisation différent, les représentations (12) et (15) sont obtenues dans [12], [1], [8], [6]. L'énoncé du théorème 6 est celui de Ershov [3].

Remarque 3 : Pour l'utilisation de l'équation (15) dans les problèmes de filtrage non linéaire et optimal et dans la statistique des processus aléatoires, voir [5], [8], [12].

BIBLIOGRAPHIE

[1] Вентцель А. Д. — Об уравнениях теории условных марковских процессов, Теория вероятностей и ее применения, X, 2, 1965.

[2] Гирсанов И. В. — О преобразовании одного класса случайных процессов с помощью абсолютно-непрерывной замены меры, Теория вероятностей и ее применения, Y, 3, 1960, 314-330.

[3] Ершов М. П. — Последовательное оценивание диффузионных процессов, Теория вероятностей и ее применения, 1970.

[4] Kallianpur G. and Striebel C. — Stochastic differential equations occuring in the estimation of continuous parameter stochastic processes, Теория вероятн. и ее применения, XIY, 4, 1969, 597-622.

[5] KALMAN R. E. and BUCY R. C. — New results in linear filtering and prediation theory, *Trans. ASME, J. Basic Engr.*, 83 D, 1961.
[6] KUSHNER H. I. — Dynamical equations for optimal nonlinear filtering, *J. Differ. equat.*, 3, 1967, p. 179-190.
[7] KUNITA H. and FUJISAKI M. — Non linear filtering problem of Markov process.
[8] Липцер Р. Ш., Ширяев А. Н. — Нелинейная фильтрация диффузионных марковских процессов, Труды Матем. ин-та им. В. А. Стеклова, 104, 1968, 135-180.
[9] Липцер Р. Ш., Ширяев А. Н. — Интерполяция и фильтрация скачкообразной компоненты марковского процесса, Изв. АН СССР, сер. матем., 33, 4, 1969, 901-914.
[10] Липцер Р. Ш., Ширяев А. Н. — О плотности вероятностных мер процессов диффзионного типа, Изв. АН СССР, сер. матем., 33, 5, 1969, 1120-1131.
[11] Скороход А. В. — Исследования по теории случайных процессов, Изд. Киевск. Унив., 1961.
[12] Стратонович Р. Л. — Условные марковские процессы и их применение к теории оптимального управления, М., Изд. МГУ, 1966.
[13] Ширяев А. Н. — О стохастических уравнениях в теории условных марковских процессов, Теория вероят. и ее примен., 11, 1, 1966, 200-206.
[14] Ширяев А. Н. — Исследования по статистическому последовательному анализу, Матем. заметки, 3, 6, 1968, 739-754.
[15] WONHAM W. M. — Some applications of stochastic differential equations to optimal nonlinear filtering, *J. SIAM Control*, Ser. A, 2, 1965, p. 347-369.

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ON THE MARTIN DECOMPOSITION OF EXCESSIVE MEASURES

by Takesi WATANABE

1. Introduction.

The theorems of unique decomposition of excessive measures and functions into extreme elements are the first theorems in the Martin boundary theory. As is well-known, the theorem of this type goes back to R.S. Martin [8]. The formulation in probability languages was given by Doob [1], Hunt [4] and the author [10] in the context of Markov chains. Since then, there have been various generalizations to more general cases ([5], [6], [7], [9] etc.).

Let E be an *LCD* space. Let $(U_\alpha)_{\alpha>0}$ be a submarkov resolvent. (We mostly follow the notation and terminology of Meyer's book, *Probability and potentials*). A σ-finite (not necessarily Radon) measure ν is called *excessive for* (U_α) if

$$\nu \geqslant \nu \cdot \alpha U_\alpha \quad \text{and} \quad \nu = \lim_{\alpha\to\infty} \uparrow \nu \cdot \alpha U_\alpha .$$

An excessive measure ν is called *extreme* if ν cannot be decomposed into the sum of two excessive measures which are not proportional to ν. Let U be the potential kernel of (U_α), i.e., $U = \lim_{\alpha\to 0} U_\alpha$.

BASIC HYPOTHESIS. — *U is a continuous kernel, that is, U maps $\mathcal{C}_c^{\cdot}(E)$, the class of continuous functions with compact support, into $\mathcal{C}_b(E)$, the class of bounded continuous functions. U is strictly positive, i.e., all of the measures $U(x\,,.)$ are not zero. Moreover, the functions Uf, $f\in\mathcal{C}_c(E)$, separate points of E.*

Let $g (\geqslant 0)$ be a continuous function such that Ug is strictly positive and continuous. Define the *Martin kernel M* (associated with g) by

$$M(x\,,A) := U(x\,,A)\,/\,Ug(x)\,. \tag{1.1}$$

There exists a unique compact metric space F such that

(i) F contains E as a dense subset and the identity mapping $E\to F$ is continuous,

(ii) each Mf, $f\in\mathcal{C}_c(E)$, admits a continuous extension $\overline{Mf}$ over F, and

(iii) the functions $\overline{Mf}$, $f\in\mathcal{C}_c(E)$, separate points of F. F is called the *entrance space for M*. For each $x\in F$, there exists a unique Radon excessive measure $\overline{M}(x,.)$ over E such that $\overline{Mf}(x) = \int f(y)\,\overline{M}(x,dy)$, $f\in\mathcal{C}_c(E)$. Let $F_{\min}$ denote the set of all $x\in F$ such that $\overline{M}(x,.)$ is extreme and $\overline{M}g(x) = 1$.

Kunita and the author [7] have proved the most general decomposition theorem of excessive measures, using the compactification of resolvents the idea of which is originally due to D. Ray.

THEOREM 1. – [7 ; Theorem 3 and 3′]. *Every excessive measure* ν *such that* $\int g(x)\,\nu(dx) < \infty$ *is an* $\bar{M}$*-potential of a unique measure* μ *supported in* $F_{\min}$, i.e. $\nu = \mu\bar{M}$.

The simplest proof of this theorem is now to use the Choquet-Meyer representation theorem of convex cones with a cap.

The purpose of this paper is to give a theorem (Theorem 9) which implies Theorem 1 and makes clear the probabilistic meaning of $F_{\min}$, the representing measure μ and the representation $\nu = \mu\bar{M}$. (All the proofs are omitted. The details will appear somewhere else). The method developed here is basically due to Hunt [4], [5]. Roughly speaking, Hunt's method does work if (U_α) is the resolvent of a semi-group admitting a "nice" Markov process. Such a process of the present case is the process, not on E but on F, which was constructed by Meyer [9].

2. Approximate processes over the entrance space.

Hereafter we consider only the case in which

$$(2.1) \qquad Ug = 1, \text{ i.e., } M = U.$$

We no longer use the letter M. F is the entrance space for U. $\bar{U}$ stands for $\bar{M}$. (The general case is reduced to this special case. In the previous work [7] we also reduced the general case to a special case, apparently similar to the present one. But the previous reduction, involving random time change, is much more complicated than the present reduction).

In the following we borrow the results of Meyer [9 ; Chap. I, § 5] without further reference. Meyer's results are too complicated to repeat here, so we only list the notation of the main objects.

- $(V_\alpha)_{\alpha>0}$ is the submarkov resolvent over F which is defined in [9] as a natural extension of (U_α). V stands for the potential kernel of (V_α). (In [9], (V_α) is also denoted by (U_α) and V, by L). D is the set of $x \in F$ such that $\alpha V_\alpha(x,.) \to \epsilon_x$ as $\alpha \to \infty$.
- $(P_t)_{t\geqslant 0}$ is a semi-group of submarkov kernels over F the resolvent of which is (V_α).
- $\mathbf{P}^\mu$ stands for the basic measure of a P_t-process (= Markov process having (P_t) as its transition function) $(X(t))_{t\geqslant 0}$ having μ as its initial measure and having sufficiently regular sample paths. $\mathbf{P}^x$ stands for $\mathbf{P}^{\epsilon_x}$, $x \in F$.

LEMMA 2. – *Every* U*-potential of a finite measure* μ *is a* $\bar{U}$*-potential of a unique measure* $\bar{\mu}$ *over* F, *supported in* D.

For each $A \in \mathcal{B}(F)$, let W_A denote the *penetration time for* A ;

$$W_A = \inf\left\{t > 0 \ ; \ \int_0^t I_A \circ X(s)\,ds > 0\right\}.$$

Define the kernel P_{W_A} over F by

(2.2) $$P_{W_A}(x, B) = \mathbf{P}^x\{X(W_A+) \in B\}, \quad B \in \mathcal{B}(E).$$

LEMMA 3. – (1) *Let f be a positive $\mathcal{B}(F)$-measurable function. Then $P_{W_A} Vf$ is the increasing limit of Vf_n such that each f_n is supported in A.*

Let ν be a σ-finite measure on E and $A \in \mathcal{B}(E)$. Define

(2.3) $$\nu L_A = \inf\{\nu' ; \nu' \geqslant \nu \text{ on } A \text{ and } \nu' \text{ is supermedian for } (U_\alpha)\}.$$

If ν is excessive, so is νL_A.

LEMMA 4. – *Let ν be a Radon excessive measure and $A \in \mathcal{B}(E)$, relatively compact in E. Then νL_A is a U-potential of a finite measure.*

From Meyer's results it follows that $\bar{U}(x, \cdot)$ is excessive for every $x \in F$. The next theorem follows from Lemma 3 and Meyer's results.

THEOREM 5. – *For every finite measure μ over F supported in $D \cup E$ and $A \in \mathcal{B}(E)$*

(2.4) $$(\mu \bar{U}) L_A = \mu P_{W_A} \cdot \bar{U}.$$

The following theorem is an extension of those results obtained by Hunt [3 ; § 9] for standard processes over E.

THEOREM 6. – *Let ν be a Radon excessive measure. Let $\{A_n\}$ be a sequence of sets in $\mathcal{B}(E)$, relatively compact in E and increasing to E. Then there exists a unique sequence $\{\mu_n\}$ of finite measures over F, each supported in D, such that*

(2.5) $$\mu_n \bar{U} \uparrow \nu, \quad \text{and} \quad \mu_n = \mu_{n+1} P_{W_{A_n}}.$$

In this case, one has $\mu_n \bar{U} = \nu L_{A_n}$.

Let us give the sketch of the proof. By Lemma 4 and 2, there exists a unique measure μ_n supported in D such that $\nu L_{A_n} = \mu_n \bar{U}$. It follows from Theorem 5 that $\mu_{n+1} P_{W_{A_n}} \bar{U} = \mu_n \bar{U}$. Since μ_n and $\mu_{n+1} P_{W_{A_n}}$ are supported in D, one gets (2.5) by Lemma 2. The uniqueness follows easily from Theorem 5.

A precise definition of approximate processes is too complicated to be stated here (see [11]). Roughly speaking, *an approximate P_t-process* is defined as the limit of random shifts of ordinary P_t-processes. Such a process is denoted by the triple $(Y(t), \alpha, \beta)$, where $(Y(t))_{t \in (-\infty, \infty)}$ denotes a random function taking values in F for $\alpha < t < \beta$. The initial time α [resp. final time β] is a random variable taking values in $[-\infty, +\infty)$ [resp. in $(-\infty, +\infty]$].

M. Weil's method [11] of constructing approximate processes applies to the present case due to Theorem 6.

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(1) Due to H. Rost, Die Stoppenverteilungen eines Markoff-Prozesses mit lokalendlichen Potential, to appear.

THEOREM 7. – *Given any Radon excessive measure ν, there exists an approximate P_t-process defined over a σ-finite measure space $(\Omega, \mathcal{F}, P)$, having ν as its potential measure of Hunt,* i.e.

$$\nu(A) = \mathbf{E}\left[\int_\alpha^\beta I_A \circ Y(t)\, dt\right], \quad A \in \mathcal{B}(E) . \tag{2.6}$$

Moreover, $(\Omega, \mathcal{F})$ can be chosen to be isomorphic to a measurable space consisting of a complete separable metric space and the σ-algebra of Borel sets.

3. A decomposition theorem.

Let ν be an excessive measure such that $\int g(x)\,\nu(dx) < \infty$. It follows from condition (2.1) that ν is a Radon measure. Let $(Y(t), \alpha, \beta)$ be an approximate P_t-process having ν as its potential measure. One has $\mathbf{P}(\Omega) = \int g(x)\,\nu(dx)$. By a martingale argument due to Hunt [4], one sees that the right limits $Y(\alpha+, \omega)$ exists in F for a.a.ω. Define

$$^{\nu}\mu(A) = \mathbf{P}\{Y(\alpha+) \in A\}, \quad A \in \mathcal{B}(F) . \tag{3.1}$$

$^{\nu}\mu$ is called the *initial measure of* $(Y(t), \alpha, \beta)$.

LEMMA 8. – *Let $(Y(t), \alpha, \beta)$, $(Y'(t), \alpha', \beta')$ be two approximate P_t-processes having the same potential measure ν. Then they have the common initial measure.*

Let F_{ess} denote the set of all $x \in F$ such that $^{\nu}\mu = \epsilon_x$ for $\nu = U(x, \cdot)$. It follows that $F_{\text{ess}} \supset D$ and $F_{\text{ess}} \in \mathcal{B}(F)$. (The points of $F_{\text{ess}} \setminus D$ correspond to what are called "passive entrance points" by Feller [2]).

One now comes to the main theorem.

THEOREM 9. – *Assume condition* (2.1).

(a) *Every excessive measure ν such that $\int g(x)\,\nu(dx) < \infty$ is the $\bar{U}$-potential of $^{\nu}\mu$, the initial measure of an approximate P_t-process having ν as its potential measure.*

(b) *The measure $^{\nu}\mu$ is supported in F_{ess}.*

(c) *Suppose that ν is a $\bar{U}$-potential of a finite measure μ supported in F_{ess}. Then $\mu = {}^{\nu}\mu$.*

The proof of (a) is a routine calculation. The proof of (c) is also not difficult. The key is to prove (b). The proof of (b) is carried out, by choosing the basic space $(\Omega, \mathcal{F})$ as a space described in the latter half of Theorem 7 and taking the *regular conditional distribution of* $\mathbf{P}$ *for* $Y(\alpha+)$ in the sense of Parthasarathy's book, *Probability measures on metric spaces.*

REFERENCES

[1] DOOB J.L. — Discrete potential theory and boundaries, *J. Math. Mech.*, 8, 1959, p. 433-458.
[2] FELLER W. — On boundaries and lateral conditions for the Kolmogorov differential equations, *Ann. of Math.*, 65, 1957, p. 527-570.
[3] HUNT G.A. — Markoff processes and potentials I, *Illinois J. Math.*, 1, 1957, p. 44-93.
[4] HUNT G.A. — Markoff chains and Martin boundaries, *Illinois J. Math.*, 4, 1960, p. 313-340.
[5] HUNT G.A. — Transformation of Markoff processes, *Proc. International Congress*, 1962, p. 531-535.
[6] KUNITA H. and WATANABE T. — Markov processes and Martin boundaries, *Illinois J. Math.*, 9, 1965, p. 485-526.
[7] KUNITA H. and WATANABE T. — Some theorems concerning resolvents over locally compact spaces, *Proc. Fifth Berkeley Symp. on Stat. and Prob.*, Vol. II, Part 2, 1965, p. 131-164.
[8] MARTIN R.S. — Minimal positive harmonic functions, *Trans. Amer. Math. Soc.*, 49, 1941, p. 137-172.
[9] MEYER P.A. — Processus de Markov : la frontière de Martin, *Lecture Notes in Math.*, 77, Berlin, Springer, 1968.
[10] WATANABE T. — On the theory of Martin boundaries induced by countable Markov processes, *Mem. Coll. Sci. Univ. Kyoto*, Ser. A, 33, 1960, p. 39-108.
[11] WEIL M. — Quasi-processus, *Séminaire de Probabilités IV*, Université de Strasbourg (*Lecture Notes in Math.*, Springer), 124, 1970, p. 216-239.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 551 à 557.

D 6 - PROBABILITÉS THÉORIE DE LA MESURE INTÉGRATION

INEQUALITIES FOR OPERATORS ON MARTINGALES

by D. L. BURKHOLDER

This is a report on some recent developments in operator extrapolation theory [1, 2]. An application to conjugate harmonic functions leading to a maximal function characterization of the Hardy class H^p is also described [3].

An operator defined on the Lebesgue class L^p $(p \geq 1)$ of a probability space may always be viewed as an operator on a family of martingales and it is this more flexible viewpoint that is convenient for us here.

The question central to the work under survey is : Suppose that $\mathcal{M}$ is a family of martingales on a probability space $(\Omega, \mathcal{A}, P)$, and U and V are operators on $\mathcal{M}$ with values in the set of nonnegative $\mathcal{A}$-measurable functions. If Φ is a nonnegative function on $[0 , \infty]$, under what conditions does

$$E\Phi(Vf) \leq c E\Phi(Uf) , f \in \mathcal{M},$$

follow from some more easily proved inequality, perhaps

$$\|Vf\|_2 \leq c \ \|Uf\|_2 \ , f \in \mathcal{M} \ ?$$

Here E denotes expectation, integration over Ω with respect to P, and the letter c denotes a positive real number, not necessarily the same number from line to line.

Assumptions.

Let $\mathcal{A}_0 , \mathcal{A}_1, \ldots$ be a nondecreasing sequence of sub-σ-fields of $\mathcal{A}$. Let $\mathcal{M}$ be the set of all martingales $f = (f_1, f_2, \ldots)$ relative to $\mathcal{A}_1, \mathcal{A}_2, \ldots$ Consider an operator T defined on $\mathcal{M}$ with values in the set of nonnegative $\mathcal{A}$-measurable functions. Examples are the maximal function operator $f^* = \sup_{1 \leq n < \infty} |f_n|$, the square function operator $S(f) = \left[\sum_{k=1}^{\infty} d_k^2 \right]^{1/2}$, and the operator

$$s(f) = \left[\sum_{k=1}^{\infty} E(d_k^2 \mid \mathcal{A}_{k-1}) \right]^{1/2} ,$$

where $d = (d_1, d_2, \ldots)$ is the difference sequence of $f : f_n = \sum_{k=1}^{n} d_k, n \geq 1$. Other examples are discussed below. The operator T is *quasi-linear* if $T(f + g) \leq \gamma (Tf + Tg)$

for some real number $\gamma \geqslant 1$ and all f and g in $\mathfrak{M}$; T is *symmetric* if $T(-f) = Tf$ and is *local* if $Tf = 0$ on the set $\{s(f) = 0\}, f \in \mathfrak{M}$. The three operators mentioned above satisfy these conditions with $\gamma = 1$. A stopping time τ is a function from Ω into $\{0, 1, \ldots, \infty\}$ such that $\{\tau \leqslant n\} \in \mathcal{A}_n$, $n \geqslant 0$. If μ and ν are stopping times, let ${}^{\mu}f^{\nu}$ denote the sequence $\sum_{k=1}^{n} I(\mu < k \leqslant \nu)\, d_k$, $n \geqslant 1$, where $I(A)$ is the indicator function of the set A ; ${}^{\mu}f^{\nu}$ is *f started at μ and stopped at ν*. Write f^{ν} for ${}^{0}f^{\nu}$; in particular, f^n is the martingale f stopped at n. Let $T_n f = Tf^n$, $0 \leqslant n \leqslant \infty$, $T^*f = \sup_{1 \leqslant n < \infty} T_n f$, and $T^{**}f = T^*f \vee Tf$. The operator T is *measurable* if $T_n f$ is $\mathcal{A}_n$-measurable, $n \geqslant 1, f \in \mathfrak{M}$. For example, $S_n(f) = \left[\sum_{k=1}^{n} d_k^2\right]^{\frac{1}{2}}$ so that $S = S^* = S^{**}$ and S is measurable.

Let Φ be a function on $[0, \infty]$ such that

$$\Phi(b) = \int_0^b \varphi(\lambda), \quad 0 \leqslant b \leqslant \infty, \tag{1}$$

for some nonnegative measurable function φ on $(0, \infty)$ satisfying the growth condition

$$\varphi(2\lambda) \leqslant c\,\varphi(\lambda), \quad \lambda > 0. \tag{2}$$

Also, always assume that $\Phi(1) < \infty$. For example, any power b^p $(0 < p < \infty)$ satisfies these conditions.

Throughout the following, if $g = (g_1, g_2, \ldots)$ is any sequence of real functions, then g^* denotes its maximal function : $g^* = \sup_{1 \leqslant n < \infty} |g_n|$.

Some results.

THEOREM 1. – *Let $0 < p_0 < \infty$. Suppose that U and V are local, quasi-linear, symmetric, and measurable operators on $\mathfrak{M}$ such that*

$$\lambda^{p_0} P(Vf > \lambda) \leqslant c\ \|U^*f\|_{p_0}^{p_0} \tag{3}$$

for all $\lambda > 0$ and f in $\mathfrak{M}$. Let Φ be a function on $[0, \infty]$ satisfying (1) and (2) and f a martingale in $\mathfrak{M}$. Let $\Delta_n = V({}^{n-1}f^n)$ and suppose that w_n is an $\mathcal{A}_{n-1}$-measurable function satisfying $U({}^{n-1}f^n) \leqslant w_n$, $n \geqslant 1$. Then

$$E\Phi(V^*f) \leqslant c\,E\,\Phi(U^*f) + c\,E\Phi(\Delta^*) + c\,E\Phi(w^*) \tag{4}$$

with the choice of c depending only on γ_U and γ_V, the quasi-linearity constants of U and V, respectively, on the growth constant $c_{(2)}$ of φ, and on p_0 and $c_{(3)}$.

The proof of Theorem 1 rests on the methods developed in [2] and may be found in [1]. The main ideas of the proof are well illustrated in the proof of Theorem 7 below. The fact that the processes $\{V_n f, n \geqslant 1\}$ and $\{U_n f, n \geqslant 1\}$ proceed in jumps accounts for the appearance of Δ^* and w^* in (4). The conclusion of Theorem 7 is simpler because of the sample function continuity of the processes studied there.

If $U(^{n-1}f^n)$ is $\mathcal{A}_{n-1}$-measurable, $n \geqslant 1$, then the term containing w^* may be deleted from (4). In this case we may let $w_n = U(^{n-1}f^n)$ and observe that $U(^{n-1}f^n) = U(f^n - f^{n-1}) \leqslant \gamma(Uf^n + Uf^{n-1}) \leqslant 2\gamma U^* f$ so that $w^* \leqslant 2\gamma U^* f$ with $\gamma = \gamma_U$. For example, let $Vf = \sum_{k=1}^{\infty} |d_k|$ and $Uf = \sum_{k=1}^{\infty} E(|d_k| \mid \mathcal{A}_{n-1})$. Then $\Delta^* = d^*$ and (3) is satisfied with $p_0 = 1$. Therefore, Theorem 1 gives

$$E\Phi\left(\sum_{k=1}^{\infty} |d_k|\right) \leqslant cE\Phi\left(\sum_{k=1}^{\infty} E(|d_k| \mid \mathcal{A}_{k-1})\right) + cE\Phi(d^*),$$

and this easily implies that if $z = (z_1, z_2, \dots)$ is any sequence of nonnegative functions on Ω such that z_k is $\mathcal{A}_k$-measurable, $k \geqslant 1$, then

$$E\Phi\left(\sum_{k=1}^{\infty} z_k\right) \leqslant cE\Phi\left(\sum_{k=1}^{\infty} E(z_k \mid \mathcal{A}_{k-1})\right) + cE\Phi(z^*). \tag{5}$$

(Let $r = (r_1, r_2, \dots)$ be the Rademacher sequence and consider the martingale $f_n(\omega, t) = \sum_{k=1}^{n} z_k(\omega) r_k(t)$, $n \geqslant 1$, on the product space $\Omega \times [0, 1)$.) If Φ is also convex, the converse inequality holds [1] :

THEOREM 2. – *Suppose that Φ is a convex function from $[0, \infty)$ into $[0, \infty)$ satisfying $\Phi(0) = 0$ and the growth condition*

$$\Phi(2\lambda) \leqslant c\,\Phi(\lambda), \lambda > 0\ ; \tag{6}$$

set $\Phi(\infty) = \lim_{\lambda \to \infty} \Phi(\lambda)$. Let $z = (z_1, z_2, \dots)$ be a sequence of nonnegative $\mathcal{A}$-measurable functions. Then

$$E\Phi\left(\sum_{k=1}^{\infty} E(z_k \mid \mathcal{A}_{k-1})\right) \leqslant cE\Phi\left(\sum_{k=1}^{\infty} z_k\right) \tag{7}$$

and the choice of c depends only on $c_{(6)}$.

A function Φ satisfying the conditions of Theorem 2 satisfies (1) and (2).

The above two theorems, together with a decomposition of martingales introduced by Davis in [4], lead to the following left-hand-side result for operators T in the case of convex Φ.

THEOREM 3. – *Let $0 < p_0 < \infty$. Suppose that T is a local, quasi-linear, symmetric, and measurable operator on $\mathcal{M}$ such that*

$$\lambda^{p_0} P(Tf > \lambda) \leqslant c\, \|f^*\|_{p_0}^{p_0} \tag{8}$$

for all $\lambda > 0$ and f in $\mathcal{M}$. If Φ is a convex function as in Theorem 2, then

$$E\Phi(T^{**}f) \leqslant cE\Phi(f^*) \tag{9}$$

for all f in $\mathcal{M}$ provided the sequence $\Delta_n = T(^{n-1}f^n)$ satisfies

$$E\Phi(\Delta^*) \leqslant cE\Phi(f^*), f \in \mathcal{M}, \tag{10}$$

(11) $$E\,\Phi(Tf) \leqslant c\,E\Big(\sum_{k=1}^{\infty} |d_k|\Big), f \in \mathfrak{M}.$$

The choice of $c_{(9)}$ *depends only on the quasi-linearity constant* γ *of* T, p_0, $c_{(6)}$, $c_{(8)}$, $c_{(10)}$, *and* $c_{(11)}$.

There is also a right-hand-side result, Theorem 2.2 of [1], with the conclusion

$$E\,\Phi(f^*) \leqslant c\,E\,\Phi(T^*f)$$

under nearly analogous conditions. There is one difference. Theorem 3 above requires only left-hand-side conditions. However, if the left-hand-side condition (2.8) were deleted from the statement of Theorem 2.2 of [1], the theorem would no longer be true.

Now suppose that X is a martingale in $\mathfrak{M}$ and x is its difference sequence : $X_n = \sum_{k=1}^{n} x_k$, $n \geqslant 1$. Let $\mathfrak{M}_X$ denote the family of all transforms of X relative to $\mathcal{A}_0, \mathcal{A}_1, \ldots$ That is, $f = (f_1, f_2, \ldots)$ belongs to $\mathfrak{M}_X$ if and only if there is a sequence $v = (v_1, v_2, \ldots)$ such that v_k is $\mathcal{A}_{k-1}$-measurable, $k \geqslant 1$, and

$$f_n = \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} v_k x_k \,.\; n \geqslant 1\,.$$

Assume that there is a positive number δ such that, for $k \geqslant 1$,

(12) $$E(|x_k| \mid \mathcal{A}_{k-1}) \geqslant \delta,$$

(13) $$E(x_k^2 \mid \mathcal{A}_{k-1}) = 1.$$

These conditions give control over the jumps of the process $\{T_n f, n \geqslant 1\}$; in particular, see Theorem 2.2 of [2]. As a consequence, Φ is a remarkably general function in the following left-hand-side result which should be compared to Theorem 3.

THEOREM 4. – *Let* $0 < p_0 \leqslant 2$. *Suppose that* X *in* $\mathfrak{M}$ *satisfies* (12) *and* (13), *and that* T *is a local, quasi-linear, symmetric, and measurable operator on* $\mathfrak{M}_X$ *such that*

(14) $$\lambda^{p_0} P(Tf > \lambda) \leqslant c\, \|f^*\|_{p_0}^{p_0}$$

for all $\lambda > 0$ *and* f *in* $\mathfrak{M}_X$. *If* Φ *is a function on* $[0, \infty]$ *satisfying* (1) *and* (2), *then*

(15) $$E\,\Phi(T^{**}f) \leqslant c\,E\,\Phi(f^*)$$

for all f *in* $\mathfrak{M}_X$ *provided the sequence* $\Delta_n = T(^{n-1}f^n)$ *satisfies*

(16) $$E\,\Phi(\Delta^*) \leqslant c\,E\Phi(f^*)\,,\; f \in \mathfrak{M}_X.$$

The choice of $c_{(15)}$ *depends only on* γ, δ, p_0, $c_{(2)}$, $c_{(14)}$, *and* $c_{(16)}$.

This theorem is proved in [2] with an extra right-hand-side condition R1. Note that if T satisfies the conditions of Theorem 4, then so does the operator $f \to f^* \vee Tf$ and the latter operator satisfies R1. Therefore, it is unnecessary to assume R1 to obtain this left-hand-side result.

For the corresponding right-hand-side result, see [2].

The two-sided inequality

$$(17) \qquad cE\Phi(S(f)) \leq E\Phi(f^*) \leq CE\Phi(S(f))$$

follows immediately : (i) If Φ is a convex function as in Theorem 2, then (17) holds for all f in $\mathcal{M}$. The choice of c and C depends only on the growth constant of Φ. (ii) If Φ merely satisfies (1) and (2), then (17) holds for all f in $\mathcal{M}_X$ provided X is in $\mathcal{M}$ and satisfies (12) and (13). Here the choice of c and C depends only on δ and $c_{(2)}$.

Similar two-sided inequalities hold for any operator of matrix type [1, 2].

The results described above are useful in the study of random walk, quadratic variation of continous parameter martingales, stochastic integration, Haar and Walsh series, to mention a few areas of application. The next section contains an application of the methods of [2] to conjugate harmonic functions.

An application to conjugate harmonic functions.

If u is a harmonic function in the unit disc $|z| < 1$ and $0 < \sigma < 1$, let

$$N_\sigma(u, \theta) = \sup_{z \in \Omega_\sigma(\theta)} |u(z)|$$

where $\Omega_\sigma(\theta)$ is the interior of the smallest convex set containing the disc $|z| < \sigma$ and the point $e^{i\theta}$. The function $N_\sigma(u) = N_\sigma(u, .)$ is the *nontangential maximal function of u.* The following theorem is one of the main results of [3].

THEOREM 5. – *Let u be harmonic in the unit disc and v its conjugate harmonic function satisfying* $v(0) = 0$. *Let* Φ *be a function on* $[0, \infty]$ *satisfying* (1) *and* (2). *Then*

$$\int_0^{2\pi} \Phi(N_\sigma(v, \theta))\, d\theta \leq c \int_0^{2\pi} \Phi(N_\sigma(u, \theta))\, d\theta .$$

The choice of c depends only on σ *and the growth constant* $c_{(2)}$.

The analogue of this result for the upper half-plane is also true but the proof is a little longer [3]. Notice that Theorem 5 implies that the analytic function $F = u + iv$ satisfies

$$(18) \qquad \sup_{0<r<1} \int_0^{2\pi} \Phi(|F(re^{i\theta})|)\, d\theta \leq c \int_0^{2\pi} \Phi(N_\sigma(u, \theta))\, d\theta .$$

By a result of Hardy and Littlewood [6], if $0 < p < \infty$ and F is in the Hardy class H^p, then the nontangential maximal function of its real part is in $L^p(0, 2\pi)$. It is also classical that the converse holds for $1 < p < \infty$. By (18), the converse is true for all $0 < p < \infty$.

Let $\{Z_t, 0 \leq t < \infty\}$ be Brownian motion in the plane starting at 0. If u is a function on the unit disc let

$$u^* = \sup_{0<t<\tau_1} |u(Z_t)|$$

where $\tau_1 = \inf\{t : |Z_t| = 1\}$. For u harmonic, this Brownian maximal function is closely related to the nontangential maximal function :

THEOREM 6. – *If u is harmonic in* $|z| < 1$, *then*

$$c\, m(N_\sigma(u) > \lambda) \leqslant P(u^* > \lambda) \leqslant C m(N_\sigma(u) > \lambda)\,, \; \lambda > 0\,. \tag{19}$$

The choice of c and C depends only on σ.

Here $m(N_\sigma(u) > \lambda)$ denotes the Lebesgue measure of the set of points θ in $[0\,, 2\pi)$ satisfying $N_\sigma(u\,,\theta) > \lambda$. Theorem 6 is proved in [3].

THEOREM 7. – *Let u be harmonic in the unit disc and v its conjugate harmonic function satisfying* $v(0) = 0$. *Let* Φ *be a function on* $[0\,,\infty]$ *satisfying* (1) *and* (2). *Then*

$$E\Phi(v^*) \leqslant c\, E\Phi(u^*)\,. \tag{20}$$

The choice of c depends only on $c_{(2)}$.

Theorem 5 follows easily from (19) and (20). Let $F = u + iv$. Here we sketch the proof of

$$E\Phi(F^*) \leqslant cE\Phi(u^*)\,. \tag{21}$$

which implies (20). To prove (21), we may assume that F is continuous on the closed disc $|z| \leqslant 1$ and that $F(0) = 0$. Since F^2 is analytic in $|z| < 1$, the process $\{F^2(z_t),\, 0 \leqslant t < \infty\}$ is a martingale (Doob [5]. Theorem 4.3) where for convenience we let $z_t = Z_{\tau_1 \wedge t}$. From standard martingale theory, it follows that if μ and ν are stopping times of the process $\{z_t,\, 0 \leqslant t < \infty\}$ and $\mu \leqslant \nu$, then

$$\|v(z_\nu) - v(z_\mu)\|_2 = \|u(z_\nu) - u(z_\mu)\|_2\,. \tag{22}$$

We now come to the key step in the proof : *Let* $\alpha \geqslant 1$ *and* $\beta > 1$. *Then*

$$P(F^* > \lambda) \leqslant cP(cu^* > \lambda) \tag{23}$$

for all $\lambda > 0$ *satisfying*

$$P(F^* > \lambda) \leqslant \alpha P(F^* > \beta\lambda) \tag{24}$$

and the choice of c depends only on α and β. This implies (21) ; see [2].

Suppose that λ satisfies (24). The stopping times

$$\mu = \inf\{t \,:\, |F(z_t)| > \lambda\},$$

$$\nu = \inf\{t \,:\, |F(z_t)| > \beta\lambda\}$$

satisfy $\mu \leqslant \nu$, $|F(z_\mu)| = \lambda$ on the set $\{\mu < \infty\} = \{F^* > \lambda\}$, and $|F(z_\nu)| = \beta\lambda$ on $\{F^* > \beta\lambda\}$. Therefore, by (22) and (24), we have

$$\int_{\{F^* > \lambda\}} [u(z_\nu) - u(z_\mu)]^2\, dP = \|u(z_\nu) - u(z_\mu)\|_2^2 = \frac{1}{2}\, \|F(z_\nu) - F(z_\mu)\|_2^2$$

$$\geqslant \frac{1}{2}(\beta\lambda - \lambda)^2\, P(F^* > \beta\lambda) \geqslant c\lambda^2\, P(F^* > \lambda)\,.$$

Also,

$$\int_{\{F^* > \lambda\}} [u(z_\nu) - u(z_\mu)]^4 \, dP \leqslant \|F(z_\nu) - F(z_\mu)\|_4^4 \leqslant c\lambda^4 P(F^* > \lambda).$$

Therefore, by a lemma of Paley and Zygmund ([7], Chapter V, 8.26),

$$P(F^* > \lambda) \leqslant c P(c \, | u(z_\nu) - u(z_\mu) | > \lambda).$$

Since $|u(z_\nu) - u(z_\mu)| \leqslant 2u^*$, we obtain (23) to complete the proof of Theorem 7.

REFERENCES

[1] BURKHOLDER D.L., DAVIS B.J. and GUNDY R.F. — Integral inequalities for convex functions of operators on martingales, *Proc. Sixth Berkeley Symp. Math. Statist. and Prob.*, to appear.

[2] BURKHOLDER D.L. and GUNDY R.F. — Extrapolation and interpolation of quasi-linear operators on martingales, *Acta Math.*, 124, 1970, p. 249-304.

[3] BURKHOLDER D.L., GUNDY R.F. and SILVERSTEIN M.L. — A maximal function characterization of the class H^p, *Trans. Amer. Math. Soc.*, 157, 1971, to appear.

[4] DAVIS B.J. — On the integrability of the martingale square function, *Israel J. Math.*, 8, 1970, p. 187-190.

[5] DOOB J.L. — Semimartingales and subharmonic functions, *Trans. Amer. Math. Soc.*, 77, 1954, p. 86-121.

[6] HARDY G.H. and LITTLEWOOD J.E. — A maximal theorem with function-theoretic applications, *Acta Math.*, 54, 1930, p. 81-116.

[7] ZYGMUND A. — *Trigonometric series*, 2nd edition, Cambridge, 1959.

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REPRESENTATION OF MEASURE TRANSFORMATIONS

by R. V. CHACON

Let $(X , \mathfrak{F} , m)$ be a measure space and let τ be an invertible anti-periodic measure preserving transformation of X into itself. A simple result, originally due to Rokhlin, is that for each n and $\epsilon > 0$ there exists a set $A = A(n , \epsilon)$ such that the sets $A , \tau A , \ldots , \tau^n A$ are pairwise disjoint and such that the measure of the complement of their union is less than ϵ. This result has led us to study measure preserving transformations by considering certain classes which admit of simple representation in these terms. We first introduce some definitions.

DEFINITION 1. – A collection ξ having union $X_\xi \subset X$ will be called a partition if the sets are in $\mathfrak{F}$ and are pairwise disjoint. We denote by $\mathfrak{F}(\xi)$ the σ-fied of subsets of X_ξ generated by ξ. If ξ is a countable partition and $A \in \mathfrak{F}$, then among the sets of $\mathfrak{F}(\xi)$ there is at least one whose symmetric difference with A has minimal measure, and we denote by $A(\xi)$ one of these sets.

DEFINITION 2. – If $F, F_n \in \mathfrak{F}$ we write $\lim_{n\to\infty} F_n = F$ provided that

$$\lim_{n\to\infty} m(F \Delta F_n) = 0 .$$

If ξ_n is a sequence of countable partitions, we write $\xi_n \to \epsilon$ if for any $A \in \mathfrak{F}$ we have that $\lim_{n\to\infty} A(\xi_n) = A$.

The simplest classes of transformations we consider are the following.

DEFINITION 3. – We say that τ is of class $\mathfrak{A}$ provided that there is a sequence

$$\xi(n) = \{A_1(n) , A_2(n) , \ldots , A_{q(n)}(n)\}$$

of partitions such that

(a) $$\xi(n) \to \epsilon$$

and such that

(b) $$\tau A_i(n) = A_{i+1}(n) , i = 1 , \ldots , q(n) - 1 .$$

We say that τ is of class $\mathfrak{B} \subset \mathfrak{A}$ if in addition we have

(c) $$\lim_{n\to\infty} q(n)\, m(B(n)) = 0$$

where $B(n) = c(\cup_{i=0}^{q(n)-1} A_i^{(n)})$.

The first problem considered for these classes is the problem of representing two commuting transformations in terms of each other. The result, obtained in collaboration with Akcoglu and Schwartzbauer is the following :

THEOREM 1. – *If τ is in class $\mathcal{A}$ and σ commutes with τ, there are two sequences X_n^1, X_n^2 of sets and two sequences of non-negative integers j_n^1, j_n^2, $n = 1, 2, \ldots$, such that $X_n^1 \cap X_n^2 = \emptyset$ and such that*

(i) *if* $A \in \mathcal{F}$ *then* $\sigma(A) = \lim_{n\to\infty} \{\tau^{j_n^1}(A \cap X_n^1) + \tau^{-j_n^2}(A \cap X_n^2)\}$,

and

(ii) *either* $\sigma = \tau^k$ *for some integer* k, *or* $\lim_{n\to\infty} j_n^1 = \lim_{n\to\infty} j_n^2 = +\infty$.

If in addition, τ is in class $\mathcal{B}$ then $X_n^2 = \phi$ so that $\sigma(A) = \lim_{n\to\infty} \tau^{j_n}(A)$.

This result can be used to simplify the construction of various examples of transformations which are either simply or strongly mixing and which have only certain specified roots.

The next circle of problems considered is concerned with the spectral type of transformations. For this purpose we needed to consider more general classes.

DEFINITION 4. – We say that τ is of class $\mathcal{C}(N)$ if there exists a sequence $\{\xi(n)\}$ of partitions such that the sets of $\xi(n)$ may be indexed as follows :

$$\xi(n) = \{A_{ij}(n) \ , i = 1, \ldots, q_j(n) \ ; j = 1, \ldots, N\},$$

and

i) $\xi(n) \to \epsilon \ , n \to \infty$

ii) $\tau A_{ij}(n) = A_{i+1\,j}(n) \ , i = 1, \ldots, q_j(n) - 1 \ ; j = 1, \ldots, N.$

If $\tau \in \mathcal{C}(N)$ we say that τ admits of simple approximation with multiplicity N.

Intuitively, we may regard $\xi(u)$ in the following way. The sets $A_{i1}(n)$, $i = 1, \ldots, q_1(n)$ are arranged in a stack, with $A_{11}(n)$ at the bottom $A_{21}(n)$ above it, and so on, with $A_{q_1(n)}(n)$ at the top. The same procedure is followed with $c_{i2}(n)$, $i = 1, \ldots, q_2(n)$, with $c_{i3}(n)$, $i = 1, \ldots, q_3(n)$, and so on, yielding N stacks of heights $q_1(n), \ldots, q_N(n)$. The action of τ is there, by ii) to map each point of each stack to the one directly above, except for those points on the top layers, where the transformation τ is not restricted by $\xi(n)$, although ultimately it is by some $\xi(j), j > n$, as we see by i).

In order to classify the spectral type of these transformations we need the following simple result for operators in Hilbert space. This result is a generalization of one given by Katok and Stepin.

THEOREM 2. – *If U is a unitary operator on a separable Hilbert space and if the spectral multiplicity of U is at least k, then there exist k orthonormal vectors $u_1, u_2, \ldots, u_k$ such that*

$$\sum_{i=1}^{k} d^2(u_i, H(w)) \geqslant k - 1$$

for any $w \in H$, where $H(w) = \{\ldots, U^{-1}w, w, Uw, \ldots\}$.

Our principal result in this connection is :

THEOREM 3. – *If τ is in class $\mathcal{C}(N)$, then the spectral multiplicity of τ is at most N.*

Proof. – Suppose that the spectral multiplicity is greater than N. Then by Theorem 2 there exist $N + 1$ orthonormal vectors $u_1, u_2, \ldots, u_{N+1}$ such that for any $w \in H$

$$\sum_{i=1}^{N+1} d^2 (u_i, H(w)) \geqslant N.$$

If τ admits of simple approximation with multiplicity N, then there exists a sequence $\{\xi(n)\}$ of partition with $\xi(n) = \{A_{ij}(n), i = 1, \ldots, q_j(n) ; j = 1, \ldots, N\}$. We let

$$w_j(n) = \chi_{A_{ij}}(n).$$

For $j = 1, \ldots, N$ we have by Theorem 2 that

$$\sum_{i=1}^{N+1} d^2 (u_i, H(w_j(n)) \geqslant N. \tag{3.2}$$

We have also that each $u_i, i = 1, \ldots, N + 1$, can be written

$$u_i(n) = u_{i1}(n) + \cdots + u_{iN}(n) + h_i(n)$$

where

(i) $\| h_i(n) \| \to 0$,

(ii) $d^2(u_{ik}(n), H(w_k(n))) = 0, k = 1, \ldots, N.$

and

(iii) $u_{ik}(n)$, $k = 1, \ldots, N$, have disjoint support.

This follows easily by taking $u_{ij}(n)$ as the linear combination of the functions $\chi_{A_{ij}(n)}, \ldots, \chi_{A_{q_j(n)j}(n)}$ closest to u_i. Next, we have that

$$d^2(u_i, H(w_j(n))) = d^2 (u_{i1}(n) + \ldots + u_{iN}(n) + h_i(n), H(w_j(n))$$
$$\leqslant \sum_{k=1}^{N} d^2(u_{ik}(n), H(w_j(n))) + \epsilon_i(n),$$

where $\epsilon_i(n) \to 0$ has $n \to \infty$. Combining this with (3.2) we obtain that

$$\sum_{k=1}^{N} \sum_{i=1}^{N+1} d^2(u_{ik}(n), H(w_j(n))) \geqslant N + \epsilon(n) \tag{3.3}$$

where $\epsilon(n) = \epsilon_1(n) + \ldots + \epsilon_{N+1}(n)$. Finally, summing over j, we have that

$$\operatorname*{Lim}_{n \to \infty} \sum_{\substack{i=1,\ldots,N+1 \\ j,k=1,\ldots,N}} d^2(u_{ik}(n), H(w_j(n))) \geqslant N^2. \tag{3.4}$$

We have by (ii) that

$$d^2(u_{ik}(n), H(w_k(n))) = 0, k = 1, \ldots, N. \tag{3.5}$$

It is clear that

$$(3.6) \qquad \|u_{ik}(n)\|^2 \geq d^2(u_{ik}(n)\ ,H(w_j(n)))\ \ ,k = 1,\dots,N,$$

and since the $u_{ik}(n)$, $k = 1,\dots,N$ have disjoint support and $\|h_i(n)\| \to 0$,

$$(3.7) \qquad \|u_i\|^2 = \lim_{n\to\infty} \sum_{k=1}^{N} \|u_{ik}(n)\|^2 .$$

Substituting (3.5) and (3.6) into (3.4) yields

$$\lim_{n\to\infty} (N-1) \sum_{\substack{i=1,\dots,N+1\\ k=1,\dots,N}} \|u_{ik}(n)\|^2 \geq N^2$$

from which we obtain, with (3.7), that

$$\lim_{n\to\infty} (N-1) \sum_{i=1,\dots,N+1} \|u_i(n)\|^2 \geq N^2$$

which is impossible since $u_i(n)$, $i = 1,\dots\dots,N+1$ have unit length.

Katok and Stepin have considered certain related classes of transformations which may be defined as follows.

DEFINITION 5. – We say that τ admits of approximation by periodic transformations with speed $f(n)$ if there exists a sequence of partitious

$$\xi(n) = \{A_1(n),\dots .A_{q(n)}(n)\}$$

and periodic transformations τ_n with $\tau_n \xi_n = \xi_n$ such that

(a) $\xi(n) \to \epsilon$

and

(b) $\sum_{i=1}^{q(n)} m(\tau A_i(n)\ \Delta \tau_n A_i(n) < f(q(n))$.

This definition may be generalized in several ways and as an example we give the following :

DEFINITION 6. – We say that τ admits of approximation with multiplicity N and speed $(f_1(n),\dots,f_n(n))$ if there exists a sequence $\xi(n)$ of partitions with $\xi(n) = \{A_{ij}(n), i = 1,\dots,q_j(n)\ ; j = 1,\dots,N\}$ such that

(a) $\xi(n) \to \epsilon$

(b) $m(A_{ij}(n) = \delta_j(n)$, $i = 1,\dots,q_j(n)$, $j = 1,\dots,N$,

(c) $\sum_{i=1}^{q_j(n)-1} m(\tau A_{ij}(A_{i+1\,j}(n)) < f_j(1/\delta_j(n))$, $j = 1,\dots,N$.

and the following extension of a result of Katok and Stepin can be obtained.

THEOREM 4. – *If τ admits of approximation with multiplicity N and speed* $(\theta_1/n, \ldots, \theta_N/n)$, $\theta_i < 2/N + 1$, $i = 1, \ldots, N$, *then the spectral multiplicity of τ is at most N.*

For these classes of transformations it is possible to obtain more information about their properties if there is some information about the sequence $q(n)$. As an example of the type of result which can be obtained in this direction we note the following. First we introduce the next definition.

DEFINITION 7. – A sequence $\{n(k)\}$ is called an m-pair sequence if

$$n(2k) = 1 + mn\,(2k - 1),\ k = 1, 2, 3, \ldots,$$

for some integer m. We say that τ admits of approximation in m pairs with speed $f(n)$ if it admits of approximation by periodic transformations with speed $f(n)$ and if the sequence $\{q(n)\}$, where $q(n)$ is the number of sets in $\xi(n)$, has a subsequence which is an m-pair sequence. We can then obtain the following results :

THEOREM 5. – *If τ admits of approximation in m pairs with speed θ/n, $\theta < 1$, then τ is weakly mixing.*

THEOREM 6. – *The set of transformations admitting of approximation in m-pairs with speed θ/n, $\theta > 1/m$* contains an everywhere dense *G_δ set (with respect to the weak topology).*

BIBLIOGRAPHY

[1] AKCOGLU M.A., CHACON R.V. and SCHWARTZBAUER T. — Commuting transformations and mixing, to appear, *Proc. Amer. Math. Soc.*

[2] CHACON R.V. — A geometric construction of measure preserving transformations, *Fifth Berkeley Symposium on Probability and Statistics,* (2) 2, 1967, p. 335-360.

[3] CHACON R.V. — *Weakly mixing transformations which are not strongly mixing,* to appear.

[4] CHACON R.V. — *Approximation of transformations with continuous spectrum,* to appear.

[5] CHACON R.V. — *Approximation and spectral multiplicity,* to appear.

[6] CHACON R.V. and SCHWARTZBAUER T. — *Commuting point transformations,* to appear.

[7] HALMOS P.R. — Lectures on ergodic theory, *Math. Soc. of Japan,* 1956, Chelsea.

[8] KATOK A.B. and STEPIN A.M. — Approximation of ergodic dynamical systems by periodic transformations (Russian), *Dokl. Akad. Nauk S.S.S.R.,* 171, p. 1268-1271, 1966, *Soviet Math. Dokl.,* 7, p. 1638-1641, 1966.

[9] KATOK A.B. and STEPIN A.M. — Approximations in ergodic theory (Russian), *Uspekhi Math. Nauk,* (5) 22, 1967, p. 81-106, (*Russian Math. Surveys* (5) 22, 1967, p. 77-102).

[10] ORNSTEIN D.S. — *A mixing transformation that commutes only with its powers,* to appear.

[11] ROKHLIN V.A. — On the fundamental ideas of measure theory (Russian), *Math. Sbor.* 67, 1949, p. 107-150, (*Amer. Math. Soc. Translation* No. 71, 1952).

[12] ROKHLIN V.A. — Lectures on the entropy theory of measure-preserving transformations (Russian), *Uspekhi Math. Nauk,* (5) 22, 1967, p. 3-56, (*Russian Math. Surveys,* (5) 22, 1967, p. 1-52).

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CENTRAL LIMIT THEOREMS FOR DEPENDENT RANDOM VARIABLES

by Aryeh DVORETZKY

1. – Limiting distributions of sums of 'small' independent random variables have been extensively studied and there is a satisfactory general theory of the subject (see e.g. the monograph of B.V. Gnedenko and A.N. Kolmogorov [2]). These results are conveniently formulated for double arrays $X_{n,k}$ ($k = 1, \ldots, k_n$; $n = 1, 2, \ldots$) of random variables where the $X_{n,k}$ ($k = 1, \ldots, k_n$), the random variables in the n-th row, are assumed mutually independent for every n. The smallness assumption is that, as $n \to \infty$, the random variables $X_{n,k}$ converge in probability to 0, uniformly in k. This assumption implies that the limit distributions of $X_{n,1} + \ldots + X_{n,k_n}$ are infinitely divisible ones, and necessary and sufficient conditions on the distributions of the summands are known for convergence to any specific infinitely divisible distribution.

Our knowledge of the corresponding theory for sums of dependent random variables is much less satisfactory. Though a great number of papers have been published on the subject, not many general results are known. A notable exception to this statement is provided by some pioneering work of P. Lévy (see [3] and the references there). Recently [1] we have shown that the classical necessary and sufficient conditions mentioned above for the independent case, remain sufficient for the most general dependent case, provided we replace in their formulation the distributions of the summands $X_{n,k}$ by their conditional distributions. In *2* we present some of these results, in *3* we state three lemmas used in proving the results of *2* while in *4* we note some applications and make some comments.

2. – The typical results which we present here are patterned after familiar results for independent random variables.

Let $(\Omega_n, \mathcal{A}_n, P_n)$, $(n = 1, 2, \ldots)$ be a sequence of probability spaces, not necessarily distinct, and let $X_{n,k}$ ($k = 1, 2, \ldots, k_n$) be any random variables defined on $(\Omega_n, \mathcal{A}_n, P_n)$. Put $S_{n,k} = \sum_{j=1}^{k} X_{n,j}$ ($k = 0, \ldots, k_n$) and let $\mathcal{F}_{n,k}$ be the σ-field generated by $S_{n,k}$. Conditional expectation and conditional probability relative to $\mathcal{F}_{n,k}$ are denoted by $E_{n,k}$ and $P_{n,k}$ respectively. $I\{.\}$ is the indicator of the set within the braces. $\mathcal{L}(.)$ is the distribution of the random variable within the brackets ; $\mathcal{N}(0, 1)$ is the standard normal law. Convergence $\to$ always refers to $n \to \infty$ and $\xrightarrow{P}$ denotes convergence in probability. We abbreviate $\sum_{k=1}^{k_n}$ to $\sum_k$ and $\max_{1 \leqslant k \leqslant k_n}$ to $\max_k$.

The first result we quote concerns asymptotic normality.

THEOREM 1. – *The conditions*

$$(1) \qquad \sum_k E_{n,k-1} X_{n,k} \xrightarrow{P} 0,$$

$$(2) \qquad \sum_k E_{n,k-1} (X_{n,k} - E_{n,k-1} X_{n,k})^2 \xrightarrow{P} 1$$

and

$$(3) \qquad \sum_k E_{n,k-1} (X_{n,k}^2 I\{|X_{n,k}| > \epsilon\}) \xrightarrow{P} 0 \quad \text{for every} \quad \epsilon > 0,$$

imply

$$(4) \qquad \mathcal{L}(S_{n,k_n}) \to \mathcal{N}(0, 1).$$

We note that (3) is a weaker condition than the standard, non-conditioned, Lindeberg condition

$$(3') \qquad \sum_k E(X_{n,k}^2 I\{|X_{n,k}| > \epsilon\}) \to 0 \quad \text{for every} \quad \epsilon > 0.$$

We also remark that (1) is automatically satisfied when $X_{n,k}$ $(k = 1, \ldots, k_n)$ are, for each n, martingale differences. The classical result about asymptotic normality (4) is, of course, a further specialization. Theorem 1 already exhibits the recurring feature of our generalizations : Replacing, where appropriate in the classical limit theorems, expectations by conditional ones relative to the preceding row sum we obtain sufficient conditions for convergence to a given limit law.

A certain generality is gained by formulating the conditions in terms of convergence in probability. However, the most important generality feature of Theorem 1 relative to the usual results on asymptotic normality for sums of dependent random variables is that no assumption beyond (2) is made about the conditional variances. Usually they are assumed to be nearly constant in some sense.

Similar results can be stated for convergence to a Poisson law. Instead of giving further special results we state one of moderate generality.

THEOREM 2. – *Assume that* $E_{n-1} X_{n,k} = 0$ *and that* $\sum_k E_{n,k-1} X_{n,k}^2$ $(n = 1, 2, \ldots)$ *are uniformly bounded in probability and let* $K(.)$ *be a bounded monotone function on the real line. Then the conditions*

$$(5) \qquad \sum_k E_{n,k-1} (X_{n,k}^2 I\{X_{n,k} < x\}) \xrightarrow{P} K(x)$$

at every continuity point x *of* K *and*

$$(6) \qquad \sum_k (E_{n,k-1} X_{n,k}^2)^2 \xrightarrow{P} 0$$

imply that $\mathcal{L}(S_{n,k_n}) \to$ *the infinitely divisible law whose characteristic function is given by* $\exp \int_{-\infty}^{\infty} \frac{e^{itx} - 1 - itx}{x^2} \, dK(x)$.

The conditioning requirements become more stringent if the conditioning σ-fields are replaced by finer ones. Thus if we denote by $E^*_{n,k}$ expectations relative to $\mathcal{F}^*_{n,k}$, the σ-field generated by $X_{n,1}, \ldots, X_{n,k}$, then the conditions (5*) and (6*) obtained on replacing E by E^* in (5) and (6), respectively, certainly imply the conclusion. We note that, given the other assumptions, the condition (6*) is implied by

$$(7) \qquad \max_k E^*_{n,k-1} X^2_{n,k} \xrightarrow{P} 0 ,$$

or by

$$(8) \qquad \max_k P^*_{n,k-1} \{ |X_{n,k-1}| > \epsilon \} \xrightarrow{P} 0 \quad \text{for every} \quad \epsilon > 0$$

($P^*_{n,k}$ is the conditional probability relative to $\mathcal{F}^*_{n,k}$).

Conditions (7) and (8) are smallness conditions similar to those used in the independent case. They are, however, expressed through finer conditionings than those relative to $\mathcal{F}_{n,k}$. But it is possible, through Lemma 1 below, to replace them by a condition which is expressed in terms of the conditional distributions of $X_{n,k}$ relative to $\mathcal{F}_{n,k-1}$ only (see (15)).

The previous theorem assumed first and second moment conditions. The following is an example of a result which dispenses entirely with such assumptions.

THEOREM 3. – *Let* a *be a real number and* K *be a bounded monotone function on the real line. Put* $a_{n,k} = E_{n,k-1} (X_{n,k} \, I\{|X_{n,k}| < 1\})$ *and* $Y_{n,k} = X_{n,k} - a_{n,k}$. *Then the conditions*

$$(9) \qquad \sum_k \left(a_{n,k} + E_{n,k-1} \frac{Y_{n,k}}{1 + Y^2_{n,k}} \right) \xrightarrow{P} 0 ,$$

$$(10) \qquad \sum_k E_{n,k-1} \left(\frac{Y^2_{n,k}}{1 + Y^2_{n,k}} \, I\{|Y_{n,k}| < x\} \right) \xrightarrow{P} K(x)$$

for every continuity point x *of* K,

$$(11) \qquad \sum_k E_{n,k-1} \left(\frac{Y^2_{n,k}}{1 + Y^2_{n,k}} \right) \xrightarrow{P} K(\infty) - K(-\infty)$$

where $K(\infty) = \lim_{x = \infty} K(x)$, $K(-\infty) = \lim_{x = -\infty} K(x)$, *and*

$$(12) \qquad \sum_k \left(E_{n,k-1} \left(\frac{Y^2_{n,k-1}}{1 + Y^2_{n,k}} \right) \right)^2 \xrightarrow{P} 0$$

imply that $\mathcal{L}(S_{n,k_n}) \to$ *the infinitely divisible law whose characteristic function is given by* $\exp\left(ita + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\, d\,K(x)\right)$.

Condition (12) is again implied by the smallness condition (15). (The theorem obtained from Theorem 3 on substituting, throughout its statement, E^* by E is, of course, weaker than Theorem 3. Then (12*), the condition obtained from (12), is implied by the smallness condition (8)).

3. – Technically the fact that the σ-fields $\mathcal{F}_{n,k}$ $(k = 1, \ldots, k_n)$ are not necessarily increasing is very cumbersome. This is overcome by the following simple but useful result.

LEMMA 1. – *Let* $S_1, S_2, \ldots S_n$ *be random variables on a probability space* $(\Omega, \mathcal{A}, P)$. *Then there exists a probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{P})$ *and random variables* $\widetilde{S}_1, \widetilde{S}_2, \ldots, \widetilde{S}_n$ *on it such that*

$$(13)\quad \widetilde{P}(\widetilde{S}_k \leqslant u\ ,\ \widetilde{S}_{k-1} \leqslant v) = P(S_k \leqslant u\ ,\ S_{k-1} \leqslant v)\ ,\quad (k = 2, \ldots, n)$$

for all real u, v *and*

$$(14)\quad \widetilde{P}(\widetilde{S}_k \mid \widetilde{\mathcal{F}}_{k-1}) = \widetilde{P}(\widetilde{S}_k \mid \widetilde{\mathcal{F}}_{k-1}^*)\ ,\quad (k = 2, \ldots, n)$$

where $\widetilde{\mathcal{F}}_k$ *and* $\widetilde{\mathcal{F}}_k^*$ *are the σ-fields generated by* $\widetilde{S}_k$ *and by* $\widetilde{S}_1, \ldots, \widetilde{S}_k$, *respectively.*

Condition (13) asserts that the distributions of the $\widetilde{S}_k$ are the same as those of the S_k and that, moreover, the conditional distributions given that $S_{k-1} = v$, respectively $\widetilde{S}_{k-1} = v$, are also the same. The point of (14) is that conditioning by $\widetilde{\mathcal{F}}_k$ is like conditioning by an increasing sequence of σ-fields. Since joint distributions of $\widetilde{S}_1, \ldots, \widetilde{S}_n$ can be obtained from the conditional distributions of the $\widetilde{S}_k$ given $\widetilde{S}_{k-1}$, i.e. of the S_k given S_{k-1}, it follows that, after obvious adaptation of notation, the condition

$$(15)\qquad \max_k \widetilde{P}_{n,k-1}\{|\widetilde{X}_{n,k}| > \epsilon\} \xrightarrow{P} 0 \quad \text{for every} \quad \epsilon > 0$$

is a smallness condition expressed in terms of the conditional distributions of the $X_{n,k}$ relative to $\mathcal{F}_{n,k-1}$. (15) is precisely the smallness condition which can replace (8) or (12).

The next two lemmas rely heavily on the fact that the conditioning σ-fields are increasing.

LEMMA 2. – *Let* $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_n$ *be σ-fields in a probability space and let* $\eta_1, \ldots, \eta_n$ *be bounded, complex-valued random variables with* η_k *measurable* $\mathcal{G}_k (k = 1, \ldots, n)$. *Put* $\varphi_k = E(\eta_k \mid \mathcal{G}_{k-1})$ *and* $\psi = \prod_{k=1}^{n} \varphi_k$. *If* ψ *is measurable* $\mathcal{G}_0$ *and* $\psi \neq 0$ *almost surely, then* $\psi = E\left(\prod_{k=1}^{n} \eta_k \mid \mathcal{G}_0\right)$.

LEMMA 3. – *Let* $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_n$ *be σ-fields in a probability space and let* $\eta_1, \ldots, \eta_n, \zeta_1, \ldots, \zeta_n$ *be bounded, complex-valued random variables with* η_k *and* ζ_k *measurable* $\mathcal{G}_k$ $(k = 1, \ldots, n)$. *Let the random variables* $\prod_{j=1}^{k-1} \zeta_j \prod_{j=k+1}^{n} \eta_j$, $(k = 1, \ldots, n)$ *be bounded by the constant c, then*

$$(16) \qquad | E \prod_{k=1}^{n} \zeta_k - E \prod_{k=1}^{n} \eta_k | \leq c \sum_{k=1}^{n} E | E(\zeta_k - \eta_k | \mathcal{G}_{k-1}) | .$$

Lemma 2 is proved by backward induction. Lemma 3 is easy and its antecedents can be traced to Lindeberg at least. When applying the above lemmas to deduce the theorems of *2* we have $|\zeta_k| \leq 1$ for all k and $|\eta_k| \leq 1$ except for the last one which is estimated, e.g. in case of Theorem 2, via the differences between the random variables on the left and the expressions on the right of (5) and (6). Through (16) we can obtain an explicit bound for the difference between the characteristic function of S_{n,k_n} and that of the limit law. With the aid of Berry-Esseen and similar estimates we can thus obtain explicit results on the rate of convergence.

4. – We refer to [1] for a number of specializations of Theorem 1 which improve various known results on asymptotic normality for dependent random variables. This can also be applied to derive a version of the three series theorem for dependent random variables. Its other applications include results on stochastic approximation, on optimal stopping and on convergence to the Brownian motion process.

The general theory can also be applied to study domains of attraction and similar questions. The generalization to m-dimensional random variables presents no difficulties.

Recently V.M. Zolotarev and others (see [4] and the references there) succeeded in developing a theory of limit laws for sums of independent random variables without assuming a 'smallness' condition. It would be interesting to extend this theory to the dependent case.

In some special cases we can show that our sufficient conditions are also necessary, but our results in this direction are either fragmentary or involve very cumbersome conditions. Further study of these problems should be of interest.

REFERENCES

[1] DVORETZKY A. — " Asymptotic normality for sums of dependent random variables ", *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley 1970 (to appear). Also the 1969 Reitz lecture of the Institute of Mathematical Statistics.

[2] GNEDENKO B.V. and KOLMOGOROV A.N. — *Limit Distributions of Sums of Independent Random Variables*, Reading, Addison-Wesley, 1954.

[3] LÉVY P. — *Théorie de l'Addition des Variables Aléatoires,* Paris, Gauthier-Villars, 1937.
[4] ZOLOTAREV V.M. — Théorèmes limites généraux pour les sommes de variables aléatoires indépendantes, *C. R. Acad. Sc. Paris,* 270, 1970, p. 899-902.

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ENTROPY IS ENOUGH TO CLASSIFY BERNOULLI SHIFTS BUT NOT K-AUTOMORPHISMS *

by Donald S. ORNSTEIN

In this talk we will concern ourselves with the problem of classifying the measure-preserving, invertible transformations of the unit interval. In fact, we will concern ourselves with the classification of the simplest examples of such transformations, namely the Bernoulli shifts.

A Bernoulli shift can be described as follows : let S be a set with a finite or countable number of points, where the i^{th} point is assigned measure $p_i > 0$ and $\Sigma p_i = 1$. Let X be the product of a doubly infinite sequence of copies of S, and put the product measure on X. Let $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ be a point in X. Define $T\{x_i\} = \{y_i\}$ where $y_{i+1} = x_i$ (that is, T shifts every sequence).

A Bernoulli shift is the simplest example of an ergodic (the only invariant sets have measure 0 or 1), invertible, measure-preserving transformations in the following sense : any ergodic, invertible, measure-preserving transformation can be represented in the above form if, instead of putting the product measure on X, we put some other measure invariant under T.

We will say that T_1 acting on X_1 is isomorphic to T_2 acting on X_2 if there are subsets $X_1' \subset X_1$ and $X_2' \subset X_2$ of measure 1 and invariant under T_1 and T_2, respectively, and if there is an invertible, measure-preserving transformation T mapping X_1' onto X_2' such that $TT_1(x) = T_2T(x)$.

There is another formulation of the problem which I believe brings out more clearly the nature of the problem. We will say that an invertible, measure-preserving transformation T on the unit interval (or a Lebesgue space) is a Bernoulli shift if there is a partition P of X consisting of a finite or countable number of sets P_i such that :

(1) the T^iP are independent (that is, $m\left(\bigcap_{-n}^{n} T^i P_{f(i)}\right) = \prod_{-n}^{n} m(P_{f(i)})$, for all n and f, where m denotes the measure of a set) ;

(2) the T^iP generate the full σ-algebra of X (that is, if E is a measurable set, then for each ϵ we can find an n and a set $\widetilde{E}$ in the algebra generated by T^iP, $-n \leqslant i \leqslant n$, such that the measure of the symmetric difference between E and $\widetilde{E}$ is less than ϵ).

T is isomorphic to a Bernoulli shift in our previous sense. (Furthermore, the sets P_i would correspond to the set of all points whose 0^{th} coordinate is the i^{th} point in S.)

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Let T be a transformation on X and P a partition such that the T^iP are independent and generate. Let $\overline{T}$ be a transformation on $\overline{X}$. The question of whether T and $\overline{T}$ are isomorphic comes to the following. Can we find a partition $\widetilde{P}$ of $\overline{X}$ such that the i^{th} set in $\widetilde{P}$ has the same measure as the i^{th} set in P and the $\overline{T}^i\widetilde{P}$ are independent and generate the full σ-algebra of $\overline{X}$?

Halmos, in his Lectures on Ergodic Theory, pointed out that it was not known whether or not all Bernoulli shifts were isomorphic. In particular, it was not known if the 2-shift (P has two sets, each of measure 1/2) is isomorphic to the 3-shift (P has three sets, each of measure 1/3). One felt that there was an important gap in our understanding of measure-preserving transformations if we could not decide whether the two simplest examples were the same.

In 1958 Kolmogorov made one of the most important advances in ergodic theory by introducing a new invariant, called *entropy*. If T is a Bernoulli shift whose independent generator P has sets P_i of measure p_i, then the entropy of T is $-\Sigma p_i \log p_i$. Since

$$\frac{1}{2}\log\frac{1}{2}+\frac{1}{2}\log\frac{1}{2} \neq \frac{1}{3}\log\frac{1}{3}+\frac{1}{3}\log\frac{1}{3}+\frac{1}{3}\log\frac{1}{3};$$

the 2-shift is not isomorphic to the 3-shift.

There are still many Bernoulli shifts with the same entropy and the question remained : which of these are isomorphic ? Was it possible that two Bernoulli shifts were isomorphic only if they were identical (that is, the $\overline{p}_i$ were simply a rearrangement of the p_i) ? Mesalkin ruled out this possibility by showing that if T and $\overline{T}$ had the same entropy, and if all of the p_i and $\overline{p}_i$ were powers of a single rational number, then T and $\overline{T}$ were isomorphic.

Sinai made substantial progress toward classifying Bernoulli shifts by proving the following theorem. Let T be a Bernoulli shift on X with a partition P, the measure of whose i^{th} set is p_i, and the T^iP are independent and generate. Let $\overline{T}$ be a Bernoulli shift on X with a partition $\overline{P}$, the measure of whose i^{th} set is $\overline{p}_i$, and the $\overline{T}^i\overline{P}$ are independent and generate. Assume that $\Sigma p_i \log p_i = \Sigma \overline{p}_i \log \overline{p}_i$ (that is, T and $\overline{T}$ have the same entropy). Then we can find a partition $\widetilde{P}$ of X, the measure of whose i^{th} set is $\overline{p}_i$, and the $T^i\widetilde{P}$ are independent. The $T^i\widetilde{P}$ do not necessarily generate. If they did, this would have shown that T and $\overline{T}$ are isomorphic.

Our main result is the following :

THEOREM 1. – *Two Bernoulli shifts with the same entropy are isomorphic.*

I would now like to discuss a class of transformations, containing the Bernoulli shifts, which was introduced by Kolmogorov (and are now called K-automorphisms). We say that T is a K-automorphism if there exists a finite partition P such that (1) T^iP generate the full σ-algebra of X and (2) $\bigcap_{n=1}^{\infty}\left(\bigvee_{n}^{\infty} T^iP\right)$ is trivial (contains only sets of measure 0 or 1). [It is easy to see that if T^iP are independent, then (2) holds.]

There is a beautiful theorem due to Sinai and Rochlin, which says that T is a K-automorphism if and only if T has no factors of entropy 0, or equivalently T is a K-automorphism if and only if for all finite partitions P, P is not included $\bigvee_{-\infty}^{-1} T^i P$ (i.e., T is very non-deterministic).

Kolmogorov conjectured that entropy was enough to classify the K-automorphisms. Our second main result is that this is false, or equivalently :

THEOREM 2. — *There is a K-automorphism which is not a Bernoulli shift.*

The ideas in the proof of Theorem 2 are very closely related to those of Theorem 1. The main thing is to find a suitable property of the pairs P, T where T is a Bernoulli shift and P is a finite partition. Such a property comes out of analyzing the proof of Theorem 1.

The main problem in this area now is to classify the K-automorphism. This is important because of the wide variety of transformations arising in other contexts (differential geometry, probability, topological groups, mechanics) that can be shown to be K-automorphisms.

I would now like to discuss some extensions of Theorem 1.

In [1] we only consider the case where P has a finite number of elements. Smorodinsky showed [5] that the argument in [1] could be modified to include the case where P is countably infinite and $\Sigma - p_i \log p_i < \infty$. In [2] we show that any two Bernoulli shifts for which $\Sigma - p_i \log p_i = \infty$ are isomorphic. Actually, in [2] we prove a little more, and to state this result we will first define a generalized Bernoulli shift as follows. Let S be a Lebesgue measure space of total measure 1. Let X be the product of a doubly infinite sequence of copies of S. For our measure on X we will take the product measure. We define T, as before, to be the shift operator. Our result is that any two generalized Bernoulli shifts with the same (finite or infinite) entropy are isomorphic. [Note that if the measure on S has a continuous part, then the entropy is infinite. Otherwise, S has a countable number of points (or, after throwing away a set of measure 0, S has a countable number of points).]

The method used in [1] can be modified to show that certain transformations (which are easily seen to be K-automorphisms) are Bernoulli shifts.

In [3] we show that a subshift of a Bernoulli shift is a Bernoulli shift. By this we mean the following : Let T be a Bernoulli shift. Let $\mathcal{A}$ be a σ-algebra of measurable sets invariant under T. We can then find a partition P (finite if the entropy of T is finite, otherwise countable) such that the $T^i P$ are independent and generate $\mathcal{A}$.

In [4] (a joint paper with N.A. Friedman) we modify the proof in [1] to obtain a condition under which the $T^i P$ generate a Bernoulli shift. We apply this to show that mixing Markov shifts are Bernoulli shifts.

A similar criterion can be applied to show that any ergodic automorphism of the 2- or 3-dimensional torus is a Bernoulli shift. For the n-dimensional torus we have the following : Any automorphism is given by a matrix with integer coefficients and determinant ± 1. If the matrix has no eigenvalues on the unit circle, then the automorphism is a Bernoulli shift. (It is ergodic if and only if it has no eigenvalues that are roots of unity.) It was previously shown by Adler and

Weiss that the ergodic automorphisms of the 2-dimensional torus were Markov shifts. These Markov shifts were of a special kind and for these Adler and Weiss showed that entropy was a complete invariant. Sinai showed that the automorphism of the m-dimensional torus with no eigenvalues on the unit circle were Markov shifts.

Some deep work of Sinai shows that our conditions apply to Anosov diffeomorphisms. Hence they are Bernoulli shifts. (The Anosov diffeomorphisms include geodesic flow on compact surfaces of negative nature.)

Smorodinsky has shown that if a Gaussian process is a K-automorphism, then it is a Bernoulli shift. Smorodinsky and H. Totoki – S. Itô – H. Murata showed that the natural extension of the continued fraction transformation is a Bernoulli shift.

Classifying the Bernoulli shifts gives information about them which at first glance one would not expect to get. For example, it was not previously known if the 2-shift had a square root. We can now show it has a square root as follows. Let T be a Bernoulli shift whose entropy is one-half that of the 2-shift. It is easy to see that T^2 will be a Bernoulli shift with the same entropy as the 2-shift. Therefore, the 2-shift has a square root. Similarly, Bernoulli shifts have roots of all orders and have lots of automorphisms that commute with them.

In [6] the above result is pushed further by showing that Bernoulli shifts can be imbedded in flows. The flow S_t (which was previously shown by Totoki to be a K-automorphism for each t [15]) can be described as follows : let T acting on X be the 2-shift. Let f be the function on X that takes on two values : α on those points of X whose first coordinate is 0 and β on those points whose first coordinate is 1. α and β are picked so that α/β is irrational. Let Y be the area under the graph of f. S_t will act on Y as follows : each point, (x, l), will move directly up at unit speed until it hits the graph of f. Then it goes to $(Tx, 0)$ and continues moving up at unit speed. S_t for each t is a Bernoulli shift of finite entropy.

Smorodinsky and Feldman constructed a flow such that S_t is a Bernoulli shift of infinite entropy.

It is not known if any two flows, S_t and $\hat{S}_t$, such that S_t and $\hat{S}_t$ are Bernoulli shifts for each t are isomorphic (after normalizing t so that the entropies of S_1 and $\hat{S}_1$ are the same).

In closing I would like to mention that there are several results, which I did not have time to talk about, due to Shields-McCabe, Parry, Bowen, and Azencott, which show that transformations arising in various contexts are Bernoulli shifts.

REFERENCES

[1] ORNSTEIN D. — Bernoulli shifts with the same entropy are isomorphic, *Adv. in Math.*, 4, 1970, p. 337-352.

[2] ORNSTEIN D. — *Bernoulli shifts with infinite entropy are isomorphic,* (to appear).
[3] ORNSTEIN D. — Factors of Bernoulli shifts are Bernoulli shifts, *Adv. in Math.*, 5, 1970, p. 349-364.
[4] ORNSTEIN D. and FRIEDMAN N. — *Markov shifts are Bernoulli shifts* (to appear).
[5] SMORODINSKY M. — *An exposition of Ornstein's isomorphic theorem,* (ibid.).
[6] ORNSTEIN D. — *Imbedding Bernoulli shifts in flows,* (ibid).
[7] ORNSTEIN D. — *A Kolmogorov automorphism that is not a Bernoulli shift,* (ibid).
[8] ADLER R.L. and WEISS B. — Entropy, a complete metric invariant for automorphisms of the torus, *Proc. Nat. Acad. Sci. U. S. A.*, 57, 1967, p. 1573-1576.
[9] BILLINGSLEY P. — *Ergodic Theory and Information,* New York, John Wiley and Sons, 1965.
[10] HALMOS P. — Recent progress in ergodic theory, *Bull. Amer. Math. Soc.*, 67, 1961, p. 70-80.
[11] KOLMOGOROV A.N. — A new metric invariant of transitive dynamic systems and automorphisms in Lebesgue spaces, *Dokl. Akad. Nauk S.S.S.R.*, 119, 1958, p. 861-864.
[12] KOLMOGOROV A.N. — On the entropy per unit time as a metric invariant of automorphisms, *Dokl. Akad. Nauk S.S.S.R.*, 124, 1959, p. 754-755.
[13] MESALKIN L.D. — A case of isomorphisms of Bernoulli schemes, *Dokl. Akad. Nauk S.S.S.R.*, 128, 1959, p. 41-44.
[14] SINAI I.G. — A weak isomorphism of transformations with an invariant measure, *Dokl. Akad. Nauk S.S.S.R.*, 147, 1962, p. 797-800 and *Soviet Math. Dokl.*, 3, 1962, p. 1725-1729.
[15] TOTOKI H. — On a class of special flows, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, 15, 1970, p. 157-167.

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SUR LES ESTIMATIONS DE LA RAPIDITÉ DE CONVERGENCE DANS LE THÉORÈME LIMITE CENTRAL CAS DE DIMENSIONS FINIE ET INFINIE

Par V. V. SAZONOV

1. Introduction.

Soit $\xi_1, \xi_2, \ldots$ une suite de variables aléatoires indépendantes et (pour simplifier) de même loi de distribution P, à valeurs dans l'espace de Hilbert séparable H. Il est bien connu (v. par ex. [1]) que si $E \|\xi_1\|^2 < \infty$, les distributions P_n des sommes normées $n^{-1/2} \sum_{i=1}^{n} (\xi_i - E\,\xi_i)$ convergent faiblement vers la loi normale Q de moyenne nulle et ayant même matrice de covariance que P. En d'autres termes

$$\int_H f(x)\ P_n(dx) \to \int_H f(x)\ Q(dx)$$

pour toute fonction f appartenant à la classe $\mathfrak{F}_Q$ des fonctions réelles boréliennes bornées définies sur H et Q-presque partout continues.

Dans ce qui suit, nous supposerons (ceci ne limite pas la généralité) que les variables ξ_i sont centrées sur leurs valeurs moyennes.

Nous nous intéresserons à la rapidité de convergence de P_n vers Q. Il est naturel de prendre pour mesure de l'écart entre P_n et Q

$$\sup_{f \in \mathfrak{F}} \left| \int_H f(x)\ (P_n - Q)\ (dx) \right| \tag{1}$$

où $\mathfrak{F} \subset \mathfrak{F}_Q$ est une "classe d'uniformité pour Q".

Rappelons qu'une classe $\mathfrak{F}$ de fonctions boréliennes bornées sur H s'appelle classe d'uniformité pour Q ([2], [3]) si

$$\sup_{f \in \mathfrak{F}} \left| \int_H f(x)\ (Q_n - Q)\ (dx) \right| \to 0$$

quand $n \to \infty$, pour toute suite de mesures de probabilité Q_n convergeant faiblement vers Q.

Dans le cas de dimension infinie, pour le moment on sait très peu de choses sur la rapidité de convergence de P_n vers Q. Ci-dessous on établira deux résultats très particuliers dans cette direction.

Nous accorderons une attention fondamentale au cas de dimension finie, c'est-à-dire au cas où les ξ_i prennent des valeurs dans $\mathbf{R}^k$. De plus, bien qu'il y ait de nombreux résultats concernant l'estimation de (1) pour des classes assez générales d'uniformité pour Q ([4], [5] et même [7]), nous nous limiterons à l'examen du cas particulier important $\mathfrak{F} = \mathfrak{C}^*$, où $\mathfrak{C}^*$ est la classe des indicateurs des sous-ensembles boréliens convexes de $\mathbf{R}^k$. En d'autres termes, seront examinées les estimations de

$$(2) \qquad \sup_{E \in \mathfrak{C}} |P_n(E) - Q(E)|$$

où $\mathfrak{C}$ est la classe des sous-ensembles boréliens convexes de $\mathbf{R}^k$. Du point de vue technique, le transfert de résultats à des classes plus générales d'uniformité pour Q se présente, en un certain sens, comme un problème accessoire et les résultats exposés ci-dessous sont vrais également pour des classes plus générales d'uniformité pour Q.

Remarquons encore que l'on a obtenu actuellement des généralisations au cas de la dimension finie des théorèmes classiques à une dimension sur la décomposition d'Edgeworth de la différence $P_n - Q$ ([2], [5], [10] ; v. également la série de travaux de A. Bikialis dans le Litovsk. Mat. Sb.) et que l'on a des résultats sur l'estimation des grandes déviations des sommes normées $n^{-1/2} \sum_i^n \xi_i$ de variables aléatoires ξ_i à valeurs dans $\mathbf{R}^k$([12], [13], [8]). Toutefois, nous ne nous occuperons pas de ces questions.

Dans ce qui suit C, $C(k)$, avec ou sans indice, désigne partout, respectivement, des constantes absolues ou des constantes dépendant seulement de la dimension k (le même symbole peut s'employer pour diverses constantes). $N_{\nu, V}$ désignera une loi normale de moyenne ν et de matrice de covariance V ; $N_T = N_{0, T^{-2}I}$, où I est la matrice unité d'ordre k. Enfin pour toute mesure réelle μ, $|\mu|$ signifie sa variation.

2. Cas de dimension finie

Pour les estimations de (2) dans le cas de dimension finie, on utilise deux méthodes – la méthode des fonctions caractéristiques et la dite méthode de composition.

Les deux méthodes utilisent le "lemme de lissage". Diverses variantes de celui-ci ont été utilisées par de nombreux auteurs ([4], [7], [15], [17], [18] etc.). Voici l'une de celles-ci.

Soient P', P'', P''' des mesures arbitraires de probabilité sur $\mathbf{R}^k$ et P_T''' la mesure déterminée par la relation $P_T'''(\cdot) = P'''(T\cdot)$ et α, β, $\frac{1}{2} < \alpha \leqslant 1$, des nombres positifs tels que

$$P'''(x : |x| < \beta) \geqslant \alpha$$

Alors pour tous $T > 0$

$$(3) \sup_{E \in \mathfrak{C}} |P'(E) - P''(E)| \leqslant (2\alpha - 1)^{-1} \left[\sup_{E \in \mathfrak{C}} |(P' - P'') * P_T'''(E)| + 2\, k^{1/2} \alpha \beta T^{-1}\right]$$

Considérons d'abord la méthode des fonctions caractéristiques (v. [2], [4], [7], [11], [19], [20]).

Soit f (resp. q) la fonction caractéristique de la loi de distribution P (resp. Q) et $f_n(\cdot) = f^n(n^{-1/2}\cdot)$ la fonction caractéristique de la loi P_n. Si H est une mesure arbitraire de probabilité de fonction caractéristique intégrable h et si $H_T(\cdot) = H(T\cdot)$, alors, d'après la formule d'inversion, pour tout ensemble borélien borné $E \subset \mathbf{R}^k$ et pour tout $T > 0$ nous avons

$$(4)\ (P_n - Q) * H_T(E) = (2\pi)^{-k} \int_{\mathbf{R}^k} \left\{ \int_E e^{-i(t,x)}\, dx \right\} (f_n(t) - q(t))\, h(T^{-1}t)\, dt$$

Pour obtenir des estimations dans le cas unidimensionel, si la condition $\rho_3 = E\,|\xi_1|^3 < \infty$ est vérifiée, on opère ensuite, comme on le sait, de la façon suivante. Supposons que la variance de P est égale à 1 (cela ne limite pas la généralité). On prend pour H dans (4) une loi satisfaisant la condition $h(t) = 0$ pour $|t| > 1$ et on pose $T = T_n = \rho_3^{-1} n^{1/2}$. Ensuite on montre que

$$|f_n(t) - q(t)| \leqslant C_2\, T_n^{-1}\, |t|^3\, e^{-t^2/4} \qquad \text{si} \qquad |t| \leqslant T_n .$$

Lorsque E est un intervalle fini, le résultat de l'intégration par rapport à x dans (4) est, en valeur absolue, au plus égal à $2\,|t|^{-1}$; il découle donc de (4) que $\sup_{E \in \mathfrak{C}} |(P_n - Q) * H_{T_n}(E)| \leqslant C_3\, T_n^{-1}$. Pour achever la démonstration, il reste à utiliser l'inégalité (3) avec $P' = P_n$, $P'' = Q$, $P''' = H$, $T = T_n$.

Un tel raisonnement ne s'étend pas au cas multi-dimensionnel, même lorsque E est un rectangle à côtés parallèles aux plans de coordonnées, puisque la fonction $|t_1 \ldots t_k|^{-1}\, |f_n(t) - q(t)|$ n'est en général pas intégrable en 0. Nous donnerons maintenant deux moyens de surmonter cette difficulté.

A cet effet, nous allons montrer comment, par exemple, en supposant $E\,|\xi_i|^3 < \infty$, on peut obtenir l'estimation

$$(5) \qquad \sup_{E \in \mathfrak{C}} |P_n(E) - Q(E)| \leqslant C(k)\, \delta_n$$

où

$$\delta_n = E\, (\Delta^{-1}\, \xi_1\,,\, \xi_1)^{3/2}\, n^{-1/2}$$

et Δ est la matrice de covariance de la loi de ξ_1.

Ces deux moyens utilisent l'idée de "tronquer" des variables aléatoires. Posons

$$\eta_i = \begin{cases} \xi_i & \text{si} \quad (\Delta^{-1}\, \xi_i\,,\, \xi_i) \leqslant n \\ 0 & \text{si} \quad (\Delta^{-1}\, \xi_i\,,\, \xi_i) > n \end{cases}$$

et soit $\overline{P}_n$ la loi de distribution de la somme normée $n^{-1/2} \sum_1^n \eta_i$ et $m_n = n^{1/2}\, E\, \eta_1$. Pour tout ensemble $E \in \mathfrak{C}$ nous avons

$$|P_n(E) - Q(E)| \leqslant |P_n(E) - \overline{P}_n(E)| + |N_{m_n,\Delta_n}(E) - N_{0,\Delta}(E)| + |\overline{P}_n(E) - N_{m_n,\Delta_n}(E)|$$
$$= \qquad I_1 \qquad + \qquad I_2 \qquad + \qquad I_3$$

où Δ_n est la matrice de covariance de η_1. On peut montrer que $I_1 \leqslant \delta_n$, et que si $\delta_n \leqslant \frac{1}{4}$ (si $\delta_n > \frac{1}{4}$, (5) est évidemment vérifiée), $I_2 \leqslant c(k)\delta_n$. En plus, si $\delta_n \leqslant \frac{1}{4}$, $E(\Delta_n^{-1}(\eta_1 - E\,\eta_1), (\eta_1 - E\eta_1)^{3/2}) \leqslant c\,E\,(\Delta^{-1}\,\xi_1, \xi_1)^{3/2}$.

De cette façon, le problème est réduit à l'obtention d'estimations du type (5) pour des variables aléatoires tronquées.

Il est ensuite facile de ramener le problème à l'obtention d'estimations du type (5) pour des variables aléatoires bornées de moyennes nulles et de matrices de covariance unité. En effet, soit A_n une matrice telle que $A_n\,\Delta_n\,A_n' = I$, $\zeta_i = A_n(\eta_i - E\eta_i)$ et $\widetilde{P}_n$ la loi de distribution de la somme normée $n^{-1/2}\sum_1^n \zeta_i$. Les variables ζ_i sont de moyenne nulle, ont des matrices de covariance unité et

$$\sup_{E\in\mathfrak{C}} |\overline{P}_n(E) - N_{m_n,\Delta_n}(E)| = \sup_{E\in\mathfrak{C}} |\widetilde{P}_n(E) - N_1(E)|$$

$$(\Delta_n^{-1}(\eta_1 - E\,\eta_1) \quad ; \quad (\eta - E\,\eta_1)) = (\zeta_1\,, \zeta_1).$$

Prenons pour H dans la formule (4) une mesure de probabilité dont la fonction caractéristique $h(t)$ vaut 0 pour $|t| > 1$ et de densité assez rapidement décroissante. Désignons par $p_{n,T}$ (resp. $q_{n,T}$) la densité (resp. la fonction caractéristique) de la mesure $(\widetilde{P}_n - N_1) * H_T$.

Le premier des moyens mentionnés ci-dessus utilise des "poids polynomiaux" (v. [7], [4], [11]). Posons

$$p(x) = 1 + \sum_{j=1}^{k} |x_j|^{2l}$$

où l est un nombre entier $\geqslant (k+2)/2$. En vertu de l'inégalité de Schwarz, pour tout ensemble borélien $E \subset \mathbf{R}^k$, nous avons

$$(6) \quad |(\widetilde{P}_n - N_1) * H_T(E)| \leqslant \int_{\mathbf{R}^k} |p_{n,T}(x)|\,dx$$

$$\leqslant \left(\int_{\mathbf{R}^k} p^{-1}(x)\,dx\right)^{1/2} \left(\int_{\mathbf{R}^k} p(x)\,p_{n,T}^2(x)dx\right)^{1/2}$$

De plus, selon la relation de Parseval

$$\int_{\mathbf{R}^k} p_{n,T}^2(x)\,dx = (2\pi)^{-k}\int_{\mathbf{R}^k} q_{n,T}^2(t)\,dt$$

$$(7) \qquad \int_{\mathbf{R}^k} |x_j|^{2l}\,p_{n,T}^2(x)\,dx = (2\pi)^{-k}\int_{\mathbf{R}^k}\left(\frac{\partial^l}{\partial t_j^l}\,q_{n,T}(t)\right)^2 dt$$

En choisissant de façon convenable T, l, en estimant $q_{n,T}(t)$ et $\frac{\partial^l}{\partial t_j^l}\,q_{n,T}(t)$ pour $|t| \leqslant T$, de (3), (6) et (7) on obtient (5).

Le deuxième procédé, dû à V. Rotar [20], consiste en ceci. Soit E un sous-ensemble borélien de $\mathbf{R}^k$, S_1 la boule unité de centre 0 dans $\mathbf{R}^k$, et $E_1 = E \cap S_1$, $E_2 = E \cap S_1^c$. En appliquant la formule d'inversion à (4), nous avons

$$(8) \qquad |(\widetilde{P}_n - N_1) * H_T(E_1)| \leqslant (2\pi)^{-k}\, V(S_1) \int_{\mathbf{R}^k} |q_{n,T}(t)|\, \mathrm{d}t$$

où $V(S_1)$ est le volume de S_1. D'autre part, de la formule d'inversion, nous obtenons, en intégrant par parties

$$(9) \quad |(\widetilde{P}_n - N_1) * H_T(E_2)| \leqslant (2\pi)^{-k} \left| \int_{E_2} \mathrm{d}x \int_{\mathbf{R}^k} e^{-i(t,x)}\, q_{n,T}(t)\, \mathrm{d}t \right|$$

$$= (2\pi)^{-k} \left| \int_{E_2} \mathrm{d}x \int_{\mathbf{R}^k} \frac{e^{-i(t,x)}}{i^{k+1}|x|^{k+1}} \frac{\partial^{k+1}}{\partial t_{\theta_x}^{k+1}}\, q_{n,T}(t)\, \mathrm{d}t \right|$$

$$\leqslant (2\pi)^{-k} \left(\sup_\theta \int_{\mathbf{R}^k} \left| \frac{\partial^{k+1}}{\partial t_\theta^{k+1}}\, q_{n,T}(t) \right| \mathrm{d}t \right) \int_{S_1^c} \frac{\mathrm{d}x}{|x|^{k+1}}$$

où $\dfrac{\partial}{\partial t_{\theta_x}}$ $\left(\text{resp. } \dfrac{\partial}{\partial t_\theta}\right)$ est la dérivée dans la direction x (resp. dans la direction θ).
En joignant (8) et (9), nous arrivons à l'inégalité

$$|(\widetilde{P}_n - N_1) * H_T(E)| \leqslant C(k) \left[\int_{\mathbf{R}^k} |q_{n,T}(t)|\, \mathrm{d}t + \sup_\theta \int_{\mathbf{R}^k} \left| \frac{\partial^{k+1}}{\partial t_\theta^{k+1}}\, q_{n,T}(t) \right| \mathrm{d}t \right]$$

Pour la suite, le raisonnement est le même qu'avec le premier procédé.

Donnons deux derniers résultats concernant l'estimation (2) obtenus par la méthode des fonctions caractéristiques.

Soit $\{ \xi_i = (\xi_{i1}, \ldots \xi_{ik}),\ i = 1, 2 \ldots \}$ une suite de variables aléatoires indépendantes centrées sur leurs espérances mathématiques, à valeurs dans $\mathbf{R}^k$ et dont les lois de distribution P_i' ne sont pas nécessairement les mêmes. Soit ζ une variable aléatoire de loi $n^{-1} \sum_1^n P_i'$, Δ la matrice de covariance correspondant à ζ et supposons que P_n a le même sens que plus haut et que $Q = N_{0,\Delta}$. Soit enfin $g(x)$ une fonction réelle définie sur $\mathbf{R}^1$ possédant les propriétés suivantes :
(1) $g(x) \geqslant 0$, (2) $g(x) = g(-x)$, (3) $g(x) \to \infty$ quand $x \to \infty$, (4) $x\, g^{-1}(x)$ est déterminée pour tous x et $g(x_1) \leqslant g(x_2)$, $x_1 g^{-1}(x_1) \leqslant x_2 g^{-1}(x_2)$ si $0 \leqslant x_1 \leqslant x_2$ (en particulier, ces propriétés sont possédées par la fonction $g(x) = |x|^\delta$, $0 < \delta \leqslant 1$).

THEOREME 1.– *Il existe une constante $C(k)$ telle que si $E\, \xi_{ij}^2\, g(\xi_{ij}) < \infty, i = 1, \ldots, n$ $j = 1, \ldots, k$ et si la matrice Δ est définie positive, alors*

$$(10) \qquad \sup_{E \in \mathcal{C}} |P_n(E) - Q(E)| \leqslant \frac{C(k)}{g(n^{1/2})}\, E\,[(\Delta^{-1}\zeta, \zeta)\, g((\Delta^{-1}\zeta, \zeta)^{1/2})]$$

Pour tout ensemble $E \subset \mathbf{R}^k$ et toute matrice carrée Δ d'ordre k définie positive, posons

(11) $$r_\Delta(E) = \inf_{x \in \partial E} (\Delta^{-1}x, x)$$

où ∂E est la frontière de E.

THEOREME 2. – *Il existe une constante $C(k)$ telle que si $E|\xi_i|^3 < \infty$, $i = 1, \ldots, n$, et si la matrice Δ est définie positive, alors pour tout $E \in \mathfrak{C}$*

(12) $$|P_n(E) - Q(E)| \leqslant \frac{C(k)}{1 + r_\Delta^3(E)} E(\Delta^{-1}\zeta, \zeta)^{3/2} n^{-1/2}$$

Le théorème 1 a été obtenu par A. Bikialis [10]. Pour sa démonstration, il utilise les poids polynomiaux. Le théorème 2 est dû à V. Rotar [20] ; pour sa démonstration, il utilise une variante du procédé d'intégration par parties indiqué plus haut. Le théorème 2 est une extension à plusieurs dimensions de l'estimation non uniforme de S. Nagaev [21], étendue au cas de termes non équidistribués par A. Bikialis [9].

Remarquons que l'on peut donner d'autres formes aux estimations (10) et (11). Ainsi par exemple, si $t_1, \ldots, t_k$ sont des éléments de $\mathbf{R}^k$ tels que les variables aléatoires (t_i, ζ), $i = 1, \ldots k$ ne sont pas correlées, alors

(13) $$E(\Delta^{-1}\zeta, \zeta)^{3/2} \leqslant k^{1/2} \sum_{i=1}^{k} \frac{E|(t_i, \zeta)|^3}{E^{3/2}(t_i, \zeta)^2},$$

et par conséquent, on peut remplacer $E(\Delta^{-1}\zeta, \zeta)^{3/2}$ dans le membre de droite de l'inégalité (12) par le membre de droite de (13).

Remarquons encore que si $\mathcal{E}$ est une classe assez riche de sous-ensembles boréliens de R^k (en particulier si $\mathcal{E} = \mathfrak{C}$), alors comme il ressort de la forme du premier terme de la décomposition d'Edgeworth de la différence $P_n - Q$, on ne peut espérer une estimation uniforme pour $E \in \mathcal{E}$ de $|P_n(E) - Q(E)|$ de meilleur ordre que $n^{-1/2}$. Toutefois, en remplissant des conditions supplémentaires, pour quelques classes d'ensembles pour lesquelles le premier terme de la décomposition d'Edgeworth de la différence $P_n - Q$ s'annule, la méthode des fonctions caractéristiques permet d'obtenir des estimations d'ordre n^{-1} ou d'ordre voisin (v. [22], [7], [5], [11]).

Passons maintenant à la deuxième méthode d'obtention des estimations qui nous intéressent, la méthode de composition. Eclaircissons sa nature sur un exemple simple.

Soient $\xi_1, \xi_2 \ldots$ des variables aléatoires réelles indépendantes de même loi de distribution, à moyennes nulles et de variances 1. Soit F_n la fonction de répartition de la somme normée $n^{-1/2} \sum_1^n \xi_i$, $F_{(n)}(\cdot) = F(n^{1/2}\cdot)$ et Φ_T la fonction de répartition normale $(0, \Gamma^{-1})$ de densité φ_T. Donnons la preuve de l'inégalité

(14) $$\sup_{x \in R^1} |F_n(x) - \Phi_1(x)| \leqslant \overline{C}\rho_3 n^{-1/2}$$

où $\rho_3 = E|\xi_1|^3$. La démonstration s'effectue par induction sur n. Pour $n = 1$ la réalisation de (14) est évidente. Ensuite il est facile de se convaincre que

$$(F_n - \Phi_1) * \Phi_T = (F_{(n)}^{*n} - \Phi_{n^{1/2}}^{*n}) * \Phi_T = \left(\sum_{i=1}^{n-1} U_i + n\, U_0\right) * H_1 ,$$

où

$$U_0 = \Phi_{\tau_0},\ U_i = H_i * \Phi_{\tau_i} ;\ i = 1, \ldots, n-1 ;$$

$$H_i = F_{(n)}^{*i} - \Phi_{n^{1/2}}^{*i} ;\ \tau_i = \left(\frac{n-i-1}{n} + T^{-2}\right)^{-1/2}$$

En développant la fonction

$$U_i(x-y) = \int_{R^1} f_i(x-z)\, \varphi_{\tau_i}(z-y)\, dz$$

où

$$f_i(u) = \begin{cases} H_i(u) & \text{si} \quad i = 1, \ldots, n-1 \\ \delta_0(u) & \text{si} \quad i = 0 \end{cases}$$

(δ_0 est la fonction de répartition correspondant à la masse unité en 0), d'après la formule de Taylor en y jusqu'aux termes de troisième ordre, en utilisant les coincidences des premiers et seconds moments correspondant de $F_{(n)}$ et $\Phi_{n^{1/2}}$ et en prenant en considération l'hypothèse d'induction, nous arrivons aux estimations suivantes

$$n\,|U_0 * H_1(x)| \leqslant c\,\rho_3\, n^{-1/2}$$

$$|U_i * H_1(x)| \leqslant \bar{c}\,\rho_3^2\, n^{-3/2}\, \tau_i^3\, i^{-1/2} \qquad i = 1, \ldots, n-1 \tag{15}$$

Tenons compte simultanément des estimations (15) $\left(\text{en employant l'inégalité } \sum_{i=1}^{n-2} \tau_i^3\, i^{-1/2} \leqslant c\, T\, n^{1/2}\right)$ et utilisons ensuite la formule (3), nous obtenons pour tout $T \leqslant n^{1/2}$

$$\sup_{x \in R^1} |F_n(x) - \Phi_1(x)| \leqslant C(\rho_3\, n^{-1/2} + \bar{C}\rho_3^2\, n^{-1}\, T + T^{-1}) \tag{16}$$

Pour $T = T_n = \bar{C}^{-1/2}\, \rho_3^{-1}\, n^{1/2}$ le membre de droite de (16) vaut

$$\bar{C}\,\rho_3\, n^{-1/2}(C\bar{C}^{-1} + 2C\bar{C}^{-1/2}) \tag{17}$$

C étant choisi tel que l'expression entre parenthèses dans (17) ne dépasse pas 1 (alors $T_n \leqslant n^{1/2}$), nous nous convainquons de l'exactitude de (14).

La méthode de composition est employée dans les travaux [14 - 17], [18] et [23]. Présentons l'un des derniers résultats obtenus à l'aide de cette méthode.

Soit $\xi_1, \xi_2 \ldots$ une suite de variables aléatoires indépendantes à valeurs dans R^k de même distribution P, ayant une matrice de covariance non dégénérée Δ et des moments de troisième ordre finis, et donnons à P_n, Q le même sens que plus haut.

THEOREME 3. – *Il existe une constante $C_1(k)$ telle que pour tout $E \in \mathfrak{C}$*

$$(18) \qquad |P_n(E) - Q(E)| \leqslant C_1(k) \frac{\overline{\nu}_3}{1 + r_\Delta^3(E)} n^{-1/2}$$

où $r_\Delta(E)$ est déterminé par la formule (11) et

$$\overline{\nu}_3 = |P - Q| \, (x : (\Delta^{-1}x , x) \leqslant 1) + \int_{(\Delta^{-1}x,x) > 1} (\Delta^{-1}x , x)^{3/2} \, |P - Q| \, (dx)$$

La constante $C_1(k) \leqslant C_1 k^5$

Faisons quelques remarques à propos du théorème 3.

Tout d'abord, si l'on enlève dans le membre de droite de (18) le facteur $(1 + r_\Delta^3(E))^{-1}$, alors on peut remplacer $C_1(k)$ par $C_2(k) \leqslant C_2 k^{5/2}$.

Ensuite, l'estimation (18), même dans le cas unidimensionnel et sans le facteur $(1 + r_\Delta^3(E))^{-1}$, est meilleure que l'estimation classique de Berri-Essen, et leurs généralisations, dans lesquelles au lieu de $\overline{\nu}_3$ il y a

$$\rho_3 = \int_{\mathbf{R}^k} (\Delta^{-1}x , x)^{3/2} P(dx)$$

En gros, ce qui se passe, c'est que si la distribution P d'un terme est voisine de Q, $\overline{\nu}_3$ est petit et ρ_3, comme le montre un calcul simple, est au moins égal à $k^{3/2}$. De plus, on a l'inégalité

$$(19) \qquad \overline{\nu}_3 \leqslant (1 + Ck^{-3/2}) \, \widetilde{\nu}_3 \leqslant (2 + C'k^{-1}) \, \rho_3$$

où

$$\nu_3 = \int_{\mathbf{R}^k} (\Delta^{-1}x , x)^{3/2} \, |P - Q| \, (dx) \quad , \quad \widetilde{\nu}_3 = \max \, (\nu_3 \, , \nu_3^{k/k+3})$$

La première des inégalités (19) montre que le théorème 3 a pour conséquence également l'estimation de $|P_n(E) - Q(E)|$ à l'aide des "pseudo-moments" ν_3 (v. [24], [23]). Cette inégalité s'établit à l'aide de la relation suivante ayant également un intérêt en elle-même : pour toute mesure de probabilité P dans $\mathbf{R}^k$ et pour tout $s > 0$

$$|P - N_1| \, (\mathbf{R}^k) \leqslant C(s) \, k^{-s/2} \left(\int_{\mathbf{R}^k} |x|^s \, |P - N_1| \, (dx) \right)^{k/k+s}$$

où $C(s)$ dépend seulement de s.

En comparant la méthode de composition et la méthode des fonctions caractéristiques, on remarque ce qui suit. En général, les estimations obtenues par la méthode des fonctions caractéristiques pour des termes équidistribués se transfèrent également sans difficulté à des termes ayant des lois de distribution diverses, alors que par la méthode de composition, on n'a pas encore trouvé d'estimations entièrement satisfaisantes lorsque les termes ne sont pas équidistribués (v. [16], [23]). Par contre, la méthode de composition permet d'obtenir des estimations assez bonnes pour les constantes $C(k)$, dépendant de la dimension (v. théorème 3 ; les constantes obtenues pour le moment par la méthode des fonctions caractéristiques ne sont pas meilleures que k^k). En outre, par la méthode de composition, il est facile d'obtenir des estimations à l'aide de la mesure $|P - Q|$, alors qu'en utilisant la méthode des fonctions caractéristiques, on n'a pu jusqu'ici réussir à obtenir des estimations satisfaisantes de ce genre.

2. Cas de dimension infinie.

Comme nous l'avons déjà remarqué dans l'introduction, pour le moment, on sait peu de choses sur la rapidité de convergence dans le théorème central limite pour les grandeurs aléatoires à valeurs dans un espace de Hilbert. Nous ne parlerons ici que de deux résultats très particuliers. Nous utiliserons ci-dessous les définitions de l'introduction.

Remarquons, avant tout, qu'à la différence du cas de dimension finie, dans le cas de l'espace de Hilbert, déjà les classes les plus simples d'ensembles, comme la classe de tous les demi-espaces ou la classe de toutes les boules, ne sont pas, en général des classes d'uniformité pour Q (v. par ex. [18]). Par conséquent dans le problème de l'estimation de la rapidité avec laquelle la quantité $\sup_{E \in \mathfrak{S}} |P_n(E) - Q(E)|$ tend vers 0 dans le cas d'un espace de Hilbert, il faut se limiter à des classes d'ensembles $\mathfrak{S}$ très spéciales.

Soit $\{e_j, j = 1, 2 \ldots\}$ une base orthonormée dans H et $\zeta_1, \zeta_2 \ldots$ des variables aléatoires indépendantes à répartition uniforme sur $[0, 1]$. Posons

$$\xi_i = \sum_{j=1}^{\infty} \frac{\sqrt{2} \cos \pi j \zeta_i}{\pi j} e_j \ ;$$

$\xi_1, \xi_2 \ldots$ sont des variables aléatoires indépendantes uniformément bornées équidistribuées à valeurs dans H. On peut montrer (v. [25]) que pour tout $\epsilon > 0$ il existe une constante $C(\epsilon)$ dépendant seulement de ϵ, telle que

$$\sup_{x \in \mathbf{R}^1} |P_n(S_x) - Q(S_x)| \leqslant C(\epsilon)\, n^{-1/6+\epsilon} \tag{20}$$

où S_x est la boule de rayon x et de centre 0.

Il est certain que le cas considéré est très particulier, mais il est intéressant pour la statistique mathématique, parce qu'on peut ramener à ce cas le problème de l'estimation de la rapidité de convergence selon le critère ω^2 (il est necessaire toutefois, de remarquer que V. Rosenkranz, en utilisant une tout autre méthode, a obtenu une meilleure rapidité de convergence selon le critère ω^2 (de l'ordre de $n^{-1/4}$ (v. [26]).

Soit ensuite $\xi_1, \xi_2 \ldots$ une suite arbitraire de variables aléatoires indépendantes équidistribuées à valeurs dans H, à opérateurs de covariance S et ayant des moments de troisième ordre finis (c'est-à-dire $E\,\|\xi_i\|^3 < \infty$). Soit A un opérateur symétrique arbitraire dans H de trace finie $\operatorname{tr} A$. Posons

$$E_x = \{y : (Ay, y) < x\} \ , \qquad x > 0$$

et soit $\lambda_1, \lambda_2 \ldots$ la suite des valeurs propres non nulles de l'opérateur AS, données par ordre décroissant et en comptant chacune avec son ordre de multiplicité. N. Bakhania et N. Kandelaki ont montré (v. [27]) que

$$(21) \qquad \sup_{x\in \mathbf{R}^1} |P_n(E_x) - Q(E_x)| \leqslant \frac{f_1(A\,,S)}{1-2\,\epsilon}\;\frac{1}{\log n} + \frac{f_2(A\,,S,\,L)}{n^{\epsilon}}$$

où ϵ, $0 < \epsilon < \frac{1}{2}$, est quelconque

$$L = \sup_{y\in H} \frac{E\,|(\xi_1\,,\,y)|^3}{(Sy\,,y)}$$

et, dans le cas où λ_1 a un ordre de multiplicité $\leqslant 2$,

$$f_1\,(A\,,S) = C_1\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \prod_{k=3}^{\infty}\left(1-\frac{\lambda_k}{\lambda_1}\right)^{-1/2}\,,\quad f_2(A\,,S\,,L) = C_2\,(\lambda_1^{-1}\ \mathrm{tr}\,A)^{1/2}L$$

(on peut écrire une expression analogue pour f_1, f_2 dans le cas où l'ordre de multiplicité de λ_1 est supérieur à 2).

On peut obtenir une estimation du type (20), avec quelques complications, également sans la condition $\mathrm{tr}\,A < \infty$.

Pour la démonstration de (20), on utilise l'estimation multi-dimensionnelles correspondante obtenue par la méthode de composition avec une constante $C(k)$ dépendant d'une façon précise de la dimension et ensuite on tient compte de la différence entre le cas de dimension finie et infinie. L'estimation (21) s'établit par la méthode des fonctions caractéristiques appliquée aux distributions de dimension infinie.

BIBLIOGRAPHIE

[1] Прохоров Ю. В. — Сходимость случайных процессов и предельные теоремы теории вероятностей, Теория вероятн. и ее примен., 1, № 2, 1956, 177-238.

[2] Ranga Rao R. — Relations between weak and uniform convergence of measures with applications, *Ann. Math. Stat.*, 33, No. 2, 1962, p. 659-680.

[3] Billingsley P., Topsøe F. — Uniformity in weak convergence, *Z. Wahrscheinlichkeitstheorie verw.*, Geb. 7, No. 1, 1967, p. 1-16.

[4] Bhattacharya R. — Rates of weak convergence for the multidimensional central limit theorem, Теория вероятн. и ее примен., 15, № 1, 1970, 69-85.

[5] Bhattacharya R. — Rates of weak convergence and asymptotic expansions for classical central limit theorems, submitted to *Ann. Math. Statist.*, 1969.

[6] Von Bahr B. — On the central limit theorem in R^k, *Arkiv för Matematik*, 7, No. 1, 1967, p. 61-69.

[7] Von Bahr B. — Multi-dimensional integral limit theorems, *Ibid.*, p. 71-88.

[8] Von Bahr B. — Multi-dimensional integral limit theorems for large deviations, *Ibid.*, p. 89-99.

[9] Бикялис А. — Оценки остаточного члена в центральной предельной теореме, Литовский матем. сб., 6, № 3, 1966, стр. 323-346.

[10] Бикялис А. — Об остаточных членах в многомерных предельных теоремах, ДАН СССР, 1968, № 4, 1966, стр. 634-635.

[11] Бикялис А. — О центральной предельной теореме в R^k, будет опубликована в Литовск. матем. сб.
[12] Richter V. — Mehrdimensionale Grenzwertsätze für grosse Abweichungen und ihre Anwendung auf die Verteilung von χ^2, Теория вероятн. и ее примен., 9, 1, 1964, стр. 31-42.
[13] Боровков А., Рогозин Б. — О центральной предельной теореме в многомерном случае, Теория вероятн. и ее примен., 10, № 1, 1965, стр. 61-69.
[14] Bergström H. — On the central limit theorem, *Skand. Aktuarietidskrift*, No. 1-2, 1944, p. 139-153.
[15] Bergström H. — On the central limit theorem in the space R_k, $k > 1$, *Ibid.*, No. 1-2, 1945, p. 106-127.
[16] Bergström H. — On the central limit theorem in the case of not equally distributed random variables, *Ibid.*, No. 1-2, 1949, p. 37-62.
[17] Bergström H. — On the central limit theorem in R^k, *Z. Wahrscheinlichkeitstheorie verw.*, Geb. 14, 1969, p. 113-126.
[18] Sazonov V. — On the multi-dimensional central limit theorem, Sankhyā, Ser. A, 30, No. 2, 1969, p. 181-204.
[19] Садикова С. — О многомерной центральной предельной теореме, Теория вероятн. и ее примен., 13, № 1, 1968, стр. 164-170.
[20] Ротарь В. — О скорости сходимости в многомерной центральной предельной теореме, будет опубликована в Теории вероятн. и ее примен.
[21] Нагаев С. — Некоторые предельные теоремы для больших уклонений, Теория вероятн. и ее примен., 10, № 2, 1965, стр. 231-254.
[22] Esseen C. — Fourier analysis of distribution functions, A mathemarical study of the Laplace-Gaussian law, *Acta Math.*, 77, 1945, p. 3-125.
[23] Паулаускас В. — Об оценке скорости сходимости в многомерной центральной предельной теореме I-II, Литовский матем. сб., 9, 1969, № 2, стр. 329-343 и № 4, стр. 791-815.
[24] Золотарев В. — О близости распределений двух сумм независимых случайных величин, Теория вероятн. и ее примен., 10, № 3, 1965, стр. 519-526.
[25] Сазонов В. — Улучшение одной оценки скорости сходимости, Теория вероятн. и ее примен., 14, № 4, 1969, стр. 667-678.
[26] Rozenkrantz W. — A rate of convergence for the Von Mises statistics, *Trans. Amer. Math. Soc.*, 139, 1969, p. 329-337.
[26] Вахания Н., Канделаки Н. — Об оценке скорости сходимости в центральной предельной теореме в пространстве Гильберта, Труды Вычисл. центра АН Гр.ССР, 9 : 1, 1969, стр. 150-160.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 589 à 593.

GROUP-VALUED OUTER MEASURES

by Maurice SION

We sketch here the highlights of a theory for group-valued measures. Most of the definitions and results are actually valid in wider contexts than are indicated (see [7]).

Notation. – Throughout this paper, S is an abstract space, H is a family of subsets of S, X is a commutative Hausdorff topological group, and μ is a function on the family of *all* subsets of S to X.

$\mathcal{B}(H)$ is the σ-field generated by H,

$A \sim B = \{s : s \in A \text{ and } s \notin B\}$ for $A, B \subset S$,

$A - B = \{(x - y) ; x \in A \text{ and } y \in B\}$ for $A, B \subset X$,

$\omega = \{0, 1, 2, \ldots\}$.

1. Outer Measures.

1.1. – DEFINITIONS. –

(1) A is *μ-measurable* iff $A \subset S$ and, for every $B \subset S$,

$$\mu(B) = \mu(B \cap A) + \mu(B \sim A).$$

$\mathfrak{M}_\mu = \{A : A \text{ is } \mu\text{-measurable}\}$.

(2) A is *μ-null* iff $A \subset S$ and, for every $B \subset A$, $\mu(B) = 0$.

$\mathfrak{N}_\mu = \{A : A \text{ is } \mu\text{-null}\}$.

(3) μ is *H-outer regular* iff for every $A \subset S$ and nbhd U of $\mu(A)$ there exists $\alpha \in H$ such that $A \subset \alpha$ and $(A \subset \beta \in H \Longrightarrow \mu(\alpha \cap \beta) \in U)$.

(4) μ is an *H-outer measure* iff μ is countably additive on $\mathcal{B}(H)$ and μ is H-outer regular.

μ is an *outer measure* iff, for some H, μ is an H-outer measure.

1.2. – FUNDAMENTAL THEOREM. – *Let μ be an H-outer measure. Then*

(i) $A_n \subset A_{n+1} \subset S$ *for* $n \in \omega \Longrightarrow \mu(\bigcup_{n\in\omega} A_n) = \lim_n \mu(A_n)$

(ii) $A_n \in \mathfrak{N}_\mu$ *for* $n \in \omega \Longrightarrow \bigcup_{n\in\omega} A_n \in \mathfrak{N}_\mu$

(iii) $\mathfrak{N}_\mu \cup \mathcal{B}(H) \subset \mathfrak{M}_\mu$

(iv) $\mathfrak{M}_\mu$ *is a σ-field and μ is countably additive on* $\mathfrak{M}_\mu$.

Thus, $\mathfrak{M}_\mu$ is complete w.r.t. μ and is the largest σ-field containing H on which μ is countably additive.

2. Generation of Outer Measures.

2.1. DEFINITIONS. – Given $\tau : H \to X$ with $\tau(\emptyset) = 0$ and $(\alpha, \beta \in H \Longrightarrow \alpha \cap \beta \in H)$, for any $A \subset S$, let

(1) $\mathfrak{P}(A) = \{P : P \text{ is countable, disjoint, } P \subset H, \ P \text{ covers } A\}$

$\mathfrak{D}(A) = \{(P, \Delta) : P \in \mathfrak{P}(A)$ and Δ is a function on $\mathfrak{P}(A)$ such that $\Delta(Q) \subset Q$ and $\Delta(Q)$ is finite for $Q \in \mathfrak{P}(A)\}$

$(P, \Delta) \prec (P', \Delta')$ iff $(P, \Delta) \in \mathfrak{D}(A)$, $(P', \Delta') \in \mathfrak{D}(A)$, P' is a refinement of P and $\Delta(Q) \subset \Delta'(Q)$ for $Q \in \mathfrak{P}(A)$.

(The condition on H insures that $\mathfrak{D}(A)$ is directed by $\prec$).

(2) $$\int_A d\tau = \text{limit} \sum_{\alpha \in \Delta(P)} \tau(\alpha) \quad \text{as } (P, \Delta) \text{ runs over } \mathfrak{D}(A).$$

2.2. THEOREM. – *Let $\tau : H \to X$ with $\tau(\emptyset) = 0$. If H is a ring and, for every $A \subset S$,*

$$\mu(A) = \int_A d\tau \in X$$

then μ is an H_σ-outer measure. If, in addition, τ is countably additive on H then μ is an extension of τ.

2.3. THEOREM. – *Let H be a ring with $S \in H_\sigma$ and $\tau : H \to X$ be countably additive. If the range of τ is contained in a complete subset of X and, for every monotone (increasing or decreasing) sequence α in H, $\lim_n \tau(\alpha_n) \in X$ then there is an H_σ-outer measure μ which extends τ.*

3. Partitionable Functions

Let μ be an outer measure (with values in X), X_1 be a topological group and $f : s \in S \to f(s) \subset X_1$.

3.1. DEFINITIONS. –

(1) $f[\alpha] = \bigcup_{s \in \alpha} f(s)$ for $\alpha \subset S$.

$f^{-1}[A] = \{s \in S : f(s) \subset A\}$ for $A \subset X_1$.

f is *single-valued* iff $f(s)$ is a singleton for $s \in S$.

(2) f is *μ-quasi bounded* iff, for every nbhd U of 0 in X_1, there exists $S' \in \mathfrak{N}_\mu$ such that a countable number of translates of U cover $f[S \sim S']$.

(3) f is *μ-partitionable* iff, for every nbhd U of 0 in X_1, there exists a countable $P \subset \mathfrak{M}_\mu$ such that $(S \sim \cup P) \in \mathfrak{N}_\mu$ and, for every $\alpha \in P$ and $x \in f[\alpha]$, $f[\alpha] \subset x + U$.

3.2. THEOREM. – *If there is a countable base for the neighborhoods of 0 in* X_1 *then the following conditions on* f *are equivalent :*

(i) f *is* μ*-partitionable*

(ii) *there exists* $S' \in \mathfrak{N}_\mu$ *such that* $f|(S \sim S')$ *is single-valued,* $f[S \sim S']$ *is separable, and, for every closed* $A \subset X_1$, $f^{-1}[A] \in \mathfrak{M}_\mu$.

(iii) *there exists* $S' \in \mathfrak{N}_\mu$ *such that* $f|(S \sim S')$ *is the uniform limit of a sequence of step functions (each with a countable number of steps).*

3.3. THEOREM. – *Let* f *be single valued.*

(i) *When* X_1 *is a vector space with the weak topology,* f *is* μ*-partitionable iff* f *is weakly* μ*-measurable.*

(ii) *When* X_1 *is a Banach space,*

f *is* μ*-partitionable iff* f *is Bochner* μ*-measurable.*

(iii) *When* X_1 *is a locally convex topological vector space, the following conditions on* f *are equivalent:*

(a) f *is* μ*-partitionable,*

(b) f *is* μ*-quasi bounded and weakly* μ*-measurable,*

(c) f *is* μ*-quasi bounded and* $f^{-1}[A] \in \mathfrak{M}_\mu$ *whenever* A *is a translate of a closed convex nbhd of 0 in* X_1.

3.4. THEOREM. – *For any sequence* f *of* μ*-partitionable functions.*

(i) $f_1 + f_2$ *is* μ*-partitionable*

(ii) *If, for every* $s \in S$, $g(s) = \lim_n f_n(s)$ *exists in the induced uniform topology on the family of subsets of* X_1, *then* g *is* μ*-partitionable.*

4. Integration.

Let X_1 and X_2 be commutative topological groups,

$$\cdot : (x_1, x_2) \in X_1 \times X_2 \to x_1 \cdot x_2 \in X \text{ be bi-additive}$$

$$f : s \in S \to f(s) \subset X_1$$

$$\phi : H \to X_2.$$

4.1. – DEFINITIONS. –

(1) ξ is a choice function iff $\xi : \alpha \in H \to \xi(\alpha) \in f[\alpha]$.

$$\tau_\xi : \alpha \in H \to \xi(\alpha) \cdot \phi(\alpha) \in X$$

(2) For any $A \subset S, \int_A f \cdot d\phi =$ the $I \in X$ (if it exists) such that, for every choice function ξ , $I = \int_A d\tau_\xi$.

4.2. DEFINITIONS. –

(1) *f behaves as a bounded function* iff, for every nbhd U of 0 in X, there exists a nbhd U_2 of 0 in X_2 such that

$$\sum_{i \in A} y_i \in U_2 \quad \text{for every} \quad A \subset \{1, \ldots, n\} \implies \sum_{i=1}^{n} f[S] \cdot y_i \subset U.$$

(2) ϕ *behaves as a bounded measure* iff ϕ is an outer measure and, for every nbhd U of 0 in X, there exists a nbhd U_1 of 0 in X_1 such that, for every finite, disjoint $\Delta \subset \mathfrak{M}_\phi$

$$\sum_{\alpha \in \Delta} U_1 \cdot \phi(\alpha) \subset U.$$

4.3. THEOREM. – *If X is complete, ϕ behaves as a bounded measure, f is ϕ-partitionable and behaves as a bounded function then, for every $A \subset S$,*

$$\int_A f \cdot d\phi \in X.$$

Hence, if $\mu(A) = \int_A f \cdot d\phi$ then μ is an outer measure.

5. Differentiation.

For $s \in S$, let $\mathfrak{F}(s)$ be a filterbase of families of subsets of S.

$$H = \bigcup_{s \in S} \bigcup_{F \in \mathfrak{F}(s)} F,$$

X be a topological vector space over the reals, and ϕ be a real-valued outer measure with $0 < \phi(\alpha) < \infty$ for $\alpha \in H$.

5.1. DEFINITIONS. –

(1) For $s \in S$, $D(s) = \bigcap_{F \in \mathfrak{F}(s)} \text{closure} \left\{ \frac{\mu(\alpha)}{\phi(\alpha)} ; \alpha \in F \right\} \subset X.$

D is the 'dérivative' of μ relative to ϕ and $\mathfrak{F}$.

(2) $\mu \ll \phi$ iff $\mathfrak{M}_\mu \cap \mathfrak{N}_\phi \subset \mathfrak{N}_\mu$.

(3) For $A \subset S$ and $J \subset H$, J is a *Vitali cover* for A iff, for every $s \in A$ and $F \in \mathfrak{F}(s)$, $J \cap F \neq \emptyset$.

$\mathfrak{F}$ is a *Vitali system* iff, for every $\epsilon > 0$, $A \subset S$ and Vitali cover J for A, there exists a countable, disjoint $J' \subset J$ which covers ϕ-almost all of A and

$$\phi(\cup J') < \phi(A) + \epsilon.$$

5.2. THEOREM. – *Suppose X is locally convex, $\mathfrak{F}$ is a Vitali system, μ and ϕ are $\mathfrak{B}$-outer measures for some $\mathfrak{B} \supset H$, $\phi(S) < \infty$ and $\mu \ll \phi$. If*

(i) *for ϕ-almost all $s \in S$, there exists $F_0 \in \mathfrak{F}(s)$ and a compact $C \subset X$ such that $\frac{\mu(\alpha)}{\phi(\alpha)} \in C$ for $\alpha \in F_0$ and*

(ii) *D is ϕ-quasi bounded*

then

(1) *$D(s) \neq \emptyset$ for ϕ-almost all $s \in S$.*

(2) *For any nbhd U of 0 in X, $D(s) - D(s) \subset U$ for ϕ-almost all $s \in S$.*

(3) *D if ϕ-partitionable.*

(4) $\mu(A) = \int_A D \cdot d\phi \quad \text{for} \quad A \subset S.$

Note :

• (i) When there is a countable base for the nbhds of 0 in X, (1) and (2) imply that D is single-valued almost everywhere. Even when this does not occur, if f is any function with $f(s) \in D(s)$ for ϕ-almost all $s \in S$ then (4) implies

$$\mu(A) = \int_A f \cdot d\phi \quad \text{for} \quad A \subset S.$$

(ii) The lifting theorem guarantees that a Vitali system $\mathfrak{V}$ always exists (for each $s \in S$, consider the lifted sets containing s and direct them by inclusion [1, 8, 9]).

(iii) When X is the dual of a Banach space and is endowed with the weak* topology, the above yields most versions of the Dunford-Pettis theorem.

(iv) When X is a Banach space, the above yields integral representations for the Bochner integral. (see [5, 9]).

(v) Extensions of the above theorem to the case when μ and ϕ are both group-valued have been obtained in [9].

REFERENCES

[1] Kölzow D. — Differentiation von Massen. *Lecture Notes in Mathematics*, No. 65. Springer-Verlag, 1968.

[2] Pettis B.J. — Differentiation in Banach spaces. *Duke Math. J.*, 5, 1939, p. 254-269.

[3] Phillips R.S. — Integration in a convex linear topological space. *T.A.M.S.*, 47, 1940, p. 114-145.

[4] Rickart C.E. — Integration in a convex topological space. *T.A.M.S.*, 52, 1942, p. 498-521.

[5] Rieffel M.A. — Radon-Nikodym theorem for the Bochner integral. *T.A.M.S.*, 131, 1968, p. 466-487.

[6] Sion M. — Outer measures with values in a topological group. *Proc. London Math. Soc.*, 19, 169, p. 89-106.

[7] Sion M. — Lectures on vector-valued measures. University of British Columbia, 1969-70.

[8] Sion M. — A proof of the lifting theorem. University of British Columbia, 1970.

[9] Traynor T.E. — Differentiation of group-valued outer measures, *Thesis*, University of British Columbia, 1969.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 595 à 600.

D 7 - FONCTIONS ANALYTIQUES D'UNE VARIABLE COMPLEXE

APPROXIMATION COMPLEXE ET PROPRIÉTÉS DES FONCTIONS ANALYTIQUES

par N. U. ARAKELJAN

Le présent exposé donne un aperçu de quelques résultats et problèmes non résolus de la théorie de l'approximation uniforme et de l'approximation avec contact à l'infini, par des fonctions entières et des fonctions analytiques, et de leurs applications.

1. Possibilité d'approximation.

Dans le plan complexe compactifié $\overline{\mathbf{C}}$, considérons un domaine D de frontière ∂D non vide. Soit une partie E de D fermée relativement à D et soit E^0 l'ensemble des points intérieurs à E. Introduisons les notations

$C(E)$ – l'ensemble des fonctions continues sur E ;

$C_A(E)$ – l'ensemble des fonctions continues sur E et analytiques sur E^0 ;

A_D – l'ensemble des fonctions analytiques dans D.

$A_D(E)$ – la fermeture sur E selon la métrique uniforme de l'ensemble A_D.

Problème essentiel : pour quels ensembles E est-il vrai que toute fonction, continue sur E et analytique dans E^0, admet une approximation uniforme sur E par des fonctions analytiques dans D ; c'est-à-dire, quand l'égalité suivante a-t-elle lieu

(1) $$A_D(E) = C_A(E)\ ?$$

Un cas particulier du problème (1) est le problème

(2) $$A_D(E) = C(E)$$

dans le cas où $\overset{\circ}{E} = \emptyset$? Deux cas particuliers des problèmes (1) et (2) présentent un intérêt spécial : les problèmes

(1′) $$A_{\mathbf{C}}(E) = C_A(E)$$

(2′) $$A_{\mathbf{C}}(E) = C(E)$$

qui concernent l'approximation uniforme par des fonctions entières.

Le problème (2′) a été tout d'abord considéré par Carleman [1]. Il a établi le résultat suivant : pour $f, \epsilon \in C(-\infty, \infty)$, $\epsilon > 0$, il existe une fonction entière g satisfaisant à l'inégalité :

$$|f(x) - g(x)| < \epsilon(x) \qquad -\infty < x < \infty$$

Ceci généralisait sur un ensemble non compact –l'axe réel $(-\infty, \infty)$– le théorème classique de Weierstrass sur l'approximation par des polynômes et mettait en évidence une particularité intéressante de l'approximation par des fonctions entières : la possibilité d'approximation avec un contact d'ordre arbitrairement grand à l'infini.

DEFINITION 1. – *Nous dirons qu'un ensemble E est un ensemble de Carleman dans le domaine D, si pour toute fonction $f \in C_A(E)$ et tout $\epsilon \in C(E)$, $\epsilon > 0$, il existe une fonction $g \in A_D$, telle que*

$$|f(z) - g(z)| < \epsilon(z) \quad pour \quad z \in E \tag{3}$$

Des exemples généraux d'ensembles de Carleman ont été construits dans les travaux de Roth [2], Bagemihl et Seidel [3], et Kaplan [4].

Pour les ensembles continus $E \subset \mathbf{C}$, satisfaisant à la condition $E^0 = \emptyset$, le problème de la description des ensembles de Carleman dans $\mathbf{C}$ (et, en particulier le problème (2')) a été complètement résolu dans le travail en collaboration de Keldyš et Lavrentev [5]. Ils ont trouvé une condition sur E (condition que nous désignerons par $K_\mathbf{C}$) qui est de caractère essentiellement topologique.

Les recherches de Keldyš [6, 7] constituent une contribution substantielle à la théorie générale des approximations uniformes et des approximations avec contact à l'infini par des fonctions entières. Les travaux connus de Mergeljan sur l'approximation uniforme par des polynômes et des fonctions rationnelles [7] ont joué un rôle important dans le développement de cette théorie. Keldyš et Mergeljan ont établi [7] que pour un continu E de $\mathbf{C}$, la condition $K_\mathbf{C}$ est suffisante pour l'égalité (1') même dans le cas $E^0 \neq \emptyset$. En outre, dans ce cas, on peut obtenir également des approximations avec contact à l'infini du type (3), mais la fonction ϵ, en général, ne peut à l'infini tendre vers 0 arbitrairement vite (Keldyš). Par exemple, on peut poser $\epsilon(z) = \epsilon \exp(-|z|^\eta)$, où $0 < \eta < 1/2$, mais on ne peut prendre $\eta = 1/2$. (pour différentes classes particulières d'ensembles E on a mis en évidence de plus grandes vitesses de décroissance de la fonction ϵ). Dans le travail de l'auteur [8] une condition intégrale : (on suppose que $\epsilon(z) = \epsilon(|z|)$)

$$\int_1^\infty r^{-3/2} \log \epsilon(r)\, dr > -\infty$$

a été obtenue, qui précise le résultat de Keldyš. On ne peut affaiblir cette condition. Des critères analogues ont été obtenus aussi pour de larges classes d'ensembles E (cf. [8], et aussi le travail de Dzrbasjan [9]). Le problème (1') et le problème plus général (1) ont été complètement résolus dans les travaux de l'auteur ([10] et [11]).

DEFINITION 2. – *L'ensemble E, fermé relativement au domaine D, sera dit K_D-ensemble ($E \in K_D$), si, D^* étant le compactifié de D par adjonction d'un point, l'ensemble $D^* \backslash E$ est connexe et localement connexe.*

THOEREME 1. – *(Solution du problème (1))*

$$A_D(E) = C_A(E) \text{ si et seulement si } E \in K_D$$

COROLLAIRE 1. – *(Solution du problème (1'))*

$A_{\mathbf{C}}(E) = C_A(E)$ si et seulement si l'ensemble $\overline{\mathbf{C}}\backslash E$ est connexe et localement connexe.

COROLLAIRE 2. – *(Solution du problème (2))*

$$A_D(E) = C(E) \text{ si et seulement si } E \in K_D \quad , \quad E^0 = \emptyset$$

COROLLAIRE 3. – *Soient f, $\varphi \in C_A(E)$, supposons de plus que φ soit bornée et que $0 \notin \varphi(E)$. Alors, il existe une fonction $g \in A_D$, telle que*

$$|f(z) - g(z)| < |\varphi(z)| \quad \text{pour} \quad z \in E.$$

On obtient à partir de là le

COROLLAIRE 4. – *Si $E \in K_D$ et $E^0 = \emptyset$, l'ensemble E est un ensemble de Carleman dans D.*

Récemment Gauthier, pour le cas $E^0 \neq \emptyset$, a obtenu quelques résultats pour le problème de la caractérisation des ensembles de Carleman dans $\mathbf{C}$ [12].

La question de la description complète des ensembles de Carleman dans D (même dans le cas $D = \mathbf{C}$) est jusqu'ici encore ouverte. Il en est de même de la question de la description des ensembles $A_D(E)$ quand $E \notin K_D$. Dans le cas $D = \mathbf{C}$, on a pu établir, au moyen d'une généralisation du principe du maximum, que les fonctions éléments de $A_{\mathbf{C}}(E)$ admettent un prolongement analytique sur celles des composantes de l'ensemble $\mathbf{C}\backslash E$ pour lesquelles l'infini n'est pas un point *frontière "accessible"*.

Un autre point intéressant serait de trouver des conditions sur le domaine D et l'ensemble E, qui assurent une approximation du type 3, fût-ce avec un "contact" arbitrairement lent ?

2. Evaluation de la croissance des fonctions réalisant l'approximation.

Dans les applications des approximations uniformes et avec contact à l'infini par des fonctions entières, l'évaluation de leur croissance joue un rôle important.

Dans les recherches de Keldyš [6] (cf. aussi [7]) des évaluations précises (et même dans certains cas des valeurs exactes) ont été obtenues dans le cas de l'approximation sur quelques ensembles canoniques (angle, bande, droite). Pour cela, dans le cas de l'approximation dans un angle ou dans une bande, la fonction à approcher est supposée holomorphe dans un angle un peu plus large, ou une bande un peu plus large, et peut avoir une croissance arbitraire. Dans le travail de l'auteur [13], un problème analogue a été résolu, avec l'hypothèse que les fonctions à approcher sont holomorphes seulement à l'intérieur de l'angle ou de la bande, et continûment différentiables à la frontière. Nous ne citerons pas toutes les nombreuses évaluations obtenues dans les travaux indiqués. Bornons-nous au cas de l'approximation sur l'axe réel. Voici un résultat obtenu récemment dans cette direction :

THEOREME 2. – *Soient* $f \in C^1(-\infty, \infty)$, $\delta > 0$, $\epsilon > 0$. *Il existe une fonction entière g telle que*

$$|f(x) - g(x)| < \epsilon \quad pour \quad x \in (-\infty, +\infty)$$

$$|g(z)| < \exp\left\{ K\left(\frac{1}{\delta}|\text{Im}\, z| + 1\right)\left[1 + \frac{\delta}{\epsilon}\mu_\delta(2|z|) + \log^+ \frac{1}{\epsilon}\mu(2|z|)\right]\right\}$$

pour $z \in \mathbf{C}$, *où K est une constante absolue et*

$$\mu_\delta(r) = \max_{\substack{|x-y|\leqslant\delta \\ |x|\leqslant r}} |f'(x) - f'(y)| \quad , \quad \mu(r) = \max_{|x|\leqslant r} |f(x)|$$

En relation avec les approximations avec contact à l'infini se présente aussi le problème suivant, qui offre un intérêt du point de vue de la théorie des fonctions entières : construire une fonction entière non triviale, décroissant avec une vitesse maximale (en un sens déterminé) sur un ensemble donné non borné, et ayant une croissance minmale sur le plan.

Ce problème a été résolu dans le cas de l'angle. Pour tout $\alpha \in (0, 2\pi)$ Keldyš a construit [7] des fonctions entières d'ordre $\rho_\alpha = \max\left(\frac{\pi}{\alpha}, \frac{\pi}{2\pi - \alpha}\right)$ qui décroissent dans l'angle $\{\Delta_\alpha = z \in \mathbf{C} : |\arg z| < \alpha/2\}$, comme les fonctions $z^{\text{const.}} \exp(-z^{\pi/\alpha})$. Le théorème suivant (cf. [14]) précise ce résultat.

THEOREME 3. – *Soit la fonction* $p \geqslant 0$ *définie sur* $[1, \infty)$ *et supposons que*

$$r^{-\pi/\alpha} p(r) \downarrow 0 \quad quand \quad r \uparrow \infty, \quad \int_1^\infty r^{-\frac{\pi}{\alpha}-1} p(r)\, dr < \infty$$

Alors il existe une fonction entière ω_α *d'ordre* ρ_α *et de type normal telle que*

$$\exp\{-K|z|^{\pi/\alpha}\} < |\omega_\alpha(z)| < \exp\{-\text{Re}\, z^{\pi/\alpha} - p(|z|)\} \; pour \; z \in \Delta_\alpha, \; |z| \geqslant 1$$

De plus on ne peut abaisser l'ordre indiqué ni le type.

Remarquons que les fonctions entières ω_α sont liées aux problèmes de quasi-analyticité de différentes classes de fonctions indéfiniment différentiables [15].

3. Quelques applications de la théorie des approximations.

Nous avons remarqué ci-dessus que dans le cas $E \in K_D$, $E^0 = \emptyset$, l'ensemble E est un ensemble de Carleman dans D, si bien que pour toutes fonctions $f, \epsilon, \in C(E)$, $\epsilon > 0$, il existe une fonction g analytique dans D telle que

$$|f(z) - g(z)| < \epsilon(z) \quad \text{pour} \quad z \in E$$

Ce théorème peut être considéré comme un schéma général de construction d'exemples de fonctions analytiques dans D, ayant un comportement pathologique à la frontière. Par des choix appropriés du domaine D, de l'ensemble E, des fonctions f et ϵ, on peut obtenir la plupart des exemples construits par Lusin et Privalov, Bagemihl et Seidel, Lehto et d'autres (voir [16]). *Par exemple, suppo-*

sons que le domaine D soit étoilé relativement au point $z = 0$, *et la fonction* $\varphi : [0,2\pi] \to \overline{\mathbf{C}}$, *mesurable pour la mesure de Lebesgue. Il existe une fonction f analytique dans D, telle que*

$$\lim_{r \to r(\theta)} f(r\, e^{i\theta}) = \varphi(\theta)$$

pour presque tout $\theta \in [0,2\pi]$, *où* $r(\theta) = Sup\ \{t > 0 : t e^{i\theta} \in D\}$. *Dans le cas* $D = \mathbf{C}$ *la fonction f est entière.*

Les méthodes et les résultats de la théorie des approximations débouchent de façon intéressante sur la théorie de la distribution des valeurs des fonctions entières et des fonctions analytiques, et en particulier sur la question du nombre de *valeurs exceptionnelles* de ces fonctions. Cette question a été examinée par de nombreux mathématiciens (cf [17]).

Les premiers exemples de fonctions méromorphes ayant un ensemble fini, donné à l'avance, de valeurs exceptionnelles ont été construits par Nevanlinna [18] et Ahlfors [19]. Goldberg a construit des fonctions méromorphes d'ordre fini, avec un ensemble infini de valeurs exceptionnelles [17]. Un exemple analogue de fonction entière d'ordre *infini* a été construit par Fuchs et Haymann [17].

Pour les fonctions entières d'ordre *fini* ρ la question du nombre des valeurs exceptionnelles est compliquée par le fait que le nombre des valeurs asymptotiques finies de telles fonctions ne dépasse pas $[2\rho]$. Nevanlinna a énoncé la conjecture [20, 21] que toute fonction entière d'ordre fini ρ possède un nombre fini (ne dépassant pas $[2\rho] + 1$) de valeurs déficientes. La justesse de cette conjecture dans le cas $\rho \leqslant 1/2$ découle d'un résultat de Edree et Fuchs [17]. Mais on montre que dans le cas $\rho > 1/2$ la conjecture de Nevanlinna n'est pas vérifiée [22].

THEOREME 4. — *Quelle que soit la suite de nombres complexes* $(a_j)_1^\infty$ *et quel que soit* $\rho > 1/2$ *il existe une fonction entière d'ordre* ρ *et de type normal, pour laquelle les valeurs* a_j, $j = 1, 2, \ldots$ *sont des valeurs exceptionnelles.*

On peut ramener le problème de la construction de la fonction entière cherchée, à un problème d'approximation avec contact à l'infini sur des ensembles formés d'une infinité d'"îles", se concentrant vers l'infini, avec évaluation de la croissance des fonctions approchantes.

Dans le cas hyperbolique on obtient le

THEOREME 5. — *Pour toute suite de nombres complexes* $(a_j)_1^\infty$ *et pour tout* $\gamma > 0$ *il existe une fonction analytique dans le cercle unité, d'ordre* γ *et de type normal, pour laquelle les valeurs* a_j, $j = 1, 2, \ldots$ *sont des valeurs exceptionnelles.*

La méthode de démonstration du théorème 4 donne des raisons de penser que pour toute fonction entière g d'ordre fini la condition suivante est remplie :

$$\sum_{a \in \overline{\mathbf{C}}} \left[\log \frac{e}{\delta(\alpha, g)}\right]^{-1} < \infty$$

Ce problème, ainsi que le problème connu de la convergence de la somme $\sum_{a \in \overline{\mathbf{C}}} \sqrt[3]{\delta(\alpha, g)}$, si g est une fonction méromorphe de type fini, ne sont pas encore résolus. En rapport avec ces problèmes, il serait intéressant d'obtenir de bonnes évaluations pour les sommes de la forme $\sum_{j=1}^{n} (z - \alpha_j)^{-m}$, en dehors d'un certain ensemble exceptionnel.

REFERENCES

[1] CARLEMAN T. — Sur un théorème de Weierstrass, *Arkiv for Mathematik, Astronomi och Fisik*, Bd. 20, No. 4, 1927, pp. 1-5.
[2] ROTH A. — Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Functionen, *Comment. Math. Helv.*, 11, 1938, pp. 77-125.
[3] BAGEMIHL F., SEIDEL W. — Some boundary properties of analytic functions, *Math. Z.*, 61, 1954, pp. 186-199.
[4] KAPLAN W. — Approximation by entire functions, *The Michigan Math. Journal*, Vol. 3, No. 1, 1955-1956, pp. 43-52.
[5] Келдыш М. В., Лаврентьев М. А. — Об одной задаче Карлемана, ДАН СССР, 23, № 8, 1939, 747-748.
[6] Келдыш М. В. — О приближении голоморфных функций целыми финкциями, ДАН СССР, 47, № 4, 1945, 243-245.
[7] Мергелян С. Н. — Равномерные приближения функций комплексного переменного, УМН, УП, вып. 2, 1952, 31-122.
[8] Аракелян Н. У. — Об асимптотическом приближении целыми фуикциями в бесконечных областях, Мат. сборник, 53 (95), № 4, 1961, 515-538.
[9] Джрбашян М. М. — Об асимптотическом приближении целыми функциями в полуплоскости, ДАН СССР, III, № 4, 1956, 749-752.
[10] Аракелян Н. У. — О равномерном приближении целыми функциями на замкнутых множествах, Изв. АН СССР, серия матем., 28, 5, 1964, 1187-1206.
[11] Аракелян Н. У. — Равномерные и касательные приближения аналитическими функциями, Математика, Изв. АН Арм. ССР, 3, № 4-5, 1968, 273-285.
[12] GAUTHIER P. — Tangential approximation by entire functions and functions holomorphic in a disc, Математика, Изв. АН Арм. ССР, т. IУ, № 5, 1969, 319-326.
[13] Аракелян Н. У. — Равномерное приближение целыми фуинкциями с оценкой их роста, Сибирский мат. журнал, т. IУ, № 5, 1963, 977-999.
[14] Аракелян Н. У. — Построение целых функций конечного порядка, равномерно убывающих в угле, Математика, Изв. АН Арм. ССР, I, 1966, № 3, 162-191.
[15] Джрабашян М. М. — Расширение квазианалитических классов Данжуа-Карлемана, Математика, Изв. АН Арм. ССР, 3, № 3, 1968, 171-248.
[16] Носиро К. — Предельные множества, М., ИЛ, 1963.
[17] Хейман У. — Мероморфные фуикции, М., 1966.
[18] NEVANLINNA R. — Über Riemansche Flächen mit endlich vielen Vindungspunkten, *Acta Math.*, 58, 1932, pp. 295-373.
[19] AHLFORS L. — Über eine in der neueren Wertverteilungstheorie betrachtete Klasse transzendenter Funktionen, *Acta Math.*, 58, 1932, pp. 375-406.
[20] NEVANLINNA R. — *Le théorème Picard-Borel et la théorie des fonctions méromorphes*, Paris, 1929.
[21] Неванлинна Р. — Однозначные аналитические функции, М.-Л., 1941.
[22] Аракелян Н. У. — Целые функции конечного порядка с бесконечным множеством дефектных значений, ДАН СССР, т. 170, № 5, 1966, 999-1002.

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SUBHARMONIC FUNCTIONS IN R^m

by W. K. HAYMAN

1. Introduction.

The study of integral functions, i.e. functions $f(z)$ regular in the plane has led to a great deal of interesting methods. I should like to draw attention particularly to those theorems which relate to the modulus and the zeros of $f(z)$. Let me first of all introduce some notation and then give some examples.

We denote by $n(r)$ the number of zeros of $f(z)$ in $|z| < r$, assume that $f(0) \neq 0$ and write

$$N(r) = \int_0^r \frac{n(t)\,dt}{t} .$$

We also write

$$B(r) = \log M(r, f) = \sup_{|z|=r} \log |f(z)| ,$$

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta ,$$

$$m(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta})}\right| d\theta ,$$

so that

$$T(r) = m(r) + N(r) + \log |f(0)| , \tag{1}$$

by Jensen's formula. If $f(z)$ is not constant, $B(r)$ and $T(r)$ tend to infinity with r and we may write

$$\delta = \delta(f) = \varliminf_{r\to\infty} \frac{m(r)}{T(r)} = 1 - \varlimsup_{r\to\infty} \frac{N(r)}{T(r)} . \tag{2}$$

We also define the order λ and lower order μ by

$$\frac{\lambda}{\mu} = \frac{\varlimsup}{\varliminf}_{r\to\infty} \frac{\log B(r)}{\log r}$$

and note that $B(r)$ can be replaced by $T(r)$ in this definition.

In many ways subharmonic (s.h.) functions $u(z)$ represent an interesting generalisation of regular functions or, more precisely, of the functions $\log |f(z)|$, where $f(z)$ is regular. However, they can also be considered in R^m for $m > 2$. We recall the definition.

A function $u(x)$ defined in an open set G in R^m is said to be s.h. in G if

(i) $-\infty \leqslant u(x) < +\infty$,

(ii) $u(x)$ is upper-semi-continuous in G,

(iii) For every $x_0 \in G$ and all small r, $u(x_0)$ does not exceed the average with respect to surface area of $u(x)$ on the hypersphere $|x - x_0| = r$.

For harmonic functions, i.e. those satisfying $\nabla^2 \phi = 0$, this average is equal to $u(x_0)$. Functions with continuous second partial derivatives are s.h. if and only if $\nabla^2 \phi \geqslant 0$. In the general case, $\nabla^2(\phi)$ exists as a distribution and is non-negative for s.h. functions. In this talk I should like to discuss to what extent the classical theory of integral functions can be extended to s.h. functions in the whole of R^m.

2. Representation Theorems.

If $f(z)$ is regular on a compact set E, then

$$\log |f(z)| = h(z) + \Sigma \log |z - z_\nu|$$

where z_ν are the zeros of $f(z)$ in E, and $h(z)$ is harmonic in the interior of E. The corresponding result for a s.h. function is due to F. Riesz [7] and states that if $u(x)$ is s.h. in a neighbourhood of a compact set E in R^m then

$$u(x) = h(x) + \int_E K(x - \xi)\, d\mu\, e_\xi\,, \tag{3}$$

where $h(x)$ is harmonic in the interior of E, μ is a mass distribution in E, and

$$\begin{aligned} K(x) &= \log |x| \quad , \text{ if } m = 2 \\ &= -|x|^{2-m}, \text{ if } m > 2\,. \end{aligned}$$

We can regard $\mu(E)$ as the analogue of the number of zeros on E, with the only difference that $\mu(E)$ can now be an arbitrary positive measure and need not be discrete, let alone integer-valued. The quantity $n(r)$ is defined as the total mass of the closed hyperball $|x| \leqslant r$.

We next suppose that $u(x)$ is s.h. in $|x| < r$ and integrate (3) with respect to $d\delta(x)$, the $(m-1)$-dimensional surface area on $|x| = R$, for $R < r$. We obtain the analogue of Jensen's formula

$$u(0) = \frac{1}{C_m R^{m-1}} \int_{|\xi| = R} u(\xi)\, d\delta(\xi) - \int_{|\xi| < R} g(0, \xi)\, d\mu\, e_\xi\,, \tag{4}$$

where C_m is the surface area of the unit sphere, and

$$\begin{aligned} g(0, \xi) &= \log (R/|\xi|) \quad , \quad m = 2 \\ g(0, \xi) &= |\xi|^{2-m} - R^{2-m}\,, \quad m > 2 \end{aligned}$$

is the Green's function in $|x| < R$.

An analogous result to (4) is the Poisson-Jensen formula which represents $u(x)$ in a domain D in terms of the values of $u(x)$ on the frontier Γ of D and the Riesz measure μ of $u(x)$ in D. Sufficient conditions (1) for the representation to be valid are that D is bounded, that Γ has m-dimensional measure zero and that $u(x)$ is s.h. on $D \cup \Gamma$.

If we split up (4) suitably we obtain the analogue

$$(1') \qquad T(R) = m(R) + N(R) + u(0)$$

of (1). Here

$$T(R) = \frac{1}{C_m R^{m-1}} \int_{|x|=R} u^+(x)\, d\delta(x),$$

$$m(R) = \frac{1}{C_m R^{m-1}} \int_{|x|=R} u^-(x)\, d\delta(x),$$

$$N(R) = d_m \int_0^R \frac{n(t)\, dt}{t^{m-1}}$$

where $u^+(x) = \max(u(x), 0)$, $u^-(x) = \max(-u(x), 0)$ and $d_2 = 1$, $d_m = m-2$, if $m > 2$.

We can also obtain the analogues of the representation theorems of Weierstrass and Hadamard for functions s.h. in R^m. We note that, for fixed ξ, $K(x-\xi)$ is real analytic in x and so can be expanded in a multiple power series

$$K(x-\xi) = \sum_0^\infty C_n(\xi) x^n$$

where x^n stands for $x_1^{n_1} x_2^{n_2} x_m^{n_m}$, and we sum over all sets of powers. The series converges absolutely for $|x| < \frac{1}{\sqrt{2}} |\xi|$ (and not in general in any larger hyperball). [6, 3].

If we now write $|n| = n_1 + n_2 + \ldots + n_m$

$$K_q(x, \xi) = K(x-\xi) - \sum_{|n| \leqslant q} C_n(\xi)\, x^n,$$

then we can obtain the following form of *Hadamard's Theorem. Suppose that* $u(x)$ *is* s.h. *in* R^m *and of finite order* λ *and let* $q = [\lambda]$. *Then*

$$(5) \quad u(x) = h_q(x) + \int_{|\xi| \leqslant 1} K(x-\xi)\, d\mu e_\xi + \int_{|\xi|>1} K_q(x-\xi)\, d\mu e_\xi,$$

where $h_q(x)$ *is a harmonic polynomial of degree at most* q.

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(1) The proof of this result and most of the others mentioned here will appear in a book, which the author is at present writing. Prof. Malliavin has informed me that the condition that Γ has measure zero can be eliminated.

In particular we see that these functions are entirely determined by their Riesz mass up to an additive harmonic polynomial. This result is thus the natural generalisation of Hadamard's representation of an integral function of finite order in terms of its zeros. Conversely, given an arbitrary measure $d\mu$ in R^m for which $N(r)$ has finite order λ in the sense that

$$\overline{\lim_{R\to\infty}} \frac{\log N(R)}{\log R} = \lambda ,$$

then (5) defines a s.h. function of order λ in R^m.

It is also possible to obtain various inequalities for functions of order λ, relating $N(r)$, $T(r)$ and $B(r)$. In particular one can obtain the sharp upper bound for functions of order $\lambda < 1$

$$\delta(u) \leqslant A(\lambda , m) ,$$

where the deficiency $\delta(u)$ is defined by (2). Equality holds for those functions whose Riesz mass is concentrated on the negative x_1 axis, and for which $N(r) = cr^\lambda$. where c is a constant. These are elementary only if m is even. E.g.

$$u(x) = cr^\lambda \cos\lambda\theta \quad , \quad \text{if} \quad m = 2$$

$$u(x) = cr^\lambda \frac{\sin(\lambda + 1)\theta}{\sin\theta} \quad , \quad \text{if} \quad m = 4 .$$

Here $r = |x|$, and θ is the angle between the vector x and the positive x_1 axis. We have correspondingly the classical result

$$A(\lambda , 2) = 0 \quad , \quad 0 \leqslant \lambda \leqslant 1/2 ,$$

$$A(\lambda , 2) = 1 - \sin \pi\lambda \quad , \quad 1/2 < \lambda \leqslant 1 .$$

Thus we see that when $m = 2$, we have $\delta(u) = 0$ for all functions for which $\lambda \leqslant 1/2$. For $m > 2$ this conclusion is only valid for functions of order zero.

3. Asymptotic Values.

In conclusion I should like to state briefly one or two results on asymptotic values. A classical theorem of Iversen [5] states that a non-constant integral function $f(z) \to \infty$ as $z \to \infty$ along a suitable path in R^m. For s.h. functions the analogous result has recently been proved by Talpur [8]. If $u(x)$ is s.h. in R^m and M is the least upper bound of $u(x)$, then

$$(6) \qquad u(x) \to M \quad \text{as} \quad x \to \infty \quad \text{along} \quad \Gamma .$$

Here Γ can be chosen to be a sectionally polygonal path if $m = 2$, in which case M is necessarily $+\infty$. If $m > 2$, M may be finite and in this case we can choose for Γ almost any straight line through the origin. However in general Talpur was able to prove (6) only with Γ an asymptotic continuum, i.e.

$$\Gamma = \bigcup_{n=1}^{\infty} \gamma_n ,$$

where γ_n is a continuum, which tends to ∞ with n and $\gamma_n \cap \gamma_{n+1} \neq 0$. If u is continuous Γ can be chosen sectionally polygonal.

Talpur and I also obtained an extension of Ahlfors' theorem [1] on asymptotic values. Suppose that $u(x)$ is s.h. in R^m and that,for some K , the set $u(x) \geqslant K$ has N components in R^m, where $N \geqslant 2$. Then the lower order μ of $u(x)$ satisfies

$$\mu \geqslant C(N, m) = \log (3/2)/\log \left\{ \frac{1 + (3/(2m))^{1/(m-1)}}{1 - (3/(2m))^{1/(m-1)}} \right\} \quad m > 2$$

and (1) $C(N, 2) = (1/2)N$. If $m = 2$, we easily deduce that an integral function with N distinct asymptotic values must have order at least $(1/2)N$. The corresponding bound for $m > 2$ is not sharp, but at least yields the right order of magnitude as $N \to \infty$.

REFERENCES

[1] AHLFORS L. — Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen, *Acta Soc. Sci. Fenn.*, series A 1, No. 9, 1930.

[2] DINGHAS A. — Das Denjoy-Carlemansche Problem für harmonische Funktionen in E^n,*Det. Kgl. Norske Videnskabers Selskabs Skrifter,* No. 7, 1962.

[3] HAYMAN W.K. — Power series expansions for harmonic functions, *Bull. London Math. Soc.,* 2, 1970, p. 152-158.

[4] HEINS M. — On a notion of convexity connected with a method of Carleman, *J. Analyse Math.,* 7, 1959, p. 53-77.

[5] IVERSEN F. — Sur quelques propriétés des fonctions monogènes au voisinage d'un point singulier, *Öfv. af-Finska Vet. Soc. Forh.,* 58, 1918.

[6] KISELMAN C.O. — Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants, *Bull. Soc. Math. France,* 97, 1969, p. 329-356.

[7] RIESZ F. — Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel, *Acta Math,* 48, 1926, p. 329-343, and 54, 1930, p. 321-360.

[8] TALPUR M.N.M. — On the sets where a subharmonic function is large, *Thesis,* London, 1967.

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(1) The case $N = 2$ is due to Heins [4]. For $N = 3$ a better constant was obtained by Talpur. A result for general N, m is due to Dinghas [2].

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D 8 - FONCTIONS ET ESPACES ANALYTIQUES COMPLEXES

E. E. LEVI CONVEXITY AND H. LEWY PROBLEM

by A. ANDREOTTI

1. Preliminaries.

Let U be an open subset of $\mathbf{C}^n$, let $\rho : U \to \mathbf{R}$ be a C^∞ function on U, we set

$$S = \{x \in U \mid \rho(x) = 0\}$$

$$U^+ = \{x \in U \mid \rho(x) \geqslant 0\}$$

$$U^- = \{x \in U \mid \rho(x) \leqslant 0\}$$

We will assume that $d\rho \neq 0$ on S so that S is a smooth hypersurface.

On U (respectively U^+, U^-) we consider the Dolbeault complex

$$(*) \qquad C^{00}(U) \xrightarrow{\bar\partial} C^{01}(U) \xrightarrow{\bar\partial} C^{02}(U) \to \cdots$$

where $C^{0s}(U)$ (resp. C^{0s} U^+), $C^{0s}(U^-)$) denote the space of C^∞ forms of type $0, s$ on U (resp U^+, U^-), and where $\bar\partial$ denotes the exterior differentiation with respect to local antiholomorphic coordinates.

We denote by $H^{0s}(U)$ (resp. $H^{0s}(U^+)$, $H^{0s}(U^-)$) the cohomology of the complex $(*)$. Note that while $H^{0s}(U)$ is the standard type of cohomology, $H^{0s}(U^+)$ and $H^{0s}(U^-)$ are not standard, as for these the differential forms are requested to be C^∞ up to the hypersurface S but not beyond it, on the respective sides.

Finally we introduce the differential ideal

$$\mathfrak{J}^{0s}(U) = \{\varphi \in C^{0s}(U) \mid \varphi = \rho\alpha + \bar\partial\rho \wedge \beta \text{ for some } \alpha \in C^{0s}(U), \beta \in C^{0s-1}(U)\}$$

Since $\bar\partial \mathfrak{J}^{0s}(U) \subset \mathfrak{J}^{0s+1}(U)$ we obtain a subcomplex of $(*)$,

$$(**) \qquad \mathfrak{J}^{00}(U) \xrightarrow{\bar\partial} \mathfrak{J}^{01}(U) \xrightarrow{\bar\partial} \mathfrak{J}^{02}(U) \to \cdots$$

We denote by

$$Q^{00}(S) \xrightarrow{\bar\partial_s} Q^{01}(S) \xrightarrow{\bar\partial_s} Q^{02}(S) \to \cdots$$

the quotient complex of $(*)$ by $(**)$. Its cohomology will be denoted by $H^{0s}(S)$. The condition $\bar\partial_S u = 0$ represents the compatibility condition on S for a form u to be the trace on S of a $\bar\partial$-closed form on one side of S. These definitions are (exept for slight simplifications) in Kohn and Rossi [6]. The considerations that follow have been prompted by the study of the papers [7] and [8] of H. Lewy and the above-mentioned paper [6].

2. The Mayer-Vietoris sequence.

We have the following

THEOREM 1. — *Under the assumptions specifed above we have an exact sequence* :

$$0 \to H^{00}(U) \xrightarrow{\alpha_0} H^{00}(U^+) \oplus H^{00}(U^-) \xrightarrow{\beta_0} H^{00}(S) \xrightarrow{\gamma_0}$$
$$\to H^{01}(U) \xrightarrow{\alpha_0} H^{01}(U^+) \oplus H^{01}(U^-) \xrightarrow{\beta_1} H^{01}(S) \xrightarrow{\gamma_1} \cdots$$

Here α is defined by restriction, β is defined by the jump, i.e. the difference of the "restrictions" (i.e. the natural maps $H^{0s}(U^+) \to H^{0s}(S)$), γ is defined as follows : for any class $\xi = [u] \in H^{0s}(S)$ we can select a representative $u^{0s} \in C^{0s}(U)$ such that $\bar{\partial} u^{0s}$ has coefficients vanishing of infinite order on S (this is possible by virtue of Whitney extension theorem) ; then $\gamma(\xi)$ is represented by the class of the form

$$\gamma(u) = \begin{cases} \bar{\partial} u \text{ in } U^+ \\ -\bar{\partial} u \text{ in } U^- \end{cases}$$

Remark. — A similar sequence holds for "cohomology" with compact supports (replacing H by H_k). For instance, we can take $S = \partial U^-$ to be the boundary of a connected domain $U^- \subset\subset \mathbf{C}^n = U$ with no bounded components in the complement. If $n \geqslant 2$ from $H^1_k(U) = 0$ one deduces the theorem of Bochner [3] : *any $\bar{\partial}_S$-closed C^∞ function on S is the trace on S of a holomorphic function C^∞ in U^-.*

COROLLARY. — *Assume that U is a domain of holomorphy, one has the short exact sequences*

$$0 \to H^{00}(U) \to H^{00}(U^+) \oplus H^{00}(U^-) \to H^{00}(S) \to 0$$
$$0 \to H^{0s}(U^+) \oplus H^{0s}(U^-) \xrightarrow{\simeq} H^{0s}(S) \to 0$$

This shows that, no matter what the shape of S is,

(a) One can always solve, for $\bar{\partial}_S$-closed functions on S, the "Riemann-Hilbert problem(1)" on U. Moreover if U and S are such that

$$U^- \subset \text{envelope of holomorphy of } U^+$$

then one gets an isomorphism $H^{00}(S) \simeq H^{00}(U^-)$. This shows that in this case we can solve in a unique way the "Cauchy problem"(2) with data on S on the side U^-. This case was first studied by H. Lewy [7].

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(1) By the Riemann-Hilbert problem for H^{0s} we mean the following : given $\alpha_S \in H^{0s}(S)$, find $u^+ \in H^{0s}(U^+)$ and $u^- \in H^{0s}(U^-)$ such that $\alpha_S = \beta(u^+ \oplus u^-)$.

(2) By the Cauchy problem for H^{0s} in U^- we mean the following: given $\alpha_S \in H^{0s}(S)$ find $u^- \in H^{0s}(U^-)$ such that $\alpha_S = \beta(0 \oplus u^-)$. Similarly for U^+. This could be called with more accuracy the Lewy problem.

(b) The "Riemann-Hilbert problem" has a unique solution for higher cohomology groups ($s > 0$).

3. Cauchy problem and vanishing theorems.

The second part of the corollary leads to a unique solution of the Cauchy problem for cohomology classes in U^- whenever we have $H^{0s}(U^+) = 0$. Thus the Cauchy problem is reduced to a theorem of vanishing for cohomology.

Let us introduce for a point $z_0 \in S$ the Levi form of S :

$$\mathcal{L}(\rho) = \left\{ \begin{array}{l} \sum \left(\frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_\beta} \right)_{z_0} u_\alpha \bar{u}_\beta \\ \sum \left(\frac{\partial \rho}{\partial z_\alpha} \right)_{z_0} u_\alpha = 0 \end{array} \right.$$

and let us assume that it has p *positive* and q *negative eigenvalues* ($p + q \leqslant n - 1$). One has the following (cf [1])

THEOREM 2. — *Under the above assumption, there exists a fundamental sequence of Stein neighborhoods U of z_0 such that,*

$$H^{0s}(U^+) = 0 \quad \textit{for} \quad \left\{ \begin{array}{l} s > n - q - 1 \\ \textit{or} \\ 0 < s < p \end{array} \right.$$

and (analogously)

$$H^{0s}(U^-) = 0 \quad \textit{for} \quad \left\{ \begin{array}{l} s > n - p - 1 \\ \textit{or} \\ 0 < s < q \end{array} \right.$$

Moreover if $p > 0$, U can be selected so that $U^- \subset$ envelope of holomorphy of U^+ (and analogously if $q > 0$ $U^+ \subset$ envelope of holomorphy of U^-)

As an illustration let us assume for simplicity that the *Levi form is non-degenerate* so that $q = n - p - 1$. Then one can have non-vanishing cohomology only for $H^{0p}(U^+)$ and $H^{0q}(U^-)$ apart from $H^{00}(U^\pm)$. One can prove that in this instance all of these groups are infinite-dimensional. In particular one deduces that

(a) The Riemann-Hilbert problem is of interest only in dimensions

$$0, p, q = n - p - 1 .$$

(b) If the three integers $0, p, q$ are distinct, then on dimension p, q one can solve the Cauchy problem for cohomology on U^+ and U^- respectively. Moreover in dimension 0 the Cauchy problem is solvable on both sides.

(c) If $p = q = \dfrac{n-1}{2}$ ($n \geqslant 3$), the Cauchy problem is not solvable in that dimension. (Note that in this dimension the system $\bar{\partial} u = f$ for $u \in C^{0p}(U)$ is a system with as many unknowns as equations).

The proof of theorem 2 requires the methods of [2] and of Hörmander [4] and the use of the regularity theorem of Kohn and Nirenberg [5].

4. An example.

Consider in a neighborhood U of the origin in $\mathbf{C}^2$ the hypersurface

$$S \equiv \left\{ \rho \equiv \frac{1}{i} (z_2 - \bar{z}_2) - 2\,|z_1|^2 = 0 \right\}$$

Setting $z_1 = x_1 + ix_2$ $z_2 = x_3 + ix_4$ we get parametric equations for S :

$$z_1 = x_1 + ix_2$$
$$z_2 = x_3 + i(x_1^2 + x_2^2)$$

This hypersurface is strongly convex i.e. $p = 0$ $q = 1$. Since we are in $\mathbf{C}^2$ the complex Q^* on S reduces to

$$Q^{00} \xrightarrow{\bar{\partial}_s} Q^{01} \to 0$$

and one may remark that locally $Q^{01} \simeq Q^{00}$ (not canonically). Making an explicit calculation one finds that in coordinates x_1, x_2, x_3 the equation $\bar{\partial}_S u = f$ gets the form

$$\frac{1}{2}\left(\frac{\partial u}{\partial x_1} + i\,\frac{\partial u}{\partial x_2}\right) - i\,(x_1 + ix_2)\,\frac{\partial u}{\partial x_3} = f$$

This is the famous equation of H. Lewy which does not admit in general any solution in any neighborhood of the origin unless f is analytic.

Now ou S we have to consider the groups $H^{00}(S)$, $H^{01}(S)$ and according to the previous theorem one has a good Cauchy problem in dimension zero

$$H^{00}(U^+) \xrightarrow{\sim} H^{00}(S)$$

and an equally good Cauchy problem in dimension 1

$$H^{01}(U^-) \xrightarrow{\sim} H^{01}(S)$$

The property of the equation of H. Lewy shows that $H^{01}(S) \neq 0$ and indeed that it is infinite-dimensional.

The content of this lecture can be considered as a progress-report on a joint paper with C. D. Hill. It is appropriate to acknowledge with gratitude the help given us in seminars and conversations by A. Huckleberry and R. Nirenberg. To the latter we are particulariy indebted for help in the proof of theorem 1 .

BIBLIOGRAPHY

[1] ANDREOTTI A. and GRAUERT H. — Théorèmes de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France,* 90, 1962, p. 193-259.
[2] ANDREOTTI A. and VESENTINI E. — Carleman estimates for the Laplace-Beltrami equation on complex manifolds, *Publ. Math. Inst. Hautes Etudes Sci.,* No. 25, 1965, p. 81-130.
[3] BOCHNER S. — Analytic and meromorphic continuation by means of Green's formula, *Ann. Math.,* s. 2, Vol. 44, 1943, p. 652-673.
[4] HÖRMANDER L. — L^2 estimates and existence theorems for the $\bar{\partial}$ operator, *Acta Math.,* Vol. 113, 1965, p. 89-152.
[5] KOHN J.J. and NIRENBERG L. — Non-coercive boundary problems. *Comm. Pure and Appl. Math.,* Vol. 18, 1965, p. 443-492.
[6] KOHN J.J. and ROSSI H. — On the extension of holomorphic functions from the boundary of a complex manifold, *Ann. Math.,* s. 2, Vol. 81, 1965, p. 451-472.
[7] LEWY H. — On the local character of the solution of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, *Ann. Math.,* s. 2, Vol. 64, 1956, p. 514-522.
[8] LEWY H. — An example of a smooth linear partial differential equation without solutions, *Ann. Math.,* s. 2, No. 66, 1957, p. 155-158.

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TOPOLOGISCHE METHODEN IN DER THEORIE STEINSCHER RÄUME

von Otto FORSTER

1. Einführung.

Ein Steinscher Raum ist ein komplexer Raum X mit folgenden Eigenschaften :

(i) X ist holomorph-separabel, d.h. sind x und y zwei voneinander verschiedene Punkte auf X, so existiert eine holomorphe Funktion $f : X \to C$ mit $f(x) \neq f(y)$.

(ii) X ist holomorph-konvex, d.h. ist $x_1, x_2, \ldots$ eine Punktfolge ohne Häufungspunkt auf X, so existiert eine holomorphe Funktion $f : X \to C$ mit lim sup $|f(x_\nu)| = \infty$.

Für die Funktionentheorie auf Steinschen Räumen stehen vorallem zwei starke Hilfsmittel zur Verfügung : Erstens das Theorem B von Cartan-Serre : Für jede kohärente analytische Garbe $\mathfrak{F}$ auf dem Steinschen Raum X und jedes $q \geqslant 1$ gilt $H^q(X, \mathfrak{F}) = 0$. Zweitens das sogenannte Okasche Prinzip, das es erlaubt, viele analytische Probleme auf Steinschen Räumen auf topologische Probleme zurückzuführen.

Ein erstes Beispiel hierfür ist die Behandlung des multiplikativen Cousin-Problems (zu finden ist eine meromorphe Funktion mit vorgegebenen Null- und Polstellenordnungen) durch Oka [24]. Oka zeigte, dass die Hindernisse gegen die Lösbarkeit des multiplikativen Cousin-Problems in Holomorphiegebieten rein topologischer Natur sind. Aufgrund des Okaschen Resultats konnte dann Stein [29] explizite topologische Bedingungen für die Lösbarkeit angeben. In der Sprache der Cohomologietheorie lässt sich das Okasche Resultat so formulieren : Ist $\mathfrak{O}^*$ die Garbe der holomorphen, und $\mathfrak{C}^*$ die Garbe der stetigen Funktionen auf dem Steinschen Raum X mit Werten in $C^* = C\backslash\{0\}$, so besteht die Isomorphie

$$H^1(X, \mathfrak{O}^*) \cong H^1(X, \mathfrak{C}^*).$$

2. Die Sätze von Grauert über das Okasche Prinzip.

Eine weitgehende Verallgemeinerung des Okaschen Resultats wurde von Grauert [14, 15, 16] erzielt (siehe auch Cartan [4]) :

Es sei $L \to X$ ein Bündel von komplexen Liegruppen über dem Steinschen Raum X. Mit $\mathfrak{L}_\omega$ bzw. $\mathfrak{L}_c$ werde die Garbe der holomorphen bzw. stetigen Schnitte von L bezeichnet. Dann ist die kanonische Abbildung

$$H^1(X, \mathfrak{L}_\omega) \to H^1(X, \mathfrak{L}_c)$$

bijektiv.

Wendet man dieses Resultat auf das triviale Bündel $X \times G$ an, so ergibt sich, dass auf einem Steinschen Raum die analytische und topologische Klassifikation von Faserbündeln mit Strukturgruppe G zusammenfallen. (Für den Fall, dass G abelsch bzw. auflösbar ist, wurde dies schon von Serre [27] bzw. Frenkel [12] bewiesen). Zum Beispiel ist eine Steinsche Mannigfaltigkeit genau dann holomorph parallelisierbar, wenn sie topologisch parallelisierbar ist.

Mit denselben Bezeichnungen gilt weiter :

Versieht man $\Gamma(X, \mathcal{L}_\omega)$ und $\Gamma(X, \mathcal{L}_c)$ jeweils mit der Topologie der kompakten Konvergenz (wir setzen der Einfachheit halber X als reduziert voraus), so induziert die kanonische Abbildung

$$\Gamma(X, \mathcal{L}_\omega) \to \Gamma(X, \mathcal{L}_c)$$

eine Bijektion zwischen den Homotopieklassen (d.h. Kurvenzusammenhangs-Komponenten) von holomorphen und stetigen Schnitten von L. Ist $Y \subset X$ Rungesch, so lässt sich ein Schnitt $f \in \Gamma(Y, \mathcal{L}_\omega)$ genau dann durch Schnitte $F_\nu \in \Gamma(X, \mathcal{L}_\omega)$ approximieren, wenn man f durch Schnitte $G_\nu \in \Gamma(X, \mathcal{L}_c)$ approximieren kann.

Diese Homotopie- und Approximationsaussagen bleiben richtig, wenn $L \to X$ ein Bündel von homogenen Räumen über X ist, auf dem ein Bündel von komplexen Lieschen Gruppen holomorph und transitiv wirkt (Ramspott [25], Grauert-Kerner [18]).

3. Vektorraumbündel über Steinschen Räumen.

Sei V ein holomorphes Vektorraumbündel vom Rang d über einem Steinschen Raum X. Wir betrachten folgende zwei Probleme :

I. – *Sei $k \leqslant d$. Gesucht sind k holomorphe Schnitte von V, die in jedem Punkt $x \in X$ linear unabhängig sind.*

II. – *Sei $r \geqslant d$. Gesucht sind r holomorphe Schnitte von V, die in jedem Punkt $x \in X$ ein Erzeugendensystem der Faser V_x bilden.*

In beiden Fällen ist das Problem gleichwertig damit, einen holomorphen Schnitt in einem dem Vektorraum zugeordneten Faserbündel zu finden, dessen Faser eine Stiefel-Mannigfaltigkeit ist. (Im Fall I ist die Faser die Stiefel-Mannigfaltigkeit der k-Beine im C^d, im Fall II die der d-Beine im C^r). Da es nach dem Okaschen Prinzip genügt, einen stetigen Schnitt zu konstruieren, kann man die bekannten Methoden aus der Hindernistheorie heranziehen. Was hier benötigt wird, ist in folgendem Satz enthalten (Steenrod [28]) :

Sei X ein CW-Komplex, Y ein abgeschlossener Unterkomplex und L ein Faserbündel über X mit zusammenhängender Strukturgruppe und typischer Faser F, die q-einfach für alle q ist. Sei s ein Schnitt von L über Y. Eine hinreichende Bedingung dafür, dass sich s zu einem Schnitt S über ganz X fortsetzen lässt, ist

$$H^{q+1}(X, Y; \pi_q(F)) = 0 \quad \text{für} \quad q \geqslant 1.$$

Bei der Anwendung dieses Satzes benützt man, dass jeder komplexe Raum mit abzählbarer Topologie triangulierbar ist (Giesecke [13], Lojasiewicz [21]) und folgende topologische Eigenschaften Steinscher Räume (Andreotti-Frankel [1], Andreotti-Narasimhan [2], Narasimhan [23], Kaup [20]) :

Für jeden n-dimensionalen Steinschen Raum X gilt $H_q(X, Z) = 0$ *für* $q > n$ *und* $H_n(X, Z)$ *ist torsionsfrei. Ist X nicht-singulär, so ist* $H_n(X, Z)$ *sogar frei.*

Daraus folgt z.B., dass für eine n-dimensionale Steinsche Mannigfaltigkeit X gilt $H^q(X, A) = 0$ für alle $q > n$ und eine beliebige Koeffizientengruppe A. Benützt man noch, dass die Stiefel-Mannigfaltigkeit der k-Beine im C^d asphärisch bis zur Dimension $2(d-k)$ ist, so ergibt sich, dass die oben genannten Probleme I und II auf einer n-dimensionalen Steinschen Mannigfaltigkeit stets lösbar sind, falls $k \leqslant d - [n/2]$ bzw. $r \geqslant d + [n/2]$.

Insbesondere existieren auf einer n-dimensionalen Steinschen Mannigfaltigkeit X stets $[(n + 1)/2]$ holomorphe Vektorfelder, die in jedem Punkt linear unabhängig sind [25]. Ist $\mathfrak{F}$ eine lokalfreie analytische Garbe auf X vom Rang d, so besitzt der Modul $\Gamma(X, \mathfrak{F})$ über $\Gamma(X, \mathcal{O})$ ein Erzeugendensystem aus $d + [n/2]$ Elementen [9].

4. Kohärente analytische Garben.

Zur Behandlung von nicht notwendig lokal-freien analytischen Garben $\mathfrak{F}$ auf einem komplexen Raum X ist es nützlich, den $\mathfrak{F}$ zugeordneten linearen Raum $V(\mathfrak{F})$ über X zu betrachten (vgl. Grothendieck [19], Grauert [17], Fischer [5]). Die Fasern von $V(\mathfrak{F})$ sind Vektorräume, die man wie folgt beschreiben kann : Für $x \in X$ sei m_x das maximale Ideal von $\mathcal{O}_{X,x}$. Dann ist $\mathfrak{F}_x/m_x\mathfrak{F}_x$ ein endlich-dimensionaler Vektorraum über C ; sein Dualraum ist kanonisch isomorph zur Faser $V(\mathfrak{F})_x$.

Die Garbe $\mathfrak{F}$ lässt sich als Garbe der holomorphen Linearformen auf $V(\mathfrak{F})$ interpretieren. Der Garbe $\mathfrak{F}$ wird eine Modulgarbe $\mathfrak{F}_c$ über der Garbe $\mathcal{C}$ der stetigen Funktionen wie folgt zugeordnet : Für eine offene Menge $U \subset X$ ist $\mathfrak{F}_c(U)$ der $\mathcal{C}(U)$-Modul der stetigen Linearformen auf $V(\mathfrak{F})|U$.

Es gilt nun folgendes Okasche Prinzip für die Anzahl der Erzeugenden :

Ueber einem Steinschen Raum X ist die minimale Anzahl der Erzeugenden von $\Gamma(X, \mathfrak{F})$ *über* $\Gamma(X, \mathcal{O})$ *gleich der minimalen Anzahl der Erzeugenden von* $\Gamma(X, \mathfrak{F}_c)$ *über* $\Gamma(X, \mathcal{C})$.

Zum Beweis benützt man eine Verallgemeinerung der Grauertschen Resultate auf sog. Okasche Paare von Garben nicht-abelscher Gruppen (Forster-Ramspott [6, 7, 8]).

Um eine Abschätzung der Anzahl der Erzeugenden für $\Gamma(X, \mathfrak{F})$ zu erhalten, hat man den linearen Raum $V(\mathfrak{F})$ genauer zu betrachten.

Für eine natürliche Zahl k sei

$$S_k = \{x \in X : \dim (\mathfrak{F}_x/m_x\mathfrak{F}_x) \geqslant k\}.$$

S_k ist eine analytische Menge in X ; beschränkt auf $S_k \backslash S_{k+1}$ ist $V(\mathfrak{F})$ ein Vektorraumbündel vom Rang k. Sei n_k die Dimension von S_k und

$$r = \sup \{k + [(n_k + 1)/2]\},$$

wobei das Supremum über alle $k \geqslant 1$ mit $S_k \backslash S_{k+1} \neq \emptyset$ zu nehmen ist. Mit Hilfe der Hindernistheorie kann man nun wie im lokal-freien Fall beweisen, dass $\Gamma(X, \mathfrak{F})$ über $\Gamma(X, \mathcal{O})$ durch r Elemente erzeugt werden kann.

Dieser Satz kann benützt werden, um folgende Verschärfung des Einbettungssatzes von Remmert-Narasimhan-Bishop [22,3] zu beweisen (Forster [11]) :

Sei X eine n-dimensionale Steinsche Mannigfaltigkeit. Für $n \geqslant 2$ *existiert eine holomorphe Einbettung* $X \to C^{2n}$ *(d.h. eine biholomorphe Abbildung von X auf eine analytische Untermannigfaltigkeit von* C^{2n}*), für* $n \geqslant 6$ *sogar eine Einbettung* $X \to C^{2n-k}$*, wobei* $k = [(n-2)/3]$.

5. Vollständige Durchschnitte.

Eine andere Anwendung des Okaschen Prinzips für die Anzahl der Erzeugenden ist das Problem der vollständigen Durchschnitte [7, 10].

Sei $\mathfrak{J}$ eine kohärente Idealgarbe auf dem Steinschen Raum X. Die Garbe $\mathfrak{J}$ heisst vom Typ D_k, wenn das Nullstellengebilde Y von $\mathfrak{J}$ in jedem Punkt $y \in Y$ die Codimension k hat und die minimale Erzeugendenzahl von $\mathfrak{J}_y$ gleich k ist. In diesem Fall ist $V(\mathfrak{J})|Y$ ein k-rangiges Vektorraumbündel, das Conormalenbündel von $\mathfrak{J}$. (Ist Y eine analytische Untermannigfaltigkeit der Steinschen Mannigfaltigkeit X und $\mathfrak{J}$ die Idealgarbe von Y, so ist das Conormalenbündel von $\mathfrak{J}$ das gewöhnliche Conormalenbündel von Y.)

Eine Idealgarbe $\mathfrak{J}$ vom Typ D_k heisst vollständiger Durchschnitt, wenn $\Gamma(X, \mathfrak{J})$ durch k Elemente erzeugt werden kann. Eine notwendige Bedingung dafür ist die Trivialität des Conormalenbündels. Eine hinreichende Bedingung ist die Trivialität des Conormalenbündels zusammen mit dem Verschwinden der Cohomologiegruppen

$$H^{q+1}(X, Y; \pi_q(S_{2k-1})) = 0 \quad \text{für} \quad q \geqslant 2k-1.$$

Dies ist z.B. der Fall, wenn Y eine Untermannigfaltigkeit der Steinschen Mannigfaltigkeit X ist mit $\dim Y < \frac{1}{2} \dim X$. Insbesondere ist jede singularitätenfreie analytische Kurve in einer mindestens 3-dimensionalen Steinschen Mannigfaltigkeit ein vollständiger Durchschnitt.

Für den Fall einer k-dimensionalen Untermannigfaltigkeit Y einer $2k$-dimensionalen Steinschen Mannigfaltigkeit X hat M. Schneider [26] folgendes Kriterium abgeleitet : Y ist vollständiger Durchschnitt genau dann, wenn das Normalenbündel von Y trivial ist und die duale Klasse von Y in $H^{2k}(X, Z)$ verschwindet.

LITERATUR

[1] Andreotti A. and Frankel T. — The Lefschetz theorem on hyperplane sections, *Ann. of Math.*, 69, 1959, p. 713-717.

[2] Andreotti A. and Narasimhan R. — A topological property of Runge pairs, *Ann. of Math.*, 76, 1962, p. 499-509.

[3] Bishop E. — Mappings of partially analytic spaces, *Amer. J. of Math.*, 83, 1961, p. 209-242.

[4] CARTAN H. — Espaces fibrés analytiques, *Symposium International de Topologia Algebraica,* Mexico, 1958, p. 97-121.
[5] FISCHER G. — Lineare Faserräume und kohärente Modulgarben über komplexen Räumen, *Arch. der Math.,* 18, 1967, p. 609-617.
[6] FORSTER O. und RAMSPOTT K.J. — Okasche Paare von Garben nicht-abelscher Gruppen, *Inventiones Math.,* 1, 1966, p. 260-286.
[7] FORSTER O. und RAMSPOTT K.J. — Analytische Modulgarben und Endromisbündel, *Inventiones Math.,* 2, 1966, p. 145-170.
[8] FORSTER O. und RAMSPOTT K.J. — Homotopieklassen von Idealbasen in Steinschen Algebren, *Inventiones Math.,* 5, 1968, p. 255-276.
[9] FORSTER O. und RAMSPOTT K.J. — Ueber die Anzahl der Erzeugenden von projektiven Steinschen Moduln, *Arch. der Math.,* 19, 1968, p. 417-422.
[10] FORSTER O. — Some remarks on parallelizable Stein manifolds, *Bull. Amer. Math. Soc.,* 73, 1967, p. 712-716.
[11] FORSTER O. — Plongements des variétés de Stein, *Comm. Math. Helv.,* 45, 1970, p. 170-184.
[12] FRENKEL J. — Cohomologie non abélienne et espaces fibrés, *Bull. Soc. Math. France,* 85, 1957, p .135-220.
[13] GIESECKE B. — Simpliziale Zerlegung abzählbarer analytischer Räume, *Math. Zeitschr.,* 83, 1964, p. 177-213.
[14] GRAUERT H. — Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen, *Math. Ann.,* 133, 1957, p. 139-159.
[15] GRAUERT H. — Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen, *Math. Ann.,* 133, 1957, p. 450-472.
[16] GRAUERT H. — Analytische Faserungen über holomorph-vollständigen Räumen, *Math. Ann.* 135, 1958, p. 263-273.
[17] GRAUERT H. — Ueber Modifikationen und exzeptionelle analytische Mengen, *Math. Ann.,* 146, 1962, p. 331-368.
[18] GRAUERT H. und KERNER H. — Approximation von Schnittflächen in Faserbündeln mit homogener Faser, *Arch. der Math.,* 14, 1963, p. 328-333.
[19] GROTHENDIECK A. — Techniques de construction en géométrie analytique V, *Exposé 12, Séminaire Henri Cartan 1960-61,* Secrétariat mathématique, Paris, 1962.
[20] KAUP L. — Eine topologische Eigenschaft Steinscher Räume, *Nachr. Akad. Wiss. Göttingen. Math.-Phys.,* Kl., Jg., 1966, p. 213-224.
[21] LOJASIEWICZ S. — Triangulation of semi-analytic sets, *Ann. Scuola Norm. Sup. di Pisa,* 18, 1964, p. 449-474.
[22] NARASIMHAN R. — Imbedding of holomorphically complete complex spaces, *Amer. J. of Math.,* 82, 1960, p. 917-934.
[23] NARASIMHAN R. — On the homology of Stein spaces, *Inventiones Math.,* 2, 1967, p. 377-385.
[24] OKA K. — Sur les fonctions analytiques de plusieurs variables, III) Deuxième problème de Cousin, *J. Sci. Hiroshima Univ.,* 9, 1939, p. 7-19.
[25] RAMSPOTT K.J. — Stetige und holomorphe Schnitte in Bündeln mit homogener Faser, *Math. Z.,* 89, 1965, p. 234-246.
[26] SCHNEIDER M. — Vollständige Durchschnitte in Steinschen Mannigfaltigkeiten, *Math. Ann.,* 186, 1970, p. 191-200.
[27] SERRE J.-P. — *Application de la théorie générale à divers problèmes globaux,* Exposé 20, Séminaire Henri Cartan (1951-52), 2e édition, Benjamin, New York. 1967.
[28] STEENROD N. — The topology of fibre bundles, *Princeton Univ. Press,* 1951.

[29] STEIN K. — Topologische Bedingungen für die Existenz analytischer Funktionen komplexer Veränderlichen zu vorgegebenen Nullstellenflächen, *Math. Ann.*, 117, 1941, p. 727-757.

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PROLONGEMENT DE FAISCEAUX ANALYTIQUES COHÉRENTS

par J. FRISCH

Soit X un espace analytique complexe, et soit S un sous-ensemble analytique de X. On connait depuis longtemps des théorèmes de prolongement de fonctions analytiques (par exemple : si X est un ouvert de $\mathbf{C}^n$ et si S est de codimension $\geqslant 2$, le morphisme de restriction $\mathcal{O}_X(X) \to \mathcal{O}_X(X - S)$ est bijectif) et de sous-ensembles analytiques (par exemple le théorème de Remmert-Stein : si un sous ensemble analytique V de $X - S$ a en chacun de ses points une dimension $>$ dim S, alors son adhérence $\overline{V}$ est un sous-ensemble analytique de X). Ce n'est qu'assez récemment qu'ont été démontrés des énoncés analogues concernant les faisceaux cohérents. Voici les deux types de problèmes évoqués cidessous.

(1) Soit $\mathfrak{F}$ un faisceau cohérent sur $X - S$. Donner des conditions suffisantes de régularité sur $\mathfrak{F}$ pour que $\mathfrak{F}$ se prolonge en un faisceau cohérent sur X, i.e. pour qu'il existe un faisceau cohérent $\widetilde{\mathfrak{F}}$ sur X dont la restriction à $X - S$ soit isomorphe à $\mathfrak{F}$.

(2) Soit $\mathfrak{F}$ un faisceau cohérent sur X, et soit sur $X - S$ un sous-faisceau cohérent $\mathfrak{G}$ de $\mathfrak{F}|_{X-S}$. Donner des conditions sur $\mathfrak{G}$ pour que $\mathfrak{G}$ se prolonge sur X en un sous-faisceau cohérent $\widetilde{\mathfrak{G}}$ de $\mathfrak{F}$ dont la restriction à $X - S$ soit égale à $\mathfrak{G}$.

Ces problèmes de prolongement sont propres à la géométrie analytique : en géométrie algébrique, tous les faisceaux cohérents sont prolongeables (2).

1. Prolongement de faisceaux cohérents à travers une sous-variété.

Notons i l'injection canonique de $X - S$ dans X. Le faisceau $i_*\mathfrak{F}$ est un prolongement naturel de $\mathfrak{F}$, mais n'est pas nécéssairement cohérent (par exemple, si $\mathfrak{F} = \mathcal{O}|_{X-S}$, c'est un faisceau prolongeable, mais si codim $S = 1$, le faisceau $i_*\mathfrak{F}$ n'est pas cohérent). Le premier théorème de prolongement a été donné par Serre dans (6).

THEOREME 1. – *On suppose X normal, et S de codimension $\geqslant 2$ en chacun de ses points. Pour un faisceau cohérent sans torsion $\mathfrak{F}$ sur $X - S$, les conditions suivantes sont équivalentes :*

(i) *le faisceau $i_*\mathfrak{F}$ est cohérent* ;

(ii) *le faisceau $\mathfrak{F}$ se prolonge en un faisceau cohérent sur X* ;

(iii) *pour tout $s \in S$, il existe un voisinage ouvert U de s dans X tel que $\mathfrak{F}(U - U \cap S)$ engendre $\mathfrak{F}_x$ pour tout $x \in U - U \cap S$.*

L'inconvénient de ce critère est que la condition (iii) n'est pas locale sur $X - S$. Avant de donner des critères de prolongement locaux sur $X - S$, rappelons quelques définitions.

Profondeur. – Soit $\mathcal{F}$ un faisceau cohérent sur X, et soit $x \in X$. Soit φ un plongement d'un voisinage de x dans un ouvert d'un espace numérique $\mathbf{C}^N$, et soit d la longueur minimale d'une résolution libre

$$0 \to \mathcal{O}_{\mathbf{C}^N}^{r_d} \to \cdots\cdots \to \mathcal{O}_{\mathbf{C}^N}^{r_0} \to \varphi_* \mathcal{F} \to 0$$

de $\varphi_* \mathcal{F}$ au voisinage de $\varphi(x)$. D'après le théorème des syzygies, on a $d \leqslant N$.

L'entier $N - d$ ne dépend pas du plongement φ ; on l'appelle la profondeur de $\mathcal{F}$ en x. Pour tout entier k, on note $S_k(\mathcal{F})$ l'ensemble des points de X où $\mathcal{F}$ est de profondeur $\leqslant k$. C'est un ensemble analytique de dimension $\leqslant k$.

Gap-sheaf. – Pour tout entier n, on associe à $\mathcal{F}$ le faisceau $\mathcal{F}^{[n]}$ défini par le préfaisceau

$$U \to \text{lim. ind.}\, \mathcal{F}(U - A)$$

où A parcourt l'ensemble des sous-ensembles analytiques de U de dimension $\leqslant n$. On a un morphisme canonique $\mathcal{F} \to \mathcal{F}^{[n]}$. Pour que ce morphisme soit un isomorphisme, il faut et il suffit que pour tout $k \leqslant n + 1$, le faisceau $\mathcal{F}$ vérifie la condition

$$(s_k) \qquad \dim S_k(\mathcal{F}) \leqslant k - 2\,.$$

THEOREME 2. – *Soit $\mathcal{F}$ un faisceau cohérent sur $X - S$. On suppose que* $\dim S \leqslant p$, *et que* $\mathcal{F}^{[p+1]} = \mathcal{F}$. *Alors il existe un faisceau $\widetilde{\mathcal{F}}$ sur X et un seul (à isomorphisme près) qui soit cohérent, prolonge $\mathcal{F}$, et vérifie la condition* $\widetilde{\mathcal{F}}^{[p+1]} = \widetilde{\mathcal{F}}$.

Si $\mathcal{F}$ vérifie seulement la condition $\mathcal{F}^{[p]} = \mathcal{F}$, il n'est pas nécéssairement prolongeable (par exemple, soit $X = C^2$ et $S = \{0\}$; posons $U_j = \{(x_1\,, x_2) \in \mathbf{C}^2\,;\ x_j \neq 0\}$, $j = 1, 2$. Le faisceau localement libre $\mathcal{F}$ obtenu en recollant $\mathcal{O}_{U_1}$ et $\mathcal{O}_{U_2}$ par multiplication par $\exp 1/x_1 x_2$ sur $U_1 \cap U_2$ n'est pas prolongeable). Toutefois, si $\mathcal{F}$ est prolongeable, il admet un prolongement cohérent $\widetilde{\mathcal{F}}$ et un seul (à isomorphisme près) vérifiant $\widetilde{\mathcal{F}}^{[p]} = \widetilde{\mathcal{F}}$, c'est $i_* \mathcal{F}$.

Notons aussi qu'un faisceau localement libre $\mathcal{F}$ sur $X - S$, ne se prolonge pas nécéssairement en un faisceau localement libre sur X, même s'il est prolongeable en un faisceau cohérent sur X (par exemple, soit $X = \mathbf{C}^3$ et $S = \{0\}$. Soit C_0 le faisceau sur X dont la fibre est C en 0, et 0 ailleurs. Soit

$$0 \to \mathcal{L}_3 \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to C_0 \to 0$$

une résolution libre de C_0 et soit $\mathcal{F}$ l'image de $\mathcal{L}_2$ dans $\mathcal{L}_1$. C'est un faisceau localement libre sur $X - S$, prolongeable en un faisceau cohérent sur X, mais n'admettant sur X aucun prolongement localement libre). Il serait intéressant d'avoir des critères de prolongement en faisceau localement libre.

2. Prolongement de faisceaux cohérents définis sur un tube.

Soit D un domaine de $\mathbf{C}^p$, et soient K et L deux polycylindres compacts de C^q tels que $L \subset \mathring{K}$. Notons Q la couronne $K - \mathring{L}$.

THEOREME 3. — *Soit $\mathcal{F}$ un faisceau cohérent sur $D \times Q$ vérifiant la condition $\mathcal{F}^{[p+1]} = \mathcal{F}$; Il existe sur $D \times K$ un faisceau cohérent $\tilde{\mathcal{F}}$ et un seul (à isomorphisme près) qui prolonge $\mathcal{F}$ et vérifie la condition $\tilde{\mathcal{F}}^{[p+1]} = \tilde{\mathcal{F}}$.*

Le théorème 2, qui est de caractère local sur S, résulte du théorème 3. Ce dernier a d'abord été démontré par Trautmann (4,12) dans le cas $p = 0$, en utilisant (1). L'idée de démonstration du cas général est la suivante. Pour chaque $t \in D$, considérons le faisceau $\mathcal{F}(t) = \mathcal{F}/\mathcal{J}.\mathcal{F}$, où $\mathcal{J}$ est l'idéal de $\{t\} \times \mathbf{C}^q$ dans $\mathbf{C}^p \times \mathbf{C}^q$. C'est un faisceau sur Q qui, pour "presque" tout t, vérifie la condition $\mathcal{F}(t)^{[1]} = \mathcal{F}(t)$, donc se prolonge en un faisceau cohérent $\mathcal{F}(t)^{\sim}$ sur K vérifiant la même condition. On montre alors (en utilisant les techniques de (3)) que la collection des $\mathcal{F}(t)^{\sim}$ varie "assez régulièrement" en fonction de t pour qu'il existe sur $D \times K$ un faisceau $\tilde{\mathcal{F}}$ tel que $\tilde{\mathcal{F}}(t) = \mathcal{F}(t)^{\sim}$. Le faisceau $\tilde{\mathcal{F}}$ est le prolongement cherché (5,8).

Dans cette direction, Siu (9) a prouvé, en utilisant le théorème 3, l'énoncé nettement plus général que voici :

THEOREME 4.— *Soit $\mathcal{F}$ un faisceau cohérent sur $D \times Q$ vérifiant la condition $\mathcal{F}^{[p]} = \mathcal{F}$. Soit A l'ensemble des $t \in D$ tels que le faisceau $\mathcal{F}(t)$ sur Q se prolonge en un faisceau cohérent sur K. Si A est épais (i.e. n'est contenu dans aucune réunion dénombrable de sous-variétés analytiques de dimension $< p$), il existe un prolongement cohérent $\tilde{\mathcal{F}}$ de $\mathcal{F}$ à $D \times K$ et un seul vérifiant la condition $\tilde{\mathcal{F}}^{[p]} = \tilde{\mathcal{F}}$.*

On peut appliquer ce résultat pour obtenir des théorèmes de prolongement d'un faisceau cohérent à travers la frontière d'un ouvert pseudo-concave. Des énoncés précis sont donnés dans (9).

3. Prolongement de sous-faisceaux.

Soient X un espace analytique, $\mathcal{F}$ un faisceau cohérent sur X et $\mathcal{G}$ un sous-faisceau cohérent de $\mathcal{F}$. Pour tout entier n, on définit le faisceau $\mathcal{G}_{[n],\mathcal{F}}$ (gap-sheaf relatif) ; c'est le faisceau

$$U \to \varinjlim \{f \in \mathcal{F}(U) \quad ; \quad f|_{U-A} \in \mathcal{G}(U - A)\}$$

où A parcourt l'ensemble des sous-ensembles analytiques de U de dimension $\leqslant n$ (cf (8)). Ce faisceau est cohérent et le support du faisceau $\mathcal{G}_{[n],\mathcal{F}}/\mathcal{G}$ est de dimension $\leqslant n$. En particulier, $0_{[n],\mathcal{F}}$ est le faisceau des sections de $\mathcal{F}$ dont le support est de dimension $\leqslant n$. On démontre que $\mathcal{F}^{[n]}$ est cohérent si et seulement si le support de $0_{[n+1],\mathcal{F}}$ est de dimension $\leqslant n$.

Cela dit, voici, pour les sous-faisceaux, les énoncés analogues aux théorèmes 2 et 4 (cf (8) et (10)). Soit X un espace analytique, soit $\mathcal{F}$ un faisceau cohérent sur X, soit S un sous-ensemble analytique de X et soit un sous-faisceau $\mathcal{G}$ de $\mathcal{F}|_{X-S}$. Soit $i : X - S \to X$ l'injection canonique.

THEOREME 5. — *Si S est de dimension $\leqslant p$ et si $\mathcal{G}$ vérifie la condition*

$$\mathcal{G}_{[p+1],\mathcal{F}} = \mathcal{G},$$

il existe un sous-faisceau $\widetilde{\mathcal{G}}$ de $\mathcal{F}$ sur X et un seul qui soit cohérent, prolonge $\mathcal{G}$ et vérifie la condition $\widetilde{\mathcal{G}}_{[p+1],\mathcal{F}} = \widetilde{\mathcal{G}}$, c'est le faisceau $i_*\mathcal{G}$.

Soient maintenant D, K, L et Q comme au paragraphe précédent. Soit $\mathcal{F}$ un faisceau cohérent sur $D \times K$ et soit sur $D \times Q$ un sous-faisceau cohérent $\mathcal{G}$ de $\mathcal{F}|_{D\times Q}$. Notons A l'ensemble des $t \in D$ tels que l'image de $\mathcal{G}(t)$ dans $\mathcal{F}(t)$ se prolonge sur K en un sous-faisceau cohérent de $\mathcal{F}(t)$.

TEOREME 6. — *Si $\mathcal{G}_{[n],\mathcal{F}} = \mathcal{G}$ et si A est épais, alors $\mathcal{G}$ se prolonge de façon unique sur $D \times K$ en un sous-faisceau cohérent $\widetilde{\mathcal{G}}$ de $\mathcal{F}$ vérifiant la condition $\widetilde{\mathcal{G}}_{[n],\mathcal{F}} = \widetilde{\mathcal{G}}$.*

Remarquons pour terminer que le théorème 5 ne redonne pas le théorème de Remmert-Stein comme cas particulier. Si V est un sous-ensemble analytique de $X - S$, le faisceau d'idéaux $\mathcal{J}_V$ défini par V est un sous-faisceau de $\mathcal{O}_X|_{X-S}$, mais l'hypothèse $\dim_x V > p$ pour tout $x \in V$ implique seulement $(\mathcal{J}_V)_{[p],\mathcal{O}_X} = \mathcal{J}_V$ ce qui ne permet pas d'appliquer le théorème 5. Ce dernier, dans le cas des faisceaux d'idéaux, donne un critère de prolongement des sous-espaces analytiques, non nécessairement réduits, de $X - S$, alors que le théorème de Remmert-Stein est essentiellement ensembliste.

BIBLIOGRAPHIE

[1] ANDREOTTI A. et GRAUERT H. — Théorèmes de finitude pour la cohomologie des espaces complexes, *Bull. Soc. Math. de France,* 90, 1962.

[2] BOREL A. et SERRE J.P. — Le théorème de Riemann-Roch, *Bull. Soc. Math. de France,* 86, 1968.

[3] DOUADY A. — Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, *Ann. Inst. Fourier,* Grenoble, 16, 1966.

[4] DOUADY A. — Prolongement de faisceaux analytiques cohérents, *Séminaire Bourbaki,* 366, 16969.

[5] FRISCH J. et GUENOT J. — Prolongement de faisceaux analytiques cohérents, *Invent. Math.,* 7, 1969.

[6] SERRE J.P. — Prolongement de faisceaux analytiques coherents, *Ann. Inst. Fourier,* Grenoble, 16, 1966.

[7] SIU Y.T. — Absolutes gap-sheaves and extension of coherent analytic sheaves, *Transl. Amer. Math. Soc.,* 141, 1969.

[8] SIU Y.T. — Extending coherent analytic sheaves, *Ann of Math.,* 90, 1969.

[9] SIU Y.T. — *A Hartogs type extension theorem for coherent analytic subsheaves,* to appear.

[10] SIU Y.T. and TRAUTMANN G. — *Extension of coherent analytic subsheaves,* to appear.

[11] THIMM W. — Lückengarben von kohärenten analytischen Modulgarben, *Math. Ann.,* 148, 1962.

[12] TRAUTMANN G. — Ein Kontinuitätssatz für die Fortsetzung kohärenter analytischer Garben, *Arch. Math.*, 18, 1957.

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SOME MULTIVARIABLE PROBLEMS ARISING FROM RIEMANN SURFACES

by R. C. GUNNING

Several problems that have been studied in the theory of functions of several complex variables originated in investigations of Riemann surfaces. The complex of problems centering about Abelian varieties is one example that comes to mind almost immediately ; and the aim of this lecture is to discuss a related set of problems that have not yet been so extensively considered.

Recall that the Picard-Jacobi variety of a compact Riemann surface M of genus $g > 0$ can be envisaged as arising as follows. The set of flat complex line bundles over M is the cohomology group $H^1(M, \mathbf{C}^*)$, which can be described equivalently as the group $\mathrm{Hom}(\pi_1(M), \mathbf{C}^*)$ of representations of the fundamental group of the surface M ; thus this group of flat line bundles is just the complex Abelian Lie group $(\mathbf{C}^*)^{2g}$ of complex dimension $2g$. Two flat line bundles are called analytically equivalent if they determine the same complex analytic line bundle ; representing the surface M as the quotient space D/Γ of the unit disc D modulo a group $\Gamma \cong \pi_1(M)$ of linear fractional transformations, the flat line bundles corresponding to representations ρ, $\sigma \in \mathrm{Hom}(\Gamma, \mathbf{C}^*)$ are thus analytically equivalent precisely when there exists a complex analytic mapping $f : D \to \mathbf{C}^*$ such that $f(TZ) = \rho(T) f(Z) \sigma(T)^{-1}$ for all elements $T \in \Gamma$ and all points $Z \in D$. It is a familiar result that this is an equivalence relation, and that analytic equivalence classes of flat line bundles are precisely the cosets of a Lie subgroup $\mathbf{C}^g \subset (\mathbf{C}^*)^{2g}$; this subgroup can be described explicitly in terms of the periods of the Abelian differential forms on M, and the quotient manifold $(\mathbf{C}^*)^{2g}/\mathbf{C}^g$ is a compact complex analytic torus of complex dimension g, the Picard-Jacobi variety of M.

An obvious direction in which to try to generalize this construction is towards the set of flat complex vector bundles over the surface M ; since line bundles have already been considered, it is natural to restrict attention to bundles of determinant 1 over surfaces of genus $g > 2$, and for simplicity, only bundles of rank 2 will be considered. Thus one is led to examine the cohomology set $H^1(M, SL(2, \mathbf{C}))$, which can be described equivalently as the quotient space $\mathrm{Hom}(\pi_1(M), SL(2, \mathbf{C}))/SL(2, \mathbf{C})$, where the group $SL(2, \mathbf{C})$ acts on the set of representations of the fundamental group of the surface M by inner automorphism. This set does not have a natural group structure ; but it does have a natural complex structure, with some rather complicated singularities which have not yet been analyzed. However, the subset V consisting of those representations having only scalar commutants has the structure of a complex analytic manifold of complex dimension $6g$-6 ; and the tangent space to V at a point corresponding to a representation ρ can be identified with the cohomology group $H^1(\pi_1(M), Ad_0\rho)$ of the group $\pi_1(M)$ with coefficients in the $\pi_1(M)$-module of 2×2 matrices of trace zero under the representation $Ad\rho$.

Two flat vector bundles are called analytically equivalent if they determine the same complex analytic vector bundle ; so that when M is represented as the quotient space D/Γ as above, the flat vector bundles corresponding to representations ρ, $\sigma \in \mathrm{Hom}(\pi_1(M), SL(2, \mathrm{C}))$ are analytically equivalent precisely when there exists a complex analytic mapping $f : D \to SL(2, \mathrm{C})$ such that $f(TZ) = \rho(T)\, f(Z)\, \sigma(T)^{-1}$ for all elements $T \in \Gamma$ and all points $Z \in D$. This is a somewhat more complicated equivalence relation than the corresponding equivalence relation for flat line bundles. The equivalence classes are complex analytic submanifolds of V, each of complex dimension $3g - 3$; but not all the equivalence classes are even topologically equivalent manifolds, and it is not clear that all are closed submanifolds. This analytic foliation of the manifold V can be described more concretely in the following terms though. For any representation $\rho \in \mathrm{Hom}(\pi_1(M), SL(2, \mathrm{C}))$ corresponding to a flat vector bundle in V, an associated generalized Prym differential for the surface $M = D/\Gamma$ is a 2 x 2 matrix $\phi(Z)$ of holomorphic differential forms on the disc D such that trace $\phi(Z) = 0$ and that $\phi(TZ) = \rho(T)\, \phi(Z)\, \rho(T)^{-1}$ for all elements $T \in \Gamma$ and all points $Z \in D$. To each such Prym differential there is naturally associated, as a form of period class, an element in the cohomology group $H^1(\pi_1(M), Ad_0\rho)$; and the period classes of all the Prym differentials for ρ form a subspace of dimension $3g - 3$ in the tangent space to V at the point corresponding to the representation ρ. The result is an analytic involutive distribution in the tangent bundle to V ; and the analytic equivalence classes of flat vector bundles are precisely the corresponding integral submanifolds of V.

The exact nature of the quotient space of V modulo the relation of analytic equivalence is not yet fully understood, but the elegant results of M.S. Narasimhan and C.S. Seshadri go a considerable way in this direction. A more detailed description of the individual leaves in the foliation of V is also a matter of some interest ; for one of the leaves, consisting of those flat bundles representing a maximally unstable analytic vector bundle, describes all the various ways in which the surface M can be uniformized. It is not at present clear whether the period classes of the generalized Prym differentials will be a useful tool in describing the foliation of V ; for detailed knowledge of the periods of even the classical Prym differentials is now lacking, and it may be easier to describe the foliations of V in other ways and use that information in turn to describe the Prym periods.

BIBLIOGRAPHY

[1] Gunning R.C. — Analytic structures on the space of flat vector bundles over a compact Riemann surface, *Several Complex Variables*, II, Maryland, 1970, (Springer, 1970).

[2] Narasimhan M.S. and Seshadri C.S. — Holomorphic vector bundles on a compact Riemann surface, *Math. Annalen*, 155, 1964, p. 69-80.

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DESINGULARIZATION OF COMPLEX-ANALYTIC VARIETIES

by Heisuke HIRONAKA

Several years ago a proof was given for the resolution of singularities of an arbitrary *algebraic* variety over a field of characteristic zero ([1]). As was pointed out there, it was readily modified to give desingularizations of complex-analytic varieties on which global meromorphic functions give a local coordinate system at every point, such as complex Stein varieties. Later, Moishezon extended the desingularization theorem to compact complex-analytic varieties bimeromorphic to algebraic varieties, simply by an ingenuity of inductive reformulation of the problem ([2]). For the case of a general complex-analytic variety, however, we must go back to the essentials of the proof of desingularization in the algebraic case, and then develop a new technique to globalize the basic ideas found there. The essential difficulty in the complex-analytic case, as is compared with the algebraic case, is in fact the lack of global meromorphic functions and the inevidence of "sufficiently many" global subvarieties. The supporting philosophy toward the complex-analytic desingularization is that the more singular is the given variety, the more global subvarieties exist there. The problem is, of course, how to find them. In the resolution of singularities of complex-analytic varieties, we look for not only global subvarieties of a given variety but also global subvarieties of its transforms by successive blowing-ups having such subvarieties as their centers.

DESINGULARIZATION THEOREM. – *Let X be a complex-analytic variety. Then there exists a proper morphism $\pi : \widetilde{X} \to X$ such that*

(i) *π is almost everywhere isomorphic, i.e., bimeromorphic, and* (ii) *$\widetilde{X}$ is smooth.*

Moreover we can choose π in such a way that every isomorphism $\alpha : X|U \to X|V$ with two open subsets U and V of X gives rise to an isomorphism β :

$$\widetilde{X}|\pi^{-1}(U) \to \widetilde{X}|\pi^{-1}(V)$$

with the commutativity $(\pi|V)\beta = \alpha(\pi|U)$.

The last property of π implies, in particular, that the automorphisms of X lift to automorphisms of $\widetilde{X}$ in a manner compatible to the bimeromorphism π. If **P** denotes the pseudo-group of all the local isomorphisms α as above, the π is obtained as the composition of a possibly infinite, but locally finite, succession of blowing-ups with closed smooth centers which are invariant under the natural liftings of **P**. Roughly speaking, the process of desingularization depends upon only the "quotient type" of X with respect to **P**. Note that π must induce an isomorphism in the smooth part of X. We also note that our desingularization process gives a result stronger than stated above. For instance, the inverse image by π of the singular locus of X can be made into normal crossings of smooth hypersurfaces.

In this article, we give an account of local theory toward the desingularization theorem. Jon the global theory, see [5].

Maximal contact and idealistic exponent.

Let X be a complex-analytic space. Pick a point $x \in X$. By localizing X around x, we assume that X is given as a complex-analytic subspace of a smooth variety Z. Every holomorphic function g in the local ring $O_{Z,x}$ induces a homogeneous polynomial function $\bar{g}$ in the tangent space $T_{Z,x}$ whose degree d is equal to the order of g at x. The formal definition of $\bar{g}$ is the residue class of g modulo $M_{Z,x}^{d+1}$, where $M_{Z,x}$ denotes the maximal ideal of $O_{Z,x}$. We call $\bar{g}$ the *initial form* of g at x. The initial forms of functions in the ideal $I_{Z,X,x}$ of X in $O_{Z,x}$ generate a homogeneous ideal on $T_{Z,x}$ and hence define a cone $C_{X,x}$ in $T_{Z,x}$. We call $C_{X,x}$ the *tangent cone* of X at x. (If we identify a neighborhood of x in Z with a neighborhood of the origin in $T_{Z,x}$ in a natural-though not unique-fashion, then the point-set of $C_{X,x}$ is the union of all the possible limits of the complex-lines connecting x with nearby points y of X as y tend to x. But note that $C_{X,x}$ is not in general reduced. For instance, for a plane curve X defined by $y^2 - x^3 = 0$, $C_{X,0}$ is the double x-axis defined by the ideal (y^2) when $T_{C^2,0}$ is naturally identified with the ambient space C^2). We define the vectorial portion of $C_{X,x}$ by $T_{Z,x} = \{v \in T_{Z,x} | C_{X,x} + v = C_{X,x}\}$, which is called the *strict tangent space* of X at x. Now pick any smooth curve E through x in Z. We then define the *contact exponent* of E with X at x as follows. Let $(z_1, z_2, \ldots, z_r, y)$ be any local coordinate system of Z with center at x, such that E is defined by

$$z_1 = z_2 = \ldots z_r = 0.$$

Take a pair of integers (α, m) with $\alpha \geqslant m > 0$ and the morphism $\lambda : Z' \to Z$ which is defined by the system of algebraic functions $(\sqrt[\alpha]{z_1}, \sqrt[\alpha]{z_2}, \ldots, \sqrt[\alpha]{z_r}, \sqrt[m]{y})$ over the coordinate neighborhood of x. We then have a unique smooth curve E' with the same point-set as $\lambda^{-1}(E)$. Let x' be the unique point of E' corresponding tc x, and let $X' = \lambda^{-1}(X)$. Now the contact exponent $\sigma_x(X, E)$ is the minimum of all the ratios α/m for those pairs (α, m) such that $T_{X',x'}$ does not contain $T_{E',x'}$. This minimum always exists and hence $\sigma_x(X, E)$ is either a rational number or infinity. Note that $\sigma_x(X, E) \geqslant 1$ always, and that $\sigma_x(X, E) = 1$ if and only if $C_{X,x}$ is not invariant by $T_{E,x}$. We next define $\sigma_x(X)$ to be the maximum of $\sigma_x(X, E)$ for all smooth curve E as above. This maximum is also attained by some E. Let us call $\sigma_x(X)$ the *characteristic exponent* of X at x. Moreover we can show that the above definitions of $\sigma_x(X, E)$ and $\sigma_x(X)$ do not depend upon the choice of the coordinate system $(z_1, z_2, \ldots, z_r, y)$.

We say that a smooth subvariety F of Z though x is *transversal* to X at x, if the ideal of $T_{F,x}$ in $T_{Z,x}$ is generated by a regular sequence of linear homogeneous polynomials for the structure sheaf of the cone $C_{X,x}$ (at the origin). We say that a complex-analytic retraction $r : Z \to W$ with a smooth subvariety W though x

is *transversal* to X at x, if $r^{-1}(x)$ is so. Let D be the unit disk, i.e.,

$$D = \{t \in \mathbf{C} \mid |t| < 1\}.$$

Given a smooth W contining x and a retraction $r : Z \to W$, we shall consider an arbitrary holomorphic map $h : D \to W$ such that $h(0) = x$, and the base extension by h as in the following diagram

$$\begin{array}{ccc} Z & \xleftarrow{\ g\ } & Z_h = Z \times_W D \\ r \downarrow \uparrow i & & r_h \downarrow \uparrow i_h \\ W & \xleftarrow{\ h\ } & D \end{array}$$

where i is the inclusion and g is the projection. We write $X_h = g^{-1}(X) = X \times_W D$, $W_h = g^{-1}(W) = i_h(D)$ and $x_h = i_h(0)$. We can prove that r is transversal to X at x if and only if $H_{X_h, x_h} = \Delta^{s-1} H_{X,x}$ for every h as above, where H denotes Samuel function, Δ is defined by $(\Delta H)(q) = H(q) - H(q-1)$ for all integers q and s denotes the dimension of W around x.

(Recall : $H_{X,x}(q) = \dim_{\mathbf{C}}(O_{X,x}/M_{M,x}^{q+1})$ for $q \geqslant 0$ and $= 0$ for $q < 0$).

DEFINITION 1. – Let $x \in X \subset Z$ be the same as above and let W be a smooth subvariety of Z through x. We say that W has *maximal contact* with X at x if, after a suitable localization around x, there exists a complex-analytic retration $r : Z \to W$ having the following properties :

(i) r is transversal to X at x,

(ii) $T_{X \cap r^{-1}(x), x} = 0$, and

(iii) for every $h : D \to W$ with $h(0) = x$, $\sigma_{x_h}(X_h, W_h) = \sigma_{x_h}(X_h)$.

Remark. – If $T_{W,x} = T_{X,x}$, then not only (ii) but also (i) is automatic.

We have the follwoing four fundamental theorems.

EXISTENCE THEOREM. – *Given $x \in X \subset Z$ as above, there exists a smooth subvariety W of Z through x such that $T_{W,x} = T_{X,x}$ and W has maximal contact with X at x.*

We classify the points of X by Samuel functions of X at them. The set of points of X with a given Samuel function is called a *Samuel stratum* of X. One can show that each Samuel stratum is a difference of two closed complex-analytic subsets of X.

CONTINUITY THEOREM. – *Let S be the Samuel stratum of X containing the point x. If W is a smooth subvariety of Z through x having maximal contact with X at x, then there exists a neighborhood N of x in S such that W contains N and has maximal contract with X at every point of N. Moreover if $r : Z \to W$ is a complex-analytic retraction having the property of Definition 1, then it has the same at every neighboring point of x in S.*

Example. – Let X be the hypersurface in $\mathbf{C}^3$ defined by $x^2 - y^2 z = 0$. Then the z-axis is a Samuel stratum of X. We can prove that, W being a smooth subvariety of $\mathbf{C}^3$ and p being a point of the z-axis,

(i) W has maximal contact with X at $p \neq 0$ if and only if either one of the following holds :

(a) W coincides with the z-axis in a neighborhood of p.

(b) Locally at p, W is a hypersurface containing the z-axis.

(ii) W has maximal contact with X at $p = 0$ if and only if W is a hypersurface containing the z-axis within a neighborhood of 0 in $\mathbf{C}^3$ and $T_{W,0} = T_{X,0}$ (which is the tangent space of the yz-plane at 0).

If X is defined by $x^2 - y^2 z^2 = 0$, then the necessary and sufficient condition for W to have maximal contact with X at 0 is that W contains a neighborhood of 0 both in the y-axis and in the z-axis.

COHERENCY THEOREM. – *Let S be a Samuel stratum of X. Let W be a smooth subvariety of Z contining S and having maximal contact with X at every point of S. Assume that we have a complex-analytic retraction $r : Z \to W$ having the property of Definition 1 at every point of S. Then every point of S has a neighborhood G in W such that there exists a pair (I, b) of a coherent ideal sheaf I on G and a positive integer b for which we have*

$$\sigma_{x_h}(X_h) = \mathrm{ord}_0 (IO_D)/b$$

for every holomorphic map $h : D \to W$ with $h(0) \in S \cap G$, where IO_D denotes the ideal sheaf generated by I by means of h and ord_0 denotes the order of the ideal at 0. (For the other symbols, see the paragraph preceeding Definition 1).

Remark. – In a certain sense, this theorem shows that the characteristic exponents are upper semi-continuous.

The property of maximal contact has certain stability with respect to certain blowing-ups. Namely, under the circumstances of Coherency Theorem, take any smooth subvariety E of S. By localizing the data if necessary, we assume that E is closed in X and Z. Let $\pi : Z' \to Z$ be the blowing-up with center E. Let X' (resp. K) be the strict transform of X (resp. $r^{-1}(E)$) by π, so that π induces the blowing-up $X' \to X$. Then the strict transform W_1 of W by π does not meet K. If $Z_1 = Z' - K$, then there exists a unique retraction $r_1 : Z_1 \to W_1$ compatible with r with respect to π. Let $X_1 = X' - K$. Let S_1 be the Samuel stratum of X' corresponding to the same Samuel function as X along S.

STABILITY THEOREM. – *S_1 is contained in X_1 and W_1. At every point of S_1, W_1 has maximal contact with X_1 with respect to the retraction r_1. If*

$$I_1 = R^{-b}(IO_{W_1})$$

with the ideal sheaf R of the exceptional divisor $\pi^{-1}(E) \cap W_1$ in W_1, then the pair (I_1, b) has the same property as in Coherency Theorem for X_1 and r_1.

Remark. – Under the conditions of Coherency Theorem, we can prove that $S = \{y \in W \mid \mathrm{ord}_y(I)/b \geqslant 1\}$. In Stability Theorem, the center E is contained in S and hence IO_{W_1} is divisible by R^b in O_{W_1}. Of course, again

$$S_1 = \{z \in W_1 \mid \mathrm{ord}_z(I_1)/b \geqslant 1\}.$$

REFERENCES

[1] Hironaka H. — Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math.*, 79, 1964, p. 109-326.

[2] Moisezon B.G. — Resolution theorems for compact complex spaces with a sufficiently large field of meromorphic functions, *Izv. Akad. N. S.S.S.R.*, 31, 1967, p. 1385-1414.

[3] Zariski O. — *Algebraic Surfaces,* Springer-Verlag, Heidelberg-Berlin, 1934, particularly Chapter 1 for references on singularities of curves and surfaces.

[4] Zariski and Abhyankar. — On resolution of singularities, especially those in *Ann. of Math.*, 1939-1944 and 1965, mainly for comparisons.

[5] Hironaka. — Gardenning of infinitely near singularities, in *the Proceedings of Nordic Summer School,* 1970, which gives more detailed and general account of the theory of Gardenning.

[6] For complete proofs of the theorems stated in this article, refer to forthcoming lecture notes based on a series of Seminars at Ecole Polytechnique, Paris, 1968-70. (Redactions by Tessier, Lejeune and others).

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GRAUERTSCHE KOHÄRENZSATZE FUR STETIGE UND DIFFERENZIERBARE FAMILIEN KOMPLEXER RÄUME

von Reinhardt KIEHL

In der Funktionentheorie mehrerer Veränderlicher spielt der Grauertsche Satz über die Kohärenz der direkten Bildgarben einer kohärenten Garbe bei eigentlichen Abbildungen komplex-analytischer Räume eine fundamentale Rolle ([1]. Es ist daher von Interesse, Sätze diesen Typs auch für relativ analytische Räume (etwa im Sinne der Variétiés mixtes) z. B. über einer differenzierbaren Mannigfaltigkeit zur Verfügung zu haben.

Allgemeiner gehe man aus von einem Frechet-geringten Grundraum (S, O_S), das ist ein lokal kompakter Raum S zusammen mit einer Garbe O_S von topologischen unitären (nicht notwendig kommutativen) Algebren über dem Körper der komplexen Zahlen, für die der Ring $O_S(U)$ der Schnitte über einer im Unendlichen abzählbaren offenen Teilmenge U ein Frechetraum ist. Für viele Beweise muß noch folgendes Axiom erfüllt sein :

Sei U offene Teilmenge von S, V in U relativ kompakte Teilmenge und B eine beschränkte Teilmenge von $O_S(U)$; dann gibt es in $O_S(V)$ eine abgeschlossene absolutkonvexe beschränkte Teilmenge $\overline{B}$, die das Bild von B in $O_S(V)$ enthält und eine Konstante c mit $B \cdot B \subseteq c B$.

Auf dem Produktraum $S \times \mathbf{C}^n$ mit dem n-dimensionalen komplexen Vektoraum $\mathbf{C}^n$ kann man dann auf natürliche Weise eine Garbe $O_{S \times \mathbf{C}^n}$ von topologischen Algebren erklären. Für eine im Unendlichen abzählbare Teilmenge U von S und eine Steinsche offene Teilmenge V von $\mathbf{C}^n$ ist der Ring der Schnitte $O_{S \times \mathbf{C}^n}(U \times V)$ kanonisch isomorph zum vollständigen Tensorprodukt von $O_S(U)$ und dem Raum der holomorphen Funktionen auf V.

Man hat eine natürliche Abbildung von Frechet – geringten Räumen

$$(S \times \mathbf{C}\ , O_{S \times \mathbf{C}^n}) \to (S, O_S)$$

Die offenen Teilmengen Y von $S \times \mathbf{C}^n$ zusammen mit der induzierten geringten Struktur O_Y und der induzierten Abbildung

$$(Y, O_Y) \to (S, O_S)$$

sind dann die lokalen Modelle von glatten analytischen Räumen

$$(X, O_X) \to (S, O_S) \quad \text{über} \quad (S, O_S).$$

Die Strukturgarben O_S sind im allgemeinen nicht mehr kohärent. Nach Grothendiek hat man deshalb an Stelle kohärenter Garben *pseudokohärente Komplexe* ([1]) zu betrachten ; das sind nach oben beschränkte Komplexe von O_S-Moduln, die lokal beliebig lange freie Auflösungen besitzen. Unter einigen zusätzlichen Voraussetzungen über (S, O_S) – S muss genügend viele für O_S azyklische offene Teilmengen besitzen und lokal endliche weiche Dimension haben – kann man zeigen :

Die Strukturabbildung

$$(X, O_X) \to (S, O_S)$$

sei eigentlich (als Abbildung der unterliegenden lokalkompakten topologischen Räume).

Dann ist das abgeleitete direkte Bild im Sinne von Verdier eines pseudokohärenten Komplexes auf X ein pseudokohärenter Komplex auf S.

Zum Beweis benötigt man eine Verallgemeinerung des Endlichkeitslemmas von Schwarz auf Komplexe von Frechetmoduln über einer Frechetalgebra.

Es ist anzunehmen, dass man unter geeigneten Einschränkungen über die Strukturgarbe O_S auch entsprechende Sätze für nicht glatte relativ analytische Räume über S mit ähnlichen Methoden beweisen kann. (*)

Aus dem Pseudokohärenzsatz ergeben sich die üblichen Folgerungen.

Etwa die Halbstetigkeitssätze von Kodaira für differenzierbare Familien komplexer Mannigfaltigkeiten, bzw. Algebraizitätssätze vom GAGA – Typ etwa für den projektiven Raum über einer Frechetalgebra.

LITERATUR

[1] Grauert. — Ein theorem der Analytischen Garbentheorie und die Modulräume Komplexer Strukturen. *Publ. Math.*, No. 11, 1961.

[2] S G A.

(*) Anmerkung bei der korrektur : Der Verf. und J. L. Verdier können inzwischen auch den nicht glatten Fall mit diesen Methoden behandeln.

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FONCTIONNELLES ANALYTIQUES

par André MARTINEAU

Le terme de fonctionnelle analytique a été introduit par le mathématicien italien Fantappié, à l'époque où Banach développait sa théorie. Les idées de Fantappié sont restées, à cette époque, assez à l'écart du développement de la théorie linéaire des espaces fontionnels.

Depuis, la notion a été remaniée pour suivre l'évolution des idées générales et des résultats de la théorie de la dualité. De façon très générale, on pourrait appeler fonctionnelle analytique linéaire une forme linéaire continue sur un espace vectoriel topologique localement convexe de fonctions holomorphes. Nous allons choisir des espaces fort naturels pour restreindre cette définition. Le choix est dicté par l'application en vue, à savoir la théorie de la croissance des fonctions entières de type exponentiel. Les résultats classiques d'Oka – Cartan [5] interviennent en plusieurs points.

Les idées développées ici ont des interprétations en théorie spectrale. Nous en donnons une idée rapide.

Enfin nous abordons la théorie des fonctionnelles non linéaires.

1. Supports

Si V est une variété analytique complexe, pour tout U ouvert de V $\mathcal{O}(U)$ l'espace des fonctions holomorphes dans U admet une topologie d'espace de Fréchet-nucléaire (*a fortiori* de Schwartz) [7b].

Si F est une partie fermée de V, $\mathcal{O}(F)$ l'espace des sections sur F du faisceau structurel peut être muni de la topologie, $\varinjlim_{U \supset F} \mathcal{O}(U)$, de la limite inductive dans la catégorie des espaces localement convexes des espaces $\mathcal{O}(U)$. Il y a d'autres topologies tout aussi naturelles [11b].

Du moins si F est compact elles coincident, et alors $\mathcal{O}(F)$ est un dual de Fréchet-nucléaire, en particulier est un espace dual de Fréchet-Schwartz (ou LN^*) [7b] [12b]. On peut appliquer la théorie de la transposition pour les applications linéaires continues.

Une fonctionnelle analytique locale, donnée au voisinage de F, est un élément du dual topologique de $\mathcal{O}(F)$. Une fonctionnelle analytique sur V est une fonctionnelle donnée au voisinage de V, c'est-à-dire un élément de $\mathcal{O}'(V)$. Soit $T \in \mathcal{O}'(V)$. On dit que T est portable par un ouvert $U \subset V$ s'il existe $\Theta \in \mathcal{O}'(U)$ qui a pour restriction à $\mathcal{O}(V)$ la fonctionnelle T. Un compact K est un porteur de T s'il existe $\Theta \in \mathcal{O}'(K)$ de restriction T à $\mathcal{O}(V)$. Si $T \in \mathcal{O}'(V)$ est portable par tout voisinage ouvert de K, alors elle est portable par K [2].

Soit $\mathcal{K}$ une famille fondamentale de compacts de V, stable par intersections décroissantes, alors toute $T \in \mathcal{O}'(V)$ admet au moins un porteur minimal dans $\mathcal{K}$. On l'appellera le $\mathcal{K}$-support de T. Par exemple dans le cas de $\mathbf{C}^n$ on peut prendre la famille des convexes compacts.

Une distribution S définie sur $\mathbf{R}^n$ à support compact définit une fonctionnelle analytique sur $\mathbf{C}^n$ par restriction à $\mathcal{O}(\mathbf{C}^n)$. Son support en tant que distribution est le plus petit compact réel portant S, mais elle peut avoir dans $\mathbf{C}^n$ d'autres supports convexes (cf. [11a], [8b] pour toutes ces questions).

On dit qu'un compact K est de Runge dans V si $\mathcal{O}(V)$ est dense dans $\mathcal{O}(K)$.

Typique est le résultat suivant : K_1 et K_2 étant deux compacts de V, $K_1 \neq K_2$, une condition nécessaire et suffisante pour que toute T portable par K_1 et par K_2 soit portable par $K_1 \cap K_2$ est que toute fonction holomorphe au voisinage de $K_1 \cap K_2$ soit différence d'une fonction holomorphe au voisinage de K_1 et d'une fonction holomorphe au voisinage de K_2, et que $K_1 \cup K_2$ soit de Runge.

Pour vérifier ces hypothèses on utilise la théorie d'Oka-Cartan. Si K_1 et K_2 sont polynomialement convexes dans $\mathbf{C}^n$ une condition nécessaire et suffisante pour qu'elles soient vérifiées est que $K_1 \cup K_2$ soit aussi polynomialement convexe. Un critère utilisable pour les convexes est celui-ci. Disons qu'un compact L de $\mathbf{C}^n$ est étoilé par rapport à un point $x_0 \in L$ si pour tout $x \in L$, le segment $[xx_0]$ est dans L. Si par tout point du complémentaire de L passe un hyperplan complexe ne rencontrant pas L, cet ensemble est polynomialement convexe.

Exemple : 1) dans $\mathbf{C}^n$ la réunion de

$$K_1 = \{z \mid |z_j| \leqslant A \quad j = 1, 2, \ldots, n\}$$

et

$$K_2 = \{z \mid y_j = 0, \ |x_j| \leqslant B, \ j = 1, 2, \ldots, n\}$$

avec $B \geqslant A$ est polynomialement convexe.

2) Si $L_1 = \left\{ z \mid \sum_{j=1}^{n} |z_j|^2 \leqslant A \right\}$, $L_2 = \left\{ z \mid y_j = 0, \ \sum_{j=1}^{n} |x_j|^2 \leqslant B \right\}$ $L_1 \cup L_2$ n'est pas polynomialement convexe si $B > A$.

On a encore le résultat général suivant : dans $\mathbf{C}^n$ si une fonctionnelle T est portable par les intersections n à n de m convexes Γ_j , alors T est portable par $\bigcap_j \Gamma_j$.

Pour la dimension 1 ceci montre, fait bien connu dans une autre terminologie [3], que T a un unique support convexe. Dans les dimensions supérieures c'est faux même quant T provient d'une distribution. Néanmoins Kiselman [8a] a prouvé que si K est un support polynomialement convexe de T, à frontière de classe C^2, alors il est son unique support. On a aussi [11e] : si K est un support convexe tel qu'en chaque point extrémal il passe un seul hyperplan complexe d'appui, alors K est seul support convexe de T. Pour d'autres théorèmes sur les supports nous renvoyons à [11a] et [8b].

2. Indicatrices.

La notion d'indicatrice a été introduite par Fantappié. Pour $n = 1$, si T est portable par K, on pose $\varphi_T(u) = T\left(z \to \frac{1}{z-u}\right)$. Ce cas a été étudié en détails [12a] [9] [7a]. On obtient, via l'intégrale de Cauchy, un isomorphisme entre $\mathcal{O}'(K)$ et l'espace des fonctions holomorphes dans $\complement K$ et nulles à l'infini.

Pour $n > 1$ l'analogue se trouve dans la cohomologie. Soit K un compact de V variété de Stein de dimension n. Le groupe $H^{n-1}(\complement K\ ;\Omega^n)$, $(n-1)^{ième}$ groupe de cohomologie du complémentaire de K à valeurs dans le faisceau Ω^n des n-formes holomorphes, admet une topologie d'espace de Fréchet non séparé. Son quotient séparé $H_s^{n-1}(\complement K\ ;\Omega^n)$ est isomorphe à $\mathcal{O}'(K)$. Si, par exemple, $H^1(K\ ;\mathcal{O}) = 0$, alors $H^{n-1}(\complement K\ ;\Omega^n)$ est séparé. Il en est ainsi si K est polynomialement convexe et alors nous avons une correspondance biunivoque entre les fonctionnelles analytiques portables par K et les éléments de $H^{n-1}(\complement K\ ;\Omega^n)$.

A une constante près on peut décrire l'accouplement comme suit : à $T \in \mathcal{O}'(K)$ correspond un élément de $H^{n-1}(\complement K\ ;\Omega^n)$ bien défini, son indicatrice ; cet élément peut être représenté par une forme différentielle φ à coefficients C^∞ et de type $(n, n-1)$. Si $f \in \mathcal{O}(K)$, elle provient d'une $\tilde{f} \in \mathcal{O}(U)$ où U est un voisinage ouvert de K. Soit $\alpha \in \mathcal{D}(U)$ identique à 1 au voisinage de K ; on a

$$T(f) = \text{cte} \int_{\mathbf{C}^n} \tilde{f}\,.\,\bar{\partial}\alpha \wedge \varphi\,.$$

Pour une explicitation de l'accouplement dans le cas de la représentation de $H^{n-1}(\complement K\ ;\Omega^n)$ par des cocycles de Čech nous renvoyons à [11h], [11k].

Une autre indicatrice qui se déduit de celle-ci peut être présentée sans cohomologie. Nous prenons le cas où V est un espace vectoriel complexe E de dimension finie n. Nous notons par E' le dual de E, par $\mathcal{E}^{(1)}$ l'espace vectoriel des formes linéaires affines sur E. Nous notons par $\mathfrak{P}(E')$ l'espace des droites de $\mathcal{E}^{(1)}$, et E' est un sous-espace de $\mathcal{E}^{(1)}$ par l'application $\zeta \to$ (droite définie par $(z \to \langle z, \zeta\rangle + 1)$).

Un point de $\mathfrak{P}(E')$ est déterminé par un couple (ζ, ζ_0) différent de $(0, 0)$, c'est la droite engendrée par $(z \to \langle z, \zeta\rangle + \zeta_0)$. Un point $\pi = (\zeta, \zeta_0)$ de $\mathfrak{P}(E')$ définit un hyperplan $\tilde{\pi}$ de $\mathfrak{P}(E)$, le lieu des points (z, z_0) tels que $\langle z, \zeta\rangle + z_0\zeta_0 = 0$. Si $K \subset \mathfrak{P}(E)$, on pose $\overset{*}{\complement} K = \{\pi \mid \tilde{\pi} \cap K = \emptyset\}$. Supposons $K \subset E$ et $T \in \mathcal{O}'(K)$. La fonction $\pi \to \varphi_T(\pi) = T\left(z \to \frac{\zeta_0}{\langle z, \zeta\rangle + \zeta_0}\right)$ est définie et holomorphe dans $\overset{*}{\complement} K$. Elle est nulle si $\zeta_0 = 0$, c'est-à-dire sur la partie à l'infini de $\overset{*}{\complement} K$. On dit que c'est l'indicatrice projective de T.

On a la propriété : si K est *convexe* $\mathcal{O}'(K)$ est isomorphe à l'espace des fonctions holomorphes dans $\overset{*}{\complement} K$ et nulles à l'infini de $\overset{*}{\complement} K$ (1) ; (11c). Mais même lorsque $\complement K = \bigcup_{\pi \in \overset{*}{\complement} K} \tilde{\pi}$, et que K est polynomialement convexe, il peut exister des fonctions holomorphes dans $\overset{*}{\complement} K$, nulles à l'infini, et qui ne sont pas des indicatrices [11d].

Pour tourner la difficulté on opère ainsi. Soit K compact défini par des inégalités : $|p_i| \leqslant 1$, $i \in I$, $(p_i - p_i(0)) \in W$, W sous-espace vectoriel de $\mathfrak{O}(E)$ qui contient au moins les coordonnées. On pose $\varphi_W(\widetilde{p}) = T\left(z \to \frac{p(0)}{p(z)}\right)$ définie si $p(z) \neq 0$ pour tout $z \in K$. L'ensemble de ces $\widetilde{p}$ est un ouvert $\overset{*}{\complement}_W K$ de $\mathfrak{R}(W)$, et T est portable par K si et seulement si φ_W, définie au voisinage de 0 dans $\mathfrak{R}(W)$, se prolonge analytiquement à $\overset{*}{\complement}_W K$.

3. Fourier-Borel.

Si $T \in \mathfrak{O}'(E)$ on pose : $FT(z') = T(z \to \exp <z, z'>)$ pour tout $z' \in E'$. Il est connu [10] que la correspondance $T \to FT$ est un isomorphisme entre $\mathfrak{O}'(E)$ et l'espace Exp (E') des fonctions entières de type exponentiel sur E'.

On pose $\lambda_{FT}(z') = \overline{\lim\limits_{t \to +\infty}}\ t^{-1} \log |FT(tz')|$. On note par Λ_{FT} la plus petite majorante semi-continue supérieurement de λ_{FT}. Pour $n = 1$ $\lambda_{FT} = \Lambda_{FT}$ [3]. Pour $n > 1$ c'est faux en général.

Soit K un convexe compact de E. On pose $h_K(z') = \sup\limits_{z \in K} \text{Re} <z, z'>$. Le théorème du diagramme conjugué de croissance [3] se généralise comme suit : T est portable par K si et seulement si $\Lambda_{FT} \leqslant h_K$ [11a] [6]. On prouve aussi que $\Lambda_{FT} \leqslant h_K$ équivaut à $\lambda_{FT} \leqslant h_K$ [11a].

La fonction Λ_{FT} est pluri-sousharmonique positivement homogène. Pour $n = 1$ il en résulte qu'elle est convexe, pour $n > 1$ nous n'avons pas de caractérisation géométrique satisfaisante, nous ne savons pas en particulier si la propriété indiquée au § 1 sur les intersections n à n de $(n + 1)$ porteurs est caractéristique.

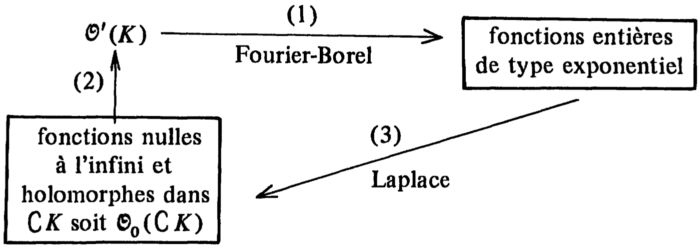
On prouve du moins [8b] [11f et g] l'analogue suivant en fonctions entières de type exponentiel de la réciproque du problème de Levi : pour toute fonction P positivement homogène et pluri-sousharmonique, donnée sur E', il existe F telle que $\Lambda_F = P$.

Les théorèmes d'intersection de supports peuvent être traduits en termes d'estimation de la plus grande minorante pluri-sousharmonique d'un certain ensemble de fonctions. Mais nous ne savons pas résoudre directement ces problèmes, par exemple le cas de l'exemple 1. Ce serait un progrès de savoir le faire car d'une part la théorie s'étendrait à l'ordre fini, d'autre part pour l'ordre 1 on pourrait sans doute étendre certains résultats au cas de fonctions pluri-sousharmoniques sur des ensembles algébriques, ce qui est utile dans la théorie générale des systèmes d'équations aux dérivés partielles à coefficients constants.

Signalons enfin le problème : est-ce qu'une fonction pluri-sousharmonique positivement homogène admet une minorante linéaire ?

4. Inversion de la transformée de Fourier-Borel : la transformation de Laplace.

Pour $n = 1$ on a le schéma suivant :



La flèche (1) a été décrite. Pour (2) nous prenons un espace d'indicatrices, soit $H^{n-1}(\complement K\,;\Omega^n)$, soit $\mathcal{O}_0(\overset{*}{\complement} K)$. Une transformation intégrale réalisant la flèche (3) sera dite transformation de Laplace.

Nous ne décrirons ici que la transformation de Laplace projective, rattachée aux travaux de Jean Leray.

Nous notons par $\mathcal{F}T(l)$ la fonction $T(z \to e^{l(z)})$ où l est une fonction linéaire affine. On a donc $\mathcal{F}T(l) = FT(l - l(0))\, e^{l(0)}$. On pose $\zeta = l - l(0)$, $\zeta_0 = l(0)$. On considère l'intégrale

$$\zeta_0 \cdot \int_0^{+\infty} \mathcal{F}T(t\,.\,l)\,.\,dt\,. = \mathcal{L}_F(l)\,.$$

Cette intégrale converge en tout point de l'ensemble ouvert Ω des l tels que $\Lambda_T(l) < 0$.

La théorie de l'intégrale de Laplace usuelle montre que $\mathcal{L}_F$ est une fonction constante sur la trace d'une droite complexe rencontrant Ω. On a en réalité défini une fonction holomorphe dans l'ouvert Ω^* de $\mathcal{R}(E')$ formé des droites de $\mathcal{E}^{(1)}$ qui rencontrent Ω. Cette fonction que nous notons encore $\mathcal{L}_F$ est dite transformée de Laplace projective de F.

Si $T \in \mathcal{O}'(\Gamma)$ où Γ est un convexe compact, on vérifie que $\Omega^* \supset \overset{*}{\complement}\Gamma$. On retrouve alors [11c] la démonstration habituellement donnée dans la dimension 1 pour le théorème du diagramme conjugué de croissance et la représentation de Polya [3].

Si Γ n'est pas convexe on procède ainsi. Nous notons par $E^{(n)}$ l'espace des polynomes sans terme constant et de degré $\leqslant n$, par $\mathcal{E}^{(n)}$ celui des polynomes de degré $\leqslant n$; $\mathcal{R}(E^{(n)})$ est l'espace des droites de $\mathcal{E}^{(n)}$.

$\varphi_n(T) = T\left(z \to \dfrac{p(0)}{p(z)}\right)$ est définie dans $\overset{*}{\complement}_n K$ si T est portable par K, où $\overset{*}{\complement}_n K = \overset{*}{\complement}_w K$ avec $W = E^{(n)}$. On introduit la transformée de Fourier non linéaire $\mathcal{F}_n T(p) = T(z \to e^{p(z)})$, $p \in \mathcal{E}^{(n)}$. La fonction $F_n T$ est la restriction de cette fonction à $E^{(n)}$. On aura : $\varphi_n(T)\,(p) = \mathcal{L}_{F_nT}(p) = p(0)\displaystyle\int_0^{+\infty} \mathcal{F}_n T\,(t\,.\,p)\,dt$, $p \in \mathcal{E}^{(n)}$.

A titre d'exemple dans $\mathbf{C}^n$, T est une combinaison finie de dérivés de masses de Dirac, c'est-à-dire que FT est une combinaison finie d'exponentielle-polynomes si et seulement si φ_n est une fraction rationnelle, ou encore si F_nT a une restriction à chaque droite qui est une combinaison finie d'exponentielle-polynomes (11l). Dans ce cas φ et φ_n admettent évidemment des développements en somme de fractions à dénominateurs puissance d'une fonction linéaire.

5. Fonctionnelles non linéaires.

Les considérations précédentes conduisent naturellement à l'étude de fonctionnelles non linéaires. Soit V une variété complexe. On considère un polynome homogène de degré n sur $\mathcal{O}(V)$ à valeurs scalaires, c'est-à-dire une application $\varphi \to P(\varphi)$ continue telle qu'il existe une application n-linéaire continue Θ sur $\underbrace{\mathcal{O}(V) \times \ldots \times \mathcal{O}(V)}_{n \text{ fois}}$ telle que $P(\varphi) = \Theta(\varphi, \varphi, \ldots, \varphi)$.

On dit que P est portable par K s'il est la restriction à $\mathcal{O}(V)$ d'un polynome continu défini sur $\mathcal{O}(K)$. Grâce au théorème de Björk [2] il revient au même de dire que pour tout ouvert ω contenant K il existe P_ω défini sur $\mathcal{O}(\omega)$ et de restriction P à $\mathcal{O}(V)$. On peut alors parler de support, support convexe, ...

On pose $F_m P(p) = P(z \to e^{p(z)})$ si $p \in E^{(m)}$. On a la propriété : si P est un polynome homogène de degré n, l'application $P \to F_m P$ est injective dans l'espace des fonctions entières de type exponentiel sur $E^{(m)}$ quand $m \geqslant n$. On introduit ensuite $h_K^m(p) = \sup\limits_{z \in K} \operatorname{Re} p(z)$, $p \in E^{(m)}$ si K est une partie de $\mathbf{C}^\nu$.

La fonctionnelle P étant toujours supposée homogène de degré n, on a les propriétés :

(a) Si $\Lambda_{F_m P} \leqslant n \,.\, h_{\mathbf{R}}^m$ pour un $m \geqslant (2n-2)$ $(n > 1)$, alors P a un support réel.

(b) Soit K un compact convexe réel.

Si $\Lambda_{F_m P} \leqslant n\, h_K^m$ pour un $m \geqslant (2n-1)$, alors P est portable par K.

(c) Soit K un compact réel. Si $\Lambda_{F_m P} \leqslant n \,.\, h_K^m$ pour un $m \geqslant (4n-2)$ alors P est portable par K. Nous renvoyons à [11i], [11j] pour les détails et pour d'autres résultats.

6. Théorie spectrale et fonctionnelles analytiques.

Soit A une algèbre de Banach commutative unifère. Si $a = (a_1, \ldots, a_n) \in A$, à $\varphi \in \mathcal{O}(\mathbf{C}^n)$ on associe $\varphi(a) = \varphi(a_1, \ldots, a_n)$. On obtient ainsi une fonctionnelle analytique linéaire dans A soit T_a, caractérisée par

$$T_a(\varphi_1 \,.\, \varphi_2) = T_a(\varphi_1) \,.\, T_a(\varphi_2),\ T_a(1) = 1 \,.$$

On peut lui appliquer les concepts de la théorie des fonctionnelles analytiques. Dans ce cadre il nous semble que les indicatrices de T_a jouent le rôle de la résolvante. La transformation de Fourier-Borel donne par exemple la "formule du rayon spectral". Soit spa le spectre simultané [4] des éléments $(a_1, \ldots, a_n)$, on a : $\sup\limits_{z \in \mathrm{spa}} \operatorname{Re} \langle z, u \rangle = \lim\limits_{t \to +\infty} t^{-1} \,.\, \log \| e^{t \langle a, u \rangle} \|$.

Nous avons appliqué ce schéma à la théorie de Fredholm (à paraître).

BIBLIOGRAPHIE

[1] AIZENBERG. — Lineinaia Vipouklort v C^n i razdelenie ocobennoctei golomorfnix founktchii. *Acad. Nauk.*, XV, 7, 1967, p. 487-494. (En russe).

[2] BJÖRK J.E. — Every compact set in C^n is a good compact set. *Ann. Inst. Fourier,* 20, fasc. 1, 1970, p.

[3] BOAS R.P. — Entire functions, New York, 1954.

[4] BOURBAKI N. — Théories spectrales, Paris, 1967.

[5] CARTAN H. — *Séminaire E.N.S.* 1951-1952. Benjamin, New York.

[6] EHRENPREIS L. — *Fourier analysis in several complex variables,* New York, 1970.

[7] GROTHENDIECK A. — a) Sur certains espaces de fonctions holomorphes. *Journal de Crelle,* 192, 1953, p. 35-64.
b) Sur les espaces (F) et (DF). *Summa Brasil. Mat.*, 3, 1954, p. 57-123.

[8] KISELMAN C.O. — a) On unique supports of analytic functionals, *Arkiv för Mat.*, 6, 1965.
b) On entire functions of exponential type and indicators of analytic functionals. *Acta Math.*, 117, 1967, p. 1-35.

[9] KÖTHE G. — Dualität in der funktionentheorie. *Journal de Crelle,* 191, 1953, p. 30-49.

[10] MALGRANGE B. — Existence et approximation des solutions des équations aux dérivés partielles et des équations de convolution. *Ann. Inst. Fourier,* 6, 1955, p. 271-354.

[11] MARTINEAU A. —
a) Sur les fonctionnelles analytiques et la transformation de Fourier-Borel. *J. Ann. Math. Jérusalem,* 9, 1963, p. 1-164.
b) Sur la topologie des espaces de fonctions holomorphes. *Math. Ann.*, 163, 1966, p. 62-88.
c) Equations différentielles d'ordre infini. *Bull. Soc. Math. France,* 95, 1967, p. 109-154.
d) Sur la notion d'ensemble fortement linéellement convexe. *Anais Acad. Brasil,* 40, 1968, p. 427-435.
e) Unicité du support d'une fonctionnelle analytique : un théorème de C.O. Kiselman. *Bull. Sc. Math.*, 2° série, 92, 1968, p. 131-141.
f) Indicatrices de croissance des fonctions entières de N-variables. *Inventiones Math.*, 2, 1966, p. 81-86.
g) Indicatrices de croissance des fonctions entières de N-variables. Corrections et compléments. *Inventiones Math.*, 3, 1967, p. 16-19.
h) Indicatrices de fonctionnelles analytiques et transformée de Laplace. Notas e communicaçoes de *Matematica. Univ. do Recife,* 10, 1965, p. 1-31.
i) Les supports des fonctionnelles analytiques, in Séminaire Lelong. *Lecture notes,* 116, Springer, 1969, p. 175-194.
j) Fonctionnelles analytiques non linéaires et représentation de Polya pour une fonction entière de N-variables de type exponentiel, in *Séminaire Lelong,* 10° année, 1970, 37 pages.
k) Indicatrices de fonctionnelles analytiques et inversion de la transformée de Fourier-Borel par la transformation de Laplace. *C.R. Acad. Sc. Paris,* 255, p. 1845-1847.
l) Sur les fonctions entières à transformée de Laplace rationnelle. *C.R. Acad. Sc. Paris,* 269, p. 696-698,

[12] SILVA JAÕ S. E. —
a) As funcões analiticas e a Analise Functional, *Port. Math.*, t. 9, 1950, p. 1-130.
b) Su certe classi di spazi localmente convessi importanti per le applicazioni. *Rend. Math. e Appl.*, (5), 14, 1955, p. 9-26.

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ALGEBRAIC VARIETIES AND COMPACT COMPLEX SPACES

by B. G. MOISHEZON

For a long time, scientists in algebraic geometry have been interested in two kinds of questions :

First, the establishing of general properties of algebraic varieties,

Second, questions of classification.

Though investigations of the first type were a great success during the last period, in the questions of classification our knowledge till now is restricted to algebraic surfaces, and there is little difference between the level of knowledge of the classics of the Italian school and of our knowledge.

The important new idea has been the consideration of algebraic surfaces as a part of the category of all compact complex analytic surfaces, and a classification of these. Here investigations of Kodaira are almost absolute [1], [2].

Apparently the consideration of varieties as a part of the category of compact complex spaces must play an important role in the classification of algebraic varieties of higher dimension.

This report is concerned mainly with problems connected with the generalization of Kodaira's works on complex spaces to higher dimension.

1. Algebraic spaces.

Let X be an irreducible compact complex space. The simplest invariant which separates X from algebraic varieties is the transcendence degree of the field of all meromorphic functions on X. Let us call this number the algebraic dimension of X and note it by a. dim X.

Kodaira has proved that if X is nonsingular and $\dim_C X = a.\dim X = 2$, then X is projective algebraic [1].

It appears that if $\dim_C X = a.\dim X > 2$, one may consider X as an algebraic variety too but in some new sense. One can generalize the conception of the abstract variety of A. Weil by substituting the etale topology of Grothendieck for the topology of Zariski. One gets the objects which M. Artin called "etale schemes" and the author called "minischemes". Later, M. Artin introduced the term "algebraic space" and I use that term in this report. One of the definitions of the algebraic space over the scheme S is the following :

One must take a pair of S-schemes X, Z where Z is a closed subscheme of $X \times_S X$, the canonical projections $p_1 : Z \to X$ and $p_2 : Z \to X$ are etale epimorphisms, and if $\sigma : X \times_S X \to X \times_S X$ is the automorphism which permutes the factors, $\gamma_{ij} : X \times_S X \times_S X \to X \times_S X$ is the canonical morphism corresponding to projections of $X \times_S X \times_S X$ on the *i-th* and *j-th* factors, then $\sigma^{-1}(Z) = Z$ and $\gamma_{13}^{-1}(Z) \supseteq \gamma_{12}^{-1}(Z) \cap \gamma_{23}^{-1}(Z)$.

Comparing to usual constructions, X corresponds to the set of local charts and Z corresponds to glueing of charts.

One can define on algebraic spaces almost all main notions of the theory of schemes, and particularly the notion of monoidal transformation [5], [15], [10]. The following is true [10] :

THEOREM. — *If V is an algebraic space of finite type over the noetherian scheme S, then there exists a monoidal transformation $\phi : V' \to V$ with nowhere dense center, such that V' is a quasiprojective S-scheme* [10].

In cases when the main theorems of resolution of singularities are true for schemes, one can prove the same theorems for algebraic spaces [10]. Moreover in these cases the following assertion is true for the algebraic space V :

There exists a finite sequence of monoidal transformations

$$\{\phi_i : V_{i+1} \to V_i,\ 0 \leqslant i < r,\ V_0 = V\}$$

such that the center of any ϕ_i is nonsingular and nowhere dense on V_i and that V_r is a quasiprojective scheme.

Analogous theorems are true for compact complex spaces X with

$$a.\dim X = \dim_C X$$

[12]. Hence one sees for example that, to identify nonsingular compact complex spaces X with $a.\dim X = \dim_C X$ with proper nonsingular algebraic spaces over C, it is sufficient to establish that the highdimensional generalization of the theorem of Castelnuovo-Enriques (on exceptional curves of the first kind), which gives the criterion of the contractibility of a subspace, preserving nonsingularity of the ambient space, is true in the category of nonsingular algebraic spaces.

With the help of an approximation theorem of M. Artin [6], the author has proved the generalization of the theorem of Castelnuovo-Enriques and simultaneously the fact that compact complex manifolds X with $a.\dim X = \dim_C X$ are algebraic spaces [11]. At the same time when M. Artin used his method more delicately, it has permitted him to prove general theorems about contractions in the category of algebraic spaces (formal possibility for a contraction is sufficient for its actual realization). Hence M. Artin concluded that all compact complex spaces X with $a.\dim X = \dim_C X$ are algebraic [7], [8].

The works of Kodaira show that there are not many compact complex analytic surfaces which are not algebraic. This gives a hope that in the high dimensional case there is a similar situation. Here is the simplest illustration when a certain non-global condition together with compactness involves the algebraicity :

Kodaira proved that a compact complex surface X is algebraic if there exists an irreducible curve C on X with $C^2 > 0$ [1]. The corresponding assertion is true for an n-dimensional irreducible compact complex space, but instead of existence of a curve C with $C^2 > 0$ one must require the existence of a complex subspace

Y on X such that the ideal of Y is locally principal and the normal bundle of Y in X is *positive*.

2. Canonical dimension.

Invariant $a.\dim X$ has an essential defect : it is not stable under deformations of the complex structure. It seems that the number which we call canonical dimension and note with $k.\dim X$ may be more successful. Let X be nonsingular. Then $k.\dim X$ is the transcendence degree of the field of meromorphic functions on X which are relations of elements of groups $H^0(X, \mathcal{O}[mK])$ for all $m > 0$ (here K is the canonical bundle on X). We shall indicate some arguments which make plausible the stability of $k.\dim X$ under deformation of complex structure of X.

First of all, one can show that this stability is a corollary of the following

Conjecture. Each plurigenus $P_m(X) = H^0(X, \mathcal{O}[mK])$ of the compact complex manifold X is stable under deformation of complex structure.

It is obvious that the conjecture is true for X with $\dim_C X = 1$, and if $\dim_C X = 2$ one can verify it by using the results of Kodaira about $P_m(X)$ when X is minimal [3], and about the stability of exceptional submanifolds under the deformations of complex structure [4]. (Cf. papers of S. Iitaka : Deformations of compact complex surfaces, I, II, in "Global Analysis, Papers in honor of K. Kodaira", Univ. of Tokyo Press and Princeton Univ. Press (1969), 267-272, and in *J. Math. Soc. Japan*, v. 22, No. 2, 1970, 247-261). It seems to us that it is possible to proceed similarly if $\dim_C X > 2$. We can remark finally that it is well known that if X is *kählerian* then $P_1(X) = p_g(X)$ is stable under local deformations.

3. Manifolds of general type.

Let us say that the compact complex manifold X is a manifold of general type if $k.\dim X = \dim_C X$. It is clear that each such manifold is an algebraic space. The conjecture of stability of the plurigenera has the consequence that an arbitrary deformation of a manifold of general type is again a manifold of general type. Let G be the class of all manifolds of general type, and G_d the part of G consisting of those with $\dim_C X = d$. One can prove that for any $X \in G$ there exists a number $c(X)$ such that the basis of the group

$$H^0(X, \mathcal{O}[c(X) K])$$

gives a birational embedding of X in a projective space. The important question is the existence of numbers $c(d)$ such that for any $X \in G_d$ the basis of the group $H^0(X, \mathcal{O}[c(d) K])$ gives a birational embedding in a projective space. It is well known that $c(1) = 3$. Some years ago the author proved that $c(2) \leqslant 9$ [13]. Then Kodaira by a precise study of the situation established that $c(2) \leqslant 6$ [3]. Recently Bombieri proved that $c(2) \leqslant 5$ [16]. If $d \geqslant 3$ the question is open. Some original interesting results gotten in the process of investigation of this question in highdimensional case belong to Tankeev [17].

4. The simplest classification of compact complex spaces.

The invariant $k.\dim X$ divides the algebraic curves into the classes of rational curves, elliptic curves and curves of general type. If $\dim_C X = 2$ one obtains by means of $k.\dim X$ the following classes of compact complex surfaces :

(i) surfaces with $k.\dim X \leqslant 0$;

(ii) surfaces of general type, that is with $k.\dim X = 2$;

(iii) surfaces with $k.\dim X = I$ which have a pencil of elliptic curves.

One can make a similar classification in the n-dimensional case :

Let us assume that the resolution theorems of Hironaka are true in the category of compact complex spaces. Then one can prove, using Grauert's results about direct image [18], that for an arbitrary compact complex space X there exists a meromorphic map $h : X \to Y$ such that Y is an algebraic space with $\dim Y = k.\dim X$, and that almost all fibers of h are irreducible compact complex spaces with $k.\dim \leqslant 0$ (one defines the canonical dimension of a singular space as the canonical dimension of its non-singular model).

One can see that all irreducible compact complex spaces are devided into three classes :

(1) spaces with $k.\dim \leqslant 0$;

(2) spaces of general type, that is with $k.\dim X = \dim_C X$;

(3) spaces with $0 < k.\dim X < \dim X$, anyone of which may be considered as a family of complex spaces such that the "general member" of the family is of $k.\dim \leqslant 0$ and the base of one is an algebraic space Y with

$$\dim_C Y = k.\dim X .$$

5. Kählerity and projectivity.

If X is a compact kählerian manifold, then X is projective algebraic [14]. Hence one sees that if $f : X \to T$ is a family of deformations of a projective algebraic complex manifold, $X_0 = f^{-1}(t_0)$, $t_0 \in T$, and if any X_t, $t \in T$, is such that $\dim_C X_t = a.\dim X_t$, then there exists a neighbourhood U of t_0 in T such that for any $t \in U$, X_t is projective algebraic.

It would be interesting to prove the corresponding algebraic assertion :

An arbitrary local deformation of a nonsingular projective algebraic variety in the category of algebraic spaces is projective.

The remark made above and the conjecture of the stability of plurigenera (§ 2) would imply that an arbitrary local deformation of a projective algebraic complex manifold of general type is projective algebraic.

In these assertions an important role is played by the condition of nonsingularity. Indeed, by using a well known example of Hironaka [9] (blowing-up of the projective plane in some points of an elliptic curve lying on it), one can construct a family of deformations of a singular projective algebraic variety, such that each member of the family except the original is a nonprojective algebraic space.

By means of this construction one can prove the following fact :

There exists a *kählerian* normal compact complex space with

$$a.\dim X = \dim_C X,$$

which is not a projective algebraic variety.

REFERENCES

[1] KODAIRA K. — On compact complex analytic surfaces, I, II, III, *Ann. Math.*, Vol. 71, 1960, p. 111-152; Vol. 77, 1963, p. 563-626; Vol. 78, 1963, p. 1-40.

[2] KODAIRA K. — On the structure of compact complex analytic surfaces, I, II, III, IV, *Amer. J. Math.*, Vol. 86, 1964, p. 751-798; Vol. 88, 1966, p. 682-721; Vol. 90, 1968, p 55-83, p 1048-1066

[3] KODAIRA K. — Pluricanonical systems on algebraic surfaces of general type, *J. Math. Soc. Japan*, 20, 1968, p. 170-192.

[4] KODAIRA K. — On stability of compact submanifolds of complex manifolds, *Amer. J. Math., Vol.* 85, 1963, p. 79-94.

[5] ARTIN M. — *The implicit function theorem in algebraic geometry, Algebraic Geometry*, Papers Presented at the Bombay Colloquium, Oxford, Bombay, 1969, p. 13-34.

[6] ARTIN M. — Algebraic approximation of structures over complete local rings, *Publ. Math. I.H.E.S.*, No. 36, 1969, p 23-58

[7] ARTIN M. — Algebraization of formal moduli : I, in *Global Analysis, Papers in Honor of K Kodaira*, Univ. of Tokio Press and Princeton Univ. Press, 1969, p. 21-71.

[8] ARTIN M. — Algebraization of formal moduli : II, Existence of modifications, *Ann. Math.*, Vol. 91, 1970, p. 88-135.

[9] ARTIN M. — Some numerical criteria for cotractability of curves on an algebraic surface, *Amer. J. Math.*, Vol. 84, 1962, p. 485-496.

[10] Мойшезон Б. Г. — Алгебраический аналог компактных комплексных пространств с достаточно большим полем мероморфных функций, I, II, III, Изв. АН СССР, сер. мат., т. 33, 1969, р. 174-238, р. 323-367, р. 506-548.

[11] Мойшезон Б. Г. — Теорема Кастельнуово-Энриквеса о стягивании для произвольной размерности, Изв. АН СССР, сер. мат., т. 33, 1969, р. 974-1025.

[12] Мойшезон Б. Г. — Георемы разрешения для компактных комплексных пространств с достаточно большим полем мероморфных функций, Изв. АН СССР, сер. мат., т. 31, 1967, р. 1385-1414.

[13] Мойшезон Б. Г. — О поверхностях основного типа, в кн. « Алгебраические поверхности », Труиды матем. ин-та им. В. А. Стеклова, т. Москва, 1965.

[14] Мойшезон Б. Г. — Об *n*-мерных компактных комплексных многообразиях, имеющих *n* алгебраически независимых мероморфных функций, I, II, III, Изв. АН СССР, сер. мат., т. 30, 1966, р. 133-174, р. 345-386, р. 621-656.

[15] KNUTSON D. — Algebraic spaces, *Thesis,* M.I.T., 1968, (to appear).
[16] BOMBIERI E. — *The pluricanonical map of a complex surface* (preprint).
[17] Танкеев С. — Об n-мерных канонически поляризованных многообразиях и многообразиях основного типа, Изв. АН СССР, сер. мат. (в печати).
[18] GRAUERT H. — Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, *Publ. Math. I.H.E.S.*, No. 5, 1960, p. 233-292

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FRACTIONS LIPSCHITZIENNES ET SATURATION DE ZARISKI DES ALGÈBRES ANALYTIQUES COMPLEXES

par Frédéric PHAM

(Exposé d'un travail fait avec Bernard TEISSIER [1])

On présente quelques remarques sur l'anneau des germes de fonctions méromorphes *continues au sens de Lipschitz,* c'est-à-dire vérifiant

$$|f(x) - f(x')| < K \, \|x - x'\|$$

sur un germe d'ensemble analytique complexe X.

0. Préliminaires sur les majorations de fonctions méromorphes.

Il n'est peut-être pas inutile de rappeler les notions suivantes (dues essentiellement, semble-t-il, à Zariski, et que nous tenons de H. Hironaka).

0.0 *Théorèmes de majoration.*

Pour tout idéal I dans un anneau (commutatif unitaire) A, on note $\bar{A}$ l'anneau normalisé, et

$$\bar{I} = \{ f \in \text{fract}\, A \mid \exists a_1 \in I,\, a_2 \in I^2, \ldots, a_k \in I^k,$$
$$f^k + a_1 f^{k-1} + a_2 f^{k-2} + \cdots + a_k = 0 \}$$

(où fract A désigne l'anneau total des fractions de A).

$\bar{I}$ est un idéal de $\bar{A}$, appelé *normalisé de l'idéal I.*

0.1 *Théorèmes de majoration.*

Soit A une *algèbre analytique complexe réduite.* On sait que $\bar{A}$ coïncide avec l'anneau des germes de fonctions méromorphes *bornées en module* sur le germe d'espace analytique X associé à A. On montre, de même, que si I est un idéal de A, engendré par $g_1, g_2, \ldots, g_p$ disons, *$\bar{I}$ coïncide avec l'idéal des germes de fonctions méromorphes f obéissant à une majoration du type*

$$|f(x)| < K \operatorname{Sup} \{ |g_1(x)|, |g_2(x)|, \ldots, |g_p(x)| \}$$

pour tout x dans un représentant assez petit du germe X (K est une constante pouvant dépendre de ce représentant ainsi que de f).

0.2 *Eclatements normalisés d'Idéaux.*

Sur un espace analytique *normal X,* on peut associer à tout Idéal (faisceau cohérent d'idéaux) $\mathfrak{I}$ un autre Idéal $\bar{\mathfrak{I}}$, le "*normalisé*" de $\mathfrak{I}$, défini par $(\bar{\mathfrak{I}})_x = (\overline{\mathfrak{I}_x})$

(au sens de 0.0). Alors, si $\bar{\omega} : \bar{Y} \xrightarrow{\nu} Y \xrightarrow{\omega} X$ désigne *l'éclatement normalisé* de l'Idéal $\mathfrak{J}$, c'est-à-dire le morphisme composé de l'éclatement ω et de la normalisation ν, on a la propriété suivante :

$$\boxed{\bar{\mathfrak{J}} = \bar{\omega}_*(\mathfrak{J}\mathcal{O}_{\bar{Y}})}$$

(où $\mathfrak{J}\mathcal{O}_{\bar{Y}}$ est une abréviation pour $\bar{\omega}^* \mathfrak{J} . \mathcal{O}_{\bar{Y}}$).

Cela signifie qu'*une fonction méromorphe f sur X est une section de $\bar{\mathfrak{J}}$ si et seulement si $f \circ \bar{\omega}$ est une section de $\mathfrak{J}\mathcal{O}_{\bar{Y}}$* .

L'intérêt de ce résultat tient au fait que $\mathfrak{J}\mathcal{O}_{\bar{Y}}$ est un *diviseur* de l'espace *normal* $\bar{Y}$, de sorte que pour vérifier qu'une fonction est une section de ce diviseur il suffit de le faire *sur un ouvert dense de chaque composante irréductible du support de ce diviseur.*

1. Saturation lipschitzienne.

1.1 *Définition algébrique* (*locale*).

Soient A une algèbre analytique complexe réduite, $\bar{A}$ sa normalisée. Désignons par I_A *l'idéal diagonal* de A dans $\bar{A} \underset{\mathbf{C}}{\otimes} \bar{A}$, c'est-à-dire le noyau du morphisme canonique

$$\bar{A} \underset{\mathbf{C}}{\otimes} \bar{A} \to \bar{A} \underset{A}{\otimes} \bar{A}$$

Explicitement, I_A est l'idéal engendré dans $\bar{A} \underset{\mathbf{C}}{\otimes} \bar{A}$ par

$$(z_1 \otimes 1 - 1 \otimes z_1 , z_2 \otimes 1 - 1 \otimes z_2 , \ldots , z_N \otimes 1 - 1 \otimes z_N) ,$$

où $(z_1, z_2, \ldots, z_N)$ désigne un système de générateurs de l'idéal maximal de A. Nous appellerons *saturée lipschitzienne* de A l'algèbre (locale et finie sur A, donc *analytique*)

$$\tilde{A} = \{f \in \bar{A} \mid f \otimes 1 - 1 \otimes f \in \bar{I}_A\}$$

où $\bar{I}_A$ est le normalisé de l'idéal I_A.

1.2 *Définition algébro-géométrique* (*globale*).

A tout espace analytique complexe réduit X on associera le diagramme (où $\bar{X}$ désigne le normalisé de X)

$$\begin{array}{ccc} D_X & \subset & E_X \\ \downarrow & & \downarrow \eta_X \\ \bar{X} \underset{X}{\times} \bar{X} & \subset & \bar{X} \times \bar{X} \end{array}$$

où η_X est *l'éclatement normalisé* de centre $\bar{X} \underset{X}{\times} \bar{X}$ et D_X le diviseur exceptionnel (*non réduit*) de cet éclatement normalisé : $D_X = \eta_X^{-1} (\bar{X} \underset{X}{\times} \bar{X})$.

THEOREME. — *Une fonction méromorphe f sur X est localement lipschitzienne si et seulement si* $(f \otimes 1 - 1 \otimes f) \mid D_X = 0$.

L'espace topologique $|X|$, muni du faisceau d'anneaux constitué par les germes de fonctions méromorphes lipschitziennes, est un espace analytique $\tilde{X}$ appelé *saturé lipschitzien* de X. Cet espace $\widetilde{X}$ est le *conoyau de la double flèche canonique* $D_X \rightrightarrows \overline{X}$.

Le morphisme canonique $\sigma : \widetilde{X} \to X$ (morphisme de "*saturation lipschitzienne*" est un homéomorphisme, et l'on a

$$\mathcal{O}_{\widetilde{X}, \tilde{x}} = (\mathcal{O}_{X, \sigma(\tilde{x})})^{\sim} \qquad \text{(au sens de 1.1)}$$

2. Couples de points infiniment voisins sur un espace analytique.

Nous introduisons ici un langage commode pour discuter la condition de Lipschitz

2.0. *Doublet* (ou *couple de points infiniment voisins*) sur X = point $\epsilon \in D_X^{\text{red}}$ (diviseur exceptionnel réduit de η_X).

• *Doublet de type* τ = point $\epsilon \in {}^{\tau}D_X^{\text{red}} - \bigcup_{\tau' \neq \tau} {}^{\tau'}D_X^{\text{red}}$, où τ est un indice numérotant les composantes irréductibles ${}^{\tau}D_X^{\text{red}}$ de D_X^{red}.

• *Multiplicité* du type τ = multiplicité du diviseur D_X en un point générique ϵ de type τ.

2.1 *Centre* du doublet ϵ = image de ϵ par le morphisme canonique $D_X^{\text{red}} \to X$

• *Lieu de confluence* du type τ = sous-espace analytique ${}^{\tau}X$ de X, image de ${}^{\tau}D_X^{\text{red}}$.

• *Types triviaux* = types des doublets ayant pour centre un point lisse de X (les lieux de confluence des types triviaux sont les composantes irréductibles de X).

2.2 *Approximations d'un doublet.*

Une suite de couples de points lisses de $X, (x_i \neq x_i')$, $i = 1, 2, 3, \ldots$ définit de façon unique une suite de points $e_i \in E_X - D_X^{\text{red}} \approx \overline{X} \times \overline{X} - |\overline{X} \underset{X}{\times} \overline{X}|$. On dira que la suite (x_i, x_i') *approche le doublet* ϵ si e_i tend dans E_X vers ϵ (cela implique évidemment que les deux points x_i, x_i' "confluent" vers le centre x de ϵ, d'où la terminologie "lieu de confluence" introduite dans 2.1).

2.3 Théorème (conséquence de 0.2)

La condition de Lipschitz n'a besoin d'être vérifiée qu'à "proximité" (au sens de 2.2) d'un doublet "générique" de chaque type. En particulier, il suffit de la vérifier au voisinage d'un point générique de chaque lieu de confluence.

2.4 *Direction d'un doublet.*

A tout plongement $z : X \to \mathbf{C}^N$ est associé canoniquement un diagramme commutatif de morphismes analytiques

$$\begin{array}{ccc} E_X & \xrightarrow{\tilde{z}} & G \\ & {\scriptstyle \text{dir}_z} \searrow & \downarrow {\scriptstyle \text{dir}} \\ & & \mathbf{P}^{N-1} \end{array}$$

où G est la variété (à $2N - 2$ dimensions) des droites affines de $\mathbf{C}^N$, "dir" l'application qui à chaque droite associe sa direction, et $\tilde{z}$ le morphisme qui à chaque couple de points distincts associe la droite qui les joint (en fait, pour voir que

$\tilde{z}$ est un morphisme analytique, on commence par construire dir_z, qui est bien défini sur l'éclatement – normalisé ou non – de l'idéal diagonal ; la connaissance de dir_z permet ensuite de construire $\tilde{z}$ comme application méromorphe et bornée, donc holomorphe sur l'espace *normal* E_X).

Pour $\epsilon \in D_X^{\mathrm{red}}$, $\mathrm{dir}_z(\epsilon)$ est appelée *direction du doublet* ϵ.

Remarque : *Il n'y a qu'un nombre fini de doublets de centre et direction donnés* (car la normalisation est un morphisme fini). Il en résulte que si l'on choisit le plongement z de façon que X ne contienne aucune droite, $\tilde{z}$ *est un morphisme fini.*

COROLLAIRE. – Si X est de dimension pure n, les lieux de confluence τ_X sont de dimensions $\geqslant 2n - N$.

3. Saturation des hypersurfaces.

Si X est une hypersurface, le corollaire 2.4 nous apprend que les lieux de confluence non triviaux sont les composantes irréductibles de codimension 1 (que nous noterons S) du lieu singulier. Nous choisirons dans $\mathbf{C}^{n+1}$ une direction de plan π et dans ce plan une direction de droite u telles que pour presque tout point lisse s de chacun des S

(1) le plan parallèle à π passant par s (noté π) soit transverse à S,

(2) la direction u soit transverse au germe de courbe singulière $C_s = \pi_s \cap X$.

LEMME. – *Alors, pour presque tout* $s \in S$, *tout doublet de centre* s *et de direction* u *peut être approché par des couples de points de* C_s *joints par des droites parallèles à* u.

COROLLAIRE (conséquence de 2.3). – Pour vérifier la condition de Lipschitz $|f(x) - f(x')| < K \|x - x'\|$ au voisinage d'un point $x_0 \in X$, il suffit de le faire pour tous les couples $x \in C_s$, $x' \in C_s$ tels que $\mathrm{dir}_z(x, x') = u$, en faisant parcourir à s un ouvert dense de chaque S adhérent à x_0.

Preuve du lemme. – Une simple considération de dimensions, jointe à la remarque 2.4, montre que D_X^{red} (ou plus exactement l'union des ${}^{\tau}D_X^{\mathrm{red}}$ ayant S comme lieu de confluence) est égal à $|\tilde{z}^{-1}(F)|$, où F désigne l'hypersurface de G formée des droites affines de $\mathbf{C}^{n+1}$ passant par S. En se restreignant au voisinage d'un ouvert dense de S on peut supposer que S est lisse et que la famille de courbes $(C_s)_{s \in S}$ est *équisingulière* [2].

(*) *Alors le lieu de ramification du morphisme fini* $\tilde{z}$ *est l'hypersurface lisse* F : en effet, par l'hypothèse d'équisingularité, les droites affines proches de u_s (la droite de direction u passant par s) ne sont jamais tangentes à X, c'est-à-dire qu'à moins de passer par S elles doivent couper X en m points distincts (m = multiplicité de X le long de S).

Il résulte de (*) que *tout champ de vecteurs holomorphe dans* G *tangent à* F *se relève de façon unique, par le morphisme fini* $\tilde{z}$, *en un champ de vecteurs holomorphe dans l'espace normal* E_X.

E_X hérite ainsi de la structure locale de produit de

$$G \overset{\text{loc}}{\approx} F \times (\pi_s/u)$$

(où π_s/u est la sous-variété à une dimension de G, ensemble des droites de π_s parallèles à u), ce qui donne

$$E_X \overset{\text{loc}}{\approx} D_X^{\text{red}} \times \widetilde{z}^{-1}(\pi_s/u)$$

prouvant ainsi le lemme.

(*Remarque* : Comme E_X est normal, donc non singulier en codimension 1, il résulte de la structure de produit ci-dessus que $\widetilde{z}^{-1}(\pi/u)$ est une *courbe lisse*).

4. Equisaturation lipschitzienne et trivialite topologique.

Soit $\pi : X \to T$ un germe de morphisme d'un germe d'espace analytique complexe réduit X sur un germe de variété analytique complexe T. Nous dirons que X est (lipschitziennement) *équisaturé* au-dessus de T s'il existe un germe d'isomorphisme α rendant commutatif le diagramme

$$\begin{array}{ccc} \widetilde{X} & \overset{\alpha}{\to} & \widetilde{X}_0 \times T \\ & \searrow_{\widetilde{\pi}} \quad \swarrow_{2^\circ \text{ proj.}} & \\ & T & \end{array}$$

où $X = \pi^{-1}(0)$, $0 \in T$, $\widetilde{X}_0$ = saturé de X_0, $\widetilde{\pi}$ = composé de π avec le morphisme de saturation.

THEOREME. – *Si X est équisaturé au-dessus de T, π est localement une fibration topologique* (lipschitzienne).

En effet, un champ de vecteurs "constant", parallèle à T, de $\widetilde{X}_0 \times T$ définit une dérivation de $\widetilde{A}$, l'algèbre analytique associée au germe $\widetilde{X}$. Par restriction à A, on obtient une dérivation de A dans $\widetilde{A}$, donc un champ de vecteurs lipschitzien tangent à X. D'après un théorème classique d'extension des fonctions lipschitziennes [3], ce champ de vecteurs s'étend, dans l'espace ambiant $\mathbf{C}^N \supset X$, en un champ de vecteurs lui aussi lipschitzien, *donc localement intégrable*, dont l'intégration réalise la trivialité topologique (et même lipschitzienne) de la paire $(\mathbf{C}^N, X)$ au-dessus de T.

5. Conclusion.

Les recherches de Zariski sur le problème de l'*équisingularité* l'ont amené à définir une opération de *saturation* des anneaux locaux [4]. A l'aide de notre § 3 ci-dessus (Corollaire), il n'est pas difficile de voir que la *saturation lipschitzienne coïncide avec la saturation de Zariski dans le cas des hypersurfaces*, ce qui nous permet, avec une définition plus simple que celle de Zariski (Def. 1.1), de retrouver ses principaux résultats, qui étaient les suivants :

(1) l'équisaturation d'une famille d'hypersurfaces implique la trivialité topologique de la famille ;

(2) l'équisaturation (c'est-à-dire l'isomorphisme des anneaux saturés) de deux germes de courbes planes *équivaut* à leur "*équisingularité*" (c'est-à-dire dans le cas irréductible, à l'égalité de leurs exposants caractéristiques de Puiseux).

Notons que dans le cas des courbes planes, le point de vue esquissé au § 2 fournit une définition intrinsèque (indépendante de tout choix de coordonnée) des *exposants caractéristiques de Puiseux* : ceux-ci apparaissent comme les exposants fractionnaires obtenus en divisant par m (multiplicité de la courbe) les "multiplicités" (au sens de 2.0) des différents "types" de doublets (démonstration très simple à partir des résultats du § 3).

REFERENCES

[1] PHAM F et TEISSIER B. — *Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski.* Centre de Mathématiques de l'Ecole Polytechnique, (Paris), Juin 1969.

[2] ZARISKI O. — Studies in Equisingularity I. Equivalent Singularities of plane algebroïd Curves. *Amer. J. Math.*, 87, 1965.

[3] BANACH S. — Introduction à la théorie des fonctions réelles. (en polonais) *Monografie Matematyczne,* Tom XVII, Polskie Towarzystwo Matematyczne, Warszawa-Wroclaw 1951 (cette référence m'a été communiquée par S. Lojasiewicz).

[4] ZARISKI O. — Studies in Equisingularity III. Saturation of local rings and equisingularity. *Amer. J. Math.*, 90, 1968.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 655 à 661.

D9 - ENSEMBLES EXCEPTIONNELS EN ANALYSE

ALMOST EVERYWHERE CONVERGENCE OF WALSH-FOURIER SERIES OF L^2 FUNCTIONS

by Richard A. HUNT *

The object of this paper is to provide some insight for the problem of a.e. convergence of Fourier series by pinpointing the central ideas of Carleson's L^2 proof. (See Carleson [2] and Hunt [3]). To accomplish this we will adapt Carleson's proof to the analogous result for Walsh-Fourier series. (See Billard [1], Sjölin [6], and Hunt and Taibleson [4]). The sharpness and relative simplicity of the Walsh case allows us to see the general form and essential points of Carleson's argument without many of the purely technical difficulties of the trigonometric case.

Given $f \in L^2$, $y > 0$, and a positive integer N we will define a set $E = E(f, y, N)$ such that

(1) $mE \leqslant C\, y^{-2} \int_0^1 |f(x)|^2\, dx$ and $x \in (0,1) - E$, $0 < n < 2^N$ implies

$$|S_n f(x)| \leqslant Cy ,$$

where $S_n f$ denotes the n^{th} partial sum of the Walsh-Fourier series of f and C is an absolute constant.

(1) implies the mapping $f \to Mf(x) = \sup\{|S_n f(x)| : n > 0\}$ is of weak type (2,2) and this implies $S_n f \to f$ a.e. for all $f \in L^2$.

The proof of (1) involves only elementary properties of Walsh functions, L^p boundedness of the mapping $f \to S_n f$, n fixed, and the Hardy-Littlewood maximal function. Let us collect these facts along with some notation.

w_n will denote the n^{th} Walsh function, $n \geqslant 0$. $w_n(x \dotplus t) = w_n(x)\, w_n(t)$, where $x \dotplus t$ denotes addition in the Walsh group 2^{ω}. ω will denote a dyadic subinterval of $(0\,,\,1)$ and ω^* will denote the unique dyadic interval with $\omega^* \supset \omega$ and $|\omega^*| = 2\,|\omega|$. If $|\omega| = 2^{-\mu}$ and $n = \sum_{j=0}^{\infty} \epsilon_j 2^j$, $\epsilon_j = 0$ or 1, we write $n(\omega) = \sum_{j=\mu}^{\infty} \epsilon_j 2^j$. $w_n = w_{n(\omega)} \cdot w_s$, where $0 \leqslant s = n - n(\omega) < 2^{\mu} = |\omega|^{-1}$. $0 \leqslant s < |\omega|^{-1}$ implies w_s is identically ± 1 on ω.

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We will use some properties of the modified Dirichlet kernel $D_n^* = w_n D_n$, $D_n = \sum_{j=0}^{n-1} w_j$. If $n = \sum_{j=0}^{\infty} \epsilon_j 2^j$, $\epsilon_j = 0$ or 1, $D_n^*(t) = \sum_{j=0}^{\infty} \epsilon_j \delta_j^*(t)$, where

$$\delta_j^* = \sum_{\nu=2^j}^{2^{j+1}-1} w_\nu = w_{2^j} D_{2^j} .$$

For fixed x, $\delta_j^*(x \dotplus t)$ is constant for $t \in \omega$ unless $x \in \omega$ and $|\omega| \geqslant 2^{-j}$. Let $T_n f(x) = \int_0^1 f(t)\, D_n^*(x \dotplus t)\, dt$. Since $T_n f(x) = w_n(x)\, S_n(w_n f)\,(x)$, Paley's theorem (see Paley [5]) yields

(2) $\|T_n f\|_p \leqslant C_p \|f\|_p, \quad 1 < p < \infty$.

Let $Hf(x) = \sup\{|S_{2^n} f(x)| : n \geqslant 0\}$. Since

$$Hf(x) = \sup\left\{ \| \omega|^{-1} \int_\omega f(t)\, dt \,| \; : \; x \in \omega \right\} ,$$

Hf is majorized by the Hardy-Littlewood maximal function of f, so

(3) $\| Hf \|_p \leqslant C_p \|f\|_p \; ; \; 1 < p \leqslant \infty$.

Let $a_n(\omega) = a_n(\omega\,; f) = |\omega|^{-1} \int_\omega f(t)\, w_n(t)\, dt$. Note $a_{n(\omega)}(\omega) = \pm\, a_n(\omega)$. We will use the numbers $A_n(\omega) = \max\{|a_n(\bar\omega)| : \bar\omega^* = \omega\}$. $\{w_{n|\omega|^{-1}}\}_{n=0}^{\infty}$ is a complete orthonormal set on ω with respect to the measure $|\omega|^{-1}\, dx$. Plancherel's formula on ω is $\sum_{n=0}^{\infty} |a_{n|\omega|^{-1}}(\omega\,; f)|^2\, |\omega| = \int_\omega |f(x)|^2\, dx$.

The main idea in the proof of (1) is to estimate $S_n f(x)$ by using predetermined estimates of certain selected sums of the form $S_{n(\omega)} f(x) - S_{n(\bar\omega)} f(x)$. We will have "enough" selected sums to write $S_n f(x) = \sum_{j=0}^{\mu} S_{n(\omega_j)} f(x) - S_{n(\omega_{j+1})}\, f(x)$, where each term $S_{n(\omega_j)} f(x) - S_{n(\omega_{j+1})} f(x)$ is the *difference* of two selected sums. In order that we have appropriate estimates of our selected sums we must avoid certain x for each selected sum. This means we cannot have "too many" selected sums.

To determine our selected sums we define a collection $G^* = \bigcup_{k=1}^{\infty} G_k^*$ of selected pairs $(n(\omega)\,, \omega)$. For each pair $(n(\omega)\,, \omega) \in G_k^*$ we define a partition

$$\Omega = \Omega(n(\omega)\,, \omega\,, k) \quad \text{of} \quad \omega\,.$$

The partition Ω determines intervals $\bar\omega$ for which $S_{n(\omega)} f(x) - S_{n(\bar\omega)} f(x)$ is a selected sum. Also, Ω is used to obtain appropriate estimates of selected sums. The selection of pairs G^*, the partitions Ω and the corresponding estimates are coordinated by use of the numbers $A_n(\omega)$.

The numbers $A_n(\omega)$ are controlled by the set

$$S^* = \cup \left\{ \omega^* : \int_\omega |f(t)|^2\, dt \geqslant y^2\, |\omega| \right\} .$$

We have

(4) $\omega \not\subset S^*$ implies $A_n(\omega) < y$ for all n.

Since we can write S^* as a union of disjoint intervals ω_j^* with

$$\int_{\omega_j} |f(t)|^2\, dt \geqslant y^2\, |\omega_j|$$

for all j, we have

(5) $mS^* = \sum_j |\omega_j^*| = 2 \sum_j |\omega_j| \leqslant 2\, y^{-2} \int_0^1 |f(x)|^2\, dx.$

For each positive integer k let G_k be the collection of all pairs $(n(\omega), \omega)$ such that

(6) $|a_{n(\omega)}(\omega)| \geqslant 2^{-k} y$ and

(7) $|\omega| = 1/2$ and $|a_{n(\omega)}(\omega)| < 2^{-k+1} y$

or

$2^{-N} \leqslant |\omega| < 1/2$ and $|a_{\bar{n}}(\bar{\omega})| < 2^{-k} y$ for all $\bar{n}, \bar{\omega}$ with $\bar{n}(\omega) = n(\omega)$ and $\bar{\omega} \supsetneq \omega$, $|\bar{\omega}| \leqslant 1/2$.

Let
$$P_k(x\,;\omega) = \sum_{\substack{(n(\bar{\omega}),\, \bar{\omega}) \in G_k \\ \bar{\omega} \supset \omega}} a_{n(\bar{\omega})}(\bar{\omega})\, w_{n(\bar{\omega})}(x)\ .$$

Note that if $(n(\omega), \omega) \in G_k$ then $P_k(x\,;\omega^*)$ contains no term $a_m w_m$ with $m(\omega) = n(\omega)$, so $\int_\omega P_k(x\,;\omega^*)\, w_{n(\omega)}(x)\, dx = 0$. Hence, $(n(\omega)\, ,\, \omega) \in G_k$ implies $a_{n(\omega)}(\omega) = a_{n(\omega)}(\omega\,;f) = a_{n(\omega)}(\omega\,; f - P_k(\cdot\, , \omega^*))$. Plancherel's formula then implies

$$\int_\omega |f(x) - P_k(x\, , \omega)|^2\, dx =$$

$$= \int_\omega |f(x) - P_k(x\, , \omega^*) - \sum_{(n(\omega),\, \omega) \in G_k} a_{n(\omega)}(\omega)\, w_{n(\omega)}(x)|^2\, dx$$

$$= \int_\omega |f(x) - P_k(x\, , \omega^*)|^2\, dx - \sum_{(n(\omega),\, \omega) \in G_k} |a_{n(\omega)}(\omega)|^2\, |\omega|\ .$$

Since

$$\sum_{\substack{\omega \\ |\omega| = 2^{-n}}} \int_\omega |f(x) - P_k(x\,;\omega^*)|^2\, dx = \sum_{\substack{\omega \\ |\omega| = 2^{-n+1}}} \int_\omega |f(x) - P_k(x\,;\omega)|^2\, dx\ ,$$

we can repeatedly apply Plancherel's formula to obtain

$$0 \leqslant \sum_{\substack{\omega \\ |\omega| = 2^{-N}}} \int_\omega |f(x) - P_k(x\, , \omega)|^2\, dx =$$

$$\int_0^1 |f(x)|^2\, dx - \sum_{(n(\omega),\, \omega) \in G_k} |a_{n(\omega)}(\omega)|^2\, |\omega|\ .$$

(6) then implies

(8) $\sum_{(n(\omega),\,\omega)\in G_k} |\omega| \leqslant 2^{2k} y^{-2} \int_0^1 |f(x)|^2\,dx.$

Let G_k^* be the collection of all pairs $(n\,,\,\omega^*)$ such that

$$n = n(\omega^*) \quad \text{and} \quad (n(\omega)\,,\,\omega) \in G_k\,.$$

Note that for each pair $(n(\omega)\,,\,\omega) \in G_k$ there correspond two pairs, $(n(\omega), \omega^*)$ and $(n(\omega) + |\omega^*|^{-1}, \omega^*)$, in G_k^*. Since $|\omega^*| = 2|\omega|$, (8) yields

(9) $\sum_{(n,\,\omega)\in G_k^*} |\omega| \leqslant 4 \cdot 2^{2k} y^{-2} \int_0^1 |f(x)|^2\,dx.$

(9) tells us G_k^* does not contain "too many" pairs. The next three observations tell us that collection $G^* = \bigcup_{k=1}^{\infty} G_k^*$ contains "enough" pairs.

(10) $2^{-\tilde{k}} y \leqslant A_{n(\omega)}(\omega) < 2^{-\tilde{k}+1} y$, $\omega = (0,1)$ implies $(n(\omega)\,,\,\omega) \in G_{\tilde{k}}^*$.

(11) If $2^{-k} y \leqslant A_{n(\omega)}(\omega)$ and $\omega \not\subset S^*$ then there exist $(\tilde{n}\,,\,\tilde{\omega}\,,\,\tilde{k})$ with $\tilde{n}(\hat{\omega}) = n(\hat{\omega})$, $2\,|\hat{\omega}| = |\omega|$, $\tilde{\omega} \supset \omega$, $1 \leqslant \tilde{k} \leqslant k$, and $(\tilde{n}(\tilde{\omega})\,,\,\tilde{\omega}) \in G_{\tilde{k}}^*$.

(10) is obvious. To see (11) note that if $(n(\omega)\,,\,\omega) \notin G_k^*$, $\omega \subsetneqq (0,1)$, then there must be $\bar{n}$, $\bar{\omega}$ with

$$\bar{n}(\hat{\omega}) = n(\hat{\omega}),\ \bar{\omega} \supsetneqq \omega, \quad \text{and} \quad A_{\bar{n}}(\bar{\omega}) \geqslant 2^{-k} y\,.$$

If $(\bar{n}(\bar{\omega})\,,\,\bar{\omega}) \notin G_k^*$ we repeat the argument until we find $(\bar{n}(\bar{\omega})\,,\,\bar{\omega}) \in G_k^*$ or reach $\bar{\omega} = (0,1)$. When $\bar{\omega} = (0,1)$ we can apply (10). Note that $\omega \not\subset S^*$ implies $\tilde{k} \geqslant 1$.

If we choose $(\tilde{n}\,,\,\tilde{\omega}\,,\,\tilde{k})$ in (11) such that $\tilde{k}$ is minimal we have

(12) $A_{\tilde{n}(\bar{\omega})}(\bar{\omega}) < 2^{-\tilde{k}+1} y$ for all $\omega \subset \bar{\omega} \subset \tilde{\omega}$.

If (12) were not true we could apply (11) to $(\tilde{n}(\bar{\omega})\,,\,\bar{\omega}\,,\,\tilde{k} - 1)$ in place of $(n(\omega)\,,\,\omega\,,\,k)$ and contradict the fact that $\tilde{k}$ is minimal.

Since $a_{n(\omega)}(\omega) = a_{n(\omega)}(\omega^*) \pm a_{n(\omega)+|\omega^*|^{-1}}(\omega^*)$, (7) implies $A_{n(\omega)}(\omega) < 2^{-k+1} y$ if $(n(\omega)\,,\,\omega) \in G_k^*$ and $|\omega| < 1$. Note that $(n, (0,1))$ might be in two sets $G_{k_1}^*$ and $G_{k_2}^*$, $k_1 < k_2$. In that case delete $(n\,, (0,1))$ from $G_{k_2}^*$. Then

(13) $(n(\omega)\,,\,\omega) \in G_k^*$ implies $A_{n(\omega)}(\omega) < 2^{-k+1} y$.

(13) insures (14), (15) and (16) below yield a collection Ω of disjoint intervals ω' with $\omega = \cup\,\omega'$, $\omega' \in \Omega$.

If $(n(\omega)\,,\,\omega) \in G_k^*$, choose $\omega' \in \Omega = \Omega(n(\omega)\,,\,\omega\,,\,k)$ if

(14) $\omega' \subsetneqq \omega$, $|\omega'| \geqslant 2^{-N}$,

(15) $|a_{n(\omega)}(\bar{\omega})| < 2^{-k+1} y$ for all $\omega \supsetneqq \bar{\omega} \supset \omega'$, and

(16) $|\omega'| > 2^{-N}$ and $A_{n(\omega)}(\omega') \geqslant 2^{-k+1} y$ or $|\omega'| = 2^{-N}$.

(Constructively, starting with $\bar{\omega} = \omega$ we split $\bar{\omega}$ in half if $A_{n(\omega)}(\bar{\omega}) < 2^{-k+1} y$ and $|\bar{\omega}| > 2^{-N}$. Otherwise, $\bar{\omega} \in \Omega$).

For each pair $(n(\omega)\,,\,\omega) \in G_k^*$ the corresponding selected sums are those sums $S_{n(\omega)}f(x) - S_{n(\bar\omega)}f(x)$, where $x \in \bar\omega$, $\omega' \subset \bar\omega \subset \omega$, $\omega' \in \Omega$. (Hence, $\omega'' \in \Omega$ implies $\omega'' \cap \bar\omega = \omega''$ or Φ).

Suppose $S_{n(\omega)}f(x) - S_{n(\bar\omega)}f(x)$ is a selected sum which corresponds to

$$(n(\omega), \omega) \in G_k^* .$$

If $n(\omega) = \sum_{j=\mu}^{\infty} \epsilon_j 2^j$, $\epsilon_j = 0$ or 1, and $n(\bar\omega) = \sum_{j=\bar\mu}^{\infty} \epsilon_j 2^j$, $\bar\mu > \mu$, then

$$S_{n(\omega)}f(x) - S_{n(\bar\omega)}f(x) = \sum_{j=\mu}^{\bar\mu-1} \epsilon_j \int_0^1 f(t)\, w_{n(\omega)}(x \dotplus t)\, \delta_j^*(x \dotplus t)\, dt ,$$

since $w_{n(\bar\omega)}(t)\, \delta_j^*(t) = w_{n(\omega)}(t)\, \delta_j^*(t)$ for $j \geqslant \bar\mu$. Define $g(t) = g(t, \Omega)$ by setting $g(t) = a_{n(\omega)}(\omega')$ for $t \in \omega'$, $\omega' \in \Omega$ and setting $g(t) = 0$ for $t \notin \omega$. Note (15) implies

(17) $|g(t)| < 2^{-k+1} y$, $t \in \omega$.

For fixed $x \in \bar\omega$, $\delta_j^*(x \dotplus t)$ is constant for $t \in \bar\omega$, $j < \bar\mu$. Hence, $\delta_j^*(x \dotplus t)$ is constant for $t \in \omega'$, where $x \in \omega'$, $\omega' \in \Omega$, $\omega' \subset \bar\omega$, $j < \bar\mu$. Since $\delta_j^*(x \dotplus t)$ is also constant for t in any interval $\omega'' \in \Omega$, with $x \notin \omega''$, and $\delta_j^*(x \dotplus t) = 0$, $t \notin \omega$, $j \geqslant \mu$, we have

$$S_{n(\omega)}f(x) - S_{n(\bar\omega)}f(x) = w_{n(\bar\omega)}(x) \sum_{j=\mu}^{\bar\mu-1} \epsilon_j \int_0^1 g(t)\, \delta_j^*(x \dotplus t)\, dt .$$

Note that $\sum_{j=\mu}^{\bar\mu-1} \epsilon_j \int_0^1 g(t)\, \delta_j^*(x \dotplus t)\, dt = S_{2^{\bar\mu}}(T_{n(\omega)}g)(x)$ and, hence,

$$|S_{n(\omega)}f(x) - S_{n(\bar\omega)}f(x)| \leqslant H(T_{n(\omega)}g)(x) .$$

Let $U^* = U^*(n(\omega), \omega, k) = \{x \in (0,1) : H(T_{n(\omega)}g)(x) > 2^{-k/2} y\}$. Noting (17) and applying (2) and (3) with, say, $p = 6$, we obtain

(18) $mU^* \leqslant (2^{-k/2} y)^{-6}\, \|H(T_{n(\omega)}g)\|_6^6 \leqslant C \cdot 2^{-3k}\, |\omega|$.

If $x \in \bar\omega$, $\omega' \subset \bar\omega \subset \omega$, $\omega' \in \Omega$, and $x \notin U^*$, then

(19) $|S_{n(\omega)}f(x) - S_{n(\bar\omega)}f(x)| \leqslant 2^{-k/2} y$.

We can now assemble the various parts of the proof.

Let $E = S^* \cup \bigcup_{k=1}^{\infty} \bigcup_{(n(\omega), \omega) \in G_k^*} U^*(n(\omega), \omega, k)$. From (5), (9) and (18) we obtain $mE \leqslant (2 + C \sum_{k=1}^{\infty} 4 \cdot 2^{-k})\, y^{-2} \int_0^1 |f(x)|^2\, dx$.

Consider $S_n f(x)$, $0 < n < 2^N$, $x \notin E$. We may assume $A_n(\omega_0) > 0$, $\omega_0 = (0,1)$. ($S_n \equiv S_m$ for some m with $n \leqslant m < 2^N$ and $A_m(\omega_0) \neq 0$ or $S_n \equiv S_{2^N}$). Then

$$2^{-\tilde k_0} y \leqslant A_n(\omega_0) < 2^{-\tilde k_0 + 1} y$$

for some $\tilde k_0 \geqslant 1$, so $(n, \omega_0) \in G_{\tilde k_0}^*$ by (10) and the partition $\Omega_0 = \Omega(n, \omega_0, \tilde k_0)$ is defined. Let ω_1 be the interval of Ω_0 which contains x. (19) implies

$$|S_n f(x) - S_{n(\omega_1)} f(x)| \leqslant 2^{-\tilde{k}_0/2} y .$$

Note $\omega_1 \subsetneqq \omega_0$. If $n(\omega_1) = 0$, we stop. If $n(\omega_1) \neq 0$, we continue with a typical step.

$n(\omega_1) \neq 0$ implies $|\omega_1| > 2^{-N}$ and so (16) implies

$$2^{-k_1} y \leqslant A_{n(\omega_1)}(\omega_1) < 2^{-k_1+1} y \quad \text{for some} \quad 1 \leqslant k_1 < \widetilde{k}_0 .$$

Choose $(\widetilde{n}_1, \widetilde{\omega}_1, \widetilde{k}_1)$ as in (11) and (12). Then $\Omega_1 = \Omega(\widetilde{n}_1(\widetilde{\omega}_1), \widetilde{\omega}_1, \widetilde{k}_1)$ is defined. Let ω_2 be the interval in Ω_1 which contains x. (12) and the definition of Ω_1 imply $\omega_2 \subsetneqq \omega_1$. Note $\tilde{n}_1(\hat{\omega}_1) = n(\hat{\omega}_1)$ and then $\omega_2 \subsetneqq \omega_1$ implies

$$\widetilde{n}_1(\omega_2) = n(\omega_2) .$$

Applying (19) twice, we obtain

$$|S_{n(\omega_1)} f(x) - S_{n(\omega_2)} f(x)| \leqslant |S_{n(\omega_1)} f(x) - S_{n(\hat{\omega}_1)} f(x)|$$

$$+ |S_{\tilde{n}_1(\tilde{\omega}_1)} f(x) - S_{n(\hat{\omega}_1)} f(x)| + |S_{\tilde{n}_1(\tilde{\omega}_1)} f(x) - S_{n(\omega_2)} f(x)| \leqslant A_n(\omega_1) + 2.2^{-\tilde{k}_1/2} y .$$

If $n(\omega_2) \neq 0$, we repeat the above typical step until we obtain

$$(0,1) = \omega_0 \supsetneqq \omega_1 \supsetneqq \cdots \supsetneqq \omega_{\mu+1} , \quad \widetilde{k}_0 > \widetilde{k}_1 > \cdots > \widetilde{k}_\mu \geqslant 1$$

with $n(\omega_j) \neq 0$, $j = 1, \ldots, \mu$, $n(\omega_{\mu+1}) = 0$ and

$$|S_{n(\omega_j)} f(x) - S_{n(\omega_{j+1})} f(x)| \leqslant 4 \cdot 2^{-\tilde{k}_j/2} y , \; j = 0, \ldots, \mu .$$

Then

$$|S_n f(x)| = |\sum_{j=0}^{\mu} S_{n(\omega_j)} f(x) - S_{n(\omega_{j+1})} f(x)| \leqslant 4 \cdot \Big(\sum_{k=1}^{\infty} 2^{-k/2}\Big) y$$

and (1) is proved.

REFERENCES

[1] Billard P. — Sur la convergence presque partout des séries de Fourier Walsh des fonctions de l'espace L^2 (0,1), *Studia Math.*, 28, 1967, p. 363-388.

[2] Carleson L. — On convergence and growth of partial sums of Fourier series, *Acta Math.*, 116, 1966, p. 135-157.

[3] Hunt R.A. — On the convergence of Fourier series, *Orthogonal expansions and their continuous analogues,* SIU Press, Carbondale, Illinois, 1968.

[4] Hunt R.A. and Taibleson M.H. — On the almost everywhere convergence of Fourier series on the ring of integers of a local field, *Notices Amer. Math. Soc.*, 15, 1968, p. 550.

[5] Paley R.E.A.C. — A remarkable series of orthogonal functions (I), *Proc. London Math. Soc.*, 34, 1932, p. 241-264.

[6] SJOLIN P. — An inequality of Paley and convergence of Walsh-Fourier series, *Arkiv för Mat.*, 7, 1968, p. 551-570.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 663 à 665.

NOMBRES DE PISOT ET ANALYSE HARMONIQUE

par Yves MEYER

1. Notations.

On désigne par $L^\infty(\mathbf{R})$ l'espace de Banach des fonctions $\varphi : \mathbf{R} \to \mathbf{C}$ mesurables et essentiellement bornées sur la droite réelle ; $\|\varphi\|_\infty = \sup_{t\in\mathbf{R}} \text{ess.}\ |\varphi(t)|$.

Le spectre de φ est le support de la distribution $\hat{\varphi}$, transformée de Fourier de φ au sens des distributions.

Un polynome trigonométrique est une somme finie $P(t) = \sum_{\lambda\in F} a_\lambda \exp 2\pi i\lambda t$; F est un ensemble fini de nombres réels et les $\lambda \in F$ tels que $a_\lambda \neq 0$ sont les fréquences de P.

Soit $\theta > 2$ un nombre réel, E_θ l'ensemble, du type Cantor, à rapport de dissection $1/\theta$; E_θ est l'ensemble de toutes les sommes $\sum_1^\infty \epsilon_k \theta^{-k}$, $\epsilon_k = 0$ ou 1.

2. Les nombres de Pisot et l'unicité.

Raphaël Salem a prouvé en 1955 que si θ est un nombre de Pisot et φ un élément de $L^\infty(\mathbf{R})$ dont le spectre est contenu dans E_θ, alors on a l'implication suivante

$$\lim_{|t|\to+\infty} \varphi(t) = 0 \Rightarrow \varphi = 0 \quad \text{identiquement.}$$

3. Les nombres de Pisot et la synthèse harmonique.

Le théorème suivant permet de décrire complètement l'espace des éléments φ de $L^\infty(\mathbf{R})$ dont le spectre est contenu dans E_θ à l'aide du sous-espace des sommes trigonométriques finies dont les fréquences appartiennent à E_θ ; ceci lorsque θ est un nombre de Pisot.

THEOREME 1. – *Soit $\theta > 2$ un nombre de Pisot. A toute fonction $\varphi \in L^\infty(\mathbf{R})$ dont le spectre est contenu dans E on peut associer une suite $(\varphi_k)_{k\geqslant 1}$ de sommes trigonométriques finies ayant les propriétés suivantes*

a) *les fréquences de φ_k appartiennent à E*

b) *φ_k dépend linéairement de φ*

c) *il existe une constante $C(\theta)$, ne dépendant que de θ, telle que*

$$\|\varphi_k\|_\infty \leqslant C(\theta)\ \|\varphi\|_\infty$$

(d) $\varphi_k(t) \to \varphi(t)$, $k \to +\infty$, *uniformément sur tout ensemble compact de nombres réels t.*

Le théorème 1 exprime de façon très précise que E_θ est un ensemble de synthèse harmonique lorsque θ est un nombre de Pisot.

La preuve du théorème 1 s'étend sur les § 4, 5 et 6.

4. Ensembles harmonieux dans un groupe abélien localement compact.

Soit G un groupe abélien localement compact, $\mathrm{Hom}(G, \mathbf{T})$ le groupe des homomorphismes continus de G dans le groupe $\mathbf{T}$ des nombres complexes de module 1. Soit G_d le groupe G muni de la topologie discrete et $\mathrm{Hom}(G_d, \mathbf{T})$ le groupe des homomorphismes de G_d dans $\mathbf{T}$. On a évidemment l'inclusion

$$\mathrm{Hom}(G, \mathbf{T}) \subset \mathrm{Hom}(G_d, \mathbf{T})$$

Une partie Λ *de* G *est un ensemble harmonieux si pour tout* $\epsilon > 0$ *et tout élément* $\chi \in \mathrm{Hom}(G_d, \mathbf{T})$, *on peut associer un élément* $\chi' \in \mathrm{Hom}(G, \mathbf{T})$ *tel que* $\sup_{\lambda \in \Lambda} |\chi(\lambda) - \chi'(\lambda)| \leqslant \epsilon$.

Exemples. — Soit $\theta > 1$, Λ l'ensemble des puissances θ^k, $k \geqslant 0$, de θ et $G = \mathbf{R}$. Alors Λ est harmonieux si et seulement si θ est un entier algébrique (soit n le degré de θ) dont les conjugués $\theta_2, \ldots, \theta_n$, autres que θ, ont une valeur absolue inférieure ou égale à 1.

Soit $\theta > 2$, Λ l'ensemble de toutes les sommes finies $\sum_{k \geqslant 0} \epsilon_k \theta^k$, $\epsilon_k = 0$ ou 1. Alors Λ est un ensemble harmonieux si et seulement si θ est un nombre de Pisot.

5. Les ensembles harmonieux et l'approximation des fonctions bornées par des fonctions presque périodiques.

THEOREME 2. — *Soit* G *un groupe commutatif localement compact,* Λ *un ensemble harmonieux de* G *et* E *un ensemble compact dans* G. *Soit* Γ *le groupe dual de* G. *On peut trouver une partie finie* F *de* E *et une constante* C *avec les propriétés suivantes : pour toute fonction* $\varphi : \Gamma \to \mathbf{C}$ *continue et bornée sur* Γ, *dont le spectre est contenu dans* $\Lambda + E$, *il existe une fonction presque périodique* ψ *sur* Γ *dont le spectre est contenu dans* $\Lambda + F$ *et telle que*

(a) $\|\psi\|_\infty \leqslant C \|\varphi\|_\infty$

(b) *l'application* $\varphi \to \psi$ *est linéaire ; on posera* $\psi = L(\varphi)$

(c) $\psi(0) = \varphi(0)$ [*si* $G = \Gamma = \mathbf{R}$, *on a, pour tout* $t \in \mathbf{R}$

$$|\psi(t) - \varphi(t)| \leqslant C|t| \; \|\varphi\|_\infty]$$

(d) *pour tout* $\lambda \in \Lambda$, *si le spectre de* φ *est contenu dans* $\lambda + E$, *celui de* ψ *est contenu dans* $\lambda + F$.

La preuve de ce résultat est trop longue pour être reproduite ici ([1]).

6. Fin de la preuve du théorème 1.

Soit θ un nombre de Pisot supérieur à 2, E l'ensemble de toutes les sommes $\sum_1^\infty \epsilon_j \theta^{-j}$, $\epsilon_j = 0,1$ et Λ l'ensemble de toutes les sommes finies $\sum_{j\geq 0} \epsilon_j \theta^j$, $\epsilon_j = 0,1$. Soit enfin Λ_k l'ensemble de toutes les sommes $\sum_{0\leq j\leq k-1} \epsilon_j \theta^j$. Alors, pour tout $k \geq 1$, $\theta^k E = \Lambda_k + E \subset \Lambda + E$.

Appelons T_k l'isométrie de $L^\infty(\mathbf{R})$ définie par $(T_k\varphi)(t) = \varphi(\theta^k t)$ si $\varphi \in L^\infty(\mathbf{R})$. Si le spectre de φ est contenu dans E, celui de $T_k\varphi$ est contenu dans $\Lambda_k + E$ et l'on peut appliquer à $T_k\varphi$ le théorème 2. Posons $\varphi_k = (T_{-k} \circ L \circ T_k)(\varphi)$. Le spectre de φ_k est contenu dans $\theta^{-k}\Lambda_k + \theta^{-k}F$, partie finie de E. On a

$$\|\varphi_k\|_\infty \leq C\,\|\varphi\|_\infty \quad \text{et} \quad |\varphi_k(t) - \varphi(t)| \leq C\theta^{-k}|t|\,\|\varphi\|_\infty$$

7. Compléments.

Par des méthodes analogues on peut "atomiser", par des procédés linéaires, les distributions appartenant à divers espaces "raisonnables" de distributions sur **R** ou sur des groupes abéliens localement compacts.

Citons un résultat précis. ([2])

Soit G un groupe localement compact commutatif et métrisable. Soit $1 < p < +\infty$ et $CV_p(G)$ l'espace de Banach des convoluteurs S de $L^p(G)$. On peut trouver une constante $C > 0$ et une suite G_k, $k \geq 1$, de parties finies de G ayant les propriétés suivantes : pour tout S dans $CV_p(G)$ il existe une suite S_k de mesures telles que

(a) S_k est portée par G_k ; $S \to S_k$ est une application linéaire

(b) $\|S_k\|_{CV_p(G)} \leq C\,\|S\|_{CV_p(G)}$

(c) pour tout f dans $L^p(G)$, $S_k * f \to S * f$ dans $L^p(G)$.

BIBLIOGRAPHIE

[1] MEYER Y. — Les nombres de Pisot et la synthèse harmonique. *Ann. Sci. E.N.S.*, 4e série, t. 3, 1970, p. 235 à 246.

[2] LOHOUÉ N. — *Thèse*. Faculté des Sciences d'Orsay.

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France

Actes, Congrès intern. Math., 1970. Tome 2, p. 667 à 678.

ALLGEMEINE ENTWICKLUNGEN UND GEMISCHTE FRAGEN

von P. ULJANOV

In diesem Vortrag berichten wir über einige Resultate, die zu folgenden Richtungen gehören :

(1) Problem der Darstellung von Funktionen durch Reihen ;

(2) Reihen nach dem Haarschen System ;

(3) Allgemeine Funktionenreihen.

An einigen Stellen formulieren wir auch Resultate, die die angeführten Fragen eng berühren.

Mein Vortrag trägt Übersichtscharakter. Ein Teil der Resultate, die ich darlegen werde, wurde von verschiedenen Autoren veröffentlicht, der andere Teil der Ergebnisse ist vollkommen neu und wurde noch nicht veröffentlicht.

Wegen der beschränkten Zeit, werden wir nur die Resultate Sowjetischer Mathematiker darlegen.

1. Problemen der Darstellung von Functionen durch Reihen.

Die Frage nach der Darstellung von Funktionen durch Reihen tauchte schon im 18. Jahrhundert auf. Erste Ergebnisse in dieser Richtung wurden für trigonometrischen Reihen erhalten und zwar von Daniel Bernoulli, Euler, Fourier und anderen.

(A) Zu Beginn des 20. Jahrhunderts führten Untersuchungen der metrischen Funktionentheorie N. Lusin zum Problem der Darstellung beliebigen messbare Funktionen durch trigonometrische Reihen.

Die ersten allgemeinen Resultate in dieser Richtung erzielte D. Menschow ;

THEOREM 1 (Menschow). – *Wenn eine Funktion $f(x)$ im Intervall $(0,2\pi)$ messbar und endlich ist, so lässt sich eine Reihe*

$$a_0/2 + \sum_K a_K \cos KX + b_K \sin Kx \tag{1}$$

finden, die fast überall im Intervalle $(0,2\pi)$ gegen $f(x)$ konvergiert.

THEOREM 2 (Menschow). – *Wenn eine Funktion $f(x)$ im Intervall $(0,2\pi)$ messbar ist, so lässt sich eine Reihe der Form (1) finden, die dem Masse nach im Intervall $(0,2\pi)$ gegen $f(x)$ konvergiert.*

Im Theorem 2 kann die Funktion $f(x)$ die Werte $+\infty$ und $-\infty$ auf Mengen mit positiven Mass annehmen.

Bis heute ist unbekannt, ob Theorem 1 ohne die Forderung nach Endlichkeit der Funktion $f(x)$ gilt. Insbesondere ist unbekannt, ob eine trigonometrische Reihe der Form (1) existiert, die auf einer gewissen Menge E mit dem Mass $|E| > 0$ gegen $+\infty$ konvergiert.

Diese Theoreme Menschows wurden auf die einen oder anderen Funktionensysteme erweitert. Es gilt zum Beispiel :

THEOREM 3 (Bari). – *Wenn eine Funktion $f(x)$ im Intervall* (0,1) *messbar und endlich ist, so lasst sich eine Reihe nach dem Haarschen System finden*

$$\sum_{m=1}^{\infty} a_m \chi_m(x) \ , \tag{2}$$

die fast überall im Intervall (0,1) *gegen $f(x)$ konvergiert.*

THEOREM 4 (Talaljan, Arutjunjan). – *Es existiert* **keine** *Reihe der Form (2), die auf einer gewissen Menge $E \subset (0,1)$ mit dem Mass $|E| > 0$ gegen $+\infty$ konvergiert.*

Die Theoreme 3 und 4 zeigen, dass beim Problem der Darstellung von Funktionen dem Wesen nach der Fall *endlicher* Funktionen vom Fall *beliebiger* Funktionen, die die Werte $+\infty$ und $-\infty$ auf einer Menge E mit dem Mass $|E| > 0$ annehmen können, getrennt werden muss.

Erinnern wir uns einer *Definition* : Das Funktionensystem $\varphi_n \in L^p(a, b)$ mit $1 \leqslant p < \infty$ heisst **Basis** des Raumes $L^p(a, b)$, wenn für jede Funktion $\mathfrak{F} \in L^p(a, b)$ nur eine Reihe

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \equiv \sum_{n=1}^{\infty} a_n(\mathfrak{F}) \varphi_n(x)$$

existiert, die in der Norm des Raumes $L^p(a, b)$ gegen $F(x)$ konvergiert. In diesem Falle sind die a_n die Fourierkoeffizienten und es ist

$$a_n \equiv a_n(\mathfrak{F}) = \int_a^b \mathfrak{F} \, \psi_n \, dt \ ,$$

wobei ψ_n adjungiertes System zu φ_n ist, dass heisst

$$\int_a^b \varphi_n \, \psi_m \, dt = \delta_{n,m} \ .$$

Zu bemerken ist, dass jedes orthonormierte in $L^2(a, b)$ vollständige Funktionensystem $\{\varphi_n\}$ Basis des Raumes $L^2(a, b)$ ist.

Was die allgemeinen orthogonalen Reihen oder Basisreihen des Raumes $L^p(a,b)$ betrifft, so gilt für sie

THEOREM 5 (Talaljan). – *Es sei $\{f_n\}$ eine Basis des Raumes $L^p(0,1)$ $(1 < p < \infty)$ (imspeziellen, $\{f_n\}$ orthonormiertes vollständiges System).*

Wenn eine Funktion $F(x)$ im Intervall (0,1) **messbar** ist, so lässt sich eine Reihe

$$\Sigma a_n f_n$$

finden, die dem Masse nach im Intevall (0,1) gegen $F(x)$ konvergiert.

Es ist klar, dass das Theorem 5 eine Verallgemeinerung des Theoremes 2 von Menschow ist.

Bezüglich Theorem 1 ist unbekannt, ob es für orthogonale Reihen gültig ist. Genauer gesagt, ist unbekannt, ob die folgende Behauptung gilt : Wenn $\{\varphi_n\}$ ein beliebiges vollständiges orthonormiertes System im Intervall (0,1) ist, so lässt sich für jede messbare und endliche Funktion $f(x)$ eine Reihe

(3) $$\Sigma a_n \varphi_n$$

finden, die fast überall in (0,1) gegen $f(x)$ konvergiert.

(B) Mit dem Problem der Darstellung von Funktionen durch Reihen ist eng die Frage nach der Existenz von *Null-Reihen* nach diesen oder jenen Systemen verbunden.

DIFINITION. – Das System f_n heisst System zur Darstellung **endlicher** messbarer Funktionen im Sinne der Konvergenz fast überall (Konvergenz dem Masse nach), wenn für eine beliebige messbare und endliche Funktion $f(x)$ (mit $x \in [0,1]$) sich eine Reihe

$$\sum_{n=1}^{\infty} a_n f_n(x)$$

finden lässt, die fast überall im Intervall (0,1) (entsprechend dem Masse nach) gegen $f(x)$ konvergiert.

Es gilt das

THEOREM 6 (Talaljan). – *Wenn $\{f_n(x)\}$ im Sinne der Konvergenz fast überall (im Sinne der Konvergenz dem Masse nach) ein System zur Darstellung endlicher messbarer Funktionen ist, so existiert nach diesem System eine Null-Reihe, das heisst es existiert eine Reihe*

$$\Sigma a_n f_n(x) \ ,$$

die fast überall (entsprechend dem Masse nach) gegen Null konvergiert, aber $\Sigma |a_n| > 0$.

Also ist die Existenz von Null-Reihen nach dem System $\{f_n\}$ automatische Folge davon, dass $\{f_n\}$ System zur Darstellung von Funktionen ist.

Deshalb folgt aus Theoreme 5 und 6 unmittelbar, dass für ein beliebiges orthonotmiertes vollständiges System $\{\varphi_n\}$ eine Null-Reihe existiert im Sinne der Konvergenz dem Masse nach.

Jedoch fehlt bisher die Antwort auf folgende Frage : Es sei $\{\varphi_n\}$ ein im Intervall (0,1) beliebiges, vollständiges und orthonormiertes Funktionensystem. Existiert dann eine Reihe

$$\Sigma a_n \varphi_n(x)$$

die fast überall in (0,1) gegen Null konvergiert, wobei aber

$$\Sigma |a_n| > 0 \ ?$$

(C) Es sei $\{\psi_n\}$ ein Schaudersches System. Für Reihen nach dem Schandersystem ist das Problem der Darstellung genügend allgemein gelösst. Es gilt, z. B.

THEOREM 7 (Talaljan). – *Wenn eine Funktion* $f(x)$ *im Intervall* $(0,1)$ *messbar ist, so lässt sich eine Reihe*

$$\Sigma a_n \psi_n(x) \tag{4}$$

finden, die fast überall im Intervall $(0,1)$ *gegen* $f(x)$ *konvergiert.*

Weiter sei $\Phi(u)$ eine gerade, stetige in $[0,\infty)$ nicht fallende Funktion, die den folgenden Bedingungen genügt : $\Phi(0) = 0$ und $\lim_{u\to\infty} \Phi(u) = \infty$. Mit $\Phi(L)$ bezeichnen wir die Menge aller im Intervall $(0,1)$ messbarer Funktionen $f(x)$, für die

$$\int_0^1 \Phi(f(x))\, dx < \infty .$$

Der Autor bewies folgendes.

THEOREM 8. – *Wenn eine Funktion* $f \in \Phi(L)$, mit $\Phi(u + \theta) \leqslant C\{\Phi(u) + \Phi(\theta)\}$, *so lässt sich eine solche Reihe der Form (4) finden, dass*

$$\lim_{N\to\infty} \int_0^1 \Phi(f(x) - \sum_{n=1}^{N} a_n \psi_n(x))\, dx = 0 .$$

Es sei bemerkt, dass man als Funktion $\Phi(u)$, z.B. $\Phi(u) = |u|^p$ mit $0 < p < \infty$, oder $\Phi(u) = ln(1 + |u|)$ nehmen kann.

Behauptungen wie Theorem 8 führten den Autor zur Frage, für welche Klassen $\Phi(L)$ das Weierstrasssche Theorem gültig ist.

Es gilt

THEOREM 9. – *Damit sich für eine beliebige Funktion* $f \in \Phi(L)$ *und für eine beliebige Zahl* $\epsilon > 0$ *ein algebraisches Polynom* $\mathcal{P}(x)$ *finden lässt, so dass*

$$\int_0^1 \Phi(f - \mathcal{P})\, dx < \epsilon ,$$

ist es notwendig und hinreichend, dass die Bedingung

$$\overline{\lim_{u\to\infty}} \frac{\Phi(u + 1)}{\Phi(u)} < \infty \tag{6}$$

erfüllt ist.

Das heisst, das Weierstrassschen Theorem ist gültig für die Klasse $\Phi(L)$ dann und nur dann, wann die Bedingung (6) erfüllt ist.

Theorem 9 gilt nicht nur für algebraische Polynome, sondern auch für trigonometrische Polynome, für Polynome nach dem Haarschen und Schauderschen System und andere.

Es sei bemerkt, dass, zum Beispiel, die Funktion $\Phi(u) = 2^{|u|} - 1$ der Bedingung (6) genügt. Allgemein gesagt, weist die Bedingung (6) darauf hin, dass die Funktionen $\Phi(u)$ nicht schneller als Exponentialfunktionen wachsen können, obwohl sie auch letztere sein können.

2. Reihen nach dem Haarschen System.

(A) Schon Haar stellte fest, dass das System $\{\chi_m\}$ Basis des Raumes $C(0,1)$ ist und dass es ein System mit Konvergenz fast überall ist das heisst dass Fourierreihe

$$(1) \qquad \sum_{m=1}^{\infty} (f, \chi_m)\, \chi_m(t)$$

fast überall in $[0,1]$ konvergiert, wenn nur $f \in L^2(0,1)$.

Erinnern wir uns der Definition : Die Reihe

$$(2) \qquad \sum_{n=1}^{\infty} \alpha_n(t) \qquad (t \in [0,1])$$

heisst *fast uberall unbedingt konvergent* im Intervall $(0,1)$, wenn sie nach einer beliebigen Umstellung ihrer Glieder fast überall in $(0,1)$ konvergiert. Dabei hängt die Ausnahme menge E mit dem Mass $|E| = 0$ von der Umstellung ab.

Dieser Begriff ist nicht gleichzusetzen mit der absoluten Konvergenz fast überall. Zum Beispiel, hat die Reihe $\Sigma \frac{1}{n} \cos nx$ keinen einzigen Punkt absoluter Konvergenz, abwohl sie nach einer beliebigen Umstellung ihrer Glieder fast überall auf der Geraden $(-\infty, \infty)$ konvergiert.

Es wurde von uns bewiesen, dass das folgende Theorem gilt.

THEOREM 1. – *Es existiert eine solche Funktion* $f \in L^2(0,1)$ (*sogar* $f \in L^p(0,1)$ *für alle* $p \in [2, \infty)$), *so dass die Fourier-Haarsche Reihe (1) nach einer gewissen Umstellung der Glieder fast überall im Intervall* $(0,1)$ *divergiert.*

Dieses Resultat erweiterten wir auf beliebige Basen $\{f_n\}$ des Raumes $L^2(0,1)$. Es wird also jede Basis $\{f_n\}$ des Raumes $L^2(0, 1)$ nach einer gewissen Umstellung $\{f_{n_k}\}$ *kein* System mit Konvergenz fast überall sein.

Es sei bemerkt, dass beim Beweis dieser Behauptung über die Basen das Theorem 1 eine zentrale Rolle spielte.

Im Zusammenhang mit den angeführten Resultaten wäre es wünschen wert, eine Antwort auf folgende Frage zu finden :

Wenn $\{f_n\}$ Basis des Raumes $L^2(0,1)$ ist (oder $\{f_n\}$ ist orthonormiertes vollständiges System), kann man dann diese Basis so umstellen, dass das neue System $\{f_{n_k}\}$ ein System mit Konvergenz fast überall ist ?

Es sei bemerkt, dass die oben angeführten Behauptungen in verschiedenen Richtungen von A. Olewski, L. Taikow, K. Tandori, F. Arutjunjan und anderen verstärkt wurden.

Zum Beispiel bewies F. Arutjunjan, dass jede Basis $\{f_n\}$ des Raumes $L^p(0,1)$ (mit $1 < p < \infty$) nach einer gewissen Umstellung $\{f_{n_k}\}$ *kein* System mit Konvergenz fast überall für die Funktionenklasse $L^p(0,1)$ sein wird.

(B) Jetzt verweilen wir noch bei der Frage der absoluten und der unbedingten Konvergenz fast überall von Reihen nach dem Haarschen System. Zusammen mit E.

Nikischin bewies der Autor, dass mit einer Genauigkeit bis zu Mengen vom Masse Null die unbedingte Konvergenz fast überall der Reihen nach dem Haarschen System mit der absoluten Konvergenz fast überall zusammenfällt (für trigonometrische Reihen ist das nicht so).

Ausserdem bewies der Autor folgendes.

THEOREM 2. –

(a) *Wenn* $1 < p \leqslant \infty$ *ist, und*

$$\omega_p(\delta, f) \equiv \sup_{0 \leqslant h \leqslant \delta} \left\{ \int_0^{1-h} |f(t+h) - f(t)|^p \, dt \right\}^{1/p} = O \left\{ \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}-\epsilon} \right\}$$

bei einem gewissen $\epsilon > 0$, *so ist*

$$\sum_m |(f, \chi_m) \chi_m(t)| < \infty$$

fast überall in $[0,1]$.

(b) *Die Behauptung a) verliert für* $p \in (1, \infty)$ und $\epsilon = 0$ *ihre Gültigkeit, das heisst es existiert eine Funktion* $f_0(t)$, *für die gilt* $\omega_p(\delta, f_0) = O\left\{\left(\log \frac{1}{\delta}\right)^{-1/2}\right\}$ *für alle* $p \in (1, \infty)$, *aber*

$$\sum_{m=1}^{\infty} |(f_0, \chi_m) \chi_m(t)| = \infty$$

fast überall in $[0,1]$.

Erst kürzlich bewies S. Botschkarjew, dass die Behauptung a) für $p = 1$ und die Behauptung b) für $p = \infty$ gilt.

Es ist nicht ausgeschlossen, das die Behauptung b) für beliebige orthonormierte vollständige Systeme gilt.

(C) Behauptungen wie das Theorem 2 führten uns zu Einbettungstheoremen. Eben wie man nach dem Fallen der Funktion $\omega_p(\delta, f)$ die Zugehörigkeit einer Funktion f zu diesem oder jenem Raum $L^\nu(0,1)$ mit $\nu > p$ chakterisieren kann.

Erinnern wir uns der Definition : Es sei $\omega(\delta)$ stetig und nicht fallend in $[0,1]$, $\omega(0) = 0$ und $\omega(\delta + h) \leqslant \omega(\delta) + \omega(h)$. Dann wird mit $H_p^{\omega(\delta)}$ die Menge aller Funktionen $\{f\}$ bezeichnet, für die $\omega_p(\delta, f) = O\{\omega(\delta)\}$ ist.

Es gilt

THEOREM 3. – *Es sei* $1 \leqslant p < \nu < \infty$. *Damit die Einbettung* $H_p^{\omega(\delta)} \subset L^\nu(0,1)$ *gilt, ist es notwendig und hinreichend, dass* $\sum_n h^{\frac{\nu}{p}-2} \omega^\nu\left(\frac{1}{n}\right) < \infty$ *ist.*

FOLGERUNG. – Damit die Einbettung $H_1^{\omega(\delta)} \subset L^2(0,1)$ gilt, ist es notwendig und hinreichend, dass $\sum_n \omega^2\left(\frac{1}{n}\right) < \infty$ *ist.*

In dieser Richtung ist uns, z.B., die notwendige und hinreichende Bedingung für die Einbettung $H_p^{\omega(\delta)} \subset l^{|L|} - 1$ nicht bekannt.

(D) Zunächst erinnern wir uns einer Definition : Die Folge $\{\omega(m)\}$ (mit $\omega(0) = 1$, $\omega(m) \uparrow$ bei $m \to \infty$) heisst Wejlscher Multiplikator für die fast überall unbedingte Konvergenz von Reihen nach dem Haarschen System

$$\sum_m a_m \chi_m(t) \ , \tag{3}$$

wenn die Reihe (3) fast überall unbedingt konvergiert im Intervall (0,1), sobald nur

$$\sum_m a_m^2 \omega(m) < \infty \ .$$

Der Autor stellte die Gültigkeit folgenden Theorems fest.

THEOREM 4. – *Damit* $\{\omega(m)\}$ *Wejlscher Multiplikator für die unbedingte Konvergenz fast überall von Reihen der Form (3) ist, ist es notwendig und hinreichend, das* $\sum_m \frac{1}{m\,\omega(m)} < \infty$.

Das heisst, die Folge $\omega(m)$ ist Wejlscher Multiplikator für Reihen der Form (3), dann und nur dann, wann $\sum_m \frac{1}{m\,\omega(m)} < \infty$.

FOLGERUNG 1. – Die Folge $(1 + \log m)^{1+\epsilon}$ ist Wejlscher Multiplikator **nur** bei $\epsilon > 0$.

FOLGERUNG 2. – Das Haarsche System besitzt *keinen genauen* Wejlschen Multiplikator für die unbedingte Konvergenz fast überall.

Es sei bemerkt, dass bis jetzt für trigonometrische Reihen Ergebnisse ähnlich Theorem 4 nicht bekannt sind.

Wir bringen noch einen unserer Sätze.

THEOREM 5. – *Es sei* $a_m \downarrow 0$. *Dann konvergiert die Reihe* $\sum_m a_m \chi_m(t)$ *fast überall im Intervall* (0,1) *dann und nur dann, wenn* $a_m \in l_2$, *das heisst, wenn* $\sum_m a_m^2 < \infty$.

Zu bemerken ist, dass bis jetzt die Frage nicht beantwortet ist, ob ein orthonormiertes *vollständiges* System $\varphi_m(x)$ existiert, nach dem die Reihen $\sum_m a_m \varphi_m(x)$ $(x \in [0,1])$ fast überall in $[0,1]$ dann und nur dann konvergieren, wenn $\{a_m\} \in l_2$.

(E) Hier noch zwei Ergebnisse für Reihen nach dem Haarschen System. Es gilt

THEOREM 6 (M. Petrowskaja). – *Die Menge* $E_1 \subset [0,1]$ *habe das Mass* $|E_1| = 0$ *und die Menge* $E_2 \subset [0,1]$ *sei abzählbar. Wenn die Reihe*

$$(4) \qquad \sum_{m=1}^{\infty} a_m \chi_m(t) \qquad \text{(mit den Koeffizienten } a_m = \bar{\bar{0}}(\sqrt{m}))$$

auf der Menge $[0,1] - E_1$ *gegen die Funktion* $f(t) \in L(0,1)$ *konvergiert und in jedem Punkt* $t \in [0,1] - E_2$ *beschränkte Partialsummen besitzt, so ist Reihe (4) die Fourier-Lebesguesche Reihe von* $f(t)$ *nach dem Haarschen System.*

Dieses Theorem ist das Analog zum bekannten Valle-Pussenschen Theorem für trigonometrische Reihen.

THEOREM 7 (F. Arutjunjan). – *Reihe (4) konvergiere überall in* $[0,1]$ *gegen die Funktion* $f(t)$, *die im Intervall* $(0,1)$ *im weiteren Sinne nach Denjoyschen integrierbar ist.*

Dann ist Reihe (4) die Fourier-Denjoyschen Reihe von $f(t)$ nach dem Haarschen System $\{\chi_m\}$.

Es sei bemerkt, dass für trigonometrische Reihen ein solches Theorem nicht bekannt ist.

3. Allgemeine Funktionenreihen.

In dieser Richtung erhielt E. Nikischin interessante Resultate.

(A) DEFINITION. – Das System im Intervall $(0,1)$ messbarer Funktionen $\{f_n(x)\}$ heisst System absoluter Konvergenz für $l_p (1 \leqslant p < \infty)$, wenn

$$\sum_n |a_n f_n(x)| < \infty$$

fast überall in $(0\,,1)$, wenn nur $\{a_n\} \in l_p$.

THEOREM 1 (Nikischin). – *Damit* $\{f_n\}$ *ein System absoluter Konvergenz für* l_p *ist, ist es notwendig und hinreichend, dass sich für jede Zahl* $\epsilon > 0$ *eine Menge* $E \equiv E(\epsilon) \subset [0,1]$ *mit dem Mass* $|E| > 1 - \epsilon$ *finden lässt so, dass*

$$(1) \qquad \begin{cases} \sum_n \left(\int_E |f_n|\, dx\right)^{\frac{p}{p-1}} < \infty & \text{bei} \quad 1 < p < \infty \\ \text{und} & \\ \operatorname{Sup} \int_E |f_n(x)|\, dx < \infty & \text{bei} \quad p = 1 \end{cases}$$

Dieser Satz ist eine Übertragung des bekannten Theorems von E. Landau für Zahlenreihen auf den Fall beliebiger Funktionenreihen.

Es sei bemerkt, dass der Hauptinhalt des Theorems 1 im Beweis der *Notwendigkeit* der Bedingung (1) besteht. Das Hinreichendsein der Bedingung (1) ist faktisch offensichtlich.

(B) Im Jahre 1927 stellte *S.* Banach folgendes Problem : Die Reihe

$$\sum_{n=1}^{\infty} f_n(x) \qquad (x \in [0,1])$$

aus messbaren Funktionen konvergiere fast überall gegen $F(x)$, und nach einer gewissen Umstellung

$$\sum_{K=1}^{\infty} f_{n_K}(x)$$

konvergiere sie fast überall gegen $\Phi(x)$. Kann man behaupten, dass für jede Funktion

$$\psi(x) \equiv \psi_\lambda(x) = \lambda F(x) + (1-\lambda)\,\Phi(x) \qquad (0 < \lambda < 1)$$

sich eine solche Umstellung m_K finden lässt, dass die Reihe $\sum\limits_K f_{m_K}(x)$ fast überall in [0,1] gegen Funktion $\psi(x)$ konvergiert ?

Die Antwort aus dieses Problem gibt

THEOREM 2 (E. Nikischin). –

(a) *Die Funktionen* $f_n(x)$ *seien messbar und es sei*

$$\sum_{n=1}^{\infty} f_n^2(x) < \infty$$

fast überall in (0,1).

Es sei Q *die Menge aller Funktionen* $\psi(x)$, *für die sich jeweils eine Umstellung* $\{n_K\}$ *finden lässt, so dass*

$$\psi(x) = \sum_{K=1}^{\infty} f_{n_K}(x)$$

fast überall in (0,1).

Dann ist die Menge Q *lienear, das heisst, wenn die Funktionen*

$$\psi_1(x) \in Q \quad \text{und} \quad \psi_2(x) \in Q\,,$$

so ist auch die Funktion

$$\psi(x) = \lambda\,\psi_1(x) + (1-\lambda)\,\psi_2(x) \in Q \quad \text{bei} \quad -\infty < \lambda < \infty\,.$$

(b) *Es existiert eine Reihe aus algebraischen Polynomen*

$$\sum_{n=1}^{\infty} \mathcal{P}_n(x) \qquad (x \in [0,1])\,, \tag{2}$$

die die folgende Eigenschaften besitzt :

(1) *Bei jedem* $\epsilon > 0$ *gilt fast überall in* (0,1)

$$\sum_{n=1}^{\infty} |\mathcal{P}_n(x)|^{2+\epsilon} < \infty\ ;$$

(2) *Die Reihe (2) konvergiert nach zwei verschiedenen Umstellungen fast überall in* (0,1) *gegen die Funktionen* $\psi_1(x)$ *und* $\psi_2(x)$;

(3) *Es existiert keine solche Umstellung der Reihe (2), so dass die Reihe nach dieser Umstellung fast überall in* (0,1) *gegen Funktion* $\frac{1}{2}\{\psi_1(x) + \psi_2(x)\}$ *konvergiert.*

Dieses Resultat scheint aus interessant. Sein Beweis ist bei weitem nicht trivial.

Theorem 2 ist das Analog zum Riemannschen Theorem für bedingt konvergente Zahlenreihen für den Fall der Funktionenreihen.

Zu bemerken ist, dass bisher unbekannt ist, wie das Analog zum Riemannschen Theorem für den Fall gleichmässig konvergenter Reihen aus stetigen Funktionen zu formulieren wäre.

(C) Die Funktion $\varphi(x)$ genüge den Bedingungen

$$\varphi(x) \in L^2(0,1),\ \varphi(x+1) = \varphi(x),\ \int_0^1 \varphi(x)\,dx = 0\,. \tag{3}$$

Betrachten wir das Funktionensystem $\{\varphi(nx)\}_{n=1}^{\infty}$.

Erdös konstruirte (1949) das Beispiel einer Funktion $\varphi(x)$, die den Bedingungen (3) genügt, aber das System $\{\varphi(nx)\}$ ist kein System mit Konvergenz fast überall. In diesem Beispiel Erdös' war die Funktion $\varphi(x)$ wesentlich unbeschränkt.

Es ist gut bekannt, dass L. Karleson bewies, dass das trigonometrische System $\varphi(nx) = \sin 2\pi nx$ $(\varphi(x) = \sin 2\pi x)$ ein System mit Konvergenz fast überall ist.

W. Gaposchkin bewies, indem er dieses Ergebnis Karleson benutzte dass $\{\varphi(nx)\}$ ein System mit Konvergenz fast überall ist, wenn die Funktion $\varphi(x)$ den Bedingungen (3) genügt und der Stetigkeitsmodul

$$\omega_2(\delta\,,\varphi) = O\left\{\delta^{\frac{1}{2}+\epsilon}\right\} \quad \text{bei gewissen} \quad \epsilon > 0\,. \tag{4}$$

Definition. Das System $\{f_n(x)\}$ heisst System der Konvergenz dem Masse nach, wenn die Reihe $\sum_n a_n f_n(x)$ $(x \in [0,1])$ dem Masse nach in [0,1] konvergiert, sobard nur $\{a_n\} \in l_2$.

In dieser Richtung gilt

THEOREM 3 (E. Nikischin). – *Die Funktion* $\varphi(x)$ *genüge den Bedingungen (3) und es sei*

$$a_K = \int_0^1 \varphi(t) \sin 2\pi\, kt\, dt\,.$$

Damit $\varphi(nx)$ *System der Konvergenz dem Masse nach ist,* ist es notwendig und hinreichend, dass die Dirichletsche Reihe

$$\sum_{K=1}^{\infty} a_K \cdot K^{-z} \tag{5}$$

im Gebiet $\mathfrak{S} \equiv \{\operatorname{Re} z > 0\}$ gegen eine gewisse Funktion $f(z)$ konvergiert, für die gilt

(6) $$|f(z)| \leqslant C = \text{const} < \infty \quad \text{für alle} \quad z \in \mathfrak{S} \ .$$

Folgerung 1. – Damit $\{\varphi(nx)\}$ System mit Konvergenz fast überall ist, ist es notwendig, dass Ungleichung (6) erfüllt ist.

Folgerung 2. – Es existiert eine **stetige** Funktion $\varphi(x)$, die den Bedingungen (3) genügt, aber $\{\varphi(nx)\}$ ist *kein* System mit Konvergenz fast überall.

Zum Beispiel ist

$$\varphi(x) = \sum_{K=2}^{\infty} (K \ln K)^{-1} \sin 2\pi K x$$

eine solche Funktion.

Folgerung 3. – Das System $\{\varphi(nx)\} \equiv \{\text{sign} \sin 2\pi n x\}$ ist **kein** System mit Konvergenz fast überall, obwohl

(7) $$\omega_2(\delta\ , \varphi) = O(\delta^{1/2}) \ .$$

Hier ist $$\varphi(x) = \text{sign}(\sin 2\pi x) \ .$$

Folgerung 3. weist darauf hin, dass im Gaposchkinschen Theorem die Bedingung (4) nicht verschärft werden kann.

Insbesondere, aus den angeführten Resultaten folgt, dass $\varphi(nx) = \sin 2\pi nx$ System mit Konvergenz fast überall ist, während $\varphi(nx) = \text{sign} \sin 2\pi nx$ schon nicht mehr System der Konvergenz fast überall ist.

LITERATUR

[1] Лузин Н. Н. — Интеграл и тригонометрический ряд, М.-Л., 1951.

[2] Меньшов Д. Е. — а) Матем. сб., 9, 1941, p. 667; б) Тр. Матем. ин-та им. В. А. Стеклова, 32, 1950.

[3] Талалян А. А., Арутюнян Ф. Г. — а) Матем. сб., 66, 1965, p. 240; б) Изв. АН СССР, 28, 1964, p. 1391.

[4] Талалян А. А. — а) Усп. матем. н., 15, № 5, 1960; б) Матем. зам., 5, № 1, 1969; в) Изв. АН Арм. СССР, 12, № 3, 1959.

[5] Ульянов П. Л. — а) Матем. зам., 7, № 4, 1970; б) Докл. АН СССР, 137, № 4, 1961; в) Усп. матем. н., 16, № 3, 1961; г) Матем. зам., 1, № 1, 1967; д) Изв. АН СССР, 32, № 3, 1968; е) Матем. сб., 60, № 1, 1963; ж) Изв. АН СССР, 28, 1964, p. 925.

[6] Олевский А. М. — а) Докл. АН СССР, 138, № 3, 1961; б) Изв. АН СССР, 27, 1963, p. 343.

[7] Арутюнян Ф. Г. — а) Докл. АН Арм. СССР, 41, № 5, 1965; б) Изв. АН СССР, 30, № 2, 1966.

[8] Никишин Е. М., Ульянов П. Л. — Усп. матем. н., 22, № 3, 1967.
[9] Бочкарев С. В. — Магем. зам., 4, № 2, 1968.
[10] Петровская М. Б. — Изв. АН СССР, 28, 1964, 773.
[11] Гапошкин В. Ф. — Матем. зам., 4, № 3, 1968.
[12] Никишин Е. М. — а) Матем. сб., 74, № 4, 1967; б) Матем. зам., 7, № 4, 1970; в) Резонансные теоремы, Дисс., МГУ, 1969.
[13] Orlicz W. — *Bull. Acad. Polon.*, 1927, p. 117-125.
[14] Carleson L. — *Acta math.*, 116, No. 1-2, 1966.

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D10 - ANALYSE FONCTIONNELLE ET ÉQUATIONS AUX DÉRIVÉES PARTIELLES LINÉAIRES

SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS

by Samuel AGMON

1. Introduction

The purpose of this talk is to report some results concerning the spectrum of self-adjoint realizations([1]) in $L_2(R^n)$ of elliptic operators which in a neighborhood of infinity differ only slightly from an elliptic operator with constant coefficients. A special example of such operators are Schrödinger operators, i.e. realizations of $-\Delta + q(x)$ where Δ is the n-dimensional Laplacian and $q(x)$ is a real function. We start by recalling some known results concerning the spectrum of such operators.

Assume that q is bounded and that $q \to 0$ as $|x| \to \infty$. Then $-\Delta + q$ admits a unique self-adjoint realization in $L_2(R^n)$ which we denote by $\mathcal{A}$. The spectrum of $\mathcal{A}$ consists of the positive axis (essential spectrum) and of a discrete bounded set of negative eigenvalues of finite multiplicity. Two problems concerning the essential spectrum are of special interest.

PROBLEM I. — *Give conditions on q which ensure that $\mathcal{A}$ has no positive eigenvalues.*

PROBLEM II. — *Give conditions on q which ensure that the positive spectrum of $\mathcal{A}$ is absolutely continuous.*

A solution to Problem I was given by Kato [7] who proved that if

$$q(x) = 0(|x|^{-\mu}) \quad \text{as} \quad |x| \to \infty \tag{1.1}$$

with $\mu > 1$, then $\mathcal{A}$ has no positive eigenvalues.

As for Problem II, there are a number of results in the literature which imply absolute continuity of the positive spectrum of $\mathcal{A}$ for q verifying (1.1) with some

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(1) A self-adjoint realization in $L_2(R^n)$ of a differential operator A is a self-adjoint operator $\mathcal{A}$ in $L_2(R^n)$ such that $A(x, D)u = \mathcal{A}u$ (distribution sense) for every u in the domain of definition of $\mathcal{A}$.

restriction on μ. Thus, Ikebe [5] proved absolute continuity (in 3-space) for $\mu > 2$. Jäger [6] proved results which implicitly yield absolute continuity for $\mu > 3/2$. Recently, Rejto [11] proved absolute continuity for $\mu > 4/3$, while Kato [8] established the same for $\mu > 5/4$. From these results one would conjecture that the positive spectrum of $\mathfrak{A}$ is absolutely continuous if q satisfies (1.1) with any $\mu > 1$. That this is indeed to case would follow from a general result which we shall describe in the following section.

2. The spectrum of general elliptic operators

Consider a formally self-adjoint elliptic operator A of order m of the form :

$$A(x, D) = A_0(D) + B(x, D) \tag{2.1}$$

where A_0 is an elliptic operator of order m with constant real coefficients and $B(x, D)$ is a "perturbation operator" of the form :

$$B(x, D) = \sum_{|\alpha| \leqslant m} b_\alpha(x) D^\alpha, \tag{2.1)'$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \; D_k = - i\,\partial/\partial x_k,$$

where $b_\alpha(x)$ are measurable bounded(²) functions such that $b_\alpha(x) \to 0$ as $|x| \to \infty$, and where the top order coefficients of B are continuous. Without loss of generality, we shall assume that the form $A_0(\xi)$ is positive for $|\xi|$ sufficiently large, $\xi \in R^n$, and set

$$a_0 = \min_{\xi \in R^n} A_0(\xi) .$$

Under the above conditions there exists a unique self-adjoint realization of $A(x, D)$ in $L_2(R^n)$ which we shall denote by $\mathfrak{A}$. A well known argument shows that spectrum $\mathfrak{A}$ is composed of the halfline $[a_0, \infty)$ together with a discrete set of eigenvalues $\{\lambda_j\}$, $j = 1, 2, \ldots$, such that $\lambda_j \uparrow a_0$. In order to state our general result on the essential spectrum of $\mathfrak{A}$ we introduce the following notion. A point λ on the real line will be called a critical value of A_0 if there exists a $\xi \in R^n$ such that

$$A_0(\xi) = \lambda \quad \text{and} \quad \operatorname{grad} A_0(\xi) = 0 .$$

THEOREM 2.1. – *Suppose that the perturbation operator B is* $0(|x|^{-1-\epsilon})$ *in the sense that*

$$b_\alpha(x) = 0(|x|^{-1-\epsilon}) \quad as \quad |x| \to \infty, \tag{2.2}$$

for some $\epsilon > 0$, $|\alpha| \leqslant m$. Let $[a, b]$ be a closed finite interval on the line which does not contain a critical value of A_0. Then spectrum $\mathfrak{A}$ has at most a finite number of eigenvalues in $[a, b]$. Each such eigenvalue has a finite multiplicity. Furthermore, the spectrum in $[a, b]$ does not contain any singular continuous part. In particular, if $[a, b]$ is free of eigenvalues, then the spectrum contained in $[a, b]$ is absolutely continuous.

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(2) The condition of boundness could be relaxed.

Note that this theorem establishes in particular the conjecture mentioned at the end of the Introduction.

The proof of Theorem 2.1 uses the limit absorption method. A crucial step in the proof is the following result.

THEOREM 2.2. – *Let* $[a, b]$ *be an interval free of critical values of* A_0. *Then, for any complex* λ *such that* $a \leqslant |\mathrm{Re}\, \lambda| \leqslant b$, $|\mathrm{Im}\, \lambda| \leqslant \delta$, $\delta > 0$ *sufficiently small, and every function* $u \in H_m(R^n)$ (*i.e.* $D^\alpha u \in L_2(R^n)$ *for* $|\alpha| \leqslant m$) *the following inequality holds* :

$$\int_{R^n} (1 + |x|)^{-1-\epsilon} \sum_{|\alpha| \leqslant m} |D^\alpha u|^2 dx \leqslant C \int_{R^n} (1 + |x|)^{1+\epsilon} |(A_0 - \lambda) u|^2 dx ,$$

where C is a constant not depending on λ *or u. (Here* ϵ *stands for an arbitrary, but fixed positive number).*

3. Uniqueness results

A problem which arises in connection with Theorem 2.1 is the following generalization of Problem I.

PROBLEM I′. – *Under what additional conditions on the differential operator A one can assert that the essential spectrum of* $\mathfrak{A}$ *does not contain any eigenvalues in a given interval* $[a, b]$.

Let us introduce the following terminology. A differential operator $P(x, D)$ on R^n will be said to possess the L_2 S-property if the assumptions that $u \in L_2(R^n)$ and that Pu has a compact support imply that u has a compact support.

Problem I′ is related to the following.

PROBLEM III. – *Give conditions on* $P(x, D)$ *so that it would possess the* L_2 *S-property.*

Known examples of operators which possess the L_2 *S*-property are :

(i) $P = \Delta + \lambda - q(x)$, λ a constant > 0 and $q(x) = 0(|x|^{-1-\epsilon})$ (Kato [7]).

(ii) More general second order elliptic operators which behave asymptotically like $\Delta + \lambda$ (Jäger [6]).

(iii) A class a higher order operators $P(D)$ having constant real coefficients (Littman [9 ; 10]).

We describe now a general class of operators with variable coefficients which possess the L_2 S-property.

THEOREM 3.1. (3). – *Let* $P(x,D)$ *be an elliptic operator of order m of the form :*

$$P(x, D) = P_0(D) + \sum_{|\alpha| \leqslant m-1} b_\alpha(x) D^\alpha$$

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(3) This theorem is a corrected (weaker) version of a result originally announced by the author.

where $P_0(D)$ *is an elliptic operator of order* m *with constant real coefficients, while* $b_a(x)$ *are bounded functions such that*

$$(3.1) \qquad b_a(x) = 0(e^{-\epsilon|x|}) \quad \text{as} \quad |x| \to \infty,$$

$\epsilon > 0$. *Suppose furthermore, that* P_0 *verifies the following conditions.* (i) *The set of real zeros of* $P_0(\xi)$ *is a* $n-1$ *dimensional manifold on which grad* $P_0(\xi) \neq 0$. (ii) *There exists a smooth path* γ *in* $R^n : [0, \infty) \ni t \to \gamma(t) \in R^n$, *such that* $\gamma(0) = 0$ and $d\gamma/dt = N_0$ *for* $t \geqslant t_0$, N_0 *a fixed non-vanishing vector. having the following property. If* $P_0(\xi + i\gamma(t)) = 0$ *for some* $\xi \in R^n$ *and* $0 < t < \infty$, *then* $\mathrm{Re}\{\mathrm{grad}_\xi P_0(\xi + i\gamma(t))\}$ and $\mathrm{Im}\{\mathrm{grad}_\xi P_0(\xi + i\gamma(t))\}$ *are linearly independent vectors.*
(iii) $\mathrm{grad}_\xi P_0'(\xi + iN_0) \neq 0$ *whenever* $P_0'(\xi + iN_0) = 0$ *(*P_0' *denotes the principal part of* P_0*).*

Under the above conditions the operator $P(x, D)$ *possesses the* L_2 *S-property.*

From Theorem 3.1, one derives the following solution to Problem III. Suppose that for every λ in $[a, b]$ the differential operator $A - \lambda$ verifies the conditions of Theorem 3.1. Suppose also that $A - \lambda$ possesses the unique continuation property (e.g. [3], [4]). Then spectrum $\mathfrak{A}$ contains no eigenvalues in $[a, b]$.

4. Concluding remarks

Returning to Theorem 2.1 one may ask whether the assumption (2.2) on the rate of decay of the coefficients of B cannot be replaced by a weaker assumption such as $b_a = 0(|x|^{-\epsilon})$ for any $\epsilon > 0$. While this may not be the case in general we do believe that a version of Theorem 2.1 holds under such weaker decay assumptions on the coefficients if additional smoothness conditions are imposed. In this connection, let us note that in the special case $A = -\Delta + q$ it is known that $\mathfrak{A}$ has no positive eigenvalues if q is a C^1 function such that

$$(4.1) \qquad q = 0(|x|^{-\epsilon}) \quad \text{and} \quad |\mathrm{grad}\, q| = 0(|x|^{-1-\epsilon})$$

for some $\epsilon > 0$ ([1 ; 2], [12]). If (4.1) holds with $\epsilon > 1/2$ it could also be shown that the positive spectrum of $\mathfrak{A}$ is absolutely continuous.

REFERENCES

[1] Agmon S. — Lower bounds for solutions of Schrödinger-type equations in unbounded domains, *Proc. Inter. Conf. on Functional Analysis and Related Topics, Tokyo*, 1969, p. 216-224.

[2] Agmon S. — Lower bounds for solutions of Schrödinger equations, *Journal d'Analyse Mathématique*, 23, 1970, p. 1-25.

[3] Calderon A.P. — Uniqueness in the Cauchy problem for partial differential equations *Amer. J. Math.*, 80, 1958, p. 16-36.

[4] Hörmander L. — *Linear Partial Differential Equations*, Springer-Verlag, 1963.

[5] Ikebe T. — Eigenfunction expansions associated with the Schroedinger operators and their Applications to Scattering Theory, *Arch. Rational Mech. Anal.*, 5 1960, p. 1-34.

[6] JÄGER W. — *Zur Theorie der Schwingungsgleichung mit variablen Koeffizienten in Aussengebieten*, *Math. Zeit.*, 102, 1967, p. 62-88.
[7] KATO T. — Growth properties of solutions of the reduced wave equation with a variable coefficient, *Comm. Pure Appl. Math.*, 12, 1959, p. 403-425.
[8] KATO T. — Some results on potential scattering, *Proc. Inter. Conf. on Functional Analysis and Related Topics*, Tokyo, 1969, p. 206-215.
[9] LITTMAN W. — Decay at infinity of solutions to partial differential equations with constant coefficients, *Trans. Amer. Math. Soc.*, 123, 1966, p. 449-459.
[10] LITTMAN W. — Decay at infinity of solutions to partial differential equations: Removal of the curvature assumption, *Israel J. Math.*, 8, 1970 (to appear).
[11] REJTO P.A. — On partly gentle perturbations III, *J. Math. Anal. Appl.*, 27, 1969, p. 21-67.
[12] SIMON B. — On positive eigenvalues of one-body Schrödinger operators, *Comm. Pure Appl. Math.*, 22, 1969, p. 531-538.

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SUR LA THÉORIE DES PROBLÈMES AUX LIMITES ELLIPTIQUES NON-FREDHOLMIENS

par A. V. BITSADZE

Etant donné un domaine $\mathcal{D}$ de l'espace Euclidien à n dimensions ($n \geqslant 2$), dont la frontière S est une hypersurface à $n - 1$ dimensions.

Considérons deux opérateurs linéaires

$$L = A^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + B^i \frac{\partial}{\partial x_i} + C \,, \; i, j = 1, \ldots, n \,, \; x \in \mathcal{D}$$

et

$$l = \alpha^i \frac{\partial}{\partial y_i} + \beta \,, \; i = 1, \ldots, n \,, \; y \in S$$

dont les coefficients sont des matrices réelles $m \times m$.

L'opérateur $\epsilon = A^{ij} \dfrac{\partial^2}{\partial x_i \partial x_j}$ est supposé uniformément elliptique.

Le problème aux limites elliptique, qui va nous occuper, est suivant : *Déterminer dans le domaine $\mathcal{D}$ un vecteur $u(x) = (u_1, \ldots, u_m)$ régulier (deux fois continuellement dérivable), qui satisfait aux conditions*

$$Lu = f(x) \,, \; x \in \mathcal{D} \tag{1}$$

$$lu = \gamma(y) \,, \; y \in S, \tag{2}$$

où $f = (f_1, \ldots, f_m)$ et $\gamma = (\gamma_1, \ldots, \gamma_m)$ sont vecteurs réels donnés.

Des cas particuliers du problème (1) - (2) sont *le problème de Dirichlet*

$$(\alpha^i = 0 \,, \; i = 1, \ldots, n \,, \; \det \beta \neq 0) \,,$$

le problème de Poincaré ($m = 1$, $\sum_{i=1}^{m} (\alpha^i)^2 \neq 0$) etc.

Nous ne traiterons que des cas où $\mathcal{D}$ est un domaine borné, S est une hypersurface de Lyapounov, toutes les données : $A^{ij}, B^i, C, f, \alpha^i, \beta, \gamma$ est le vecteur inconnu $u(x)$ sont assez régulières.

Le problème aux limites elliptique

$$Lu = 0 \,, \; x \in \mathcal{D}, \tag{3}$$

$$lu = 0 \,, \; y \in S \tag{4}$$

sera appelé : *problème homogène correspondant à* (1) - (2).

Le problème aux limites elliptique comme il a été posé plus haut n'est pas en général toujours résoluble d'une façon unique. *Construire une théorie du problème aux limites elliptique signifie, d'une part,* établir *les degrés de son indétermination ou de sa surdétermination et, d'autre part, ce qui est l'essentiel, trouver* pour (2) *les conditions aux limites complémentaires à imposer à* $u(x)$ *(si le problème est indéterminé) où les conditions de compatibilité à imposer aux données* f *et* γ (si le problème est surdéterminé) *nécessaires et suffisantes pour l'existence et l'unicité de solution.*

Par degré x_1 de l'indétermination du problème (1) - (2) nous convenons de désigner la dimension de l'espace des solutions du problème aux limites homogène (3) - (4) correspondant à (1) - (2) et par degré x_2 de sa surdétermination - la dimension d'un certain espace de fonctionnelles linéaires associées d'une façon naturelle au problème envisagé dont l'annulation par les seconds membres (1) et (2) représente une condition de compatibilité sur f et γ, assurant l'existence d'une solution. Pour réaliser le but essentiel de notre étude il devient nécessaire d'étudier en détail le caractère structural de ces espaces.

Il est bien connu, que, dans le cas du problème de Dirichlet pour l'équation uniformément elliptique (1), si $m = 1$, les deux nombres x_1 et x_2 sont finis et égaux entre eux. Ou bien, comme il est encore d'usage de dire, le problème de Dirichlet pour l'équation uniformément elliptique (1), si $m = 1$, est un problème *fredholmien.*

Si (1) *représente un système d'équations* ($m > 1$) *l'hypothèse de l'ellipticité uniforme de l'opérateur* ϵ *ne suffit pas à garantir, en général, la fredholmité du problème de Dirichlet pour l'équation* (1).

Parmi des problèmes aux limites elliptiques non-fredholmiens d'un grand intérêt sont ceux, dont les degrés de l'indétermination x_1 et de la surdétermination x_2 sont des nombres finis, non nécessairement égaux. Il est d'usage de les qualifier de *noetheriens.* Signalons enfin l'importance considérable de certains problèmes non-Fredholmiens, qui ne sont pas neotheriens, mais qui peuvent être décrits en termes d'alternatives de Hausdorff bien connus dans la théorie des équations linéaires dans les espaces linéaires abstraits.

Si $m = 1$, le caractère fredholmien ou noetherien du problème aux limites elliptique ne dépend que des parties principales de l'équation (1) et de la condition aux limites (2). Dans le cas où $m > 1$, les coefficients B^i et C dans (1) peuvent exercer une influence essentielle sur le caractère du problème (1) - (2), [1, 2].

Le problème (1) - (2) en cas de deux variables indépendantes ($n = 2$), quand $A^{ii} = E$, $A^{ij} = 0$, $i \neq j$, où E est la matrice (diagonale) unité et

$$\det(\alpha^1 + i\alpha^2) \neq 0 \tag{5}$$

a été étudié par nous en 1944 [1, 3, 4]. Nous avons établi, que :

(a) *les nombres* x_1 *et* x_2 *sont finis*

(b) *la différence* $x_1 - x_2 = x$ *(l'indice du problème* (1) - (2)) *est égale à* $2(p + m)$, *où* $2\pi p$ *est l'accroissement de* arg det $(\alpha^1 - i\alpha^2)$ *le long du bord* S *du domaine* $\mathcal{D}$ *parcouru dans le sens positif.*

(c) *les conditions nécessaires et suffisantes de résolubilité du problème* (1) - (2) *sont construites explicitement.* En même temps il fut démontré, que *si la condition* (5) *n'est pas satisfaite le coefficient* β *du premier membre* (2) *a une influence essentielle sur le caractère noetherien du problème* (1) - (2). Ensuite tous ces résultats furent généralisés aux classes plus larges des systèmes elliptiques à deux variables indépendantes [1, 5, 6].

Les classes vastes des problèmes aux limites elliptiques non-fredholmiens bidimensionnels sont analysées dans les monographies [7, 8, 9].

Il faut mentionner que, dans le cas $n = 2$, l'étude du problème a pour base la théorie très commode des équations intégrales singulières ayant des noyaux de Cauchy ou de Hilbert.

Quand le nombre des variables indépendantes $n > 2$, l'étude du problème (1) - (2) présente des difficultés considérables. Ici l'absence d'une théorie générale des équations intégrales singulières multidimensionnelles se fait sentir.

Un cas particulier très important du problème (1) - (2) est le problème de la dérivée oblique

$$(6) \qquad l(y) \text{ grad } u = f(y) \,, y \in S \,,$$

où le vecteur $l = (l_1, \ldots, l_n)$, qui ne s'annule pas, et la fonction f sont donnés sur S et où $u(x)$ est la fonction inconnue, harmonique dans le domaine $\mathcal{D}$; lorsque $n > 2$, l'analyse de ce problème est loin d'être achevée.

A condition que le vecteur l n'a pas des points de contact avec S le caractère fredholmien du problème de la dérivée oblique pour les fonctions harmoniques ainsi que pour les solutions d'une équation uniformément elliptique (1) à $m = 1$ fut établi par les travaux [10] et [11].

Le problème (6) ne peut donc être non-fredhomien que si l'ensemble M des points de contact du vecteur l avec S n'est pas vide.

Dans le cas $n = 2$, le nombre entier p exprimé par l'indice du problème (6) coïncide avec l'indice de Kronecker du champ vectoriel $(l_1 - l_2)$; de plus, si $p \geqslant 0$ le problème (6) est inconditionnellement résoluble et le degré de son indétermination est égal à $2p + 2$; si $p \leqslant -1$ le degré de surdétermination du problème (6) est égal à $-2p - 1$ et le degré de l'indétermination est un [12].

Si $n > 2$, le rôle de la rotation du champ vectoriel $l = (l_1, \ldots, l_n)$ dans les investigations consacrées à l'analyse du problème (6) est encore difficile à saisir.

Il est assez facile de vérifier la justesse des affirmations suivantes :

(1) *si l'ensemble M consiste en k points, le degré de l'indétermination du problème* (6) *ne peut pas dépasser k,*

(2) *si l'ensemble M consiste en k arcs, mutuellement disjoints à tangentes continues, en chaque point desquels la direction de la tangente à l'arc coïncide avec la direction l, le degré de l'indétermination du problème* (6) *ne peut pas dépasser k.*

Sous l'hypothèse que les composantes du vecteur $l(y)$ sont des polynômes des coordonnées du point $y = (y_1, \ldots, y_n)$ de la frontiere S, nous allons désigner par $l(y)$ le prolongement de ce polynome au domaine $\mathcal{D}$. Soit m le degré du polynôme $l(x)$.

La fonction $\nu(x) = l(x)$ grad $u(x)$ dans le domaine $\mathcal{D}$ est une solution régulière de l'équation polyharmonique $\Delta^{n+1}\ \nu = 0$ qui satisfait à la condition aux limites

$$\nu(y) = f(y)\ ,\ y \in S. \tag{7}$$

Cette condition définit $\nu(x)$ de façon unique si et seulement si $m = 0$.

Le problème (6) est ainsi réduit à la détermination des solutions harmoniques de l'équation linéaire aux dérivées partielles du premier ordre.

$$l(x) \text{ grad } u(x) = \nu(x)\ , \tag{8}$$

où $\nu(x)$ est une fonction polyharmonique dans $\mathcal{D}$ satisfaisante à la condition aux limites (7).

Si l'on connaît $n - 1$ intégrales premières holomorphes indépendantes

$$\xi_k(x)\ ,\ k = 1, \ldots, n-1$$

du système des équations différentielles ordinaires

$$dx - l(x)\, dt = 0 \tag{9}$$

et une solution particulière holomorphe $u_0(x)$ de l'équation (8), – alors la solution holomorphe générale dans $\mathcal{D}$ de cette équation peut être écrite en forme

$$u(x) = \phi(\xi) + u_0(x)\ , \tag{10}$$

où ϕ est une fonction arbitraire holomorphe des variables $(\xi_1, \ldots, \xi_{n-1}) = \xi$.

Les questions de l'existence des intégrales premières indépendantes du système (9) et de la solution particulière holomorphe de l'équation (8) furent étudiées par Poincaré [13] à condition que le vecteur $l(y) \neq 0$ partout sur S. Cidessous, nous allons supposer cette condition remplie.

Si $\mathcal{D}$ est un domaine convexe simplement connexe contenant l'origine des coordonnées, l'équation (8) peut être écrite de la façon suivante

$$l(x) \,.\, \text{grad } u(x) = \sum_{j=0}^{m} |x|^{2j}\ \nu_j(x) \tag{11}$$

où les $\nu_j, (j = 0, \ldots, m)$ sont des fonctions harmoniques arbitraires dans $\mathcal{D}$.

Dans le cas où $l(x) \neq 0$ partout dans le domaine fermé $\overline{\mathcal{D}}$ (dans ce cas l'indice de Kronecker, qui caractérise la rotation du champ vectoriel $l(y)$, est égal à zéro) les intégrales premières holomorphes $\xi_k(x)\,, k = 1, \ldots, n-1$ du système (9) et une solution particulière holomorphe $u_0(x)$ de l'équation (11) existent ; par conséquent, la solution générale de cette équation peut être mise sous la forme (10). Pour que la fonction holomorphe $u(x)$ représentée par la formule (10) soit harmonique dans le domaine $\mathcal{D}$, la fonction $\phi(\xi)$ doit satisfaire à une équation aux dérivées partielles de deuxième ordre à $n - 1$ variables indépendantes. Par conséquent *dans le cas envisagé le problème de la dérivée oblique* (6) *est toujours résoluble et le degré de son indétermination est défini par des éléments arbitraires, contenus* *dans le second membre de l'équation* (11) *et dans la solution générale holomorphe de l'équation aux dérivées partielles imposée à* $\phi(\xi)$. En se basant sur l'étude de la structure de l'espace des solutions du problème homogène correspondant à (6) on peut trouver des conditions aux limites complémentaires pour la résolubilité unique du problème (6). Des sous-variétés de l'hypersurface S doivent porter ces données complémentaires.

Si l'indice de Kronecker du champ vectoriel $l(y)$ est différent du zéro, alors $\mathcal{D}$ contient nécessairement des points, où $l(x) = 0$. Ces points représentent des points singuliers du système (9), qu'on classifie selon le caractère des racines de l'équation $\det(A - E\lambda) = 0$, où $A(x) = \left\| \dfrac{\partial l_i}{\partial x_x} \right\|$, $i, k = 1, \ldots, n$, et où E est la $n \times n$ matrice unité.

Quand $x = x_0 \in \mathcal{D}$ est le seul point singulier du système (9) dans $\mathcal{D} \cup S$, les degrés de l'indétermination et la surdétermination du problème (8) dépendent essentiellement du type de ce point singulier. C'est à cause de cela, qu'au cours de l'accroissement de n l'analyse du problème (6) devient de plus compliquée.

La formule de Hilbert bien connue

$$\frac{\partial u}{\partial x_2} = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{\theta - \varphi}{2} \quad \frac{\partial u}{\partial y_1}\, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial y_2}\, d\theta \ , x_1 + ix_2 = e^{i\varphi} \ , y_1 + iy_2 = e^{i\theta},$$

connectant entre eux les valeurs aux limites sur la circonférence $S : |x| = 1$ des dérivées partielles $\partial u/\partial x_1$ et $\partial u/\partial x_2$ de la fonction harmonique $u(x)$ de la classe $C^{(1,h)}(\mathcal{D} \cup S)$ dans le cercle $\mathcal{D} : |x| < 1$, permet, dans le cas $n = 2$, de ramener le problème de la dérivée oblique (6) à l'équation intégrale singulière

$$l_1 \mu(\varphi) + \frac{l_2}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{\theta - \varphi}{2}\, \mu(\theta) d\theta = f(\varphi) + Cl_2 , \tag{12}$$

où $C = -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{\partial u}{\partial y_2}\, d\theta$ est la constante inconnue, tandis que $\mu = \partial u/\partial x_1$ est la fonction inconnue sur la circonférence S.

La réduction du problème (6) à une équation intégrale singulière multidimensionnelle est possible aussi quand $n > 2$. Ainsi, quand $n = 3$, les valeurs limites sur la sphère $S : |x| = 1$ des dérivées partielles $\partial u/\partial x_k$, $k = 1, 2, 3$, dans la boule $\mathcal{D} : |x| < 1$ de la fonction harmonique $u(x)$ de la classe $C^{(1,h)}(\mathcal{D} \cup S)$ sont liées entre elles par les formules

$$\frac{\partial u}{\partial x_k} = -\frac{1}{2\pi}\iint \left[\frac{x_1 y_k - y_1 x_k}{\delta R^{1/2}} + \frac{(x_1 + y_1)(x_k - y_k)}{\delta^2} R^{1/2} \right] \frac{\partial u}{\partial y_1}\, ds_y + \frac{\partial \eta}{\partial x_k}, \ k = 2, 3, \tag{13}$$

où $\delta = (x_2 - y_2)^2 + (x_3 - y_3)^2$, $R = \delta + (x_1 - y_1)^2$, tandis que $\eta(x_2, x_3)$ est la fonction harmonique dans le cylindre $x_2^2 + x_3^2 < 1$, définie par la relation

$$\eta(x_2, x_3) + u(0, 0, 0) + 2 \operatorname{Re} u\left(0 , \frac{x_2 + ix_3}{2}, \frac{x_2 + ix_3}{2i}\right) = 0 .$$

En substituant les valeurs aux limites $\partial u/\partial x_2$ et $\partial u/\partial x_3$ de (13) dans le premier membre de la condition (6), nous avons

$$(14)\ l_1\psi - \frac{1}{2\pi}\sum_{k=2}^{3} l_k \iint\left[\frac{x_1 y_k - y_1 x_k}{\delta R^{1/2}} + \frac{(x_1+y_1)(x_k-y_k)}{\delta^2} R^{1/2}\right]\psi\, ds_y = f - \sum_{k=2}^{3} l_k \frac{\partial \eta}{\partial x_k},\ x \in S,$$

où $\psi = \partial u/\partial x_1$. Ainsi *le problème de la dérivée oblique* (6) *dans le cas envisagé se réduit à une équation intégrale singulière bidimensionnelle* (14), *dont le noyau est un analogue bidimensionnel du noyau Hilbert* ctg $(\theta - \varphi)/2$ *de l'équation intégrale* (12).

REFERENCES

[1] BITSADZE A.V. — *Boundary Value Problemes for sec. ordr. ellipt. Equations,* 1966.
[2] SAKS R.S. — *Dokl. Akad. Nauk S.S.S.R.,* 177, No. 4, 1967, p. 786-789.
[3] BITSADZE A.V. Soob. — *Akad. Nauk Gruz. S.S.R.,* 5, No. 8, 1944, p. 761-770.
[4] VEKUA N.P. — *Systems of sing. integr. Equations,* 1950.
[5] VOLPERT A.I. — *Trudy Mos. Mat. ob.,* 10, 1961, p. 41-87.
[6] TOVMASYAN N.E. — *Diff. Uravn.,* 2, No. 1, 1966, p. 3-23; No. 2, 1966, p. 163-171.
[7] VEKUA I.N. — *New Meth. for Solv ellipt. Equations,* North-Holland, 1967.
[8] VEKUA I.N. — *General Anal. Funktions,* 1959.
[9] MUSHELIŠVILI N.I. — *Singul. integr. Equations,* 1962.
[10] GIRAUD G. — *Bull. Sc. Math.,* 56, 1932, p. 248-272; p. 281-312; p. 316-352.
[11] BOULIGAND G., GIRAUD G. et DELENS P. — *Le problème de la dérivée obl.,* Publ. sous la direct. de M.E. Cartan, N° 6, 1935, p. 5-78.
[12] JACOB C. — *Bull. Math. Soc. Roumaine,* 42, 1940, p. 9-47.
[13] POINCARÉ H. — *Œuvres,* Paris, 1928.

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UNICITÉ DU PROBLÈME DE CAUCHY ET HYPOELLIPTICITÉ POUR UNE CLASSE D'OPÉRATEURS DIFFÉRENTIELS

par Jean-Michel BONY

1. Introduction et notations

Dans la théorie générale des opérateurs différentiels à coefficients constants, un rôle essentiel est joué par le polynome caractéristique et l'ensemble des zéros de celui-ci. Dans le cas où les coefficients sont variables, certaines propriétés sont liées plus précisément à l'algèbre de Lie engendrée par l'idéal caractéristique. Nous démontrons d'abord un résultat général d'unicité du problème de Cauchy pour des opérateurs à coefficients analytiques. Nous établissons ensuite un théorème d'hypoellipticité lié à cette algèbre de Lie, bien que des conditions supplémentaires portant sur les termes d'ordre inférieur soient nécessaires en général.

Nous désignerons par P un opérateur différentiel d'ordre p, à coefficients C^∞, défini dans un ouvert de $\mathbf{R}^n$

$$P\left(x, \frac{\partial}{\partial x}\right) = \sum_{|\alpha| \leqslant p} a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha ,$$

et par P_p son polynome caractéristique :

$$P_p(x, \xi) = \sum_{|\alpha|=p} a_\alpha(x)\, \xi^\alpha$$

Introduisons la *variété caractéristique* : ensemble des couples (x, ξ) tels que l'on ait $P_p(x, \xi) = 0$; et l'*idéal caractéristique* : ensemble des polynomes $Q(x, \xi)$ homogènes en ξ et à coefficients C^∞ en x qui s'annulent sur la variété caractéristique. Nous noterons $\mathcal{K}$ cet idéal.

Rappelons la définition du crochet de Poisson :

$$[Q_1, Q_2](x, \xi) = \sum_i \left(\frac{\partial Q_1}{\partial \xi_i}\frac{\partial Q_2}{\partial x_i} - \frac{\partial Q_1}{\partial x_i}\frac{\partial Q_2}{\partial \xi_i}\right)$$

$[Q_1, Q_2](x, \xi)$ est le polynome caractéristique de l'opérateur différentiel :

$$Q_1\left(x, \frac{\partial}{\partial x}\right) \circ Q_2\left(x, \frac{\partial}{\partial x}\right) - Q_2\left(x, \frac{\partial}{\partial x}\right) \circ Q_1\left(x, \frac{\partial}{\partial x}\right)$$

La notion suivante sera fondamentale dans cet exposé.

DEFINITION 1. – *L'algèbre de Lie engendrée par l'idéal caractéristique, notée Lie* ($\mathcal{K}$) *est le plus petit idéal (gradué) contenant* $\mathcal{K}$ *et stable par le crochet de Poisson. Pour que* $R(x, \xi)$ *appartienne à Lie* ($\mathcal{K}$), *il faut il suffit qu'il soit combinaison linéaire de termes du type :*

$$[Q_{i_1}, [Q_{i_2}, [\ldots, Q_{i_l}] \ldots]] \quad \textit{où les } Q_{i_k} \textit{ appartiennent à } \mathcal{K}.$$

Remarque. – Lorsque la variété caractéristique est pour chaque x de dimension $n - 1$ en ξ, l'idéal caractéristique est principal (au moins sous des hypothèses d'analyticité ou de constance du rang). On a alors Lie ($\mathcal{K}$) = $\mathcal{K}$. La notion n'a donc d'intérêt que dans le cas où cette variété a une dimension strictement inférieure à $n - 1$ (opérateurs "elliptiques dégénérés"). Nous verrons sur des exemples que Lie ($\mathcal{K}$) peut être alors beaucoup plus grande que $\mathcal{K}$.

2. Unicité du problème de Cauchy

Nous supposerons dans tout ce paragraphe que les coefficients de P sont analytiques. Nous allons énoncer dans ce cas une extension du théorème classique de Holmgren, et nous bornerons à donner une idée des démonstrations, en renvoyant à [2] pour une preuve plus complète.

Rappelons qu'une surface S (de classe C^1) est *caractéristique* en un point x_0 si on a $P_p(x_0, \nu) = 0$ en désignant par ν la normale à S en x_0.

Rappelons également le théorème de Holmgren (voir [3] chapitre 5) : Soient S une surface non caractéristique en x_0, et u une distribution vérifiant $P\left(x, \frac{\partial}{\partial x}\right) u = 0$ et nulle d'un coté de S ; alors u est nulle au voisinage de x_0.

DEFINITION 2. – *Une surface S de classe C^1 sera dite fortement caractéristique en x_0 si on a $R(x_0, \nu) = 0$ pour chaque R appartenant à Lie* ($\mathcal{K}$), *en désignant par ν la normale à S en x_0.*

THEOREME 1. – *Soit S une surface de classe C^1 non fortement caractéristique en x_0, et soit u une distribution vérifiant $P\left(x, \frac{\partial}{\partial x}\right) u = 0$ et nulle d'un coté de S. Alors, on a $u = 0$ au voisinage de x_0.*

COROLLAIRE. – (Unicité du prolongement des solutions)

Supposons que pour chaque couple (x, ξ) tel que ξ soit non nul, il existe R appartenant à Lie ($\mathcal{K}$) *tel que $R(x, \xi) \neq 0$. Alors, si une solution u de $P\left(x, \frac{\partial}{\partial x}\right) u = 0$ est nulle au voisinage d'un point, elle est nulle dans la composante connexe de ce point.*

Le corollaire se déduit du théorème par un argument classique. Le théorème résulte de la proposition suivante, pour laquelle nous introduisons quelques notations.

Nous dirons qu'un vecteur ν est *normal* à un fermé F en un de ses points x_0 s'il existe une sphère ne rencontrant F qu'en x_0 et de normale ν en ce point. On a alors :

PROPOSITION 1. – *Soit F un fermé, et soient* $Q_1(x, \xi)$ et $Q_2(x, \xi)$ *deux fonctions homogènes en* ξ *et de classe* C^1 *pour* $\xi \neq 0$. *Supposons que pour chaque point x de F et chaque normale* ν à F en x, on ait :

$$Q_1(x_0, \nu) = Q_2(x, \nu) = 0 .$$

On a alors :

$$[Q_1, Q_2]\,(x, \nu) = 0 .$$

Si on désigne par F le support d'une solution u de

$$P\left(x, \frac{\partial}{\partial x}\right) u = 0 ,$$

on voit aisément que le théorème de Homgren assure que l'on a $Q(x, \nu) = 0$ pour chaque point x de F, chaque normale ν en ce point et chaque Q appartenant à l'idéal caractéristique, tandis que la conclusion du théorème 1 équivaut à la même assertion pour tous les Q appartenant à Lie $(\mathcal{H})$. La proposition entraîne ainsi immédiatement le théorème.

La démonstration de la proposition est purement géométrique. Le résultat en est bien connu lorsque F est limité par une surface de classe C^2. Le résultat essentiel à la démonstration du cas général est le suivant : sauf sur un ensemble de mesure aussi petite qu'on le veut, la fonction distance de x à F coïncide avec une fonction de x dont les dérivées secondes appartiennent à L^∞ (voir [1] et [2]).

Exemples. – Un cas typique où les résultats précédents s'appliquent est celui d'un opérateur elliptique dégénéré du second ordre P, à coefficients analytiques qui se décompose sous la forme suivante :

$$Pu = \sum_{i=1}^{r} X_i^2 u + X_0 u + cu ,$$

où les X_i sont des opérateurs différentiels homogènes du premier ordre, et à coefficients réels pour $i = 1, \ldots, r$.

Alors, si l'algèbre de Lie engendrée par $X_1, \ldots, X_r$ est de rang n en chaque point, on a la propriété de prolongement unique des solutions.

On peut donner bien d'autres exemples, ainsi $P = M(X_1, \ldots, X_r)$ où M est un polynome elliptique en r variables et où les X_i vérifient la condition précédente. D'autre part, on peut substituer aux X_i des opérateurs Q_i d'ordre q tels que pour chaque (x, ξ) avec $\xi \neq 0$, il existe un R, appartenant à l'algèbre de Lie engendrée par les Q_i tel que l'on ait $R(x, \xi) \neq 0$.

3. Inégalités

Il n'est pas possible de donner un résultat aussi général que le précédent sur la régularité des solutions. Nous montrerons en effet, sur une classe particulière d'opérateurs, que des résultats d'hypoellipticité doivent faire intervenir non seulement la partie principale, par l'intermédiaire de Lie $(\mathcal{H})$, mais encore les termes d'ordre inférieur.

Nous commençons par énoncer deux inégalités générales. La première, due à Kohn et Radkevitch, permet de préciser la régularité de Ru, où R appartient à Lie ($\mathcal{K}$), à partir de celle des Qu pour Q appartenant à $\mathcal{K}$. La seconde, très insuffisante, donne une relation entre la régularité des Qu, pour Q appartenant à $\mathcal{K}$, et celle de Pu.

PROPOSITION 2. – *Soient Q_1 et Q_2 deux opérateurs différentiels d'ordre respectif q_1 et q_2. Alors, quels que soient s réel et ϵ tel que* $0 < \epsilon \leqslant 1/2$, *il existe une constante C telle que l'on ait*

$$\|[Q_1 , Q_2]\|_{s-q_1-q_2+1+\epsilon} \leqslant C(\|Q_1 u\|_{s-q_1+2\epsilon} + \|Q_2 u\|_{s-q_2+1} + \|u\|_s)$$

pour chaque u de classe C^∞ et à support compact, en désignant par $\|\,\|_t$ la norme de l'espace de Sobolev H^t.

Pour une démonstration, voir [5] et [6].

COROLLAIRE. – *Si R appartient à l'algèbre de Lie engendrée par des opérateurs $Q_1, \ldots, Q_r$, il existe $\epsilon > 0$ tel que pour chaque s, on ait une majoration :*

$$\|Ru\|_{s-r+\epsilon} \leqslant C\left(\sum_{i=1}^{r} \|Q_i u\|_{s-q_i+1} + \|u\|_s\right)$$

valable pour les u de classe C^∞ à support compact, en désignant par r et q_i les ordres respectifs de R et Q_i.

COROLLAIRE. – *Supposons que pour chaque (x , ξ), il existe R appartenant à l'algèbre de Lie engendrée par $Q_1, \ldots, Q_r$, tel que $R(x , \xi) \neq 0$. Alors il existe $\epsilon > 0$ tel que, pour chaque s, on ait une majoration :*

$$\|u\|_{s+\epsilon} \leqslant C\left(\sum_{i=1}^{r} \|Q_i u\|_{s-q_i+1} + \|u\|_s\right)$$

Le premier corollaire se démontre par récurrence, tandis que le second résulte du fait que l'ensemble des opérateurs R de l'algèbre de Lie constitue un système surdéterminé elliptique.

PROPOSITION 3. – *Supposons les coefficients de P analytiques. Soit Q, d'ordre q, appartenant à l'idéal caractéristique, à coefficients analytiques. Il existe alors un entier k tel que l'on ait une majoration*

$$\|Q^k u\|_{s-kq} \leqslant C(\|Pu\|_{s-p} + \|u\|_{s-1})$$

valable pour les u de classe C^∞ à support compact.

Comme on le verra sur une classe particulière d'opérateurs, c'est essentiellement la présence de l'entier k qui ne permet pas d'utiliser cette majoration. La démonstration repose sur le résultat algébrique suivant :

Soient $f(x , \xi)$ et $g(x , \xi)$ deux polynomes en ξ à coefficients germes de fonctions analytiques, tels que g s'annule sur les zéros réels de f. On peut alors trouver des polynomes a_i et b_j en nombre fini, tels que l'on ait

$$f^2\left(\sum_i a_i^2\right) = g^{2k} + \sum_j b_j^2 .$$

Il suffit ensuite d'appliquer ce résultat, rendu homogène, à P et Q et d'intégrer par parties pour conclure.

4. Hypoellipticité

Nous nous limiterons aux opérateurs P du type suivant :

$$P = \sum_{i=1}^{r} Q_i^{2k} + L$$

où les Q_i sont des opérateurs différentiels à coefficients C^∞ réels, de même ordre q, et où L est un opérateur différentiel d'ordre $2kq - 1$.

On suppose en outre que pour chaque (x, ξ) avec $\xi \neq 0$, il existe R appartenant à l'algèbre de Lie engendrée par les Q_i tel que $R(x, \xi) \neq 0$.

(a) Si $k = 1$ et si L est à coefficients réels, P est hypoelliptique. C'est un cas particulier des résultats de Radkevitch (voir [6]). Dans le cas où les Q_i sont du premier ordre, le résultat est dû à Hörmander (voir [4]). Les théorèmes démontrée sont d'ailleurs plus généraux, faisant intervenir en plus la partie principale de L.

(b) Si $k = 1$ et si L est à coefficients complexes, P n'est plus nécessairement hypoelliptique comme le montre l'exemple suivant.

$$P = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} - i \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial x} + ix \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} - ix \frac{\partial}{\partial y}\right)$$

P se décompose en produit de deux facteurs qui ne sont pas hypoelliptiques, il n'y a pas en effet résolubilité locale pour leurs adjoints. Donc P ne peut être hypoelliptique.

Remarquons que $\overline{P}P = \left(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}\right)^2 + \frac{\partial^2}{\partial y^2}$ n'est pas hypoelliptique, bien qu'il soit à coefficients réels et que Lie ($\mathcal{K}$) contienne tous les opérateurs différentiels.

(c) Nous allons montrer que pour k quelconque, si les termes d'ordre inférieur sont suffisamment dominés par la partie principale, P est hypoelliptique.

THEOREME 2. — *On suppose que L est de la forme suivante :*

$$L = \sum_{i=1}^{r} \sum_{l=1}^{2k} Q_i^{2k-1} R_{i,l} + R'$$

où les $R_{i,l}$ sont des opérateurs différentiels à coefficients complexes d'ordre $l(q-1)$ et où R' est d'ordre $2k(q-1)+1$ et à coefficients réels. Alors P est hypoelliptique.

Les idées de la démonstration sont voisines de celles de [4]. On déduit de la proposition 2 la majoration suivante :

$$\|u\|_{k(q-1)+\epsilon} \leqslant C \left(\sum_i \|Q_i^k u\|_0 + \|u\|_{k(q-1)}\right)$$

On introduit la norme $|||u|||^2 = \sum_i \|Q_i^k u\|_0^2 + \|u\|_{k(q-1)}^2$ et on note $|||'$ la norme duale. L'étape essentielle est le résultat suivant :

PROPOSITION 4. – *Si u appartient à $H^{k(q-1)}$ et est à support compact, et si $|||Pu|||' < \infty$, alors, u appartient à $H^{k(q-1)+\epsilon}$.*

Cela s'établit d'abord pour u de classe C^∞, puis dans le cas général par un procédé de régularisation. On montre ensuite que si u appartient localement à $H^{t+k(q-1)}$ et si Pu appartient localement à $H^{t-k(q-1)}$, alors u appartient localement à $H^{t+k(q-1)+\epsilon}$, d'où résulte l'hypoellipticité.

On voit que l'on "perd" presque $2k$ dérivées par rapport au cas elliptique, ce qui rend naturel le fait que les conditions portent sur les termes d'ordre supérieur à $2k(q-1)+1$.

BIBLIOGRAPHIE

[1] BONY J.-M. — *C. R. Acad. Sc. Paris*, 268, 1969, p. 1015-1017.
[2] BONY J.-M. — *C. R. Acad. Sc. Paris*, 268, 1969, p. 1103-1106.
[3] HÖRMANDER L. — *Linear partial differential operators*, Springer (1963).
[4] HÖMANDER L. — Hypoelliptic second order differential equations, *Acta Math. Uppsala*, 119, 1967, p. 147-171.
[5] KOHN J.J. — Pseudo-differential operators and non elliptic problems, *C.I.M.E.*, Sept. 1968, p. 159-165.
[6] RADKEVITCH. — *Uspeki Mat. Nauk*, 24-2, 1969, p. 233-234.

66 rue Gay-Lussac
Paris 5ème

Actes, Congrès intern. Math., 1970. Tome 2, p. 697 à 701.

UNE ALGÈBRE ASSOCIÉE AUX PROBLÈMES AUX LIMITES ELLIPTIQUES

par Louis BOUTET de MONVEL

Nous nous proposons de décrire une méthode d'étude des problèmes aux limites elliptiques. Classiquement, il s'agit de résoudre un système d'équations :

$$(1) \qquad \begin{aligned} Pf &= g \qquad \text{dans } X \\ B_j f &= u_j \qquad \text{sur } \partial X \end{aligned}$$

où X est une variété à bord, de bord ∂X, P est un opérateur différentiel elliptique sur X, et les B_j sont des opérateurs différentiels.

Sous cette forme, le problème a été étudié par de très nombreux auteurs, et finalement résolu dans le cas le plus général par Agmon, Douglis et Nirenberg (1959).

On peut généraliser le problème en remplaçant l'opérateur P de (1) par un opérateur pseudo différentiel : ce nouveau problème a été étudié et résolu par Visik et Eskin dans une série d'articles commençant en 1964.

Dans le même ordre d'idées, on a remarqué et utilisé le fait que sur le bord ∂X, les dérivées successives d'une solution de l'équation $Pf = 0$ sont reliées entre elles par des opérateurs pseudo différentiels (Calderon, Seeley, Agranovitch - Dynin, Hörmander).

La théorie décrite ci-dessous rend bien compte de ces résultats.

1.

Afin d'éclairer ce qui suit, nous commençons par rappeler comment les opérateurs pseudo différentiels (Calderon - Zygmund, 1952 ; Kohn - Nirenberg, 1965) permettent de décrire la situation quand il n'y a pas de bord. Ces opérateurs forment une algèbre, ie. $P + Q$, $P \circ Q$ sont des opérateurs pseudo différentiels si P et Q en sont. En outre, ils donnent lieu à un calcul symbolique : si P est un tel opérateur, on définit son degré, puis son symbole (principal) $\sigma(P)$.

$\sigma(P)$ est une fonction sur l'espace T^*X des vecteurs cotangents non nuls (quand on a affaire à un système d'opérateurs, il est plus commode de l'interpréter comme un homomorphisme de fibrés vectoriels sur T^*X). σ respecte l'addition et la multiplication ; et on a le résultat suivant :

THEOREME. — *P est elliptique*

équivant à : *$\sigma(P)$ est inversible*

ou : *P possède une parametrix de degré* $-$ deg P (*i.e. il existe un opérateur pseudo différentiel Q de degré* $-$ deg P *tel que* $P \circ Q - 1$, $Q \circ P - 1$ *soient des opérateurs à noyau C^∞*)

Ceci permet de rendre compte très agréablement de la théorie des opérateurs elliptiques.

2. L'algèbre des opérateurs de Green

Nous proposons maintenant de développer une théorie analogue pour les problèmes aux limites elliptiques. A cette fin, on introduit toute une classe d'opérateurs : les opérateurs de Green. Un tel opérateur A opère sur l'espace $C^\infty(X) + C^\infty(\partial X)$ ($C^\infty(X)$ désigne l'espace des fonctions indéfiniment dérivables, dont toutes les dérivées ont une limite au bord). Il a une matrice :

$$A = \begin{pmatrix} P_X + G & K \\ T & Q \end{pmatrix} \tag{2}$$

Nous commençons par décrire chaque terme de cette matrice.

(a) Q est un opérateur pseudo différentiel sur le bord ∂X.

(b) P_X est défini comme suit : soit Y une variété voisinage de X, et P un opérateur pseudo différentiel sur Y. Si $f \in C^\infty(X)$, on pose

$$P_X f = P(\tilde{f})/X$$

où $\tilde{f}$ est la fonction qui prolonge f par 0 hors de X ($P\tilde{f}/X$ est la restriction à X)

L'opérateur P_X ainsi défini n'est en général pas continu : $C^\infty(X) \to C^\infty(X)$, et pour qu'il le soit, il faut imposer à P une condition (transmission le long du bord ∂X)

(Si X est le demi espace $x_n \geqslant$ dans R^n, et si P est défini par la formule

$$Pf(x) = (2\pi)^{-n} \int e^{ix.\xi}\, p(x\,,\,\xi) \quad f(\xi)\, d\xi$$

P vérifie la condition de transmission si son symbole $p(x\,,\,\xi)$ et chacune de ses dérivées admet, pour chaque x du bord ($x_n = 0$) et chaque ξ' fixés, un développement asymptotique en puissances entières de ξ_n quand ξ_n tend vers l'infini :

$$p(x\,,\,\xi'\,,\,\xi_n) \sim \sum_{k \text{ entier} \geqslant -N} a_k(x\,,\,\xi') \quad \xi_n^{-k}$$

(c) K (opérateur de Poisson) est continu : $C^\infty(\partial X) \to C^\infty(X)$, de la forme

$$Ku = P(u\, \delta_{\partial X})/X$$

où P est un opérateur pseudo différentiel défini au voisinage de X, qui a comme ci-dessus la propriété de transmission, et $\delta_{\partial X}$ désigne une mesure de densité C^∞ sur le bord ∂X.

(d) T (opérateur trace) est continu : $C^\infty(X) \to C^\infty(\partial X)$, de la forme

$$Tf = \sum_1^N Q_k\, (P_X^k\, f \,/\, \partial X)$$

où les Q_k sont des pseudo différentiels sur le bord, et les P^k comme dans (b)

(e) Enfin G (opérateur de Green singulier) est continu $C^\infty(X) \to C^\infty(X)$, de la forme

$$G = \sum_1^\infty K_n\, T_n$$

où K_n, T_n sont deux suites à décroissance rapide d'opérateurs de Poisson (b–) (resp. trace, (c–)).

Les opérateurs de Green forment une algèbre (autrement dit $A + B$, $A \circ B$ sont des opérateurs de Green si A et B en sont).

Il est commode de les généraliser un peu pour les faire opérer sur les sections C^∞ de fibrés vectoriels.

3. Calcul symbolique

Si A est un opérateur de Green, on définit son degré, puis son symbole (principal). En fait il y a deux symboles :

– le symbole intérieur $\sigma_X(A)$, qui est simplement le symbole de l'opérateur pseudo différentiel P qui intervient dans la définition de A (2.b) : $\sigma_X(A)$ est donc une fonction (homomorphisme de fibrés) sur T^*X.

– le symbole bord $\sigma_{\partial X}(A)$. C'est un objet plus compliqué, et je me contente d'en donner une image grossière (et malheureusement pas tout à fait correcte si le degré de A est positif) :

soit $H = L^2(R)$ l'espace des fonctions de carré sommable sur R.

soit $H^+ \subset H$ le sous espace des fonctions qui admettent un prolongement holomorphe (et pas trop grand) dans le demi plan complexe inférieur im $t \leqslant 0$ (H^+ est l'espace des transformées de Fourier des fonctions L^2 portées par la demi droite positive). Enfin soit p^+ la projection orthogonale sur H^+.

Si f est une fonction bornée sur R, l'opérateur p^+f (qui transforme $\varphi \in H^+$ en $p^+(f\varphi)$) est continu : $H^+ \to H^+$. Si f et g sont continues sur R, y compris à l'infini, $p^+fg - p^+f\,p^+g$ est un opérateur compact : $H^+ \to H^+$.

Soient E, E', F, F' quatre espaces vectoriels de dimension finie. Appelons opérateur de Wiener-Hopf un opérateur

$$a : H^+ \times E + F \to H^+ \times E' + F'$$

de matrice

$$a = \begin{pmatrix} p^+f + g & k \\ t & q \end{pmatrix} \qquad (3)$$

où f est une matrice à coefficients continus (y compris à l'infini), g est un opérateur compact, k, t, q sont arbitraires (de rang fini).

Le symbole bord $\sigma_{\partial X}(A)$ fait correspondre à tout vecteur cotangent $\xi' \neq 0$ sur le bord un opérateur de Wiener-Hopf $a(\xi')$.

On a alors les résultats suivants :

THEOREME – *Les deux symboles σ_X, $\sigma_{\partial X}$ sont des "homomorphismes d'algèbre" (autrement dit $\sigma(A + B) = \sigma(A) + \sigma(B)$, $\sigma(A \circ B) = \sigma(A) \circ \sigma(B)$)*

THEOREME. — *A est elliptique*

équivant à : $\sigma_X(A)$ *et* $\sigma_{\partial X}(A)$ *sont inversibles*

ou : *A possède une parametrix B de degré — deg A (i.e. B est un opérateur de Green de degré — deg A, et AB — 1, BA — 1 sont des opérateurs à noyau* C^∞)

4. Indice d'un opérateur de Green elliptique.

Le problème est de relier l'indice de A : $\chi(A) = \dim \ker A - \dim \operatorname{coker} A$ à un invariant topologique du symbole de A. Je ne donne pas ici de formule, celle-ci demandant une construction trop longue de K-théorie. Je me contenterai d'indiquer une méthode qui y conduit.

— *premier cas* : $A = \begin{pmatrix} P_X & 0 \\ 0 & Q \end{pmatrix}$ où P est un opérateur de degré 0, qui coïncide avec l'opérateur identité près du bord ∂X. L'opérateur A est alors bien décomposé : $\chi(A) = \chi(P_X) + \chi(Q)$, et chacun des deux indices $\chi(P_X)$, $\chi(Q)$ est donné par la formule de Atiyah et Singer.

— *deuxième cas* : appelons déformation de A une famille continue A_t $(0 \leqslant t \leqslant 1)$ d'opérateurs de Green, avec $A_0 = A$. Si A est un opérateur de Green elliptique, et si on peut déformer A en un des opérateurs du premier cas, il est encore possible de calculer son indice par la formule de Atiyah et Singer.

— En général, il n'est pas possible de déformer A comme ci-dessus (même si son degré est nul, il y a une obstruction topologique à l'existence d'une telle déformation). On est alors conduit à introduire toute une collection d'opérateurs B, pour lesquels on prouve directement $\chi(B) = 0$ (très grossièrement, ces opérateurs sont construits à partir de facteurs de l'opérateur qui correspond au problème de Dirichlet). On constate alors que pour tout A elliptique, il existe un B tel que $A \circ B$ puisse être déformé comme ci-dessus. Ceci conduit tout naturellement à une formule de l'indice dans le cas général.

Le problème aux limites classique (1) correspond à un opérateur de Green dont la matrice est une colonne. Dans ce cas, le calcul symbolique décrit ci-dessus redonne très simplement les résultats connus, y compris la formule de l'indice de Atiyah et Bott.

BIBLIOGRAPHIE

(Je ne donne pas ici de références pour les problèmes aux limites classiques)

— Sur les opérateurs pseudo différentiels :

CALDERON A.P. et ZYGMUND A. — Singular integral operators and differential equations, *Amer. Journ. Math.*, Vol. 79, 1957, p. 901-921.

KOHN J.J. et NIRENBERG L. — An algebra of pseudo differential operators, *Comm. Pure Appl. Math.*, 18, 1965, p. 269-305.

HORMANDER L. — Pseudo differential operators, *Comm. Pure Appl. Math.*, 18, 1965, p. 501-517.

HORMANDER L. — Pseudo differential operators and hypoelliptic equations, *Amer. Math. Soc. Proc. Symp. Pure Appl. Math.*, 10, 1968, p. 138-183.

— Relations pseudo-différentielles entre les dérivées des solutions de Pf = 0.

CALDERON A.P. — Boundary value problems for elliptic equations, *Outline of the joint Soviet-American symposium on partial differential equations,* Novosibirsk 1963, p. 303-304.

SEELEY R.T. — Singular integrals and boundary problems, *Amer. J. Math.*, 88, 1966, p. 781-809.

HORMANDER L. — Pseudo differential operators and non elliptic boundary problems, *Ann. Math.*, 83, 1966, p. 129-209.

AGRANOVITCH M.S. et DYNIN A.S. — Problèmes aux limites généraux pour un système elliptique, *Dokl.*, 146, n° 3, 1962, p. 511-514, (traduction : *Soviet Math.*, Vol. 3, n° 5, p. 1323-1327).

— Problèmes aux limites pour un opérateur pseudo différentiel.

VISIK M.I. et ESKIN G.I. — Equations en convolution dans un domaine borné.
— *Uspeki Mat. Nauk,* 20, 1965, n° 3 (123), p. 89-152.
— *Mat. Sbornik,* 69 (111), 1966, p. 65-110.
— *Uspeki Mat. Nauk,* 22, 1967, n° 1 (133), p. 15-76.
— *Trudy Mosc. Mat. Obsc.*, 16, 1967, p. 25-50.
— *Mat. Sbornik,* 74 (116), 1967, n° 3, p. 326-356.

BOUTET DE MONVEL L. — Comportement d'un opérateur pseudo différentiel sur une variété à bord, *J. d'analyse,* Jérusalem, 17, 1966, p. 241-304.
— Opérateurs pseudo différentiels analytiques et problèmes aux limites elliptiques, *Ann. Inst. Fourier,* 19, 2 (1969), p. 169-268.
Le travail présenté ici doit paraître dans *Acta Math.*, 126, 1971, n° 1-2.

— Indice des problèmes elliptiques.

ATIYAH M.F. et SINGER I.M. — The index of elliptic operators, *Ann. Math.*, 87, 1968, p. 434-530.

ATIYAH M.F. et BOTT R. — The index problem for manifolds with boundary, *Coll. differential analysis,* Tata Inst., Bombay, Oxford University Press, 1964, p.175-186.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 703 à 715.

SUR UNE CLASSE DE FONCTIONNELLES INTÉGRALES A DOMAINE VARIABLE D'INTÉGRATION

par I. I. DANILJUK

L'objet du présent exposé est, en fait, une classe de problèmes non linéaires aux limites libres, qui constituent une généralisation lointaine d'écoulements en jets, avec cavitation, ou en vagues dans un champ de forces extérieur. Les fonctionnelles intégrales à domaine variable d'intégration, dont on expose ici le calcul variationnel, constituent une méthode efficace d'étude de cette famille de problèmes de la physique mathématique, mais aussi un objet intéressant du point de vue de la mathématique pure ; leur théorie, à côté de ses aspects propres, touche des domaines d'études contemporains comme la théorie des équations intégrales singulières, des variétés de dimension infinie, le calcul des variations "global" etc.

1. Problèmes non linéaires aux limites libres ; leur nature variationnelle.

Considérons une courbe Γ fermée, sans point double, suffisamment régulière, du plan de la variable $z = x + iy$, et orientée, par exemple, dans le sens positif trigonométrique. Appelons G_γ le disque à un trou, ayant pour frontière Γ et une courbe γ, ("la frontière variable"), et situé à gauche de la courbe orientée Γ. On peut prendre γ et G_γ aussi bien dans un domaine du plan complexe que sur les feuillets de Riemann d'une application. Soit $Q(x, y)$ une fonction réelle, positive, suffisamment régulière des variables réelles x et y. Nous allons étudier la classe suivante de problèmes non linéaires aux limites libres γ : on cherche un domaine G_γ, du type défini plus haut, tel que sa fonction-courant harmonique ψ vérifie, sur γ, une condition généralisée de Bernoulli dans le champ de forces extérieur grad Q^2, i.e. :

$$(1) \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{dans} \quad G_\gamma\,,$$
$$\psi = 0 \quad \text{sur} \quad \Gamma\,,\ \psi = 1 \quad \text{sur} \quad \gamma\,,\ |\operatorname{grad} \psi| = Q \quad \text{sur} \quad \gamma\,.$$

Dans les cas concrets de fonctions de forces Q, les problèmes de ce type définissent des écoulements en jets, avec cavitation, ou par vagues, d'un liquide incompressible idéal, et leur théorie comporte à l'heure actuelle une série de résultats classiques. Rappelons, à ce propos, la méthode de Leray-Schauder, élaborée lors de l'étude du contournement d'obstacles courbes avec décollement du jet, la méthode du petit paramètre de Nekrasov-Levi-Civita, la méthode de variation d'applications conformes de Lavrentiev, la méthode de théorie des fonctions de Friedrichs-Jerbe, etc., qui ont permis de résoudre les problèmes d'existence et d'unicité de cette sorte de problèmes d'écoulement. Récemment A. Beurling [3] a élaboré une méthode de théorie des ensembles, qui a fourni une approche nouvelle du problème général [(1)]. La méthode que nous développons ci-dessous, remonte au principe variationnel de Riabouchinsky, grâce auquel K. Friedrichs [19] a pu étudier l'unicité d'écoulements par jets plans et à symétrie axiale, et Garabedian, Spenser, Lewy, Schiffer [5] [6], l'existence de ces écoulements.

Considérons l'ensemble des domaines possibles du type G_γ, et, pour chacun, l'ensemble des fonctions ψ régulières par morceaux, vérifiant les conditions de "fonction courant" : $\psi = 0$ sur Γ, et $\psi = 1$ sur γ. Considérons sur l'ensemble de ces couples "permis", (G_γ , ψ), la fonctionnelle intégrale :

$$\mathfrak{J}(\gamma , \psi) = \iint_{G_\gamma} \{\psi_x^2 + \psi_y^2 + Q^2 (x , y)\} \, dx \, dy . \tag{2}$$

Si l'"argument" de cette fonctionnelle, à savoir le couple (G_γ , ψ), possède les propriétés classiques de régularité, la première variation peut toujours s'écrire

$$\delta \mathfrak{J}(\gamma , \psi \, ; \delta z, \delta \overline{\psi}) = -2 \iint_{G_\gamma} \delta \overline{\psi} \, . \, \Delta \psi \, dx \, dy + \int_\gamma [Q^2 (x , y) - |\operatorname{grad} \psi|^2] \operatorname{Re} (i \delta z \, d\bar{z}), \tag{3}$$

où $\delta \overline{\psi}$ est la variation de ψ relative à un domaine constant, et δz, la variation correspondant au passage de γ à une courbe permise suffisamment voisine.

Ceci nous amène à la proposition [7] :

LEMME FONDAMENTAL. – *Supposons que le couple permis (G_γ , ψ) possède les propriétés classiques de régularité assurant la validité de la formule (3). Pour qu'il soit critique pour la fonctionnelle (2), i.e., par définition, pour qu'il annule la première variation (3), il faut et il suffit qu'il constitue une solution du problème (1).*

La condition non linéaire du problème (1), (la "condition généralisée de Bernoulli") constitue ainsi une "condition aux limites naturelles" pour le problème variationnel (1) lié à la fonctionnelle. A partir de maintenant, notre objet principal d'étude sera la fonctionnelle $\mathfrak{J}(\gamma , \psi)$. Notons qu'il existe une proposition correspondante pour les problèmes analogues dans le cas de milieux compressibles.

2. Variété des éléments permis. Variété tangente.

Considérons la couronne G_ρ : $\rho < |\tau| < 1$, pour un $p \in]0,1[$ fixé, et notons $H_2^{(1)} (G_\rho)$, l'ensemble des fonctions $z(\tau)$, analytiques dans G_ρ, et vérifiant la la condition :

$$\sup_{\rho < r < 1} \int_0^{2\pi} |z' (r \exp i\sigma)|^2 \, d\sigma < + \infty , \; z'(\tau) = \frac{dz}{d\tau} . \tag{4}$$

Chacune de ces fonctions peut être prolongée de façon naturelle au bord ∂G_ρ, de telle sorte que z (exp $i\sigma$), et z (ρ exp $i\sigma$) constituent des fonctions absolument continues dont les dérivées appartiennent à $L_2(0,2\pi)$: la dernière condition entraîne que ces fonctions vérifient la condition de Hölder, au moins avec l'exposant

1/2. Transformons le domaine G_γ en une couronne G_ρ. En vertu de l'invariance conforme de l'intégrale de Dirichlet, on peut prendre pour Γ le cercle unité : $|z| = 1$. Alors la normalisation $z(1) = 1$ définit complètement la fonction $z_\gamma(\tau)$, qui réalise la transformation conforme de G_ρ en G_γ, et la courbe γ a pour représentation analytique : $z = z_\gamma(\rho \exp i\sigma)$, $0 \leqslant \sigma \leqslant 2\pi$. Nous l'appellerons paramétrisation "holomorphe" de la courbe permise γ. Nous considèrerons uniquement des courbes γ pour lesquelles la fonction correspondante $z_\gamma(\tau)$ vérifie une condition de la forme (4).

Soit $\{c_k\}$ la suite des coefficients du développement de $z(\tau)$ en série de Laurent. La condition (4) équivaut à la suivante :

$$(5)\qquad |c_0|^2 + \sum_{k=1}^{\infty} k^2 |c_k|^2 + \sum_{k=1}^{\infty} k^2 \rho^{-2k} |c_{-k}|^2 < +\infty .$$

Si l'on munit l'ensemble de ces suites de nombres complexes, considéré comme un espace vectoriel réel, du produit scalaire réel correspondant, on obtient un espace de Hilbert : $h_2^{(1)}(\rho)$. Le développement de Laurent nous donne alors, à partir de l'ensemble $\overset{\circ}{H}{}_2^{(1)}(G_\rho)$, considéré comme un espace vectoriel réel, un espace de Hilbert (isométrique à $h_2^{(1)}(\rho)$). Nous noterons $\mathrm{C}\overset{\circ}{W}{}_2^{(1)}(0\,,2\pi)$ (resp. $\mathrm{R}\overset{\circ}{W}{}_2^{(1)}(0{,}2\pi)$) l'ensemble des fonctions complexes (resp. réelles), définies sur le segment $[0{,}2\pi]$, absolument continues, de période 2π, dont les dérivées premières appartiennent à $L_2(0\,,2\pi)$. Chacun de ces ensembles possède une structure d'espace de Hilbert.

THEOREME 1 [8]. – *L'ensemble $\mathfrak{U}_\rho$ des fonctions non ramifiées, $z \in H_2^{(1)}(G_\rho)$, qui appliquent le cercle $|\tau| = 1$ dans le cercle $|z| = 1$, et la couronne G_ρ, sur un domaine du type G_γ, et qui vérifient la condition : $z(1) = 1$, constitue une variété connexe de dimension infinie et sans bord de l'espace de Hilbert $\overset{\circ}{H}{}_2^{(1)}(G_\rho)$. En chaque point $z \in \mathfrak{U}_\rho$, l'espace tangent $T_z\,\mathfrak{U}_\rho$ est isomorphe au produit direct $\mathbf{R} \times \mathrm{R}\overset{\circ}{W}{}_2^{(1)}(0{,}2\pi)$, (où $\mathbf{R}$ est la droite réelle), et l'isomorphisme s'écrit :*

$$(6)\quad (c\,,\eta) \to h(\rho \exp i\sigma\,;\rho\,,z) \equiv c\,.\,h^{(0)}(\rho \exp i\sigma\,;\rho\,,z) + \eta(\sigma) + \frac{i}{2\pi}\int_0^{2\pi} \eta(s)\left[\operatorname{cotg}\frac{s-\sigma}{2} + L(s\,,\sigma\,;\rho\,,z)\right] ds,$$

où : $c \in \mathbf{R}$,

$\eta \in \mathrm{R}\overset{\circ}{W}{}_2^{(1)}(0{,}2\pi)$,

$h^{(0)}(\tau\,;\rho\,,z)$ *est une fonction analytique déterminée, qui prend, sur le cercle* $|\tau| = \rho$ *des valeurs imaginaires pures,*

et L *un noyau déterminé, réel et régulier dans le carré* : $0 \leqslant s\,,\sigma \leqslant 2\pi$).

La réunion des espaces de Hilbert $H_2^{(1)}(G_\rho)$, pour $\rho \in\,]0\,,1[$, n'est pas un espace de Hilbert, par conséquent la réunion $\mathfrak{U}$ des variétés $\mathfrak{U}_\rho$, pour $\rho \in\,]0\,,1[$, qui est une représentation de l'ensemble des éléments permis, ne peut pas se caractériser de la même façon géométrique. Supposons qu'une paramétrisation holomorphe $z_\gamma(\rho \exp i\sigma)$ d'une courbe permise γ appartient à l'espace $\mathrm{C}\overset{\circ}{W}{}_2^{(1)}(0\,,2\pi)$, en même temps que sa dérivée première $dz_\gamma(\rho \exp i\sigma)/d\sigma$. En un tel point "régulier", l'ensemble $\mathfrak{U}$ possède, par rapport à la métrique de l'espace $\mathbf{R} \times \mathbf{R} \times \mathrm{R}\overset{\circ}{W}{}_2^{(1)}(0{,}2\pi)$ un "espace tangent", dont l'élément le plus général est donné par la formule :

$$(7)\qquad \delta z(\sigma) = h(\rho \exp i\sigma\,;\rho\,,z) - \frac{i\,\delta^{\rho}}{\rho}\frac{d}{d\sigma} z(\rho \exp i\sigma)\,,\ \delta\rho \in \mathbf{R}\,.$$

Sur cet ensemble de variations $\delta z(\sigma)$, exprimant le passage d'une courbe permise γ à une courbe voisine, l'expression (3) de la première variation de la fonctionnelle (2) devient :

$$\delta\mathcal{J}(\gamma\,,\psi\,;\delta z(\sigma)\,,\delta\overline{\psi}) = -2\iint_{G_\gamma}\delta\overline{\psi}\,.\,\Delta\psi\,dx\,dy$$

$$(8)\quad -\int_0^{2\pi} Q^2[z_\gamma(\rho \exp i\sigma)]\,\mathrm{Re}\left[\,ih(\exp i\sigma\,;p\,,z_\gamma)\frac{d}{d\sigma}\overline{z}_\gamma(\rho \exp i\sigma)\right]d\sigma$$

$$+\,\rho\,\delta\rho\left(\frac{2\pi}{\mathrm{Log}^2\rho} - \int_0^{2\pi} Q^2[z_\gamma(\rho \exp i\sigma)]\left|\frac{d}{d\sigma}z_\gamma(\rho \exp i\sigma)\right|^2 d\sigma\right),$$

où : $Q^2(z) \equiv Q^2(x\,,y)$.

Nous dirons qu'un couple permis $(G_\gamma\,,\psi)$ est *critique* pour la fonctionnelle (2), si le second membre de la formule (8) est nul, quels que soient $\delta\overline{\psi}$, et le couple $(h\,,\delta\rho)$, avec : $h \in T_{z_\gamma}\mathfrak{U}_\rho\,,\ \delta\rho \in \mathbf{R}$.

3. Conditions nécessaires et suffisantes pour qu'une courbe admissible soit critique.

Du fait que les variations $\delta\overline{\psi}$ et $\delta z(\sigma)$ sont indépendantes, et que le second membre de (8) s'annule, il faut, tout d'abord, que $\psi(x\,,y)$ soit harmonique à l'intérieur du domaine G_γ. Substituons, dans le deuxième terme du second membre de (8), le second membre de la formule (6), intervertissons l'ordre d'intégration, ce qu'autorisent les hypothèses faites, et tenons compte du fait que la somme de la quantité obtenue et du dernier terme de (8) doivent s'annuler, pour un triplet $(\delta\rho\,,c\,,\eta)$ arbitraire. Ceci nous donne le système d'équations :

$$W_1(z_\gamma\,,\rho) \equiv \int_0^{2\pi} Q^2[z_\gamma(\rho \exp i\sigma)\left|\frac{d}{d\sigma}(z_\gamma(\rho \exp i\sigma)\right|^2 d\sigma - \frac{2\pi}{\mathrm{Log}^2\rho} = 0\,,$$

$$W_2(z_\gamma\,,\rho) \equiv \int_0^{2\pi}\frac{dx(\sigma)}{d\sigma}Q^2[z_\gamma(\rho \exp i\sigma)]\,\mathrm{Im}\,h^{(0)}(\rho \exp i\sigma\,;\rho\,,z_\gamma)\,d\sigma = 0,$$

(9)

$$W_3(z_\gamma\,,\rho) \equiv Q^2[z_\gamma(\rho \exp i\sigma)]\frac{dy(\sigma)}{d\sigma}$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\frac{dx(s)}{ds}Q^2[z_\gamma(\rho \exp is)]\left[\mathrm{cotg}\frac{s-\sigma}{2} - L(\sigma\,,s\,;\rho\,,z_\gamma)\right]ds$$

où l'on note :

$$z_\gamma(\rho \exp i\sigma) \equiv x(\sigma) + iy(\sigma).$$

Nous arrivons ainsi à la proposition [8] :

THEOREME 2. — *Soient G_γ un domaine permis et ψ sa fonction courant harmonique. Supposons qu'une paramétrisation holomorphe $z_\gamma(\rho \exp i\sigma)$ de la frontière* *libre* γ, *ainsi que sa dérivée première* $dz_\gamma(\rho \exp i\sigma)/d\sigma$ *appartiennent à l'espace* $\mathbf{C}\, \overset{0}{W}{}_2^{(1)}(0,2\pi)$. *Pour que le couple* (G_γ, ψ) *soit critique pour la fonctionnelle* (2), *il faut et il suffit que le système* (9) *soit vérifié pour un* $\rho \in]0,1[$.

Chaque équation du système (9) comporte au premier membre un opérateur intégro-différentiel non linéaire. Les deux premiers sont des fonctionnelles, dans le troisième figure un opérateur intégral avec des valeurs principales de Cauchy. Les opérateurs du système (9) sont définis univoquement par le potentiel du champ de forces extérieur Q^2 et les fonctions particulières $h^{(0)}$ *et* L, par conséquent le système d'équations non linéaires (9) donne un procédé analytique pour déterminer la frontière libre γ, de même que l'équation d'Euler (qui, dans notre cas, se réduit à celle de Laplace) permet de déterminer les fonctions permises critiques ψ. Le système (9) constitue le moyen d'attaque principal des problèmes locaux ainsi que globaux des fonctionnelles du type (2). Il est non linéaire même en l'absence de champ de forces ($Q^2(x, y) = $ cte).

4. Propriétés différentielles de la frontière libre.

Il est évident que tous les couples de fonctions $x(\sigma), y(\sigma)$, choisies au hasard dans l'espace $\mathbf{R}\, \overset{0}{W}{}_2^{(1)}(0,2\pi)$ ne constituent pas une paramétrisation holomorphe $z = z(\sigma) \equiv x(\sigma) + iy(\sigma)$ d'une courbe γ, même si γ appartient a l'ensemble défini plus haut. Nous allons donner une condition nécessaire, portant sur le couple de fonctions $x(\sigma), y(\sigma)$, pour que ce soit le cas.

Soient $\zeta(u)$ la fonction de Weierstrass (de demi période $\omega = \pi$, bien connue en théorie des fonctions elliptiques), $\omega' = i \operatorname{Log} \rho$, $\rho \in]0,1[$, $\eta(\rho) = \zeta(\pi)$, et $\zeta_0(u) = \zeta(u) - \frac{u}{\pi}\eta(\rho) - \frac{1}{2}\operatorname{cotg}\frac{u}{2}$ la partie régulière périodique de la fonction $\zeta(u)$, pour des u réels. Si la fonction $x(\sigma) + iy(\sigma)$ réalise une paramétrisation holomorphe d'une courbe permise γ, le couple de fonctions $x(\sigma), y(\sigma)$ vérifie l'équation non linéaire intégro-différentielle :

$$(10)\quad \frac{dx(\sigma)}{d\sigma} + \frac{1}{2\pi}\int_0^{2\pi} \frac{dy(s)}{ds}\operatorname{cotg}\frac{s-\sigma}{2}\, ds - \frac{1}{\pi}\int_0^{2\pi}\frac{dy(s)}{ds}\zeta_0(\sigma - s)\, ds$$
$$+ \frac{1}{\pi}\int_0^{2\pi} \mathrm{R}(x, y)(s)\, \operatorname{Re}\zeta(\sigma - s + i\operatorname{Log}\rho)\, ds - \frac{\eta(\rho)}{\pi}\int_0^{2\pi} y(s)\, ds = 0,$$

où $R(x, y)$ est un opérateur sur $x(\sigma), y(\sigma)$, bien déterminé, régulier, non linéaire, de type intégral.

Juxtaposons maintenant la dernière équation du système (9) et l'équation (10). Le système obtenu est *linéaire* par rapport aux dérivées et appartient à une classe bien connue d'équations intégrales singulières [16]. Si, le long de la solution considérée $z_\gamma(\rho \exp i\sigma)$ du système (9), la condition : $Q^2[z_\gamma(\rho \exp i\sigma)] > 0$, $\sigma \in [0,2\pi]$ est vérifiée, ce couple d'équations en $x'(\sigma), y'(\sigma)$ appartiendra à ce que l'on appelle le type *normal*. Utilisant la théorie de ce type d'équations et la dérivabilité de tout ordre des termes réguliers de l'équation (10) et du noyau L, nous arrivons à la proposition [8] :

THEOREME 3. – *Supposons que la fonction donnée* $Q^2(x, y)$ *appartient à la classe* $C_\alpha^{(n)}$, $0 < \alpha < 1$, $n \geqslant 0$, *dans tout le plan* $\mathbf{R}^2$. *Si, le long d'une courbe critique* γ, *de paramétrisation holomorphe* $z = z_\gamma(\rho \exp i\sigma)$, *on a l'inégalité :* $Q^2 [z_\gamma (\rho \exp i\sigma)] > 0$, $\sigma \in [0, 2\pi]$, *alors la frontière libre* γ, *ou, plus exactement, sa représentation paramétrique* $z_\gamma(\rho \exp i\sigma)$ *appartient à l'espace* $C_\alpha^{(n+1)}(0, 2\pi)$.

5. **Critère d'unicité locale des frontières libres** (condition pour qu'elles soient isolées).

Comme plus haut, nous allons utiliser une paramétrisation holomorphe des frontières libres, et partir du système non linéaire (9). Comme d'habitude, pour voir si une certaine solution est isolée, on se ramène au "système varié", obtenu par linéarisation du système initial en la solution considérée $z_\gamma(\rho \exp i\sigma)$. Pour cela, donnons au module conforme ρ du domaine G_γ un accroissement $\delta\rho$, à la variable $z_\gamma(\rho \exp i\sigma)$ la variation (7), où h est définie par la formule (6), substituons les expressions obtenues dans le premier membre des équations (9), et ne gardons que les termes "importants". Du fait que W_1 et W_2 sont des fonctionnelles, les opérateurs "linéarisés", que nous obtenons ainsi, seront des fonctionnelles linéaires sur le produit direct $\mathbf{R} \times \mathbf{R} \times \mathbf{R}\, \overset{\circ}{W}{}_2^{(1)}(0, 2\pi)$:

$$(11) \quad \delta W_i(z_\gamma, \rho\,;\delta\rho, c, \eta) = \int_0^{2\pi} \eta(s) A_i(s\,; z_\gamma, \rho)\, ds + B_i(z_\gamma, \rho)c + C_i(z_\gamma, \rho)\delta\rho,$$

où A_i, B_i, C_i ont des valeurs bien déterminées. La linéarisation du troisième opérateur W_3 conduit à un opérateur linéaire intégro-différentiel d'argument $(\delta\rho, c, \eta)$. Après quelques manipulations, on obtient une équation équivalente dont le premier membre a la forme d'un opérateur intégral linéaire singulier agissant sur η. Il a un index nul et sa partie caractéristique est un opérateur singulier à noyau de Hilbert. Conformément aux formules d'inversion de Hilbert, cette équation est équivalente à un système de deux équations : la première a la forme $\delta W_0 = 0$, où l'opérateur δW_0 a la forme (11), et la seconde peut s'écrire :

$$(12) \quad \delta W_3(z_\gamma, \rho\,; \delta\rho, c, \eta) \equiv$$

$$\eta(\sigma) - \int_0^{2\pi} \eta(s) A_3(\sigma, s\,; z_\gamma, \rho)\, ds + B_3(\sigma\,; z_\gamma, \rho) c + C_3(\sigma\,; z_\gamma, \rho)\,\delta\rho,$$

si $Q^2 [z_\gamma(\rho \exp i\sigma)] > 0$ sur le segment $[0, 2\pi]$. L'opérateur intégral de la formule (12) est compact dans $\mathbf{R}\overset{\circ}{W}{}_2^{(1)}(0, 2\pi)$. Ainsi, nous arrivons à la proposition [8] :

THEOREME 4. – *Supposons que* $(z_\gamma(\rho \exp i\sigma), \rho)$ *est une solution fixée du système intégro-différentiel non linéaire (9), le long de laquelle*

$$Q^2 [z_\gamma(\rho \exp i\sigma)] > 0, \ \sigma \in [0, 2\pi],$$

et considérons le "système varié" linéaire :

$$(13) \qquad \delta W_i(z_\gamma, \rho\,; \delta\rho, c, \eta) = 0, \ i = 0, 1, 2, 3,$$

où δW_0, δW_1, δW_2, *(resp.* δW_3*) ont la forme (11) (resp. (12)).*

Si, dans l'espace $\mathbf{R} \times \mathbf{R} \times \mathbf{R}\, W_2^{(1)}(0\,,2\pi)$ *le système (13) n'a que la solution identiquement nulle,* $(z_\gamma(\rho \exp i\sigma),\, \rho)$ *constitue une solution de (9) isolée pour la métrique de l'espace* $\mathbf{R} \times \mathbf{R} \times \mathbf{R}\, \overset{\circ}{W}{}_2^{(1)}(0\,,2\pi)$.

Combiné à ce qui précède, ce dernier théorème nous donne une condition pour que soient isolés les éléments critiques de la fonctionnelle (2), et pour que le problème non linéaire aux limites (1) ait une solution localement unique. Il est intéressant de remarquer que les questions d'unicité locale dépendent, par conséquent, du spectre d'un certain opérateur linéaire compact, et que les transformations analytiques qui donnent le système (13) sont équivalentes, au fond, à celles qui donnent l'expression de la deuxième variation de la fonctionnelle initiale (2).

Dans ce cas des problèmes hydrodynamiques réels, la fonction de forces Q^2 peut dépendre aussi d'un certain nombre de paramètres réels. Le théorème à nous donne également une condition de "ramification" des solutions par rapport à ces paramètres : ces derniers doivent être, en effet, "spectraux" pour l'opérateur compact de plus haut, i.e. le système (13) doit avoir une solution non nulle. Dans le cas où : $Q^2(x\,,y) = q^2(\rho)$, $\rho^2 = x^2 + y^2$, ces conditions s'explicitent en fonction de $q(\rho)$ et de ses dérivées [2].

6. Propriétés globales de l'ensemble des courbes critiques.

Introduisons la notation :

$$(14)\qquad \mathfrak{J}(\psi\,, z_\gamma\,, \rho) \equiv \iint_{G_\rho} \left\{ \psi_\xi^2 + \psi_\eta^2 + Q^2(z) \left| \frac{dz_\gamma}{d\tau} \right|^2 \right\} d\xi\, d\eta\,,$$

conformément à laquelle, une application $z_\gamma : G_\rho \to G_\gamma$ nous donnera :

$$\mathfrak{J}(\gamma\,,\psi) = \mathfrak{J}(\widetilde{\psi}\,, z_\gamma\,,\rho) \quad \text{ou} \quad \widetilde{\psi}(\xi\,,\eta) = \psi(x\,,y)\,,\ \tau = \xi + i\eta\,.$$

L'ensemble des triplets $(\psi\,, z_\gamma\,, \rho)$, dont dépend la fonctionnelle (14), correspond biunivoquement aux couples permis $(\gamma\,,\psi)$, dont dépend la fonctionnelle (2). Il est facile de voir que l'étude de la fonctionnelle (14) peut s'effectuer de la manière suivante : on y fait varier ψ, pour z, ρ fixés, et on trouve les valeurs critiques ψ. Du fait que ces fonctions sont harmoniques dans la couronne G_ρ, que $\psi = 0$ pour $|\tau| = 1$ et $\psi = 1$ pour $|\tau| = \rho$, on a $\psi = \operatorname{Log}|\tau| / \operatorname{Log}\rho$. Substituant cette valeur pour ψ dans (14), on obtient la fonctionnelle :

$$(15)\qquad \mathfrak{J}_1(z\,,\rho) \equiv -\frac{2\pi}{\operatorname{Log}\rho} + \iint_{G_\rho} Q^2(z) \left|\frac{dz}{d\tau}\right|^2 d\xi\, d\eta\,,$$

qui ne dépend que du couple $(z\,,\rho)$. Il suffit alors de chercher les couples critiques de la fonctionnelle (15). Si l'on y fait varier, d'abord z, puis ρ, on obtient, après quelques manipulations, le terme de la variation (8) qui s'exprime au moyen d'intégrales curvilignes (cf [7]). De cette façon, nous revenons au système (9).

Dans ce paragraphe, nous allons étudier la fonctionnelle (15), pour des valeurs fixées de $\rho \in \,]0\,,1[$. Dans ce cas, les paramétrisations holomorphes des courbes critiques ont pour propriété caractéristique d'être solution des deux dernières équations du système (9). Le domaine naturel de définition de la fonctionnelle (15) est la variété $\mathcal{U}_\rho$ construite au théorème 1. Quant à la fonctionnelle (2), elle n'a pour domaine de référence, que les G_γ, conformément équivalents à la couronne G_ρ relative au même ρ ; en d'autres termes, on considère l'ensemble des courants possédant la même circulation : $\nu = -2\pi/\text{Log}\,\rho$. Par conséquent, nous ne pouvons pas espérer que la condition de Bernoulli sera remplie au sens de la formule (1). Il se trouve que, à chaque point critique $z(\tau) \in \mathfrak{U}_\rho$ de la fonctionnelle (15), ρ étant fixé, correspond un domaine permis G_γ, image de la couronne G_ρ par l'application $z = z(\tau)$; sur la frontière libre γ de cette dernière on a la condition de Bernoulli, au changement près de $Q^2(x, y)$ en $Q^2(x, y)/c_1^2$, où $c_1 \neq 0$ est la "divergence" du courant (qui dépend, en général, de la fonction $z(\tau)$).

Désignons par N_ρ l'ensemble des points critiques de la fonctionnelle (15) sur la variété $\mathfrak{U}_\rho$, et par $N_\rho(R)$ le sous ensemble de N_ρ inclus dans la boule de rayon R de l'espace de Hilbert $H_2^{(1)}(G_\rho)$. Nous allons formuler le théorème principal de ce paragraphe [10] :

THEOREME 5. — *Supposons donnée, dans tout le plan, une fonction réelle $Q^2(x, y)$, continûment différentiable et vérifiant la condition :*

$$(16) \qquad Q^2(x, y) \geqslant Q_0^2 > 0 \;, \; Q_0 = \text{cte}$$

Alors, pour tout $R < +\infty$, le sous ensemble $N_\rho(R)$ est (fortement) compact dans l'espace de Hilbert $H_2^{(1)}(G_\rho)$.

La fonctionnelle continûment différentiable (15) a un gradient $\text{grad}_z\, \mathfrak{I}_1\,(z, \rho)$ le long de la variété $\mathfrak{U}_\rho$. Exprimé en fonction des variables de l'espace $\mathbf{R} \times \mathbf{R}\overset{\circ}{W}{}_2^{(1)}(0, 2\pi)$, (isomorphe, d'après la formule (6), à l'espace tangent $T_z\mathfrak{U}_\rho$), le champ de vecteurs non linéaire du gradient est donné par les formules :

$$\text{grad}_z\, \mathfrak{I}_1(z, \rho) \equiv (-W_2(z, \rho), - W_3^*(z, \rho)) ,$$

$$(17) \quad W_3^*(z, \rho) \equiv \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\exp ik(\sigma - s)}{k^2} \right\} W_3(z, \rho)(s)\, ds ,$$

où W_2 et W_3 sont définis en (9). La fonction $W_3^*(z, \rho)(s)$ appartient à l'espace $\mathbf{R}\overset{\circ}{W}{}_2^{(2)}(0, 2\pi)$, dont l'opérateur non linéaire $\text{grad}_z \mathfrak{I}_1(z, \rho) : \mathfrak{U}_\rho \to \mathbf{R} \times \mathbf{R}\overset{\circ}{W}{}_2^{(1)}(0, 2\pi)$, est *compact*. On a, en effet, une proposition plus générale que le théorème 5 : toute partie bornée de l'image inverse d'un compact de $\mathbf{R} \times \mathbf{R}\,\overset{\circ}{W}{}_2^{(2)}(0, 2\pi)$ par l'application engendrée par l'opérateur $\text{grad}_z\, \mathfrak{I}_1(z, \rho)$ est compacte pour la métrique de l'espace ambiant.

7. Points critiques non dégénérés de la fonctionnelle $\mathfrak{I}_1$.

Récemment, à la suite des travaux de Smale et Palais [17], [18], sur le calcul des variations "global", la théorie classique de Morse a été largement généralisée, pour traiter des fonctionnelles intégrales correspondant à des problèmes aux limites généralisés pour des équations elliptiques non linéaires. Dans le même ordre d'idées, on trouve les généralisations de Ja. B. Lopatinskij [13]. Quant aux fonctionnelles de la forme (15), il se trouve qu'elles ne rentrent pas dans le schèma général et abstrait de Smale-Palais, plus précisément, elles ne vérifient pas la condition "C" de ces auteurs.

Il existe une autre propriété des fonctionnelles du type (15) qui les fait sortir du cadre de la théorie en question, elle a trait à la notion de point critique non dégénéré. La partie quadratique du développement de Taylor de la fonctionnelle engendre, au voisinage d'un point critique, un opérateur auto adjoint $\mathfrak{A}$. D'après la définition de Smale-Palais, un point critique est dit non dégénéré si l'opérateur inverse $\mathfrak{A}^{-1}$ existe et est continu. Aucun point critique de la fonctionnelle (15) ne peut être non dégénéré au sens de cette définition, du fait que l'opérateur $\mathfrak{A}$, engendré par la partie principale du champ de vecteurs (17), est, en même temps que lui, compact. Aussi, nous nous tiendrons à la définition suivante : Soit, dans l'espace tangent $T_x \mathfrak{U}$, une variété linéaire $\hat{T}_x \mathfrak{U}$, qui constitue un espace de Hilbert par rapport à son produit scalaire. Supposons que le gradient de la fonctionnelle correspond à une application régulière de $\mathfrak{U}$ dans $\hat{T}\mathfrak{U}$, et que l'opérateur $\mathfrak{A}$, relatif à un point critique $x \in \mathfrak{U}$, possède, en tant qu'application de $T_x \mathfrak{U}$ dans $\hat{T}_x \mathfrak{U}$, un inverse continu. Nous dirons alors que le point critique x est *non dégénéré.* Dans le cas où les espaces de Hilbert $T_x \mathfrak{U}$ et $\hat{T}_x \mathfrak{U}$ coïncident, nous retrouvons la définition de Smale-Palais. On a la proposition :

THEOREME 6 [9]. – *Supposons que la fonction de forces $Q^2(x, y)$ vérifie les conditions du théorème 5, y compris l'inéquation (16). Pour qu'un point critique $z \in \mathfrak{U}_\rho$ de la fonctionnelle (15) soit non dégénéré, il est nécessaire et suffisant que le système linéaire :*

$$\delta W_i (z_\gamma , \rho \ ; 0 , c , \eta) = 0 , i = 0, 2, 3, \tag{18}$$

obtenu à partir du système (13) par suppression de la deuxième équation ($i = 1$), et pour $\delta\rho = 0$, n'ait que la solution nulle. Chaque point critique non dégénéré de la fonctionnelle (15) est isolé, dans l'ensemble des points critiques, sur la variété $\mathfrak{U}_\rho$. Si tous les points critiques de la fonctionnelle (15) sont non dégénérés dans $\mathfrak{U}_\rho$, chaque partie bornée $N_\rho(R)$, de l'ensemble N_ρ des points critiques, est finie.

La dernière proposition de ce théorème se déduit facilement de la seconde et du théorème 5.

8. Théorie de Morse généralisée.

Cette théorie a pour thème les caractéristiques qualitatives et quantitatives des points critiques des fonctionnelles et leurs relations avec les invariants d'homologie et d'homotopie de la variété $\mathfrak{U}$ où sont définies ces fonctionnelles. On ébauche dans ce paragraphe, la théorie abstraite d'une classe de fonctionnelles, contenant les fonctionnelles du type (15). La méthode générale de la théorie de Morse, remontant, en partie, aux travaux de ce dernier, tout en ayant été passablement modernisée au cours des dernières recherches, en particulier de Smale et Palais, peut se généraliser à cette classe de fonctionnelles.

Notons H un espace de Hilbert séparable de dimension infinie, et soit $\mathfrak{U} \subset H$ une variété suffisamment régulière, localement isomorphe à un espace de Hilbert de dimension infinie : H_1. Supposons que $\mathfrak{U}$ constitue un sous ensemble faiblement fermé de H, complet relativement à sa métrique riemannienne "interne". Soit $\mathfrak{J} : \mathfrak{U} \rightarrow \mathbf{R}$ une fonctionnelle réelle suffisamment régulière. Voici en quoi consiste la "condition C" qui, comme on l'a dit plus haut, se trouve à la base de la théorie de Smale-Palais : Soit S un sous ensemble de la variété $\mathfrak{U}$, dans lequel la fonctionnelle $\mathfrak{J}$ est bornée, mais a son gradient arbitrairement petit en norme. Alors la fermeture de S comporte, au moins, un point critique de la fonctionnelle $\mathfrak{J}$. Comme on l'a remarqué, cette condition C n'est pas vérifiée par la fonctionnelle (15), considérée sur la variété $\mathfrak{U}_\rho \subset H_2^{(1)}(G_\rho)$. Elle vérifie plutôt la condition "A" : si $S \subset \mathfrak{U}$ est *borné* pour la métrique de H, alors, sous les hypothèses de la condition C, la fermeture *faible* de S comporte, au moins, un point critique de la fonctionnelle. La condition A est vérifiée pour toutes les fonctionnelles régulières dont le champ des gradients engendre une application compacte de $\mathfrak{U}$ dans H. Cependant, notre généralisation de la théorie de Morse va s'appuyer sur d'autres axiomes.

Supposons que la fonctionnelle $\mathfrak{J} : \mathfrak{U} \to \mathbf{R}$ vérifie la *"condition B"*. Notons N l'ensemble de tous ses points critiques dans $\mathfrak{U}$. Alors chaque partie $N(R)$ de N, bornée (pour la métrique de H), est compacte dans H. La fonctionnelle (15) vérifie cette condition sous les hypothèses du théorème 5. Nous entendons la non dégénérescence au sens de la définition du § 7. On a alors des analogues des première et troisième propositions du théorème 6 : sous les hypothèses de la condition B les points critiques non dégénérés sont isolés, et si la fonctionnelle n'a que des points critiques non dégénérés, il y en a un nombre fini dans chaque boule de rayon $R < +\infty$.

Rajoutons encore, dans notre étude abstraite, une hypothèse sur la fonctionnelle $\mathfrak{J} : \mathfrak{U} \to \mathbf{R}$ qui la rapproche de la fonctionnelle (15) ; nous supposons remplie la condition "D" ; la fonctionnelle $\mathfrak{J} : \mathfrak{U} \to \mathbf{R}$ est faiblement continue, mais son gradiant $\nabla\mathfrak{J} : \mathfrak{U} \to T\mathfrak{U}$, est complètement continu. On sait que, dans un domaine convexe, la compacité du gradient entraîne la faible continuité de la fonctionnelle ([4], § 8). La complète continuité du gradient de la fonctionnelle (15) découle des formules (17) et (9), et sa faible continuité se démontre immédiatement.

Au voisinage d'un point critique non dégénéré x d'une fonctionnelle suffisamment régulière, il existe une carte dans laquelle la fonctionnelle coïncide, à une constante additive près, avec une forme quadratique. En dimension finie, c'est le Lemme de Morse. (cf. par ex., [15], § 2). Pour la non dégéneresence à la Smale-Palais, on trouvera la généralisation du lemme de Morse en dimension infinie dans [1], § 8. On a une proposition analogue si l'on définit, comme plus haut, la non dégénérescence d'un point critique : soit $x = 0$, un point critique et non dégénéré. Il existe un voisinage U du point $x = 0$, des voisinages $V, W \subset U$, et un C^1-difféomorphisme $\psi : W \to V$, $\psi(0) = 0$ tels que, quel que soit $y \in W$:

$$\mathfrak{J}[\psi(y)] = \mathfrak{J}(0) + (\mathfrak{A}y, y)_{T_0\mathfrak{U}}, \tag{19}$$

où $\mathfrak{A}$ est un opérateur continu auto adjoint de l'espace tangent $T_x\mathfrak{U}$ au point $x = 0$. Sous la condition D, l'opérateur $\mathfrak{A}$ sera *compact.* Notons $\{\lambda_i^-\}$(resp. $\{\lambda_i^+\}$) l'ensemble des valeurs absolues des valeurs propres négatives (resp. positives) de l'opérateur $\mathfrak{A}$, numérotées, pour fixer les idées, dans l'ordre croissant. On sait qu'on peut alors traduire (19) :

$$\mathfrak{J}[\psi(y)] = \mathfrak{J}(0) - \sum_j \lambda_j^- \overset{(-)}{U}{}_j^2 + \sum_j \lambda_j^+ \overset{(+)}{U}{}_j^2, \tag{20}$$

$$\overset{(-)}{U}_j = (y, x_j^-)_{T_0\mathfrak{U}}, \quad \overset{(+)}{U}_j = (y, x_j^+)_{T_0\mathfrak{U}},$$

où $\{x_j^-, x_j^+\}$ est la base orthonormée correspondante de l'espace tangent $T_0\mathfrak{U}$. Le nombre (éventuellement infini) des termes de la première somme de la formule (20) s'appelle *index de Morse* du point critique non dégénéré $x = 0$ de la fonctionnelle $\mathfrak{J}$.

Utilisons les notations habituelles :

$$\mathfrak{J}^{a,b} = \{x : x \in \mathfrak{U}\ , a \leqslant \mathfrak{J}(x) \leqslant b\}\ , \mathfrak{J}^b = \{x : x \in \mathfrak{U}\ , \mathfrak{J}(x) \leqslant b\}.$$

Nous appellerons le nombre réel c valeur critique de la fonctionnelle $\mathfrak{J}$, si l'ensemble $\mathfrak{J}^{-1}(c) = \mathfrak{J}^{c,c}$ en contient des points critiques. Si la condition B est remplie, et les points critiques non dégénérés, l'ensemble de ces points est, au plus, *dénombrable*. Par conséquent, il en est de même pour l'ensemble des valeurs critiques de la fonctionnelle. Sous nos hypothèses, cet ensemble des valeurs critiques peut, en général, avoir un point d'accumulation à distance finie. Utilisant des raisonnements de [18], ainsi que la méthode générale de la théorie moderne de Morse (cf, par ex., [15], chap. I), on arrive aux propositions suivantes [10] :

THEOREME 7. – *Supposons que la fonctionnelle $\mathfrak{J} : \mathfrak{U} \to \mathrm{R}$ appartient à la classe $C^2(\mathfrak{U})$, vérifie les axiomes B et D, et a tous ses points critiques non dégénérés au sens ci-dessus (§ 7), chacun d'entre eux ayant un index de Morse fini. Soient a et b, avec $a < b$, des valeurs régulières (i.e. non critiques) de la fonctionnelle $\mathfrak{J}$. Si l'on fait correspondre à chaque point critique de $\mathfrak{J}^{a,b}$ une cellule de dimension égale à l'index de Morse de ce point, en recollant toutes ces cellules à l'ensemble $\mathfrak{J}^a$, on obtient un ensemble du même type d'homotopie que $\mathfrak{J}^b$.*

Notons R_k le nombre de Betti d'ordre k de la variété $\mathfrak{U}$, et C_k le nombre des points critiques de la fonctionnelle $\mathfrak{J}$ dont l'index de Morse égale k (chacun des nombres R_k et C_k, ou les deux ensembles pouvant être infini)

THEOREME 8 [10]. – *Rajoutons aux hypothèses du théorème 7 que la fonctionnelle $\mathfrak{J}$ est bornée inférieurement. Alors la variété $\mathfrak{U}$ a le type d'homotopie d'un complexe cellulaire où, à chaque point critique d'index de Morse k, correspond une cellule de dimension k. On a les inégalités de Morse :*

$$R_k \leqslant C_k \tag{21}$$

et, si les C_k sont finis, pour $k = 0, 1, 2, \ldots, m$,

$$\sum_{k=0}^{m} (-1)^{m-k} R_k \leqslant \sum_{k=0}^{m} (-1)^{m-k} C_k \tag{22}$$

9. Théorème d'existence.

Pour résoudre le grand problème de la théorie des fonctionnelles du type (2), la question de l'existence des points critiques, nous utilisons des raisonnements plus "classiques" que les méthodes des précédents paragraphes. Le point de départ en est le fait que nos fonctionnelles sont bornées inférieurement. La construction de suites minorantes nous permet de trouver un élément qui en réalise le minimum absolu. On peut surmonter toutes les difficultés qui se présentent en utilisant les méthodes de la théorie des fonctions de la variable complexe, par conséquent, d'après le lemme fondamental (§ 1), on peut démontrer un théorème d'existence dans la théorie des problèmes du type (1), sans faire aucune hypothèse fonctionnelle sur la fonction de forces $Q^2(x, y)$. Dans la mise en oeuvre de la méthode de minoration de la fonctionnelle (2), on rencontre en germe toutes les idées et méthodes fondamentales qu'on vient d'expliquer. Ainsi l'idée de ces constructions se ramène à établir l'existence de suites minorantes *bornées* (au sens de la métrique ambiante) d'éléments permis. L'hypothèse essentielle, qui permet de résoudre le probléme d'existence, a la forme :

$$(23) \quad d = \operatorname*{Inf}_{(G_\gamma, \psi)} \iint_{G_\gamma} \{\psi_x^2 + \psi_y^2 + Q^2(x, y)\} < \iint_G Q^2(x, y)\, dx\, dy,$$

où G désigne un domaine fini, borné par une courbe fixe Γ. Par son contenu, la condition (23) est parente de la condition classique de Douglas dans le problème de Plateau (cf, par ex., [12], chap. IV), et, dans les problèmes concrets d'hydrodynamique, cette condition peut toujours être satisfaite si l'on joue sur les paramètres (comme, par exemple, la "divergence" du courant).

THEOREME 9 [7]. (Théorème d'existence). – *Supposons que la fonction de forces $Q^2(x, y)$ a des dérivées partielles premières continues au sens de Hölder, et vérifie les conditions (16) et (23). Alors il existe un couple permis (G_γ, ψ), ayant les propriétés classiques de différentiabilité, et réalisant le minimum absolu de la fonctionnelle (2).*

Notons que, sous les hypothèses de ce théorème, le système intégro-différentiel non linéaire (9) admet, au moins, une solution.

Nous n'avons pas eu pour propos d'élargir au maximum les hypothèses de nos propositions, et on peut en améliorer certaines, par exemple en affaiblissant la régularité de la fonction de forces. Le caractère plan de nos problèmes a joué, dans des moments délicats, un rôle déterminant, et le passage aux problèmes analogues dans l'espace à trois dimensions requerra, sans aucun doute, des méthodes et raisonnements nouveaux.

BIBLIOGRAPHIE

[1] ABRAHAM R. — *Lectures of Smale on differential topology, mimeogr.* (cf. aussi l'appendice III à la traduction en russe du livre de S. Lang, 14).

[2] BAZALIJ B.V. et DANILJUK I.I. — Sur la ramification des points critiques des fonctionnelles définies par des intégrales à domaine d'intégration variable, *D.A.N. U.R.S.R.*, Sér. A, n° 1, 1970.

[3] BEURLING A. — On free boundary problems for Laplace equation, *Sem. Analyt. Funct.*, Vol. 1, N.-J., 1958, Inst. Adv. Stud.

[4] VAJNBERG M.M. — *Méthodes variationnelles d'étude des opérateurs non linéaires,* GITTL M., 1956.

[5] GARABEDIAN P.R. et SPENSER D.C. — Extremal methods in cavitational flow, *J. Rat. Mech. Anal.*, 1, 1952, p. 359.

[6] GARABEDIAN P.R., LEWY H. et SCHIFFER M. — Axially symmetric cavitational flow, *Ann. Math.*, 56, 1952, p. 560.

[7] DANILJUK I.I. — Théorème d'existence dans un problème non linéaire aux limites libres, *U.M.J.*, 20, n° 1, 1968, p. 25.

[8] DANILJUK I.I. — Etude d'une classe de fonctionnelles définies par des intégrales à domaine d'intégration variable, *D.A.N. U.R.S.R.*, Sér. A, n° 9, 1969, p. 783.
[9] DANILJUK I.I. — Sur la méthode de l'antigradient dans la théorie d'une classe de fonctionnelles, *D.A.N. U.R.S.R.*, Sér. A, n° 10, 1970, p. 876.
[10] DANILJUK I.I. — Généralisation de la théorie de Morse à une classe de fonctionnelles, *D.A.N. U.R.S.R.*, Sér. A, n° 1, 1971, p. 16.
[11] DANILJUK I.I. — Sur les fonctionnelles intégrales à domaine variable d'intégration, *Trudy Matem. Inst. A.N. S.S.S.R.*, t. 118 (sous presse).
[12] COURANT P. — *Principe de Dirichlet, applications conformes et surfaces minimales*, M., 1953.
[13] LOPATINSKIJ Ja. B. — Sur une application des méthodes de Morse en théorie des équations aux dérivées partielles du type elliptique, *D.A.N. U.R.S.S.*, Sér. A, n° 6, 1968, p. 515.
[14] LANG S. — *Introduction à la théorie des variétés différentiables*, M., 1966 (cf. aussi J. Wiley, 1962 et Dunod, 1967).
[15] MILNOR J. — *Théorie de Morse*, M., 1965 (cf. aussi Princeton, 1963).
[16] MUSHELIŠVILI N.I. — *Equations intégrales singulières*, M., 1962.
[17] PALAIS R.S. et SMALE S. — A generalized Morse theory, *Bull. Amer. Math. Soc.*, 70, 1, 1964, p. 165.
[18] SMALE S. — Morse theory and non-linear generalization of the Dirichlet problem, *Ann. of Math.*, Vol. 80, No. 2, 1964, p. 382.
[19] FRIEDRICHS K.O. — Über ein Minimumproblem für Potentialströmungen mit freien Rande, *Math. Ann.*, Vol. 109, 1933.

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Actes, Congrès intern. Math., 1970, Tome 2, p. 717 à 722.

ON THE LOCAL SOLVABILITY OF PSEUDO-DIFFERENTIAL EQUATIONS

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The construction of the complete theory of the boundary-value problems for elliptic equations is one of the most important achievments of the theory of partial differential equations for the last twenty years. This achievment would be of course impossible without the theory of distributions, and it can be considered rightfully as an attainment of this last theory. But when we pass to some wider classes of equations, we collide at once with the following dilemma: we must either extend the class of the generalized solutions, oversteping the limits of the theory of distributions, or study the conditions when the considered equations can be solved in the class of distributions.

In regard to the first way it has been followed only in particular cases, for example, in oblique derivative problem for the second order elliptic equation. Here we shall talk about the investigations concerning the second approach. We shall limit ourselves to the case of the scalar linear pseudo-differential equations of principal type. This class of equations is apparently next in simplicity to the class of elliptic equations. While the problem of finding solutions of an elliptic equation is reduced to the solving of an algebraic equation, the solving of a principal type equation can be reduced to the integration of a first order differential equation.

1. – The typical situation arising at the investigation of the solvability of pseudo-differential equations of principal type, can be seen from the following result of L. Hörmander.

THEOREM (L. Hörmander [2]). – *If in each point* $(x, \xi) \in T^*(\Omega)$, *where* $p^0(x, \xi) = 0$, *we have*

$$C_1^0(x, \xi) \equiv \frac{1}{2i} \{\overline{p^0}(x, \xi), p^0(x, \xi)\} < 0$$

(*here* { } – *the Poisson brackets* : $\{f, g\} \equiv \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right)$), *then the equation*

$$(1) \qquad P(x, \mathcal{D})u = f$$

can be solved always for $f \in \mathcal{D}'(\Omega)$ (Ω *is a domain in* $\mathbf{R}^n$). *Moreover, if* $f \in \mathcal{H}_s(\Omega)$, *then there exists a solution* $u \in \mathcal{H}_{s+m-1/2}(\Omega)$. *But if we have* : $c_1^0(x, \xi) > 0$ *at a point* $(x, \xi) \in T^*(\Omega)$, *where* $p^0(x, \xi) = 0$, *then there exist such smooth functions* $f \in \mathcal{D}(\Omega)$, *that the equation (1) has no solution in the class* $\mathcal{D}'(\Omega)$.

2. – The first example of such an equation has been constructed by H. Lewy (1957) (see [1]).

The results I shall talk about develop the famous results of L. Hörmander and L. Nirenberg - F. Treves (See [2], [5]).

As an example of such results I state the following.

THEOREM. – *Let $P(x, \mathcal{D})$ be a differential operator of order m with smooth (C^∞) coefficients. Let $p^0(x, \xi)$ be its principal symbol, $\operatorname{Im} p^0(x, \xi) = a_1(x, \xi)$, $\operatorname{Re} p^0(x, \xi) = a_2(x, \xi)$. Let $I = (i_1, \dots i_k)$ be a finite sequence of integers i_j which are equal either to 1 or to 2, and $C_I(x, \xi) = \{\dots\{a_{i_1}, a_{i_2}\}, \dots, a_{i_k}\}$. We put $|I| = k - 1$ and $k(x, \xi) = |I_0|$, if $p^0(x, \xi) = 0$, $C_I(x, \xi) = 0$ for $|I| < |I_0|$, but $C_{I_0}(x, \xi) \neq 0$. If $p^0(x, \xi) \neq 0$, we set $k(x, \xi) = 0$. Suppose that*

$$\sup k(x, \xi) = k < \infty.$$

Then the equation (1) can be solved for all $f \in \mathcal{D}'(\Omega)$ if and only if the function $k(x, \xi)$ has only even values. If it is so and $f \in \mathcal{H}_s(\Omega)$; then (1) has a solution $u \in \mathcal{H}^{\mathrm{loc}}_{s+m-\frac{k}{k+1}}(\Omega)$.

This theorem can be generalized for general pseudo-differential operators (see [12]).

3. – The following statement plays very important role in the proof of the above results.

LOCALISATION THEOREM. – *The estimate*

$$\|u\|_s \leqslant C_{K,s}\,(\|Pu\|_{s-m+\frac{k}{k+1}} + \|u\|_{s-1}), \quad u \in C_0^\infty(K).$$

(K is a compact in Ω), holds if and only if for all $x \in K$, $\xi \in S^{n-1}$, $\lambda \geqslant 1$ and $\psi \in C_0^\infty(\mathbf{R}^n)$ the estimate

$$(2)\qquad \|\psi\|_{L_2} \leqslant C \left\{ \|T^k_{(x,\xi)}\, p^0(x + y\lambda^{-\frac{1}{k+1}}, \xi + \mathcal{D}\lambda^{-\frac{k}{k+1}})\,\psi\|_{L_2}\,\lambda^{-m+\frac{k}{k+1}} + \right.$$
$$\left. + \lambda^{-\epsilon} \sum_{|\alpha+\beta| \leqslant N} \|y^\beta \mathcal{D}^\alpha \psi\|_{L_2}\,\lambda^{-|\alpha|\frac{k-1}{k+1}} \right\}$$

is valid. Here $\epsilon > 0$, $N \geqslant 0$ are some numbers $\left(\text{in the statement about necessity } N \geqslant k+1,\ \epsilon \leqslant \dfrac{1}{k+1}\right)$, and $T^k_{(x,\xi)}\, f(x + y, \xi + \eta)$ is a segment of the Taylor expansion for the function $f(x + y, \xi + \eta)$ in a point (x, ξ) with respect to (y, η).

This theorem has been proved by L. Hörmander for the case $k = 1$ (See [2]) and later it has been generalized for any k by L. Hörmander (in some different form ; see [3]) and by myself [8].

4. – The following two theorems about canonical transformations are very important for our theory. The first theorem states that the conditions (2) are invariant relative to any canonical transformations of cotangent bundle $T^*(\Omega)$. We would remind that canonical transformation is a mapping : $(x, \xi) \to (x', \xi')$ preserving the values of Poisson brackets $\{f, g\}$ for all pairs of functions $f, g : T^*(\Omega) \to \mathbf{C}$. This theorem permits to make the imaginary part of principal symbol of our operator equal to ξ_1 in a neighbourhood of a considered point (x^0, ξ^0), where $p^0(x^0, \xi^0) = 0$. It allows to simplify the investigation of such operators essentially.

The second theorem about canonical transformations states that for any pseudo-differential operator P and for any real function $S(x, \xi)$ such that

$$S(x, \lambda\xi) = \lambda S(x, \xi)$$

for $\lambda > 0$ and $\det \|\frac{\partial^2 S}{\partial x_i \partial \xi_j}\| \neq 0$, there exists such a pseudo-differential operator Q that $P\phi = \phi Q$, where

$$\phi u = \int \widetilde{u}(\xi)\, e^{iS(x,\xi)}\, d\xi$$

and $q^0(x', \xi') = p^0(x, \xi)$, if (x', ξ'), is the image of a the point (x, ξ) at the canonical transformation :

$$\xi = \operatorname{grad}_x S(x, \xi'), x' = \operatorname{grad}_{\xi'} S(x, \xi').$$

(G. Eskin had proved that ϕ is bounded and has a bounded inverse operator).

5. – The investigation of the estimates (2) can be reduced to the question of the validity of the inequalities of the following type

$$\|u\|_{L_2} \leqslant C \,\|\sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j} + Q(x, \mu)\, u\|_{L_2}, \forall\, u \in C_0^\infty(\mathbf{R}^n),$$

where $a = (a_1, \ldots, a_a)$ is a constant complex vector and $Q(x, \mu)$ is a polynomial of degree k. Precise necessary and sufficient conditions are obtained here for the homogeneous polynomials $Q(x, \mu)$. These conditions are sufficient too and if Q is not homogeneous. I shall not state these conditions here (see [9]). Note only that its forms are different for real a and for a, which is not proportional to a real vector.

6. – Our main theorem about necessary conditions of solvability is proved for operator of principal type : $\operatorname{grad}_{x,\xi} p^0(x, \xi) \neq 0$ if $p^0(x, \xi) = 0$. Let us note that such a theorem has been proved independently by L. Nirenberg and F. Treves [6], but at somewhat stronger conditions : $\operatorname{grad}_\xi p^0(x, \xi) \neq 0$, if

$$p^0(x, \xi) = 0.$$

This theorem states that if $k(x^0, \xi^0)$ is odd and (after due normalisation) the value of the first non-vanishing Poisson bracket $C_1(x^0, \xi^0) = (\operatorname{ad} a_1)^k\, C_2\, (x^0, \xi^0)$

is negative, then the equation (1) is unsolvable in the class of the distributions even locally.

This theorem can be formulated in the terms of the null bicharacteristics. We would remind that the bicharacteristic corresponding to the function $f(x, \xi)$ is the curve $x = x(t)$, $\xi = \xi(t)$ such that

$$\frac{dx_j}{dt} = \frac{\partial f(x, \xi)}{\partial \xi_j}, \quad \frac{d\xi_j}{dt} = -\frac{\partial f(x, \xi)}{\partial x_j}$$

The bicharacteristic is named null-bicharacteristic of $f(x, \xi)$, if $f(x(t_0), \xi(t_0)) = 0$ (and hence $f(x(t), \xi(t)) \equiv 0$). The above formulated condition of solvability means that

(A) along the null bicharacteristics of $a_1(x, \xi)$ the function $a_2(x, \xi)$ cannot change its sign from plus to minus.

7. – This condition (A) is close to a sufficient one. It becomes a sufficient condition if the following conditions B or B′ and C (or C′) are fullfilled.

THEOREM. – *The condition* A + B *or* A + B′ + C (*or* C′) *are sufficient for solvability.*

We suppose that either

(B) $$\sup k(x, \xi) = k < \infty$$

or

(B′) $$\operatorname{grad}_\xi p^0(x, \xi) \neq 0, \quad \text{if } p^0(x, \xi) = 0.$$

The conditions (C) and (C′) concern those points $(x, \xi) \in T^*(\Omega)$ only, in which $p^0(x, \xi) = 0$ and the vector grad $p^0(x, \xi)$ is proportional to a real vector. Let (x^0, ξ^0) be such a point and $\varphi = \arg \operatorname{grad} p^0(x^0, \xi^0)$, $q = p^0 e^{-i\varphi}$ so that the vector grad $q(x^0, \xi^0)$ is real.

(C) If the function Im $q(x, \xi)$ changes its sign on the manifold $Re\, q(x, \xi) = 0$ in any neighbourhood of the point (x^0, ξ^0), then the null-bicharacteristics of $Re\, q(x, \xi)$ are transversal to the manifold S, on which this changement of sign is realized. The manifold S is smooth.

The condition (C) is not necessary and can be replaced for example by a following one :

(C′) If (x^0, ξ^0) is such a point as above and the function Im $q(x, \xi)$ changes its sign on the manifold $Re\, q(x, \xi) = 0$ in any neighbourhood of the point (x^0, ξ^0), then there exists a smooth function $r(x, \xi)$ in some neighbourhood ω of the point (x^0, ξ^0) such that $r(x, \lambda\xi) = \lambda^{m-1'} r(x, \xi)$ for $\lambda > 0$, $\xi \neq 0$ and

$$C_1^0(x, \xi) \leqslant Re\, r(x, \xi)\, p^0(x, \xi) \quad \text{if } Re\, q(x, \xi) = 0$$

If the conditions (A) and (B) are fullfilled, then there exists such a solution of (1), that

$$\|u\|_s \leqslant C\,(\|f\|_{s+\frac{k}{k+1}-m} + \|u\|_{s-1}),$$

if A + B′ + C (or C′) — then the solvability of (1) is proved for a small neighbourhood ω_0 of the point x_0 and

$$\|u\|_s \leqslant C_1\,\|f\|_{s-m+1} + C_2\,\|u\|_{s-1}.$$

where $C_1 \to 0$ if diam $\omega_0 \to 0$. (see [13]).

A close theorem for the differential operators with analytic coefficients has been proved by L. Nirenberg and F. Freves.

At the conclusion of my lecture I should make some remarks about the nearest perspectives in this domain.

(1) The theorem about necessary conditions of solvability is true apparently without the supplementary supposition about the finitness of $k(x\,,\,\xi)$.

(2) We can hope to obtain the algebraic conditions for which the estimates

$$\|u\|_s \leqslant C(K\,,s)\,(\|Pu\|_{s-m+\delta} + \|u\|_{s-1}),\ \forall\ u \in C_0^\infty(K)$$

are valid with $1 \leqslant \delta < 2$. In any case it is possible to foretell these conditions.

(3) The theory of the boundary – value problems for operators of principal type can be constructed in the near future apparently so complete as it has been constructed for elliptic equations.

(4) At last, I think that the full clearness in the theory of solvability will be attained only after the solving of such a problem for the systems of pseudo-differential operators.

BIBLIOGRAPHY

[1] Lewy H. — An example of a smooth linear partial differential equation without solution, *Ann. of Math.*, (2) 66, 1957, p. 155-158.

[2] Hörmander L. — Pseudo-differential operators and non-elliptic boundary problems, *Ann. of. Math.*, 83, 1, 1966, p. 129-209.

[3] Hörmander L. — Pseudo-differential operators and hypoelliptic equations, *Proc. Symp. in pure Math.*, X, 1967, p. 138-183.

[4] Hörmander L. — *Linear Partial Differential Operators*, Springer-Verlag, 1963.

[5] Nirenberg L. and Treves F. — Solvability of a first order linear partial differential equation, *Comm. Pure Appl. Math.*, 16, 1963, p. 331-351.

[6] Nirenberg L. and Treves F. — On local solvability of linear partial differential equations. Part I : Necessary conditions, *Comm. Pure Appl. Math.*, 23, 1970, p. 1-38.

[7] Nirenberg L. and Treves F. — On local solvability of linear partial differential equations. Part : Sufficient conditions, *Comm. Pure Appl. Math.*, 23, 1970, p. 459-510.

[8] Egorov Ju. V. — On pseudo-differential operators of the principal type, *Dokl. Akad. Nauk S.S.S.R.*, 171, 1966, p. 778-780 [English translation : *Soviet Math. Dokl.*, 7, 1966, p. 1534-1536].

[9] Egorov Ju. V. — A priori estimates for first order differential operators, *Dokl. Akad. Nauk S.S.S.R.*, 185, 1969, p. 507-508 [English translation : *Soviet Math. Dokl.*, 10, 1969, p. 368-370].

[10] Egorov Ju. V. — О канонических преобразованиях псевдодифференциальных операторов, Upsehi Mat. Nauk, 24, № 5, 1969, 235-236.

[11] Egorov Ju. V. — Об условиях разрешимости псевдодифференциальных уравнений, Dokl. Akad. Nauk SSSR, 187, № 6, 1969, 1232-1234.

[12] Egorov Ju. V. — О субэллиптических псевдодифференциальных операторах, Dokl. Akad. Nauk SSSR, 188, № 1, 1969, 20-22.

[13] Egorov Ju. V. — О достаточных условиях локальной разрешимости псевдодифференциальных уравнений, Uspehi Mat. Nauk, XXV, 3, 1970, 269-270.

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SHARP FRONTS AND LACUNAS

by Lars GÅRDING

1. Introduction.

The theory of lacunas of fundamental solutions of hyperbolic operators is a spectacular link between the theory of partial differential operators and algebraic geometry, in particular the topology of plane sections of algebraic hypersurfaces. The theory was created in 1945 by I.G. Petrovsky [10] and carried further in a recent paper [1] by M.F. Atiyah, R. Bott and myself. The aim of this lecture is to outline the present state of the theory. Most of the material is from our joint paper, but I will also present some new results.

The various kinds of linear free wave propagation that occur in the mathematical models of classical physics are of the following general type. There is an elastic $(n-1)$-dimensional medium whose deviation from rest position is described by a function $u(x)$ with values in some R^N and defined in some open subset O of R^n, one of the coordinates being time. When there are no exterior forces, u satisfies a system $P(x, \partial/\partial x)u = 0$ of N linear partial différential equations with smooth coefficients. Further, a unit pulse applied at some point y produces close to y a movement that propagates with bounded velocity in all directions. This movement is smooth outside a system of possibly criss-crossing wave fronts and vanishes outside the propagation cone $K(P, y)$, a conical region with its vertex at y and bounded by the fastest fronts. Mathematically, the movement is described by a distribution $E = E(P, x, y)$ which is defined when x is close to y, vanishes when x is outside $K(P, y)$ and satisfies

$$P(x, \partial/\partial x)\; E(P, x, y) = \delta(x-y)$$

so that E is a (right) fundamental solutions of P. Under these circumstances we say that P is a hyperbolic operator. Briefly, P is hyperbolic if it has a fundamental solution with conical support as described above. When $P = P(\partial/\partial x)$ has constant coefficients, its fundamental solution E is defined for all x, y, it is unique and depends only on the difference $x - y$. It will then be denoted by $E(P, x-y)$ and we have

$$P(\partial/\partial x)\, E(P, x) = E(P, x)\, P(\partial/\partial x)\, \delta(x) = \delta(x).$$

The general situation just described covers among other things the free propagation of light in refracting media and the free propagation of waves in linear elasticity theory and linear magneto-hydrodynamics. The notions of wave fronts, sharp fronts and lacunas will be applied to fundamental solutions E but they are quite general. When u is a distribution defined in O, let Su denote the support of u and SSu its singular support, i.e. the complement of the largest open subset of O where u is a C^∞-function. When u describes a wave, parts of SSu are called wave fronts.

Let L be a component of $O - SSu$ and let $y \in \partial L$. We say that u is sharp at y from L or has a sharp front at y from L if u has a C^∞-extension from L to $\overline{L} \cap M$ for some neighbourhood M of y. Of course, sharpness is an exception. In general, the wave or derivatives of it tend to ∞ as one approaches the front. When u is sharp from L at all points of ∂L, L is said to be a lacuna for u. A lacuna L is said to be strong if $u = 0$ in L. Petrovsky [10], who seems to have introduced the word lacuna, only considers strong lacunas. The definition given here was suggested by L. Hörmander. Note that if u has homogeneity q, i.e. if $u(\lambda x) = \lambda^q u(x)$ when $\lambda > 0$ and if L contains the origin, then u is sharp hom L at the origin if and only if u is a polynomial in L. In particular, L is a strong lacuna for u unless q is a non-negative integer. The classical lacuna is the interior of the light-cone under free propagation of light in space-time. Another example is the part bounded by the slowest waves in free propagation of light in a general crystal. The outside of the propagation cone is trivially a strong lacuna for the fundamental solution of a hyperbolic operator.

2. Hyperbolic operators with constant coefficients.

Hyperbolic operators with constant coefficients can be characterized algebraically. Taking the scalar case $N = 1$ first, write $P = P(D)$ where $D = \partial / i\partial x$ is the imaginary gradient operator and associate with P its characteristic polynomial $P(\xi) = a(\xi) + b(\xi)$ where $a(\xi)$ is the principal part of P. Then P has a fundamental solution E with support in the union of the origin and the half-space $x\theta > 0$ if and only if $a(\theta) \neq 0$ and $P(\xi + \tau\theta) \neq 0$ for all real ξ when $|\operatorname{Im} \tau| \geqslant$ some constant c. Let hyp (θ) be the class of these operators, hyp (θ, m) the set of $P \in$ hyp (θ) of order m and write Hyp (θ), Hyp (θ, m) for the corresponding subclasses of homogeneous operators. Both hyp (θ) and Hyp (θ) are closed under multiplication and the taking of factors. Taking the principal part is a surjective map hyp $(\theta) \to$ Hyp (θ). A homogeneous operator $a(D)$ belongs to Hyp (θ, m) if and only if the equation $a(\xi + \tau\theta) = 0$ has m real roots τ for every real ξ. If all these roots are separate when ξ is not proportional to θ, a is said to be strongly or strictly hyperbolic. The corresponding class is denoted by Hyp^0 (θ, m). Addition of lower terms is an injective map

$$\text{Hyp}^0 (\theta, m) \to \text{hyp} (\theta, m).$$

The non-scalar case is simple, P is hyperbolic if and only if det P is hyperbolic, and we shall restrict ourselves to the scalar case. There is an explicit formula for the fundamental solution $E = E(P, \theta, x)$ of $P \in$ hyp (θ), viz.

$$E(P, \theta, x) = (2\pi)^{-n} \int P(\xi + i\eta)^{-1} e^{ix(\xi+i\eta)} d\xi \tag{1}$$

where $\eta = - c\theta$ with a large enough c and the integral is taken in the distribution sense. Let A be the complex hypersurface $a(\xi) = 0$ and let

$$\Gamma = \Gamma(P, \theta) = \Gamma(A, \theta)$$

be the component of $R^n - \operatorname{Re} A$ that contains θ. It is an open convex cone and $P \in$ hyp (η) for all $\eta \in \pm \Gamma$. More precisely, $P(\xi \pm i\eta) \neq 0$ when η belongs to a certain convex open subset Γ_1 of Γ such that every ray in Γ through the origin meets Γ_1 in an infinite interval. Also, the integral (1) is independent of the choice of $\eta \in - \Gamma_1$ so that, by the Paley-Wiener-Schwartz theorem, E vanishes outside the propagation cone

$$K = K(P, \theta) = K(A, \theta) = \{x \,;\, x\Gamma(A, \theta) \geqslant 0\}.$$

More precisely, K is the convex hull of the support of E. Developing $P^{-1} = (a+b)^{-1}$ in a geometrice series, we can write (1) in the form

$$(1') \qquad E(P, \theta, x) = \sum_0^\infty (-1)^k\, b(D)^k\, E(a^{k+1}, \theta, x).$$

It turns out that the sum converges in the distribution sense.

3. The wave front surface and the Herglotz-Petrovsky-Leray formulas.

We shall now describe the singular support of E, assuming for simplicity that a is strongly hyperbolic, $a \in \mathrm{Hyp}^0(\theta)$. The general case presents complications partly dealt with in [1]. Let ${}^0\mathrm{Re}\, A$ be the real dual of $\mathrm{Re}\, A$, i.e. the set of real x such that $\mathrm{Re}\, X$ is tangent to $\mathrm{Re}\, A$ along some real ray. Here X denotes the complex hyperplane $x\xi = 0$. Then $E(P, \theta, \cdot)$ and all

$$(2) \qquad E(a^k, \theta, x) = (2\pi)^{-n} \int a(\xi + i\eta)^{-k}\, e^{ix(\xi + i\eta)}\, d(\xi + i\eta)$$

with η as in (1) have the same singular support, namely the wave front surface of a defined by

$$W = W(A, \theta) = {}^0\mathrm{Re}\, A \cap K(A, \theta).$$

Outside W they are locally holomorphic. One way of seeing this is to replace the constant vector field $\xi \to \eta$ in (1), (2) by a non-constant smooth real vector field $\xi \to \nu(\xi)$. By Stokes' formula this does not change the integral provided certain safety measures are observed. There are only two requirements, that $P(\xi + i\nu(\xi)) \neq 0$ and that $x\nu(\xi)$ be bounded from below. We can meet the first by taking $\nu(\xi) = \eta$ for small ξ, $\nu(\xi)$ small compared to ξ for large ξ and see to it that $\nu(\xi)$ and $-\theta$ point to the same side of $\mathrm{Re}\, A$ when ξ is close to $\mathrm{Re}\, A$. If then $\mathrm{Re}\, X$ is nowhere tangent to $\mathrm{Re}\, A$ we can still control the sign of $x\nu(\xi)$ and, letting $x\nu(\xi) \geqslant \epsilon\, |\xi|$, $\epsilon > 0$, for large ξ, there is absolute convergence in (1) and (2) and the integrals turn out to be holomorphic in x. In the latter case, the integrand is homogeneous which makes it possible to perform a radial integration. In addition to the requirements above, the vector field ν is then taken to be homogeneous of degree 1, $\nu(\xi) = \nu(-\xi)$, $\nu(\xi) \in \mathrm{Re}\, X$ when ξ is close to $\mathrm{Re}\, A$ and finally, $a(\xi + is\nu(\xi)) \neq 0$ when $0 < s \leqslant 1$. Let $V(A, X, \theta)$ be the class of these vector fields. We get the Herglotz-Petrovsky-Leray formulas

$$(3) \qquad D^\beta E(a^k, \theta, x) = \mathrm{const} \int_{a^*} (x\xi)^q\, \xi^\beta a(\xi)^{-k}\, \omega(\xi)\,, \quad q > 0$$

$$(3') \qquad D^\beta E(a^k, \theta, x) = \mathrm{const} \int_{t_x \partial a^*} (x\xi)^q\, \xi^\beta\, a(\xi)^{-k}\, \omega(\xi)\,, \quad q \leqslant 0$$

valid when $x \in K(A, \theta) - W(A, \theta)$. Here const $\neq 0$, $q = mk - |\beta| - n$ is the homogeneity of the left side, m being the degree of a and

$$\omega = \Sigma (-1)^{k-1}\, \xi_k\, d\xi_1 \ldots \widehat{d\xi_k} \ldots d\xi_n,$$

The integrands are closed rational $(n-1)$-forms on $(n-1)$-dimensional complex projective space Z^* with poles on A^* and $A^* \cup X^*$ respectively where A^*, X^* are the images of A, X in Z^*. The forms are integrated over certain homology classes $\alpha^* = \alpha(A, \theta, x)^*$ and $t_x \partial\alpha^*$. The class α^* belongs to $H_{n-1}(Z^* - A^*, X^*)$ and $2\alpha^*$ is simply the class of the image in projective space of the map

$$\xi \to \xi + iv(\xi), \; v \in V(A, X, \theta),$$

oriented by $x\xi\omega(\xi) > 0$. Its boundary $\partial\alpha^* \in H_{n-2}(X^* - X^* \cap A^*)$ is an absolute class. The tube operation $t_x : H_{n-2}(X^* - X^* \cap A^*) \to H_{n-1}(Z^* - X^* \cup A^*)$ is generated by the boundary of a small 2-disk in the normal bundle of X^* when its center moves on X^*. The formulas (3), (3′) are essentially due to Herglotz [5] and Petrovsky [10]. The formulas of Petrovsky are obtained by taking one residue onto A^* in (3) and two successive residues onto $A^* \cap X^*$ in (3′) when $q = -1$. These operations are well-defined only when A^* and $A^* \cap X^*$ are non-singular. The class α^* was introduced by Leray [8]. With suitable definitions of W and α^*, (3), (3′) hold also in the general case $a \in \mathrm{Hyp}(\theta)$ (see [1]).

4. Topological criteria for lacunas and sharp fronts.

Let L be a component of $K(A, \theta) - W(A, \theta)$ and let $y \in \partial L$. According to the Herglotz-Petrovsky-Leray formulas, the behaviour of the cycle $\partial\alpha^*(A, \theta)$ as x approaches y determines the behaviour of the fundamental solutions $E = E(a^k, \theta, x)$ as x approaches y. Let us assume that E is sharp from L at y. When $y = 0$ this means that L is a lacuna for E and we consider this case first. We have

THEOREM 1. – *Let* $a \in \mathrm{Hyp}^0(\theta)$. Then x belongs to a lacuna for all $E(a^k, \theta, .)$ if and only if

$$\partial\alpha(A, \theta, x)^* = 0 \quad \text{in} \quad H_{n-2}(X^* - A^* \cap X^*). \tag{4}$$

The condition (4) was invented by Petrovsky and will be called the Petrovsky condition. If it holds for one $x \in L$, then, by (3′), all sufficiently high derivatives of $E = E(a^k, \theta, \cdot)$ vanish at x and hence E is a polynomial of homogeneity $mk - n$ in L. In particular, L is a lacuna for E and a strong lacuna if $mk - n < 0$. Hence the direct part of the theorem is an immediate consequence of (3′). To prove the converse part one has to use a basic result in the topology of algebraic manifolds. It follows from a well-known theorem by Grothendieck [3] that the cohomology of $Z^* - A^* \cup X^*$ is spanned by all rational forms with poles on $A^* \cup X^*$. Now all such forms appear in (3′) when k and ν vary and this proves the converse part of the theorem. When A^* is non-singular it is not difficult to see that all our forms with a given k span the cohomology so that the Petrovsky condition holds whenever x is in a lacuna for some $E(a^k, \theta, \cdot)$. In this form, Theorem 1 generalizes Petrovsky's main result in [10]. Components L of $K(A, \theta) - W(A, \theta)$ for which (4) holds for some and hence for all x will be called Petrovsky lacunas for a. They are then lacunas for all $E(a^k, \theta, \cdot)$. Petrovsky lacunas are stable. If $b \in \mathrm{Hyp}^0(\theta, m)$ is close to a $\mathrm{Hyp}^0(\theta, m)$ and L is a Petrovsky lacuna for a then the corresponding component L_b of $K(B, \theta) - W(B, \theta)$ is a Petrovsky lacuna for b.

In the general case $y \neq 0$, there is as yet no complete analogue of Theorem 1.

The corresponding direct part would of course be that if, as $x \to y$, $\partial\alpha\,(A\,,\theta,x)^*$ does not intersect a neighbourhood of the set where Y^* is tangent to A^*, then E is sharp from L at y. In fact, under these circumstances, there is a cycle independent of x which represents $t_x\partial\,\alpha\,(A\,,\theta\,,x)^*$ when x is close to y so that, in view of (3′), E has a holomorphic extension across ∂L at y. In precise terms, our condition amounts to the following.

$$\partial\alpha(A\,,\theta\,,x)^* \in H_{n-2}(Y^* - Y^* \cap A^*)\,. \tag{5}$$

Here the right side means the image of the map

$$H_{n-2}(Y^* - Y^* \cap A^*) \to H_{n-2}\,(X^* - X^* \cap A^*)$$

defined by a projection $Y^* \to X^*$ when x is close to y. Projections also induce isomorphisms between the right sides for all $x \in L$. Note that if $y = 0$ then Y^* is empty and the right side of (5) vanishes. Hence (5) generalizes the Petrovsky condition (4). When $n = 3$, it is easy to see that (5) is both necessary and sufficient for E to be sharp from L at y. Let T be the isomorphism of

$$H_{n-2} = H_{n-2}\,(X^* - X^* \cap A^*)$$

that we get by letting x make a loop T at y around the complexification of W and denote by the superscript inv the part of H_{n-2} that is invariant under all T. When E has a holomorphic extension across ∂L at y, then $\partial\alpha^* \in H_{n-2}^{\text{inv}}$ and one is led to ask whether the map

$$H_{n-2}(Y^* - Y^* \cap A^*) \to H_{n-2}^{\text{inv}}\,(X^* - X^* \cap A^*)$$

is surjective. This problem is connected with the so-called invariant cycle conjecture which is of current interest in algebraic geometry.

5. Sharp fronts and lacunas for hyperbolic operators with variable coefficients.

Let $P(x\,,D)$ be a differential operator of order m with smooth variable coefficients and let $a\,(x\,,D)$ be its principal part. Suppose that there is a θ such that $a\,(0\,,D) \in \text{Hyp}^0\,(\theta\,,m)$. Then, as is well known, P is hyperbolic and every P^k has a fundamental solution $E\,(P^k\,,x\,,y)$ defined for small x and y such that the support of the distributions

$$x \to E\,(P^k\,,\,x\,,\,y)\,,\,k = 1,\,2,\,\ldots \tag{6}$$

is contained in some conical neighbourhood of $y + K$ where $K = K\,(A_0\,,\theta)$, $a_y = a(y\,,D_x)$. They all have the same curved propagation cone $K(P\,,y) = K(P\,,\theta\,,y)$ with its vertex at y, tangent to $y + K\,(A_y\,,\theta)$ at y, and their singular support is a curved wave front surface $W\,(P,y) = W\,(P\,,\theta\,,y) \subset K\,(P,y)$ tangent to $y + W\,(A_y\,,\theta)$ at y. To every component L_0 of $K\,(A_0\,,\theta) - W\,(A_0\,,\theta)$ whose boundary has codimension 1 everywhere there is a corresponding component L of $K\,(P,y) - W\,(P,y)$ reducing to L_0 when $y = 0$. It can be shown that if L_0 is a Petrovsky lacuna for a_0, then L is a lacuna for the distributions (6) at least when L_0 has a regular boundary outside the origin. This last restriction may not be necessary. Also, roughly speaking, if $E\,(a_0\,,\theta\,,\cdot)$ is sharp from L_0 at parts of ∂L_0, (6) is sharp from L at the corresponding parts of ∂L. In the simplest case $m = 2$, this result is implicit in the formulas for fundamental solutions due to Hadamard [4] and M. Riesz [11]. The general result is local, the proof using Lax's local formulas [7] for the fundamental solutions. Global versions of them have been given recently by Hörmander [6] (see also Maslov [9]). They make it possible to study the behaviour of the distributions (6) near wave fronts which are not very close to the origin, but the general picture is far from clear.

6. Historical note.

It was discovered by Fredholm [2] (1900) that homogeneous elliptic differential operators with constant coefficients in three variables have fundamental solutions which can be written as abelian integrals. Between 1911 and 1921, Fredholm's work was extended by Nils Zeilon ([12], [13], [14]) to hyperbolic operators in three and four variables. His papers actually give the Herglotz-Petrovsky-Leray formulas in special cases. This was before the time of distributions and his proof that the function E that they represent is the fundamental solution with support in a cone is a bit vague. It uses the behaviour of E near the wave fronts in an essential way. In the long sprawling paper [14], dedicated to Vito Volterra, Zeilon constructs the fundamental solution of the system of crystal optics and gives a very complete analysis of its support, singular support and behaviour near the wave fronts. This paper must have been nearly impossible to read at the time but it makes good sense against the background of present-day theory. Fredholm's and Zeilon's work did not make much impact in Sweden and was almost entirely overlooked abroad. Gustav Herglotz started from scratch.

REFERENCES

[1] Atiyah M.F., Bott R. and Gårding L. — Lacunas for hyperbolic differential operators with constant coefficients I, *Acta Math.*, 124, 1970, p. 109-189.

[2] Fredholm I. — Sur les équations de l'équilibre d'un corps solide élastique, *Acta Math.*, 23, 1900, p. 1-42.

[3] Grothendieck A. — On the de Rham cohomology of algebraic varieties, *Publ. I.H.E.S.*, 29, 1966, p. 351-359.

[4] Hadamard J. — *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques,* Paris, 1932.

[5] Herglotz G. — Über die Integration linearer partieller Differentialgleichungen. I, II, III, Leipzig, *Ber. Sächs. Akad. Wiss. Math. Phys. Kl.*, 78, 1926, p. 93-126; 80, 1928, p. 6-114.

[6] Hörmander L. — *Fourier Integral Operators I, II,* Mimeographed, Lund, 1970.

[7] Lax P. — Asymptotic solutions of oscillatory initial value problems, *Duke Math. J.*, 24, 1957, p. 627-646.

[8] Leray J. — Un prolongement de la transformation de Laplace... (Problème de Cauchy IV), *Bull. Soc. Math. France,* 90, 1962, p. 39-156.

[9] Maslov V.P. — Theory of perturbations and asymptotic methods, *Moskov. Gos. Univ.*, Moscou, 1965 (Russian).

[10] PETROVSKY I.G. — On the diffusion of waves and lacunas for hyperbolic equations, *Mat. Sb.*, 17, 59, 1945, p. 289-370.
[11] RIESZ M. — L'intégrale de Riemann-Liouville et le problème de Cauchy, *Acta Math.*, 81 1949, p. 1-223.
[12] ZEILON N.O. — Das Fundamentalintegral der allgemeinen partiellen Differentialgleichung mit konstanten Koeffizienten, *Arkiv f. Mat. Astr. o. Fys.*, 6, No. 38, 1911, p. 1-32.
[13] ZEILON N.O. — Sur les intégrales fondamentales des équations à caractéristique réelle de la Physique Mathématique, *ibid.*, 9, n° 18, 1913, p. 1-70.
[14] ZEILON N.O. — Sur les équations aux dérivées partielles à quatre dimensions et le problème optique des milieux biréfringents; I, II, *Nova Acta Reg. Soc. Sc. Upsaliensis*, Ser. IV, Vol. 5, No. 3, 1919, p. 1-55; *ibid.*, Ser. IV, Vol. 5, No. 4. 1921, p. 1-128.

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ÉQUATIONS OPÉRATIONNELLES ABSTRAITES ET PROBLÈMES AUX LIMITES DANS DES DOMAINES NON RÉGULIERS

par P. GRISVARD

1. Equations opérationnelles abstraites :

Le but de cette première partie est d'étudier la "somme" de deux opérateurs fermés A et B (à domaines D_A et D_B denses) dans un espace de Banach (complexe) E. Cette somme L est définie par $Lu = Au + Bu$ pour $u \in D_L = D_A \cap D_B$. Les propriétés de L sont bien connues lorsqu'on considère B comme une perturbation de A (c.f. par exemple le livre de Kato [10] où $D_A \subset D_B$ avec B relativement borné par rapport à A). La situation qu'on considère ici est complètement différente puisqu'on ne suppose pas qu'il y a une inclusion entre les domaines D_A et D_B, par contre en vue des applications aux équations aux dérivées partielles, on supposera que les deux opérateurs commutent en un certain sens qui sera précisé plus loin. En général L n'est même pas fermé, mais il admet une fermeture $\widetilde{L}$ sous des hypothèses assez faibles. On cherchera à préciser $D_{\widetilde{L}}$ et on étudiera le spectre $\sigma_{\widetilde{L}}$ de $\widetilde{L}$: On peut espérer étendre à cette situation le "théorème spectral" en prouvant l'inclusion $\sigma_{\widetilde{L}} \subset \sigma_A + \sigma_B$ ou ce qui revient au même de prouver que $\widetilde{L}$ est inversible lorsque $\sigma_A \cap \sigma_{-B} = \emptyset$. On donnera une condition suffisante permettant d'obtenir cette propriété ; pour cela on utilisera les hypothèses suivantes qui sont commodes dans les applications :

(i) L'ensemble résolvant ρ_A de A contient le secteur $\{\lambda\ ;\ |\arg \lambda| < \theta_A\}$ et $\|(A - \lambda I)^{-1}\| \leqslant C_A(\theta)\,|\lambda|^{-1}$ pour $\arg \lambda = \theta$, $|\theta| < \theta_A$.

(ii) L'ensemble résolvant ρ_B de B contient le secteur $\{\lambda\ ;\ |\arg \lambda| < \theta_B\}$ et $\|(B - \lambda I)^{-1}\| \leqslant C_B(\theta)\,|\lambda|^{-1}$ pour $\arg \lambda = \theta$, $|\theta| < \theta_B$

(iii) $[(A - \lambda I)^{-1}\ ;\ (B - \mu I)^{-1}] = 0$ pour $\lambda \in \rho_A$ et $\mu \in \rho_B$ (cette hypothèse de "commutativité" peut être affaiblie).

THEOREME 1. — *Sous les hypothèses* (i) (ii) (iii) *avec* $\theta_A + \theta_B > \pi$ *et* $\sigma_A \cap \sigma_{-B} = \emptyset$ *l'opérateur* $\widetilde{L}$ *est inversible.*

On démontre ce résultat en construisant explicitement $\widetilde{L}^{-1}$ sous la forme d'une intégrale de Cauchy

$$- 1/(2\pi i) \int_\gamma (B + \lambda I)^{-1} (A - \lambda I)^{-1}\, d\lambda$$

où γ est une courbe joignant $\infty e^{-i\theta_0}$ à $\infty e^{i\theta_0}$ avec $\pi - \theta_B < \theta_0 < \theta_A$, γ demeurant hors de σ_A et de σ_{-B}. Il est évident que l'essentiel dans cette construction est de pouvoir trouver une courbe γ "séparant" σ_A de σ_{-B} et sur laquelle

$$\|(B + \lambda I)^{-1} (A - \lambda I)^{-1}\|$$

décroît comme $|\lambda|^{-2}$; cette idée permet de modifier notablement les hypothèses (i) et (ii).

On peut préciser $D_{\widetilde{L}}$ puisque c'est l'image de l'opérateur $\widetilde{L}^{-1}$ qui est explicite ; pour cela on est amené à introduire de nouveaux espaces liés à A et B et construits au moyen de la K-théorie de l'interpolation (c.f. Peetre [15] et sa bibliographie) : On munit $D_A m$ ($m \in \mathbf{N}$) de sa norme naturelle d'espace de Banach et on pose par définition

$$D_A(s\ ;p) = (D_A m\ ; D_A n)_{\theta,p;K}\ ,\ 0 < \theta < 1\ ,\ 1 \leqslant p \leqslant +\infty\ ,$$

avec $s = m(1-\theta) + n\theta$. On montre que pour s et p donnés cet espace ne dépend pas du choix particulier de m et n et plus précisément on montre dans [4] le

THEOREME II. — *Sous l'hypothèse* (i), $x \in D_A(s\ ;p)$ *si et seulement si*

$$\int_0^{+\infty} \| t^s A^m (A - tI)^{-m} x \|^p\, dt/t < +\infty ,$$

pour n'importe quel $m \in \mathbf{N}$, $m > s$ (avec la modification habituelle pour $p = +\infty$).

Ces espaces seront faciles à expliciter dans les cas concrets (c.f. plus loin). Dans le cas hilbertien on rappelle que

$$D_A(s\ ;2) = (D_A m\ ; D_A n)_{\theta,2;K} = [D_A m\ ; D_A n]_\theta$$

où les crochets désignent l'interpolation complexe de [2]. L'espace $D_A(s\ ;p)$ est une fonction croissante de p, décroissante de s et

$$D_A(m\ ;1) \subset D_A m \subset D_A(m\ ;\infty), \qquad \text{pour } m \in \mathbf{N}$$

On peut évidemment introduire des espaces analogues relatifs à B ; et on prouve dans [5] le

THEOREME III. — *Sous les hypothèses du théorème I on a :*

$$D_{\widetilde{L}} \subset D_A(1\ ;\infty) \cap D_B(1\ ;\infty)$$

Cet espace est très "voisin" de D_L (c.f. [4]). Dans [5] on démontre aussi le

THEOREME IV. — *Sous les hypothèses du théorème I l'opérateur* $\widetilde{L}^{-1}$ *est linéaire continu de* $D_A(s\ ;p)$ *dans* $D_A(s+1\ ;p)$ *pour tout* $s > 0$ *et tout* $p \geqslant 1$.

On en déduit que L est un isomorphisme de $\{u \in D_L\ ; Au\, ,\, Bu \in X\}$ sur X où X est l'un quelconque des espaces $D_A(s\ ;p)$ et $D_B(s\ ;p)$; en particulier la "restriction" de L à X est fermée.

Dans le cas où $E = H$ est un espace de Hilbert, on peut améliorer les résultats précédents (c.f. [12]) :

THEOREME V. — *Si E est un espace de Hilbert, on fait les hypothèses* (i) (ii) (iii) *et on suppose de plus qu'il existe* $s > 0$ *tel que*

$$D_A(s\ ;2) = D_{A^*}(s\ ;2)$$

alors L est fermé et inversible.

On verra que dans les applications la nouvelle hypothèse est facile à vérifier. La démonstration utilise le fait que d'après [14] E est interpolé entre $D_A(s\ ;\ 2)$ et son dual.

Une variante du problème étudié ci-dessus consiste à déterminer l'image de $\widetilde{L}$ dans certains cas où $\sigma_A \cap \sigma_{-B} \neq \emptyset$: On ne considérera que le cas plus simple où σ_A contient un nombre fini de valeurs propres de $-B$ (pour d'autres situations c.f. Doubinski [3]) : Plus précisément on remplacera l'hypothèse (ii) par la suivante :

(ii)′ Il existe $\lambda_1, \ldots \lambda_k$ tels que $\rho_B \supset \{\lambda\ ;\ |\arg \lambda| < \theta_B\ , \lambda \neq \lambda_j\ , j = 1, 2 \ldots k\}$, $(B - \lambda I)^{-1}$ a un pôle simple en chacun des points λ_j et $\|(B - \lambda I)^{-1}\| \leqslant C_B(\theta)\ |\lambda|^{-1}$ pour $\arg \lambda = \theta$, $|\theta| < \theta_B$, $|\lambda|$ assez grand. On introduit les projecteurs

$$P_j = -\ 1/(2\pi i) \int_{\gamma_j} (B - \lambda I)^{-1}\ d\lambda$$

où γ_j est une courbe simple d'indice un par rapport à λ_j et zéro par rapport à tout autre point de σ_B. Ceci posé on a le

THEOREME VI. – *Sous les hypothèses* (i) (ii)′ (iii) *avec* $\theta_A + \theta_B > \pi$ *l'image de* $\widetilde{L}$ *est le sous-espace de* E *formé des* f *tels que* $P_j f \in (A + \lambda_j I)\ (D_A)$ *pour* $j = 1, 2, \ldots k$. *Si de plus* $f \in D_A(s\ ;p)$, *il existe* $u \in D_L$ *(non unique) solution de* $Lu = f$ *avec* Au, $Bu \in D_A(s\ ;p)$. *Enfin dans le cas hilbertien et si il existe* $s > 0$ *tel que* $D_A(s\ ;\ 2) = D_{A^*}(s\ ;\ 2)$ *l'image de* L *est le sous-espace de* E *défini par les conditions* $P_j f \in (A + \lambda_j I)\ (D_A)$ *pour* $j = 1, 2, \ldots k$.

En particulier lorsque $(A + \lambda_j I)$ est à image fermée pour $j = 1, 2, \ldots k$, l'image de E est fermée.

Pour le cas où l'hypothèse (i) n'est vérifiée que sur un sous-espace fermé F de E c.f. [5] (On résoud $Lu = f$ pour $f \in F$ seulement).

II. Applications :

(a) Dans les exemples on aura à déterminer les espaces $D_A(s\ ;\ p)$ lorsque $D_A = W^k_{p,M}(\Omega) = \{u \in W^k_p(\Omega)\ ; M_j u = 0 \text{ sur } \partial\Omega\ , j = 1, \ldots l\}$, $1 < p < +\infty$ où Ω est un ouvert borné et régulier de R^n, M une famille d'opérateurs M_j d'ordre $\leqslant k - 1$ et $W^k_p(\Omega)$ est l'espace de Sobolev usuel d'ordre k relatif à $L_p(\Omega) = E$ (On renvoie à [13] pour la définition précise de tous les espaces fonctionnels utilisés dans ce qui suit).

Pour s assez petit $(0 < s < 1/kp)$ il résulte de [14] que $D_A(s\ ;p) = W^{ks}_p(\Omega)$ sans hypothèse sur M. Dans [6] [7] on détermine $D_A(s\ ;\ p)$ pour tout s en supposant que le système M est *normal* (i.e. $\partial\Omega$ est non caractéristique pour M_j pour tout j et les M_j sont tous d'ordres différents ; on notera m_j l'ordre de M_j).

THEOREME VII. – *On suppose que le système* M *est normal et qu'aucun des opérateurs* M_j *n'est d'ordre* $ks - 1/p$; *on suppose de plus que* ks *n'est pas entier lorsque* $p \neq 2$, *alors pour* $0 < s < 1$, *on a*

$$D_A(s\ ;p) = W^{ks}_{p,M}(\Omega) = \left\{ u \in W^{ks}_p(\Omega)\ ; M_j u = 0 \ \text{ sur } \ \partial\Omega\ ,\ m_j < ks - \frac{1}{p} \right\}$$

Plus généralement on a pour $0 < s \leqslant 1$ et $m_j \neq ks - 1/p$, $j = 1, \ldots l$

$$D_A(s\ ;\ q) = \{u \in B^{ks}_{p,q}(\Omega)\ ;\ M_j u = 0 \text{ sur } \partial\Omega \text{ pour } m_j < ks - 1/p\}$$

où $B^{ks}_{p,q}$ désigne l'espace de Besov et on a aussi

$$D_A(s\ ;\ \infty) = \{u \in H^{ks}_p(\Omega)\ ;\ M_j u = 0 \text{ sur } \partial\Omega \text{ pour } m_j < ks - 1/p\}$$

où $H^{ks}_p(\Omega)$ désigne l'espace de Nikolski. On renvoie à [6] [7] pour les autres cas qui nécessitent l'emploi d'espaces avec poids. Ces résultats sont encore vrais si on remplace les fonctions numériques par des fonctions à valeurs dans un espace de Banach X.

Récemment Seeley [16] a étendu ces résultats au cas de l'interpolation complexe.

(b) On va détailler l'application de la partie I à l'étude du premier problème aux limites pour l'équation de la chaleur. Les résultats ne seront pas nouveaux mais serviront à éclairer dans un cas concret la signification des théorèmes abstraits de la partie I : On pose

$$Q =]0\,,T[\ \times \Omega$$

où Ω est un ouvert borné régulier de R^n ; puis on pose

$$E = L_p(Q)\ ,\ D_A = \{u \in W^{1,0}_p(Q)\ ;\ u = 0 \text{ pour } t = 0\}\ ,\ Au = -\ \partial u/\partial t$$

$$D_B = \{u \in W^{0,2}_p(Q)\ ;\ u = 0 \text{ sur }]0\,,T[\ \times \partial\Omega\}\ ,\ Bu = \Delta_x u\ ,$$

où $W^{j,k}_p(Q)$ désigne l'espace des fonctions qui ont leurs dérivées jusqu'à l'ordre j par rapport à $t \in]0\,,T[$ et jusqu'à l'ordre k par rapport à $x \in \Omega$ dans $L_p(Q)$. Alors l'équation $Lu = f$ avec $u \in D_L$ s'explicite comme suit :

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_x u = f \text{ dans } Q \\ u = 0 \text{ pour } t = 0 \text{ et } u = 0 \text{ sur }]0\,,T[\ \times \partial\Omega \end{cases}$$

Le théorème III signifie que pour $f \in L_p(Q)$ ce problème admet une unique solution (faible) $u \in H^{1,2}_p(Q)$ (espace de Nikolski) ; ensuite le théorème IV montre que si de plus $f \in W^{s,2s}_p(Q)$ avec $0 < s < 1/2p$, le problème (1) admet une solution (forte) $u \in W^{s+1,2s+2}_p(Q)$, enfin le théorème V montre que pour $f \in L_2(Q)$, le problème (1) admet une solution (forte) $u \in W^{1,2}_2(Q)$: Dans ce dernier cas ($p = 2$) on a

$$D_A \ = \{u \in W^{1,0}_2(Q)\ ;\ u = 0 \text{ pour } t = 0\}$$

$$D_{A^*} = \{u \in W^{1,0}_2(Q)\ ;\ u = 0 \text{ pour } t = T\}$$

d'où $D_A(s\ ;\ 2) = D_{A^*}(s\ ;\ 2) = W^{s,0}_2(Q)$ pour $s < 1/2$; ce qui donne un exemple typique où l'hypothèse du théorème V est vérifiée sans que $D_A = D_{A^*}$.

On a ainsi retrouvé des résultats classiques de Solonnikov [17], Lions [12]. L'application de ces techniques à la résolution des problèmes mixtes paraboliques de degré quelconque est développée dans [7]. On résoud de même les problèmes elliptiques pondérés de Agmon-Nirenberg [1] ; dans ce cas l'opérateur A est la dérivation $\partial/\partial t$ dans $Q = R \times \Omega$ donc A est autoadjoint lorsque $p = 2$.

(c) Voici à présent un exemple moins classique : L'étude d'un problème aux limites elliptique dans un cône. Pour simplifier l'exposé, on considérera le problème le plus simple de ce type : Le problème de Dirichlet pour l'équation de Laplace dans un cône (tronqué) : $\Omega = \{x = r\omega \; ; \; 0 < r < 1, \; \omega \in G\}$ où G désigne un ouvert régulier de S^{n-1} (la sphère à $n - 1$ dimensions). On pose naturellement

$$E = \{u \; ; \; u/r^2 \in L_p(\Omega)\}\,, \; Au = r^2 \frac{\partial^2 u}{\partial r^2} + (n-1)\, r \frac{\partial u}{\partial r}$$

pour $u \in D_A = \left\{ u \; ; \; u \,, \; r \dfrac{\partial u}{\partial r}, \; r^2 \dfrac{\partial^2 u}{\partial r^2} \in E, \; u = 0 \text{ pour } r = 1 \right\}$

et $\quad Bu = \Lambda u$ (l'opérateur de Laplace-Beltrami sur S^{n-1})

pour $u \in D_B = \{u \; ; \; D^\alpha_\omega u \in E\,, \; |\alpha| \leqslant 2$ et $u = 0$ pour $x = r\omega, \omega \in \partial G\}$.

Le résultat suivant est conséquence directe du théorème VI lorsque $p = 2$ (et plus indirecte lorsque $p \neq 2$) :

THEOREME VIII. – *On suppose $p > 2n/(n+2)$ alors pour $f \in L_p(\Omega)$, il existe u telle que $r^{|\alpha|-2} D^\alpha u \in L_p(\Omega)$ pour $|\alpha| \leqslant 2$, solution de*

$$(2) \qquad \begin{cases} \Delta u = f & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

si et seulement si $(f \; ; \; v) = 0$ pour toute v telles que $r^{|\alpha|} D^\alpha v \in L_{p'}(\Omega)$ solution de

$$(3) \qquad \begin{cases} \Delta v = 0 & \text{dans } \Omega \\ v = 0 & \text{dans } \partial\Omega \end{cases}$$

à condition que soit vérifiée l'hypothèse (v.p.) qui suit :

(v.p.) Les valeurs propres de Λ avec conditions de Dirichlet sur G sont toutes différentes de $(n/p')\left(\dfrac{n}{p} - 2\right)$

On est ici dans la situation où $\sigma_A \cap \sigma_{-B} \neq \emptyset$, cet ensemble contenant un nombre fini de pôles simples de $(B - \lambda I)^{-1}$, l'opérateur $(A + \lambda I)$ étant pour chacune de ces valeurs injectif et à image fermée. Les fonctions v considérées dans l'énoncé du théorème VIII forment un espace vectoriel dont la dimension est le nombre de valeurs propres de Λ avec condition de Dirichlet sur G qui sont supérieures à $\dfrac{n}{p'}\left(\dfrac{n}{p} - 2\right)$. Cet espace est donc réduit à 0 (et la condition (v.p.) est automatiquement vérifiée) lorsque $p = 2$ et $n \geqslant 4$; en dimension 3, si on ne considère que des cônes circulaires (i.e. si G est une calotte sphérique) l'espace des v est réduit à 0 si Ω est convexe ; il en est de même en dimension 2 pour un secteur convexe. Dans le cas où $p < n$ utilisant un théorème d'immersion, on voit que si $n > p > 2n/(n+2)$, alors le problème (2) admet pour $f \in L_p(\Omega)$ une solution unique $u \in W^2_p(\Omega)$ si et seulement $(f \; ; \; v) = 0$ pour toute $v \in L_{p'}(\Omega)$

solution du problème (3) à condition que l'hypothèse (v.p.) soit vérifiée (On peut montrer que ce résultat est encore vrai pour $n = p = 2$).

Enfin on peut aussi montrer que ces résultats restent vrais si on remplace Δ par $\Delta - \lambda$ (λ non réel négatif) ce qui permet en ajoutant des variables supplémentaires, d'étudier le même problème dans un dièdre. Pour $p = 2$ les résultats ci-dessus sont proches de ceux de Kondratiev [11] et de Hanna-Smith [9] ; pour les démonstrations c.f. [8].

BIBLIOGRAPHIE

[1] AGMON-NIRENBERG. — *C.P.A.M.*, Vol. 16, 1963, p. 121.
[2] CALDERON. — *Studia-Math.*, t. 24, 1964, p. 113.
[3] DOUBINSKI. — *Mat. Sbornik*, t. 79, n° 121, 1969, p. 91.
[4] GRISVARD. — *J. Math. pures et appliquées*, t. 45, 1966, p. 143.
[5] GRISVARD. — *Annali S.N.S. Pisa*, t. 21, 1967, p. 307.
[6] GRISVARD. — *Archive for rat. mec. and anal.*, t. 25, 1967, p. 40.
[7] GRISVARD. — *Annales Sc. de l'E.N.S. Paris*, t. 2, 1969, p. 311.
[8] GRISVARD. — Article à paraître sur les problèmes aux limites elliptiques dans des domaines non réguliers.
[9] HANNA-SMITH. — *C. P. A. M.*, vol. XX, n° 3, p. 575-593. Août 1957.
[10] KATO. — *Die Grundlehren der Math.*, 132, Springer-Verlag.
[11] KONDRATIEV. — *Trudy Mockovckovo Mat. Ob.*, t. 16, 1967, p. 209.
[12] LIONS. — *Die Grundlehren der Math.*, 111 (2e édit.), Springer-Verlag.
[13] LIONS-MAGENES. — *Annali S.N.S., Pisa*, t. 15, 1961, p. 39 et p. 311; t. 16, 1962, p. 1.
[14] LIONS-PEETRE — *Publications de l'I.H.E.S.*, t. 19, 1964, p. 5.
[15] PEETRE. — Communication à ce Congrès.
[16] SEELEY. — *Amer. J. of Math.*, à paraître, et communication à ce Congrès.
[17] SOLONNIKOV. — *Trudy Mat. Inst. Steklov*, t. 83, 1965, p. 1.

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LES PROBLÈMES AUX LIMITES DÉGÉNÉRÉS ET LES OPÉRATEURS PSEUDO-DIFFÉRENTIELS

par V.V. GRUSHIN

1. Introduction.

Nous nous proposons de donner ici des conditions d'hypoellipticité et d'étudier les problèmes aux limites pour une certaine classe d'opérateurs différentiels et pseudodifférentiels elliptiques dégénérés. Si l'étude des problèmes aux limites elliptiques se ramène finalement à celle des opérateurs différentiels ordinaires à coefficients constants sur une demi-droite, dans le cas des équations elliptiques dégénérées, il apparait indispensable d'étudier les propriétés d'une certaine classe d'opérateurs différentiels à coefficients variables dans R^n. C'est pourquoi, nous commencerons par formuler les théorèmes de résolubilité des opérateurs dans R^n.

2. Opérateurs pseudodifférentiels dans R^n avec des symboles bornés.

Désignons par S^m l'ensemble de tous les symboles

$$p(x, \xi) \in C^\infty(R^n \times R^n),$$

tels que

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m-|\alpha|}, \quad p^{(\alpha)}_{(\beta)} = \frac{\partial^{\alpha+\beta}}{\partial \xi^\alpha \, \partial x^\beta} p.$$

Comme d'habitude, à chaque symbole correspond un opérateur pseudodifférentiel

$$p(x, \mathcal{D})\, u(x) = \frac{1}{(2\pi)^n} \int p(x, \xi)\, \widetilde{u}(\xi)\, e^{i(x,\xi)}\, d\xi, \quad u \in C_0^\infty(R^n).$$

D'après L. Hörmander [1] et H. Kymano-go [2], tout symbole de la classe S^m détermine un opérateur borné

$$p(x, \mathcal{D}) : H_{s+m}(R^n) \to H_s(R^n) \quad \forall s. \tag{2.1}$$

Le théorème suivant, établi dans [3], donne des conditions suffisantes pour que chaque opérateur soit d'indice fini, c'est-à-dire

$$\dim \operatorname{Ker} p(x, \mathcal{D}) < \infty \quad \text{et} \quad \dim \operatorname{Coker} p(x, \mathcal{D}) < \infty.$$

THEOREME 2.1. – *Soit* $p(x, \xi) \in S^m$ *et supposons*

$$p_{(\beta)}(x, \xi)/(1 + |\xi|)^m \to 0 \quad \forall \beta \neq 0 \quad \text{pour } \xi \to \infty$$

uniformément pour $x \in R^n$. Alors l'opérateur (2.1) *a un indice fini si et seulement si*

$$\varliminf_{(x,\xi)\to\infty} |p(x,\xi)|/(1+|\xi|)^m > 0.$$

3. Opérateurs différentiels à coefficients polynomiaux dans R^n.

Soit maintenant m un entier positif et soit $\delta > 0$ tel que $m\delta$ soit aussi entier. Considérons dans R^n l'opérateur différentiel

$$L(x,\mathcal{D}) = \sum_{\alpha\leqslant m,\ |\gamma|\leqslant(m-|\alpha|)\delta} a_{\alpha\gamma}\, x^\gamma \mathcal{D}^\alpha. \tag{3.1}$$

où α et γ sont des multi-indices d'entiers positifs ou nuls. Posons

$$L_0(x,\xi) = \sum_{|\alpha|\leqslant m,\ |\gamma|=(m-|\alpha|)\delta} \alpha_{\alpha\gamma}\, x^\gamma \xi^\alpha.$$

Désignons par $H_{(m,\delta)}(R^n)$ l'espace des fonctions $u(x)$ de norme

$$\|u\|_{(m,\delta)} = \Big(\sum_{|\alpha|\leqslant m} \int (1+|x|)^{2(m-|\alpha|)\delta} |\mathcal{D}^\alpha u(x)|^2\, dx\Big)^{1/2} < \infty\ .$$

THEOREME 3.1. – *L'opérateur*

$$L(x,\mathcal{D}) : H_{(m,\delta)}(R^n) \to L_2(R^n)$$

est d'indice fini si et seulement si

$$L_0(x,\xi) \neq 0 \qquad \forall\, (x,\xi)\in R^n \times R^n,\ |x|+|\xi|\neq 0. \tag{3.2}$$

Démonstration. Prenons une fonction monotone $t(r)\geqslant 1$ telle que $t(r)=1$ pour $r<1$, $t(r)=r^\delta$ pour $r>2$ et $t(r)\in C^\infty(R^1)$. Dans l'équation

$$L(x,\mathcal{D})\,u(x) = f(x), \tag{3.3}$$

effectuons le changement de fonctions

$$u_1(x) = t^{m-n/2}(|x|)\,u(x)\ ,\ f_1(x) = t^{-n/2}(|x|)\,f(x)$$

et le changement de variable $y = x\,t(|x|)$. L'équation (3.3) est transformée en l'équation

$$p(y,\mathcal{D}_y)\,u_1(y) = f_1(y)$$

et pour l'opérateur

$$p(y,\mathcal{D}_y) : H_m(R^n_y) \to L_2(R^n_y)$$

on peut appliquer le théorème 2.1. Revenant aux anciennes variables, on obtient la démonstration du théorème 3.1.

THEOREME 3.2. – *Supposons que l'opérateur* (3.1) *vérifie* (3.2) *; alors*

$$u\in S'(R^n),\ L(x,\mathcal{D})\,u \in S(R^n) \Rightarrow u\in S(R^n).$$

COROLLAIRE. — Si la condition (3.2) est réalisée, alors les affirmations suivantes sont équivalentes :

(i) Ker $L(x, \mathcal{D}) = 0$,

(ii) l'équation $L(x, \mathcal{D})\, u = 0$ n'a pas de solution non triviale $u \in S(R^n)$,

(iii) si on considère $L(x, \mathcal{D})$ comme un opérateur non borné dans $L_2(R^n)$, alors 0 n'est pas valeur propre de $L(x, \mathcal{D})$.

4. Critères d'hypoellipticité d'une classe d'opérateurs différentiels.

Considérons maintenant l'espace R^N, où $N = k + n$ et répartissons dans R^N les variables en deux groupes, $x = (x', y)$, $x' \in R^k$, $y \in R^n$. Désignons par $\mathfrak{M}_0$ l'ensemble des triplets d'entiers

$$\mathfrak{M}_0 = \{(\alpha, \beta, \gamma) : |\alpha| + |\beta| \leqslant m,\ |\gamma| = |\alpha| + (1 + \delta)|\beta| - m\}.$$

Soit donné dans R^N un opérateur $p(x, \mathcal{D})$ de la forme

$$p(x, \mathcal{D}) = \sum_{\mathfrak{M}_0} a_{\alpha\beta\gamma}\, y^\gamma \mathcal{D}_{x'}^\beta \mathcal{D}_y^\alpha \tag{4.1}$$

Supposons que $p(x, \mathcal{D})$ est elliptique pour $y \neq 0$. Alors, pour $\xi \neq 0$, les conditions du théorème 3.1 sont satisfaites pour l'opérateur $p(y\,;\, \xi, \mathcal{D}_y)$ et par suite, Ker $p(y, \xi, \mathcal{D}_y) < \infty$.

THEOREME 4.1. — *Soit un opérateur $p(x, \mathcal{D})$ de la forme* (4.1) *elliptique pour $y \neq 0$. Alors $p(x, \mathcal{D})$ est hypoelliptique si et seulement si*

$$\text{Ker}\ p(y\,;\, \xi, \mathcal{D}_y) = 0 \qquad \forall\, \xi \in R^n,\ |\xi| = 1. \tag{4.2}$$

Démonstration. — Si (4.2) n'est pas satisfait, alors on peut trouver $v(y) \in S(R^n)$, $v(y) \not\equiv 0$ et $\omega \in R^n$, $|\omega| = 1$, tels que

$$p(y\,;\, \omega, \mathcal{D}_y)\, v(y) = 0.$$

Alors la fonction

$$u(x', y) = \int_0^\infty v\left(t^{\frac{1}{1+\delta}} y\right) e^{it(x',\omega)} (1 + t)^{-l}\, dt$$

vérifie l'équation $p(x, \mathcal{D})\, u(x) = 0$ pour tout $l > 1$. Puisque la fonction $v(y)$ est solution d'une équation elliptique à coefficients analytiques, cette fonction $v(y)$ est analytique et, par suite, $\mathcal{D}_y^\alpha v(0) \neq 0$ pour un certain multiindice α. Il est facile de voir qu'alors $\mathcal{D}_y^\alpha u(x', 0) \notin C^\infty(R^k)$ et par suite l'opérateur $p(x, \mathcal{D})$ n'est pas hypoelliptique.

La démonstration de la suffisance de la condition (4.2) est beaucoup plus complexe. L'étape fondamentale de cette démonstration est que, d'après la condition (4.2), l'opérateur $p(y\,;\, \xi, \mathcal{D}_y)$ a un inverse à gauche pour tout $\xi \neq 0$ et, par suite, on peut, en utilisant la transformation de Fourier par rapport aux variables x', construire une parametrix à gauche pour $p(x, \mathcal{D})$. Nous ne donnerons pas ici les détails de cette démonstration.

Exemple. L'opérateur

$$p^{(x,\mathcal{D})} = \frac{\partial^2}{\partial x_2^2} + x_2^2 \frac{\partial^2}{\partial x_1^2} + i\lambda \frac{\partial}{\partial x_1} \tag{4.3}$$

est hypoelliptique si et seulement si $\pm\lambda$ ne sont pas valeurs propres de l'opérateur ordinaire $x_2^2 - \frac{d^2}{dx_2^2}$. Comme on le sait, les valeurs propres de cet opérateur sont les nombres $2k - 1$, $k = 1, 2, \ldots$. Par suite, l'opérateur (4.3) est hypoelliptique si et seulement si $\lambda \neq \pm (2k - 1)$. Remarquons qu'il ressort des résultats [4] de L. Hörmander que l'opérateur (4.3) est hypoelliptique pour $\mathrm{Re}\,\lambda = 0$. Notre résultat sur l'hypoellipticité de l'opérateur (4.3) pour $\mathrm{Re}\,\lambda \neq 0$ et $\lambda \neq \pm (2k - 1)$ est nouveau. On peut également démontrer que l'opérateur (4.3) est localement résoluble dans un voisinage de l'origine des coordonnées si et seulement si

$$\lambda \neq \pm (2k - 1).$$

On peut obtenir des conditions suffisantes d'hypoellipticité pour des opérateurs plus généraux. Soit

$$\mathfrak{M} = \{(\alpha, \beta, \gamma) : |\alpha| + |\beta| \leqslant m , m\delta \geqslant |\gamma| \geqslant |\alpha| + (1 + \delta)|\beta| - m\}.$$

Considérons un opérateur de la forme

$$p(x, \mathcal{D}) = \sum_{\mathfrak{M}} q_{\alpha\beta\gamma}(x, \mathcal{D}) \,.\, y^\gamma \mathcal{D}_{x'}^\beta \mathcal{D}_y^\alpha \tag{4.4}$$

où les $q_{\alpha\beta\gamma}(x, \mathcal{D})$ sont des opérateurs pseudodifférentiels "classiques" d'ordre zéro. avec $q_{\alpha 00}(x, \mathcal{D}) = q_{\alpha 00}(x)$ pour $|\alpha| = m$. Soit

$$q^0_{\alpha\beta\gamma}(x ; \xi, \eta), (\xi, \eta) \in R^k \times R^n,$$

la partie de plus haut degré du symbole $q_{\alpha\beta\gamma}$. Posons

$$L_0(y ; \xi, \mathcal{D}_y) = \sum_{\mathfrak{M}_0} q^0_{\alpha\beta\gamma}(0 ; \xi, 0) y^\gamma \xi^\beta \mathcal{D}_y^\alpha .$$

THEOREME 4.2. – *Supposons que*

$$L_0^0(y ; \xi, \eta) = \sum_{\mathfrak{M}_0^0} q^0_{\alpha\beta\gamma}(0 ; \xi, 0) y^\gamma \xi^\beta \eta^\alpha \neq 0$$

pour tout $y \neq 0$, $|\xi| + |\eta| \neq 0$, *où* $\mathfrak{M}_0^0 = \{(\alpha, \beta, \gamma) \in \mathfrak{M}_0 : |\alpha| + |\beta| = m\}$. *Si Ker* $L_0(y ; \xi, \mathcal{D}_y) = 0$ *pour tout* $\xi \in R^k$, $|\xi| = 1$, *alors l'opérateur* $p(x, \mathcal{D})$ *est hypoelliptique dans un certain voisinage de l'origine des coordonnées.*

Pour $n = 1$, ce résultat a été obtenu dans [5].

5. Problèmes aux limites pour les équations elliptiques dégénérées.

Dans le travail [6], on a étudié les problèmes aux limites pour les équations elliptiques dégénérées. Soit, dans un domaine borné Ω de frontière différentiable $\partial\Omega$, un opérateur différentiel $L(x, \mathcal{D})$ d'ordre $2m$, elliptique pour $x \in \Omega$, l'ellipticité n'étant plus satisfaite pour $x \in \partial\Omega$. Supposons donné, dans un certain voisinage de l'ensemble $\partial\Omega$ un champ de vecteurs ν transversal à $\partial\Omega$. Dans un certain voisinage V d'un point quelconque $x_0 \in \partial\Omega$, choisissons un système de coordonnées telles que les caractéristiques du champ de vecteurs ν soient données par les équations $x_1 = C_1, \ldots, x_{n-1} = C_{n-1}$ et que l'ensemble $\Omega \cap V$ soit défini par l'inégalité $x_n > 0$. Supposons que, pour un tel système de coordonnées, l'opérateur $L(x, \mathcal{D})$ est de la forme

$$(5.1) \qquad L(x, \mathcal{D}) = \sum_{|\alpha| \leqslant 2m} a_\alpha(x) x_n^{l_\alpha} \mathcal{D}_\alpha$$

où $\qquad l_\alpha = \min(0, |\alpha_n| + (1+\delta)|\alpha'| - 2m), \alpha' = (\alpha_1, \ldots, \alpha_{n-1})$,

où les coefficients $a_\alpha(x)$ sont continus dans V (les indices l_α ne sont pas ici nécessairement des entiers).

Désignons par $H_{(2m,\delta)}(\Omega)$ l'espace des fonctions $u(x) \in H_{2m}^{\text{loc}}(\Omega)$ telles que pour tout voisinage V avec des coordonnées satisfaisant aux conditions ci-dessus et toute $\varphi(x) \in C_0^\infty(V)$,

$$\varphi(x)\, x_n^{l_\alpha}\, \mathcal{D}^\alpha\, u(x) \in L_2\,(V \cap \Omega) \qquad \forall \alpha,\ |\alpha| \leqslant 2m.$$

Posons

$$L_0(x_n, \mathcal{D}) = \sum_{\substack{|\alpha| \leqslant 2m \\ |\alpha_n| + (1+\delta)|\alpha'| \geqslant 2m}} a_\alpha(x_0)\, x_n^{l_\alpha} \mathcal{D}^\alpha$$

Supposons que, pour tout point $x_0 \in \partial\Omega$, l'opérateur $L_0(x_n, \mathcal{D})$ est ellpitique pour $x_n > 0$. On peut alors montrer que, pour $n > 2$, l'équation :

$$(5.2) \qquad L_0^0(1\,;\xi', \zeta) = \sum_{|\alpha| = 2m} a_\alpha(x_0)\,(\xi')^{\alpha'} \zeta^{\alpha_n} = 0$$

pour $\xi' \neq 0$, a exactement m racines avec $\operatorname{Im} \zeta > 0$. Dans le cas $n = 2$, nous ferons l'hypothèse que l'équation (5.2), pour $\xi' \neq 0$, a encore exactement m telles racines.

LEMME 5.1. – L'équation $L_0(x_n; \xi', \mathcal{D}_n)\, v(x_n) = 0$, pour $\xi' \neq 0$, a exactement m solutions linéairement indépendantes dans $S(R_+^1)$, où $R_+^1 = \{x_n : x_n > 0\}$.

Soient donnés m opérateurs bornés $B_j(x, \mathcal{D})$, qui s'écrivent dans toute carte locale V sous la forme

$$B_j(x, \mathcal{D}) = \sum_{|\alpha_n| + (1+\delta)|\alpha'| \leqslant m_j} b_{j\alpha}(x) \mathcal{D}^\alpha$$

avec $m_j < 2m$. Posons

$$B_j^0(\mathcal{D}) = \sum_{|\alpha_n| + (1+\delta)|\alpha'| = m_j} b_{j\alpha}(x_0)\, \mathcal{D}^\alpha$$

THEOREME 5.1. – *Soit donné un opérateur $L(x, \mathcal{D})$ satisfaisant aux conditions ci-dessus et tel que, pour tout point $x_0 \in \partial\Omega$, le problème aux limites sur le demi-axe R^1_+*

$$L_0(x_n ; \xi', \mathcal{D}_n)\, v(x_n) = 0 \,, B^0_j(\xi', \mathcal{D}_n)\, v(0) = 0 \,, 1 \leqslant j \leqslant m,$$

pour $|\xi'| = 1$, n'ait pas de solution non triviale dans $S(R^1_+)$. Alors, l'opérateur

$$(L(x, \mathcal{D}), \gamma B_1(x, \mathcal{D}), \ldots, \gamma B_m(x, \mathcal{D})),$$

où γ est l'opérateur de restriction des fonctions à $\partial\Omega$, définit une application continue de $H_{(2m,\delta)}(\Omega)$ dans $L_2(\Omega) \times \prod_{j=1}^{m} H_{s_j}(\partial\Omega)$, pour

$$s_j = \left(2m - m_j - \frac{1}{2}\right) / (1 + \delta),$$

et est d'indice fini.

Remarquons que la surface $\partial\Omega$ n'est pas caractéristique pour de tels opérateurs. Dans le travail [7], on a obtenu des résultats analogues pour certaines classes d'équations pour lesquelles la surface $\partial\Omega$ est caractéristique. Dans [8] et [9], V.P. Glouchke a étudié, avec des méthodes analogues, de tels problèmes aux limites pour des équations du second ordre. On peut, de la même manière, étudier les opérateurs pseudo-différentiels elliptiques dans une variété formée Ω, dégénérés sur une sous-variété Γ, qui, au voisinage de tout point $x_0 \in \Gamma$, pour un système convenable de coordonnées locales, sont de la forme (4.4), en se donnant sur Γ des opérateurs limites et colimites. Dans le cas Codim $\Gamma = 1$, de tels problèmes sont examinés dans [5]. Par manque de place, nous ne nous attarderons pas ici sur les applications de ces résultats aux problèmes aux limites non elliptiques du type du problème avec dérivée oblique, traitées dans [10].

BIBLIOGRAPHIE

[1] HÖRMANDER L. — Pseudo-differential operators and hypoelliptic equations, *Amer. Math. Soc. Symp. on Singular Integral Operators*, 1966, p. 138-183.

[2] KYMANO-go H. — Remarks on pseudo-differential operators, *J. Math. Soc. Japan*, 21, No. 3, 1969, p. 413-439.

[3] Грушин В. В. — Псевдодифференциальные операторы в R^n с ограниченными символами. Функциональный анализ и его приложения, 4, № 3, 1970, 37-50.

[4] HÖRMANDER L. — Hypoelliptic second order differential equations, *Acta matematica*, 119, 1967, p. 147-171.

[5] Вишик М. И., Грушин В. В. — Эллиптические псевдодифференциальные операторы на замкнутом многообразии, вырождакщиеся на подмногообразии. Доклади АН СССР, 189, № 1, 16-19.

[6] Вишик М. И., Грушин В. В. — Об одном классе вырождающиеся эллиптических уравнений высших порядков. Матем. сборник, 79, № 1, 3-36, 1969.

[7] Вишик М. И., Грушин В. В. — Краевые задачи для эллиптических уравнений, вырождающихся на границе области. Матем. сборник, 80, № 4, 455-591, 1969.
[8] Глушко В. П. — Коэрщитивность в L_2 общих граничных задач для вырождающегося эллиптического уравнения второго порядка. Фуикциональный анализ и его приложения, 2, № 3, 87-88, 1968.
[9] Глушко В. П. — Априорные оценки в L_2 решений обших граничных задач для вырождающихся эллиптических уравнений второго порядка. Доклады АН СССР, 187, № 2, 245-248, 1969.
[10] Вишик М. И., Грушин В. В. — Эллиптические краевые задачи, вырождающиеся на подмногообразии границы. Доклады АН СССР, 190, № 2, 255-258, 1970 г.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 745 à 753.

DÉVELOPPEMENTS EN FONCTIONS PROPRES DES EXTENSIONS ARBITRAIRES AUTOADJOINTES NON-NÉGATIVES DE QUELQUES OPÉRATEURS ELLIPTIQUES

par V. A. IL'INE

Le présent travail étudie les développements spectraux correspondant aux extensions arbitraires autoadjointes non-négatives de quelques opérateurs elliptiques dans un domaine arbitraire G(1) (borné ou non-borné) dans l'espace E_N.

On propose des méthodes universelles d'étude de ces développements, qui embrassent le cas de n'importe quel spectre (ponctuel, continu ou mixte).

En gros, ces méthodes résultent de la confrontation de mes travaux [1] – [5] et du théorème de L. Gårding [6] sur la représentation ordonnée de l'espace $L_2(G)$ par rapport à une extension arbitraire autoadjointe de l'opérateur elliptique.

Ces méthodes permettent d'obtenir :

(1) Les conditions de convergence uniforme des développements spectraux et leurs moyennes de Riesz, qui sont définitives dans chaque classe de fonctions de Sobolev W_p^α, de Nicolscky H_p^α, de Liouville L_p^α, de Bessov $B_{p,\theta}^\alpha$ et de Zygmund-Hölder C^α(2).

(2) Les conditions de localisation des moyennes de Riesz ci-dessus indiquées au sens classique (c'est-à-dire de convergence uniforme dans un voisinage du point donné), ainsi qu'au sens généralisé (c'est-à-dire, de convergence presque partout, dans un voisinage de ce point).

Les résultats obtenus sont nouveaux même dans le cas des séries multiples trigonométriques de Fourier et des intégrales multiples de Fourier dans l'espace E_N.

Nous nous permettons de formuler d'abord tous les résultats dans le cas simple de l'opérateur de Laplace et ensuite nous tâchons de donner des indications sur l'extension de ces résultats aux autres opérateurs elliptiques.

1. Quelques remarques sur le caractère du spectre ponctuel d'une extension autoadjointe non-négative de l'opérateur de Laplace.

THEOREME 1. — *(Les valeurs propres ne peuvent se raréfier*(3)*). Soit G un domaine arbitraire de l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace dans le domaine G, ayant un spectre ponctuel.*

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(1) Il est possible d'employer aussi l'espace plus grand $L_2(\Omega)$, où $G \subseteq \Omega \subseteq E_N$.

(2) Pour la définition de toutes ces classes voir l'article [7].

(3) Voir l'article [8], V.A. Il'in.

Alors, il existe des nombres positifs r et α, tels que le nombre des valeurs propres λ_k sur le segment $\mu^2 \leqslant \lambda_k \leqslant (\mu + r)^2$ vérifie l'inégalité

$$(1) \qquad \sum_{\mu^2 \leqslant \lambda_k \leqslant (\mu+r)^2} 1 \geqslant \alpha \cdot \mu^{N-1} .$$

L'estimation (1) démontre que les valeurs propres ne peuvent se raréfier sur l'axe numérique : pour tout $\mu > 0$ le nombre de valeurs propres appartenant au segment $[\mu^2 , (\mu + r)^2]$ est supérieur à $\alpha \cdot \mu^{N-1}$.

Notons deux corollaires du théorème 1.

COROLLAIRE 1. – *Soit $n(\lambda)$ – le nombre des valeurs propres n'excédant pas λ. Alors, pour une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace à spectre ponctuel, l'inégalité*

$$(2) \qquad n(\lambda) \geqslant \alpha \cdot \lambda^{N/2}$$

est vérifiée.

COROLLAIRE 2. – *Supposons ceci :* (1) *une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace a un spectre discret (c'est-à-dire : les valeurs propres n'ont pas de valeur d'accumulation finie),* (2) *le numérotage des valeurs propres est tel que $\lambda_{k+1} \geqslant \lambda_k$ $(k = 1, 2, \ldots)$.*

Alors on a l'estimation(1)

$$(3) \qquad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = O(1) .$$

THEOREME 2. – (*Sur l'ensemble de points d'accumulation des valeurs propres* (2)).

Soit $N \geqslant 2$, $\{\lambda_n^0\}$ un ensemble arbitraire de nombres non-négatifs, donné d'avance sans point d'accumulation à distance finie. Pour un domaine borné dans l'espace E_N(3) *il y a une extension autoadjointe non-négative de l'opérateur de Laplace dans ce domaine, à spectre ponctuel, pour laquelle l'ensemble $\{\lambda_n^0\}$ est une partie de l'ensemble des valeurs propres. De plus, si, de l'ensemble des valeurs propres, on supprime l'ensemble $\{\lambda_n^0\}$, le nombre $\widetilde{n}(\lambda)$ des autres valeurs propres, qui n'excèdent pas λ, vérifie la formule :*

$$(4) \qquad \widetilde{n}(\lambda) = O(\lambda^{N/2}) .$$

Il en résulte, que dans un domaine borné il existe une extension autoadjointe non-négative de l'opérateur de Laplace à spectre ponctuel, dont les valeurs propres

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(1) Si, pour démontrer l'inégalité (3), nous employons la formule de R. Courant pour $n(\lambda)$ (voir [9], p. 364-377), alors dans la partie droite (3) au lieu de $O(1)$ nous aurons, généralement, $O(ln\, \lambda_k)$.

Cela résulte de ce que dans la formule de R. Courant pour $n(\lambda)$ on n'a réussi à supprimer le logarithme figurant à droite que pour les parallélépipèdes orthogonaux à N-dimensions et pour quelques domaines plats exotiques.

(2) Voir l'article [10], V.A. Il'in. A.Ph. Philippov.

(3) Dans ce cas, on peut prendre, comme domaine, une sphère à N-dimensions ou un cylindre à N-dimensions égal au produit d'un segment et d'un domaine arbitraire borné à $(N$-1)-dimensions.

ont des points d'accumulation couvrant soit tout le demi-axe $\lambda \geqslant 0$, soit n'importe quel sous-ensemble fermé de ce demi-axe.

Il en résulte encore la présence de valeurs propres de multiplicité infinie (par exemple, on peut constater que pour $N \geqslant 2$ il y a une extension autoadjointe non-négative de l'opérateur de Laplace dans la sphère à N-dimensions, pour laquelle tout nombre rationnel positif est une valeur propre de multiplicité infinie).

Enfin, il en résulte la constation suivante :

THEOREME 3. – (*Sur le terme principal de la formule asymptotique pour le nombre des valeurs propres*(1)).

Soit $N \geqslant 2$, $f(\lambda)$ *une fonction arbitraire, donnée d'avance, croissante sur le demi-axe* $\lambda \geqslant 0$. *Pour un domaine borné dans l'espace* E_N, *il y a une extension autoadjointe non-négative de l'opérateur de Laplace dans ce domaine, dont le spectre est discret, et dont le nombre* $n(\lambda)$ *des valeurs propres n'excèdant pas* λ, *vérifie la formule*

$$n(\lambda) = f(\lambda) + O(\lambda^{N/2}) . \tag{5}$$

Ainsi, pour $N \geqslant 2$, le terme $n(\lambda)$ peut être asymptotiquement égal à toute fonction croissante plus vite que $\lambda^{N/2}$.

2. **Etude de la convergence uniforme et de la localisation (au sens classique) des développements spectraux correspondant à l'extension arbitraire autoadjointe non-négative de l'opérateur de Laplace à spectre arbitraire.**

Soit G un domaine arbitraire dans l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace dans G, E_λ la décomposition spectrale correspondant à cette extension et $\Theta(x, y, \lambda)$ le noyau de E_λ, nommé fonction spectrale de l'extension $\mathfrak{L}$.

On examinera le développement spectral

$$E_\lambda f(x) = S_\lambda(x, f) = \int_G \Theta(x, y, \lambda) f(y)\, dy \tag{6}$$

pour toute fonction de la classe $L_2(G)$ et ses moyennes de Riesz d'ordre s

$$\overset{s}{\sigma}_\lambda(x) = \overset{s}{\sigma}_\lambda(x, f) = \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s dS_t(x, f) . \tag{7}$$

Le n° 2 étudiera les moyennes de Riesz (7) dont les ordres s vérifient l'inégalité

$$0 \leqslant s < \frac{N-1}{2} . \tag{8}$$

Pour $s = 0$ les moyennes de Riesz (7) coïncident avec le développement spectral (6), qui est donc englobé par notre étude.

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(1) Voir l'article [10], V.A. Il'in, A. Ph. Philippov.

Désignons par le symbole G_h l'ensemble de points du domaine G, dont la distance à la frontière G est supérieure au nombre $h > 0$ (1).

Examinons les conditions, qui ne garantissent ni la convergence uniforme ni la localisation des moyennes de Riesz d'ordre s (vérifiant l'inégalité (8)).

THEOREME 4. – *(Les conditions qui ne garantissent pas la localisation des moyennes de Riesz d'ordre $0 \leqslant s < \frac{N-1}{2}$ dans les classes de Zygmund-Hölder* (2)*).*

Soit $N \geqslant 2$, G un domaine arbitraire dans l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace dans G, x_0 un point arbitraire fixe à l'intérieur de G, α un nombre arbitraire fixe vérifiant l'inégalité

$$0 < \alpha < \frac{N-1}{2} - s . \tag{9}$$

Alors, il existe une fonction $f(x)$ vérifiant les conditions suivantes :

(1) *$f(x)$ s'annule dans un voisinage $\mathcal{O}$ du point x_0 et hors de l'ensemble G_h pour un certain $h > 0$,*

(2) $f(x) \in C^\alpha(G)$,

(3) *les moyennes de Riesz (7) du développement spectral de $f(x)$ n'ont pas de limite au point x_0 lorsque $\lambda \to \infty$.*

COROLLAIRE DU THEOREME 4. – *(Les conditions qui ne garantissent pas la localisation des moyennes de Riesz d'ordre $0 \leqslant s < \frac{N-1}{2}$ dans chaque classe W_p^α, H_p^α, L_p^α et $B_{p,\theta}^\alpha$* (3)*). Dans le théorème 4 au lieu de la classe $C^\alpha(G)$ on peut prendre chacune des classes $W_p^\alpha(G)$, $H_p^\alpha(G)$, $L_p^\alpha(G)$ ou $B_{p,\theta}^\alpha(G)$ du même ordre de différentiation α et de n'importe quel degré de sommation $p \geqslant 1$ et (dans le cas de la classe de Bessov) pour n'importe quel $\theta \geqslant 1$.*

Ainsi, l'appartenance de la fonction $f(x)$ à chacune des cinq classes données ci-dessus avec un ordre de différentiation α vérifiant l'inégalité (9), ne garantit pas la localisation des moyennes de Riesz d'ordre s (quel que soit le degré de sommation $p \geqslant 1$ et dans le cas de la calsse de Bessov quel que $\theta \geqslant 1$).

THEOREME 5. – *(Les conditions qui garantissent la convergence uniforme des moyennes de Riesz d'ordre $0 \leqslant s < \frac{N-1}{2}$ dans les classes de Nicolsky* (4)*). Soit $N \geqslant 2$, G un domaine arbitraire dans l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe, non-négative de l'opérateur de Laplace dans G, $f(x)$ une fonction*

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(1) Quand G coïncide avec tout E_N, l'ensemble G_h coïncide avec G pour tout $h > 0$. Dans ce cas il existe une extension unique autoadjointe non-négative de l'opérateur de Laplace, dont le développement correspondant est l'intégrale multiple de Fourier.

(2) Voir les articles [11] et [12], V.A. Il'in, et Ch. A. Alimov.

(3) Pour la définition de toutes ces classes, voir [7].

(4) Voir les articles [11] et [12].

arbitraire vérifiant les conditions suivantes :

(1) *$f(x)$ devient nulle en dehors de l'ensemble G_{h_0} pour un certain $h_0 > 0$,*

(2) *dans tout le domaine G la fonction $f(x)$ appartient à la classe H_2^α pour un certain α vérifiant l'inégalité*

$$\alpha \geqslant \frac{N-1}{2} - s\,, \tag{10}$$

(3) *dans un domaine $\mathcal{D}$, contenu dans G*(1), *la fonction $f(x)$ appartient à la classe H_p^α pour certains α et p vérifiant les conditions*

$$\alpha \geqslant \frac{N-1}{2} - s\,, \quad p \cdot \alpha > N\,, \quad p \geqslant 1\,. \tag{11}$$

Alors, pour chaque $h > 0$ les moyennes de Riesz (7) du développement spectral de $f(x)$ convergent vers $f(x)$ (lorsque $\lambda \to \infty$) uniformément sur l'ensemble $\mathcal{D}_h$.

COROLLAIRE DU THEOREME 5. — *(Les conditions qui garantissent la convergence uniforme des moyennes de Riesz d'ordre $0 \leqslant s < \frac{N-1}{2}$ dans les classes de Sobolev, de Liouville, de Bessov et de Zygmund-Hölder). Dans le théorème 5 au lieu de la classe H_p^α on peut prendre chacune des classes W_p^α, L_p^α, $B_{p,\theta}^\alpha$ ou C^α du même ordre de différentiation α et du même degré de sommation p et (dans le cas de la classe de Bessov) pour n'importe quel $\theta \geqslant 1$.*

Ainsi, dans chacune des cinq classes indiquées, nous avons établi l'ordre définitif de différentiation

$$\alpha \geqslant \frac{N-1}{2} - s \tag{10}$$

qui garantit la convergence uniforme des moyennes de Riesz d'ordre s.

La condition du degré de sommation

$$p \cdot \alpha > N \tag{12}$$

est aussi définitive, puisque dans chacune des trois classes $B_{p,\theta}^\alpha$, W_p^α, L_p^α pour $p > 1$, $\Theta > 1$ et dans la classe H_p^α même pour $p \geqslant 1$, la condition

$$p \cdot \alpha \leqslant N \tag{13}$$

permet la présence d'une fonction non-bornée, pour laquelle les moyennes de Riesz (7) ne convergent pas uniformément.

Il en résulte une condition de localisation des moyennes de Riesz (7) d'ordre s (vérifiant l'inégalité (8)), définitive pour chacune des classes W_2^α, H_2^α, L_2^α, $B_{2,\theta}^\alpha$ et C^α. La condition est exprimée par l'inégalité (11).

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(1) Naturellement, le domaine $\mathcal{D}$ peut coïncider avec G.

Ces conditions de localisation et de convergence uniforme sont nouvelles, même pour l'intégrale multiple de Fourier et pour la série multiple trigonométrique de Fourier dans l'espace E_N (avec des sommes partielles sphériques).

Les autres conditions, moins communes, de localisation et de convergence uniforme du développement en fonctions propres et leurs moyennes de Riesz sont données dans des travaux de E.C. Titchmarsh, de S. Bochner, de E. Stein, de B.M. Levitan et de S. Minakshisundaram (voir les notes (13) – (18)).

3. **Etude de la localisation (au sens classique) des moyennes de Riesz (7) d'ordre s, vérifiant l'inégalité**

$$s \geqslant \frac{N-1}{2} \tag{14}$$

On sait(1), que pour les moyennes de Riesz (7) d'ordre s, vérifiant l'inégalité (14), dans la classe $L_2(G)$ le principe classique de localisation est vérifié.

La méthode de mes travaux (voir [19] et [11]) permet de démontrer les résultats suivants :

THEOREME 6. — *(Sur l'estimation locale uniforme des moyennes de Riesz d'ordre $s \geqslant \frac{N-1}{2}$). Soit G un domaine arbitraire dans l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace dans G, $f(x)$ une fonction arbitraire, qui appartient à la classe L_2 dans tout G et qui devient nulle dans une certaine région $\mathcal{D}$, contenue dans G. Alors, pour chaque $h > 0$, pour les moyennes de Riesz (7) du développement spectral de $f(x)$ d'ordre s, vérifiant l'inégalité (14), uniformément dans l'ensemble $\mathcal{D}_h$ l'estimation*

$$\overset{s}{\sigma}_\lambda(x, f) = 0\left(\frac{1}{\sqrt{\lambda}^{\,s-\frac{N-1}{2}}}\right) \tag{15}$$

est vérifiée.

THEOREME 7. — *(L'estimation locale des moyennes de Riesz ne peut pas être améliorée). Soit G un domaine arbitraire dans l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace dans G, x_0 un point arbitraire à l'intérieur de G, $\alpha(\lambda)$ une fonction arbitraire donnée d'avance, décroissante sur le demi-axe $\lambda \geqslant 0$ qui converge vers zéro, lorsque $\lambda \to \infty$. Alors, il existe une fonction $f(x)$, vérifiant les conditions suivantes :*

(1) *$f(x)$ devient nulle dans un certain voisinage $\mathcal{D}$ du point x_0,*

(2) *$f(x) \in L_\infty(G)$,*

(3) *la quantité*

$$\overset{s}{\sigma}_\lambda(x_0, f) \cdot \sqrt{\lambda}^{\,s-\frac{N-1}{2}}\ [\alpha(\lambda)]^{-1} \tag{16}$$

n'a pas de limite lorsque $\lambda \to \infty$.

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(1) Voir l'article [14] pour le cas de la série multiple trigonométrique et l'article [15] pour le cas de fonctions propres de l'opérateur de Laplace.

Il en résulte, que l'estimation (15) des moyennes de Riesz d'ordre $s \geqslant (N-1)/2$ est définitive (même dans la classe des fonctions bornées dans G).

Le résultat obtenu est nouveau, même pour l'intégrale multiple de Fourier et la série multiple trigonométrique de Fourier dans l'espace E_N (avec des sommes partielles sphériques).

4. **Condition de localisation des moyennes de Riesz (7) de tout ordre arbitraire positif s au sens généralisé** (c'est-à-dire au sens de la convergence presque partout dans un voisinage du point donné).

THEOREME 8. — (*Sur l'estimation locale des moyennes de Riesz d'ordre* $s > 0$ (1) *presque partout*). *Soit G un domaine arbitraire dans l'espace E_N, $\mathfrak{L}$ une extension arbitraire autoadjointe non-négative de l'opérateur de Laplace dans G, $f(x)$ une fonction arbitraire, appartenant à la classe L_2 dans tout domaine G et qui devient nulle presque partout dans une certaine région $\mathfrak{D}$, contenue dans G. Alors, pour les moyennes de Riesz (7) du développement spectral de $f(x)$ de tout ordre positif s presque partout dans la région $\mathfrak{D}$ l'estimation*

$$\overset{s}{\sigma}_\lambda (x, f) = 0\left(\frac{1}{\sqrt{\lambda}^s}\right) \tag{17}$$

est vérifiée.

Le théorème 8 est nouveau même pour le cas l'intégrale multiple de Fourier et la série multiple trigonométrique de Fourier dans E_N (avec des sommes partielles sphériques).

5. **Série multiple trigonométrique de Fourier.**

A côté des sommes partielles sphériques de cette série, on examine souvent les sommes partielles orthogonales (dites de Prinsheim).

Comme comparaison, examinons les conditions de localisation classique des sommes partielles orthogonales, qui sont définitives dans les calsses de Nic̆olsky (ces conditions ont été récemment déterminées dans mon article [22]).

THEOREME 9. — *(Sur les conditions de localisation classique des sommes orthogonales de série multiple trigonométrique de Fourier dans les classes de Nicolsky). Soit $f(x) \in \hat{H}_p^\alpha$, où $\hat{H}_p^\alpha$ est la classe périodique de Nicolsky dans l'espace E_N*(2)*, les nombres α et p vérifiant les conditions $\alpha > 0$, $p \geqslant 1$. Alors, à condition que $\alpha \cdot p > N - 1$, le principe classique de localisation des sommes partielles orthogonales de la série multiple trigonométrique de Fourier de la fonction $f(x)$, dans E_N, est vérifié, mais si $\alpha \cdot p = N - 1$ ce principe ne s'applique pas en général.*

Ajoutons que dans les classes $\hat{H}_2^\alpha$ les conditions de localisation des sommes partielles sphériques de la série multiple trigonométrique dans E_N sont moins

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(1) Voir les articles [20] et [21], V.A. Il'in.

(2) Définition de cette classe, voir l'article [7].

strictes, que les conditions de localisation des sommes partielles orthogonales de cette série (pour $\alpha = (N - 1)/2$ le principe de localisation des sommes partielles sphériques est vérifié, mais celui-ci ne s'appliquera pas aux sommes partielles orthogonales).

6. Opérateurs elliptiques autoadjoints.

Nous avons formulé les résultats pour l'opérateur elliptique le plus simple : l'opérateur de Laplace.

Mais les résultats relatifs à l'estimation du nombre des valeurs propres et aux conditions de convergence uniforme et de localisation des moyennes de Riesz du développement spectral valent non seulement pour l'opérateur de Laplace, mais aussi pour l'opérateur de Beltrami dans un domaine de l'espace E_N et même pour l'opérateur de Beltrami – Laplace dans l'espace compact de Riemann avec ou sans frontière(1) (voir [4] et [23]) et aussi pour l'opérateur polyharmonique (voir [24]).

Tout récemment, un élève de l'auteur de ce rapport, E.J. Maujsseev, a étendu la plupart de ces résultats au cas de l'opérateur général autoadjoint elliptique du deuxième ordre.

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

[1] Ильин В. А. — Ядра дробного порядка, Матем. сборник, 41, (83) : 4, 1957, p. 459-480.

[2] Ильин В. А. — О сходимости разложений по собственным функциям оператора Лапласа, Успехи матем. наук, 13, № 1, 1958, p. 87-180.

[3] Ильин В. А. — Проблемы локализации и сходимости для рядов Фурье по фундаментальным системам функций оператора Лапласа, Успехи матем. наук, 23, № 2, 1968, p. 61-120.

[4] Ильин В. А. — О рядах Фурье по фундаментальным системам функций оператора Бельтрами, Дифф. уравнения, 5, № 11, 1969, p. 1940-1978.

[5] Ильин В.А., Алимов Ш.А. — Условия равномерной риссовской суммируемости рядов по Фурье по произвольной ФСФ оператора Лапласа, окончательные в классах Соболева, Никольского, Бесова, Лиувилля и Зигмунда-Гельдера, Доклады АН ССР, 193, № 2, 1970.

[6] Gårding L. — Application of the theory direct integrals of Hilbert spaces to some integral and differential operators, The Institute for Fluid Dynamics and Applied Mathematics, *Lecture series*, 11, 1954.

[7] Никольский С. М. — Приближение функций многих переменных и теоремы вложения, « Наука », М., 1969.

[8] Ильин В. А. — Оценка снизу суммы квадратов фундаментальных функций и числа фундаментальных значений для произвольной ФСФ оператора Лапласа, Дифф. уравнения, 6, № 1, 1970, p. 7-16.

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(1) Dans le cas de l'espace de Riemann sans frontière la convergence est uniforme dans tout l'espace.

[9] Курант Р., Гильберт Д. — Методы математической физики, том I, Гостехиздат, М, 1951.
[10] Ильин В. А., Филиппов А. Ф. — О характере спектра самосопряженного расширения оператора Лапласа в ограниченной области (Фундаментальные системы функций с произвольной наперед заданной подпоследователь ностью фундаментальных чисел), Доклады АН СССР, 191, № 2, 1970, p. 267-270.
[11] Алимов Ш. А., Ильин В. А. — О спектральных разложениях, отвечающих произвольному неотрицательному самосопряженному расширению оператора Лапласа, Доклады АН СССР, 193, № 1, 1970, p. 9-12.
[12] Il'in V.A., Alimov Ch. A. — Conditions exactes de convergence uniforme des développements spectraux et de leurs moyennes de Riesz pour une extension autoadjonctive arbitraire non-négative de l'opérateur de Laplace, *Comptes Rendus de l'Académie des Sciences*.
[13] Титчмарш Э. Ч. — Разложения по собственным функциям, связанные с дифференциальными уравнениями второго порядка, том 2, ИЛ, М, 1961.
[14] Bochner S. — Summation of multiple Fourier series by spherical means, *Trans. Amer. Math. Soc.*, 40, 1936, p. 175-207.
[15] Stein E.M. — Localisation and summability of multiple Fourier series, *Acta math.*, 100, No. 1-2, 1958, p. 93-147.
[16] Minakshisundaram S. — Summability of eigenfunction expansions, *Math. Z.*, 83, No. 4, 1964, p. 321-335.
[17] Левитан Б. М. — О разложении по собственным функциям оператора Лапласа, Матем. сборник, 35 : 2, 1954, p. 267-316.
[18] Левитан Б. М. — О суммировании кратных рядов и интегралов Фурье, Доклады АН СССР, 102, № 6, 1955, p. 1073-1077.
[19] Ильин В. А. — О суммировании рядов Фурье по собственным функциям средними Рисса, Чезаро и Пуассона-Абеля, Дифф. уравнения, 2, № 6, 1966, p. 816-827.
[20] Ильин В. А. — Обобщенный принцип локализации для риссовых средних ряда Фурье по произвольной ФСФ оператора Лапласа с дискретным спектром, Дифф. уравнения, 6, № 7, 1970.
[21] Ильин В. А. — Обобщенный принцип локализации для риссовских средних, отвечающих произвольному самосопряженному неотрицательному расширению оператора Лапласа. Дифф. уравнения, 6, № 7, 1970.
[22] Ильин В. А. — Условия локализации прямоугольных частичных сумм кратного тригонометрического ряда Фурье в классах С. М. Никольского, Матем. заметки, том 7, № 10, 1970.
[23] Ильин В. А. — О характере спектра самосопряженных неотрицательных расширений эллиптических операторов и о точных условиях сходимости и риссовской суммируемости рядов Фурье различных классов функций (Доклад на Общем собрании Отделения математики Академии Наук СССР), Матем. заметки, том 7, № 4, 1970, p. 515-523.
[24] Ильин В. А., Шишмарев И. А. — Ряды Фурье по фундаментальным системам функций полигармонического оператора, Доклады АН СССР, 189, № 4, 1969, p. 19-21.

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THE CHARACTERISTICS OF PSEUDO-DIFFERENTIAL OPERATORS AND DIFFERENCE SCHEMES

by V. MASLOV

I would like to begin by stating new results, but as my old results of 1961-1965 are now used and partially reproduced in the literature on pseudodifferential operators and are not understood well enough, I shall begin first by these old ones (§ 1).

I started from two physical problems which were similar but not unified in a single formal scheme. These are : the problem of quasiclassical asymptotics in quantum mechanics on one side, and the problem of propagation of discontinuities in the theory of hyperbolic equations on the other. The local solution of the first problem was known in the physical literature and the problem of the propagation of discontinuities and of the construction of regularisator was solved by Sobolev [10] in 1930, following Hadamard's work.

I succeeded in unifying these two problems by means of one single formalism. This formalism gives the possibility to extend results from partial differential equations to difference schemes and to solve these latter problems "in the large" i.e. globally.

1. The canonical operator

(1) Following R. Feynman ordering convention we shall use the following notations :

$$L\left(\overset{②}{x}, -i\overset{①}{\frac{\partial}{\partial x}}\right)u(x) \overset{\text{def}}{=} F^*_{p\to x}\, L(x,p)\, F_{x\to p}\, u(x)\,, x \in R^N$$

$$L\left(\overset{①}{x}, -i\overset{②}{\frac{\partial}{\partial x}}\right)u(x) \overset{\text{def}}{=} F^*_{p\to x}\, [F_{x\to p}\, L(x\,,p)\, u(x)]$$

where the indices ① and ② indicate which of the operators $x\,, -i\,\dfrac{\partial}{\partial x}$ in the differential expression

$$L\left(x\,, -i\,\frac{\partial}{\partial x}\right)$$

is the first to operate and which is the second. For instance, if $L(x\,,p) = xp$, then

$$L\left(\overset{\textcircled{2}}{x}, -i \overset{\textcircled{1}}{\frac{\partial}{\partial x}}\right) u(x) = -ix \frac{\partial u(x)}{\partial x} \quad ;$$

$$L\left(\overset{\textcircled{1}}{x}, -i \overset{\textcircled{2}}{\frac{\partial}{\partial x}}\right) u(x) = -i \frac{\partial x u(x)}{\partial x} \quad .$$

(2) Let $L\left(\overset{\textcircled{2}}{x}, -i \overset{\textcircled{1}}{\frac{\partial}{\partial x}}\right)$ be a pseudodifferential operator as defined in invariant form by L. Hörmander in 1965. We have

$$(1)\quad \lambda^{-S_0} L\left(\overset{\textcircled{2}}{x}, -i \overset{\textcircled{1}}{\frac{\partial}{\partial x}}\right) e^{i\lambda S(x)} \varphi(x) = e^{i\lambda S(x)} L^0\left(x, \frac{\partial S}{\partial x}\right) \varphi(x) + 0\left(\frac{1}{\lambda}\right)$$

where S_0 and λ are positive numbers and $L^0(x, p)$ is an homogenous function of p. We suppose that $\varphi(x)$ and $S(x)$ are differentiable, and that $\varphi(x)$ has a compact support such that grad $S(x) \neq 0$ on supp φ.

In my book "The Theory of Perturbations and Asymptotic Methods" published in 1965 I considered a wider class of pseudodifferential operators.

In particular, if $L\left(\overset{\textcircled{2}}{x}, -\lambda^{-1} \overset{\textcircled{1}}{\frac{\partial}{\partial x}}, \lambda^{-1}\right)$ is a pseudo-differential operator in this sense [7], acting in a space of functions of $x \in R^N$ and $\lambda \in R_+$, then for every $\varphi(x, \lambda)$, C^∞ with respect to x and such that all the x-derivatives are square-integrable with respect to λ, the following equality holds

$$L\left(\overset{\textcircled{2}}{x}, -\left(\lambda^{-1} \overset{\textcircled{1}}{\frac{\partial}{\partial x}}\right), \lambda^{-1}\right) e^{i\lambda S(x)} \varphi(x) = e^{i\lambda S(x)} L\left(x, \frac{\partial S}{\partial x}, 0\right) \varphi(x) + 0\left(\frac{1}{\lambda}\right)$$

where $0\left(\frac{1}{\lambda}\right)$ denotes a function such that $\lambda 0\left(\frac{1}{\lambda}\right) \in L_2[R^N \times R_+]$ and where $L(x, p, h)$ belongs to $C^\infty[R^{2N+1}]$.

In this expressian, $L\left(x, \frac{\partial S}{\partial x}, 0\right)$ is not in general, a homogenous function of $\frac{\partial S}{\partial x}$. The so defined "$\lambda$-speudodifferential operator" will be invariant with respect to λ-Fourier transform. This is defined by

$$F_{\lambda, x \to p}\, u(x) \overset{\text{def}}{=} \frac{\lambda^{N/2}}{(2\pi)^{N/2}} \int e^{ipx\lambda} u(x)\, dx \quad , \quad x \in R^N$$

(in particular $F_{1, x\to p} = F_{x \to p}$). Hence,

$$F_{\lambda,x\to p}\,\overset{②}{L}\left(x\,,\left\{-i\lambda^{-1}\,\overset{①}{\frac{\partial}{\partial x}}\right\},\lambda^{-1}\right)F^*_{\lambda,p\to x}=\overset{②}{L}\left(i\lambda^{-1}\,\frac{\partial}{\partial p},\overset{①}{p},\lambda^{-1}\right)$$

and

$$\overset{②}{L}\left(\left\{i\lambda^{-1}\,\frac{\partial}{\partial p}\right\},\overset{①}{p},\lambda^{-1}\right)e^{i\lambda\tilde{S}(p)}\varphi(p)=e^{i\lambda\tilde{S}(p)}L\left(-\frac{\partial\tilde{S}}{\partial p},p\,,0\right)\varphi(p)+0\left(\frac{1}{\lambda}\right)$$

wit $\varphi(p)\in C_0^\infty$ and $\tilde{S}(p)\in C^\infty$.

(3) Consider now an abstract Hilbert space H and a self-adjoint positive invertible operator A.

Let $\mathcal{D}(A)$ be the domain of definition of A in the space $L_2[R^N,H]$, equipped with the norm

$$\|f\|^2_{L_2[R^N,H]}=\int\|f(x)\|^2_H\,dx$$

Substitute now formally the operator A instead of the variable λ in the relations (2) and (4) ; then, one replaces $0\left(\frac{1}{\lambda}\right)$ by a function $\psi(x)\in\mathcal{D}(A)\cap\mathcal{D}\left(\frac{\partial}{\partial x}\right)$ and, respectively, $\psi(p)\in\mathcal{D}(A)\cap\mathcal{D}\left(\frac{\partial}{\partial p}\right)$ and considers that $\varphi(x)\,,\varphi(p)$ are elements of $C^\infty[R^N,H]$ and that $u(x)\in L_2[R^N,H]$. We obtain

$$(2^*)\quad \overset{②}{L}\left(x\,,-\left\{iA^{-1}\,\overset{①}{\frac{\partial}{\partial x}}\right\},A^{-1}\right)e^{iAS(x)}\varphi(x)=e^{iAS(x)}L\left(x\,,\frac{\partial S}{\partial x},0\right)\varphi(x)+\psi(x)$$

$$\psi(x)\in\mathcal{D}(A)\cap\mathcal{D}\left(\frac{\partial}{\partial x}\right)$$

$$(3^*)\qquad F_{A,x\to p}\,u(x)\overset{\text{def}}{=}\frac{A^{N/2}}{(2\pi)^{N/2}}\int e^{ipXA}\,u(x)\,dx\quad,\quad x\in R^N$$

$$F_{A,x\to p}\,\overset{②}{L}\left(x\,,-\left(iA^{-1}\,\overset{①}{\frac{\partial}{\partial x}}\right),A^{-1}\right)F^*_{A,p\to x}=\overset{②}{L}\left(\left\{iA^{-1}\,\frac{\partial}{\partial p}\right\},\overset{①}{p},A^{-1}\right)$$

$$\overset{②}{L}\left(\left\{iA^{-1}\,\frac{\partial}{\partial p}\right\},\overset{①}{p},A^{-1}\right)e^{iAS(p)}\varphi(p)=e^{iAS(p)}L\left(-\frac{\partial S}{\partial p},p\,,0\right)\varphi(p)+\psi(p)$$

$$\psi(p)\in\mathcal{D}(A)\cap\mathcal{D}\left(\frac{\partial}{\partial p}\right)$$

The formulas so obtained give the definitions of *A-Fourier transform* and A-pseudodifferential operator.

(4) In the Euclidean $2N$-dimensional phase space we now consider the manifold $\Lambda^N:\{x=x(\alpha)\,,p=p(\alpha)\}\,,\alpha\in R^N\}$ such that the Lagrange brackets for the functions $x_j\,,p_j\,,j=1\,,\ldots,N$ are equal to 0 :

$$\sum_{k=1}^{N} \left(\frac{\partial x_k}{\partial \alpha_i}\frac{\partial p_k}{\partial \alpha_j} - \frac{\partial x_k}{\partial \alpha_j}\frac{\partial p_k}{\partial \alpha_i}\right) = 0 \quad , \quad ij = 1 , \ldots , N$$

(in other words : the form pdx is closed on Λ^N). I called such a manifold a Lagrangean manifold.

A Lagrangean manifold Λ^N has the following important property : it can be covered by patches with the local coordinates

$$(5) \qquad p_{i_1}, \ldots , p_{i_k}, x_{i_{k+1}}, \ldots , (m \neq n) \Rightarrow (i_m \neq i_n)$$

where these local coordinates are a subset of the global coordinates which define the $2N$-phase space.

(5) Let λ^0 be a point of Λ^N and let σ be some smooth measure on Λ^N.

We assume that the form pdx is exact on Λ^N and the patch U_j has the local coordinates (5) ; hence the projection map of U_j to the n-dimensional plane

$$x_{i_1} = \cdots = x_{i_k} = p_{i_{k+1}} = \cdots = p_{i_N} = 0$$

is a diffeomorfism. Then we may define the following function corresponding to U_j :

$$S_j(\lambda) = S(\lambda) - \sum_{\ell=1}^{k} p_{i_\ell} x_{i_\ell} \quad , \quad \lambda = (x , p) \in \Lambda^N$$

where $S(\lambda) = \int_{\lambda^0}^{\lambda} pdx$ is the integral of the exact form pdx from λ^0 to λ along any path on Λ^N. In the above-mentioned book [7] the index $\gamma_{\ell m}$ in the intersection of the patches U_ℓ and U_m was defined as the difference of the signatures of the hessians of the functions S_ℓ and S_m : if U_ℓ and U_m have the local coordinates

$$(p_{i_1}, \ldots , p_{i_k}, x_{i_{k+1}}, \ldots , x_{i_N}) \overset{\text{def}}{=} (\tilde{\alpha}_1, \ldots , \tilde{\alpha}_N)$$

and

$$(p_{j_1}, \ldots , p_{j_s}, x_{j_{s+1}}, \ldots , x_{j_N}) \overset{\text{def}}{=} (\tilde{\tilde{\alpha}}_1, \ldots , \tilde{\tilde{\alpha}}_N)$$

respectively then

$$\gamma_{\ell m} = sgn \left\| \frac{\partial^2 S_\ell(\lambda(\tilde{\alpha}))}{\partial \tilde{\alpha}_i \, \partial \tilde{\alpha}_j} \right\|_{i,j=1}^{k} - sgn \left\| \frac{\partial^2 S_m(\lambda(\tilde{\tilde{\alpha}}))}{\partial \tilde{\tilde{\alpha}}_i \, \partial \tilde{\tilde{\alpha}}_j} \right\|_{i,j=1}^{s}$$

where $\lambda(\tilde{\alpha}) = \lambda(\tilde{\tilde{\alpha}}) \in U_\ell \cap U_m$.

Therefore, one can be naturally defined the index γ for an arbitrary chain of patches ; hence, some 1-dimensional characteristic cohomology class of Λ^N is defined.

(6) Assume that the characteristic class corresponding to the index γ is equal to 0. Then let γ_j be the index of any chain of patches

$$(U_0, U_{i_1}, U_{i_2}, \ldots , U_{i_s}, U_j) \quad ; \quad \lambda^0 \in U_0$$

joining U_0 to U_j.

Let $\varphi(\lambda)$, $\lambda \in \Lambda^N$ be a C^∞-function with values in the Hilbert space H such that

$$\operatorname{supp} \varphi \subset U_j$$

We define the canonical operator $K^{\lambda^0}_{\Lambda^N}$ in the patch U_j with the local coordinates (5) by :

(6) $$[K^{\lambda^0}_{\Lambda^N} \varphi](x) = i^{\gamma_j} F_{A,(p_{i_1},\ldots,p_{i_k}) \to (x_{i_1},\ldots,x_{i_k})} \times$$

$$\times \left[e^{iS_j(\lambda(p_{i_1},\ldots,x_{i_N}))A} \left| \frac{dp_{i_1}\ldots dx_{i_N}}{d\sigma} \right|^{-\frac{1}{2}} \times \varphi(\lambda(p_{i_1},\ldots,p_{i_k}, x_{i_{k+1}},\ldots,x_{i_N})) \right]$$

We shall consider the canonical operator as an operator mapping $C^\infty[\Lambda^N, H]$ into the algebraical quotient space $L_2[R^N, H]/D(A) \cap D\left(\frac{\partial}{\partial x}\right)$; (here, R^N has the global coordinates x). Using partition of unity

$$\{e_j(\lambda)\} \quad , \quad \operatorname{supp} e_j \subset U_j$$

the canonical operator is now defined for any C^∞-smooth function $\varphi(\lambda)$, $\lambda \in \Lambda^N$ with values belonging to H by :

(7) $$K^{\lambda^0}_{\Lambda^N} \varphi = \sum_j K^{\lambda^0}_{\Lambda^N} e_j \varphi$$

In the book (7) it was proved that *the canonical operator is independent of the choice of patches and of the choice of the partition of unity.*

Note that if the form pdx is not exact on Λ^N and the characteristic class corresponding to the index γ is not equal to 0, then we must impose additional conditions on the spectrum of A (the quantum conditions) to define the canonical operator by (6) and (7).

(7) We also proved the following formula :

$$L\left(\overset{②}{x}, -iA^{-1}\overset{①}{\frac{\partial}{\partial x}}\right) K^{\lambda^0}_{\Lambda^N} \varphi = K^{\lambda^0}_{\Lambda^N} L(x, p)\varphi + \psi$$

where

$$\psi \in D(A) \cap D\left(\frac{\partial}{\partial x}\right) \ ;$$

(for simplicity, we omit the third argument of L in the closing part of § 1).

Let the Lagrangian manifold Λ^N be embedded in M^{2N-1}, which is defined by the equation $L(x, p) = 0$, Let σ be an invariant measure for the dynamical system

$$\dot{x} = \frac{\partial L(x, p)}{\partial p} \quad , \quad \dot{p} = -\frac{\partial L(x, p)}{\partial x}$$

on Λ^N. This system will be called the A-bicharacteristic system. In this case $L(x(\lambda), p(\lambda)]) = 0$ and

$$\overset{②}{L}\left(x, -iA^{-1}\overset{①}{\frac{\partial}{\partial x}}\right) K^{\lambda^0}_{\Lambda^N}\varphi(\lambda) = A^{-1} K^{\lambda^0}_{\Lambda^N}\left[i\dot{\varphi} -- \sum_{i=1}^{N} L_{x_i p_i}(x, p)\varphi(\lambda)\right] + \psi(x)$$

where $\dot{\varphi} = \varphi_x \frac{\partial L}{\partial p} - \varphi_p \frac{\partial L}{\partial x}$ and $\psi(x) \in D(A^2) \cap D\left(\left[\frac{\partial}{\partial x}\right]^2\right)$. For simplicity we deal with the Euclidean phase space. But in fact, the phase space may be the contangent bundle of any smooth manifold.

(8) The canonical operator, with the above definition, allows to solve some difficult and old problems. For example, the principle of correspondence in the Quantum mechanics, and the discontinuity propagation problem in the large for the linear hyperbolic systems. The solutions of these latter problems were not known in the physical literature.

The generalisation of the fundamental concept of a characteristic made it possible to obtain the trajectories of a classical particle interacting with quantized particles. This is very important for Quantum chemistry.

The generalization of the concept of caracteristic for A-pseudo differential operator also allowed to define characteristics for difference schemes. It is known that the discontinuity (or more exactly an approximation of the discontinuity) of the solution of the difference scheme is followed by a tail, consisting of rapid oscillations. The characteristics of the difference scheme allows to find the support of this tail.

From the very construction of the canonical operator one can see that the order of the singularities of the projection map of the Lagrangean manifold into the X-space is important. It determines the number k of p-type coordinates to be used to define the local patch.

In fact, these singularities and some of their topological invariants provide, for example, the minimum number of those steps in the difference scheme for which one may neglect the round-of-errors.

They play also an essential role for eigenfunction expansions. In the two-dimensional case, for the absolute convergence of the Fourier double series of a function having a weak discontinuity on a curve, it turned out that the evolute of the curve and its singular points are essential [9].

Since 1965 the canonical operator has been used by a few Soviet authors, mathematicians as well as physicists [1 , 2 , 4].

A similar but a more elaborate construction of the canonical operator was given by the author to prove the same results modulo the infinitely differentiable func- [8].

In a special case my student [2] succeed to define the canonical operator modulo the holomorphic functions.

2. Discontinuities of solutions of pseudodifferential equations with complex symbol and of difference schemes with complex coefficients

For generalization to the complex case, the classical work by Leray [5] was very helpful (see also the work [3]).

(1) Note that if $S(x) = f(x) + iF(x)$ and if $F(x) \geqslant 0$ then the right part of formula (2) may cease to tend to 0 as $\lambda \to \infty$ only in a small neighbourhood of the surface $F(x) = 0$. This makes us to expect that we should not consider the complete complex analog of the whole theory but only a variant such that we could expand all the quantities in powers of the small imaginary part of their arguments. In fact, we will restrict ourselves "a priori" to the coefficients of these expansions.

In this paragraphe, we consider some applications.

(2) Consider the pseudodifferential equation

$$(8) \qquad i\frac{\partial \psi}{\partial t} - L\left(\overset{1}{i\frac{\partial}{\partial x}}, \overset{2}{x}, t\right)\psi = 0 \quad , \quad x \in R^N, t \in R^1$$

$$\psi \mid t = 0 = \psi_0(x)$$

where $L(p, x, t) \in C^\infty(R^{2N+1})$ for $p \neq 0$ and having the following properties :

(1) $L(p, x, t) = L_1(p, x, t) + L_2(p, x, t)$ where $L_1(p, x, t)$ and $L_2(p, x, t)$ as functions of the parameter p are positively homogeneous of the first order and of the null order respectively.

(2) $\operatorname{Im} L_1(p, x, t) \overset{\text{def}}{=} \widetilde{H}(p, x, t) \leqslant 0$

(3) The solution of the system

$$(9) \qquad \dot{x} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x}$$

with initial values

$$(10) \qquad \begin{aligned} x(0) &= x^0 \\ p(0) &= p^0 \\ |p^0| &\neq 0 \end{aligned}$$

where $H(p, x, t) = \operatorname{Re} L_1(p, x, t)$, exists in the large.

THEOREM 1. — *If the conditions* 1-3 *are satisfied and* $\psi_0(x) \in H_S(R^N)$, *there exists a unique solution* $\psi(x, t)$ *of the problem* (8) *which is continuous with respect to* t, *and such, that* $\psi(x, t) \in H_S(R^N)$ *for every fixed* t.

(3) DEFINITION. — Let $X(x^0, p^0, t)$, $P(x^0, p^0, t)$ be the solution of the problem (9), (10) satisfying the condition

$$\widetilde{H}(P(x^0, p^0, t), X(x^0, p^0, t), t) = 0$$

for every $t \in [0, T]$. We shall call the curve $x = X(x^0, p^0, t)$, $0 \leqslant t \leqslant T$ a bicharacterictic of the equation (8).

Let's denote the set of points, where the function u is not infinitely differentiable, by sing. supp. u.

Let's also denote by $\psi[\psi_0]$ the solution of the Cauchy problem (8).

Let $M(\Omega)$ be the following subset of R^{N+1} : $(\xi , \tau) \in M(\Omega)$ if and only if there exists a point (x^0 , p^0) satisfying the conditions $x^0 \in \Omega$, $|p^0| = 1$, $\xi = X(x^0, p^0, \tau)$ is a bicharacteristic of the equation (8)).

THEOREM 2(1). — *Assume that the conditions* 3 *are satisfied. Then*

(a) *sing supp* $\psi[\psi_0] \subset M(\Omega)$ *where* $\Omega =$ *sing. supp.* ψ_0 *and* (b) *for any point* $(x , t) \in M(\Omega)$ *there exists an initial datum* ψ_0 *such that sing. supp.* $\psi_0 \subset M$ *and* $(x , t) \in$ *sing. supp.* $\psi[\psi_0]$.

Thus $\overline{\bigcup_{\psi_0 \in \Phi} \text{sing. sup. } \psi[\psi_0]}$ *where* $\Phi = \{f \mid \text{sing. supp. } f \in \Omega\}$ *coincides with the union of all bicharacteristics starting from the set* Ω $(x^0 \in \Omega)$.

Theorems 1 and 2 partially generalize my theorem (cf. [8]) to the complex case.

(4) We see that the family of bicharacteristics starting from every point of sing. supp. ψ_0 defines the support of the solution discontinuities. However, it turns out that if the initial discontinuity has a representation by means of the canonical operator(2) then the support of the solution discontinuity can be defined with the help of the Huygens' principle. This means that in this case we can consider only the boundary of the bicharacteristics union. But I'm not going to give the exact wording of this theorem, but pass to the question of the metamorphosis of the discontinuity.

Suppose that the initial datum for a pseudo-differential equation has a finite discontinuity on the sub-manifold M^{N-1}. For hyperbolic equations, it was known that the solution remains finite for small t. In my above-mentioned book it is proved that the solution of the pseudo-differential equation of hyperbolic type will be unbounded precisely at the points of the bicharacteristics starting normally to M^{N-1}, where the Morse index is odd. (This, I call the metamorphosis of the discontinuity).

In the case of the previous equation (8), the behaviour of the discontinuity is determined not only by bicharacteristics $x = X(x^0, p^0, t)$, $0 \leqslant t \leqslant T$, but also by supplementary imaginary terms $iY(x^0, p^0, t) = iy(t)$ which can be obtained from the following generalised variation system :

$$(11) \qquad \begin{cases} \dot{y}_k = \sum_l (H_{p_k p_l} \eta_l + H_{p_k x_l} y_l) + \widetilde{H}_{pk} \\ \dot{\eta}_k = \sum_l (H_{x_k p_l} \eta_l + H_{x_k x_l} y_l) - \widetilde{H}_{xk} \\ y(0) = \eta(0) = 0 \quad , \quad k = 1 , \dots , N \end{cases}$$

----

(1) Note that it is easy to prove theorem 1 and the statement (a) of theorem 2, using only the construction of regularisator "in the small", without using the canonical operator (cf. the remarks at the end of [6]).

(2) i.e. the discontinuity belongs to the image of the canonical operator.

Here, $x = X(x^0, p^0, t)$, $p = P(x^0, p^0, t)$, where $x^0 \in M^{N-1}$ and $(p^0, \ell) = 0$ for any vector tangent l to M^{N-1} at the point x^0.

Denote

$$I(x^0, t) \overset{\text{def}}{=} \det \left\| \frac{\partial X_k(x^0(\beta), p^0(\beta), t) + i Y_k(x^0(\beta), p^0(\beta), t)}{\partial \beta} \right\|$$

where $\beta_1, \dots, \beta_{N-1}$ are local coordinates on M^{N-1} and $\beta_N = t$.

The solution of this problem will be unbounded only at the points of the bicharacteristics where any. of the following the three conditions is fullfilled :

(1) $\mathrm{Im} I(x^0, t) \neq 0$ or (2) $|I| = 0$ or (3) a certain index similar to the Morse index is odd.

Thus in the complex case the solution may be unbounded even for small t.

So we consider an entirely new situation from the point of view of the Hamiltonian formalism, a situation which is neither purely real nor purely complex, but a mixture : one could also say that it is real, but with a complex germ.

Note that here occurs now a new integral one dimensional cohomology class which will be defined below.

(5) Let's now consider a difference scheme, having $(k + 1) -$ steps with respect to t. Let's consider a one-parameter family (with parameter h) of nets with mesh h with respect to $x \in R^N$ and with mesh $\tau = \alpha h$ with respect to t $(0 < h < 1)$. Functions u^n_m on the net (where m is a multy index $(m_1, \dots, m_N)$) shall be extended smoothly to function $u^n(x)$ $(u^n(mh) = u^n_m)$.

We shall represent the operator

$$T_{x_1} u^n_m = u^n_{m_1+1, m_2, \dots, mN}$$

as a shift $T_{x_1} u^n(x) = \exp(h\partial/\partial x_1) u^n(x)$. Hence any "smooth" difference operator can be written down as

$$\sum_{j=0}^{c} a_j \Big(\overset{①}{ih \frac{\partial}{\partial x}}, x, \overset{②}{\tau(n + j)}, h \Big) u^{n+j} \overset{\text{def}}{=} Pu^n$$

where $a_j(p, x, t, h) \in C^\infty(R^{2N+2})$ for $p \neq 0$. Now let's consider the Cauchy difference problem

(12) $\qquad Pu^n = 0 \quad , \quad u^i = f^i(x) \quad i = 0, \dots, k-1$

Let's call the following function the main symbol :

$$\rho(\lambda, p, x, t) = \sum_{j=0}^{k} a_j(p, x, t, 0) \lambda^j$$

Suppose that

(1) The moduli of the roots $\lambda_i(p, x, t)$, $i = 1, \ldots, k$ of the equation $\rho(\lambda, p, x, t) = 0$ do not exceed unity (for $0 < \rho \leqslant \pi$).

(2) In the neighbourhood of the manifold $|\lambda_i| = 1$ the function

$$H^i(hp, x, t) \stackrel{\text{def}}{=} \frac{1}{\alpha} \arg \lambda_i (hp, x, t) \in C^\infty(R^{2N+2})$$

for $p \neq 0$, and equals 0 for $h = 0$; and for $p \neq 0$ the surfaces $H^i(hp, x, t) = 0$ do not intersect each other for all $0 \leqslant h < 1$.

(Note that this condition is similar to the condition of absence of multiple characteristics for hyperbolic equations).

(3) The solution of the system

$$(13) \qquad \begin{cases} \dot{x} = \dfrac{\partial H^i}{\partial p} & \dot{p} = -\dfrac{\partial H^i}{\partial x} \\ x(0) = x^0 \quad , & p(0) = p^0 \neq 0 \end{cases}$$

exists in the large.

DEFINITION. — Let $X^i(x^0, p^0, t), P^i(x^0, p^0, t)$ be the solution of the problem of the problem (13) satisfying the condition

$$|\lambda_i(P^i(x^0, p^0, t), X^i(x^0, p^0, t), t)| = 1$$

for all $0 \leqslant t \leqslant T$ is called a bicharacteristic of the difference operator P.

THEOREM 3. — *Under the condition* 1-3, *the difference scheme* (12) *is stable in the sense of the* $\mathfrak{L}_2$*-norm on the net, and the solution* $u_h(x, t)$, $t = n\tau$ *of the problem* $Pu_h = 0$, $u_h^i = f^i(x)$, $i = 0, \ldots, k-1$ *satisfies the estimate*

$$\|u_h(x, t)\|_{\mathfrak{L}_2} \leqslant M(t)\| f(x)\|_{\mathfrak{L}_2} \quad , \quad f(x) = \{f^0(x), f^1(x), \ldots, f^{k-1}(x)\},$$
$$x = mh \quad , \quad t = n\tau$$

where the function $M(t)$ *is independent of* h.

(6) Let's call a point (x, t) a regular point for the function $u_h(x, t)$ depending from the paramete $h > 0$ which is a smooth extension of a function on the net with mesh h for x and mesh αh for t, if there exists a neighbourhood of this point such that every difference derivative of $u_h(x, t)$ is uniformly bounded in this neighbourhood when $h \to 0$. The complement of the set of the regular points of $u_h(x, t)$ is denoted by sing. supp. u_h. Let $u[f]$ be the solution of (12)

THEOREM 4. — *If the conditions* 1-3 *are satisfied then* $\overline{\bigcup_{f \in \Phi} \text{sing. supp. } u[f]}$, *where* $\Phi = \{f \mid \text{sing. supp. } f \subset \Omega\}$, *is equal to the closure of the union of all* ($i = 1, \ldots, k$) *bicharacteristics* $x = X(x^0, p, t)$ *starting from the set* Ω *(that is* : $x^0 \in \Omega$*) and such that* $\pi \geqslant p_j^0 > 0$ *for every* j.

This theorem may be generalized to the case when the A-pseudo differential operator P is defined on a manifold M^N with the condition that it transforms a function on a net into a function on the same net (this means that, for any function f defined on a net contained in the manifold M, the value of the operator $P\tilde{f}$ for points of the net is independant of the choice of the smooth extension $\tilde{f}$ of f).

3. The Lagrangean manifold with a complex germ. and the canonical operator for the complex case

(1) We use a generalisation of the method of stationary phase in the case when the phase is complex to prove the theorem formulated below. In addition, we use the following relationship :

$$|\nabla F(x)|^n e^{-F(x)A} g(x) \in D(A^{\frac{n}{2}})$$

where

$$F(x) \in C^\infty \quad , \quad F(x) \geqslant 0 \quad , \quad g(x) \in C^\infty[R^N, H] \quad .$$

(2) Let Λ^N be a Lagrangean manifold in $2N$-dimensional Euclidean phase space with coordinates (p, q) and let $\sigma \in C^\infty$ be a measure on Λ^N. On this manifold we consider a smooth vector field

$$(y(\lambda) \; , \; \eta(\lambda)) \; , \; y \in R^N \; , \; \eta \in R^N \; , \; y \in \Lambda^N \; ,$$

vanishing on the subset Γ, such that the following relations are valid on Γ :

$$\sum_{j=1}^{N} \left(\frac{\partial y_i}{\partial \alpha_k} \frac{\partial \eta}{\partial \alpha_\ell} - \frac{\partial y_j}{\partial \alpha_\ell} \frac{\partial \eta_j}{\partial \alpha_k} \right) = 0 \quad ; \quad \alpha, k, \ell = 1, 2, \ldots, N$$

$$\sum_{j=1}^{N} \frac{\partial \eta_j}{\partial \alpha_k} \frac{\partial x_j}{\partial \alpha_\ell} - \frac{\partial \eta_j}{\partial \alpha_\ell} \frac{\partial x_j}{\partial \alpha_k} = \sum_{j=1}^{N} \frac{\partial y_i}{\partial \alpha_k} \frac{\partial \xi_j}{\partial \alpha_\ell} - \frac{\partial y_j}{\partial \alpha_\ell} \frac{\partial \xi_j}{\partial \alpha_k} \quad , \quad \alpha, k, \ell = 1, 2, \ldots, N$$

where $(\alpha_1, \ldots, \alpha_N)$ are the local coordinates on Λ^N and $q = x(\alpha), p = \xi(\alpha)$ are the local equations of Λ^N.

We shall call this field a complex germ. Let $F(\alpha)$ be a function on Λ^N, such that in the vicinity of Γ

$$dF = \eta\, dx - y\, d\xi + [O(y^2) + O(\eta^2)](|dx| + |d\xi|) \quad ,$$

F is not negative and vanishes only on Γ. Let $f(x)$ be a function on Λ^N in a neighbourhood of Γ such that

$$df = y\, d\eta + [O(y^2) + O(\eta^2)]|d\eta|$$

and $f(\lambda)|_\Gamma = 0$. We shall call the pair $(\Lambda^N, (y(\lambda), \eta(\lambda)))$ a Lagrangean manifold with complex germ.

(3) The manifold Λ^N may be covered by patches having coordinates

$$\beta_{k,\{i_\nu\}} = \{q_{i_1}, q_{i_2}, \ldots, q_{i_k}, p_{i_{k+1}}, \ldots, p_{i_N}\}$$

where $i_\nu \neq i_\mu$ when $\nu \neq \mu$.

It is convenient to define two types of action.

(1) We define, in the patch U_j with coordinates $\beta_{k,\{i_\nu\}}$, the S-action by the following formula

$$S^c_{U_j}(\lambda) \stackrel{\text{def}}{=} \int_{\lambda^0}^{\lambda} p\,dq + f(\lambda) - \sum_{\nu=1}^{N-k} [p_{i_{k+\nu}} q_{i_{k+\nu}} + \eta_{i_{k+\nu}}(\lambda)\, y_{i_{k+\nu}}(\lambda)]$$

where λ^0 is a fixed point of Λ^N (the function $S^c_{U_j}(\lambda)$ is multy-valued in general, depending on the path c of integration.

(2) We define, in the patch U_j with coordinates $\beta_{k,\{i_\nu\}}$, the μ-action by the expression (cf. [11])

$$\mu_{U_j}(\lambda) = -\frac{1}{2}\,(y(\lambda) \quad , \quad BC^{-1} y(\lambda))$$

where $(.\,,.)$ is the scalar production in R^N and the matrices are given by the following formlae :

$$B_{\ell_j} = \begin{cases} \dfrac{\partial\,(\xi_\ell(\alpha) + i\eta_\ell(\lambda\,(\alpha)))}{\partial \alpha_j} & \text{for} \quad \ell = i_1, \dots, i_k \quad , \\[2ex] \dfrac{\partial\,(x_\ell(\alpha) + i y_\ell(\lambda\,(\alpha)))}{\partial \alpha_j} & \text{for} \quad \ell = i_{k+1}, \dots, i_N \ , \end{cases}$$

$$C_{\ell_j} = \begin{cases} \dfrac{\partial\,(x_\ell(\alpha) + i y_\ell(\lambda\,(\alpha)))}{\partial \alpha_j} & \text{for} \quad \ell = i_1, \dots, i_k \quad , \\[2ex] -\dfrac{\partial\,(\xi_\ell(\alpha) + i\eta_\ell(\lambda\,(\alpha)))}{\partial \alpha_j} & \text{for} \quad \ell = i_{k+1}, \dots, i_N \ , \end{cases}$$

Here, $\alpha_1, \dots, \alpha_N$ are local coordinates on Λ^N and $(q\,,p) = \lambda\,(\alpha) = (x\,(\alpha)\,,\xi\,(\alpha))$ is a local equation of Λ^N. We shall define the Jacobian as follows

$$I_{k;i_1,\dots,i_N}(\lambda) \stackrel{\text{def}}{=} (-1)^{N-k}\,\frac{d\alpha_1 \dots d\alpha_N}{d\sigma}\,\det C$$

We shall define the phase argument $\operatorname{Arg}^c I_{k;i_1,\dots,i_N}(\lambda)$ as follows; this argument depending on the path c on Λ^N joining λ^0 with λ and can be made a well defined number in the range $(-\infty\,,\infty)$ (not only modulo 2π) with the help of the following procedure. In the intersection of two patches : U_l with the coordinates

$$q_{i_1}, \dots, q_{i_k}, p_{i_{k+1}}, \dots, p_{i_N}$$

and U_j with the coordinates $q_{i_1}, \dots, q_{i_{k-n}}, p_{i_{k-n+1}}, \dots, p_{i_N}$ we consider the corresponding Jacobians $I_{k;i_1,\dots,i_N}(\lambda)$ and $I_{k-n;i_1,\dots,i_N}(\lambda)$ and get

$$v_{i_{k-\nu}}(t) = (q_{i_{k-\nu}} + i y_{i_{k-\nu}}) \cos t + (p_{i_{k-\nu}} + i\eta_{i_{k-\nu}}) \sin t \quad , \quad \nu = 0, \dots, n-1$$

By substituting in $I_{k;i_1,\ldots,i_N}(\lambda)$

$$x_{i_{k-\nu}} + iy_{i_{k-\nu}} \to v_{i_{k-\nu}}(t)$$

we obtain a Jacobian $I(t)$, which is equal to $I_{k;i_1\ldots i_N}$ at $t = 0$ and it is equal to $I_{k-n;i_1,\ldots,i_N}$ at $t = \pi/2$.

Let us draw a path, in the complex plane t, from 0 to $\pi/2$ such that is leaves the real zeroes of $I(t)$ on its left (in the direction of increasing $\arg I(t)$. One can see that this procedure really provides a uniquelly defined $\mathrm{Arg}^c I_{k;i_1,\ldots,i_N}(\lambda)$ (cf. [8]).

Define the *reduced* $\arg I$ in such a way that $0 \leqslant \arg I \leqslant 2\pi$. Then, we obtain the definition of the index of a path c which joins points λ^0 and λ, belonging to the patche with $\nu = N$ by putting

$$\mathrm{Ind}\, c = \frac{1}{2\pi}\left[\mathrm{Arg}^c I(\lambda) - \arg I(\lambda)\right]$$

(4) Let $\varphi(\lambda)$ be an infinitely differentiable function with support contained in the patch U_j with coordinates $\beta_{k,\{i_\nu\}}$ and with values in some Hilbert space H. Let A be a see adjoint invetible operator in H. Let K_s be the Hilbert space of functions $f(q)$ with values in H with the norm

$$\|f\|^2_{K_s} = \int \|(\sqrt{-\Delta + A^2 q^2} + i)^s f\|_H \, dq$$

The canonical operator $K^{\lambda^0,\Gamma}_{\Lambda^N}$ maps infinitely differentiable functions with values in H into the algebraic quotient space K_s/K_{s+1} according to the following formula

$$K^{\lambda^0,\Gamma}_{\Lambda^N}\varphi(\lambda) = F^*_{A,p_{i_{k+1}}\to q_{i_{k+1}},\, p_{i_{k+2}}\to q_{i_{k+2}},\ldots,p_{i_N}\to q_{i_N}} \times$$

$$\times \frac{\cos\left[\frac{1}{2}\mathrm{Arg}^c I_{k,i_1,\ldots,i_N}(\lambda)\right] + i\sin\left[\frac{1}{2}\mathrm{Arg}^c I_{k,i_1,\ldots,i_N}(\lambda)\right] Sgn\, A}{\sqrt{|I_{k;i_1,i_2,\ldots,i_N}(\lambda)|}}$$

$$\times \exp\{iA\,[S^c_{U_j}(\lambda) + \mathrm{Re}\,\mu_{U_j}(\lambda)] - |A|\,[F(\lambda) + \mathrm{Im}\,\mu_{U_j}(\lambda)]\}\,\varphi(\lambda) \text{ modulo } K_{s+1},$$

where $\lambda(q,p) \in U_j \subset \Lambda^N$. Here, the A-Fourier-transform is defined by (3) by substituting $\lambda \to A$, where $A^{\frac{N}{2}}$ is defined by

$$A = \int \mu\, dE_\mu \quad , \quad A^{\frac{N}{2}} = \int \mu^{\frac{N}{2}}\, dE_\mu \quad , \quad \mu^{\frac{1}{2}} = \mu_+^{\frac{1}{2}} + i\mu_-^{\frac{1}{2}}$$

Finally, using partition of unity, we define the $K^{\lambda^0,\Gamma}_{\Lambda^N}.\ \varphi$ for all $\varphi \in C^\infty(\Lambda^N, H)$.

THEOREM 5. — *The canonical operator $K^{\lambda^0,\Gamma}_{\Lambda^N}$ does not depend on the choice of the covering if and only if any eigenvalue μ of the operator A satisfies the relations*

$$\mu \oint_\gamma p\,dx = \frac{1}{2} \operatorname{Ind}\gamma\ [\operatorname{Mod} 2\pi] + 0\left(\frac{1}{\mu}\right)$$

where γ is an arbitrary cycle on Γ, and Ind γ is the index of the path along this cycle.

(5) If we specialize the theorem to the cases coundered in § 2, all $\oint pdx$ and ind γ vanish and therefore there is no condition on the spectrum of A. The Lagrangean manifold associated to the problem (8) is defined as a solution $\{X(\alpha, t), P(\alpha, t)\}$ of the problem (9), (10), where $x^0 = x^0(\alpha), p^0 = p^0(\alpha), |p^0(\alpha)| = 1$ and the complex germ $\{y(\alpha, t), \eta(\alpha, t)\}$ is a solution of the system (11).

We construct a regularisator for the problem (3) by using the canonical operator for $H = L_2[R^1], A = i\frac{\partial}{\partial\tau}$ and the obtained Lagrangean manifold with its complex germ. The boundedness of the regularizator is proved with the use of a shift in time-reversed direction along the bicharacteristics ; the shift is realized with the help of Weyl's pseudodifferential operator

$$i\frac{\partial}{\partial t} + H\left(i\frac{\overset{②}{\partial}}{\partial x}, \frac{\overset{①}{x} + \overset{②}{x}}{2}, t\right)$$

which preserves norms.

We construct a regularizator for the problem (12) by taking as space H the space of functions $g(\tau, h)$ of two variables $\tau \in R^1$ and $h \in (0, 1]$ such that

$$\int_{R^1} e^{i\eta\tau} g(\tau, h)\,d\tau = 0 \quad \text{for} \quad |\eta| > \frac{\pi\sqrt{N}}{h}$$

with the norm

$$\|g\|_H^2 = \int_0^1 dh \int_{-\infty}^{\infty} |g(\tau, h)|^2\,d\tau$$

and taking as operator A the operator $i\frac{\partial}{\partial\tau}$.

REFERENCES

[1] Buslaev V. — Invariant description of the canonical Maslov's operator. *Dokl. Akad. Naúk S.S.S.R.*, tom 184, No. 1, 1969, p. 59-62.

[2] Dubnov V. — The Canonical Maslov's operator for hyperbolic equations. *Dokl. Akad. Nauk S.S.S.R.*, tom 183, 1968, No. 4.

[3] Gårding L., Kotake T., Leray J. — Problème de Cauchy (I bis et VI). *Bull. Soc. Math. France,* 92, 1964, p. 263-361.

[4] Kravtsov Ju. — Concerning two new asymptotical methods in the theory of wave propagation in inhomogeneous medias, *Acoust. Jorn.*, tom XIV, No. 1, 1968, p. 1-24.

[5] LERAY J. — Probleme de Cauchy, I, *Bull. Soc. math. France,* 85, 1967, p. 389-429.
[6] LUDWIG D. — Exact and asymptotic solutions of the Cauchy problem. *Communs. pure and applied Math.,* 13 N 3, 1960, p. 473-508.
[7] Maslov V. — Теория возмущений и асимптотические методы, Москва, 1965.
[8] MASLOV V. — Regularization of the Cauchy problem for pseudodifferential equations. *Dokl. Akad. Nauk S.S.S.R.,* tom 8, 1967, No. 6.
[9] MASLOV V. — On the properties of absolute convergence of multidimensional Fourier series from the point of view of the geometry of the contact discontinueties of the expanded functions. *Dokl. Akad. Nauk S.S.S.R.,* tom 191, 1970, No. 2.
[10] Sobolev S. — Волновое уравнение для неоднородной среды, Труды сейсм. ин-та, 6 (1930), № 57.
[11] NOVIKOV. — Algebraic construction and properties of hermitean analogs of K-theory on a ring with involution, from the point of view of the Hamiltonian formalism, Изв. А. Н. С. С. Р., Ser. Math., Т. 34, 1970, p. 253-288.

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ON LINEAR SECOND ORDER EQUATIONS WITH NON-NEGATIVE CHARACTERISTIC FORM

by O. A. OLEINIK

The equation

$$L(u) \equiv a^{kj}(x)\, u_{x_k x_j} + b^k(x)\, u_{x_k} + c(x)\, u = f(x) \tag{1}$$

is called a second order equation with non-negative characteristic form at the set G, if at every point x belonging to G, $a^{kj}(x)\, \xi_k\, \xi_j \geqslant 0$ for all $\xi = (\xi_1, \ldots, \xi_m)$. (Here and everywhere in the paper summation with respect to repeated indices is carried out from 1 to m).

Interest in equations, which do not preserve their type in a considered domain, has existed for a long time, principally after the results of *F*. Tricomi [1]. The works of M.V. Keldyš [2] and G. Fichera [3] were the beginning of a long series of papers about the second order equations with non-negative characteristic form. Numerous results have been obtained for a special class of equations (1), namely for second order equations, which are elliptic in a domain and degenerate on its boundary (see, for example, [4], [5]). General second order equations with non-negative characteristic form are investigated in the papers [3], [6] – [18] and others.

Let us consider equation (1) in a domain Ω with boundary Σ and suppose for simplicity that the coefficients of equation (1), the function f and the boundary Σ are sufficiently smooth. In the paper [3] the following boundary value problem for equation (1) was posed. Let $n = (n_1, \ldots, n_m)$ be the interior normal vector to the boundary Σ of Ω. We denote by Σ^0 the set of points Σ, where $a^{kj} n_k n_j = 0$. At the points of Σ^0 we consider the function

$$b = (b^k - a^{kj}_{x_j})\, n_k\ . \tag{2}$$

We denote by Σ_0, Σ_1, Σ_2 the sets of points of Σ^0 where $b = 0$, $b > 0$, $b < 0$ respectively. The set of points of Σ where $a^{kj} n_k n_j > 0$ is denoted by Σ_3.

The first boundary value problem is to find a function $u(x)$ such that

$$L(u) = f \quad \text{in} \quad \Omega\ , \tag{3}$$

$$u = g \quad \text{on} \quad \Sigma_2 \cup \Sigma_3 \tag{4}$$

where f and g are given functions.

The function $u(x)$ is called a weak solution of the first boundary value problem (3), (4) if for any $v \subset C^{(2)}(\Omega \cup \Sigma)$, which is equal to zero on $\Sigma_1 \cup \Sigma_3$ the integral identity

$$\int_\Omega uL^*(v)\, dx = \int_\Omega vf dx - \int_{\Sigma_3} g\, \frac{\partial v}{\partial \gamma}\, d\sigma + \int_{\Sigma_2} bgv d\sigma \tag{5}$$

holds where $\dfrac{\partial}{\partial \gamma} \equiv a^{kj} n_k \dfrac{\partial}{\partial x_j}$, $d\sigma$ is the area element on the surface Σ,

$$L^*(v) \equiv a^{kj} v_{x_k x_j} + b^{*k} v_{x_k} + c^* v, \qquad b^{*k} = 2a^{kj}_{x_j} - b^k, \qquad c^* = a^{kj}_{x_k x_j} - b^k_{x_k} + c .$$

Suppose that in the neighbourhood of a point x which belongs to the boundary Σ of the domain Ω, the set Σ is given by the equation

$$\mathfrak{F}(x_1, \ldots, x_m) = 0, \quad \text{grad}\, \mathfrak{F} \neq 0, \quad \mathfrak{F} > 0 \quad \text{in} \quad \Omega$$

and consider on Σ the function $\beta \equiv L(\mathfrak{F})$. Let Γ^0 be the boundary of $\Sigma_2 \cup \Sigma_0$ on Σ.

THEOREM 1. – *Suppose that $c(x) \leqslant -c_0 = const < 0$ in Ω, $\beta \leqslant 0$ at the internal points of*

$$\Sigma_2 \cup \Sigma_0 \; ; \; f \subset \mathcal{L}_\infty(\Omega) \quad \text{and} \quad g \subset \mathcal{L}_\infty(\Sigma_2 \cup \Sigma_3).$$

Then there exists a weak solution $u(x)$ of problem (3), (4) *which belongs to $\mathcal{L}_\infty(\Omega)$ and satisfies the inequality (maximum principle) :*

$$|u| \leqslant \max \left(\sup \frac{|f|}{c_0}, \sup |g|\right). \tag{6}$$

If in addition $c^*(x) < -c_1 =$ const < 0 in Ω, $f(x) \subset \mathcal{L}_p(\Omega)$, $1 \leqslant p < \infty$, $g = 0$, then there exists a weak solution $u(x)$ of the problem (3), (4) *which belongs to $\mathcal{L}_p(\Omega)$.*

A weak solution $u(x)$ of problem (3), (4) can be obtained by the regularisation method which means that $u(x)$ can be obtained as a limit as $\epsilon \to 0$ and $N \to \infty$ of a sequence $u^N_\epsilon(x)$ of solutions of the elliptic equations

$$\epsilon \Delta u + L(u) = f_N, \quad \epsilon = \text{const} > 0, \tag{7}$$

with the boundary condition $u = g_N$ on Σ, where f_N and g_N are sequences of smooth functions which converge to f and g as $N \to \infty$ in Ω and on $\Sigma_2 \cup \Sigma_3$ respectively. One can find numerous applications of the regularisation method in the books [19], [20].

THEOREM 2. – *Suppose that $c^* \leqslant -c_1 = const < 0$ in Ω, $\beta^* \equiv L^*(\mathfrak{F}) \leqslant 0$ at the points of $\Sigma_1 \cup \Sigma_0$ and also in a neighbourhood of $\overline{\Sigma}_3 \cap (\Sigma_1 \cup \Sigma_0)$ on Σ. Assume that the coefficients a^{kj} of equation* (3) *can be extended to a neighbourhood of Σ_2 with the same smoothness as in Ω and with $a^{kj}\xi_k \xi_j \geqslant 0$ for $\xi \in R^m$. Let Γ be the boundary of Σ_2 on Σ and $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is a finite number of smooth $(m-1)$-dimensional manifolds and the area of a δ-neighbourhood of Γ_2 on Σ has order δ^q, $q \geqslant 2$. Then a weak solution of problem* (3), (4) *is unique in the class of functions which belong to $\mathcal{L}_2$ in Ω and also to $\mathcal{L}_3$ in a neighbourhood of Γ_1.*

A weak solution $u(x)$ of problem (3), (4) *is unique in the class $\mathcal{L}_\infty(\Omega)$, if the area of the δ-neighbourhood of Γ on Σ as order δ^q, $q > 0$.*

Examples show that under the conditions of Theorem 2 the class of functions for the uniqueness of a weak solution of problem (3), (4) can not be enlarged. A weak solution of problem (3), (4) can be non-unique in the class $\mathcal{L}_p(\Omega)$ for $p < 3$, if Γ_1 is not empty, and in the class $\mathcal{L}_p(\Omega)$ for $p < 2$, if Γ_2 is not empty. One can construct such examples by considering of the heat equation $u_t = \Delta u$ in a domain such that its boundary coincides in a neighbourhood of the origin with the surface $t = |x|^{2+\epsilon}$, $\epsilon = \text{const} > 0$ and by taking g equal to a fundamental solution of this equation with a singularity at the origin.

In the paper [14] an example of problem (3), (4) is constructed which shows that a weak solution may be non-unique in the class of bounded mesurable functions if the area of the δ-neighbourhood of Γ on Σ does not tend to zero as $\delta \to 0$.

Using the elliptic regularisation one can prove a uniqueness theorem similar to that of *R*. Phillips and L. Sarason [14], who used methods of the theory of symmetrical systems.

The question arises, under which conditions a weak solution of problem (3), (4) is a smooth function in the closed domain $\Omega \cup \Sigma$. The same question can also be asked with regard to smoothness in a neighbourhood of any given point of Ω.

The last question is also connected with the problem of finding conditions under which equation (1) is hypoelliptic.

For equations of the form

$$L(u) = -\sum_{j=1}^{N} X_j^2 u + iX_0 u + cu = f, \tag{8}$$

where $X_j(x, \mathcal{D}) \equiv a_j^e(x)\, \mathcal{D}_e$, $j = 0,1, \dots, N$; $\mathcal{D}_e = -i \dfrac{\partial}{\partial x_e}$; and coefficients $a_j^e(x)$, $c(x)$ are real functions in $C^\infty(\Omega)$, sufficient conditions for the hypoellipticity were given by L. Hörmander [15], using the Lie algebra theory. It is easy to show that there exist equations (1) with real coefficients in C^∞ which can not be written in the form (8). Using the theory of pseudodifferential operators one can prove the following results for equations (8) and also for general equations (1).

Let us introduce some notations. Consider the system of operators

$$(X_0, X_1, \dots, X_N)$$

defined by equation (8). For any multi-index $\mathfrak{I} = (\alpha_1, \dots, \alpha_k)$ where α_e are integers in the range 0 to N, we set

$$|\mathfrak{I}| = \sum_{e=1}^{k} \lambda_e$$

with $\lambda_e = 1$, if $\alpha_e = 1, \dots, N$, and $\lambda_e = 2$, if $\alpha_e = 0$. We define the operator

$$X \;= ad\, X_{\alpha_1} \dots ad\, X_{\alpha_{k-1}} X_{\alpha_k} .$$

Here, as usually, ad $AB = AB - BA$ for any operators A, B ; $\mathcal{H}_s$ is a space of distributions in $S'(R^m)$, for which

$$\|u\|^2 \equiv \int_{R^m} (1 + |\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi < \infty, \quad s \subset R^1 ,$$

$\hat{u}(\xi)$ is the Fourier transform of $u(x)$. Let us denote by M a set of points which is contained in $\mathcal{M}$, where $\mathcal{M}$ is a finite set of closed $(m-1)$-dimensional, smooth manifolds and $\mathcal{M} \subset \Omega$.

THEOREM 3. – *Suppose that for every point $x_0 \subset \Omega \setminus M$ there exists an integer $R(x_0)$ such that*

$$\sum_{|\mathfrak{J}| \leqslant R(x_0)} |X_{\mathfrak{J}}(x_0, \xi)| > 0 \quad \text{for} \quad |\xi| \neq 0 \tag{9}$$

where $X_{\mathfrak{J}}(x, \xi)$ is a symbol of the operator $X_{\mathfrak{J}}$. Suppose that at every point $x_0 \subset M$

$$\sum_{j=1}^{N} |a_j^l \, \Phi_{x_l}| + |L(\Phi)| \neq 0 \tag{10}$$

where $\Phi(x_1, \ldots, x_m) = 0$ is an equation for $\mathcal{M}$ in a neighbourhood of the point x_0 with grad $\Phi \neq 0$. Then equation (8) *is hypoelliptic, that is $u \subset C^\infty(\Omega)$, if $L(u) \subset C^\infty(\Omega)$ and $u \subset \mathcal{D}'(\Omega)$. In addition, if u is a distribution in $\mathcal{D}'(\Omega)$ and $\varphi L(u) \subset \mathcal{H}_s$ for any function $\varphi \subset C_0^\infty(\Omega)$, then $\varphi u \subset \mathcal{H}_s$ and the following inequality is valid :*

$$\|\varphi u\|_s^2 \leqslant C\{\|\varphi_1 L(u)\|_s^2 + \|\varphi_1 u\|_\gamma^2\} \tag{11}$$

where φ, $\varphi_1 \subset C_0^\infty(\Omega)$, $\varphi_1 \equiv 1$ on the support of φ and either supp $\varphi \cap M = \emptyset$ or $\varphi \equiv 1$ on M ; $\gamma =$ const $< s$, *C is a constant dependant on γ and φ, φ_1.*

Let us write equation (1) in the form

$$L(u) \equiv - \mathcal{D}_j(a^{kj}(x) \, \mathcal{D}_k u) + iQu + cu = f \tag{12}$$

where $Qu = (b^k - a^{kj}_{x_j}) \mathcal{D}_k u$. Suppose that

$$a^{kj}, \, b^k, \, c \subset C^\infty(\Omega), \; L^0(x, \xi) \equiv a^{kj}(x) \, \xi_k \, \xi_j, \; L^{0(j)}, \; L^0_{(j)}$$

for $j = 1, \ldots, m$ are differential operators with the symbols $\partial L^0/\partial \xi_j$, $\partial L^0/\partial x_j$ respectively ; $\mathcal{E}_{-1}$ is a pseudodifferential operator with the symbol $\varphi(x)\,(1 + |\xi|^2)^{-1/2}$ where $\varphi \subset C_0^\infty(\Omega)$, $\varphi(x) \geqslant 0$, $\varphi(x) \equiv 1$ on a compact set $K \subset \Omega$.

Consider the system of operators $\{Q_0, Q_1, \ldots, Q_{2m}\}$ where $Q_0 = Q$, $Q_j = L^{0(j)}$ for $j = 1, \ldots, m$ and $Q_j = \mathcal{E}_{-1} L^0_{(j-m)}$ for $j = m+1, \ldots, 2m$. For any multi-index $\mathfrak{J} = (\alpha_1, \ldots, \alpha_k)$ where α_e are integers in the range 0 to $2m$, we set

$$|\mathfrak{J}| = \sum_{e=1}^{k} \lambda_e$$

with $\lambda_e = 1$, if $\alpha_e = 1, \ldots, 2m$, and $\lambda_e = 2$, if $\alpha_e = 0$. For every $\mathfrak{J}$ we define the operator

$$Q_{\mathfrak{J}} = ad\, Q_{\alpha_1} \ldots ad\, Q_{\alpha_{k-1}} \, Q_{\alpha_k} \, .$$

According to the theory of pseudodifferential operators, the operator $Q_{\mathfrak{J}}$ has the form

$$Q_{s} = Q^0_{s} + T \quad ,$$

where the operator T_s has order less or equal to zero and Q^0_s is a pseudodifferential operator of the first order with a symbol $q^0_s(x, \xi)$.

THEOREM 4. – *Suppose that for any compact set $K \subset \Omega \setminus M$ there exists a number $R(K)$ and a positive constant $C(K)$ such that the inequality*

$$1 + \sum_{|s| \leqslant R(K)} |q^0_s(x, \xi)|^2 \geqslant C(K)\,(1 + |\xi|^2)$$

holds for all $x \subset K$ and $\xi \subset R^m$. Suppose that at every point $x_0 \subset M$

(13) $$a^{kj}(x)\,\Phi_{x_k}\Phi_{x_j} + |L(\Phi)| \neq 0 \;.$$

Then equation (1) *is hypoelliptic in Ω, that is $u \subset C^\infty(\Omega)$, if $L(u) \subset C^\infty(\Omega)$ and $u \subset \mathcal{D}'(\Omega)$. In addition, if u is a distribution in $\mathcal{D}'(\Omega)$ and $\varphi L(u) \subset \mathcal{H}_s$ for any function $\varphi \subset C_0^\infty(\Omega)$, then $\varphi u \subset \mathcal{H}_s$ and the inequality* (11) *holds.*

The proofs of theorems 1-4 are given in [21], (see also [11], [16], [17]).

We note that estimate (11) is also valid when the coefficients of equations (8) and (12) are sufficiently smooth.

For equations (1) with analytic coefficients and with

$$\sum_{k=1}^{m} (|a^{kk}| + |b^k|) \neq 0 \quad \text{in } \Omega,$$

the necessary and sufficient condition for the hypoellipticity is given in [25]. For equations of the form (8) such a theorem is proved by M. Derridj.

Results about the smoothness of weak solutions of problem (3), (4) in the closed domain $\Omega \cup \Sigma$ are obtained in [10] – [13], [21]. We can not formulate all these results here, but we note the following case. Let $C_\mu(\Omega)$ be the class of functions with bounded derivatives in Ω up to the order μ. If we suppose that the coefficients of equation (3), can be extended outside of Ω with the same smoothness and with the condition $a^{kj}\xi_k \xi_j \geqslant 0$, then for the existence of a solution $u(x)$ of problem (3), (4) in the class $C_\mu(\Omega)$, it is sufficient, for example, to require that the coefficients of (3) and f belong to $C_\mu(\Omega)$, the boundary Σ and the function g are sufficiently smooth, the intersection of any two of the sets Σ_3, Σ_2, $\Sigma_0 \cup \Sigma_1$ is empty, and an inequality between $c(x)$ and $b^k(x)$, $a^{kj}(x)$ and their derivatives of the first and second orders is fulfiled at the points where $\det \|a^{kj}\| = 0$. (This inequality is satisfied, for example, if $c < -c_0 = \text{const} < 0$ and c_0 is sufficiently large, (see [10], [12]).

Examples show that all these conditions are essential. The solution $u(x)$ in this case can be obtained as a limit as $\epsilon \to 0$ of the solutions $u_\epsilon(x)$ of a boundary value problem for elliptic equations of the form (7) in a domain which contains Ω.

The case of an intersection of $\overline{\Sigma}_3$ and $\overline{\Sigma}_2$ is considered in [13]. The smoothness of solutions of problem (3), (4) without the assumption about the extention of the coefficients is investigated in [13] and [21].

Second order equations with non-negative characteristic form appear in boundary layer theory, the theory of filtration, in problems of Brownian motion, the theory of probability and in other cases. Problem (3), (4) was also studied by the methods of the theory of probability using K. Ito's stochastic equations (see, for example, [22]).

For second order equations with non-negative characteristic form there are many open problems. We mention some of them.

The structure of spectrum of problem (3), (4) has not been studied, also conditions for a finite index and for the Fredholm properties has not been found.

It is of interest to investigate in more detail conditions for smoothness and for non-smoothness of weak solutions of problem (3), (4) and to find out what kind of singularities exists for weak solutions of problem (3), (4) and under which conditions they arise. (Let us notice that the smoothness of weak solutions of problem (3), (4) has not been completely investigated even for the heat equation. This question is discussed in detail in [13].

An open problem is to find the classes of equations (1) with analytic coefficients and with analytic functions f which have only analytic solutions in Ω.

It is also of interest to describe well posed boundary value problems for equations (1).

It is very important to study quasilinear second order equations with non-negative characteristic form. Such equations arise in boundary layer theory, in gas dynamic problems and in other cases of physical importance (see [23], [24]).

REFERENCES

[1] Tricomi F. — Sulle equazioni lineari alle derivate parziali di 2 ordinate di tipo misto, *Memorie della R. Accademia Nazionale dei Lincei,* serie V, 14, 1923, Fasc. 7.

[2] Keldyš M.V. — On certain cases of degeneration of elliptic type equations on the boundary of a domain, *Dokl. Akad. Nauk S.S.S.R.,* 77, 1951, p. 181-183.

[3] Fichera G. — Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine, *Atti Accad. Naz. Lincei. Mem. Cl. sci. fis., mat. Nat. ser.,* 1 (8), 1956, p. 5.

[4] Smirnov M.M. — *Degenerate elliptic and hyperbolic equations,* Moscow, Nauka, 1966.

[5] Vishik M.I., Grushin V.V. — On a class of degenerate elliptic equations of higher order, *Matem. Sbornik,* 79, 1969, No. 1, p. 3-36.

[6] Fichera G. — *On a unified theory of boundary value problem for elliptic-parabolic equations of second order,* Madison, Wisc., Univ. of Wisconsin Press, 1960, p. 97-120.

[7] Alexandrov A.D. — Investigation on maximum principle. I, *Izv. vis. uchebn. zaved., Matematika,* No. 5, 1958, p. 126-157.

[8] ALEXANDROV A.D. — Investigation on maximum principle, V. *Izv. vis. uchebn. zaved., Matematika,* No. 5, 1960, p. 16-26.
[9] OLEINIK O.A. — On a problem of G. Fichera, *Dokl. Akad. Nauk S.S.S.R.,* 157, 1964, p. 1297-1300, *Soviet Mathem. Dokl.,* 5, 1964, p. 1129-1133.
[10] OLEINIK O.A. — On the smoothness of the solutions of degenerate elliptic and parabolic equations, *Dokl. Akad. Nauk S.S.S.R.,* 163, 1965, p. 577-580, *Soviet Mathem. Dokl.,* 6, 1965, p. 972-976.
[11] OLEINIK O.A. — Linear equations of second order with non-negative characteristic form, *Matem. Sbornik,* 69, 1966, No. 1, p. 111-140.
[12] KOHN J.J., NIRENBERG L. — Non coercive boundary value problems, *Communs. Pure and Appl. Mathem.,* 18, 1965, p. 443-492.
[13] KOHN J.J., NIRENBERG L. — Degenerate elliptic-parabolic equations of second order, *Communs. Pure and Appl. Mathem.,* 20, 1967, No. 4, p. 797-872.
[14] PHILLIPS R.S., SARASON L. — Elliptic-parabolic equations of the second order, *J. of Mathem. and Mech.,* 17, 1968, No. 9, p. 891-918.
[15] HORMANDER L. — Hypoelliptic second order differential equations, *Acta Mathem.,* 119, 1967, p. 147-171.
[16] RADKEVICH E.V. — On a theorem of L. Hörmander, *Uspekhi Mathem. Nauk,* 24, 1969, No. 2, p. 233-234.
[17] RADKEVICH E.V. — A priori estimates and hypoelliptic operators with multiple characteristics, *Dokl. Akad. Nauk S.S.S.R.,* 187, 1969, No. 2, p. 271-274.
[18] BONY J.-M. — Principle du maximum, inegalité de Harnack et unicité du problem de Cauchy pour les opérateurs elliptiques dégénérés, *Ann. Inst. Fourier,* 19, 1969, No. 1, p. 277-304.
[19] LIONS J.L. — *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Paris, Dunod, Gauthier-Villars, 1969.
[20] LATTES R., LIONS J.-L. — *Méthode de quasi-reversibilité et applications,* Paris, Dunod, 1967.
[21] OLEINIK O.A., RADKEVICH E.V. — *Second order equations with non-negative characteristic form,* Moscow, VINITI Acad. Nauk S.S.S.R., 1971.
[22] FREIDLIN M.I. — Markov processes and differential equations, In : « *Theory of Probability, Mathematical Statistics, Theoretical Cybernetics* », Moscow, VINITI Acad. Nauk S.S.S.R., 1967, p. 7-58.
[23] OLEINIK O.A. — Mathematical problems of boundary layer theory, *Uspekhi Mathem. Nauk,* 23, 1968, No. 3, p. 3-65.
[24] VOLPERT A.I., HUDIAEV S.I. — On the Cauchy problem for quasilinear degenerate parabolic second order equations, Matem. Sbornik, 78, 1969, No. 13, p. 374-396.
[25] OLEINIK O.A., RADKEVICH E.V. — On the local smoothness of weak solutions and the hypoellipticity of differential second order equations, *Uspekhi Matem. Nauk,* 26, 1971, № 2, p. 265-281.

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SCATTERING THEORY FOR HYPERBOLIC SYSTEMS

by Ralph PHILLIPS

I should like to review some of the recent developments in scattering theory for hyperbolic systems. In line with my own interests I shall limit my remarks to two topics : (1) the dependence of the "exterior" eigenvalues on the geometry of the obstacle for the acoustic equation in an exterior domain and (2) scattering theory for first order symmetric hyperbolic systems of partial differential equations.

Since a good part of the motiviation for this work comes from the approach to scattering theory developed by Lax and Phillips [2], I shall begin with a brief resumé of this approach. For definiteness we consider the acoustic equation :

$$u_{tt} = \Delta u \tag{1}$$

over an exterior domain G with initial data : $u(x, 0) = f_1(x)$ and $u_t(x, 0) = f_2(x)$ and Dirichlet boundary conditions on f_1. Let H denote the Hilbert space of all initial data $f = \{f_1, f_2\}$ with finite energy, normed by the energy norm :

$$\|f\|^2 = \frac{1}{2} \int_G [\,|\partial f_1|^2 + |f_2|^2\,]\, dx . \tag{2}$$

Let ρ be chosen large enough so that the obstacle lies inside of the ball $\{x : |x| < \rho\}$. We call a solution *outgoing* if it is zero for $|x| \leqslant t + \rho$, $t \geqslant 0$, and incoming if it is zero for $|x| \leqslant \rho - t$, $t \leqslant 0$. The set of initial data for all outgoing [incoming] solutions we denote by D_+ $[D_-]$. Let $U(t)$ denote the operator taking initial data into the solution data at time t ; it is clear that the $\{U(t)\}$ form a one-parameter group of unitary (energy conserving) operators on H.

Next we define the operators

$$Z(t) = P_+ U(t) P_- , \qquad t \geqslant 0 , \tag{3}$$

where $P_+ [P_-]$ is the orthogonal projection onto the orthogonal complement of $D_+ [D_-]$. The effect of the projection P_- is to remove signals which might be coming in from far away and the effect of P_+ is to remove that part of the signal which has already been converted into an outgoing wave and no longer interacts with the obstacle. Since data in D_+ and D_- are zero inside the ball $\{|x| < \rho\}$, we see that, for data f with support in this ball,

$$[Z(t) f](x) = [U(t) f](x) \tag{4}$$

for all x with $|x| < \rho$; thus for such data the two sides of (4) are equal near the obstacle.

In an odd number of space dimensions the subspaces D_+ and D_- are orthogonal ; this and the fact that for $t > 0$, $U(t)$ maps D_+ into itself implies that the operators $Z(t)$ annihilate D_+ and D_- and form a *semi-group* of operators on the subspace $K = H \ominus (D_+ \oplus D_-)$. The infinitesimal generator B of this semigroup has a pure point spectrum $\{\lambda_k\}$ with corresponding eigenfunctions $\{w_k\}$. Since the operators $Z(t)$ are contractions and tend to zero strongly, Re $\lambda_k < 0$. In particular if $Z(t)$ is compact for some $t > 0$, then for every f in K one can express $Z(t) f$ asymptotically as

$$(5) \qquad Z(t)f \sim \sum c_k e^{\lambda_k t} w_k .$$

The parameter ρ is arbitrary ; happily however the eigenvalues $\{\lambda_k\}$ do not depend on ρ, and neither do the eigenfunctions for $|x| < \rho$. In fact the w_k obtained in this way for various values of ρ converge as $\rho \to \infty$ to an eigenfunction of Δ, which we also denote by w_k, with λ_k^2 as eigenvalue :

$$(6) \qquad \Delta w_k = \lambda_k^2 w_k \quad \text{in } G.$$

The eigenfunction w_k behaves asymptotically like $|x|^{-1} \exp(-\lambda_k |x|)$ for large $|x|$ and hence lies outside of H, however it does satisfy an outgoing radiation condition,

There is a close relation between the semi-group $\{Z(t)\}$ and the scattering matrix $\mathfrak{S}(z)$ for the acoustic equation (1) ; in fact the spectrum of B consists precisely of those points λ for which $\mathfrak{S}(i\bar{\lambda})$ is not invertible. We recall that for this problem $\mathfrak{S}(z)$ is meromorphic in the complex plane, holomorphic in the lower half-plane, and has a pole at $\bar{z}$ in the upper half-plane if and only if $\mathfrak{S}(z)$ is not invertible. As a consequence the semi-group $\{Z(t)\}$ provides us with a convenient tool for studying the singularities of the scattering matrix.

To connect properties of $\{Z(t)\}$ with the geometry of the obstacle $\mathfrak{O}$, we introduce the following notation : Consider all rays starting on the sphere of radius ρ which proceed toward the obstacle and are continued according to the law of reflection whenever they impinge on $\mathfrak{O}$, until they finally leave the ball $\{|x| < \rho\}$. We call $\mathfrak{O}$ *confining* if there are arbitrarily long paths of this kind ; otherwise $\mathfrak{O}$ is called *nonconfining*.

Surmising that sharp signals propagate along rays Lax and Phillips conjectured (see pp. 155-157 of [2]) that $Z(t)$ is eventually compact if and only if $\mathfrak{O}$ is nonconfining. The principle result in this direction is due to Ludwig and Morawetz [7] (see also Phillips [8]) who have shown that if $\mathfrak{O}$ is convex then $Z(t)$ is eventually compact. We note that when $\{Z(t)\}$ is eventually compact then the operators decay exponentially in norm ; that is, there exist positive constants C and α such that $|Z(t)| \leqslant C \exp(-\alpha t)$. While it has not been proved that $\{Z(t)\}$ is eventually compact for star-shaped objects which are of course nonconfining, still credance is lent to the above conjecture by the Lax-Morawetz-Phillips result (see [2]) which establishes exponential decay for such reflecting objects.

For a given ray path of length L in the ball $\{|x| < \rho\}$, J. Ralston [9] has constructed solutions to the acoustic equation which depend on a frequency parameter and as the frequency increases these solutions tend to remain within the ball for the entire time L ; as a result he proves that $\|Z(L)\| = 1$. In particular for confining objects $\|Z(t)\| = 1$ for all $t > 0$ and hence $Z(t)$ can not be eventually compact. Another possibility that arises in the case of confining $\mathcal{O}$ is that a ray path closes on itself ; this provides an open cavity for resonances (which correspond to poles of the scattering matrix) with decays which diminish for the higher harmonics. At the present time it is believed that these resonances will occur only for the dynamically stable closed paths ; however as yet even the existence of such resonances is in question. Intuitively one would expect from the above that corresponding complex eigenvalues are very sensitive to the surface details of $\mathcal{O}$.

The situation is quite different for the real exterior eigenvalues associated with the purely decaying modes in (5). In fact, according to a recent result of Lax and Phillips [3], valid for both the Dirichlet and the Neumann boundary conditions, the real eigenvalues *depend monotonically on the obstacle* and hence are influenced only by the bulk properties of the obstacle. A more precise statement of this result follows : For a given scatterer $\mathcal{O}$, order the real eigenvalues as $0 > -\sigma_1(\mathcal{O}) \geqslant -\sigma_2(\mathcal{O}) \geqslant \ldots$. An unqualified monotonicity assertion holds for the fundamental real decaying mode : $\sigma_1(\mathcal{O}_1) \geqslant \sigma_1(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$. For the higher order real decaying modes however the relation $\sigma_n(\mathcal{O}_1) \geqslant \sigma_n(\mathcal{O}_2)$ has been established only when $\mathcal{O}_1 \subset \mathcal{O}_2$ and $\mathcal{O}_1$ is star-shaped.

The exterior problem for a sphere (of radius R) is the only problem which has been solved in detail. Denoting by $C(\sigma)$ the number of real eigenvalues $\{-\sigma_k\}$ which are less than or equal to σ in absolute value, it has been shown that

$$C(\sigma) \sim \frac{1}{2}\left(\frac{\sigma R}{\gamma_0}\right)^2, \quad \gamma_0 = 0.66274\ldots .$$

Combining this with the previously mentioned results, one obtains the following asymptotic estimates : suppose the scatterer $\mathcal{O}$ contains a sphere of radius R_1 and is contained in a sphere of radius R_2. Let $C(\sigma)$ be defined as before but now for $\mathcal{O}$. Then

$$\liminf_{\sigma \to \infty} \frac{C(\sigma)}{\sigma^2} \geqslant \frac{1}{2}\left(\frac{R_1}{\gamma_0}\right)^2 \tag{7}$$

and if, in addition, $\mathcal{O}$ is star-shaped then

$$\limsup_{\sigma \to \infty} \frac{C(\sigma)}{\sigma^2} \leqslant \frac{1}{2}\left(\frac{R_2}{\gamma_0}\right)^2 . \tag{8}$$

It is tempting to surmise that $\lim C(\sigma)/\sigma^2$ exists, if so there remains the problem of determining this limit.

We turn next to a discussion of eigenvalue-free regions in the left half-plane for the acoustic equation in an exterior domain. When $\{Z(t)\}$ is eventually compact then it can be shown (see [2]) that there are only a finite number of eigenvalues in any half-plane of the form $\operatorname{Re} \lambda > \beta$. However if the range of $Z(t)$ is contained in the domain $D(B)$ of its infinitesimal generator B for some t_1, we obtain a much stronger statement. In this case $BZ(t_1)$ is a bounded operator by the closed graph theorem so that its spectral points satisfy $|\lambda_k \exp(t_1 \operatorname{Re} \lambda_k)| \leqslant C$ for all k. We deduce from this that

(9) $$\operatorname{Re} \lambda_k \leqslant a + b \log |\operatorname{Im} \lambda_k|$$

for some real a, b with $b < 0$. Again employing the Ludwig-Morawetz [7] result, Lax and Phillips [5] (see also [4]) have shown that (9) holds for convex $\mathfrak{O}$ with Dirichlet boundary conditions. Similar results have been obtained for the Schrödinger operator with a bounded potential of compact support (cf. A.G. Ramm [10]).

Finally one can pose the problem : Determine the region in the left half-plane which is free of eigenvalues for all scattering objects $\mathfrak{O}$ contained in a sphere say of radius 1. Lax and Phillips (unpublished) have proved that this region contains the circle of radius 1/2 about the point (– 1/2, 0) for all of the familiar boundary value problems. This is a consequence of the following general result : If for a given $\mathfrak{O}$, $\|\mathcal{S}'(x)\| = \alpha$ at some point x on the real axis, then there are no eigenvalues in the circle of radius $1/\alpha$ about the point $(-1/\alpha, -ix)$.

A large class of wave propagation phenomena in classical physics can be subsumed under first order symmetric hyperbolic systems first studied by K.O. Friedrichs. These systems are of the form

(10) $$u_t = E(x)^{-1} \left(\sum_{j=1}^{n} A^j(x)\, \partial_j u + B(x)u \right) \equiv Lu \,, \quad x \text{ in } G \subset R^n \,,$$

where u is an m-vector valued function and the $m \times m$ matrix-valued functions $E(x)$ and $A^j(x)$ are Hermitian and C^1 in x, $E(x)$ positive definite. With the energy norm

(11) $$\|u\|^2 = \int_G u \,.\, Eu \, dx$$

defining the Hilbert space H, such a system is energy conserving if

$$B(x) + B(x)^* - \sum_{j=1}^{n} \partial_j A^j(x) \equiv 0$$

and if the boundary conditions are chosen to be energy conserving (see p. 198 of [2]).

In 1967 Lax and Phillips [2] treated the spectral and scattering problem for these systems within the framework of their approach, assuming (a) that L was elliptic, that is $\lambda E(x) - \Sigma A^j(x)\,\xi_j$ has no zero λ-eigenvalues for $\xi \neq 0$; (b) that the boundary conditions were coercive (see pp. 200-205 of [2]) as well as energy conserving ; (c) that the coefficient matrices were constant for $|x| > \rho$; and (d) that n was odd. An essential ingredient in their proof was the *local energy decay* :

(12) $$\lim_{|t| \to \infty} \int_{G_R} u(x\,,t) \,.\, E(x)\, u(x\,,t)\, dx = 0$$

for every solution with initial data orthogonal to the null space $N(L)$ of L ; here $G_R = G \cap \{|x| < R\}$. Recently N. Iwasaki [1] has established local energy decay under the less restrictive assumptions : (a), (b)′ coercive dissipative boundary conditions, (c), and (d)′ n even or odd.

The problem becomes considerably more difficult if either the ellipticity of L or the coercivity of the boundary conditons is not assumed. In attacking problems of this sort one hopes that $N(L)$ is sufficiently large so that the operator acting in $H \ominus N(L)$ is again coercive. G. Schmidt [11] was able to show in 1968 that this was indeed the case for the Maxwell's equations in an exterior domain if he imposed boundary conditions of the form : $\alpha e + \beta m$ normal to the obstacle ; here e denotes the electric field, m the magnetic field, and α, β are real constants not both zero. As an indication of how delicate the general problem is, we note that J. Ralston [Stanford University thesis, 1968] showed that this limited coercive property could be lost if α/β were allowed to vary along the boundary in the above problem.

Recently J.R. Schulenberger and C.H. Wilcox [12] have treated the initial value problem ($G = R^n$) for *uniformly propagative systems* ; that is systems for which the A^j's are constant, $|E(x) - E_0| = 0(|x|^{-n-\epsilon})$ for some $\epsilon > 0$, $E_0 - \Sigma A^j \xi_j$ are of constant rank for all $\xi \neq 0$, and the ordered λ-roots of det $(\lambda E_0 - \Sigma A^j \xi_j) = 0$ are of constant multiplicity independent of $\xi \neq 0$. Again the coercivity of the system in $D(L) \ominus N(L)$ plays a crucial role in their proof. K. Mochizuki [6] has shown that if one assumes coercivity in $D(L) \ominus N(L)$ then the boundary value problem for non-elliptic uniformly propagative isotropic systems can also be solved.

It is worth noting that coercivity is actually not required in the Lax-Phillips approach ; instead it suffices to show that the set $\{U(t)f \; ; -\infty < t < \infty\}$ is compact in the local energy norm for each f in $D(L) \ominus N(L)$. This fact is utilized in work now in progress by Phillips and Sarason on the neutrino equation in the exterior of a torus. In this problem the operator L is itself elliptic but the boundary conditions are not coercive (see p. 206 in [2]). Never-the-less for a large class of energy conserving boundary conditions it is possible to establish the local energy decay and, from this, the spectral and scattering theory for the system.

REFERENCES

[1] Iwasak N. — Local decay of solutions for symmetric hyperbolic systems with dissipative and coercive boundary conditions in exterior domains, *R.I.M.S.*, Kyoto University, Vol. 5, 1969, p. 193-218.

[2] Lax P.D. and Phillips R.S. — *Scattering theory*, Academic Press, New York, 1967.

[3] Lax P.D. and Phillips R.S. — Decaying modes for the wave equation in the exterior of an obstacle, *Comm. Pure Appl. Math.*, Vol. 22, 1969, p. 737-787.

[4] Lax P.D. and Phillips R.S. — Scattering theory, *Rocky Mountain Jr. Math.*, to appear.

[5] Lax P.D. and Phillips R.S. — Regions free of poles for the scattering matrix, to appear.

[6] Mochizuki K. — Spectral and scattering theory for symmetric hyperbolic systems in an exterior domain, *R.I.M.S.*, Kyoto University, Vol. 5, 1969, p. 219-258.

[7] Morawetz G.S. and Ludwig D. — Generalized Huyghens principle for reflecting bodies, *Comm. Pure Appl. Math.*, Vol. 22, 1969, p. 189-206.

[8] Phillips R.S. — A remark on the preceding paper of C.S. Morawetz and D. Ludwig, *Comm. Pure Appl. Math.*, Vol. 22, 1969, p. 207-211.

[9] Ralston J. — Solutions of the wave equation with localized energy, *Comm. Pure Appl. Math.*, Vol. 22, 1969, p. 807-823.

[10] Ramm A.G. — Regions free of resonance poles in the scattering problem for a three-dimensional potential, *Soviet Physics-Doklady*, Vol. 11, 1966, p. 114-116.

[11] Schmidt G. — Spectral and scattering theory for Maxwell's equations, *Arch. Rat. Mech. Anal.*, Vol. 28, 1968, p. 284-322.

[12] Schulenberger J.R. and Wilcox C.H. — Completeness of the wave operators for perturbations of uniformly propagative systems, *O.N.R. Technical Summary Report*, No. 4, University of Denver, 1970.

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REGULARITY OF HYPERFUNCTION SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

by Mikio SATO

1.

The theory of hyperfunctions seems proving its usefulness in analysis thanks to the works by Profs. Martineau and Komatsu, and a number of young mathematicians.

Hyperfunctions are defined as follows in case of dimension 1. Take real axis $\mathfrak{R}$ embedded in the complex plane $\mathfrak{C}$. Take an open set I of $\mathfrak{R}$ and take an open set D of $\mathfrak{C}$ that contains I as a closed set. Such D will be called a complex neighborhood of I. Then by definition the space of hyperfunctions on I, $\mathcal{B}$(I), is the quotient group of spaces of holomorphic functions on $D-I$ and D :

$$\mathcal{B}(I) = \mathcal{O}(D - I)/\mathcal{O}(D).$$

Incidentally, this is equivalent to saying that $\mathcal{B}(I)$ is the 1^{st} cohomology group of D relative to $D - I$, with $\mathcal{O}$ as its coefficient group : $\mathcal{B}(I) = H^1(D \bmod D-I)$, $\mathcal{O}$) ; namely, a hyperfunction is nothing but a 1^{st} relative cohomology class of $D \bmod D-I$ of holomorphic functions. It is shown that (i) $\mathcal{B}(I)$ is inherent to I, being independent of the choice of its complex neighborhood D, (ii) the presheaf $I \to \mathcal{B}(I)$ is a sheaf : $\mathcal{B}(I) = \Gamma(I, \mathcal{B})$ and, (iii) The sheaf $\mathcal{B}$ is flabby.

Now the definition of hyperfunctions on an oriented real analytic manifold M of arbitrary dimension n : Let X be a complex neighborhood or a complexification of M. Then the space of hyperfunctions on M is :

$$\mathcal{B}(M) =_{\text{def}} H^n(X \bmod X - M, \mathcal{O}),$$

with $\mathcal{O}$ = sheaf of holomorphic functions on X. Again, our notion of hyperfunctions enjoys the properties just mentioned : $\mathcal{B}(M)$ is inherent to M ; the presheaf $U \to \mathcal{B}(U)$ is a flabby sheaf on M. $\mathcal{B}$ naturally contains as a subsheaf the sheaf of distributions and hence the sheaf $\mathcal{A}$ of real analytic functions on M which is the restriction of sheaf $\mathcal{O}$ onto M.

1

2.

Recently there has been a new development of hyperfunction theory ([1], [2], [3]) which makes it possible to describe and analyse in detail the structure of a hyperfunction by means of the cotangential *sphere* bundle S^*M. Here S^*M is the quotient space of T^*M — (the zero section) divided by $\mathfrak{R}^+$, the group of *positive* real numbers. (It is important that we deal with the S^*M constructed in this way, and not with the cotangential *projective* bundle obtained by division by all non-zero real numbers). S^*M is a $(2n-1)$-dimensional manifold equipped with a projection map $\pi : S^*M \to M$ whose fibers are $(n-1)$-spheres S^{n-1}.

Consider first the case $n = 1$. Here each fiber of $\pi : S^*M \to M$ is a 0-sphere S^0 which consists just of 2 points. Hence $S^*M = M \cup M$, the direct union or the disjoint union of 2 copies of M. On the other hand an implication of our definition of hyperfunctions of dimension 1 is that a hyperfunction is expressed as a sum of two "ideal" boundary values of a holomorphic function

$$\varphi \in \mathcal{O}(D - I) : f(x) = \varphi(x + i0) + (-\varphi(x - i0)).$$

Hence, if we denote with $\tilde{\mathcal{A}}_+$ and $\tilde{\mathcal{A}}_-$ the sheaves over $\mathfrak{R}$ consisting of "ideal" boundary values from the upper and the lower half plane respectively, we have

$$\mathcal{B} = \tilde{\mathcal{A}}_+ + \tilde{\mathcal{A}}_-, \quad \text{and} \quad \tilde{\mathcal{A}}_+ \cap \tilde{\mathcal{A}}_- \simeq \mathcal{A}$$

or equivalently, $0 \to \mathcal{A} \to \tilde{\mathcal{A}}_+ \oplus \tilde{\mathcal{A}}_- \to \mathcal{B} \to 0$ where $\mathcal{A}$ denotes the sheaf of real analytic functions on $\mathfrak{R}$. This exact sequence yields at once

$$\mathcal{B}/\mathcal{A} \simeq \tilde{\mathcal{A}}_+/\mathcal{A} \oplus \tilde{\mathcal{A}}_-/\mathcal{A}.$$

This means the sheaf $\mathcal{B}/\mathcal{A}$ that measures the degree of irregularity of hyperfunctions, can be decomposed into two independent components. If one restricts the above observations to an appropriate subsheaf of $\mathcal{B}$, say the sheaf of locally L_p functions with $p > 1$, then we have a decomposition of the sheaf of such functions into two components ; and this is what is known as the function space of class H_p in Fourier analysis. The case $p = 1$ is excluded because this class of functions (or rather, of hyperfunctions) is not stable under this decomposition. Similarly the sheaves $\mathcal{E}$ (C^∞ functions) and $\mathcal{D}'$(distributions) are both subsheaves of $\mathcal{B}$ which are stable under this decomposition ; we can talk about decomposition of the quotient sheaves $\mathcal{E}/\mathcal{A}$, $\mathcal{D}'/\mathcal{A}$ and hence, also about decomposition of $\mathcal{D}'/\mathcal{E} = (\mathcal{D}'/\mathcal{A})/(\mathcal{E}/\mathcal{A})$.

All these things are elementary. We mention however the following points : 1^{st}, it is the quotient sheaf $\mathcal{B}/\mathcal{A}$, and not the sheaf $\mathcal{B}$ itself, which is subject to a decomposition in a natural way independent of the choice of coordinate system. 2^{nd}, the flabbiness of $\mathcal{B}$ together with the cohomological trivialness of $\mathcal{A}$ immediately implies the flabbiness of $\mathcal{B}/\mathcal{A}$. 3^{rd}, in higher dimensions, S^*M has a connected fiber of sphere. This means that decomposition of the sheaf $\mathcal{B}/\mathcal{A}$ is not described as a mere direct sum. We need a new language to describe it ; and this new language is provided by the notion of *direct images* of sheaves.

Let $\mathcal{G}$ be a sheaf on a space Y and let $f : Y \to X$ be a morphism or a continuous map. Then the (0-th) direct image of $\mathcal{G}$ is by definition the sheaf $f_*\mathcal{G}$ over X characterized by the formula $\Gamma(U, f_*\mathcal{G}) = \Gamma(f^{-1}U, \mathcal{G})$ valid for every open set U of X. Since the functor $f_* : \mathcal{G} \to f_*\mathcal{G}$ is left exact, it is natural to introduce the (q-th) right derived functor $\mathfrak{R}^q f_*$ of f_*, and this is nothing but to introduce the sheaf $\mathfrak{R}^q f_* \mathcal{G} = \mathcal{H}^q_f \mathcal{G}$ over X called q^{-th} direct image of $\mathcal{G}$ which is obtained from the presheaf $U \to H^q(f^{-1}U, \mathcal{G})$. We shall say that the map f is purely r-dimensional with respect to $\mathcal{G}$ if $\mathfrak{R}^q f_* \mathcal{G} = 0$ unless $q = r$. For instance f is purely 0-dimensional with respect to any flabby sheaf over Y, and the 0-th direct image is again flabby.

3.

Now the decomposition of $\mathcal{B}/\mathcal{A}$ is attained in the following manner. A sheaf over S^*M, which we shall call sheaf $\mathcal{C}$, will be constructed in a natural manner (as will be described in 4) and in such a way that the 0-th direct image of $\mathcal{C}$ by the projection map $\pi : S^*M \to M$ is canonically isomorphic to $\mathcal{B}/\mathcal{A}$. In other words, we have an exact sequence of natural homomorphisms

$$0 \to \mathcal{A} \overset{\alpha}{\to} \mathcal{B} \overset{\beta}{\to} \pi_* \mathcal{C} \to 0.$$

Moreover, the map π is purely 0-dimensional with respect to $\mathcal{C}$, and the image $\pi_* \mathcal{C}$ is a flabby sheaf on M. Far beyond these facts, M. Kashiwara established the decisive result that the sheaf $\mathcal{C}$ itself is a flabby sheaf.

Taking the cross sections of the above sequence we have an exact sequence

$$0 \to \mathcal{A}(M) \overset{\alpha}{\to} \mathcal{B}(M) \overset{\beta}{\to} \Gamma(M, \pi_* \mathcal{C}) \to 0.$$

The third term is rewritten as $\Gamma(S^*M, \mathcal{C})$ by the definition of direct image. Hence, for each hyperfunction $u \in \mathcal{B}(M)$, the image βu (i.e. the residue class of u modulo analytic functions) may be viewed either as a section of $\pi_* \mathcal{C}$ over M or as a section of $\mathcal{C}$ over S^*M. Accordingly, the notion of support of βu also admits two interpretations, either as a closed set of M or as that of S^*M. The former is the singular support of u in the customary sense (in notation : $\mathrm{S.S.}_{\mathrm{M}}\, u$) while the latter is that of u in a sharpened sense (in notation : $\mathrm{S.S.}_{\mathrm{C}} u$). Clearly $\mathrm{S.S.}_{\mathrm{M}}\, u = \pi(\mathrm{S.S.}_{\mathrm{C}} u)$ and $\pi^{-1}(\mathrm{S.S.}_{\mathrm{M}}\, u) \supset \mathrm{S.S.}_{\mathrm{C}} u$, and flabbiness of $\mathcal{C}$ implies that any closed set of S^*M can actually appear as a singular support of some hyperfunction. Thus $\mathrm{S.S.}_{\mathrm{C}} u$ gives us more detailed information about the irregularity of u than $\mathrm{S.S.}_{\mathrm{M}}\, u$. If $\bar{u}$ denotes the complex conjugate of u, then $\mathrm{S.S.}_{\mathrm{C}} \bar{u} = (\mathrm{S.S.}_{\mathrm{C}} u)^a$ where a signifies antipodal points on S^*M.

A few examples of $\mathrm{S.S.}_{\mathrm{C}} u$ in the case of $n = 2$ are illustrated in Figure 1.

As is shown in [3] one can develop a calculus on the sheaf $\mathcal{C}$ with applications to the calculus for hyperfunctions. For example :

Multiplication. — The product of 2 hyperfunctions $u_1, u_2 \in \mathcal{B}(M)$ is well-defined if $\mathrm{S.S.}_{\mathrm{C}} u_1$ and $(\mathrm{S.S.}_{\mathrm{C}} u_2)^a$ are disjoint to each other. For example, $\delta(x_1)\, \delta(x_2)$ or more generally, $f_1(x_1) f_2(x_2)$ is well-defined.

Specialization (or *restriction*). – Let N be an oriented submanifold of M. The conormal sphere bundle S_N^*M is naturally considered to be a submanifold of S^*M. Then the specialisation $f|_N \in \mathcal{B}(N)$ is always well-defined for a hyperfunction $f \in \mathcal{B}(M)$ whose singular support $\mathrm{S.S.}_{\mathrm{C}} f$ is disjoint to S_N^*M. ([3])

For example, if N is a hypersurface which is non-characteristic with respect to a differential operator P, and if $u \in \mathcal{B}(M)$ satisfies $Pu \in \mathcal{A}(M)$, then by the theorem below u as well as any (higher) derivatives of u can be specialized onto N. This means that the notion of initial data makes sense for a hyperfunction solution of linear differential equation.

Generally speaking, if $f : N \to M$ is a morphism between oriented real analytic manifolds of dimension n' and n respectively, and if $\rho : S^*M \times_M N - S_N^*M \to S^*N$ and $\sigma : S^*M \times_M N - S_N^*M \to S^*M$ denote 2 morphisms naturally induced by f, then we have as a generalization of the notion of specialization the following sheaf homomorphism over S^*N

$$f^* : \rho_! (\sigma^{-1} \mathcal{C}_M) \to \mathcal{C}_N,$$

where $\rho_!$ stands for the 0-th direct image with *proper support.* We have, on the other hand, the following sheaf homomorphism over S^*M as the *integration* in $\mathcal{C}$:

$$\sigma_! (\rho^{-1} \mathcal{C}_N^{(n')}) \to \mathcal{C}_M^{(n)},$$

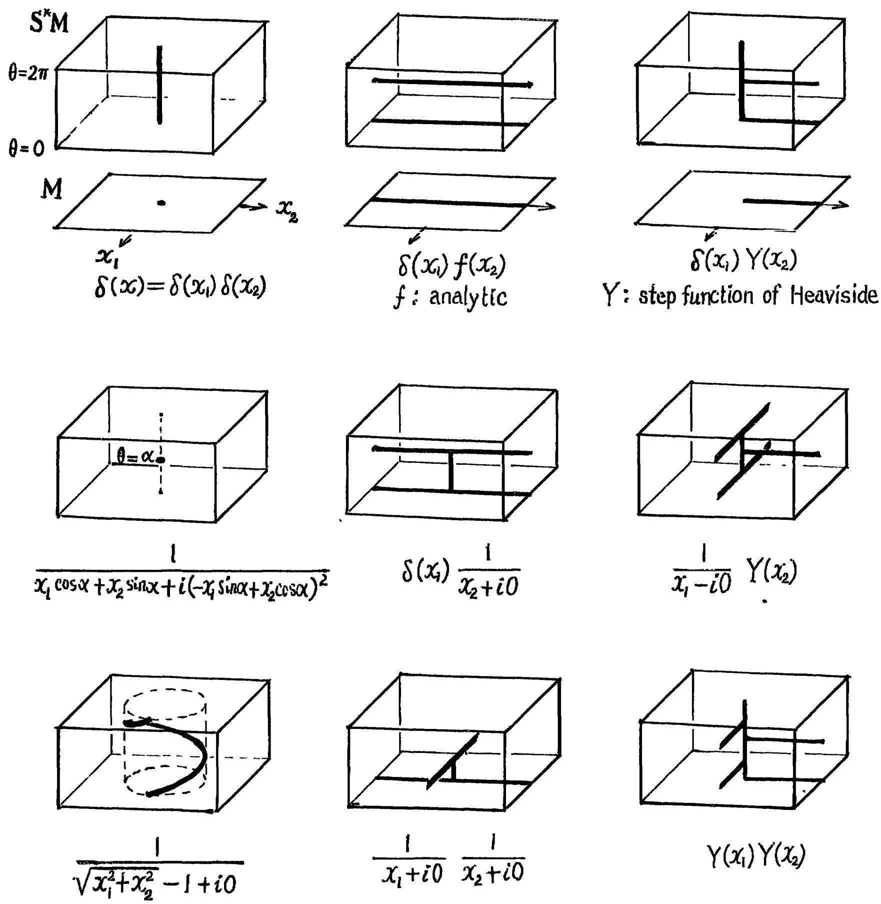

Figure 1

where $\mathcal{C}_N^{(n')}$ (resp. $\mathcal{C}_M^{(n)}$) means the sheaf $\mathcal{C}$ of n'-forms (resp. n-forms). ([3], §6).

Combining the method of F. John [5] with the theory of $\mathcal{C}$, we can easily derive the following

THEOREM ([1] ; [2] ; [3] § 8). – *Let $P(x, D)$ be a differential operator on M with the principal symbol P_m, and let $F = \{(x, \overline{\eta}) \in S^*M \,|\, P_m(x, \eta) = 0\}$. Then the sheaf endomorphism of $\mathcal{C}$ induced by $P(x, D)$ is bijective on $S^*M - F$.*

More specifically, P_m is *invertible* on $S^*M - F$ in the sheaf of rings $\mathcal{P}$ over S^*M consisting of "pseudo-differential operators" operating on $\mathcal{C}$. (The sheaf $\mathcal{P}$ is defined to be $\mathrm{Dist}^0(S^*M, \mathcal{C}_{M\times M}^{(0,n)})$. Here $\mathcal{C}_{M\times M}^{(0,n)}$ stands for the sheaf $\mathcal{C}$ over $S^*(M \times M)$ that behaves as n-forms on the 2^{nd} copy of M. S^*M is regarded to be a submanifold of $S^*(M \times M)$ by "anti-diagonal" embedding. See [3] § 6). I note that, besides generalities about $\mathcal{C}$, the only fact we need to prove this theorem is the theorem of Cauchy-Kowalewski.

COROLLARY. – *If $P(x, D)$ is elliptic then every hyperfunction solution of the equation $Pu = 0$ is analytic.*

Proof. Since F is empty in this case, we have the isomorphism $P : \mathcal{C} \to \mathcal{C}$ valid on the wole S^*M, and hence the isomorphism $P : \pi_*\mathcal{C} \overset{\sim}{\to} \pi_*\mathcal{C}$ on M. On the other-hand, $P : \mathcal{A} \to \mathcal{A}$ is surjective (Cauchy-Kowalewski theorem) ; i.e. we have an exact sequence

$$0 \to \mathcal{A}^P \to \mathcal{A} \overset{P}{\to} \mathcal{A} \to 0,$$

where $\mathcal{A}^P$ denotes the sheaf of analytic solutions of $Pu = 0$. Now we observe the following diagram of exact sequences

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & \mathcal{A}^P & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \pi_*\mathcal{C} & \to & 0 \\ & & P\downarrow & & P\downarrow & & P\downarrow\wr & & \\ 0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \pi_*\mathcal{C} & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

and conclude that $0 \to \mathcal{A}^P \to \mathcal{B} \overset{P}{\to} \mathcal{B} \to 0$ is exact. (q.e.d.)

M. Morimoto mentioned that a theorem of Bargman-Haag-Wightman on Jost points in the quantum field theory is also an easy corollary of the above theorem.

The improvement of the above theorem is now being worked out by T. Kawai, M. Kashiwara, and the present speaker along the lines of Lewy-Hörmander-Egorov-Nirenberg-Treves. (See also [6], [7]). The problem is to determine supports of the kernel and cokernel sheaves of $P : \mathcal{C} \to \mathcal{C}$. (These may be substantially smaller than F). Or rather, to determine the kernel and cokernel sheaves themselves. For example, if $P = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) - i(x_1 + ix_2)\frac{\partial}{\partial x_3}$ is the operator of H. Lewy, and if Q (resp. $\overline{Q}$) is a pseudo-differential operator (which induces a well-defined *sheaf* homomorphism $\mathcal{C} \to \mathcal{C}$ on S^*M) defined by means of the kernel function

$$K(x, x')\, dx' = (x_3 - x_3' + 2(x_2 x_1' - x_1 x_2') + i((x_1 - x_1')^2 + (x_2 - x_2)^2))^{-1} dx_1' \, dx_2' \, dx_3'$$

(resp. the complex conjugate of $K(x, x')\, dx'$), then it is shown that

$$\mathcal{C} \xrightarrow{Q} \mathcal{C} \xrightarrow{P} \mathcal{C} \xrightarrow{\overline{Q}} \mathcal{C}$$

is an exact sequence. This implies that $Pu = f$ is solvable if and only if $\overline{Q}f = 0$ in $\mathcal{C}$, and that the supports of $\mathrm{Ker}_C P$ and $\mathrm{Coker}_C P$ are given by the supports of the operators Q and $\overline{Q}$, which are quite easy to determine.

Propagation of singularities. Let $P(x, D)$ be such that the principal symbol $P_m(x, \eta)$ is real and of principal type on M. Kawai and Kashiwara proved that a closed set $F \subset S^*M$ can be a $\mathrm{S.S.}_C u$ for some $u \in \mathcal{B}(M)$ such that $Pu \in \mathcal{A}(M)$ or even $Pu = 0$, if and only if F is a union of bicharacteristic *strips.* An easy corollary of this is that a closed set of M can be a $\mathrm{S.S.}_M u$ for some hyperfunction solution of $Pu = 0$ if and only if it is a union of bicharacteristic *curves.* For example, there exists a solution of $((\partial/\partial x_1)^2 - (\partial/\partial x_2)^2 - (\partial/\partial x_3)^2)u = 0$ for which $\mathrm{S.S.}_M u = \{x \in \mathfrak{R}^3 \,|\, x_2^2 + x_3^2 \geqslant 1\}$. (F. John).

I-hyperbolicity. Kawai introduced the notion of I-hyperbolic operators as an interesting generalization of hyperbolic operators.

4. – Construction of the sheaf C ([2], [3])

4.1. *Relative cohomology groups in generalized sense.* Let $\mathcal{F}$ denote a sheaf over a space X and let $f : Y \to X$ be a continuous map. Then the sheaf $f^{-1}\mathcal{F}$ over X called the inverse image of $\mathcal{F}$ is defined to be the fiber product over X of Y and $\mathcal{F}$. This functor $\mathcal{F} \to f^{-1}\mathcal{F}$ is an exact one. Now the relative cohomology groups in the generalized sense, $H^p(X \leftarrow Y, \mathcal{F})$, are defined in a natural way ([2], [3]) so that we have an exact sequence

$$\ldots \to H^{p-1}(X, \mathcal{F}) \to H^{p-1}(Y, f^{-1}\mathcal{F}) \to H^p(X \leftarrow Y, \mathcal{F}) \to H^p(X, \mathcal{F}) \to H^p(Y, f^{-1}\mathcal{F}) \to \ldots$$

If $f : Y \hookrightarrow X$ is the natural embedding of an open subset our $H^p(X \leftarrow Y, \mathcal{F})$ reduces to the (ordinary) relative cohomology group $H^p(X \bmod Y, \mathcal{F})$ and if $Y = \phi$, the empty set, this reduces further to $H^p(X, \mathcal{F})$.

If we have still another space Z and another continuous map $Z \to Y$, we have the following exact sequence (which reduces to the above one when $Z = \phi$).

$$\ldots \to H^{p-1}(X \leftarrow Z, \mathcal{F}) \to H^{p-1}(Y \leftarrow Z, f^{-1}\mathcal{F}) \to H^p(X \leftarrow Y, \mathcal{F}) \to H^p(X \leftarrow Z, \mathcal{F}) \to H^p(Y \leftarrow Z, f^{-1}\mathcal{F}) \to \ldots$$

4.2. *Real monoidal transform.* Monoidal transforms, which are the most essential in desingularizing analytic spaces, are described as follows in the case of a (nonsingular) complex analytic manifold X and a submanifold Y of arbitrary codimension in X. One removes Y from X and instead inserts the *normal projective bundle*

$P_Y X$ over Y defined by $P_Y X = (T_Y X -$ (zero-section))/(non-zero complex numbers). Here $T_Y X$ denotes the (tangential) normal vector bundle over Y defined as follows by means of tangent vector bundles TX and TY :

$$0 \to TY \to Y \times_X TX \to T_Y X \to 0.$$

This replacement of Y by $P_Y X$ or the blowing up is a natural one so that the transform $\widetilde{X} = (X - Y) \cup P_Y X$ acquires a natural complex analytic structure and the natural projection map $\tau : \widetilde{X} \to X$ becomes a proper morphism of analytic manifolds. The inverse image of Y by τ coincides with $P_Y X$ and $P_Y X$ lies in $\widetilde{X}$ as a hypersurface or a submanifold of codimension 1.

What we need in construction of $\mathcal{C}$ is the real analytic version of the monoidal transform. Take a real analytic manifold M and a submanifold N of arbitrary codimension in M. Then the real monoidal transform of M at N is by definition

$$\widetilde{M} = (M - N) \cup S_N M$$

where the normal sphere bundle $S_N M$ over N is defined by

$$S_N M = (T_N M - \text{(zero-section)})/\text{(positive real numbers)}.$$

$\widetilde{X}$ naturally acquires the structure(1) of real analytic manifold *with* boundary at $S_N M$, and the natural projection $\tau : \widetilde{M} \to M$ becomes a proper morphism (Fig. 2). The inverse image by τ of N coincides with $S_N M$ and $S_N M$ is of codimension 1 in $\widetilde{M}$.

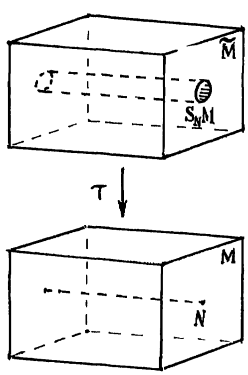

Figure 2

4.3. *The sheaf $\mathcal{Q}$ over SM.* Now return to the n-dimensional oriented real analytic manifold M and its complexification X. Since X has a structure of real analytic manifold of dimension $2n$, we can talk about the real monoidal transform $\widetilde{X}$ of X at M. Here, furthermore, we can naturally identify the normal

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(1) If antipodal pairs of points on each fiber of $S_N M$ are identified, $S_N M$ shrinks to the normal projective bundle $P_N M$ and $\widetilde{M}$ shrinks to a real analytic manifold *without* boundary in which $P_N M$ lies as a hypersurface. We mention however that what we need in what follows is the topological structure of M rather than the analytic structure thereof.

bundle $T_M X$ (resp. $S_M X$) with the tangent bundle TM (resp. SM) — or rather with $\sqrt{-1}$ times TM (resp. SM) —, because we have

$$0 \to TM \to M \times_X TX \to T_M X \to 0 \qquad \text{(by the definition)}$$

and

$$M \times_X TX = TM \otimes_R \mathbb{C} = TM \oplus \sqrt{-1}\, TM.$$

Hence we can write $\widetilde{X} = (X - M) \cup SM$. The natural map $\tau : \widetilde{X} \to X$ is proper. On $\tau^{-1}M = SM$ the map τ gives the fiber structure $\tau : SM \to M$ while it reduces to a homeomorphism outside SM, $\tau : \widetilde{X} - SM = X - M$.

Now we apply the exact sequence of 4.1. to the triple $\widetilde{X} - SM \hookrightarrow \widetilde{X} \xrightarrow{\tau} X$ with the structure sheaf $\mathcal{O} = \mathcal{O}_X$ as coefficients, and obtain

$$\ldots \to H^p(X \leftarrow \widetilde{X}, \mathcal{O}) \xrightarrow{\alpha} H^p(X \leftarrow \widetilde{X} - SM, \mathcal{O}) \xrightarrow{\beta} H^p(\widetilde{X} \hookleftarrow \widetilde{X} - SM, \tau^{-1}\mathcal{O}) \to \ldots.$$

We have however $H^p(X \hookleftarrow \widetilde{X}, \mathcal{O}) = \mathcal{A}(M)$ (for $p = n$), $= 0$ (for $p \neq n$) and $H^p(X \hookleftarrow \widetilde{X} - SM, \mathcal{O}) = H^p(X \bmod \widetilde{X} - SM, \mathcal{O}) = \mathcal{B}(M)$ (for $p = n$), $= 0$ (for $p \neq n$). The latter is nothing but the fundamental fact in hyperfunction theory while the former is, as Kashiwara mentioned, quite an easy consequence of the elementary fact that $H^p(pt \leftarrow S^{n-1}, \mathfrak{Z}) = \mathfrak{Z}$ (for $p = n$), $= 0$ (for $p \neq n$). ("*pt*" denotes the space consisting of a single point and $\mathfrak{Z}$ denotes rational integers).

These vanishing theorems, together with the fact that $\alpha : \mathcal{A}(M) \to \mathcal{B}(M)$ is injective, immediately imply that the exact sequence mentioned above yields

$$0 \to \mathcal{A}(M) \to \mathcal{B}(M) \to H^n(\widetilde{X} \bmod \widetilde{X} - SM, \tau^{-1}\mathcal{O}) \to 0$$

and

$$H^p(\widetilde{X} \bmod \widetilde{X} - SM, \tau^{-1}\mathcal{O}) = 0 \qquad \text{if } p \neq n.$$

According to the hyperfunction theory we know that M is purely n-codimensional in X with respect to the structure sheaf $\mathcal{O}_X$: $\mathrm{Dist}^p(M, \mathcal{O}_X)$, $= \mathcal{B}$ (for $p = n$), $= 0$ (for $p \neq n$). This includes at the same time the definition of the sheaf $\mathcal{B}$ of hyperfunctions. On the other hand, it is easy to show that the hypersurface SM in X is purely 1-codimensional with respect to $\tau^{-1}\mathcal{O}_X$; i.e. we have $\mathrm{Dist}^p(SM, \tau^{-1}\mathcal{O}_X) = 0$ unless $p = 1$. Defining the sheaf $\mathcal{Q}$ over SM by $\mathcal{Q} = \mathrm{Dist}^1(SM, \tau^{-1}\mathcal{O}_X)$, we can deduce from this and the above facts that

$$H^p(SM, \mathcal{Q}) = \begin{cases} H^n(\widetilde{X} \bmod \widetilde{X} - SM, \tau^{-1}\mathcal{O}), & p = n - 1, \\ 0 & \text{if } p \neq n - 1, \end{cases}$$

and

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{H}_\tau^{n-1}\mathcal{Q} \to 0.$$

4.4. *The sheaf $\mathcal{C}$ over S^*M*. The next and final step to construction of the sheaf $\mathcal{Q}$ from SM to the cosphere bundle S^*M. Define the fiber bundle DM over M by

$$DM = \{(x, \bar{\xi}, \bar{\eta}) \in SM \times_M S^*M \mid \xi \in T_xM - \{0\}, \eta \in T_x^*M - \{0\}, <\xi, \eta> \geqslant 0\}.$$

(We mention that each point in the fiber product $SM \times_M S^*M$ is a pair of tangent and cotangent vectors at the same point in M so that we can talk of inner product of them). We have two chains of fiberings :

$$DM \overset{\pi'}{\to} SM \overset{\tau}{\to} M \quad \text{and} \quad DM \overset{\tau'}{\to} S^*M \overset{\pi}{\to} M,$$

consisting of proper maps. Also important is the fact that the fiberings π' and τ' have both contractible fibers of (closed) hemispheres $\frac{1}{2}S^{n-1}$, for this fact implies in particular that the map π' is purely 0-dimensional with respect to the inverse image sheaf $\pi'^{-1}\mathcal{Q}$ over DM, and the 0-th direct image coincides with the original sheaf $\mathcal{Q}$:

$$\mathcal{H}^p_{\pi'}(\pi'^{-1}\mathcal{Q}) = \begin{cases} \mathcal{Q}, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

On the other hand, the map τ' is shown to have pure dimension $n-1$ with respect to $\pi'^{-1}\mathcal{Q}$: $\mathcal{H}^p_{\tau'}(\pi'^{-1}\mathcal{Q}) = 0$, $(p \neq n-1)$. This is a fact equivalent to a result of M. Morimoto [4] about the edge-of-the-wedge theorems. Now we define the sheaf $\mathcal{C}$ over S^*M by

$$\mathcal{C} =_{\text{def}} \mathcal{H}^{n-1}_{\tau'}(\pi'^{-1}\mathcal{Q}),$$

and obtain the formulae

$$\mathcal{H}^{p-1}_{\tau}\mathcal{Q} = \mathcal{H}^{p-1}_{\tilde{\omega}}(\pi'^{-1}\mathcal{Q}) = \mathcal{H}^{p-n}_{\pi}\mathcal{C} = \begin{cases} \mathcal{B}/\mathcal{A}, & p = n, \\ 0 & p \neq n, \end{cases}$$

$$0 \to \mathcal{A} \to \mathcal{B} \to \pi_*\mathcal{C} \to 0,$$

where $\tilde{\omega}$ is the abbreviation of $\tau \circ \pi' = \pi \circ \tau' : DM \to M$. (We understand the cohomology group for negative dimension is always 0).

We mention that a further consideration gives us two diagrams consisting of exact sequences of sheaves over SM and S^*M respectively :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to \tau^{-1}\mathcal{A} & \to & \tilde{\mathcal{A}} & \to & \mathcal{Q} & \to & 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to \tau^{-1}\mathcal{A} & \to & \tau^{-1}\mathcal{B} & \to & \tau^{-1}\pi_*\mathcal{C} & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \pi'_*\tau'^{-1}\mathcal{C}^a & = & \pi'_*\tau'^{-1}\mathcal{C}^a & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to \pi^{-1}\mathcal{A} & \to & \mathcal{A}^{*a} & \to & \mathcal{Q}^{*a} & \to & 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to \pi^{-1}\mathcal{A} & \to & \pi^{-1}\mathcal{B} & \to & \pi^{-1}\pi_*\mathcal{C} & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{C} & = & \mathcal{C} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Here $\tilde{\mathcal{A}}$ is the sheaf of "ideal boundary values" of holomorphic functions defined as follows : Define a sheaf $\tilde{\mathcal{O}}$ over X to be the 0-th direct image of $\mathcal{O}_{\tilde{X}\text{-}SM} = \mathcal{O}_{X\text{-}M}$ by the natural embedding $\tilde{X} - SM \hookrightarrow \tilde{X}$. Then $\tilde{\mathcal{A}}$ is the restriction of $\tilde{\mathcal{O}}$ onto SM i.e. the inverse image of $\tilde{\mathcal{O}}$ by the natural embedding $SM \hookrightarrow \tilde{X}$. (We omit the definitions of sheaves $\mathcal{A}*$ and $\mathcal{Q}*$. See [3]). The symbol "a" on the right shoulder stands for the direct (and at the same time inverse) image by the antipodal mapping on SM or S^*M. The exact sequences $0 \to \tilde{\mathcal{A}} \to \tau^{-1}\mathcal{B} \to \dots$ and $\dots \to \pi^{-1}\mathcal{B} \to \mathcal{C} \to 0$ in the above diagrams are of great importance in further study and applications of hyperfunction theory.

For complete accounts and proofs the reader is referred to [3].

REFERENCES

[1] Sato M. — Hyperfunctions and partial differential equations, *Proc. Int. Conf. on Functional Analysis,* Tokyo, 1969.

[2] Sato M. and Kawai T. — Structure of hyperfunctions, *Symposium on Algebraic geometry and hyperfunction theory,* Katata, 1969 (in japanese).

[3] Sato M. and Kashiwara. — Structure of hyperfunctions, *Sûgaku no Ayumi,* 15 (1970), p. 9-72 (in Japanese).

[4] Morimoto M. — Sur la décomposition du faisceau des germes de singularités d'hyperfonctions, *J. Fac. Sci. Univ. of Tokyo,* 17, 1970, p. 215-240.

[5] John F. — Plane waves and spherical means applied to partial differential equations, *Interscience,* 1955.

[6] Schapira P. — Solutions hyperfonctions des équations aux dérivées partielles du premier ordre. *Bull. Soc. math. France,* 97, 1969, p. 243-255.

[7] Suzuki H. — Local existence and regularity of solutions for linear differential equations in two independent variables, to appear in *J. Math. Soc. Japan.*

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Actes, Congrès intern. Math., 1970. Tome 2, p. 795 à 801.

FRACTIONAL POWERS OF BOUNDARY PROBLEMS

by R. SEELEY

Consider an elliptic system A of C^∞ differential operators in a domain G ; a system B of operators defined on the boundary of G ; and the corresponding unbounded operator A_B defined in $L^p(G)$, $1 < p < \infty$, with domain given by $Bu = 0$. We have the following chain of results :

(I) An asymptotic expansion of the resolvent $R_\lambda = (A_B - \lambda)^{-1}$ for large negative λ, under natural algebraic conditions on A and B.

(II) An expansion of the complex powers $A_B^z = \frac{i}{2\pi}\int_\Gamma \lambda^z R_\lambda d_\lambda$ (Cauchy integral over an infinite contour).

(III) Formulas for the residues at the poles of the meromorphic function Trace (A_B^z), and the values of this function at $z = 0, 1, 2, \ldots$

(IV) An estimate of the norm $\|A_B^{iy}\|$, y real.

(V) A concrete description of the domains of the fractional powers A_B^θ, $0 < \theta < 1$, (with immediate application to the convergence of eigenfunction expansions in norms stronger than L^p).

The relations in this chain are

$$\text{(I)} \Longrightarrow \text{(II)} \Longrightarrow \text{(III)}\ ,\quad \text{(I)} \Longrightarrow \text{(IV)} \Longrightarrow \text{(V)}.$$

Other similar results (particularly those that have provided useful hints) are cited below.

I. The resolvent.

The standard model of a boundary problem is the Dirichlet problem in the upper half plane. Let $x = (x', t) \in R^n$, $x' \in R^{n-1}$, $t \geqslant 0$. Let $\Delta = -\Sigma\partial^2/\partial x_j^2$, and let Δ_0 denote the corresponding operator with domain defined by $u(x', 0) = 0$. The resolvent is

$$(0)\ (\Delta_0 - \lambda)^{-1} f(x', t) = (2\pi)^{-n} \iint e^{ix'\xi' + it\tau} \frac{1}{|\xi'|^2 + \tau^2 - \lambda} \hat{f}(\xi', \tau)\, d\xi'\, d\tau$$

$$+ \text{const} \iint e^{ix'\xi'} \frac{\exp(-\sqrt{|\xi'|^2 - \lambda}(t+s))}{|\xi'|^2 - \lambda} \acute{f}(\xi', s)\, d\xi' ds,$$

where $\hat{f}$ is the full Fourier transform, and $\acute{f}$ the tangential. It is not hard to check that for $|\arg \lambda| > \epsilon$, $\|(\Delta_0 - \lambda)^{-1}\| \leqslant C_\epsilon/|\lambda|$.

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In the general case, we have not an exact formula, but an asymptotic expansion of the same general form. To state precisely the conditions for this expansion, we need the usual cast of characters :

G = an open, smooth, bounded subset of R^n (or, a compact C^∞ manifold with boundary)

$A = \sum_{|\alpha|\leqslant\omega} a_\alpha D^\alpha$, where a_α are C^∞, $q \times q$ matrices (or, A may be a differential operator on sections of a vector bundle)

$B_j = \sum_{\nu=0}^{\omega_j} b_j^\nu D_t^{\omega_j-\nu}$, where D_t is a normal derivative along ∂G ; b_j^ν is an $r_j \times q$ system of differential operators on ∂G or order $\leqslant \nu$; and $\omega_1 < \omega_2 < \ldots < \omega_k < \omega$. (The boundary operators of a given order ω_j have been collected into a single system B_j, for reasons that will appear below).

$L^p_{B,\omega}$ = q-tuples of fuctions in the Bessel potential space $L^p_\omega(G)$, satisfying $B_j u = 0$ on ∂G for $j = 1, \ldots, k$.

A_B = the operator A with domain $L^p_{B,\omega}$

$$\sigma_\omega(A)(x\,;\xi) = \sum_{|\alpha|=\omega} a_\alpha(x)\xi^\alpha \quad , \quad x \text{ in } G,\ \xi \text{ in } R^n$$

$$\sigma_{\omega_j}(B_j)(x'\,;\xi', D_t) = \sum_{\nu=0}^{\omega_j} (b_j^\nu)(x', \xi')\, D_t^{\omega_j-\nu},$$

(where (x', t) are local coordinates near a point of ∂G, and $t > 0$ in G).

The algebraic conditions guaranteeing a good resolvent R_λ for $\lambda < 0$ are given in Agmon [1] :

(1) $\sigma_\omega(A)(x, \xi) - \lambda I$ is invertible for $\lambda \leqslant 0$, $(\xi, \lambda) \neq 0$.

(2) The boundary problem (in ordinary differential equations)

(i) $\sigma_\omega(A)(x', 0\,; \xi', D_t)\,u(t) = \lambda u(t)$, $t > 0$

(ii) $\sigma_{\omega_j}(B_j)\,u(0) = c_j$, $j = 1, \ldots, k$

(iii) $u(+\infty) = 0$

has a unique solution u for each choice of c_j in $\mathbf{C}^{r_j}$, $\lambda \leqslant 0$, $(\xi', \lambda) \neq 0$.

These conditions, with $\lambda = 0$, say simply (1) that A is elliptic and (2) the B_j are "well posed" or "covering". With $\lambda \neq 0$ and $\xi' = 0$, condition (2ii) implies that in B_j, the coefficient b_j^0 of the highest power of the normal derivative is an $r_j \times q$ matrix *of rank* r_j. A boundary system satisfying this condition at every point should be called *normal,* since for $q = 1$ this is equivalent to the usual definition ; and for $q > 1$ it appears to give the results usually associated with normal systems.

The resolvent then has the following expansion for $\lambda \to -\infty$:

(3) $$(A_B - \lambda)^{-1} \sim \Sigma\, O_p(c_{-\omega-j}) + \Sigma\, O_p''(d_{-\omega-j})$$

where

$$O_p(c)\, f(x) = (2\pi)^{-n} \int_{R^n} e^{ix\xi}\, c(x\,,\, \xi)\, \hat{f}(\xi)\, d\xi$$

$$O_p''(d)\, f(x'\,,t) = (2\pi)^{1-n} \int_0^\infty \int_{R^{n-1}} e^{ix'\xi'}\, d(x'\,,t\,,\xi'\,,s)\, \acute{f}(\xi'\,,s)\, d\xi' ds$$

$\hat{f}(\xi)$ = Fourier transform of $f(x)$, $\acute{f}(\xi'\,,s)$ = Fourier transform of $f(x'\,,s)$

$c_{-\omega} = \dfrac{1}{\sigma_\omega(A)(x\,,\,\xi) - \lambda}$, $c_{-\omega-j}(x\,,\xi\,,\lambda)$ is homogeneous in $(\xi\,,\lambda^{1/\omega})$, of degree $-\omega - j$.

$d_{-\omega-j}(x'\,,t\,,\xi'\,,s\,,\lambda)$ homogeneous in $(\xi\,,\lambda^{1/\omega}\,,t^{-1}\,,s^{-1})$, of degree $1 - \omega - j$.

The example (0) has this form, with $\omega = 2$; all terms but c_{-2} and d_{-2} drop out. The meaning of (3) is this ; for $\omega + k \geqslant n$ (= the dimension of G), the difference

$$(A_B - \lambda)^{-1} - \sum_0^K O_p(c_{-\omega-j}) - \sum_0^K O_p''(d_{-\omega-j})$$

has a kernel $K_\lambda(x\,,y)$ which is of class $C^{\omega+k-n}$ in x and y, and $O\left(|\lambda|^{-1-\frac{K-n}{\omega}}\right)$ as $\lambda \to -\infty$. The functions $c_{-\omega-j}$ and $d_{-\omega-j}$ are computed more or less explicitly from the coefficients of A and B. The analytic nature of $d_{-\omega-j}$ is given by

(4) $$|d_{-\omega-j}| \leqslant C \exp[(|\xi| + |\lambda|^{1/\omega})\,.\,(t+s).(\text{const})]\,(|\xi| + |\lambda|^{1/\omega})^{1-\omega-j},$$

together with similar estimates on the derivatives.

II. The fractional powers.

When A_B has a complete orthonormal set of eigenfunctions φ_j with eigenvalues λ_j, we can set

$$A_B^z f = \Sigma\, \lambda_j^z\, (f\,,\varphi_j)\, \varphi_j\,,$$

and when $\mathrm{Re}(z) < n/\omega$, this has the kernel

$$K_z(x\,,y) = \Sigma\, \lambda_j^z\, \varphi_j(x)\, \overline{\varphi}_j(y)\,.$$

(When we have a system, then $\varphi_j = (\varphi_j^1, \ldots, \varphi_j^q)$, and $\varphi_j(x)\,\overline{\varphi}_j(y)$ denotes the matrix $(\varphi_j^k(x)\,\overline{\varphi}_j^l(y))$. The trace of A_B^z is

$$\int_G \text{trace}\, K_z(x\,,x)\, dx = \int_G \Sigma\, \lambda_j^z\, |\varphi_j(x)|^2\, dx = \Sigma\, \lambda_j^z.$$

Two questions to consider here are (i) the analytic continuation of $K_z(x\,,x)$, and (ii) the analytic continuation of $\int_G K_z(x\,,x)\,dx$. We study these by representing A_B^z as a Cauchy integral,

$$A_B^z = \frac{i}{2\pi} \int_\Gamma \lambda^z\, (A_B - \lambda)^{-1}\, d\lambda\,.$$

THEOREM 1. – *For $x \notin \partial G$, $K_z(x, x)$ is meromorphic in z, with simple poles at $z = (j-n)/\omega$, $j = 0, 1, 2, \ldots$. These are the only singularities of K_z, and those at $z = 0, 1, 2, \ldots$ are removable. The residue $\gamma_j(x)$ at $z = (j-n)/\omega$ depends only on the term $c_{-\omega-j}$ in (3). When $z = (j-n)/\omega = 0, 1, 2, \ldots$, the value of $K_z(x, x)$ is explicitly determined by $c_{-\omega-j}$.*

THEOREM 2. – *$\int_G K_z(x, x)\, dx$ has the singularities in Theorem 1, except that the residue at $z = (j-n)/\omega$ is*

$$\int_G \gamma_j(x)\, dx + \int_{\partial G} \delta_j(x')\, dx',$$

where δ_j is determined explicitly from the $d_{-\omega-j+1}$ in (3). Similar results hold for the values at $z = 0, 1, 2, \ldots$

The only thing involved here is the *homogeneity* of the *c's* and *d's* ; the integral $C_{-\omega-j} = \int_\Gamma \lambda^z c_{-\omega-j}\, d\lambda$ is homogeneous of degree $\omega z - j$, and so

$$\int_{|\xi|\leqslant 1} C_{-\omega-j}\, d\xi = \left(\int_{|\xi|=1} C_{-\omega-j}\, d\Sigma_\xi\right) \int_1^\infty r^{\omega z - j + n - 1}\, dr,$$

which gives a simple pole when $\omega z - j + n = 0$. The *d's* have no real effect away from the boundary, because of their exponential decay. But $\int_0^\infty d_{-\omega-j}(x', \xi', t, t, \lambda)\, dt$ is homogeneous in $(\xi', \lambda^{1/\omega})$, so these terms show up as boundary integrals when we integrate over G. For details, see Seeley [1], [2] ; closely related results are due to Greiner [1], [2].

Of course, the general theory derives its interest from the examples, and the most interesting example is the Laplacian and related operators. In this case, the most complete results have been obtained by McKean and Singer [1], studying $e^{-z\Delta}$ for $z \to 0$, by constructing $\left(\frac{\partial}{\partial z} + \Delta\right)^{-1}$. Extending well-known results of Minakshisundaram and Pleijel, and of Kac, they find the following results (where C denotes a dimensional constant, possibly different in each occurrence) :

	$\gamma_j(x)$ or γ_j'(interior term)	$\delta_j(x')$ or $\delta_j'(x')$ (boundary term)
$z = -\frac{n}{2}$	$C\,\nu(x)$ (volume)	0
$z = \frac{1-n}{2}$	0	$C\,\nu'(x')$ ("volume" on ∂G)
$z = \frac{2-n}{2}$	$C\,K(x)$ (curvatura integra)	$CJ(x')$ (mean normal curvature)
$z = \frac{3-n}{2}$	0	?
$z = \frac{4-n}{2}$	(a polynomial in curvature)	?

This is for Δ acting on functions ; when G is a compact manifold without boundary, the boundary terms are of course absent.

A recent very difficult result has been obtained by Patodi [1]. Let Δ_p be the Laplacian acting on p-forms, and let $K_{z,p}(x, y)\,dy$ be the kernel of $(\Delta_p)^z$, Re $(z) < -n/2$. Then the analytic continuation of the alternating sum

$$\sum_{p=0}^{n} (-1)^p \text{ trace } K_{z,p}(x, x)\,dx$$

vanishes at $z = 0$, when n (= dim G) is odd ; but when n is *even*, and G is orientable, the sum is precisely the Chern polynomial $c(x)$. (When G has no boundary, $\int_G c(x)dx$ is the Euler characteristic ; it was already known that $\int_G \sum_{p=0}^{n} (-1)^p$ trace $K_{0,p}(x, x)\,dx$ is also the Euler characteristic, and this suggests the possibility that the integrands may be equal.)

III. **Pure imaginary powers.**

The motive for studying $\|A_B^{iy}\|$ is the Phragmen-Lindelof theorem : if $f(z)$ is analytic and bounded for z in the strip $S\,\{0 \leqslant \mathrm{Re}(z) \leqslant 1\}$, then the maximum principle applies ; $|f|$ is bounded by the supremum of $|f(iy)|$ and $|f(1+iy)|$. This carries over to Banach spaces, and leads to the simplest of all interpolation methods, known as the "complex method" (Calderon [1], Lions [2]). This will be our tool in studying the domain of A_B^θ, $0 < \theta < 1$.

It is easy to check that

$$\|O_p(c_{-\omega-j})\| = O(|\lambda|^{-1-j/\omega}) = \|O_p''(d_{-\omega-j})\|,$$

so

(5) $$\frac{i}{2\pi}\int_\Gamma \lambda^{iy}\, O_p(c_{-\omega-j})\,d\lambda$$

and

(6) $$\frac{i}{2\pi}\int \lambda^{iy}\, O_p''\,(d_{-\omega-j})\,d\lambda$$

are absolutely convergent for $j = 1, 2, \ldots$; the problem is reduced thus to studying $j = 0$. Since $c_{-\omega} = (\sigma_\omega(A) - \lambda)^{-1}$, (5) becomes $O_p(\sigma_\omega(A)^{iy})$ for $j = 0$, and this can be reduced to Mihlin's multiplier theorem, by standard expansions. The norm is $O(e^{\gamma|y|})$, where γ is any constant larger than sup $|\arg\lambda|$, λ ranging over the eigenvalues of $\sigma_\omega(A)$.

The term (6) with $j = 0$ is less significant, but more complicated. We use the estimate (4) and Mihlin's theorem in the *tangential* variables, which reduces the problem to an estimate of the operator

$$Tf(t) = \int_0^\infty f(s)\,\frac{ds}{s+t},$$

and this is essentially the Hilbert transform ! This method shows that the norm of (6) is $O(e^{\gamma|y|})$ like (5), where now γ is larger than

$$\sup\,\{|\arg\lambda| \text{ ; condition (2) above fails}\}.$$

Hence, finally

$$\|A_B^{iy}\| \leqslant C e^{\gamma |y|}, \tag{7}$$

where $\gamma > \sup \{|\arg \lambda| : \text{either (1) or (2) fails}\}$.

IV. The domains.

Fujiwara [1] has described the domains of fractional powers in special cases, suggesting the general case. The estimate (7) shows easily that the domain of A_B^θ is the complex interpolation space

$$[L^p(G)\ ,\ L^p_{B,\omega}(G)]_\theta .$$

The hard part is to figure out just what this space is. Fortunately, Grisvard [1] has published the result for $p = 2$ and $q = 1$ (A not a *system*). He uses "real" interpolation methods, but the results, and parts of the proof, work as well for complex interpolation and general p and q. Thus, denoting the order of B_j by ω_j, we find that

$$[L^p(G)\ ,\ L^p_{B,\omega}(G)]_\theta = \{u \text{ in } L^p_{\theta\omega}(G) : B_j u = 0 \text{ on } \partial G \text{ for } \omega_j < \theta\omega - 1/p\},$$

if $\theta\omega - 1/p$ is different from $\omega_1, \ldots, \omega_k$. In the exceptional case where $\theta\omega - 1/p = \omega_l$, we have to add to the restrictions on u the following :

$$\text{the function } v = \begin{cases} B_l u & \text{in } G \\ 0 & \text{outside } G \end{cases} \quad \text{is in } L^p_{1/p}(R^n).$$

This says that $B_l u$ vanishes on ∂G, in a certain weak sense. The space $L^p_s(R^n)$ is the "Bessel potential" space of Aronszajn and Smith, and Calderon [2] ; and $L^p_s(G)$ is its restriction to G. The boundary conditions B_j with $\omega_j < \theta\omega - 1/p$ (the ones retained in $[L^p(G), L^p_{B,\omega}(G)]_\theta$) are precisely those that can be defined in $L^p_{\theta\omega}$, according to the standard restriction theorems (e.g. Stein [1]).

This result requires that $B_1, \ldots, B_k$ form a *normal* system, as described above. Normality enters in two ways ; first, when $D_t^j u$ is defined on ∂G for $j < \omega_m$, and satisfies $B_l u = 0$ on ∂G for $l < m$, then normality makes it possible to define $D_t^{\omega_m} u$ so that $B_m u = 0$ on ∂G, as well. Second, when passing an exceptional value $\omega\theta - 1/p = \omega_m$, normality makes it possible to split u into two parts, one that will have to satisfy $B_m u = 0$ (in some sense), and another that is free of such restrictions.

REFERENCES

AGMON S. — On the eigenfunctions and on the eigenvalues of general elliptic boundary problems, *Comm. Pure Appl. Math.*, Vol. 15, 1962, p. 119-147.

CALDERON A.P. — 1) Lebesgue spaces of differentiable functions and distributions, *Proceedings of Symposia in Pure Math.*, Vol. 4, A.M.S., Providence.

2) Intermediate spaces and interpolation, the complex method, *Studia Math.*, Vol. 24, 1964, p. 113-190.

FUJIWARA D. — Concrete characterization of the domains of fractional powers..., *Proc. Japan Academy*, Vol. 43, 1967, p. 82-86.

GREINER P. — 1) Asymptotic expansions for the heat equation, *Summer Institute on Global Analysis*, Berkeley, 1968, p. S5 A1 - S5 A3.

2) *On zeta functions connected with elliptic differential operators*, University of Toronto preprint.

GRISWARD P. — Caractérisation de quelques espaces d'interpolation, *Arch. Rat. Mech. Anal.*, Vol. 25, 1967, p. 40-63.

LIONS J.L. — *C. R. Acad. Sci. Paris*, Vol. 251, 1960, p. 1853-1855.

Mc KEAN H. and SINGER I.M. — Curvature and the eigenvalues of the Laplacian, *Journ. Diff. Geom.*, Vol. 1, 1967, p. 43-70.

PATODI V.K. — *Curvature and the eigenforms of the Laplace operator*, preprint, Tata Institute, Bombay.

SEELEY R. — 1) The resolvent of an elliptic boundary problem, *Amer. Journ. Math.*, Vol. 91, 1969, p. 889-920.

2) Analytic extension of the trace associated with boundary problems, *Amer. Journ. Math.*, Vol. 91, 1969, p. 963-983.

3) *Norms and domains of the complex powers A_B^z*, to appear.

4) *Interpolation in L^p with boundary conditions*, to appear.

STEIN E.M. — The characterization of functions arising as potentials, II, *Amer. Math. Soc. Bull.*, Vol. 68, 1962, p. 577-582.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 803 à 811.

HAMILTONIAN FIELDS, BICHARACTERISTIC STRIPS IN RELATION WITH EXISTENCE AND REGULARITY OF SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

by F. TREVES

This is a brief report on some recent progress in the theory of linear partial differential equations with variable coefficients. The new results fit well in what can be regarded as the long-range program of the general PDE theory, namely : to translate in terms of the form of the differential equation (i.e., of its "symbol") all the significant properties of the solutions (existence, regularity, approximations, integral representations, etc.). Admittedly the progress has been essentially limited to the equations with simple real characteristics (that is, of principal type). But this is a good place where to begin. Progress of the same nature in problems involving multiple real characteristics will, very likely, have to wait the clarification in the simple characteristics case. Because of this, and also because of my own limited competence, I shall restrict myself to the latter.

We shall deal with a (linear partial) differential operator P of order $m > 0$, with C^∞ coefficients, in an open subset Ω of R^N (the coefficients are complex-valued) :

$$\mathbf{P} = P(x, D) = \sum_{|\alpha| \leqslant m} c_\alpha(x) D^\alpha \qquad (D = -\sqrt{-1}\ \partial/\partial x).$$

Let P_m denote the principal symbol of $\mathbf{P}$:

$$P_m(x, \xi) = \sum_{|\alpha| = m} c_\alpha(x)\ \xi^\alpha\ ;$$

it defines a complex-valued C^∞ function on the cotangent bundle $T^*(\Omega)$ over Ω. We recall that $\mathbf{P}$ is said to be of *principal type* if, at every point $x \in \Omega$, its *real characteristic cone*

$$C(P, x) = \{\xi \in R_N\ ;\ \xi \neq 0\ ,\ P_m(x, \xi) = 0\}$$

has no singular points or equivalently, if

(1) *for all* $x \in \Omega$, *all* $\xi \in R_N$, $\xi \neq 0$, *we have* $d_\xi P_m(x, \xi) \neq 0$

(the equivalence follows from Euler's homogeneity relation). Roughly speaking one may say that the differential operators of principal type are those operators whose significant properties are completely determined by their principal symbol. Among the classical types, *elliptic* and *hyperbolic* equations are of principal type ; *parabolic* are not.

We recall that the operator P is elliptic if, for every x, the cone $C(P, x)$ is empty. Now, elliptic equations $Pu = f$ are always locally solvable. Moreover, if the right hand side f is C^∞ so must be the solution u ; if the coefficients of P are analytic and if f is analytic, the same must be true of the solution. One might ask whether it is possible to extend some of these three properties *(local solvability, hypoellipticity, analytic hypoellipticity)* to some nonelliptic equations of principal type and, more ambitiously, whether it is possible to classify with respect to them –in a simple enough fashion– *all* PDEs of principal type. We hope to show in this lecture that this is not far-fetched.

The striking feature, in the current treatment of these and many other questions, is the relevance of the classical Hamilton-Jacobi theory of characteristics and bicharacteristics. Neither the reason for its role, nor the exact extent of it are yet well understood and we shall have to content ourselves with presenting the evidence, in the hope that further progress will yield a satisfactory "explantatiön".

In practically every problem in this area, much depends on whether one can find certain types of solutions w of the *characteristic equation*

$$P_m(x\,,\,\mathrm{grad}\ w(x)) = 0. \tag{2}$$

The integration of Eq. (2) calls into play the Hamilton-Jacobi notion of *bicharacteristic strips* (of P) : these are the curves $t \to (x(t)\,,\,\xi(t))$ in $T^*(\Omega)$ defined by the system of ordinary differential equations

$$\frac{dx}{dt} = \mathrm{grad}_\xi\ P_m(x\,,\,\xi)\ .\ \frac{d\xi}{dt} = -\ \mathrm{grad}_x\ P_m(x\,,\,\xi)\,. \tag{3}$$

We encounter here a difficulty : (3) defines x and ξ by complex equations, since –at least in general– the coefficients of P_m are not real. But then, if x is given a complex value, how do we substitute it in these coefficients ? It can be canonically done if the coefficients are analytic (and if x is near enough to the real space). But what if they are C^∞ ? In practice one usually relies on suitable approximations, analytic where it is needed, of P_m, w, $x(t)$, $\xi(t)$. Note also that, when they make sense, the bicharacteristic strips are the integral curves of the *Hamiltonian field* of P_m. The latter is the following vector field on $T^*(\Omega)$:

$$H_{P_m} = \sum_{j=1}^{N} \frac{\partial P_m}{\partial \xi_j}\frac{\partial}{\partial x^j} - \frac{\partial P_m}{\partial x^j}\frac{\partial}{\partial \xi_j}\,.$$

The Hamiltonian field always makes sense, even when its integral curves do not.

We must now pay attention to the complex structure : we must distinguish between the real and imaginary parts of P_m and w. Why this is necessary is not entirely clear. One senses the influence of the Paley-Wiener theorem. Indeed, we shall be interested in distributions solutions of the equation $Pu = f$, when the right-and side is itself a distribution or a function. If we reason locally we may assume that the supports of u and f are compact and introduce their Fourier transforms. By the Paley-Wiener theorem distributions are characterized among, say, analytic functionals, by the property that their Fourier transforms, when restricted to the *real* space, have a slow growth at infinity. If we succeed in expressing u in terms of f by an integral formula and want to prove that u, thus defined, is a distribution, the slow growth property must be preserved. This is best seen on the following example of a differential operator in R^2 :

$$\text{(4)} \qquad \mathbf{P} = D_1 + \sqrt{-1}\, b(x^1)\, D_2 \ , \ D_j = -\sqrt{-1}\ \partial/\partial x^j \ , j = 1, 2,$$

where $b(x^1)$ is a smooth real-valued function. Let us denote by $\hat{f}(x^1, \xi_2)$ the Fourier transform of f with respect to x^2. To obtain a solution of $\mathbf{P}u = f$ we may try the formula :

$$u(x) = \sqrt{-1}\,(2\pi)^{-1} \iint e^{iw(x,y^1,\xi_2)}\, h(x^1 - y^1, \xi_2)\, \hat{f}(y^1, \xi_2)\, dy^1\, d\xi_2 ,$$

with

$$w(x, y^1, \xi_2) = \xi_2 \left\{ x^2 - i \int_{y^1}^{x^1} b(t)\, dt \right\} , \quad i = \sqrt{-1},$$

and $(\partial/\partial t)\, h(t, \xi_2) = \delta(t)$, the Dirac measure.

The question, then, is whether we can choose h in such a way that the Fourier transform of u with respect to x^2 will be tempered. This is easy enough to do when $b(t)$ keeps the same sign (on a neighborhood of the support of $f(y^1, t)$). Suppose for instance that $b(t) \geqq 0$. We may take, in this case,

$$h(t, \xi_2) = 0 \quad \text{if} \quad t\xi_2 > 0, \quad h(t, \xi_2) = -1 \quad \text{if} \quad t\xi_2 \leqq 0 .$$

If now $b(t)$ changes sign at some point t_0, which means that $b(t)$ takes >0 and <0 values in every neighborhood of t_0, one can find, in the vicinity of t_0, a point t_1 such that the primitive $B(t)$ of $b(t)$ vanishing at t_1 has the following property : $B(t)$ keeps the same sign in some neighborhood of t_1 ; given any $r > 0$, $B(t)$ does not vanish identically on $)t_1, t_1 + r($ nor on $)t_1 - r, t_1($. Under these circumstances, it is easy to check that a choice of h such that $\hat{u}(x^1, \xi_2)$ is tempered with respect to ξ_2 is *not* possible. For a more systematic study of this question see [10]. Note that w is the solution of the characteristic equation (2) —when $\mathbf{P}$ is given by (4)— which satisfies :

$$w = \xi_2 x^2 \qquad \text{when} \qquad x^1 = y^1 .$$

The deciding factor in the viability of our integral representation of u has been the nature of the range of Im w as a function of x^1, that is, *along the bicharacteristic curve* of Re P_m : near every point $x^1 = y^1$, the range of Im w, viewed as a germ of set, is connected. The representation works when, for every y^1, either the range is an open interval, containing the origin, or else is reduced to $\{0\}$. It does not when, for some y^1, the range is a semi-open interval $(0, r($, $r > 0$. In terms of $b(x^1)$, the first case means that b does not change sign, the second case that it does.

Suitably interpreted, these considerations extend to the general case. Let $x_0 \in \Omega$, $\xi^0 \in R^N \backslash \{0\}$ be fixed arbitrarily. By virtue of (1) we have $(\partial/\partial \xi_j) P_m \neq 0$ at (x_0, ξ^0) for some j (depending on (x_0, ξ^0)). For simplicity let us assume $j = N$

and write $\xi' = (\xi_1, \ldots, \xi_{N-1})$. By the implicit functions theorem we obtain, in an open neighborhood $U \times \Gamma$ of (x_0, ξ^0),

$$P_m(x, \xi) = Q(x, \xi)(\xi_N - \tau(x, \xi')) \tag{5}$$

where Q and τ are C^∞ with respect to x and analytic with respect to ξ. We may suppose that Γ is an open cone, that Q and τ are homogeneous with respect to ξ, of degree $m - 1$ and 1 respectively, and that Q does not vanish anywhere in $U \times \Gamma$. We are interested in the solutions (or, possibly, in certain approximate solutions) of (2), satisfying

$$w(x_0) = 0, \qquad (\text{grad } w)(x_0) = \xi^0. \tag{6}$$

In view of the factorization (5) this is equivalent with solving

$$D_N w = \tau(x, D'w) \tag{7}$$

under Condition (6). Note that this makes sense even when w is not real, since τ is analytic in ξ' and (when w exists) $D'w$ remains close to the real vector ξ'^0 when x is close to x_0. Wishing to get some information about the range of w we do as suggested by the examples (4) : we look at the restriction of $\text{Im}\,\tau(x, \xi')$ to the bicharacteristic strips of $\xi_N - \text{Re}\,\tau(x, \xi')$ (in $U \times \Gamma$). It should be noticed that the function $\xi_N - \text{Re}\,\tau$ itself remains constant along each one of its bicharacteristic strips.

Let (y, η) be an arbitrary point in $U \times \Gamma$. If $P_m(y, \eta) \neq 0$, $P_m(x, \xi)$ is an *elliptic symbol* near (y, η) and no trouble should come from there. Let us therefore assume that $P_m(y, \eta) = 0$. Then, both $\xi_N - \text{Re}\,\tau$ and $\text{Im}\,\tau$ vanish at (y, η). The former vanishes identically on its bicharacteristic strip through (y, η) ; for this reason we refer to it as a *null bicharacteristic strip* of $\xi_N - \text{Re}\,\tau$. If we look at the restriction of $\text{Im}\,\tau$ to this strip, we distinguish the following two possibilities :

either (**case I**) $\text{Im}\,\tau$ *keeps the same sign in some small arc of curve centered at* (y, η),

or (**case II**), it does not, i.e., $\text{Im}\,\tau$ *takes* > 0 *and* < 0 *values arbitrarily near* (y, η) (always along the bicharacteristic strip of $\xi_N - \text{Re}\,\tau$).

Accessorily, may also consider

case III : $\text{Im}\,\tau$ *vanishes identically on a small arc containing* (y, η).

We may now state the properties which prove to be meaningful for our purpose :

($\mathfrak{P}$) *there is an open neighborhood* $\mathcal{N} \subset U \times \Gamma$ *of* (x_0, ξ^0) *at every point* (y, η) *of which, either* $P_m \neq 0$ *or else Case* I *is realized* ;

($\mathfrak{Q}$) *there is an open neighborhood* $\mathcal{N}$ *of* (x_0, ξ^0) *at no point of which Case* III *is realized.*

Remarkably, both ($\mathfrak{P}$) and ($\mathfrak{Q}$) are independent of the factorization (5). As a matter of fact, they remain unchanged if we multiply P_m by any complex C^∞ function, nowhere vanishing in a neighborhood of (x_0, ξ^0) (see [9], Appendix). In particular, in order to state the **Cases I, II, III** above, we dont have to rely on a factorization such as (5) : it suffices to study the changes of sign (or the zeros), if any, of $\mathrm{Im}\,(zP_m)$ along the null bicharacteristic strips of $\mathrm{Re}\,(zP_m)$ – after having chosen for z any complex number such taht $d_\xi \,\mathrm{Re}\,(zP_m) \neq 0$ in a neighborhood of (x_0, ξ^0).

Conjectures

We now relate Properties $(\mathcal{P})$ and $(\mathcal{Q})$ to certain nonformal properties of the differential operator **P**. We introduce a few definitions concerning the equation :

$$\mathbf{P}u = f. \tag{8}$$

DEFINITION 1. – *The equation* (1) *is said to be locally solvable at the point* x_0 *if there is an open neighborhood* ω *of* x_0 *in* Ω *such that, to every* $f \in C_c^\infty(\omega)$ *there is a distribution* $u \in \mathcal{D}'(\omega)$ *satisfying* (8) *in* ω.

DEFINITION 2. – *We shall say that the equation* (1) *is hypoelliptic (resp. analytic-hypoelliptic) at the point* x_0 *if there is an open neighborhood* $\omega \subset \Omega$ *of* x_0 *in which the differential operator* **P** *is hypoelliptic (resp. analytic-hypoelliptic).*

We recall that **P** is said to be hypoelliptic (resp. analytic-hypoelliptic) in ω if, given any open subset U of ω and any distribution u in U, $\mathbf{P}u \in C^\infty(U) \Rightarrow u \in C^\infty(U)$ (resp. $\mathbf{P}u \in C^\infty(U) \Rightarrow u \in C^\infty(U)$ and, moreover, if $\mathbf{P}u$ is analytic in U, this is also true of u).

We may now formulate the main conjectures :

CONJECTURE 1 : *The equation* (8) *is locally solvable at* x_0 *if and only if Property* $(\mathcal{P})$ *holds whatever* $\xi^0 \in R_N \backslash \{0\}$.

(This conjecture was first made in [7]).

CONJECTURE 2 : *Eq.* (8) *is hypoelliptic at* x_0 *if and only if both Properties* $(\mathcal{P})$ *and* $(\mathcal{Q})$ *hold whatever* $\xi^0 \in R_N \backslash \{0\}$.

CONJECTURE 3 : *When the coefficients of* **P** *are analytic, Eq.* (8) *is analytic-hypoelliptic at* x_0 *if and only if* $(\mathcal{P})$ *and* $(\mathcal{Q})$ *hold for all* $\xi^0 \in R_N \backslash \{0\}$.

Let us repeat that **P** is assumed to be of principal type. Otherwise these conjectures would be absurd ! Conjecture 3, combined with Conjecture 2, means that every hypoelliptic differential operator of principal type is analytic-hypoelliptic if its coefficients are analytic.

Results

At the present time (and to my knowledge) only *Conjecture* 3 *has been completely proved* (in [13], [15]). The results obtained so far, concerning the first two conjectures, although incomplete, leave little room for doubting their validity. We begin by the local solvability.

THEOREM 1. – *Suppose that the differential operator* **P** *(of principal type) satisfies at least one of the following three conditions :*

(a) **P** *is a first-order operator* ;

(b) *the coefficients of* P_m *are analytic* ;

(c) **P** *is a differential operator in two independent variables.*

In any of these three cases, if Property ($\mathcal{P}$) *holds for all* $\xi^0 \in R_N \backslash \{0\}$, *the equation* $\mathbf{P}u = f$ *is locally solvable at* $x_0 \in \Omega$.

The sufficiency of ($\mathcal{P}$) in the case (a) has been proved in [7] ; in the case (b), in [9] ; in the case (c), in [11]. In all these cases, one succeeds in proving the existence of local solutions with optimal regularity properties. More precisely, to every real number s there is an open neighborhood ω_s of x_0 such that

(9) *to every* $f \in H^s(\omega_s)$ *there is* $u \in H^{s+m-1}(\omega_s)$ *satisfying* (8) *in* ω_s.

(in the first order case, (a), the proof of this fact can be found in [12]). That (9) is optimal, in so far as smoothness of the solution is concerned, is clear when one looks at *hyperbolic* equations.

The implication in the opposite direction, namely that *if* ($\mathcal{P}$) *is not true then Eq.* (8) *is not locally solvable at* x_0, has been proved for general operators of principal type, under an additional assumption : *the zero,* (y, η), *at which the change of signe of* $\mathrm{Im}\,(zP_m)$ *occurs, along the null bicharacteristic strip of* $\mathrm{Re}\,(zP_m)$ *through* (y, η) (we assume $d_\xi \,\mathrm{Re}\,(zP_m)\,(y, \eta) \neq 0$), *is supposed to be of finite* (hence *odd*) *order.* For the proof, see [8]. The most general and precise results to date in this direction, the necessity of Property ($\mathcal{P}$) for local solvability, have been obtained by Yu. V. Egorov (see [2]). His results apply to a wide class of pseudodifferential operators, not all of which need be of principal type. For pseudodifferential operators one must adapt the statement of Property ($\mathcal{P}$) : it is not simply a change of signe of $\mathrm{Im}\,(zP_m)$ that one must assume (in order to show that (8) is not locally solvable) but a change of sign from *minus* to *plus* (see [8] ; note that the bicharacteristic strips are *oriented* curves) Egorov's results are in fact related to Conjecture 2. But before going into this question, we should point out that, when the coefficients of P_m are analytic, the changes of sign of $\mathrm{Im}(zP_m)$ must perforce occur at zeros of finite odd order. Thus :

THEOREM 2. — *Let* **P** *be a differential operator of principal type in* Ω, *of order m, whose principal part* $P_m(x, D)$ *has analytic coefficients. Then, the equation* $\mathbf{P}u = f$ *is locally solvable at* x_0 *if and only if Property* ($\mathcal{P}$) *holds for all* $\xi^0 \in R_N \backslash \{0\}$.

Let us also mention, for the record, that the necessity of ($\mathcal{P}$) has been proved in a fairly large number of cases, when the order of **P** is *one, even when the above finiteness assumption is not satisfied.*

We come now to Conjecture 2. Let us go back for a moment to the operators (4) and, more precisely, to the integral representation of $u(x)$. It can be rewritten

$$u(x) = \int K(x, y)\, f(y)\, dy,$$

setting

$$K(x, y) = (2\pi i)^{-1}\, pv \left\{ \int_{y^1}^{x^1} b(t)\, dt + i(x^2 - y^2) \right\}^{-1}$$

(*pv* stands for *principal value*). Suppose now that ($\mathcal{P}$) and ($\mathcal{Q}$) both hold. Here it means that (1) $b(t)$ does not change sign, (2) $b(t)$ does not vanish on any (nonempty) open interval. But then it is evident that the kernel-distribution $K(x, y)$ is a C^∞ function of (x, y) outside the diagonal ; it is also easy to check that $f \to \int K(.,y) f(y)\, dy$ and $g \to \int K(x,.)\, g(x)\, dx$ both map continuously C^∞ into C^∞. According to a classical theorem of Laurent Schwartz, this implies that P is hypoelliptic. Moreover, when $b(t)$ is analytic, $K(x, y)$ is an analytic function of (x, y) outside the diagonal, and this implies that **P** is analytic-hypoelliptic. All this is in complete agreement with Conjectures 2 and 3. Notice also that the assumption 2) above means, when $b(t)$ is analytic, that its zeros have finite order. In extending the preceding results to differential operators of principal type, in any number N of variables, of any order m (with C^∞ coefficients), we shall adapt this assumption, We consider the property :

($\mathcal{R}$) *There is an open neighborhood $\mathfrak{N}$ of (x_0, ξ^0) in $T^*(\Omega)$ and an integer $k \geqq 0$ such that the following is true : given any $(y, \eta) \in \mathfrak{N}$, any complex number z such that*

$$P_m(y, \eta) = 0 \ , \ d_\xi \operatorname{Re}(zP_m)(y, \eta) \neq 0 , \tag{10}$$

the restriction of $\operatorname{Im}(zP_m)$ *to the bicharacteristic strip of* $\operatorname{Re}(zP_m)$ *through (y, η) has a zero of order $\leqq k$ at this point.*

When the coefficients of P_m are analytic, Properties ($\mathcal{Q}$) and ($\mathcal{R}$) are equivalent. In fact, we have :

THEOREM 3. – *Let* **P** *be a differential operator of principal type in Ω, of order m, with analytic coefficients. The following properties are equivalent* :

(i) **P** *is hypoelliptic at* x_0 ;

(ii) **P** *is analytic-hypoelliptic at* x_0 ;

(iii) *Properties ($\mathcal{P}$) and ($\mathcal{Q}$) both hold, for all* $\xi^0 \in R_N \backslash \{0\}$.

The proof of Th. 3 can be found in [13] and [15].

Let us now look at the case of C^∞ coefficients. When the neighborhood $\mathfrak{N}$ in Property ($\mathcal{R}$) shrinks, the *smallest* integer k which is admissible decreases and, after a while, stays constant, at a value $k(x_0, \xi^0)$. If Property ($\mathcal{P}$) holds, $k(x_0, \xi^0)$ must be *even.* Let us denote by $k(x_0)$ the largest of the numbers $k(x_0, \xi^0)$ as ξ^0 ranges over $R_N \backslash \{0\}$ (or, equivalently, over the unit sphere of R_N). The following is a particular case of Egorov's results (see [3] ; also, [14]) :

THEOREM 4. – *Let P be a differential operator of principal type in Ω, of order m, with C^∞ coefficients. The following properties are equivalent* :

(i) *Properties ($\mathcal{P}$) and ($\mathcal{R}$) hold for every* $\xi^0 \in R_N \backslash \{0\}$;

(ii) *Property ($\mathcal{R}$) holds and P is hypoelliptic at* x_0 ;

(iii) *there is a number $\delta > 0$ such that, to every s real there is an* open *neighborhood ω_s of x_0 and a constant $C_s > 0$ such that* :

(11) *for all* $\varphi \in C_c^\infty(\omega_s)$, $\quad \|\varphi\|_{s+m-1+\delta} \leqq C_s \ \|\mathbf{P}\varphi\|_s$.

Furthermore, if these properties hold and if the neighborhood ω_s *in* (iii) *are sufficiently small, the largest admissible number* δ *is equal to* $(k(x_0) + 1)^{-1}$.

We have denoted by $\| \ \|_s$ the norm in the Sobolev space H^s. Estimates of the kind (11) are called *subelliptic*. They are easily seen to imply the hypoellipticity of **P** at x_0. By a straightforward application of the closed graph theorem one sees that if a differential operator **P**, be it of principal type or not, is hypoelliptic, its formal transpose ${}^t\mathbf{P}$ is locally solvable (say at a point x_0). Now, if $(\mathcal{R})$ were to hold but not $(\mathcal{P})$, $\operatorname{Im}(zP_m)$ would have a zero of (finite odd) order at (y, η) along the bicharacteristic strip of $\operatorname{Re}(zP_m)$ through that point (for a suitable choice of z and (y, η) such that (10) holds). The necessary conditions of solvability stated above, which do not distinguish between **P** and ${}^t\mathbf{P}$, would then imply that ${}^t\mathbf{P}$ is not locally solvable at (y, η) – hence **P** could not be hypoelliptic at the point y arbitrarily close to x_0, hence at x_0. This proves the implication (ii) ⇒ (i) in Th. 4.

There is an alternate statement of Property $(\mathcal{R})$ which makes use of the Hamiltonian fields instead of their integral curves, the bicharacteristic strips. Let us call H_{P_m} (resp. $H_{\overline{P}_m}$) the Hamiltonian field defined by P_m (resp. by the complex conjugate $\overline{P}_m$). We may then form a *Poisson bracket*

$$H_{P_m}^{\alpha_1} H_{\overline{P}_m}^{\overline{\alpha}_1} \dots H_{P_m}^{\alpha_r} H_{\overline{P}_m}^{\overline{\alpha}_1} P_m(x, \xi) \tag{12}$$

where the α_j and the α_j's are nonnegative integers. Property $(\mathcal{R})$ is equivalent with stating that *for each* (y, η) *in the neighborhood* $\mathcal{N}$ *of* (x_0, ξ^0), *there is a Poisson bracket* (12) *with length* $\alpha_1 + \overline{\alpha}_1 + \dots + \alpha_r + \overline{\alpha}_r \leqslant k$ *which does vanish at* (y, η). The conjunction of $(\mathcal{P})$ and $(\mathcal{R})$ asserts that *if we choose suitably the neighborhood* $\mathcal{N}$, *the brackets* (12) *of smallest length which do not vanish at* (y, η) *have even length.* For a proof of this equivalence, see [8].

When one lifts the restriction $(\mathcal{R})$ little is known about Conjecture 2. Let us only mention that it is not difficult to prove its validity in the first order case – provided that we assume the equation under study satisfies $(\mathcal{P})$ for all $\xi^0 \neq 0$.

We conclude with two examples of operators in $\mathbf{R}^3$:

Example 1. – $L = D_1 + \sqrt{-1}\,(D_2 + x^1 D_3)$ is not locally solvable at any point of $\mathbf{R}^3$ ($\sqrt{-1}\,L$ is equal to the Lewy's operator modulo a coordinates change).

Example 2. – $\mathbf{P} = D_1^2 + D_2^2 - D_3^2 + \sqrt{-1}\,(D_2 - x^2 D_3)^2$ is analytic-hypoelliptic in $\mathbf{R}^3$ but is nowhere elliptic.

Bicharacteristic strips and Hamiltonian fields play a crucial role also in global problems : for instance, in the propagation of singularities and the related (global) existence theorems (see [1], [4], [5], [6]). At the moment the results apply only to special classes of differential operators of principal type, e.g., those with *real* principal symbol, but it is very likely that they will be generalized in the not too distant future.

Since this lecture was delivered, Conjecture $\mathcal{Q}$ was almost completely proved by the author (see [16]). Namely it is now proved that the conjunction of $(\mathcal{P})$ and $(\mathcal{Q})$ implies the hypoellipticity of **P** and that if **P** is hypoelliptic $(\mathcal{Q})$ must hold. Concerning the necessity of $(\mathcal{P})$, cf. remarks above.

REFERENCES

[1] ANDERSSON C.G. — *Propagation of analyticity of solutions of partial differential equations with constant coefficients,* to appear in Arkiv for Mat.

[2] EGOROV Yu. V. — *Soviet Math.*, Vol. 10, No. 4, 1969, p. 1020-1022.

[3] EGOROV Yu. V. — On subelliptic pseudodifferential operators, *Doklay Akad. Nauk. U.S.S.R.*, Vol. 188, 1969, p. 20-22.

[4] GRUSHIN V.V. — The extension of smoothness of solutions of differential equations of principal type, *Doklady Akad. Nauk. U.S.S.R.*, Vol. 148, 1963, p. 1241-1244.

[5] HÖRMANDER L. — *On the singularities of solutions of partial differential equations,* to appear.

[6] HÖRMANDER L. — *Fourier Integral Operators,* Part II, to appear.

[7] NIRENBERG L. and TREVES F. — *Solvability of a First Order Linear Partial* Differential Equation, *Comm. Pure Applied Math.*, Vol. XVI, No. 3, 1963, p. 331-351.

[8] NIRENBERG L. and TREVES F. — On local solvability of linear partial differential equations — Part I: Necessary conditions, *Com. Pure Applied Math.*, Vol. XXIII, No. 1, 1970, p. 1-38.

[9] NIRENBERG L. and TREVES F. — On local solvability of linear partial differential equations — Part II: Sufficient conditions, *Comm. Pure Applied Math.*, Vol. XXIII, No. ?, 1970, p. ??-??.

[9*a*] NIRENBERG L. and TREVES F. — A correction to « On local solvability — Part II », *Comm. Pure Applied Math.*, to appear.

[10] NIRENBERG L. and TREVES F. — Remarks on the solvability of linear equations of evolution, *Symposium on Evolution Equations,* Rome (Italy), May 1970.

[11] TREVES F. — On the local solvability of linear partial differential equations in two independent variables, *Amer. J. Math.*

[12] TREVES F. — Local solvability in L^2 of first order linear PDEs, *Amer. J. Math.*

[13] TREVES F. — Hypoelliptic PDEs of principal type with analytic coefficients, *Comm. Pure Applied Math.*

[14] TREVES F. — A new method of proof of the subelliptic estimates, *Comm. Pure Applied Math.*

[15] TREVES F. — *Analytic-hypoelliptic PDEs of principal type,* to appear in Comm. Pure Applied Math.

[16] TREVES F. — *Hypoelliptic PDEs of principal type. Necessary conditions and sufficient conditions.* To appear in Comm. Pure Applied Math.

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D 11 - ANALYSE FONCTIONNELLE ET ÉQUATIONS AUX DÉRIVÉES PARTIELLES NON LINÉAIRES

GEOMETRIC MEASURE THEORY AND ELLIPTIC VARIATIONAL PROBLEMS

by F. J. ALMGREN, JR.

For the past thirty or forty years mathematicians have increasingly been attracted to problems in the calculus of variations in higher dimensions and codimensions. However, prior to 1960 (with one or two notable exceptions) there was relatively little fundamental progress in the calculus of variations in higher dimensions and codimensions and essentially no progress on the so called parametric problems (those which are of an essentially geometric character). Beginning about ten years ago, however, (in particular with the work of De Giorgi, Federer, Fleming, and Reifenberg) new ideas began to be introduced into the subject with surprising success in these higher dimensions and codimensions. Indeed, in these higher dimensions and codimensions, the calculus of variations seems to have passed from a classical period in its development into a modern era. Many of these new methods and ideas are included in the collection of mathematical results known as *geometric measure theory* (see, in particular, the treatise [$F1$]). This article is intended as a very brief discussion of several problems in the calculus of variations as an indication of the perspective from which these problems are now being studied.

The parametric boundary value problem.

Parametric boundary value problems arise in the following way : Suppose k and n are positive integers with $k \leqq n$ and one is given a reasonably nice function $F : R^n \times G_k^n \to R^+$ where G_k^n denotes the Grassmann manifold of all unoriented k plane directions in R^n (which can be regarded as the space of all unoriented k dimensional planes through the origin in R^n). If S is a reasonably nice surface of dimension k in R^n, one defines the integral $\mathbf{F}(S)$ of F over S by setting $\mathbf{F}(S) = \int_{x \in S} F(x\,,\, \mathbf{S}(x))\, d\mathbf{H}_k(x)$ where $\mathbf{S}(x)$ denotes the tangent k plane direction to S at x and $\mathbf{H}_k$ denotes k dimensional Hausdorff measure on R^n. Hausdorff k dimensional measure gives a precise meaning to the notion of k dimensional area in R^n and is the basic measure used in defining a theory of integration over k

dimensional surfaces in R^n which may have singularities. The Hausdorff k dimensional measure of a smooth k dimensional submanifold of R^n agrees with any other reasonable definition of the k area of such a manifold. With this terminology, the problem can be stated :

PROBLEM. – *Among all k dimensional surfaces S in R^n having a prescribed boundary, is there one minimizing* $\mathrm{F}(S)$? *And, if there is, how nice is it* ?

To make this problem precise, there are, of course, several questions to be answered : (1) What is a surface ?, (2) What is the boundary of a surface ?, and (3) What are reasonable conditions to put on F ? To see what is involved in answering these questions, one needs to study the phenomena which arise. Indeed, even for the case $F \equiv 1$ (i.e. the problem of minimizing k dimensional area–often called Plateau's problem), examples [$A3$] show : (1) In order to solve the problem of least area–and really achieve the least area–one sometimes has to admit surfaces of infinite topological type into competition (even for two dimensional surfaces whose boundaries are piecewise smooth simple closed curves) ; (2) Complex algebraic varieties are surfaces of least oriented area, so that, in particular, at least all the singularities of complex algebraic varieties occur in solutions to Plateau's problem ; (3) In some cases there are topological obstructions to surfaces of least area being free of singularities ; (4) Sometimes surfaces of least area do not span their boundaries in the sense of algebraic topology ; and (5) The realization of certain soap films as mathematical "minimal surfaces" requires that the boundary curves have positive thickness.

One approach to the study of variational problems in the generality suggested by the phenomena which arise is based on a correspondence between suitable surfaces and measures on appropriate spaces. Indeed, the natural setting for parametric (1) problems in the calculus of variations seems to be that in which surfaces are regarded as intrinsically part of R^n (in particular as measures on spaces associated with R^n) rather than that in which surfaces are regarded as mappings from a fixed k dimensional manifold, even though with this approach one is not able to use the traditional methods of functional analysis for showing the existence of solutions. The principal reasons for formulating the problem this way are indicated in [$A3$]. The most important measure theoretic surfaces are indicated in the following :

(1) *Rectifiable sets.* – A set $S \subset R^n$ is k rectifiable if and only if $\mathbf{H}_k(S) < \infty$ and $\mathbf{H}_k([S \sim f(A)] \cup [f(A) \sim S]) = 0$ for some measurable set $A \subset R^k$ and some Lipschitzian function $f : R^k \to R^n$.

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(1) Traditionally one has considered surfaces as mappings f from a fixed k dimensional manifold M into R^n and attempted to minimize the integral of a suitable integrand. If the integrand depends only on $x \in M$, $f(x)$, and $Df(x)$ and if the integral of the integrand is independent of the parametrization of M (as is the case for the area integrand, but is not the case for most "energy" integrands), the variational problem is said to be in *parametric form.* Problems in parametric form are precisely those problems for which the necessary integration can be performed over the image $f(M)$ in R^n.

(2) *Variation measures.* – If S is k rectifiable, the variation measure $\|S\|$ associated with S is given by the formula $\|S\| = \mathbf{H}_k \cap S$ (i.e. $\|S\|(A) = \mathbf{H}_k(S \cap A)$ for $A \subset R^n$). $\|S\|$, of course, determines S $\mathbf{H}_k$ almost uniquely, but it is difficult to evaluate $\mathbf{F}(S)$ from S alone.

(3) *Integral varifolds.* – If S is k rectifiable, the integral varifold $|S|$ is given by the formula $|S| = \varphi_\#(\|S\|)$, where $\varphi : R^n \to R^n \times G_k^n$, $\varphi(x) = (x , \mathbf{S}(x))$ for $\|S\|$ almost all x in R^n. Note that $\mathbf{F}(S) = \int F \, d\,|S|$.

(4) *Integral currents.* – If S is an oriented k rectifiable subset of R^n, the current (continuous linear functional on differential k forms) associated with S is given by integration of k forms over S. If ∂S (defined by exterior differentiation of $k - 1$ forms) is $k - 1$ rectifiable, then S is an integral current.

(5) *Flat chains modulo ν.* – The flat chains modulo ν have the same relationship to singular chains with coefficients in the integers modulo ν that the integral currents have to singular chains with coefficients in the integers.

In general the procedure for finding an $\mathbf{F}$ minimal surface is the following : First take a minimizing sequence $S_1, S_2, S_3, \ldots$ of surfaces having a prescribed boundary so that $\mathbf{F}(S_i)$ approaches its minimum value. Second, associate with each surface a measure, obtaining, say, $|S_1|, |S_2|, |S_3|, \ldots$. Third, choose a convergent subsequence of the measures which converges to a limit measure V (this is easy to do because spaces of measures have strong compactness properties in the weak topology). Finally, show that V corresponds to a nice surface, say $V = |S|$ (this is where the real work comes). S is the desired $\mathbf{F}$ minimal surface.

As the variety of different measure theoretic surfaces suggests, there are a number of different ways in which the problem of finding an $\mathbf{F}$ minimal surface having a prescribed boundary can be formulated. The following formulation is an especially fundamental one.

DEFINITIONS. –

(a) *A surface* S is a compact k rectifiable subset of R^n.

(b) *A boundary* B is a compact $k - 1$ rectifiable subset of R^n.

(c) $H_{k-1}(B\,;G)$ denotes the $k - 1$ dimensional Vietoris homology group of B with coefficients in a finitely generated abelian group G. For $\sigma \in H_{k-1}(B\,;G)$ (intuitively σ is a hole in B), we say that S *spans* σ if and only if $i_*(\sigma) = 0$ where $i_* : H_{k-1}(B\,;G) \to H_{k-1}(B \cup S\,;G)$ is induced by the inclusion $i : B \to B \cup S$.

(d) An integrand $F : R^n \times G_k^n \to R^+$ is called *elliptic with respect to* G if and only if there is a continuous positive function $c : R^n \to R^+$ such that for each $x \in R^n$, each k disk D in R^n, and each surface S which spans some nonzero σ in $H_{k-1}(\partial D\,;G)$, $\mathbf{F}^x(S) - \mathbf{F}^x(D) \geq c(x)\,[\mathbf{H}_k(S) - \mathbf{H}_k(D)]$ where for $y \in R^n$, $\pi \in G_k^n$, $F^x(y , \pi) = F(x , \pi)$. If the codimension $n - k$ equals 1, the ellipticity of F with respect to any G is equivalent to the uniform convexity of each F^x. The set of elliptic integrands contains a (computable) convex neighborhood of the k area integrand $F \equiv 1$ in the $C^{(2)}$ topology. Also, if $f : R^n \to R^n$ is a diffeomorphism, then $f_\# F$ is elliptic if and only if F is. This fact extends the results of the following theorem from R^n to compact n dimensional Riemannian manifolds without boundary of class $j + 1$. Finally, the ellipticity of F implies that the various Euler equations which arise are strongly elliptic systems of partial differential equations, and "in the small" the ellipticity of F is equivalent to the ellipticity of these systems.

THEOREM [$A2$]. – *Let B be a boundary, G be a finitely generated abelian group, and $\sigma \in H_{k-1}(B\,; G)$. Suppose $F : R^n \times G_k^n \to R^+$ is an integrand of class $j \geqq 3$ which is elliptic with respect to G and which is bounded away from 0. Then there exists a surface S such that S spans σ, $\mathbf{F}(S) \leqq \mathbf{F}(T)$ whenever T is a surface which spans σ, and, except possibly for a compact singular set of zero $\mathbf{H}_k$ measure, S is a k dimensional submanifold of R^n of class $j - 1$.*

A second class of variational problems.

So far we have been concerned with boundary value problems for a wide class of integrands. A second class of problems relates to the study of a particular integrand –the area integrand– under conditions where the surfaces in question are subject to distortions, constraints, or other influences as well as being permitted to have arbitrary topological type and essential singularities. Solutions to these problems usually are not minimal surfaces, nor surfaces minimal for any integrand. To be precise, we need some more terminology.

By a *varifold* we mean a Radon measure on $R^n \times G_k^n$. By the *area* $\mathbf{W}(V)$ of a varifold V we mean $V(R^n \times G_k^n)$. Each smooth diffeomorphism $f : R^n \to R^n$ induces a map $f_\#$ of varifolds in a natural way characterized by requiring that $f_\#|S| = |f(S)|$ whenever S is k rectifiable. By the *first variation* δV of a varifold V we mean the distribution on R^n of type R^n and order 1 given by $\delta V(g) = (d/dt)\mathbf{W}(G_{t\#}V)|_{t=0}$ (first variation of area) for each vectorfield $g : R^n \to R^n$ where $G_t : R^n \to R^n$ is the deformation given by $G_t(x) = x + tg(x)$ for $x \in R^n$. For example, if M is a smooth submanifold of R^n, then $\delta\,|M|(g) = -\int_M g \cdot \vec{mc}\, d\mathbf{H}_k + \int_{\partial M} g \cdot \vec{n}\, d\mathbf{H}_{k-1}$ where $\vec{mc}$ is the mean curvature vectorfield on M, and $\vec{n}$ is the exterior normal vectorfield on ∂M tangent to M. One says that the first variation δV *is a measure* if and only if δV is of order 0, i.e. there is a covector valued measure μ on R^n such that $\delta V(g) = \int g\, d\mu$ for each vectorfield g. Finally, by the *k density* $\Theta^k(\|V\|, p)$ of a varifold V at a point $p \in R^n$ we mean

$$\lim_{r \to 0+} r^{-k}\, V(\{x : |x - p| < r\} \times G_k^n).$$

A variety of physical and biological phenomena have mathematical representation by varifolds whose first variation distributions are measures ; for example, spider webs (here the tension in each strand corresponds to the density of the corresponding varifold), soap bubbles as well as soap films, liquid-liquid interfaces in equilibrium, and partitioning surfaces of least weighted area (such as occur as interfaces in the stable states of free living cells). These phenomena remain representable by varifolds whose first variations are measures when subject to gravitational fields, wind pressures, etc. As a representative simple mathematical example we have the following :

PARTITIONING PROBLEM. – *Let $m_i > 0$ for $i = 1, 2, \ldots, j$. Among all disjointed regions $A_1, A_2, \ldots, A_j$ in R^n such that $L_n(A_i) \geq m_i$ for each i, are there regions for which $H_{n-1}(\cup_i \partial A_i)$ attains a minimum value ? (if so one can verify that $\delta |\cup_i \partial A_i|$ is a measure).*

The varifold setting seems both a natural and a powerful way to study a variety of geometric and variational problems, including those suggested by the physical and biological phenomena above. The following results are suggestive of the present state of the theory.

THEOREM. – *If V is a varifold and δV is a measure, then $\{p : \Theta^k(\|V\|, p) > \epsilon\}$ is a k rectifiable set for each $\epsilon > 0$.*

ISOPERIMETRIC INEQUALITY. – *If V is a varifold, $W(V) < \infty$ and δV is a measure, then $W(V)^{(k-1)/k} \inf \{\Theta^k(\|V\|, p) : p \in \text{support } \|V\|\}^{1/k} \leq c \, \|\delta V\|$. For example, if M is a minimal surface, $H_k(M) \leq c_1 \, H_{k-1}(\partial M)^{k/(k-1)}$. Here c and c_1 are constants depending only on n.*

By an *integral varifold* one means a varifold which can be represented $\Sigma_i \, |S_i|$ corresponding to rectifiable sets $\{S_i\}_i$.

THEOREM. – *The space of integral varifolds with locally uniformly bounded areas and first variations which are measures is compact in the weak topology.*

THEOREM. – *On each compact n dimensional Riemannian manifold without boundary there exists at least one nonzero k dimensional integral varifold V with $\delta V = 0$. V is thus a k dimensional "minimal surface". (The proof is by Morse Theory methods).*

REGULARITY THEOREM [*AL*]. – *If S is a k rectifiable set and $\delta |S|$ is integrable to the $k + \epsilon$ power, then, except possibly for a compact singular set of zero H_k measure, S is a smooth k dimensional submanifold of R^n with first derivatives which are locally Holder continuous with exponent $\epsilon/(k + \epsilon)$.*

SOLUTION TO THE PARTITIONING PROBLEM. – *The theory of integral currents guarantees a solution to the partitioning problem such that $\cup_i \partial A_i$ is $n - 1$ rectifiable and $\delta |\cup_i \partial A_i|$ is bounded. The regularity theorem implies that H_{n-1} almost everywhere $\cup_i \partial A_i$ is a smooth submanifold. The regular part of $\cup_i \partial A_i$ has locally constant mean curvatures, hence is an analytic manifold.*

Estimates on singular sets.

Very little is known at the present time about the structure of the singular sets of solutions to general elliptic variational problems (except for their existence). However, for the area integrand there has been substantial progress. For example, we have the following two representative theorems which generalize immediately to manifolds.

THEOREM [*F*2]. – *For every unoriented boundary $B \subset R^n$ of dimension $k - 1$ (any k), there exists an unoriented minimal surface (flat chain modulo 2) S with $\partial S = B$ of least k dimensional area. The interior singular set of S has Hausdorff dimension at most $k - 2$. The regular part of S is a real analytic submanifold of R^n.*

THEOREM [$F2$]. – *For every oriented boundary $B \subset R^n$ of dimension $n - 2$ there exists an oriented minimal surface (integral current) S with $\partial S = B$ of least $n - 1$ dimensional area (counting multiplicities). The interior singular set of S has Hausdorff dimension at most $n - 8$. In particular, there are no interior singularities if $n \leqq 7$. The regular part of S is a real analytic submanifold of R^n.*

Examples show that both of these results are the best possible (at least in terms of Hausdorff dimension).

EXAMPLE. – *The unoriented 2 dimensional surface $S = \{x : x_3 = x_4 = 0$ and $x_1^2 + x_2^2 \leqq 1\} \cup \{x : x_1 = x_2 = 0$ and $x_3^2 + x_4^2 \leqq 1\} \subset R^4$ is of least area among all unoriented 2 dimensional surfaces having boundary $B = \{x : x_3 = x_4 = 0$ and $x_1^2 + x_2^2 = 1\} \cup \{x : x_1 = x_2 = 0$ and $x_3^2 + x_4^2 = 1\}$. The origin 0 is the singular set of S.*

EXAMPLE [BDG]. – *Let S be the 7 dimensional oriented cone $0(S^3 \times S^3)$ over $S^3 \times S^3 \subset R^4 \times R^4 = R^8$. Then S has less 7 dimensional area than any other oriented hypersurface T in R^8 with $\partial T = S^3 \times S^3$. The origin 0 is the singular set of S.*

The results above for oriented minimal hypersurfaces are intimately connected with the possibility of extending Bernstein's theorem (that a globally defined nonparametric minimal hypersurface must be a hyperplane) to higher dimensions. We have the following.

THEOREM [$F1$ 5.4.18] [BDG]. – *If $n = 2, 3, \ldots, 7$ and if the graph of $f : R^n \to R$ is a minimal hypersurface in R^{n+1}, then the graph of f is a hyperplane. On the other hand, for each $n \geqq 8$ there exist functions $g : R^n \to R$ with graphs which are minimal hypersurfaces but which are not hyperplanes.*

REFERENCES

For a complete set of references to geometric measure theory one should consult the Bibliography of [F1].

[A2] ALLARD W.K. — *The theory of varifolds* (preprint), Princeton University, 1971.

[A1] ALMGREN F.J. Jr. — *The theory of varifolds. A variational calculus in the large for the* k *dimensional area integrand* (multilithed notes), Princeton, 1965.

[A2] ALMGREN F.J. Jr. — Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure, *Ann. of Math.*, (2), 87, 1968, p. 321-391.

[A3] ALMGREN F.J. Jr. — Measure theoretic geometry and elliptic variational problems, *Bull. Amer. Math. Soc.*, 75, 1969, p. 285-304.

[BDG] BOMBIERI E., DE GIORGI E. and GIUSTI E. — Minimal cones and the Bernstein theorem, *Invent. Math.*, 7, 1969, p. 243-269.

[F1] FEDERER H. — *Geometric measure theory,* Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, Berlin and New York, 1969.

[F2] FEDERER H. — The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension, *Bull. Amer. Math. Soc.*, 76, 1969, p. 767-771.

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RECENT RESULTS IN NONLINEAR FUNCTIONAL ANALYSIS AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

by Felix E. BROWDER

The central theme of the development of nonlinear functional analysis in the past six or seven years has been its breaking out of the classical framework of compact and contractive operators (the Picard contraction principle and its consequences, the Schauder fixed point theorem, the Leray-Schauder degree theory for maps f of the form $I - C$ with C compact) into a much broader terrain of non-compact operators with many interesting applications to partial differential equations and integral equations. In the present discussion, we give a brief survey of some of the central concepts of this development, giving no historical references and obviously not pretending to completeness. For more detailed treatments, we refer to the general papers and their bibliographies cited in the bibliography below. In particular, we refer for the pure functional-analytical development to the writer's very lengthy treatment in [3] and to the individual papers in the Symposium volume [4] to which [3] is related. For extensive treatment of the applications to partial differential equations and boundary value problems, we refer to [5], [9], [11], and for the related treatment of eigenvalue problems to [5] and [6].

The topics which we shall treat are (1) Monotone operators and their generalizations ; (2) Applications of monotone and pseudomonotone operators to nonlinear boundary value problems ; (3) Nonexpansive, condensive, and asymptotically compact mappings ; (4) Accretive operators and nonlinear semigroups ; (5) Generalizations of the topological degree ; (6) A-proper and related mappings ; (7) Variational theory of eigenvalue problems ; (8) Normally solvable mappings.

Section 1 : Monotone operators and their generalizations.

In the present section, X is a reflexive Banach space, X^* its dual space with the pairing between u in X and w in X^* written as (w, u). If T is a mapping from X to 2^{X^*}, its domain $D(T) = \{u \mid T(u) \neq \emptyset\}$, $R(T) = \bigcup_{u \in X} T(u)$.

T is said to be *monotone* if for $w \in T(u)$, $z \in T(x)$, we have

$$(w - z, u - x) \geqslant 0.$$

If f is a once Gateaux differentiable function from X to the reals, its derivative f' is a single-valued mapping from X to X^*. Then f' is monotone if and only if

f is convex. Thus the theory of monotone operators is a generalization to the context of mappings in Banach spaces of the basic ideas of the calculus of variations on convex sets in Banach spaces, i.e. of convex programming.

T is said to be *maximal monotone* if it is maximal in the sense of inclusion of graphs among monotone maps from X to X^*. If $D(T) = X$ and T is single-valued and continuous from each line segment in X to the weak topology of X^*, then T is maximal monotone. If f is a lower-semi-continuous convex function from X to $R^1 \cup \{+\infty\}$, $(f \neq +\infty)$, the *subgradient* f of f is a maximal monotone map of X into 2^{X^*} given by

$$w \in \partial f(u) \Longleftrightarrow f(x) \geqslant f(u) + (w, x-u) \quad \text{for all } x \text{ in } X.$$

If g is a convex function on the reals, $f(x) = g(\|x\|)$, the subgradient $\partial f = J_g$, is a maximal monotone map of X into 2^{X^*} called a *duality mapping* of X. If $g(r) = \frac{1}{2} r^2$, J is the *normalized* duality map. Some of the main results of the theory of monotone mappings are the following :

(a) T maximal monotone, $0 \in D(T) \Rightarrow R(T + J) = X^*$.

(b) T maximal monotone implies that $R(T) = X^*$ if and only if T^{-1} is locally bounded.

(c) If T and T_1 are maximal monotone, $0 \in D(T) \cap D(T_1)$, and if T is quasi-bounded (i.e. for some continuous φ and all w in $T(u)$, $\|w\| \leqslant \varphi(\|u\|, (w, u))$), then $T + T_1$ is maximal monotone. In particular, this will be true if $0 \in \mathrm{Int}(D(T))$.

(d) If L is linear, closed, densely defined, and monotone, then L is maximal monotone if and only if L^* is monotone.

Let T be a map of X into 2^{X^*} with $D(T) = X$ such that $T(u)$ is closed and convex and T is upper-semi-continuous on each finite dimensional subspace of X with respect to the weak topology of X^*. Then T is said to be *pseudo-monotone* if for any pair of sequences $\{u_j\}$ in X, $\{w_j\}$ in X^* with $w_j \in T(u_j)$ for each j, u_j converging weakly to u in X, it follows from the assumption that $\overline{\lim}\, (w_j, u_j - u) \leqslant 0$, that for each v in X, there exists w in $T(u)$ such that $\underline{\lim}\, (w_j, u_j - v) \geqslant (w, u - v)$. Pseudomonotone maps turn out to be the most useful extensions of the class of monotone maps for applications. In particular, if for a differentiable f, f' is pseudomonotone, then f is lower-semi-continuous with respect to the sequential weak topology on X.

(e) Let T be pseudo-monotone, T_1 maximal monotone from X to 2^{X^*} with at least one quasi-bounded, $0 \in D(T_1)$. Suppose that T is *subcoercive*, i.e. there exists a constant k such that for w in $T(u)$, we have $(w, u) \geqslant -k\|u\|$. Then if $(T + T_1)^{-1}$ is bounded, it follows that $R(T + T_1) = X^*$.

We note that if $0 \in D(T)$, a sufficient condition for T^{-1} to be bounded (i.e. map bounded sets of X^* into bounded sets in X) is that T be *coercive* i.e.

$$\|u\|^{-1} (w, u) \to +\infty \text{ for } w \in T(u), \; \|u\| \to +\infty.$$

2. Applications of monotone and pseudo monotone mappings.

The most direct applications are to elliptic and parabolic boundary value problems of arbitrary even order, though other applications have been given to nonlinear Hammerstein integral equations, dissipative nonlinear wave equations, symmetric hyperbolic systems, and periodic solutions of various equations of evolution.

Let Ω be a bounded, smoothly bounded open set in R^n, and consider

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u)) \tag{1}$$

on Ω with $A_\alpha(x, \xi)$ a function of x in Ω and $\xi = \{\xi_\alpha \| \alpha| \leq m\}$ in R^{n_m}. Corresponding to tne differential operator A in the representation (1), we may define the generalized Dirichlet form :

$$a(u, v) = \sum_{|\alpha| \leq m} (A_\alpha(x, \xi(u)), D^\alpha v).$$

Suppose that each A_α is continuous and satisfies the inequality

$$|A_\alpha(x, \xi)| \leq c(1 + |\xi|^{p-1})$$

for a real number p with $1 < p < +\infty$. We may define a nonlinear boundary value problem for A on a closed subspace V of the Sobolev space $W^{m,p}(\Omega)$ by asking for a given f in $V^*$for an element u of V such that

$$a(u, v) = (f, v), (v \in V).$$

More generally, we may consider the *variational inequality* for a convex closed subset C of $W^{m,p}(\Omega)$: $u \in C$, $a(u, v - u) \geq (f, v - u)$ for all v in C.

Suppose that : (1) $\sum_{|\alpha| = m} (A_\alpha(x, \zeta, \gamma) - A_\alpha(x, \zeta', \gamma)) (\zeta_\alpha - \zeta'_\alpha) > 0$ for $\zeta \neq \zeta'$ and any γ, (where we have divided ξ into terms ζ of order m and γ of lower order) : (2) $\sum_{|\alpha| \leq m} A_\alpha(x, \xi) \xi_\alpha \geq c|\xi|^p - c_1$, with $c > 0$. Then both of the above problems have at least one solution u for each f in V^*. The argument applies the fact that $a(u, v) = (T(u), v)$ where T is a continuous single-valued mapping of X into V^* which is pseudo-monotone and coercive. We note that indeed T satisfies the stronger condition : $(S)_+$. If u_j converges weakly to u in X and if

$$\overline{\lim} \, (T(u_j), u_j - u) \leq 0,$$

then u_j converges strongly to u. (The similar condition S is obtained by replacing $\overline{\lim}$ by lim).

Under similar conditions, solutions u in $L^p([0, M] ; V) \cap C^0([0, M] ; L^2(\Omega))$ are obtained for mixed initial-boundary value problems for the parabolic equation : $u' + A(t, u) = f(t)$, where for each t in $[0, M]$, $A(t, u)$ is an strongly elliptic operator of the type described above.

3. Nonexpansive, condensive, and asymptotically compact maps.

Let X be a uniformly convex Banach space, C a closed bounded convex subset of X (or more generally, C is a weakly compact convex subset of a general Banach space having normal structure). Let U be a mapping of C into X which is *nonexpansive*, i.e. $\|Ux - Uu\| \leq \|x - u\|$ for x, u in C. Suppose that U maps the boundary of C in X into C. Then U has a fixed point in C. More generally, if 0 lies in Int (C), then U has a fixed point in C if $Ux \neq \xi x$ for x in bdry(C), $\xi > 1$.

For a closed subset A of X, let $\gamma(A)$ be the infimum of $\epsilon > 0$ such that A can be covered by a finite numbers of sets of diameter $\leq \epsilon$. If C is a bounded closed convex set in the Banach space X, U a continuous map of C into X, then U is said to be *weakly condensing* if $\gamma(U(A)) \leq \gamma(A)$ for any closed subset A of C, condensing if $\gamma(U(A) < \gamma(A)$ for $\gamma(A) \neq 0$. If U maps the boundary of C into C and $(I - U)$ (C) is closed in X where U is weakly condensing and X is any Banach space, then U has a fixed point in C. (More generally, this is true if 0 lies in Int(C) and $Ux \neq \xi x$ for x in bdry(C), $\xi > 1$). If U is condensing, $I - U$ is proper and hence maps closed sets of C on closed sets of X. Hence, the above fixed point theorem holds for condensing maps. The class of weakly condensing maps includes all maps of the form $U_0 + R$ with U_0 non-expansive and R compact. More generally, it includes *semi-contractive* mappings, i.e. maps U of C into X such that $U(x) = S(x, x)$ where S maps $C \times C$ into X, $S(., v)$ is non-expansive from C to X for each fixed v, and the map $v \to S(., v)$ is a compact map from C into the space of non-expansive maps from C to X.

There is a close link between the theory of condensing maps and the *asymptotically compact* maps f for which compactness assumptions are made on high iterates f^n or their limits to obtain fixed point theorems for f. For example, let C be a closed convex subset of a Banach space X, f a continuous map of C into C such that f is locally compact (or locally condensing), each point of C has a compact orbit under f, and $\bigcap_n f^n(C) \neq \emptyset$ and is relatively compact in C. Then f has a fixed point in C. The theory of asymptotically compact maps rests on the Lefschetz fixed point theorem, while the theory of nonexpansive and condensing maps rests upon the following simple result : If C is a weakly compact convex subset of a Banach space, f a continuous map of C into C such that every minimal non-empty closed convex subset C_0 of C is compact, then f has a fixed point in C.

4. Accretive mappings and nonlinear semigroups.

If X is a Hilbert space H, and U is a non-expansive mapping of the closed bounded convex subset C of H into H, then $T = I - U$ is a monotone map of C into H and the theory of monotone operators yields the fixed point theorems for U. There is a second and deeper connection between the theories of monotone and nonexpansive mappings in Hilbert space, namely the fact that if T is maximal monotone, the solutions of the initial value problem $u(0) = v$, $(v \in D(T))$, for the equation

$$\frac{du}{dt}(t) + T(u(t)) = 0,$$

generate a continuous semigroup $\{U(t), t \geq 0\}$, of nonexpansive mappings by setting $U(t)v = u(t)$, where $u(t)$ is the solution of the differential equation.

The generalization of the class of monotone maps in Hilbert space from the point of view of this property is the class of *accretive* mappings where the map T from X to 2^X is said to be accretive if for $w \in T(u)$, $z \in T(x)$, there exists y in $J(u - x)$ such that $(w - z, y) \geqslant 0$, (i.e. $(J(u - x), Tu \quad Tx) \geqslant 0$). T is said to be *hypermaximal accretive* (m-accretive) if in addition $(\xi I + T)$ for $\xi > 0$ has all of X as its range. Equivalently, this condition becomes : $\|(\xi T + I)^{-1}\|_{\text{Lip}} \leqslant 1$ for all $\xi > 0$. Hypermaximal accretive mappings generate semigroups of nonexpansive operators in the above sense, either if X^* is uniformly convex or if T is single-valued with $D(T) = X$ and T is locally uniformly continuous. From the theory of semigroups, one obtains mapping theorems for accretive maps like those for monotone maps, e.g. : Let X be uniformly convex, T m-accretive and T^{-1} bounded. Then $R(T) = X$. If T is continuous and accretive, T is hypermaximal accretive. As a consequence, one obtains fixed point theorems for $U = (I - T)$ with T m-accretive, generalizing nonexpansive mappings.

5. Generalizations of the topological degree.

Among the classes of nonlinear mappings we have considered, several of the most important (monotone, accretive, maps of the form $f = I - U$ with U nonexpansive) have the important property of being uniformly approximable on their domains by a sequence of homeomorphisms or, more generally, mappings having continuous inverses defined on the whole of the range space. For a given pair of Banach spaces X and Y, we consider a class A of such approximating mappings and define an extension of the Leray-Schauder degree function for maps f of the following form : Let G be an open subset of the Banach space X, f a mapping of G into Y such that for a point a of Y, $f^{-1}(a)$ is a compact subset of G. Suppose that there exists an open subset W of $X \times X$ which contains the diagonal over G and a mapping S of W into Y with $f(v) = S(v, v)$ for v in G. Suppose that for fixed v in G, the mapping S_v of the slice of W over the second coordinate v is an invertible mapping in the class A, and that the map $v \to S_v$ is a locally compact map of G into the family of mappings S_v. We may then define a generalized degree function for f depending upon the representation map S and written as $\deg([f, S], G, a)$ where $\deg([f, S], G, a) = \deg(C_a, G_a, a)$, with the degree on the right being a modification of the usual degree function for the locally compact map C_a given by $C_a(v) = S_v^{-1}(a)$. This degree function is invariant under homotopies of S as long as $f^{-1}(a)$ remains in a compact subset of G and is additive on G, and $f^{-1}(a)$ is non-empty if the degree is different from zero.

If we apply the hopothesis that the family A of invertible mappings is convex, however, as is true in the cases mentioned earlier, the degree is independent of S, is invariant under homotopies of f rather than S, and can be extended to the uniform limits of mappings f having such representations. In this fashion, we obtain a general degree theory for semimonotone, semi-accretive, and mappings $I - f$ with f semicontractive. For the latter case, the degree function obtained coincides with one obtained geometrically. The degree function depending on S yields a interesting specialization for proper Fredholm mappings of index zero.

6. A-proper and related mappings.

Another important extension of the definition of the degree is obtained by calculating a degree function using finite dimensional approximations of a generalized Galerkin type, along lines familiar in the early stages of the theory of monotone maps. Let X and Y be Banach spaces, $\{X_n\}$ and $\{Y_n\}$ two sequences of finite dimensional Banach spaces, with each X_n and Y_n oriented Euclidean spaces of the same dimension. Let G be a closed subset of X, P_n and Q_n maps of X_n into X, Y into Y_n, resp.. If f is a mapping of G into Y, we may construct the approximants $f_n = Q_n f P_n$ of $G_n = P_n^{-1}(G)$ into Y_n. For a given a in Y, we may sometimes form $\deg(f_n, G_n, Q_n(a))$ provided that we have $Q_n\, a \in f_n(\mathrm{bdry}(G_n))$. A multi-valued degree function $\mathrm{Deg}(f, G, a)$ may be defined as the set of all limit points in $Z \cup \{+\infty\} \cup \{-\infty\}$ of $\deg(f_n, G_n, Q_n(a))$ provided that f is *A-proper,* i.e. for any sequence $\{x_{n_j}\}$ with $x_{n_j} \in G_{n_j}$, $n_j \to \infty$ such that

$$\| f_{n_j}(x_{n_j}) - Q_{n_j}(a) \| \to 0,$$

we may find an infinite subsequence such that $P_{n_j} x_{n_j}$ converges strongly to x in G, with $f(x) = a$. Strongly monotone and strongly accretive maps are examples of A-proper maps with respect to appropriate approximation schemes, but the most important example is that in which $Y = X^*$ for a reflexive Banach space X, $\{X_n\}$ is an increasing sequence of finite dimensional subspaces of X whose union is dense, $Y_n = X_n^*$, P_n is injection and Q_n is projection, while the mapping f satisfies the condition (S) described in Section 2 above.

The multivalued degree is invariant under homotopy and additive in a suitable sense. If the definition is restricted to a convex subclass of the A-proper mappings, it may be extended to the uniform limits of that convex class. For the classes of strongly monotone maps, strongly accretive maps, and maps satisfying condition $(S)_+$, we obtain thereby a useful degree theory for monotone, accretive, and pseudomonotone mappings respectively. In the latter case, for example, we obtain the following useful result for application to elliptic boundary problems : Let X be reflexive, f a continuous pseudomonotone mapping of X into X^*. Suppose that we are given a uniformly continuous homotopy of f through mappings f_t which are pseudomonotone so that for each bounded set B, $\bigcup_t f_t^{-1}(B)$ is bounded. Suppose that f_1 is odd, $f = f_0$. Then $R(f) = X^*$.

This is an extension to pseudo-monotone maps of the Borsuk-Ulam theorem and may be generalized to the case in which f_1 is covariant under a freely acting finite group of mapping acting on $X - \{0\}$ and $X^* - \{0\}$.

7. Variational methods for nonlinear eigenvalue problems.

An interesting complement to the existence theory for solutions of the boundary value problem for $A(u) = f$ where $A(u)$ is the strongly elliptic operation given in Section 2 above is the corresponding theory for the nonlinear eigenvalue problem $A(u) = \lambda B(u)$ under corresponding nonlinear boundary conditions. Here, $A(u)$ is of order $2m$ as above, B is lower order, and the two are the Euler-Lagrange operators, respectively, of the functionals f and g given by

$$f(u) = \int_\Omega F(x, \xi(u))\, dx \quad , \quad g(u) = \int_\Omega G(x, \gamma(u(x)))\, dx.$$

If f and g are even functionals, A satisfies the above conditions so that f' is a mapping of V into V^* satisfying condition (S), and g' is compact, the existence of infinitely many eigenfunctions can be established from an extension of the Lusternik-Schnirelman theory which applies to the critical points of the functional f on the level surface $g(u) = c$, or to the critical points of g on the level surface $f(u) = c$. In particular, under rather minimal hypotheses, one also obtains the approximability of the eigenfunctions by the corresponding higher order eigenfunctions in finite dimensional problems obtained by Galerkin or Rayleigh-Ritz approximation to the given problem. The corresponding results hold as well when the hypothesis that f and g are even is replaced by their invariance under other finite groups of transformations.

8. Normally solvable nonlinear mappings.

The theory of monotone mappings is based upon the application of separation arguments, weak compactness, and fixed point arguments for maps of compact convex sets. The theory of accretive mappings rests upon the construction of semigroups of nonexpansive maps and the fixed point theory of nonexpansive mappings. If one looks for alternative methods for extending the existence theorems obtained in these two cases to more general classes of mappings, for example the *φ-accretive* mappings f of X into a Banach space Y for a given $\varphi : X \times X \to Y^*$, where we assume that $(\varphi(x, u), f(x) - f(u)) \geqslant c\, \|x - u\|^2$, one of the two alternative methods for treating such maps for the case of f locally Lipschitzian is the theory of *normal solvability* of mappings f of a topological space X into a Banach space Y. This theory uses sharper results in the geometry of Banach spaces to treat the structure of the image set $f(X)$ under the assumption that f is *normally solvable*, i.e. $f(X)$ is closed in Y. Two results of this theory are the following :

(a) Let X be a topological space, Y a Banach space, f a mapping of X into Y with $f(X)$ closed in Y. Suppose that y is a given point of Y, and that there exist $r > 0$ and $p < 1$ such that $f^{-1}(B_r(y))$ is non-empty, while for each x in $f^{-1}(B_r(y))$, there exists a sequence $\{u_j\}$ in X such that $f(u_j) \neq f(x)$, $f(u_j) \to f(x)$, while

$$\|(f(u_j) - f(x)) - (y - f(x)\| \leqslant p\, \|f(u_j) - f(x)\| \cdot \|y - f(x)\|.$$

Then y lies in $f(X)$.

(b) Let X be a topological space, Y a Banach space with Y^* uniformly convex, f a mapping of X into Y with $f(X)$ closed in Y. Suppose that y is a given point of Y, and that there exist $r > 0$ and $\delta > 0$ such that $f^{-1}(B_r(y))$ is non-empty, while for each x in $f^{-1}(B_r(y))$, there exists a sequence $\{u_j\}$ in X such that $f(u_j) \neq f(x)$, $f(u_j) \to f(x)$, while

$$(J(y - f(x)), f(u_j) - f(x)) \geqslant \delta\, \|y - f(x)\| \cdot \|f(u_j) - f(x)\|.$$

Then y lies in $f(X)$.

Corresponding theorems can be given when Y is a manifold modelled on a Banach space, and other results in which the conditions on the points x in (a) and (b) are assumed to hold only on $X - N$, for N satisfying some neglibility condition.

9.

The results of the various theories which we have mentioned above hold under somewhat diverse structural hypotheses. The theory of monotone mappings is related in its basic definition to the duality of a locally convex topological vector space and its dual space and is not naturally extendable to manifolds. The theories of nonexpansive and accretive mappings are metric in character, with the accretive theory presupposing a manifold structure. The generalized degree theory of Section 6 and the asymptotic fixed point theory are topological in character and can be founded in the theory of absolute neighborhood retracts. The variational theory of Section 7 based upon the use of topological arguments concerning the structure of variational problems by constructing deformations of the underlying manifold is intrinsically a part of analysis on manifolds, i.e. global analysis. The diversity of hypotheses from the axiomatic point of view is compensated by the common point of application, and by the basic drive of nonlinear functional analysis : To find new classes of nonlinear mappings which are sufficiently cohesive to have a non-trivial theory and sufficiently broad to have interesting classes of analytical applications.

BIBLIOGRAPHY

[1] Browder F.E. — Existence and uniqueness theorems for solutions of nonlinear boundary value problems, *Proc. Symposia on App. Math.*, Vol. 17, *Amer. Math. Soc.*, 1965, p. 24-49.

[2] Browder F.E. — Nonlinear operators and nonlinear equations of evolutions in Banach spaces, Nonlinear functional analysis, *Proc. Symposia Pure Math.*, Vol. 18, Part 2; *Amer. Math. Soc.*, 1970, to appear.

[3] Browder F.E. (Editor). — Nonlinear functional analysis, *Proc. Symposia in Pure Math.*, Vol. 18, Part 1; *Amer. Math. Soc.*, 1970.

[5] Browder F.E. — Existence theorems for nonlinear partial differential equations, Global analysis, *Proc. Symposia in Pure Math.*, Vol. 16; *Amer. Math. Soc.*, 1970, p. 1-60.

[6] Browder F.E. — *Nonlinear eigenvalue problems and group invariance, Functional analysis and related fields*, Springer-Verlag, 1970, p. 1-58.

[7] Dubinski Ya. A. — Quasilinear elliptic and parabolic equations of arbitrary order, *Uspekhi Mat. Nauk*, 23, 1968, p. 45-90.

[8] Kachurovski R.I. — Nonlinear monotone operators in Banach spaces, *Uspekhi Mat. Nauk*, 23, 1968, p. 121-168.

[9] Lions J.L. — *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod-Gauthier Villars, 1969.

[10] OPIAL Z. — Non expansive and monotone mappings in Banach spaces, *Lecture Notes*, Division of App. Math., Brown University, 1967.
[11] STAMPACCHIA G. — Variational inequalities, Theory and applications of monotone operators, *Proc. of N.A.T.O. Institute at Venice*, June 1968, p. 101-192.

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NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF EVOLUTION

by Peter D. LAX

1. Introduction.

The equations of the title are of the form

$$u_t = K(u) \tag{1.1}$$

where K is some nonlinear partial differential operator, and the functions $u = u(x, t)$ are vector variables defined for x in G n and $t \geqslant 0$, subject to some boundary conditions on ∂G. We are interested in the *initial value problem* for (1.1), i.e. in finding solutions whose state at $t = 0$ is specified :

$$u(x, 0) = u_0(x) \quad \text{given} . \tag{1.2}$$

Much work in the past has been directed at proving the existence of solutions, and at showing that solutions are uniquely determined by their initial values. Current research, in both linear and nonlinear problems is turning toward the much more interesting question of determining qualitative and quantitative properties of solutions, such as their asymptotic behavior as $t \to \infty$, the statistical distribution of solutions, and others.

The following three, deceptively simple looking equations are prototypes of three important classes of equations in mathematical physics :

$$u_t + uu_x = 0 , \tag{1.3}$$

$$u_t + uu_x - u_{xx} = 0 , \tag{1.4}$$

$$u_t + uu_x + u_{xxx} = 0 . \tag{1.5}$$

Equation (1.3), sometimes called Hopf's equation, is the prototype of hyperbolic systems of conservation laws, i.e. systems of the form

$$u_t^j + f^j(u)_x = 0 , \; j = 1 , \ldots n . \tag{1.3}$$

The main phenomena for equations of this type is *the formation and decay of shock waves.*

Equation (1.4) is the prototype of dissipative equations, such as the Navier Stokes equations. The main phenomena here are the *formation and decay of turbulence.*

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Equation (1.5) is the prototype of equations with dispersion ; the main phenomenon here seems to be *the existence of infinitely many conserved quantities, and consequently a high degree of reversibility.*

Each of these three types of equations exhibits an entirely different range of phenomena ; this shows that there can be no unified theory of equations of form (1.1). The only common thread in recent investigations of equations of type (1.3) - (1.5) is that numerical experimentation plays an increasingly important role in studying properties of solutions.

In section 2 we shall set forth some recent results about properties of hyperbolic equations of type (1.3). In section 3 we shall describe a method for studying the dispersive equation (1.5). Dissipative equations will not be discussed.

2. **Formation and decay of shock waves.**

The equation

$$u_t + uu_x = 0 \tag{2.1}$$

says that u has zero directional derivative along the curves

$$\frac{dx}{dt} = u \, , \tag{2.2}$$

i.e. that u is constant along such a curve. These curves are called characteristics. From the constancy of u along characteristics it follows that the characteristics are straight lines. If the initial values of u are prescribed along $t = 0$, this determines a characteristic line issuing from each point of the x-axis. These characteristics will, in general, intersect ; at a point of intersection u would have to be double valued, which shows that no continuous solution can exist in a domain containing a point of intersection.

The above analysis shows that it is impossible to solve the initial value problem for all time within the class of smooth solutions. It is however possible if we admit discontinuous solutions ; the development of discontinuities in initially smooth solutions is called the *formation of shock waves.*

To develop a theory of discontinuous solutions we have to go back to the physical principle underlying the differential equation (2.1) ; this principle is a *conservation law,* which asserts that the rate of change of the total amount of u in any interval is equal to the flux f across the boundary. At points where u and f are differentiable, this conservation law implies

$$u_t + f_x = 0 \, , \tag{2.3}$$

which is equation (2.1) for $f(u) = \frac{1}{2} u^2$, so that (2.1) can be written as

$$u_t + \left(\frac{1}{2} u^2\right)_x = 0 \tag{2.4}$$

In general we require that the integral form of the conservation law hold ; this means that the differential form (2.4) holds in the sense of distribution theory. At points of discontinuity this implies the following relation :

$$s[u] = [f] \tag{2.5}$$

where $[g]$ denotes the difference in the values of g across the discontinuity, and s is the speed with which the discontinuity propagates.

It is easy to show by simple examples that solutions which satisfy the integral conservation law are not uniquely determined by their initial values ; consider e.g. the functions u_1 and u_2, defined for $t \geqslant 0$ by

$$u_1(x, t) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ \dfrac{x}{t} & \text{for} \quad 0 < x < t \\ 1 & \text{for} \quad t \leqslant x \end{cases}$$

and

$$u_2(x, t) = \begin{cases} 0 & \text{for} \quad x < t/2 \\ 1 & \text{for} \quad t/2 < x \ . \end{cases}$$

Both u_1 and u_2 satisfy (2.4) in the distribution sense, yet they are equal at $t = 0$. To restrict the class of admissible solutions we appeal once more to physics and eliminate those solutions which violate the entropy condition ; for equation (2.3) the *entropy condition* requires that *forward drawn* characteristics on either side of a discontinuity intersect the line of discontinuity ; for equation (2.4) this means

$$u_l > s > u_r \ , \tag{2.6}$$

where u_l is the state on the left, u_r on the right of the discontinuity. Discontinuities satisfying both (2.5) and (2.6) are called *shocks.*

Note that the shock condition explicitly distinguishes the forward from the backward time direction.

It can be shown that equation (2.4) has exactly one distribution solution in $t \geqslant 0$ with arbitrarily prescribed bounded measurable initial values all whose discontinuities are shocks.

We turn now to the decay of shock waves. Let $x_1(t)$ and $x_2(t)$ be any pair of characteristics of a continuous solution u. Since u is constant along characteristics, we deduce that u has the same variation along all intervals $[x_1(t), x_2(t)]$, $t > 0$.

Suppose there are shocks present ; since it has been stipulated that characteristics run into shocks as t increases, and since $u_l > u_r$ across a shock it follows that the increasing variation of u along $[x_1(t), x_2(t)]$ is a *decreasing function of time.* It is this decrease of variation which indicates the decay of shock waves ;

the rate of decrease can be estimated quantitatively :

Denote $x_2(t) - x_1(t)$ by $D(t)$; since the characteristics are straight lines which travel with speed u, it follows that

$$D(T) = D(0) + T\Delta u\,, \tag{2.7}$$

where $\Delta u = u_1 - u_2$ and u_1, u_2 denote the constant values of u along x_1, respectively x_2.

Let u be any solution with shocks ; denote by $L(t)$ the interval at time t contained between two curves which are either characteristics or shocks for u. It is not difficult to show that almost all the increasing variation of u on $L(T)$ is contained in a finite number of intervals $L_j(T)$ of the form $[x_j(T), y_j(T)]$, where $x_j(t), y_j(t)$ are a pair of characteristics such that the strip bounded by them is free of shocks for $0 \leqslant t \leqslant T$ and such that $\Delta_j u = u(y_j) - u(x_j)$ is almost equal to the increasing variation of u along $L_j(T)$. Applying (2.7) to the jth strip we get

$$D_j(T) - D_j(0) + T\Delta_j u\,,$$

where $D_j(T)$ is the length of $L_j(T)$. Summing with respect to j and noting that the intervals $L_j(T)$ are disjoint and contained in $L(T)$ we conclude that

$$|L(T)| \geqslant T \begin{Bmatrix} \text{Total increasing variation} \\ \text{of } u \text{ on } L(T) \end{Bmatrix} \tag{2.8}$$

Suppose in particular that $u(x, t)$ is periodic in x with period p. We choose $L(0)$ to have length p and take $L(t)$ to be the interval contained between the characteristics-shock curves issuing from the endpoints of $L(0)$. By periodicity $L(t)$ has length p for all t, so from (2.8) we conclude that

$$\text{Total variation of } u \text{ per period} < \frac{p}{T}\,, \tag{2.9}$$

i.e. that the total variation of periodic solutions decay like p/T. It is remarkable that the upper bound for this total variation is universal, i.e. independent of the solution in question.

We turn now to systems of two conservation laws :

$$u_t + f_x = 0, \quad v_t + g_x = 0\,, \tag{2.10}$$

f and g nonlinear functions of u and v, subject to the following conditions :

(a) The system (2.10) is hyperbolic

(b) The system (2.10) is genuinely hyperbolic in the sense of [4].

Under these conditions Glimm, [2], proved the existence for all $t \geqslant 0$ of solutions of (2.10) in the distribution sense with prescribed initial values, provided that the oscillation of the initial values is small enough and their total variation is not too large. In [3] Glimm and the speaker showed, under a very mild additional restriction, this

DECAY THEOREM

(A) *The initial value problem for* (2.10) *can be solved for all* $t \geqslant 0$ *provided that the oscillation of the initial data is small enough, but without requiring that the data be of bounded variation.*

(B) *If the initial data of* u, v *are periodic, the total variation of the solution per period at time* T *is bounded by* pH/T, H *a constant depending on the equation but not the solution.*

The method of proof combines ideas in [2] with those explained here for the simple equation (2.4). For a system of two equations there are two families of characteristic curves, two kinds of shocks, and two so-called *Riemann invariants,* functions of u and v which are constant along characteristics of the corresponding families. It follows that for smooth solutions the variation of each Riemann invariant between two characteristics of the same family is independent of t. It is further true that the effect of shocks of each kind is to diminish the increasing variation of the Riemann invariant of the same family. To prove however that there is an overall decrease in variation we also have to estimate the effect of shocks on the variation of Riemann invariants of the opposite family, and we have to estimate the influence of the total variation of the Riemann invariant of one family on the width of characteristics strips of the other family. These difficulties are handled with the aid of conservation laws for waves of each kind ; also we make essential use of the fact that across weak shocks the change in the Riemann invariant of the same family.

Detailed statement of results and proof are given in [3].

3. Integrals of nonlinear equations of evolution.

In this section we present a method developed in [4] for finding invariant and exponentially varying functionals for nonlinear evolution equations. These integrals and functionals were discovered for the Korteweg-de Vries equation (1.5) by Kruskal, Miura, Gardner and Greene.

We start with a criterion for a differentiable one-parameter family of selfadjoint operators $L(t)$ to be unitarily equivalent. Unitary equivalence means that there is a one-parameter family of *unitary* operators $U(t)$ such that

$$U^*(t)\, L(t)\, U(t)\, U(t) \tag{3.1}$$

is independent of t. Suppose that U depends differentiably on t ; then the constancy of (3.1) can be expressed by setting its t-derivative equal to 0 :

$$U_t^* LU + U^* L_t U + U^* L U_t = 0 \tag{3.2}$$

A differentiable one-parameter family of unitary operators satisfies a differential equation of the form

$$U_t = BU \tag{3.3}$$

where $B(t)$ is antisymmetric :

$$B^* = -B\,. \tag{3.4}$$

Taking the adjoint of (3.3) and using (3.4) gives

$$U_t^* = U^*B^* = -U^*B \ ;$$

substituting this and (3.3) into (3.2) we get, after multiplication by U on the left, U^* on the right and rearranging terms, that

$$L_t = BL - LB = [B, L] \ . \tag{3.5}$$

Let's take for L the Schrödinger operator

$$L = D^2 + \frac{1}{6}u, \quad D = \frac{d}{dx} \tag{3.6}$$

over the infinite interval $(-\infty, \infty)$, the potential u a function of t. Then L_t is multiplication by $u_t/6$, and equation (3.5) calls for an antisymmetric operator B whose commutator with L is multiplication. It is not hard to show that there is such a differential operator B of each odd order $2q + 1$. The first order operator B_0 is D ; in this case $U(t)$ is translation, along the x axis and

$$u(x, t) = u(x + t/6) \ .$$

The third order operator

$$B = -4D^3 - uD - \frac{1}{2}u_x \tag{3.7}$$

leads to a much more interesting result ; in this case

$$[B, L] = -\frac{1}{6}u_{xxx} - \frac{1}{6}uu_x$$

so that we conclude from equation (3.5) that if $u(x, t)$ satisfies

$$u_t + uu_x + u_{xxx} = 0 \ , \tag{3.8}$$

the operators

$$L(t) = D^2 + \frac{1}{6}u(t)$$

are all unitarily equivalent !

Equation (3.8) is the Korteweg-de Vries equation (1.5) listed in section 1. It is easy to show, using the energy method, see [4], that solutions which tend to zero fast enough as $|x| \to \infty$ are uniquely determined by their initial values. Existence of solutions with prescribed initial values was proved by Sjöberg, [7].

The unitary equivalence of all $L(t)$ implies

THEOREM 3.1. – *The eigenvalues of the Schrödinger operator* (3.6) *are invariant functionals (integrals) of solutions of the Korteweg-de Vries equation* (3.8). This remarkable fact was discovered by Gardner, Kruskal and Miura [6].

The present derivation can be used to determine more directly than in [6] how the eigenfunctions of $L(t)$ vary with t. Denote by ϕ_0 an eigenfunction of $L(0)$:

$$L(0)\,\phi_0 = \lambda\phi_0 .$$

Since by construction of U, (3.1) is independent of t, we have (assuming that $U(0) = I$) that

$$U^*(t)\,U(t)\,\phi_0 = L(0)\,\phi_0 = \lambda\,\phi_0$$

for all t. Multiplying by $U(t)$ and introducing

$$\phi(t) = U(t)\,\phi_0 \tag{3.9}$$

we deduce that

$$L(t)\,\Phi(t) = \lambda\,\phi(t) ,$$

which shows that the eigenfunctions $\phi(t)$ of $L(t)$ are related to those of $L(0)$ by (3.9). Differentiating (3.9) with respect to t and using the differential equation (3.3) satisfied by $U(t)$ we deduce that

$$\phi_t = B\phi . \tag{3.9$_t$}$$

For the case at hand when B is given by (3.7) it follows that ϕ satisfies

$$\phi_t + 4\phi_{xxx} + u\phi_x + \frac{1}{2}\,u_x\phi = 0 \tag{3.10}$$

Now ϕ itself satisfies the eigenvalue equation

$$\phi_{xx} + \frac{1}{6}\,u\phi = \lambda\phi .$$

Differentiating this with respect to x, multiplying by 4 and subtracting from (3.10) gives

$$\phi_t + \;4\lambda + \frac{1}{3}\,u\;\;\phi_x - \frac{1}{6}\,u_x\phi = 0 , \tag{3.11}$$

a first order equation satisfied by ϕ.

On the infinite interval $(-\infty\,, \infty)$ and for a function u which tends to 0 rapidly as $x \to \pm\infty$ the operator L given by (3.6) has a finite number of positive eigenvalues and a continuous spectrum of multiplicity 2 covering the negative reals. With each point $-k^2$ of the continuous spectrum we can associate two *improper eigenfunctions*, each a solution of

$$\phi_{xx} + \frac{1}{6}\,u\phi + k^2\phi = 0 . \tag{3.12}$$

We saw earlier that the proper eigenfunctions ϕ of $L(t)$ satisfy the differential equation (3.11). It is not hard to show the converse :

THEOREM 3.2. – *Let ϕ be a solution of*

$$\phi_t + \left(\frac{1}{3}\,u - 4k^2\right)\phi_x - \frac{1}{6}\,u_x\phi = 0 \tag{3.13}$$

which at $t = 0$ satisfies

$$(L(0) + k^2)\,\phi(0) = 0$$

Then ϕ satisfies

$$(L(t) + k^2)\,\phi(t) = 0$$

for all t.

Proof. – Define χ by

$$\chi = (L + k^2)\,\phi\ .$$

Equation (3.13) can be rewritten as

$$\phi_t - B\phi - 4\,D_\chi = 0 \tag{3.14}$$

where B is defined in (3.7). We calculate now χ_t ; using (3.14) we get

$$\chi_t = L_t\phi + L\phi_t + k^2\,\phi_t = L_t\phi + LB\phi + 4LD\chi + k^2B\phi + 4k^2D\chi\ .$$

Using the identity (3.5) according to which $L_t = BL - LB$, we get, after grouping terms, that

$$\chi_t = B(L + k^2)\,\phi + 4(L + k^2)\,D\chi = B\chi + 4(L + k^2)\,D\chi$$

This equation can be rewritten as

$$\chi_t + \left(\frac{1}{3}\,u - 4k^2\right)\chi_x + \frac{1}{2}\,u_x\,\chi = 0\ . \tag{3.15}$$

Since by assumption $\chi(0) = 0$, and since solutions of (3.15) are uniquely determined by their initial values, it follows that $\chi(t) = 0$ for all t, as asserted in the theorem.

It is well known that if u tends to zero fast enough as $|x| \to \infty$, then solutions of (3.12) behave like linear combinations of e^{ikx} and e^{-ikx}. Suppose we normalize $\phi(0)$ so that it consists of a plane wave of unit strength coming from, say, the left scattered by the potential $u(0)$:

$$\phi(x\,,0\,,k) \simeq \begin{matrix} e^{ikx} + Re^{-ikx} & \text{for } x \text{ near } -\infty \\ Te^{ikx} & \text{for } x \text{ near } +\infty\ . \end{matrix} \tag{3.16}$$

The quantity $R = R(k)$ is called the *reflection coefficient*, $T = T(k)$ the *transmission coefficient.*

How does the asymptotic behavior of ϕ near $|x| = \infty$ vary with time ? For $|x|$ large we can write

$$\phi(x\,,t) \simeq A(t)\,e^{ikx} + B(t)\,e^{-ikx}$$

and we can neglect u and u_x in (3.13). We get

$$(A_t - 4k^3iA)\,e^{ikx} + (B_t + 4k^3iB)\,e^{-ikx} = 0\ ,$$

which implies that

$$A(t) = e^{4ik^3t}\ ,\ B(t) = e^{-4ik^3t}\ .$$

Taking the initial values (3.16) of ϕ into account we conclude that

$$\psi = e^{-4ik^3t}\,\phi$$

represents, for each t, a plane wave of unit strength coming from the left and scattered by the potential $u(t)/6$:

$$\psi(x\,,t\,,k) \cong \begin{cases} e^{ikx} + e^{-8ik^3t}\,R(k)\,e^{-ikx} & \text{for } x \text{ near } -\infty\,. \\ T(k)\,e^{ikx} & \text{for } x \text{ near } +\infty\,. \end{cases}$$

This asymptotic description of ψ shows

THEOREM 3.3. – *The transmission coefficient of $L(t)$ is independent of t, and the reflection coefficient R varies exponentially in t :*

$$R(k\,,t) = R(k\,,0)\,e^{-8ik^3t}\,. \tag{3.17}$$

This result is due to Gardner, Greene, Kruskal and Miura, [1]. In their paper the authors point out the following method based on relation (3.17) for solving the initial value problem for the Korteweg-de Vries equation (3.8) :

Given the initial value $u(x\,,0)$ one can determine the point eigenvalues as well as the transmission and reflection coefficients of the operator $L(0)$. Using the previous results we deduce that $L(t)$ has the same point eigenvalues as $L(0)$, and its reflection coefficient is given by formula (3.17). Using the Gelfand-Levitan procedure for solving the inverse problem it is possible to reconstruct the potential $u(x\,,t)/6$ appearing in the operator $L(t)$.

BIBLIOGRAPHY OF § 2

[1] CONWAY E. and SMOLLER J. — Global solutions of the Cauchy problem for quasi-linear first-order equations in several space variables, *Comm. Pure Appl. Math.*, Vol. 19, No. 1, Feb. 1966, p. 95-105.

[2] GLIMM J. — Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, Vol. 18, No. 4, Nov. 1965, p. 697-715.

[3] GLIMM J. and LAX P.D. — Decay of solutions of systems of nonlinear hyperbolic conservation laws, *Memoirs of the Amer. Math. Soc.*, No. 101, 1970.

[4] LAX P.D. — Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.*, Vol. 10, 1957, p. 537-556.

[5] NISHIDA T. — *Global solution for an initial boundary value problem of a quasi-linear hyperbolic system*, Proc. Japan Academy, Vol. 44, No. 7 (1968), pp. 642-646.

[6] OLEINIK O.A. — Discontinuous solutions of nonlinear differential equations, *Uspehi Mat. Nauk.*, Vol. 12, 1957, p. 3-73.

BIBLIOGRAPHY OF § 3

[1] Gardneer C.S., Greene J.M., Kruskal M.D. and Miura R.M. — A method for solving the Korteweg - de Vries equation, *Phys. Rev. Letters*, Vol. 19, 1967, p. 1095-1097.

[2] Gardner C.S. and Morikawa G.K. — *Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves*, New York Univ., Courant Inst. Math. Sci., Res. Rep. NYO-9082, 1960.

[3] Korteweg D.J. and de Vries G. — *On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary wave, Philos. Mag.*, Vol. 39, 1895, p. 422-443.

[4] Lax P.D. — Integrals of nonlinear equations of evolution and solitary waves, *C.P.A.M.*, Vol. XXI, 1968, p. 467-490.

[5] Miura R.M. — Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, *J. Math. Phys.*, Vol. 9, 1968, p. 1202-1204.

[6] Miura R.M., Gardner C.S. and Kruskal M.D. — Korteweg-de Vries equations and generalizations. II. Existence of conservation laws and constants of motion, *J. Math. Phys.*, Vol. 9, 1968, p. 1204-1209.

[7] Sjöberg A. — On the Korteweg-de Vries equation, *Journal of Math. Anal. and Applications*, Vol. 29, 1970, p. 569-579.

[8] Whitham G.B. — Nonlinear dispersives waves, *Proc. Roy. Soc.*, Ser. A, Vol. 283, 1965, p. 238-261.

[9] Zabusky N.J. — A synergetic approach to problems of nonlinear dispersive wave propagation and interaction, *Nonlinear Partial Differential Equations*, Academic Press, New York, 1967.

[10] Zabusky N.J. and Kruskal M.D. — Interaction of solutions in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Letters*, Vol. 15, 1965, p. 240-243.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 841 à 851.

INÉQUATIONS VARIATIONNELLES D'ÉVOLUTION

par J. L. LIONS

On présente dans ce rapport

– différents problèmes d'inéquations variationnelles d'évolution intervenant dans les applications (n° 1 et 2) ;

– quelques outils généraux utiles dans la résolution de ces problèmes (n° 3).

1. Inéquations d'évolution. Formulation générale. Premiers exemples.

1.1. *Formulation générale.*

Dans un espace de Hilbert $\mathcal{H}$ (sur **R**) soient $\mathcal{A}$ et $\mathcal{B}$ deux opérateurs non bornés, *linéaires ou non* (1). Soit par ailleurs $\mathcal{K}$ un sous-ensemble (2) convexe de $\mathcal{H}$, fermé dans une topologie convenable (on précisera cela plus loin).

Soit t la variable de temps, $t \geqslant 0$.

On se donne une fonction $t \to f(t)$ de $t \geqslant 0 \to \mathcal{H}$ et u_0 tel que $\mathcal{B}u_0 \in \mathcal{K}$.

On cherche une fonction $t \to u(t)$ à valeurs dans les domaines de $\mathcal{A}$ et de $\mathcal{B}$ telle que

$$\mathcal{B}u(t) \in \mathcal{K} \qquad \forall t \geqslant 0, \tag{1.1}$$

$$\left(\frac{\partial u(t)}{\partial t} + \mathcal{A}u(t) - f(t), v - \mathcal{B}u(t)\right) \geqslant 0 \qquad \forall v \in \mathcal{K}\ (^3), t > 0, \tag{1.2}$$

$$u(0) = u_0. \tag{1.3}$$

C'est la "formulation générale" (et vague (4)) des *inéquations variationnelles d'évolution.*

Remarque 1.1.

Si $\mathcal{B}$ = identité, $\mathcal{K} = \mathcal{H}$, (1.1) est sans objet et (1.2) se réduit à *l'équation d'évolution*

$$\frac{\partial u(t)}{\partial t} + \mathcal{A}u(t) - f(t) = 0.$$

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(1) Les opérateurs $\mathcal{A}$ et $\mathcal{B}$ sont dans les applications des opérateurs différentiels ou intégro-différentiels. Ils peuvent dépendre de t.

(2) Dépendant éventuellement de t.

(3) (φ, ψ) = produit scalaire dans $\mathcal{H}$.

(4) Il faudra préciser ce qu'on entend par "*solution*".

1.2. *Un exemple simple.*

Soient V, H deux Hilbert, $V \subset H$, V dense dans H ; on identifie H à son dual ; alors le dual V' de V s'identifie à un sur-espace de H de sorte que

$$V \subset H \subset V'. \tag{1.4}$$

Soit $a(u, v)$ une forme bilinéaire continue sur V, définissant $A \in \mathcal{L}(V ; V')$ par (1)

$$a(u, v) = (Au, v). \tag{1.5}$$

Soit enfin

$$K = \text{ensemble convexe fermé de } V. \tag{1.6}$$

Alors on prend dans le "problème général" :

$$\mathcal{H} = H \quad , \quad \mathcal{K} = K \quad , \quad \mathcal{A} = A \quad , \quad \mathcal{B} = \text{identité}.$$

On obtient ainsi le problème suivant, introduit dans [29] : trouver une fonction u telle que

$$u(t) \in K, \tag{1.7}$$

$$\left(\frac{\partial u(t)}{\partial t}, v - u(t)\right) + a(u(t), v - u(t)) \geqslant (f(t), v - u(t)) \quad \forall v \in K, \text{(2)} \tag{1.8}$$

$$u(0) = u_0 \ (u_0 \text{ donné dans } K). \tag{1.9}$$

On a montré [29] [3] [24] (3) : *Si l'on suppose a coercif, au sens*

$$\left\{\begin{array}{l} \text{il existe } \lambda \in \mathbf{R}, \ \alpha > 0, \text{ tels que} \\ a(v, v) + \lambda \|v\|_H^2 \geqslant \alpha \|v\|_V^2, \qquad \forall v \in V \end{array}\right. \tag{1.10}$$

le problème (1.7) (1.8) (1.9) *admet une solution unique,* où la solution est entendue (selon les hypothèses) en un sens *fort* ((1.7) (1.8) ont lieu *p.p.* en t) ou *faible* (on prend dans (1.8) $v = v(t)$, *on intègre en* t sur $[0, T]$ — et, éventuellement, on intègre par parties en t).

Exemple 1.1.

Prenons $H = L^2(\Omega)$ où Ω est un ouvert borné de $\mathbf{R}^n$ de frontière "régulière" Γ, $V = H^1(\Omega)$ (4),

$$a(u, v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx, \tag{1.11}$$

$$K = \{v \mid v \in H^1(\Omega) \ , \ v \geqslant 0 \ p.p. \text{ sur } \Gamma\}. \tag{1.12}$$

---

(1) On désigne par (,) le produit scalaire dans H et entre V et V'.

(2) Equivaut à $\left(\frac{\partial u}{\partial t}(t) + Au(t) - f(t), v - u(t)\right) \geqslant 0 \quad \forall v \in K$.

(3) Les *outils* principaux pour les démonstrations sont donnés au n° 3 ci-après.

(4) Espace de Sobolev du 1er ordre construit sur $L^2(\Omega)$.

Alors le problème (1.7) (1.8) (1.9) équivaut à

$$(1.13)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f\,, x \in \Omega\,,\ t > 0\,, \\ u \geqslant 0\,, \dfrac{\partial u}{\partial n} \geqslant 0\ (^1)\,, u\dfrac{\partial u}{\partial n} = 0 \text{ sur } \Gamma \times \{t > 0\}\,, u(x\,,0) = u_0(x)\,, x \in \Omega\,. \end{cases}$$

Exemple 1.2.

$$V = H_0^1(\Omega)\ (^2)\,,\ a(u\,,v) \text{ comme en } (1.11)\,,$$

$$(1.14)\qquad K = \{v \mid v \in H_0^1(\Omega)\,,\ v \geqslant 0 \text{ p.p. dans } \Omega\}\,.$$

Alors la solution de (1.7) (1.8) (1.9) est la solution de

$$(1.15)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - f \geqslant 0 \quad,\quad u \geqslant 0 \quad,\quad u\left(\dfrac{\partial u}{\partial t} - \Delta u - f\right) = 0 \\ u = 0 \quad \text{si} \quad x \in \Gamma \quad,\quad u(x\,,0) = u_0(x). \end{cases}$$

Remarque 1.2.

Dans les deux exemples précédents (et *c'est absolument général*) on voit l'analogie (en fait, il y a dans certains cas *identité*) des problèmes d'inéquations variationnelles et des problèmes à **frontière libre**.

Dans (1.15) par exemple, il y a une région où $u = 0$, une région où

$$\frac{\partial u}{\partial t} - \Delta u - f = 0$$

et avec une "*interface*" *non donnée à priori.*

Le problème de *la régularité* de ces "*interfaces*" est encore très largement ouvert dans les problèmes d'*évolution* (pour certains cas elliptiques, cf. [22] [32]).

Signalons aussi l'analogie avec les "*surfaces de commutation*" apparaissant dans la théorie du **contrôle optimal** de systèmes gouvernés par des équations aux dérivées partielles [23].

1.3 *Inéquations variationnelles et équations avec opérateurs multivoques*

Dans le cadre du n° 1.2, introduisons la fonction ψ_K :

$$(1.16)\qquad \psi_K(v) = \begin{cases} 0 & \text{si} \quad v \in K\,, \\ +\infty & \text{si} \quad v \notin K\,. \end{cases}$$

Alors (1.7) (1.8) équivalent à

$$(1.17)\quad \left(\frac{\partial u(t)}{\partial t}, v - u(t)\right) + a(u(t)\,, v - u(t)) + \psi_K(v) - \psi_K(u(t)) \geqslant (f(t)\,, v - u(t)) \quad \forall v \in V.$$

---

(1) $\partial/\partial n$ = dérivée normale à Γ dirigée vers *l'extérieur de* Ω.

(2) Sous-espace des fonctions de $H^1(\Omega)$ nulles au bord.

Mais soit $\partial \psi_K(u)$ = *sous différentielle de* ψ_K *en* u =

$$= \{\chi \mid \chi \in V' \,,\, \psi_K(v) - \psi_K(u) - (\chi\,, v - u) \geqslant 0 \quad \forall v \in V\}\,;$$

$\partial\psi_K$ est un opérateur (monotone) **multivoque** et (1.17) équivaut à

$$0 \in \frac{\partial u(t)}{\partial t} + Au(t) - f(t) + \partial\psi_K(u(t)). \tag{1.18}$$

2. **Exemples.**

2.1. *Problèmes paraboliques (I).*

Nous avons déjà vu les Exemples 1.1 et 1.2. Donnons un autre exemple [13] [14].

Exemple 2.1.

Soit $v \to j(v)$ une fonction de $V \to \mathbf{R}$ continue $\geqslant 0$ *non différentiable* ; on cherche $u = u(t)$ solution, avec $u(0) = u_0$, de

$$\left(\frac{\partial u(t)}{\partial t}, v - u(t)\right) + a(u(t)\,, v - u(t)) + j(v) - j(u(t)) \geqslant (f(t)\,, v - u(t)) \quad \forall v \in V. \tag{2.1}$$

On montre encore *l'existence et l'unicité de u* (sous l'hypothèse (1.10)). Si l'on prend *par ex.* $V = H^1(\Omega)$, $H = L^2(\Omega)$, $a(u\,, v)$ donné par (1.11) et

$$j(v) = g\int_\Gamma |v|\, d\Gamma \quad , \quad g > 0\,, \tag{2.2}$$

alors (2.1) équivaut à

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = f \quad , \quad x \in \Omega \quad , \quad t > 0\,, \\ &\left|\frac{\partial u}{\partial n}\right| \leqslant g \quad , \quad u\,\frac{\partial u}{\partial n} + g|u| = 0 \ \text{ si } x \in \Gamma \quad , \quad t > 0 \quad , \quad u(x\,,0) = u_0(x). \end{aligned}\right. \tag{2.3}$$

Remarque 2.1.

On peut donner des exemples où l'on a un opérateur parabolique *d'ordre quelconque.*

Problème ouvert : Quels sont les problèmes d'inéquations qui sont "bien posés" pour les opérateurs paraboliques au sens de Petrowsky [31] ?

2.2. *Problèmes paraboliques (II).*

Un certain nombre d'applications [13] [14] imposent l'étude du problème suivant où, avec les notations générales du n° 1.1, **$\mathcal{B}$ n'est plus l'opérateur identité**

$$\left(\mathcal{B}u(t) = \frac{\partial u(t)}{\partial t}\right).$$

On cherche une fonction $u = u(t)$ telle que $u(0) = u_0$ et

(2.4) $\frac{\partial u(t)}{\partial t} \in K,$

(2.5) $\left(\frac{\partial u(t)}{\partial t}, v - \frac{\partial u(t)}{\partial t}\right) + a\left(u(t), v - \frac{\partial u(t)}{\partial t}\right) \geqslant \left(f(t), v - \frac{\partial u(t)}{\partial t}\right) \quad \forall v \in K.$

Si l'on suppose $a(u, v)$ coercif au sens (1.10) *et symétrique* :

(2.6) $$a(u, v) = a(v, u) \quad \forall u, v \in V$$

alors le problème admet une solution unique.

Exemple 2.2.

V, a, K comme dans l'Exemple 1.1. Alors

(2.7) $$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f \ , \ x \in \Omega \ , \ t > 0 \ , \\ \frac{\partial u}{\partial t} \geqslant 0 \ , \ \frac{\partial u}{\partial n} \geqslant 0 \ , \ \frac{\partial u}{\partial t}\frac{\partial u}{\partial n} = 0 \text{ sur } \Gamma \times \{t > 0\}, u(x, 0) = u_0(x), x \in \Omega. \end{cases}$$

2.3. *Inéquations pour des opérateurs du type "Navier-Stokes"* (1).

On introduit

$$V = \{v \mid v \in (H_0^1(\Omega))^n \ , \ \text{Div}\, v = 0\},$$

$$a(u, v) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx \ , \ (\varphi, \psi) = \sum_{i=1}^{n} \int_\Omega \varphi_i \psi_i \, dx \, ,$$

$$b(u, v, w) = \sum_{i,j=1}^{n} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx \ , \ j(v) = g \int_\Omega \left(\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right)^{1/2} dx \ , \ g > 0.$$

On cherche une fonction $t \to u(t)$ de $t \geqslant 0 \to V$ telle que $u(0) = u_0$ et

(2.8) $$\left(\frac{\partial u(t)}{\partial t}, v - u(t)\right) + a(u(t), v - u(t)) + b(u(t), u(t), v - u(t)) + j(v) - j(u(t)) \geqslant (f(t), v - u(t)) \ (2).$$

Si $g = 0$, on retrouve les *équations de Navier-Stokes* [20].

On montre [14] *des résultats tout-à-fait analogues à ceux déjà connus pour les équations de Navier-Stokes* :

si $n = 2$ il y a existence et unicité de la solution $(u = u_g)$;

si $n \geqslant 3$ il y a existence globale d'une solution "faible", l'unicité étant alors un problème ouvert.

En outre, si $n = 2$, $u_g \to u$, *lorsque* $g \to 0$, *u solution des équations de Navier-Stokes usuelles.*

---

(1) On les rencontre dans l'écoulement des fluides rigides visco-plastiques de Bingham ; cf. [13] [14].

(2) On écrit le terme (nul) $b(u(t), u(t), u(t))$ pour la symétrie.

2.4. *Problèmes pour opérateurs hyperboliques.*

Avec les notations introduites en 1.2, on cherche $u = u(t)$ telle que

$$\frac{\partial u(t)}{\partial t} \in K, \tag{2.9}$$

$$\left(\frac{\partial^2 u(t)}{\partial t^2}, v - \frac{\partial u(t)}{\partial t}\right) + a\left(u(t), v - \frac{\partial u(t)}{\partial t}\right) \geqslant \left(f(t), v - \frac{\partial u(t)}{\partial t}\right) \quad \forall v \in K, t \geqslant 0, \tag{2.10}$$

$$u(0) = u_0 \quad , \quad \left.\frac{\partial u(t)}{\partial t}\right|_{t=0} = u_1 \quad , \quad u_0 \quad , \quad u_1 \quad \text{donnés.} \tag{2.11}$$

Ce type de problème, introduit dans [25], intervient dans de nombreuses applications [14].

Ecrivant (2.10) comme *un système du 1er ordre,* on aboutit à une équation du "type général" introduit en 1.1.

On montre, sous les hypothèses (1.10) *et* (2.6), *l'existence et l'unicité de la solution* [25] [6] [5].

Exemple 2.3.

V, a, K comme dans l'Exemple 1.1. Alors (2.9) (2.10) (2.11) équivalent à

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f \quad , \quad x \in \Omega \quad , \quad t > 0 \, , \\[2ex] \dfrac{\partial u}{\partial t} \geqslant 0 \quad , \quad \dfrac{\partial u}{\partial n} \geqslant 0 \quad , \quad \dfrac{\partial u}{\partial t} \cdot \dfrac{\partial u}{\partial n} = 0 \quad \text{sur} \quad \Gamma \times \{t > 0\} \, , \\[2ex] u(x, 0) = u_0(x) \quad , \quad \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \quad , \quad x \in \Omega \, . \end{cases} \tag{2.12}$$

Remarque 2.2.

Le problème analogue au précédent, mais où l'on remplace (2.9) par "$u(t) \in K$" et $v - \dfrac{\partial u(t)}{\partial t}$ par $v - u(t)$ est *probablement généralement* mal posé ; pour un cas particulier où on arrive ainsi à un problème bien posé, cf. [27].

2.5. — *Problèmes pour systèmes d'opérateurs hyperboliques du 1er ordre.*

Le problème suivant intervient dans l'étude de l'électromagnétisme de milieux polarisables [13] [14] : on cherche $\vec{D}$ et $\vec{B}$ (avec les notations habituelles dans les équations de Maxwell) solutions de

$$\begin{cases} \dfrac{\partial \vec{D}}{\partial t} - \operatorname{Rot}\left(\dfrac{1}{\mu}\vec{B}\right) + \vec{J} = \vec{G}_1 \, , \\[2ex] \dfrac{\partial \vec{B}}{\partial t} + \operatorname{Rot}\left(\dfrac{1}{\epsilon}\vec{D}\right) = \vec{G}_2 \, , \end{cases} \tag{2.13}$$

où $\vec{J}$ est donné de la façon suivante :

$$(2.14)\quad \begin{cases} \text{si } |\vec{D}| < \mathcal{D}_0 \text{(constante positive “de claquage”) alors (loi d'Ohm } \textit{usuelle}) \\ \qquad \vec{J} = \frac{\sigma}{\epsilon} \vec{D} \,; \\ \text{si } |\vec{D}| = \mathcal{D}_0 \text{ alors il existe } \lambda \geqslant 0 \text{ tel que} \\ \qquad \vec{J} = \left(\frac{\sigma}{\epsilon} + \lambda\right) \vec{D}. \end{cases}$$

Ce problème se formule encore *sous forme d'inéquation variationnelle* [14] ; *on montre l'existence et l'unicité de* $\{\vec{D}, \vec{B}\} = \{\vec{D}^{\mathcal{D}_0}, \vec{B}^{\mathcal{D}_0}\}$ *solution de* (2.13) (2.14) (avec les conditions aux limites et initiales habituelles).

En outre si $\mathcal{D}_0 \to +\infty$, $\{\vec{D}^{\mathcal{D}_0}, \vec{B}^{\mathcal{D}_0}\} \to \{\vec{D}, \vec{B}\}$ *solution des équations de Maxwell usuelles.*

2.6. *Conclusion. Orientation.*

Faute de place, on ne donne pas d'autres exemples (cf. [12] [13] [14]) ; **il semble que l'on puisse dire que, du point de vue des applications, il y a “presque autant” de situations de la Physique Mathématique où l'on rencontre des inéquations que des équations** (1).

On donne ci-après quelques notions sur les *principaux outils disponibles* pour l'instant pour obtenir *l'existence* de solutions.

3. Outils principaux pour l'existence de solutions.

3.1. *Régularisation elliptique.*

L'idée est de ramener, par approximation puis passage à la limite, les cas “paraboliques” aux cas “elliptiques” résolus par ailleurs, pour des classes très générales d'opérateurs (les opérateurs pseudo-monotones de [3], axiomatisation importante de [21]). Cette idée conduit à approcher (1.8) par

$$\int_0^T \left(-\epsilon \frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial u_\epsilon}{\partial t} + A u_\epsilon - f \;,\; v - u_\epsilon\right) dt \geqslant 0$$

qui est, pour $\epsilon > 0$, de nature elliptique.

Détails techniques : [24].

3.2. *Régularisation parabolique.*

Une fois résolu le cas “parabolique” (par ex. par 3.1) il est naturel d'essayer de ramener les cas “hyperboliques” aux cas “paraboliques” par régularisation (ou *viscosité*). C'est ce qui a été fait dans [6].

---

(1) Les solutions des *équations* apparaissent d'ailleurs comme des *cas limites* de certaines *inéquations.*

3.3. *Pénalisation.*

3.3.1. *Cas de l'Exemple* 1.2.

Soit β un opérateur monotone de $V \to V'$ (i.e. $(\beta(u) - \beta(v), u - v) \geqslant 0$) borné hémi-continu, tel que

$$\text{Ker}\,\beta = K. \tag{3.1}$$

Exemple : $\beta = J(I - P_K)$, J = isomorphisme canonique de $V \to V'$, P_K = projection de $V \to K$.

On "remplace" (1.8) par *l'équation pénalisée*

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + Au_\epsilon + \dfrac{1}{\epsilon}\,\beta(u_\epsilon) = f\,, \\ u_\epsilon(0) = u_0\,. \end{cases} \tag{3.2}$$

On résout (3.2) par les méthodes "usuelles" de monotonie [30] [9] [24] ... On établit des estimations a priori indépendantes de ϵ.

Soit ensuite $v \in K$; alors – comme $\beta(v) = 0$ – on déduit de (3.2) que

$$\left(\frac{\partial u_\epsilon(t)}{\partial t} + Au_\epsilon(t) - f(t)\,, v - u_\epsilon(t)\right) = \frac{1}{\epsilon}\,(\beta(v) - \beta(u_\epsilon(t))\,, v - u_\epsilon(t)) \geqslant 0, \tag{3.3}$$

ce qui permet de faire $\epsilon \to 0$.

3.3.2. *Cas des Exemples des n° 2.2 et 2.4.*

Les équations pénalisées sont, respectivement,

$$\frac{\partial u_\epsilon}{\partial t} + \frac{1}{\epsilon}\,\beta\left(\frac{\partial u_\epsilon}{\partial t}\right) + Au_\epsilon = f\ ,\ u_\epsilon(0) = u_0\ ; \tag{3.4}$$

$$\frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{1}{\epsilon}\,\beta\left(\frac{\partial u_\epsilon}{\partial t}\right) + Au_\epsilon = f\ ,\ u_\epsilon(0) = u_0\ ,\ \frac{\partial u_\epsilon}{\partial t}(0) = u_1\,. \tag{3.5}$$

3.4. *Semi-groupes non linéaires. Approximation Yosida.*

L'approximation de Yosida [35] consiste à "approcher" l'équation

$$\frac{\partial u}{\partial t} + Du = f\,, \tag{3.6}$$

où $(-D)$ est générateur infinitésimal d'un semi-groupe (linéaire), par

$$\frac{\partial u_\epsilon}{\partial t} + \frac{1}{\epsilon}\,(I - (\epsilon D + I)^{-1})\,u_\epsilon = f\ ,\ (\epsilon > 0)\,. \tag{3.7}$$

Cette technique s'étend à certains opérateurs multivoques et c'est essentiel dans l'étude des semi-groupes non linéaires.

(cf. [4] [7] [10] [11] [18] [19]).

Approchons dans (1.18) $\partial\psi_K$ par cette méthode, K étant un convexe fermé de H. On vérifie que

$$(\epsilon\, \partial\psi_K + I)^{-1} = P_K \tag{3.8}$$

de sorte que l'on approche (1.18) par

$$\frac{\partial u_\epsilon}{\partial t} + Au_\epsilon + \frac{1}{\epsilon}(I - P_K)\, u_\epsilon = f\ ; \tag{3.9}$$

on retrouve un cas particulier de la pénalisation.

Remarque 3.1.

On arrive à préciser, par la technique des semi-groupes non linéaires, le comportement en t de la solution $u(t)$ et, surtout, *à préciser à quel élément de l'ensemble* $\partial\psi_K(u(t))$ appartient $-\left(\frac{\partial u(t)}{\partial t} + Au(t) - f(t)\right)$. Cf. [5] [18] [19].

3.5. *Approximation par régularisation.*

Dans le cadre 2.3, *on régularise* $j(v)$ en

$$j_\epsilon(v) = g \int_\Omega \left(\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right)^{\frac{1+\epsilon}{2}} dx \quad , \quad \epsilon > 0\ ; \tag{3.10}$$

on résout (1)

$$\left(\frac{\partial u_\epsilon}{\partial t}\ ,\ v\right) + a(u_\epsilon\ ,\ v) + b(u_\epsilon\ ,\ u_\epsilon\ ,\ v) + (j'_\epsilon(u_\epsilon)\ ,\ v) = (f\ ,\ v) \tag{3.11}$$

puis l'on passe à la limite en ϵ.

3.6. *Approximation par différences finies.*

Bornons-nous à (1.8). Désignons par u^n une "approximation" de u à l'instant $n\,\Delta t$. On "approche" alors (1.8) par

$$\left(\frac{u^{n+1} - u^n}{\Delta t}\ ,\ v - u^{n+1}\right) + a(u^{n+1}\ ,\ v - u^{n+1}) \geqslant (f^{n+1}\ ,\ v - u^{n+1}) \quad \forall v \in K, \tag{3.12}$$

ce qui définit u^{n+1} à partir de u^n par la résolution d'une *inéquation elliptique.* On peut alors passer à la limite. cf. [28] [16].

Remarque 3.2.

Il s'agit là du premier pas –d'ailleurs le plus facile !– pour *l'approximation numérique de la solution des inéquations variationnelles* [16].

---

(1) Il y a si $n \geqslant 3$ une difficulté technique obligeant (?) à une régularisation supplémentaire [14].

4. Compléments bibliographiques.

cf. [3] [14] [24] pour des résultats *d'unicité* ;

[5] [8] [22] [26] [33] pour l'étude de la *régularité* des solutions ;

[10] [34] pour l'étude d'inéquations dans des espaces *non réflexifs* ;

[2] pour l'étude (par adaptation des méthodes de [1]) *des solutions presque périodiques* des inéquations d'évolution ;

[17] [24] [5] [14] pour l'étude de *propriétés particulières de la solution* (principe du maximum, comparaison des solutions, etc...).

BIBLIOGRAPHIE

[1] AMERIO L. et PROUSE G. — *Abstract almost periodic functions and functional analysis,* Van Nostrand, New York, 1970.

[2] BIROLI M. — A paraître.

[3] BREZIS H. — *Annales Institut Fourier,* 18, 1968, p. 115-175.

[4] BREZIS H. — *Séminaire Alta Matematica,* Rome, mai 1970.

[5] BREZIS H. — *Inéquations variationnelles,* Paris, 1970.

[6] BREZIS H. et LIONS J.L. — *C.R. Acad. Sc. Paris,* 264, 1967, p. 928-931.

[7] BREZIS H. et PAZY A. — *J. of Functional Analysis,* 6, 1970, et *Israel J. of Math.,* à paraître.

[8] BREZIS H. et STAMPACCHIA G. — *Bull. Soc. Math. France,* 96, 1968, p. 153-180.

[9] BROWDER F. — *Proc. Symp. on non-linear Functional Analysis,* Chicago 1968, Publication A.M.S. 1970.

[10] CRANDALL M.G. et LIGGETT T.M. — A paraître.

[11] CRANDALL M.G. et PAZY A. — *J. Functional Analysis,* 3, 1969, p. 376-418.

[12] DUVAUT G. — Ce volume.

[13] DUVAUT G. et LIONS J.L. — *C. R. Acad. Sc. Paris,* septembre 1969, décembre 1969, janvier 1970, juin 1970.

[14] DUVAUT G. et LIONS J.L. — *Sur les inéquations en mécanique et en physique,* Dunod, 1971.

[16] Travaux de CEA J., GLOWINSKI R., HAUGAZEAU Y., NEDELEC J.C., BOURGAT, FORTIN, GOURSAT, MARROCCO, TRÉMOLIÈRES, M^lle^ VERCOUSTRE, VIAUD, et GLOWINSKI R., LIONS J.L. et TRÉMOLIÈRES R. — *Résolution numérique des inéquations de la mécanique et de la physique,* à paraître.

[17] HAUGAZEAU Y. — *C. R. Acad. Sc. Paris,* 1967.

[18] KATO T. — *J. Math. Soc. Japan,* 19, 1967, p. 508-520.

[19] KOMURA Y. — *J. Math. Soc. Japan,* 19, 1967, p. 493-507.

[20] LERAY J. — *J. Pures et Appliquées,* XII, 1933, p. 1-82; XIII, 1934, p. 331-418.

[21] LERAY J. et LIONS J.L. — *Bull. Soc. Math. France,* 93, 1965, p. 97-107.

[22] LEWY H. et STAMPACCHIA G. — *Comm. Pure Applied Math.,* 22, 1969, p. 153-188.

[23] LIONS J.L. — *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles,* Dunod-Gauthier Villars, 1968.

[24] LIONS J.L. — *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod-Gauthier Villars, 1969.

[25] LIONS J.L. — *Séminaire J. Leray*, Vol. II, 1965-66, p. 17-33.
[26] LIONS J.L. — Functional analysis and related topics, Tokyo, 1969; *M. S. of Japan*, 1970, p. 269-282.
[27] LIONS J.L. — *Sem. Alta Matematica*, Rome, mai 1970.
[28] LIONS J.L. — *C. R. Acad. Sc. Paris*, 263, 1966, p. 55-57.
[29] LIONS J.L. et STAMPACCHIA G. — *Comm. Pure Applied Math.*, XX, 1967, p. 493-519.
[30] MINTY G. — *Duke Math. J.*, 29, 1962, p. 341-346.
[31] PETROWSKY J.G. — *Mat. Sbornik*, 2, 44, 1937, p. 815-870.
[32] STAMPACCHIA G. — Ce volume.
[33] TARTAR L. — A paraître au *J. of Funct. Analysis*.
[34] TEMAM R. — A paraître.
[35] YOSIDA K. — *Functional analysis*, Grundleheren B. 123, Springer, 1965.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 853 à 858.

NOUVEAUX RÉSULTATS POUR LES HYPERSURFACES MINIMALES

par Mario MIRANDA

Je parlerai des hypersurfaces de dimension $n-1$ dans un espace euclidien à n dimensions dont l'aire, c'est-à-dire la mesure à $n-1$ dimensions, est minimale par rapport aux hypersurfaces ayant le même bord. Les problèmes que je considèrerai en détail sont la régularité à l'intérieur pour de telles hypersurfaces, le problème de Bernstein pour les fonctions de n variables et le problème de Dirichlet pour l'équation des hypersurfaces minimales. Ensuite je donnerai des indications sur des problèmes encore largement ouverts comme la régularité jusqu'au bord des solutions du problème de Plateau, la régularité des hypersurfaces dont la courbure moyenne est bornée et l'étude des hypersurfaces minimales en présence d'obstacles. La bibliographie que je donne est celle strictement nécessaire pour la compréhension du texte. Pour plus d'information sur les sujets traités je renvoie aux articles de MM. Almgren et Osserman qui ont paru dans le *Bull. A.M.S.* de 1969 et surtout au livre de Federer "Geometric measure theory" publié par Springer en 1969.

1. Un problème, qui est intéressant en soi et pour ses liaisons avec la théorie des hypersurfaces minimales, auquel on a beaucoup travaillé dans les dernières années est le problème des cônes minimaux. Un cône à $n-1$ dimensions dans R^n est l'ensemble des demi-droites issues de l'origine et s'appuyant sur une variété orientée compacte M de dimension $n-2$, contenue dans la sphère unité S^{n-1}. On dira qu'un tel cône est minimal si, en chacun de ses points différents de l'origine, sa courbure moyenne est nulle. Dans R^3 les seuls cônes minimaux sont les plans ; mais déjà à partir de R^4 la famille des cônes minimaux contient des cônes singuliers, dont l'exemple plus simple est donné par

$$\{x\ ,\ x \in R^4\ ,\ x_1^2 + x_2^2 = x_3^2 + x_4^2\ \}$$

à savoir la projection du tore

$$M = \{x\ ,\ x \in R^4\ ,\ x_1^2 + x_2^2 = 1/2\ ,\ x_3^2 + x_4^2 = 1/2\}$$

Un problème très intéressant dans la théorie des hypersurfaces minimales est de savoir si la partie du cône engendrée par la variété M et contenue dans la boule unité est d'aire minimale dans la classe des variétés de bord M ; c'est-à-dire si le cône est solution du problème de Plateau avec donnée M. Evidemment les hyperplans ont cette propriété ; la question est de savoir s'il y a aussi des cônes singuliers ayant la même propriété. La réponse à cette question est d'autant plus interessante que la non-existence de cônes singuliers d'aire minimum entraîne la régularité intérieure pour toutes les solutions du problème de Plateau, comme cela avait été remarqué par Federer, Fleming, Reifenberg, De Giorgi et Almgren. Mais il y a encore un autre problème qui est strictement lié à la non-existence de cônes singuliers d'aire minimum, à savoir le problème de Bernstein. Bernstein a prouvé pour $n = 2$, l'énoncé suivant :

"Si $f \in C^2(R^n)$ est solution de l'équation des hypersurfaces minimales dans tout R^n, alors f est un polynôme de degré $\leqslant 1$".

Beaucoup d'autres auteurs ont donné des démonstrations du théorème de Bernstein. En 1962 Fleming a montré que la non-existence de cônes singuliers d'aire minimum dans R^{n+1} entraine la validité de l'énoncé de Bernstein dans R^n. Comme il n'y a pas de cônes minimaux dans R^3 Fleming avait obtenu en particulier une nouvelle démonstration du théorème de Bernstein. En 1965 De Giorgi avait remarqué que pour la validité de l'énoncé de Bernstein dans R^n il suffisait de ne pas avoir de cônes singuliers d'aire minimum dans R^n, il obtenait ainsi l'extension du théorème de Bernstein à $n = 3$.

On voit alors comment le problème des cônes singuliers solutions du problème de Plateau était devenu un des problèmes principaux dans la théorie des hypersurfaces minimales. Ce problème a été complètement résolu dans les années 1966-1969 par les articles de Almgren, Simons et Bombieri – De Giorgi – Giusti. D'abord Almgren en 1966 a demontré la non-existence de cônes singuliers d'aire minimum dans R^4, ensuite Simons en 1968 a prouvé le même résultat jusqu'à la dimension 7. Finalement Bombieri – De Giorgi – Giusti en 1969 ont demontré que le cône

$$\left\{x \; ; \; x \in R^8 \; , \; \sum_{i=1}^{4} x_i^2 = \sum_{i=5}^{8} x_i^2\right\}$$

déjà considéré par Simons, est d'aire minimum. Ils ont ainsi prouvé que le résultat de Almgren-Simons concernant la régularité intérieure des solutions du problème de Plateau est le meilleur possible. (Récemment H. Blaine Lawson a trouvé d'autres cônes singuliers d'aire minimum). Dans le même article Bombieri – De Giorgi – Giusti montrent que même l'énoncé de Bernstein est faux à partir de $n = 8$, en prouvant l'existence d'une solution de l'équation des hypersurfaces minimales dans R^8 qui croit à l'infini comme $|x|^3$.

A ce point il est intéressant de remarquer que des énoncés plus faibles que celui de Bernstein sont vrais en toute dimension. Par exemple la conclusion de Bernstein est valable pour les solutions non négatives de l'équation des hypersurfaces minimales ou plus généralement pour les solutions vérifiants une inégalité du type : $f(x) \geqslant -M - K|x|$, $\forall x \in R^n$, avec M et K constantes positives. Ce résultat est conséquence d'une remarque de Moser sur la validité du théorème de Bernstein pour les solutions à dérivées premières bornées et de la majoration à priori du gradient des solutions de l'équation des hypersurfaces minimales prouvée par Bombieri – De Giorgi – Miranda.

En ce qui concerne encore le problème de Bernstein, précisons que la remarque de Fleming déjà signalée permet d'étendre le théorème de Bernstein à des hypersurfaces qui ne sont pas nécessairement des graphes ; par exemple pour les hypersurfaces de De Giorgi on a le théorème suivant :

"Soit $n \leqslant 7$, $E \subset R^n$ mesurable avec $\partial E \neq \emptyset$, et tel que sa fonction caractéristique soit de gradient minimal. Alors E est un demi-espace".

Un tel énoncé n'est plus valable dans R^8. Comme pour l'énoncé de Bernstein la conclusion est valable pour toute dimension si on ajoute l'hypothèse que E contienne un demi-espace (v. Miranda [19]).

Pour terminer cette première partie je veux signaler un intéressant résultat de Federer concernant les singularités des solutions du problème de Plateau. Federer a démontré que pour $n \geqslant 8$ la dimension de l'ensemble singulier ne peut pas dépasser $n - 8$.

2. Le deuxième sujet dont je veux parler est le problème de Dirichlet pour l'équation des hypersurfaces minimales. Ce problème dans le cas bidimensionnel a une solution classique qui est la suivante (v. Bernstein, Haar et surtout Rado) :

"Soit Ω un ouvert convexe et borné dans R^2 et g une fonction continue sur $\partial\Omega$. Il existe une seule fonction f dans $C^2(\Omega) \cap C(\bar{\Omega})$ solution de l'équation des hypersurfaces minimales dans Ω et prenant la valeur g sur $\partial\Omega$". La condition de convexité pour Ω est aussi nécessaire (v. Finn et Jenkins-Serrin) en ce sens que pour tout ensemble non convexe Ω il existe une fonction continue g sur $\partial\Omega$ pour laquelle le problème de Dirichlet n'a pas de solutions.

Le cas $n \geqslant 3$ a été considéré et complètement résolu ces dernières années. Le premier résultat a été obtenu en faisant des hypothèses qui assuraient une majoration a priori des dérivées premières dans tout le domaine. Une telle hypothèse est la B.S.C. (Bounded Slope Condition) qui est équivalente pour $n = 2$ à la condition des trois points de Haar et en dimension supérieure, à la condition des $n + 1$ points. En utilisant la B.S.C. on démontre l'existence d'une solution lipschitzienne du problème de Dirichlet par un simple argument variationnel. La régularité à l'intérieur de cette solution est assurée par les résultats de De Giorgi – Nash, Morrey etc. Des conditions suffisantes entrainant la B.S.C. sont : Ω uniformément convexe et g de classe C^2(v. Gilbarg, Miranda [18] et Stampacchia). En 1968 Jenkins et Serrin ont prouvé que la condition naturelle (nécessaire et suffisante) pour le domaine Ω est que la courbure moyenne de $\partial\Omega$ soit de signe constant. Jenkins et Serrin conservent l'hypothèse que g soit de classe C^2. La majoration à priori du gradient déjà dite et un simple procédé d'approximation permettent l'extension du résultat de Jenkins et Serrin au cas de g continue. On a ainsi l'extension du resultat classique à toutes les dimensions.

Remarquons que la dite majoration du gradient permet d'obtenir la régularité intérieure pour les solutions faibles (dans un sens a preciser) de l'équation des hypersurfaces minimales.

3. Je veux maintenant donner des indications sur le problème de la régularité jusqu'au bord des solutions du problème de Plateau. Comme d'habitude je ne considererai pas les résultats spéciaux relatifs au cas bidimensionnel. Le seul résultat général (en n variables) de ma connaissance est celui de Allard. Ce résultat, annoncé dans le *Bull. A.M.S.* en 1969, est le suivant :

"Si B est une variété orientée compacte de dimension $m - 1$ dans $R^n (0 < m < n)$ de classe $p \geqslant 2$ (analytique) contenue dans la frontière d'un ensemble uniformément convexe et si T est une solution du problème de Plateau correspondant à B (par example T peut-être un courant au sens de Federer-Fleming) alors dans un voisinage de B le courant T est une variété m-dimensionnelle de bord B et de classe $p - 1$ (analytique)".

Je veux encore donner des indications sur deux directions de recherche dont le point de départ est la théorie des hypersurfaces minimales. Dans la première direction, l'étude des hypersurfaces a courbure moyenne donnée, il faut indiquer un résultat de Allard, non encore publié, sur la régularité presque partout à l'intérieur des varifolds (surfaces généralisées au sens de Almgren) dont la courbure moyenne est bornée. Ce résultat est une généralisation des résultats de Reifenberg, De Giorgi et Almgren pour les hypersurfaces minimales. Dans cette même direction il y a les recherches de Bakel'man et Serrin (v. la conférence de Serrin) sur l'équation differentielle

$$\sum_{i=1}^{n} D_i \left(\frac{D_i f}{\sqrt{1 + |Df|^2}}\right) = A \ , \qquad (*)$$

ou A est une fonction donnée qui peut dépendre de x, f et Df. On doit tenir compte du fait que le premier membre de l'équation (*) représente la courbure moyenne du graphe de f multipliée par le nombre des variables.

La deuxième direction de recherche est l'étude des hypersurfaces minimales en présence d'obstacles, à savoir du problème de Plateau dans un espace qui ne soit pas R^n tout entier mais seulement une partie de celui-ci. Une telle formulation générale du problème de Plateau se trouve déjà dans l'article de Federer-Fleming. Cette formulation conduit à des situations nouvelles surtout lorsqu'on s'occupe de la régularité des solutions. En effet, si l'espace n'est pas convexe la solution peut s'appuyer sur le bord de l'espace et cela a comme conséquence, même dans le cas d'obstacles très réguliers, une limitation de la régularité de la solution. Avec des exemples simples on s'aperçoit qu'on ne peut pas attendre en général une régularité meilleure que $C^{1,1}$. Mais les obstacles ont aussi une influence favorable sur la régularité. Rappelons-nous que pour le problème sans obstacles, des singularités très importantes pouvaient apparaître (v. le cône de Bombieri – De Giorgi – Giusti). Un tel type de singularités ne se présente pas au voisinage des obstacles de classe C^1. Nous avons en effet le résultat suivant de Miranda [19] : "Soit L un ensemble dont la frontière est de classe C^1 et E un ensemble mesurable contenant L et de gradient minimal dans un ouvert Ω de R^n (parmi les ensembles contenant L). Alors il existe un ouvert $A \supset \Omega \cap \partial L$ tel que $[\mathrm{supp}(D\varphi_E)] \cap A$ soit une variété a $n - 1$ dimensions de classe C^1". (φ_E est la fonction caractéristique de E).

Même dans le cas des obstacles les choses changent complètement lorsqu'on passe au problème non paramétrique. Dans ce cas les obstacles, déjà considérés par d'autres auteurs pour d'autres équations, (v. la conférence de Stampacchia) sont introduits sous forme d'inégalités devant être vérifiées par les fonctions cherchées. Pour le cas de l'équation des hypersurfaces minimales le problème a été considéré par Kinderlehrer, Nitsche et Tomi en deux variables. Pour le cas de n variables il y a les résultats de Lewy-Stampacchia, Giaquinta-Pepe et Giusti. Tous ces résultats pour le cas de n variables sont à paraître.

BIBLIOGRAPHIE

[1] ALLARD W.K. — On the boundary regularity for Plateau's problem, *Bull. A.M.S.*, may 1969, p. 522-523.

[2] ALMGREN F.J. Jr. — Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem, *Ann. of Math.*, 1966, p. 277-292.

[3] ALMGREN F.J. Jr. — Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure, *Ann. of Math.*, 1968, p. 321-391.

[4] ALMGREN F.J. Jr. — Measure theoretic geometry and elliptic variational problems, *Bull. A.M.S.*, March 1969, p. 285-304.

[5] BOMBIERI E., DE GIORGI E. et GIUSTI E. — Minimal cones and the Bernstein problem, *Inv. Math.*, 1969, p. 243-268.

[6] BOMBIERI E., DE GIORGI E. et MIRANDA M. — Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche, *Arch. Mat. Mech. Anal.*, 1969, p. 255-267.

[7] DE GIORGI E. — Frontiere orientate di misura minima, *Sem. Mat. S.N.S.*, Pisa, 1961, p. 1-56.

[8] DE GIORGI E. — Una estensione del teorema di Bernstein, *Ann. S.N.S.*, Pisa, 1965, p. 79-85.

[9] FEDERER H. et FLEMING W.H. — Normal and integral currents, *Ann. of Math.*, 1960, p. 458-520.

[10] FEDERER H. — *Geometric measure theory*, Springer-Verlag, New York, 1969.

[11] FEDERER H. — The singular sets of area minimizing rectifiable currents with codimension 1 and of area minimizing flat chains modulo two with arbitrary codimension, *Bull. A.M.S.*, 1970.

[12] FINN R. — Remarks relevant to minimal surfaces, *Techn. Rep.*, No. 120, Standard 1963.

[13] FLEMING W.H. — On the oriented Plateau problem, *Rend. Circ. Mat.*, Palermo, 1962, p. 69-90.

[14] FLEMING W.H. — Flat chains over a finite coefficient group, *Trans. A.M.S.*, 1966, p. 160-186.

[15] GILBARG D. — Boundary value problems for non-linear equations in n variables, *Non-linear-problems*, Univ. of Wisc. Press, Madison Wisc. 1963, p. 151-159.

[16] JENKINS H. et SERRIN J. — Minimal surface equation in higher dimensions, *J. Reine u. Ang. Math.*, 1968, p. 170-187.

[17] LAWSON Blaine H. — *The equivariant Plateau problem and interior regularity*, (è paraître).

[18] MIRANDA M. — Un teorema di esistenza e unicità per il problema dell' area minima in n variabili, *Ann. S.N.S.*, Pisa, 1965, p. 233-250.

[19] MIRANDA M. — *Frontiere minimali con ostacoli* (à paraître).

[20] MOSER J. — On Harnack's theorem for elliptic differential equations, *Comm. P.A.M.*, 1961, p. 577-591.

[21] OSSERMAN R. — Minimal varieties, *Bull. A.M.S.*, nov. 1969, p. 1092-1120.

[22] RADO T. — On the problem of Plateau, *Eg. d. Math.*, Berlin, 1933.

[23] REIFENBERG E.R. — Solution of the Plateau problem for *m*-dimensional surfaces of varying topological type, *Acta Math.*, 1960, p. 1-92.

[24] SIMONS J. — Minimal varieties in Riemannian manifolds, *Ann. of Math.*, 1968, p. 62-105.

[25] STAMPACCHIA G. — On some regular multiple integral problems in the Calculus of Variations, *Comm. P.A.M.*, 1963, p. 383-421.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 859 à 866.

DIFFERENTIABILITY THEOREMS FOR NON-LINEAR ELLIPTIC EQUATIONS *

by Charles B. MORREY, Jr.

1. Introduction.

I shall discuss equations of the form

$$(1) \qquad \int_G \sum_{i=1}^{N} \sum_{|\alpha| \leqslant m_i} \zeta^i_{,\alpha}(x) A_i^\alpha[x, Dz(x)]\, dx = 0 \ , \text{ for all } \zeta \in C_C^\infty(G)$$

where $C_C^\infty(G)$ denotes the set of functions of class C^∞ with compact support in $G, x = (x^1, \ldots, x^\nu), \zeta = (\zeta^1, \ldots, \zeta^N)$, and $z = (z^1, \ldots, z^N)$ are vector functions, $\alpha = (\alpha_1, \ldots, \alpha_\nu)$ denotes a *multi-index* in which each α_i is a non-negative integer, $|\alpha| = \alpha_1 + \cdots + \alpha_\nu$, and

$$\phi_{,\alpha}(x) \text{ and } D^\alpha \phi(x) \text{ stand for } \frac{\partial^{|\alpha|}\phi}{(\partial x^1)^{\alpha_1} \ldots (\partial x^\nu)^{\alpha_\nu}},$$

and Dz stands for all the derivatives $D^\alpha z^i$ for $i = 1, \ldots, N$ and $0 \leqslant |\alpha| \leqslant m_i$ (of course if $|\alpha| = 0$, $D^\alpha z = z$).

Equations of the form (1) were discussed in my paper "Partial regularity theorems for elliptic systems" which appeared in the January 1968 issue of the Journal of Mathematics and Mechanics [17] where it was assumed that the A_i^α are of class $C_\mu^2 (0 < \mu < 1)$ in their arguments and satisfy

$$|A(x, p)| \ , \ |A_x(x, p)| \leqslant MV^{k-1} \ ; \ |A_p| \, , |A_{px}| \, , |A_{pp}| \leqslant MV^{k-2} \ ;$$

$$(2) \qquad \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} A^\alpha_{i p^j_\beta}(x, p)\, \pi^i_\alpha \pi^j_\beta \geqslant m^* V^{k-2} |\pi|^2 \ , \ m^* > 0 \, ,$$

$$k \geqslant 2 \ , \ V^2 = 1 + \sum_{i=1}^{N} \sum_{|\alpha| \leqslant m_i} (z^i_{,\alpha})^2 \quad (p = \{p^i_\alpha\} \, , \ 0 \leqslant |\alpha| \leqslant m_i)$$

where

$$(3) \qquad |A_p|^2 \text{ means } \sum_{i,j=1}^{N} \sum_{|\alpha| \leqslant m_i} \sum_{|\beta| \leqslant m_j} (A^\alpha_{i p^j_\beta})^2 \ , \text{ etc.}$$

I and a student have obtained similar results under somewhat more general hypotheses on the A_i^α (see below).

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It is to be noticed that if the A_i^α and z^i are sufficiently differentiable, the terms of (1) involving the derivatives of ζ^i may be integrated by parts so that (1) becomes

$$(1') \quad \int_G \sum_i \zeta^i \left\{ \sum_{|\alpha| \leqslant m_i} (-1)^{|\alpha|} D^\alpha A_i^\alpha [x, Dz(x)] \right\} dx = 0, \text{ for all } \zeta \in C_C^\infty(G).$$

However, the equations (1) make sense if the z^i are merely in the Sobolev spaces $H_k^{m_i}$ (see below) in which case we say that z is a *weak solution* of the equations $\{ \}_i = 0$ in (1′).

It is easy to see that the left side of (1) is just the so-called *first variation* of an integral of the form

$$(4) \qquad I(z) = \int_G f[x, Dz(x)]\, dx$$

if we set

$$(5) \qquad A_i^\alpha(x, p) = f_{p_\alpha^i}(x, p).$$

Thus a discussion of the regularity of the solutions of (1) includes that of the extremals of multiple integral problems in the calculus of variations. Of course the equations (1) are more general since the A's for a variational problem would satisfy (5) and hence

$$(6) \qquad A^\alpha_{ip^j_\beta} = A^\beta_{jp^i_\alpha}$$

which we do not assume. The inequality in the middle of (2) just involves the symmetric part of the A_p matrix. If A satisfies (5), this inequality implies that f is convex in the p-variables.

In the case where all the $m_i = 1$ and (5) holds, (1) becomes

$$(1'') \qquad I_1(z, \zeta; G) \equiv \int_G (\zeta^i_{,\alpha} f_{p^i_\alpha} + \zeta^i f_{z^i})\, dx$$

where repeated indices are summed (i from 1 to N, α from 1 to ν) and $\zeta^i_{,\alpha}$ now means $\partial\zeta^i/\partial x^\alpha$, $\alpha = 1, \ldots, \nu$. If z, ζ, and $f \in C^1(G)$,

$$(7) \qquad I_1(z, \zeta; G) = \phi'(0) \quad \text{where} \quad \phi(\lambda) = I(z + \lambda\zeta; G)$$

and $I(z, G)$ is given by (4) which reduces in this case to

$$(4') \qquad I(z, G) = \int_G f(x, z, \nabla z)\, dx, \quad \nabla z = \{z^i_{,\alpha}\}.$$

In case we wish to discuss the minimizing character of a critical solution, it is desirable to discuss the *second variation* defined by

$$(8) \quad \phi''(0) \equiv I_2(z, \zeta; G) = \int_G (f_{p^i_\alpha p^j_\beta} \zeta^i_{,\alpha} \zeta^j_{,\beta} + 2 f_{p^i_\alpha z^j} \zeta^i_{,\alpha} \zeta^j + f_{z^i z^j} \zeta^i \zeta^j)\, dx$$

in case f is of class C^2. If $I_2(z, \zeta; G) \geqslant 0$ for every $\zeta \in C^1(\overline{G})$ which vanishes on ∂G, it follows that (see [16], p. 11)

$$(9) \qquad f_{p^i_\alpha p^j_\beta} \lambda_\alpha \lambda_\beta \xi^i \xi^j \geqslant 0 \, \forall \, \lambda, \xi .$$

In case $\nu = 1$ or $N = 1$ and (9) holds for all (z, x, p) it follows that *f is convex in the p's* for each (x, z). However, if $\nu > 1$ and $N > 1$, this is no longer true. A variational problem for which the inequality holds in (9) for all $\lambda \neq 0$ and $\xi \neq 0$ is called *regular*. In this case the Euler equations

$$(10) \quad \frac{\partial}{\partial x^\alpha} f_{p^i_\alpha} = f_{z^i} \text{ or } f_{p^i_\alpha p^j_\beta} z^j_{,\alpha\beta} + f_{p^i_\alpha z^j} z^j_{,\alpha} + f_{p^i_\alpha x^\alpha} = f_{z^i}, \; i = 1, \ldots, N$$

(which are satisfied weakly by any critical solution) are *strongly elliptic* in the sense of Nirenberg ([22], [16], § 6.5). The condition in the middle of (2) corresponds in the present case to assuming that the quadratic form

$$(11) \qquad f_{p^i_\alpha p^j_\beta} \pi^i_\alpha \pi^j_\beta$$

is positive definite which implies that f be convex in the p's for each (x, z). Many of our results have this requirement and therefore do not include regularity results for the most general regular variational problems. The number k in (2) is seen in our case to be the degree of f at ∞ as a function of p, since our assumptions imply that

$$(12) \quad f(x, z, p) \geqslant f(x, z, 0) + f_{p^i_\alpha}(x, z, 0)\, p^i_\alpha + |p|^2 \int_0^1 [1 + |z|^2 + t^2 |p|^2]^{(k-2)/2} \, dt$$

When discussing a variational problem of this sort, it is customary to adjoin the condition

$$(13) \qquad m\,|p|^k - K \leqslant f(x, z, p) \leqslant M\,|p|^k + K, \quad 0 < m \leqslant M.$$

In the case where the A^α_i satisfy (5), the *existence* of a solution can be proved by using the so-called direct methods of the calculus of variations developed by Tonelli and others to show the existence of a minimizing function; if f satisfies the conditions above, the minimizing vector satisfies (1). The idea of the direct methods is to show (i) that the integral to be minimized is lower semicontinuous with respect to some kind of convergence, (ii) that it is bounded below in some class of "admissible functions" and (iii) that there is a "minimizing sequence," i.e., a sequence for which the integral tends to its infimum, which converges in the sense required to some admissible function.

For the one dimensional problems ($\nu = 1$) with all the $m_i = 1$, Tonelli found it expedient to allow absolutely continuous functions as admissible and to use uniform convergence. This comes about roughly as follows: Suppose that

$$(14) \qquad f(x, z, p) \geqslant m|p|^r - K, \quad r > 1, \quad m > 0$$

(which is not unreasonable since f is convex in p) or satisfies (13). Then

$$(15) \qquad \int_a^b |z_n'(x)|^r \, dx \leq L \quad , \quad n = 1, 2, \ldots$$

in any minimizing sequence $\{z_n\}$. From (15), one sees from the Hölder inequality that any minimizing sequence is equi-continuous. Moreover if (a subsequence of) z_n converges uniformly to some function z on $[a, b]$, then z is absolutely continuous and

$$\int_a^b |z'(x)|^r \, dx \leq \liminf \int_a^b |z_n'(x)|^r \, dx.$$

Then z would be minimizing if the integral were lower-semicontinuous with respect to that type of convergence.

Unfortunately the equicontinuity of minimizing sequences is not implied by (14) $\left(|p|^2 = \sum_{i,\alpha} (p_\alpha^i)^2\right)$ unless $r > \nu$. This fact led me in the Fall of 1937 to introduce function spaces, now to be identified with the so-called "Sobolev spaces", in order to carry through the program for cases where $\nu > 1$. We assume that the reader is familiar with these spaces. We denote by $H_p^m(G)$ those "functions" (i.e., distributions) whose (distribution) derivatives up to the m-th order are in L_p on G. These functions are defined and discussed in the author's book ([16], Chapter 3). We recall here that $H_{p0}^m(G)$ is the closure in $H_p^m(G)$ of the set $C_C^\infty(G)$.

Using these functions, we may state a lower semicontinuity theorem and an existence theorem as follows (see [16], 22-24) :

THEOREM (Lower semicontinuity). – *Suppose $f = f(x, z, p)$ and the $f_{p_\alpha^i}$ are continuous with $f(x, z, p) \geq 0$ for all (x, z, p), suppose f is convex in p for each (x, z), and suppose $z_n \rightharpoonup$ (tends weakly to) z in $H_1^1(D)$ for each $D \subset G$. Then*

$$I(z, G) \leq \liminf_{n \to \infty} I(z_n, G).$$

THEOREM (Existence). – *Suppose f satisfies the conditions of the preceding theorem as well as* (14), *suppose G is bounded, $z^* \in H_r^1(G)$, and $I(z^*, G) < \infty$. Then there exists a $z \in H_r^1(G)$ such that $z - z^* \in H_{r0}^1(G)$ and z minimizes $I(z, G)$ among all such functions.*

More general existence theorems have been proved, of course. And recently existence theorems have been proved for equations of the general type (1) ; these involved the theory of monotone operators and its extensions developed by Visik, Minty, Browder, Leray, Lions and others (see [16], § 5.12). But these yield only the conclusion that each z^i of a solution belongs to $H_r^{m_i}(G)$.

The principal purpose of this paper is to present some results which state that solutions (possibly weak) of certain elliptic systems have additional differentiability properties. One of the first results of this sort was that due to *S.* Bernstein in 1904, who proved that any solution of class $C^{(3)}(G)$ of an analytic non-linear elliptic equation of the second order in one unknown function and two independent variables ($N = 1$, $\nu = 2$) is analytic. His proof was long and many others (including himself) gave simpler proofs and extended his results. These analyticity results have been extended to very general elliptic systems and include results concerning analytic extensions across an analytic boundary of solutions satisfying general regular boundary conditions (in the sense of Agmon, Douglis, and Nirenberg [1]) ; for references see the author's presidential address [18] p. 688, Bull. Amer. Math. Soc., 75 (1969), p. 688.

A somewhat different series of generalizations of Bernstein's result was begun by L. Lichtenstein [10] when he showed in 1912 that a solution of class $C^2(G)$, $G \subset R^2$, of an analytic variational problem ($\nu = 2$, $N = 1$) is of class C''' and hence analytic. This result was extended in 1929 by E. Hopf [8] to the case where the solution was required only to be of class $C^1_\mu(G)$ for some μ, $0 < \mu < 1$. The author [11] extended this result further in January 1938 to the case where the solution was required merely to satisfy a Lipschitz condition. Somewhat earlier, Haar [7] showed the existence and uniqueness of a Lipschitz solution of any variational problem in which $\nu = 2$, $N = 1$, $f = f(p)$, ∂G is strictly convex, and the given boundary values satisfy a three point condition (i.e., $\exists$ an $M \ni$ any plane intersecting the boundary curve in three points has slope $\leqslant M$) ; this solution is analytic by the author's result.

During the year 1937-38, the author proved in the case where f satisfies conditions a little more general than the corresponding conditions in (2) with $k = \nu = 2$, N arbitrary, m_i all $= 1$, that the solution vector $z \in C^n_\mu(G)$ if $f \in C^n_\mu$ and $n \geqslant 3$. These results were presented in the seminar of Marston Morse at the Institute for Advanced Study during the Spring of 1938 ; the notes were written by H. Busemann and are in the library at the Institute under Busemann's name. They were also presented in an invited address before the American Mathematical Society in the Fall of 1939 [12]. A greatly simplified account of this work is to be found in the author's Pisa lectures [13], especially Chapter 4.

In the work above the author used a certain "Dirichlet growth" principle which did not generalize to more than two variables. No progress was made on this problem until the famous results of De Giorgi [3] and Nash [19] who proved independently that any solution u in $H^1_2(G)$ of an equation of the form

$$(16) \qquad \int_G \zeta_{,\alpha} a^{\alpha\beta}(x) u_{,\beta}\, dx = 0 \qquad \zeta \in H^1_{20}(G)$$

in which the $a^{\alpha\beta}$ are bounded and measurable and satisfy

$$(17) \qquad m|\lambda|^2 \leqslant a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \leqslant M|\lambda|^2 \quad , \quad 0 < m \leqslant M \quad , \quad x \in G \, ,$$

are Hölder continuous on interior domains. De Giorgi used this to show that if z is a solution $\in H^1_2(G)$ of a problem in which $N = 1$, $k = 2$, ν arbitrary, $f = f(p)$, then $z \in C^n_\mu(G)$ (analytic, C^∞) if f is. During the year 1959-60, the author and a student E. R. Buley [14], [15] and concurrently O. A. Ladyzenskaya and N. N. Ural'tseva [9] established corresponding regularity results for equations of the form (1) in which $N = 1$ and all the $m_i = 1$ but ν and k are arbitrary. The

A_i^α were required to satisfy conditions closely resembling (2) ; the exact conditions for the work of this author and his student are to be found in the author's book [16], p. 33.

For several years there were no results for the cases where $N > 1$ except some results on higher order systems in $\nu = 2$ variables by J. Neças [20], [21]. Meanwhile, Reifenberg [23], [24], [25], Almgren [2], and others have shown the existence of solutions of the *parametric* problem in higher dimensions which solutions were each the union of smooth open manifolds with a locally compact set of (ν-dimensional) measure 0. Their methods are entirely different from those involved in the non-parametric problem. However, the author was able to adapt some of Almgren's theorems to prove the following theorem [17] :

THEOREM. — *Suppose the A_i^α are of class C_μ^2 and satisfy* (2), *suppose each $z^i \in H_k^{m_i}(G)$, and suppose z is a solution of* (1). *Then each $z^i \in C_\mu^{m_i+2}(D)$ where $D = G - Z$ and Z is locally compact, and of measure zero.*

In view of the discovery by E. Giusti and M. Miranda [5] of an analytic variational problem which has the unique extremal

$$u^i = |x|^{-1}\, x^i \quad , \quad x \in B(0\,,1)$$

the theorem above is of some interest. This example and the theorem above focus interest on the properties of the set Z. In a recent paper, Giusti [4] has shown under assumptions related to ours that Z has Hausdorff $(\nu - 1)$ dimensional measure zero. In another recent paper [6], Giusti and Miranda have proved the following theorem :

THEOREM. — *Suppose the a's are bounded and uniformly continuous for all $(x\,,u)$ and satisfy*

$$a_{ij}^{\alpha\beta}(x\,,u)\ \pi_\alpha^i \pi_\beta^j \geqslant |\pi|^2 \tag{18}$$

for all $(x\,,u\,,\pi)$. Suppose that $u \in H_p^1(\Delta)$ for some $p \geqslant 2$ and all domains $\Delta \subset\subset G$ and suppose u is a solution of

$$\int_G \zeta^i_{,\alpha} a_{ij}^{\alpha\beta}(x\,,u) u^j_{,\beta}\, dx = 0\ \forall\, \zeta \in \mathrm{Lip}_c(G) \tag{19}$$

Then u is Hölder continuous on each compact subset of a domain $D = G - Z$ where Z is locally compact with Hausdorff $(\nu - p)$-dimensional measure zero. ($\mathrm{Lip}_c G$ is all Lipschitz functions with compact support in G).

BIBLIOGRAPHY

[1] AGMON S., DOUGLIS A. and NIRENBERG L. — Estimates near the boundary for the solutions of elliptic differential equations satisfying general boundary values. II, *Comm. Pure Appl. Math.*, 17, 1964, p. 35-92.

[2] Almgren F.J. Jr. — Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure, *Ann. of Math.*, 87, 1968, p. 321-391.
[3] De Giorgi E. — Sulla differenziabilitá e analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino A. Sci. Fis. Mat. Nat.* (3) 3, 1957, p. 25-43.
[4] Giusti E. — *Sulla regolarità parziale delle soluzione di sistemi ellitici quasi-lineari di ordine arbitrario* (preprint).
[5] Giusti E. and Miranda M. — Un esempio di soluzione discontinue per un problema di minimo relativo ad un integrale del calcolo delle variazioni, *Boll. Un. Mat. Ital.*, (4) 2, 1968, p. 1-8.
[6] Giusti E. and Miranda M. — Sulla regolarità delle soluzioni deboli di una class di sistemi ellittici quasi-lineari, *Archive Rat'l. Mech. Anal.*, 31, 1968, p. 173-184.
[7] Haar A. — Uber das Plateausche Problem, *Math. Ann.*, 97, 1927, p. 127-158.
[8] Hopf E. — Zum analytischen Charakter der Lösungen regulärer zwei dimensionaler Variationsprobleme, *Math. Z.*, 30, 1929, p. 404-413.
[9] Ladyzenskaya O.A. and Ural'tseva N. — On the smoothness of weak solutions of quasi-linear equations in several variables and of variational problems, *Comm. Pure Appl. Math.*, 14, 1961, p. 481-495.
[10] Lichtenstein L. — Uber den analytischen Charakter der Lösungen zweidimensionaler Variationsprobleme, *Bull. Acad. Sci. Cracovie, Cl. Sci. Math. Nat.* A, 1912, p. 915-941.
[11] Morrey C.B. Jr. — On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, 43, 1938, p. 126-166.
[12] Morrey C.B. Jr. — Existence and differentiability theorems for the solutions of variational problems for multiple integrals, *Bull. Amer. Math. Soc.*, 46, 1940, p. 439-458.
[13] Morrey C.B. Jr. — Multiple integral problems in the calculus of variations and related topics, *Ann. Scuola Norm.*, Pisa (III) 14, 1960, p. 1-61.
[14] Morrey C.B. Jr. — Extensions and applications of the De Giorgi-Nash results, *Proc. Sympos. Pure Math.*, Vol. IV, Amer. Math. Soc., Providence, R.I., 1961.
[15] Morrey C.B., Jr. — *Existence and differentiability theorems for multiple integrals. Partial differential equations and continuum mechanics*, Univ. of Wisconsin Press, Madison, 1961, p. 241-270.
[16] Morrey C.B., Jr. — *Multiple integrals in the calculus of variations*, Die Grundlehren der Math. Wissenschaften, Band 130, Springer-Verlag, New York, 1966.
[17] Morrey C.B., Jr. — Partial regularity results for non-linear elliptic systems, *J. Math. Mech.*, 17, 1968, p. 649-670.
[18] Morrey C.B., Jr. — Differentiability theorems for weak solutions of non-linear elliptic differential equations, *Bull. Amer. Math. Soc.*, 75, 1969, p. 684-705.
[19] Nash J. — Continuity of the solutions of parabolic and elliptic equations, *Amer. J. Math.*, 80, 1958, p. 931-954.
[20] Neças J. — Les méthodes directes en théorie des équations elliptiques, *Academia*, Prague, 1967.
[21] Neças J. — Sur la régularité des solutions variationelles des équations elliptiques non-linéaires d'ordre 2k en deux dimensions, *Ann. Scuola Norm. Sup.*, Pisa, 21, 1967, p. 427-457.
[22] Nirenberg L. — Remarks on strongly elliptic partial differential equations, *Comm. Pure Appl. Math.*, 8, 1955, p. 648-674.
[23] Reifenberg E.R. — Solution of the Plateau problem for m-dimensional surfaces of varying topological type, *Acta Math.*, 104, 1960, p. 1-92.

[24] Reifenberg E.R. — An epiperimetric inequality related to the analyticity of minimal surfaces, *Ann. of Math.*, 80, 1964, p. 1-14.

[25] Reifenberg E.R. — On the analyticity of minimal surfaces, *Ann. of Math.*, 80, 1964, p. 15-21.

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BOUNDARY CURVATURES AND THE SOLVABILITY OF DIRICHLET'S PROBLEM

by James SERRIN

In recent years the theory of the Dirichlet problem for second order quasilinear elliptic differential equations has been brought to a significant level of completeness, and in particular it has been found that certain invariants of the equation play a central role in the question of existence of solutions. Moreover, in certain cases, governed by the behavior of these invariants, it turns out to be necessary to impose curvature restrictions on the boundaries of the underlying domains in order for the Dirichlet problem to be generally solvable for given smooth boundary data. The purpose of this note is to give a very brief outline of this theory, and to present a number of examples illustrating its application to particular situations.

We consider throughout quasilinear elliptic equations of the form

$$\mathfrak{A}(x\,,u\,,Du)\,D^2u \;=\; \mathfrak{B}(x\,,u\,,Du) \tag{1}$$

where $x = (x_1, \ldots, x_n)$ denotes points of E^n, Du and D^2u denote respectively the gradient vector and Hessian matrix of the dependent variable $u = u(x)$, and $\mathfrak{A}$ and $\mathfrak{B}$ are respectively a given symmetric matrix and a given scalar function of the variables indicated. The multiplication convention on the left hand side is the natural contraction $\mathfrak{A}_{ij} D^2_{ij}\, u$.

The structure of (1) is determined by the functions $\mathfrak{A}(x\,,u\,,p)$ and $\mathfrak{B}(x\,,u\,,p)$, which are assumed to be defined and continuously differentiable for all real u and p, and for all points x in the closure of any domain under consideration. Ellipticity is then expressed by the condition $\xi\,\mathfrak{A}\,\xi > 0$ for non-vanishing real vectors $\xi = (\xi_1, \ldots, \xi_n)$. The Dirichlet problem for (1) in a domain Ω consists in the determination of a solution u of class $C^2(\Omega)$, which continuously takes on assigned values f on the boundary of Ω.

As a typical example, consider the equation

$$[(1 + |Du|^2)I - DuDu]\,D^2u = n\Lambda\,(1 + |Du|^2)^{3/2} \tag{2}$$

which governs surfaces of constant mean curvature Λ. Here the following result holds [17], [19], [20], [9].

Let Ω be a bounded domain in n-dimensional Euclidean space, whose boundary is of class C^2. Then the Dirichlet problem for equation (2) *in Ω is solvable for arbitrarily given continuous boundary data if and only if*

$$H \geqslant \frac{n}{n-1}\,|\Lambda| \tag{3}$$

at each point of the boundary of Ω, where H denotes the mean curvature of the boundary surface.

When $n = 2$ and $\Lambda = 0$, (2) reduces to the minimal surface equation ; the curvature condition (3) then implies that Ω should be convex, a well known result of Bernstein and Finn. In connection with this example, it is also of interest to consider the equation

$$(4) \qquad [I + Du\,Du]\,D^2u = c(1 + |Du|^2)^{3/2}$$

where c is a constant. The coefficient matrix has exactly the same eigenvalues as before (though with different multiplicities when $n > 2$), but now the Dirichlet problem is solvable for all C^2 domains and C^2 data ; this contrasting result emphasizes the delicate relation between the structure of (1) and the need for boundary curvature restrictions.

1.

The basic existence theorem for the Dirichlet problem can be stated as follows ([12], [15] ; for the particular form given, see [19], Section 1).

Assume that the boundary of Ω and the boundary data f are of class C^3. Let τ be any real number in [0, 1]. *Suppose there exists a number M, independent of τ, such that if v is a member of $C^2(\overline{\Omega})$ with*

$$\mathcal{A}(x, v, Dv)\,D^2v = \mathcal{B}(x, v, Dv\,;\tau) \quad \text{in } \Omega, \quad v = \tau f \quad \text{on } \partial\Omega$$

then $\sup_\Omega(|v| + |Dv|) \leqslant M$. Here $\mathcal{B}(x, u, p\,;\tau)$ is assumed to be a continuously differentiable function of all its arguments satisfying the conditions

$$\mathcal{B}(x, u, p\,;0) = 0 \quad \textit{when} \quad |p| \leqslant 1, \qquad \mathcal{B}(x, u, p\,;1) = \mathcal{B}(x, u, p).$$

Then the Dirichlet problem for equation (1) in Ω has at least one solution $u \in C^2(\overline{\Omega})$ corresponding to the given boundary data f.

This proposition reduces the question of existence to that of obtaining appropriate *a priori* estimates ; accordingly, the existence of solutions of the Dirichlet problem for smooth domains and smooth data can be established by the following succession of steps :

(A) Obtain an estimate for the maximum absolute value of the solution v in Ω.

(B) Assuming the result of (A), obtain an estimate for the gradient of the solution at the boundary of Ω.

(C) Assuming the results of (A) and (B), obtain an estimate for the gradient of the solution in all of Ω.

In this program we are of course obliged to consider a *family* of Dirichlet problems corresponding to all values of the homotopy parameter τ, and the estimates must be independent of τ. Despite this it will suffice to discuss the various steps in terms of the original Dirichlet problem, since the details stand out more clearly there and can easily be carried over to the case when the homogeneous parameter is present.

There are a variety of methods available for treating Part (A) of the existence program, based on the maximum principle and other devices. Perhaps the simplest result in this regard is the following : *Suppose that* $u\mathcal{B}(x, u, 0) \geqslant 0$ *for large* $|u|$. *Then Part (A) of the existence program can be carried out.* While there is no guarantee that this or any other of these methods will apply to a particular problem under consideration, we are at least fortunate that many applications are presently covered, including all those to be discussed in Section 2.

Part (B) is the source of the boundary curvature restrictions discussed earlier. We introduce the orthogonal invariants

$$\mathcal{C}(x, u, p) = |p|^{-1}\, \mathcal{B}(x, u, p), \qquad \mathcal{E}(x, u, p) = p\, \mathcal{A}(x, u, p) p,$$

where it is assumed for the moment all that the equation is normalized so that Trace $\mathcal{A} = 1$. The following result is proved in [19].

THEOREM 1. – *Suppose there exists an increasing continuous function* $\Phi(\rho)$ *such that*

$$\frac{\mathcal{E}}{1 + |\mathcal{C}|} \geqslant \Phi(|p|), \qquad \int^{\infty} \Phi(\rho) \frac{d\rho}{\rho^2} = \infty .$$

Then Part (B) of the existence program can be carried out (1).

Suppose on the contrary that there exists a continuous function $\Psi(\rho)$ *such that*

$$\frac{\mathcal{E}}{1 + |\mathcal{C}|} \leqslant \Psi(|p|), \qquad \int^{\infty} \Psi(\rho) \frac{d\rho}{\rho^2} < \infty .$$

Then the following conclusions hold : I. If $|\mathcal{C}| \to \infty$ *as* $p \to \infty$ *then for any smooth domain there exists smooth boundary data for which the Dirichlet problem is not solvable. II. If* $\mathcal{C}$ *is bounded, then there exist smooth domains and smooth boundary data for which the Dirichlet problem has no solution.*

When the hypotheses of the first part of the theorem hold we say that (1) is *regularly elliptic.* When the hypotheses of the second part hold, the equation is called *irregularly elliptic* in case I, and *singularly elliptic* in case II. It is easy to show that this classification is invariant under arbitrary C^3 diffeomorphisms of E^n. If (1) is irregularly elliptic we cannot expect the Dirichlet problem to be well-set for any domain ; conversely, if the equation is singularly elliptic, Part (B) of the existence program can be carried out for certain domains, provided the boundary satisfies appropriate curvature restrictions. Because of space limitations we must refer the reader to [19] for a discussion of these conditions.

In any case, it should be noted that equation (2) is singularly elliptic and correspondingly requires a boundary curvature condition for the solvability of the Dirichlet problem (that is, inequality (3)), while equation (4) is regularly elliptic and is consequently solvable for all smooth domains and data. It may also be observed that any uniformly elliptic equation for which $\mathcal{B}(x, u, p) = O(|p|^2)$ is regularly elliptic. (Order relations here and later in the paper are assumed to apply as $p \to \infty$, uniformly for (x, u) in any compact set).

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(1) For this conclusion it is actually enough to have such a function $\Phi(\rho)$ corresponding to each compact set of values (x, u).

In connection with Theorem 1 it would be valuable to prove that the Dirichlet problem is not well-posed in the sense of Hadamard for any irregularly elliptic equation, or for any singularly elliptic equation unless the boundary curvature condition is satisfied.

For Part (C) of the existence program to be carried out it seems necessary to assume a special form for the matrix $\mathcal{A}$. Whether this can be avoided remains an open problem, though, as we shall see, a number of important situations are covered by the theory as it now stands. In any event, we shall specifically assume here that $\mathcal{A}$ has the form

$$\mathcal{A}'(x, u, p) + p\mathcal{C}(x, u, p) + \mathcal{C}(x, u, p)p$$

where $\mathcal{A}'$ is a positive definite matrix and $p\mathcal{C}$ and $\mathcal{C}p$ are dyadics. Here $\mathcal{A}'$ and $\mathcal{C}$ are assumed to be continously differentiable, and $\mathcal{A}'$ should additionally obey at least one of the following conditions :

(I) $\mathcal{A}'$ depends only on p,

(II) $\mathcal{A}'_x = O(\sqrt{(\lambda\mathcal{E})})$, $\quad \mathcal{A}'_u , p \cdot \mathcal{A}'_p = O(\sqrt{(\lambda\mathcal{E})}/|p|)$

where λ is the smallest eigenvalue of $\mathcal{A}'$ and $p \cdot \mathcal{A}'_p$ denotes the contraction $p_i \, \partial \mathcal{A}'/\partial p_i$.

THEOREM 2. – *Part (C) of the existence program can be carried out in the following cases :*

(1) *Condition (I) holds and* $|p|^{-1} p \cdot \mathcal{C}_x + |p| \mathcal{C}_u + \mathcal{C}^2/\text{Trace}\, \mathcal{A}' \geqslant 0$ *for large*

(2) *Condition (II) holds (with O replaced by o for* $\mathcal{A}'_x$ *and* $\mathcal{A}'_u$*),*

$$p . \mathcal{E}_p \leqslant O(\mathcal{E})$$

and

$$\mathcal{B}, \mathcal{B}_x, \mathcal{E}_x, \mathcal{E}_u = O(\mathcal{E}), \; - \mathcal{B}_u \leqslant o(\mathcal{E}), \; p \cdot \mathcal{B}_p \leqslant O(\mathcal{E}).$$

(3) *Condition (II) holds and (with O replaced by o for* $p . \mathcal{A}'_p$*), and*

$$\mathcal{A}' = o(\sqrt{(\lambda\mathcal{E})}/|p|),$$

$$p . \mathcal{E}_p \leqslant O(\mathcal{E}),$$

while also

$$\mathcal{B}, \mathcal{B}_x, \mathcal{E}_x, \mathcal{E}_u = O(\mathcal{E}), \qquad -\mathcal{B}_u , p \; \mathcal{B}_p \leqslant O(\mathcal{E}).$$

(4) *Condition (II) holds (with O replaced by o for* $p \cdot \mathcal{A}'_p$*), and*

$$p \cdot \mathcal{E}_p \leqslant \mathcal{E} + o(\mathcal{E}),$$

while also

$$\mathcal{C}_x, \mathcal{E}_x, \mathcal{E}_u = O(\mathcal{E}), \qquad - \mathcal{C}_u , p \cdot \mathcal{C}_p \leqslant O(\mathcal{E}/|p|).$$

Case (1) is proved in [19], Section 13 ; the remaining parts will appear in a forthcoming paper. Note that when $n = 2$ any positive definite matrix can be written in the form $\mathcal{G}(I + p\mathcal{C} + \mathcal{C}p)$ so that conditions (I) and (II) are automatically satisfied. We observe moreover that case (1) covers variational problems $\delta \int F(Du)\,dx = 0$, cases (2) – (4) apply to variational problems $\delta \int F(x, u, |Du|)\,dx = 0$, and case (2) can be used for uniformly elliptic equations if we set $\mathcal{A}' = \mathcal{A}$, and $\mathcal{C} = 0$. (For related work, see [13], [16], [21]).

The existence program outlined above applies to smooth boundary data. By compactness arguments it is possible to reduce the degree of smoothness required of the data, provided that appropriate interior estimates are available for the gradient of solutions. Estimates of this type are discussed in the paper of Uraltseva included in this volume ; in particular, these estimates can be used to lighten the smoothness requirements on the data in some of the examples given in the following section.

2.

In this section we consider a number of examples which can be treated by the theory outlined above. When specific references are not given, the result is new.

1. Let M denote the minimal surface operator (i.e. the differential operator on the left side of (2)), and consider the equations

$$Mu = n\Lambda(1 + |Du|^2)^{\theta/2}, \qquad Mu = cu(1 + |Du|^2)^{\theta/2}$$

where $\Lambda \neq 0$ and $c > 0$ are constants (1). Then the corresponding Dirichlet problem in a domain Ω with C^2 boundary is solvable for arbitrarily given boundary values of class C^2 in the following circumstances : when $\theta \leqslant 2$, if and only if $H \geqslant 0$ at each boundary point ; when $2 < \theta < 3$, if $H > 0$ at each boundary point ; when $\theta = 3$, if and only if $H \geqslant [n/(n-1)]\,|\Lambda|$ and $H \geqslant c|f|/(n-1)$, respectively. When $\theta > 3$ both equations are irregularly elliptic and the Dirichlet problem is not generally solvable for any domain [3], [18], [19].

2. Let $M'u = [I + DuDu]\,D^2u$ and consider the equations

$$M'u = c(1 + |Du|^2)^{\theta/2}, \qquad M'u = cu(1 + |Du|^2)^{3/2}.$$

Then the corresponding Dirichlet problem in a domain with C^2 boundary is solvable for arbitrarily given C^2 boundary values in the following circumstances : for the first equation, if and only if $\theta \leqslant 3$; for the second equation, if and only if $\theta \leqslant 4$. (Both equations are regularly elliptic for $\theta \leqslant 4$ and irregularly elliptic for $\theta > 4$; for the first equation, however, Part (A) of the existence program can be treated for arbitrary domains only when $\theta \leqslant 3$: if $3 < \theta \leqslant 4$ the domain must have a suitably small diameter [18]).

3. Suppose (1) is uniformly elliptic (with Trace $\mathcal{A} = 1$) and

$$\mathcal{A}_x,\, p\cdot\mathcal{A}_p = O(1), \qquad \mathcal{A}_u = o(1)$$

$$\mathcal{B},\, \mathcal{B}_x = O(|p|^2), \qquad p\cdot\mathcal{B}_p \leqslant O(|p|^2), \qquad \mathcal{B}_u \geqslant -o(|p|^2).$$

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(1) Included in this example are soap films under the influence of gravity (first equation with $\theta = 2$) and the upper surface of a fluid under the combined action of gravity and surface tension (second equation with $\theta = 3$).

Moreover, assume that $u\mathcal{B}(x, u, 0) \geqslant 0$ for all sufficiently large $|u|$. Then the Dirichlet problem for (1) is solvable for all smooth domains and data. (A weaker version of this result appears in [12]).

4. Consider the Euler-Lagrange equation associated with the variational problem

$$\delta \int \phi(u) \sqrt{a^2 + |Du|^2 \, dx} = 0$$

where $\phi(u)$ and $a = a(x, u)$ are positive functions of class C^2, with $u(a\,\phi)_u \geqslant 0$ for all sufficiently large $|u|$. Then the corresponding Dirichlet problem in a domain with C^2 boundary is solvable for arbitrarily given C^2 boundary values if and only if $H \geqslant 0$ at each point of the boundary.

5. Consider the variational problem $\delta \int F(|Du|)\,dx = 0$, where $F(t)$ is of class C^3 and $F'' > 0$ (it is assumed that $F'(0) = F'''(0) = 0$ for consistency). Let $f = tF''/F'$. Suppose first $\int^{\infty} f\,dt = \infty$. If $f \geqslant \Phi(t)$, where

$$\Phi \searrow 0 \;, \quad \int^{\infty} \Phi \, dt = \infty \,,$$

then the Euler-Lagrange equation is regularly elliptic and the Dirichlet problem is solvable for arbitrarily given C^2 boundary data on any domain of class C^2. (It should be noted in passing that $\lim t^{-1}F = \infty$ implies $\int^{\infty} f\,dt = \infty$).

Suppose next that $\int^{\infty} f\,dt < \infty$. If $tf \geqslant \Phi(t)$, where Φ satisfies the same condition as before, then the Dirichlet problem is solvable for arbitrarily given C^2 boundary data if and only if $H \geqslant 0$ at each point of the boundary. Finally, whatever may be the asymptotic behavior of f, the Dirichlet problem is solvable for smooth domains and data if $H > 0$ at each point of the boundary.

6. Let S be an oriented n-dimensional surface embedded in E^{n+1}, with unit normal vector $\vec{n}$, and consider the regular parametric variational problem

$$\delta \int_S \mathfrak{F}(\vec{n})\, dA = 0 \,,$$

where $\mathfrak{F}$ is of class C^3 on the unit sphere. The corresponding Euler-Lagrange equation for a non-parametric extremal is singularly elliptic, and boundary curvature restrictions are required for the associated Dirichlet problem. It is interesting to note that the required conditions depend *only* on the form of the reduced function $\mathfrak{F}(\vec{n}_0)$, where $\vec{n}_0$ is restricted to be orthogonal to the particular projection under consideration [11].

7. Consider equation (2), but with the mean curvature $\Lambda = \Lambda(x, u\,; \vec{n})$ a prescribed continuously differentiable function of the position (x, u) and the unit normal $\vec{n}$ on the surface $z = u(x)$. We assume that Ω is contained in a ball of radius R and that its boundary is of class C^2. Suppose finally

(5) $\Lambda_u \geqslant 0\,, \qquad 1 + R\Lambda \operatorname{sign} u \geqslant 0 \qquad$ for $|u|$ suitably large.

Then the Dirichlet problem in Ω is solvable for given C^2 boundary values f provided

$$H \geqslant \pm \frac{n}{n-1} \Lambda(y\,, f\,;\, \pm \vec{\nu}) \tag{6}$$

at each point y of the boundary, where $\vec{\nu}$ denotes the inner normal vector to the boundary. The solution is unique if it exists.

If Λ is independent of u, then condition (6) is necessary for the Dirichlet problem to be solvable for arbitrarily given smooth boundary values. On the other hand, if Λ depends only on u, the second condition of (5) can be dropped ; if $\Lambda = \Lambda(u\,;\vec{n})$ this condition can again be omitted provided the equality sign does not hold in (6).

8. A similar conclusion can be obtained for surfaces of prescribed mean curvature in central projection, that is, when the surfaces can be represented in the form $r = \nu(x)$, where x denotes points on the unit sphere S^n and r is the distance from the origin. In fact, let Ω be a smoothly bounded domain on the unit sphere whose closure is contained in the upper hemisphere $x_{n+1} > 0$. Let the mean curvature $\Lambda = \Lambda(x\,, r\,;\vec{n})$ be a prescribed continuously differentiable function on $E^{n+1} \times S^n$, such that $(r\,\Lambda)_r \geqslant 0$. Then the corresponding Dirichlet problem over Ω is solvable for (positive) radial values $\nu = f$ of class C^2 assigned on the boundary of Ω, provided that

$$H_g \geqslant \frac{n}{n-1} f(y)\,\Lambda(y\,, f\,;\vec{\nu})$$

at each point y of the boundary, where H_g is the geodesic mean curvature of the boundary and $\vec{\nu}$ is the unit inner normal (as a vector on S^n).

9. In connection with the preceding example, Alexandrov [1] has proposed relations of the type

$$\Lambda(x\,, r\,;\vec{n}) = \frac{\gamma(x\,,\vec{n})}{r} \qquad \text{and} \qquad \Lambda(x\,, r\,;\vec{n}) = \frac{\epsilon(x\,,\vec{n})}{s}$$

where γ and ϵ are continuously differentiable functions and $s = rx \cdot \vec{n}$ is the distance from the tangent plane at $(x\,, \nu)$ to the origin. (Note that Alexandrov's functions are invariant under similarities about the origin). The above formulae do not satisfy the smoothness requirements of the preceding case and so require a separate treatment. In this regard we shall in fact consider the more general relations

$$\Lambda(x\,, r\,;\vec{n}) = \frac{\gamma(x\,, r\,;\vec{n})}{r} \quad \text{and} \quad \Lambda(x\,, r\,;\vec{n}) = \frac{\epsilon(x\,, r\,;\vec{n})}{s}\,.$$

For the first relation, assume that γ is continuously differentiable and satisfies the conditions

$$(7) \qquad \frac{\partial\gamma}{\partial r} \geqslant 0\,, \qquad \frac{n}{n-1}\ |\gamma(x\,,0\,;\vec{n})| < \frac{x_{n+1}}{\sqrt{1-x_{n+1}^2}}$$

Then the associated Dirichlet problem over Ω is solvable for radial values f of class C^2 assigned on the boundary of Ω provided that

$$(8) \qquad H_g \geqslant \pm \frac{n}{n-1}\ \gamma(y\,,f\,;\pm\vec{\nu})$$

at each point y of the boundary. If γ is independent of r, then condition (8) is necessary if the Dirichlet problem is to be solvable for arbitrarily given smooth boundary values. If γ is independent of x, the second condition of (7) can be omitted provided the equality sign does not hold in (8).

When $\Lambda = \epsilon/s$ the relevant differential equation is irregularly elliptic (assuming $\epsilon \neq 0$). Consequently, for any domain there exist smooth boundary values for which the associated Dirichlet problem is not solvable.

10. The geometric problems discussed in examples 7, 8, and 9, lead one to propose similar boundary value problems when the Gauss curvature, or other combinations of the principal curvatures of a surface, are prescribed functions of the space variables and the surface normal. The theory of such problems should form an interesting area of research.

In conclusion, note that the list of references contains several relevant papers not mentioned earlier due to limitation of space.

REFERENCES

[1] ALEXANDROV A.D. — Uniqueness theorems for surfaces in the large. *Vestnik Lenin Univ.*, 11, 1956, p. 5-17.

[2] BAKKELMAN I.J. — Mean curvature and quasilinear elliptic equations. *Siberskii Mat. Z.*, 9, 1968, p. 1014-1040.

[3] BERNSTEIN S. — Conditions nécessaires et suffisantes pour la possibilité du problème de Dirichlet. *C.R. Acad. Sci. Paris,* 150, 1910, p. 514-515.

[4] BERNSTEIN S. — Sur les surfaces définies au moyen de leur courbure moyenne et totale. *Ann. Sci. Ecole Norm. Sup.*, 27, 1910, p. 233-256.

[5] BERNSTEIN S. — Sur les équations du calcul des variations. *Ann. Sci. Ecole Norm. Sup.*, 29, 1912, p. 431-485.

[6] FINN R. — Remarks relevant to minimal surfaces and to surfaces of constant mean curvature. *J. d'Anal. Math.*, 14, 1965, p. 139-160.

[7] GILBARG D. — *Boundary value problems for nonlinear elliptic equations in n variables. Nonlinear problems.* Madison : Univ. of Wisc. Press, 1963, p. 151-159.

[8] HARTMAN P. & STAMPACCHIA G. — On some nonlinear elliptic differential functional equations. *Acta Math.*, 115, 1966, p. 271-310.

[9] HEINZ E. — To appear.

[10] JENKINS H. & SERRIN J. — The Dirichlet problem for the minimal surface equation in higher dimensions. *J. Reine Ang. Math.*, 229, 1968, p. 170-187.
[11] JENKINS H. & SERRIN J. — *The Dirichlet problem for regular parametric variational problems.* To appear.
[12] LADYZHENSKAYA O.A. & URALTSEVA N.N. — *Linear and quasilinear elliptic equations.* Moscow, 1964.
[13] LADYZHENSKAYA O.A. & URALTSEVA N.N. — Global estimates of the gradient of solutions of quasilinear elliptic and parabolic equations. *Sem. Mat. Inst. Steklov,* 14, 1969, p. 127-151.
[14] LERAY J. — Discussion d'un problème de Dirichlet. *J. Math. Pures Appl.,* 18. 1939, p. 249-284.
[15] LERAY J. & SCHAUDER J. — Topologie et équations fonctionelles. *Ann. Sci. Ecole Norm. Sup.,* 51, 1934, p. 45-78.
[16] OSKOLKOV A.P. — A priori estimates for the first derivatives of solutions of the Dirichlet problem for non-uniformly elliptic quasilinear equations. *Trudy Mat. Inst. Steklov,* 102, 1967, p. 105-127.
[17] SERRIN J. — The Dirichlet problem for surfaces having prescribed mean curvature. *Notices Amer. Math. Soc.,* 14, 1967, p. 135.
[18] SERRIN J. — The Dirichlet problem for quasilinear equations with many independent variables. *Proc. Nat. Acad. Sci.,* 58, 1967, p. 1829-1835.
[19] SERRIN J. — The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Phil. Trans. Roy. Soc. London,* A 264 1969, p. 413-496.
[20] SERRIN J. — The Dirichlet problem for surfaces of constant mean curvature, *Proc. London Math. Soc.,* 21, 1970, p. 361-384.
[21] TRUDINGER N.S. — Some existence theorems for quasilinear non-uniformly elliptic equations in divergence form. *J. Math. Mech.,* 17, 1968.

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VARIATIONAL INEQUALITIES

by Guido STAMPACCHIA

In these last few years the theory of variational inequalities, is being developed very fast, having as model the variational theory of boundary value problems for partial differential equations. The theory of variational inequalities represents, in fact, a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied Mathematics, such as Mechanics, Physics, the Theory of convex programming and the Theory of control.

While the variational theory of boundary value problems has its starting point in the method of orthogonal projection, the theory of variational inequalities has its starting point in the projection on a convex set.

The first existence theorem for variational inequalities [1] was proved in connection with the theory of second order equations with discontinuous coefficients [2] in order to bring together again, as it was at the beginning, potential theory and the theory of elliptic partial differential equations.

It turned out that many other problems could be fitted in this theory and that many other theories were closely related.

Let X be a reflexive Banach space over the reals with norm $\| \ \|$ and denote by X' its dual and by $< \ , \ >$ the pairing between X and X'. Let A be an operator from X into X' ; fix a closed convex subset $\mathbf{K}$ of X and consider the following problem :

PROBLEM 1. – *To find $u \in \mathbf{K}$ such that*

$$< Au , v - u > \geqslant 0 \quad \text{for all } v \in \mathbf{K} \tag{1}$$

Problem 1 is what is called a variational inequality and any element $u \in \mathbf{K}$ which satisfies (1) is called a solution.

Note that in the case when $\mathbf{K} = X$ (or u is an interior point of $\mathbf{K}$), (1) reduces to the equation $Au = 0$, since then the $(v - u)$ ranges over a neighborhood of the origin in X.

When A is a coercive linear operator from an Hilbert space H to its dual H', the existence and uniqueness of the solution was proved in [1] and [3]. In [3] also the case when A is assumed to be only positive or semicoercive was considered. This last case includes the problem of Signorini [4].

When A is a non linear operator the existence theorem of the solution of Problem 1 was proved by Hartman-Stampacchia [5] and Browder [6], assuming that A is a monotone hemicontinuous operator.

We recall that A is a *monotone* operator from X into X' if the condition

(2) $$< Au - Av , u - v > \geqslant 0 \qquad \text{for all} \quad u , v \in X$$

holds. If in (2) equality holds only for $u = v$ the operator is called *strictly monotone.* The operator A is hemicontinuous if the map $t \to < A((1 - t)u + tv) , w >$ is continuous from $[0 , 1]$ into R^1 for u , v , w in X.

The theorem mentioned above is the following

THEOREM 1. – *Let A be a monotone, hemicontinuous operator from X into X', and $\mathbf{K}$ a bounded, closed convex set of X. Then there exists at least one solution of Problem* 1.

Moreover, the set of all solutions of Problem 1 *is a closed convex subset of* $\mathbf{K}$, *which reduces to a single point of* $\mathbf{K}$ *if A is strictly monotone.*

When $\mathbf{K}$ is unbounded, consider the closed ball Σ_R in X with center in the origin and radius R and the closed convex set

$$\mathbf{K}_R = \mathbf{K} \cap \Sigma_R$$

Then

THEOREM 2. – *Problem* 1 *has solution if and only if there exists an* $R > 0$ *such that at least one solution of the problem*

$$u_R \in \mathbf{K}_R \ : < Au_R , v - u_R > \geqslant 0 \qquad \text{for all} \quad v \in \mathbf{K}_R$$

(*which exists because of Theorem* 1) *satisfies the inequality*

$$\| u_R \| < R.$$

A sufficient condition in order that the condition of Theorem 2 hold is the *coerciveness* of the operator A ; i.e. we assume that there exists $v_0 \in \mathbf{K}$ such that

(3) $$< Av , v - v_0 > / \| v \| \to + \infty \quad \text{as} \quad \| v \| \to + \infty , \ v \in \mathbf{K}.$$

The variational inequalities generalize the theory of equations ; on the other hand a variational inequality can be reduced to an equation with the following device.

Define the multivalued map χ from $\mathbf{K}$ into X' in the following way : set for $u \in \mathbf{K}$ $\chi(u) = 0$ and for $u \in \partial\mathbf{K}$ let $\chi(u)$ be the set of elements of X' such that

(4) $$< \chi(u) , v - u > \geqslant 0 \qquad \text{for all} \quad v \in \mathbf{K} \ ;$$

(4) defines the set of all supporting planes to $\mathbf{K}$ in u.

Then (1) can be written as

$$A(u) \in \chi(u)$$

or

$$A(u) - \chi(u) \ni 0.$$

In fact (1) means that $< Au, v - u > = 0$ is a supporting plane of $\mathbf{K}$ at u, and the convex $\mathbf{K}$ is in the half space $X^+(u)$ where $< Au , v - u > \geqslant 0$.

In general the variational inequality is satisfied, not only by the v's in K, but by all the v's in $X^+(u)$.

Assume that A is a strictly monotone hemicontinuous operator from X into X' and coercive, i.e. (3) is satisfied. Let u_0 be the solution of the equation

$$A(u_0) = 0.$$

Let K be a closed convex set of X and u the solution of the related variational inequality.

If $u_0 \in \mathrm{K}$, then $u = u_0$; if $u_0 \notin \mathrm{K}$ then $u \in \partial\mathrm{K}$ and u belongs to that part of $\partial\mathrm{K}$ where at least one of the supporting planes separates u_0 from K. In other words this means that u can be seen from u_0 in $X - \mathrm{K}$ or that the segment joining u_0 to u is completely outside of K.

Let $u_0 \notin \mathrm{K}$; if the fact just mentioned were not true, we would have

$$< A(u), u_0 - u > \leqslant 0$$

and thus

$$< A(u_0) - A(u), u_0 - u > \leqslant 0$$

which implies $u_0 = u$, contradicting $u \in \mathrm{K}$ and $u_0 \notin \mathrm{K}$.

An analogous relation holds between the solutions related to two convex sets K_1, K_2 such that $\mathrm{K}_1 \supset \mathrm{K}_2$.

These facts have been used in order to compute solutions of variational inequalities in finite dimensional spaces [7].

Another approach to variational inequalities is to consider Problem 1 as limit for $\epsilon \to 0$ of a sequence u_ϵ of solutions of monotone equations. This approach has been used by H. Lewy and G. Stampacchia [8] in a special case. J.L. Lions [9] has shown that this can be done in a very general situation ; it is enough that the norm of X and that of X' are strictly convex.

In this case it is possible to reduce Problem 1 to the sequences of problems

$$A(u_\epsilon) + \frac{1}{\epsilon}\,\beta(u_\epsilon) = 0$$

where β is called a "penalization", namely a bounded, hemicontinuous monotone operator such that

$$\{v \mid v \in X \quad , \quad \beta(v) = 0\} \equiv \mathrm{K}.$$

This method of penalization can also lead to theorems of regularization of which we shall speak later on.

A very important tool for variational inequalities is a lemma due essentially to Minty [5], [6].

Lemma 1. – *Let A be a monotone hemicontinuous operator ; u is a solution of the variational inequality* (1) *if and only if*

$$u \in \mathrm{K}, \quad < Av, v - u > \geqslant 0 \quad \text{for all} \quad v \in \mathrm{K}.$$

Many of the results I have mentioned above hold under a more general assumption about A, namely that A is pseudo-monotone. This notion is due to Brezis [10]. It requires that (i) A is bounded and (ii) if $u_i \to u$ weakly in X and if

$$\limsup < A(u_i), u_i - u > \leqslant 0 \quad \text{then}$$

$$\liminf < A(u_i), u_i - v > \geqslant < A(u), u - v > \quad \text{for all} \quad v \in X.$$

2. Following the thought that the theory of variational inequalities generalizes the theory of boundary value problems, the next problem which appears to be natural is the problem of the regularity of the solutions of variational inequalities.

This problem has been studied from an abstract point of view by H. Brezis and Stampacchia [11]. Let us write our variational inequality (1) in the form

$$(5) \qquad u \in \mathbf{K} \qquad < Au - f, v - u > \geqslant 0 \qquad \text{for all} \quad v \in \mathbf{K}$$

where f is an element of X'. Assuming that f belongs to a subspace W of X', when can we assure that : $A(u) \in W$?

Assume that W is a reflexive Banach space such that

$$(6) \qquad \text{(i) } W \subset X' \ , \quad \text{(ii) } \| \ \|_{X'} \leqslant \text{const } \| \ \|_W \ , \quad \text{(iii) } W \text{ is dense in } X'.$$

Let J be a duality map between W and W' (dual of W) and assume that for any $u \in \mathbf{K}$ and $\epsilon > 0$ there exists $u_\epsilon \in \mathbf{K}$ such that $Au_\epsilon \in W$ and

$$u_\epsilon + \epsilon J(Au_\epsilon) = u.$$

Then the existence theorem 1 and its generalization for the variational inequality (1) give

$$Au \in W.$$

Theorems on the regularity of the solutions of variational inequalities have been considered for many special problems. In order to describe some of these results, I would like to confine myself to special operators.

Let $a(p)$ be a continuous vector field defined on $\mathbf{R}^N$. The field is called *monotone* if for any vectors p, q in $\mathbf{R}^N$, the following condition holds

$$(6) \qquad (a(p) - a(q)) \cdot (p - q) \geqslant 0.$$

If in (6) equality holds only if $p = q$, the field $a(p)$ is called *strictly monotone.* We shall say that the vector field $a(p)$ is *locally coercive* if, for any compact set C of R^N, there exists a positive constant $\nu(C)$, such that

$$(7) \qquad (a(p) - a(q)) \cdot (p - q) \geqslant \nu(C)\, |p - q|^2 \qquad \text{for all} \quad p, q \in C.$$

We consider, formally, the operator

$$(8) \qquad Au = - \frac{\partial}{\partial x_i}\, a_i(u_x)$$

where u_x denotes the gradient of a function u defined in a bounded open set Ω. The operator (8) is defined on the Sobolev spaces $H^{1,\alpha}(\Omega)$ of functions u such that $u \in L^\alpha(\Omega)$ and $u_x \in (L^\alpha)^N$, $1 < \alpha \leqslant + \infty$, only if some conditions on the growth of the $a_i(p)$ are fulfilled. For instance, A is defined on the space $H^1 = H^{1,2}$ if the $a_i(p)$ are linear functions or on the space $H^{1,\alpha}(\Omega)$ if the $a_i(p)$ are bounded by const. $|p|^{\alpha-1}$ $(\alpha > 1)$. Only in the case of Lipschitz functions (that we will denote by $H^{1,\infty}(\Omega)$) no conditions on the growth of the $a_i(p)$ are needed. In any case the operator is defined in the sense of distributions and in its domain it is monotone and hemicontinuous. Denote by $H^{1,\alpha}(\Omega)$ the domain of the operator (8) and consider the following example of variational inequality.

Let $\mathbf{K}_\alpha$ be the closed convex subset of $H_0^{1,\alpha}(\Omega)$ (functions of $H^{1,\alpha}(\Omega)$ vanishing on the boundary of Ω)

(9) $$\mathbf{K}_\alpha \equiv \{v(x) \in H_0^{1,\alpha}(\Omega) \quad ; \quad v(x) \geqslant \psi(x) \text{ in } \Omega\}$$

where $\psi(x)$ is a given function of the domain of A subject to the condition of being negative on $\partial\Omega$. For $\alpha > 1$, it follows from theorem 1 and theorem 2 that there exists a solution u such that

(10) $$u \in \mathbf{K}_\alpha \quad ; \quad \int_\Omega a_i(u_x)\,(v-u)_{x_i}\,dx \geqslant 0 \qquad \text{for all} \quad v \in \mathbf{K}_\alpha$$

provided that A satisfies a coerciveness condition ; for istance,

(11) $$a_i(p)\,p_i \geqslant c\,|p|^\alpha \qquad\qquad c > 0.$$

Recently H. Lewy and G. Stampacchia [12] have proved that in the case $\alpha = +\infty$ condition (11) can be dropped if Ω is supposed to be convex. We have proved

THEOREM 3. – *There exists, for any monotone field $a(p)$ a function $u \in H_0^{1,\infty}(\Omega)$ such that*

(12) $$u \in \mathbf{K}_\infty \quad ; \quad \int_\Omega a_i(u_x)\,(v-u)_{x_i}\,dx \geqslant 0 \qquad \text{for all} \quad v \in \mathbf{K}_\infty$$

The Lipschitz coefficient of u is no greater than that of the obstacle ψ. Such a solution u is unique provided that the field $a(p)$ is strictly monotone.

This result contains, as special case, the problem of minimizing the integral

$$\int_\Omega f(\text{gradv})\,dx$$

where f is a convex function in $\mathbf{R}^N$ and v ranges in $\mathbf{K}_\infty$. The special case

$$f(p) = (1 + p^2)^{1/2}$$

deals with the surface of minimal area among all surfaces $u(x)$ with given boundary values which stay above the obstacle represented by ψ. In the same paper [12] we have proved that the solution of theorem 3 has Hölder continuous first derivatives provided that $a(p)$, Ω and ψ are suitably smooth.

A similar problem, for linear $a(p)$, was treated in a previous paper by the same Authors [8]. Other results were obtained in [11] in the case of a non linear field $a(p)$. All these papers deal with coercive problems. Problems of the type considered in theorem 3 for the case of minimal surfaces have been the subject of investigation by several Authors. Minimal surfaces with obstacles have been studied by

J.C.C. Nitsche [13] for $N = 2$. Also for $N = 2$, Kinderlehrer [14] has considered the interesting question of the nature of the set where the solution and the obstacle coincide. M. Miranda [15] solves for all N the *parametric* minimal surface problem with obstacles.

In the special case of minimal surface the condition of convexity on Ω could be weakend with a condition on the mean curvature on $\partial\Omega$ as was done by Serrin [16] in the case of Dirichlet problem.

Other interesting problems of the same type arise when the obstacle does not belong to the domain of A. The case of linear $a(p)$ has been treated in a paper just appeared [17] and the case of continuous ψ in the situation of theorem 3 is treated in [12]. It turns out that the solution is the greatest lower bound of all smooth supersolutions which are $\geqslant \psi$ in Ω and non negative on $\partial\Omega$.

For lacking of space I cannot treat many other interesting examples of variational inequalities. I refer to the book of Lions [9] and to the expository papers [18] [19] [22]. I would like to add only that recently some achievements have been obtained on the problem of Signorini of unilater constrains on the boundary by H. Beirao da Veiga [20] and H. Brézis [21].

For more general results and for the problems of evolution, I refer to the lecture of J.L. Lions [23].

REFERENCES

[1] Stampacchia G. — Formes bilinéaires coercitives sur les ensembles convexes. *C.R. Acad. Sc. Paris*, 258, 1964, p. 4413-4416.

[2] Stampacchia G. — Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier*, 15, 1965, p. 189-259.

[3] Lions J.L. and Stampacchia G. — Variational inequalities, *Comm. Pure Applied Math.*, 20, 1967, p. 493-519.

[4] Fichera G. — Problemi elastostatici con vincoli unilateri: il problema di Signorini con ambigue condizioni al contorno. *Mem. Accod. Naz. Lincei.*, ser. 8, Vol. 17, 1964, p. 91-140.

[5] Hartman Ph. and Stampacchia G. — On some non linear elliptic differential functional equations. *Acta Math.*, 115, 1966, p. 271-310.

[6] Browder F. — Non linear monotone operators and convex sets in Banach spaces. *Bull. Amer. Math. Soc.*, 71, 1965, p. 780-785.

[7] Mancino Q. and Stampacchia G. — Convex programming and variational inequalities (to appear).

[8] Lewy H. and Stampacchia G. — On the regularity of the solution of a variational inequality, *Comm. Pure Appl. Math.*, 22, 1969, 153-188.

[9] Lions J.L. — *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars. Paris (1969).

[10] Brézis H. — Equations et inéquations non linéaires dans les espaces vectoriel en dualité. *Ann. Inst. Fourier*, 18, 1968, p. 153-180.

[11] Brézis H. et Stampacchia G. — Sur la régularité de la solution d'inéquations elliptiques. *Bull. Soc. Math. France*, 96, 1968, p. 153-180.

[12] LEWY H. and STAMPACCHIA G. — *On existence and smoothness of solutions of some non-coercive variational inequalities*, to appear in *Archive for Rational Mechanics and Analysis.*
[13] NITSCHE J.C.C. — Variational problem with inequalities as boundary conditions. *Arch. Rational Mech. Anal.*, 35, 1969, p. 83-113.
[14] KINDERLEHERER D.S. — The coincidence set of solutions of certain variational inequalities. *Archive for Rational Mechanics and Analysis*, 40 (1971), p. 231-250.
[15] MIRANDA M. — See this volume.
[16] SERRIN J. — See this volume.
[17] LEWY H. and STAMPACCHIA G. — On the smoothness of superharmonics which solves a minimum problem, *Journal d'Analyse Mathématique*, 23, 1970, p. 227-236.
[18] STAMPACCHIA G. — Regularity of solutions of some variational inequalities. *Symposium on non-linear functional Analysis*, Chicago (1968).
[19] STAMPACCHIA G. — On the regularity of solutions of variational inequalities, *Proceedings of the International Conference on Functional Analysis and related topics*, Tokyo, April 1969.
[20] BEIRAO DA VEIGA H. — *Sur la régularité des solutions de l'équation div* $A(x, u, u) = B(x, u, u)$ *avec des conditions aux limites unilatérales et mêlées.* To appear.
[21] BRÉZIS H. — *Problèmes linéaires avec constraintes unilatérales sur le bord.* To appear.
[22] STAMPACCHIA G. — *Variational inequalities*, Theory and Applications of Monotone Operators. Proceeding of a Nato Advances Study Inst. Venice, Italy, June 1968 pubbl. Oderisi, Gubbio (Italy).
[23] LIONS J.L. — *Inéquations variationnelles d'evolution*, this volume.

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ON THE NON-UNIFORMLY QUASILINEAR ELLIPTIC EQUATIONS

by N. N. URALTSEVA

In this report we deal with quasilinear elliptic equations of the second order though some of statemens are true for parabolic equations too. In general case they have the following form

$$(1) \qquad L(u) \equiv \sum_{i,j=1}^{n} a_{ij}(x, u, u_x)\, u_{x_i x_j} + a(x, u, u_x) = 0 \,.$$

Here $x = (x_1, \dots, x_n) \in \Omega$, Ω is a domain in Euclidian space R^n,

$$u_x = (u_{x_1}, \dots, u_{x_n}) \,.$$

We call equation (1) uniformly elliptic if the inequalities

$$(2) \quad \nu(u)\, \xi^2 \leqslant (A\xi, \xi) \equiv \sum_{i,j=1}^{n} a_{ij}(x, u, p)\, \xi_i \xi_j \leqslant \mu(u)\, \xi^2, \quad \xi^2 = \sum_{i=1}^{n} \xi_i^2 \,,$$

hold for any $x \in \Omega$, $u \in R^1$, $p \in R^n$ and $\xi \in R^n$, where $\nu(u)$ and $\mu(u)$ are continuous positive functions of u. The standard conditions of ellipticity is

$$(3) \qquad \nu(u, p)\, \xi^2 \leqslant (A\xi, \xi) \leqslant \mu(u, p)\, \xi^2$$

where $\nu(u, p)$ and $\mu(u, p)$ are continuous positive functions of their arguments. Here it is not supposed that $\mu(u, p)/\nu(u, p)$ is bounded when $|u| + |p| \to \infty$.

Next we shall discuss the question of finding the a priori estimates of max $|u_x|$ of solutions $u(x)$ of the equation (1). In the most of cases we suppose the value of $\max_\Omega |u| = M$ to be known, so we may suppose that inequalities (2) and (3) hold only for $|u| \leqslant M$.

It is well known that about the middle of 30^{th} [11] the questions of solvability of the classical boundary value problems for equations (1) were reduced essentially to the questions of obtaining the estimates for Hölder norms

$$|u|^{(1+\alpha)} \equiv \max_\Omega |u| + \max_\Omega |u_x| + \langle u_x \rangle_\Omega^{(\alpha)}$$

$$(\langle v \rangle_\Omega^{(\alpha)} \equiv \max_{x, x' \in \Omega} |v(x) - v(x')|\, |x - x'|^{-\alpha})$$

for solutions of equations (1).

Near the 1960 the methods of estimating of the Hölder constant for u_x were developed for the case when $M = \max_\Omega |u|$ and $M_1 = \max_\Omega |u_x|$ are known [6].

As for the estimation of Hölder constant for u_x the both cases (2) and (3) mentioned above are the same because for $|u| \leqslant M$ and $|p| \leqslant M_1$, the functions ν and μ in inequalities (2) and (3) can be replaced by constants $\nu_0 = \min\limits_{\substack{|u| \leqslant M \\ |p| \leqslant M_1}} \nu > 0$ and $\mu_0 = \max\limits_{\substack{|u| \leqslant M \\ |p| \leqslant M_1}} \mu > 0$ respectively. However, for the a priori estimates of $\max\limits_{\Omega} |u_x|$ the cases (2) and (3) are quite different. The case (2) was the main subject of studies up to the 60^{th}.

In this direction first results were obtained for the equations in two independent variables. An important step in the following investigations was the paper [5] where a method of estimation (global and local) of $\max |u_x|$ for any $n \geqslant 2$ was given. This approach was applied in different ways to study different classes of elliptic and parabolic equations. It gave the possibility of investigating the most general cases of uniformly elliptic and parabolic equations. It has proved that this method is applicable to non-uniformly elliptic (and parabolic) equations too.

In papers [7,8, 13] various classes of non-uniformly elliptic equations were singled out for which the method mentioned above permitted to obtain the global inner estimates of $\max\limits_{\Omega} |u_x|$. These classes include also all the classes of elliptic equations considered in the very interesting memoir by J. Serrin [14].

We shall describe now in a few words this method of estimation of $\max\limits_{\Omega} |u_x|$ under the hypothesis that $u(x) \in C^2(\bar{\Omega})$ and $M = \max\limits_{\Omega} |u|$ and $M_2 = \max\limits_{\partial\Omega} |u_x|$ are known. We introduce instead of $u(x)$ a new function $\nu(x)$ defined by $u = \varphi(\nu)$, $\varphi'(\nu) > 0$, and instead of $\max u_x^2$ evaluate $\max\limits_{\Omega} \nu_x^2$. One can consider function $w = \nu_x^2$ as a weak solution of class $C^1(\bar{\Omega})$ of the equation

$$(4) \qquad \sum_{i,j=1}^{n} \frac{d}{dx_i} (a_{ij}(x, u, u_x) w_{x_j}) + \sum_{i=1}^{n} b_i w_{x_i} = F,$$

where b_i are bounded in $\bar{\Omega}$, $F = 2\varphi'(A\nu_{xx}, \nu_{xx}) + B$, B is linear function of ν_{xx} and nonlinear function of u_x. For the weak solutions of elliptic equations of form (4) the maximum principle takes place and it can be applied to (4) if one knows, that $F > 0$ in points of maximum of $w(x)$. So the question about the estimation of $\max w$ reduces to the question of finding a function $\varphi(\nu)$, for which $F > 0$ in points of maximum of w and $\varphi'(\nu) > 0$.

The analysis of this question leads to the following ordinary differential equation for the function φ :

$$(5) \qquad -\left(\frac{\varphi''}{\varphi'}\right)' + c_1 \left(\frac{\varphi''}{\varphi'}\right)^2 = c\varphi'^2 + c_2 |\varphi''|, \quad c > 0,$$

under the condition

$$(6) \qquad \int_{-\infty}^{\infty} \varphi'(\nu)\, d\nu > 2M, \quad \varphi'(\nu) > 0.$$

So one can see that the restriction [8] to be put on the functions $a_{ij}(x\ u, p)$ and $a(x, u, p)$ is the requirement of solvability of the problem (5), (6). The last holds in two cases :

(1) if $c_1 \geqslant 1$, then c and c_2 may be arbitrary ;

(2) if $c_1 < 1$, then for solvability of (5), (6) it is necessary and sufficient that the inequality

$$\int_0^\infty \frac{d\beta}{\sqrt{2\beta}\,[2(1-c_1)\beta + c_2\sqrt{2\beta} + c]} > 2M \tag{7}$$

is satisfied. (7) holds for instance, if c or M is sufficiently small.

Following the same line of arguments we succeeded in getting the most general results (up to now) concerning the global inner estimates for non-uniformly elliptic equations. For example for Euler equations corresponding to the regular variational problems in parametric form and for the equations of Geometry mentioned above the quadratic form $(A\xi, \xi)$ connected with the equation shall not degenerate on the vectors ξ which are orthogonal to $u_x = (u_{x_1}, \ldots, u_{x_n})$. It permits to satisfy the inequality $F > 0$ in points of maximum of w. For the equation

$$\sum_{i,j=1}^{n} [(1 + u_x^2)\,\delta_i^j - u_{x_i}u_{x_j}]\,u_{x_ix_j} = n\mathscr{H}(1 + u_x^2)^{3/2}, \tag{8}$$

determining the mean curvature $\mathscr{H}$ of a surface $u = u(x)$ our restrictions on $\mathscr{H} = \mathscr{H}(x, u, x_x)$ are the following ones :

$$|\mathscr{H}(x, u, p)| \leqslant c \ ; \ |\mathscr{H}_x| \leqslant c \ ; \ \mathscr{H}_u \geqslant 0 \ ; \ \left|\sum_{i=1}^{n} p_i \mathscr{H}_{p_i}\right| \leqslant c\,|p|^{-1} \ ; \tag{9}$$
$$x \in \bar{\Omega}, \ |u| \leqslant M, \ |p| > 1 .$$

Let us consider now the question of local estimates of $\max\limits_{\Omega} |u_x|$, $\bar{\Omega}' \subset \Omega$. In the paper [5] a method for obtaining local estimates of u_x was also given, which does not require the knowledge of $\max\limits_{\partial\Omega} |u_x|$. With the help of this method such estimates were obtained for all the classes of uniformly elliptic equations (see [6]). It turned out that the method is applicable in all cases for which we are able to find the global inner estimates. The only difference is that now one more condition should be fulfilled : the degree of degeneration of the form $(A\xi, \xi)$ must be less than 2 (That is $\dfrac{\mu(u, p)}{p^2\nu(u, p)} \to 0$ when $|p| \to \infty$).

But this last condition may be really restrictive : it does not hold, for instance, for the equations (8) (even when $\mathscr{H} \equiv 0$) and for the Euler equations for the parametric functional

$$\mathscr{J}(S) = \int_\Omega \mathscr{F}(X, \mathscr{Q})\,d\alpha\ , \quad X \in R^{n+1}\ , \quad \Omega \subset R^n\ .$$

Here $X = X(\alpha)$ gives the parametric representation of hypersurface S,

$$\mathcal{D}_i = \det_{j \neq i} \left\| \frac{\partial X_j}{\partial \alpha_k} \right\| .$$

The order of degeneracy of quadratic forms in these cases equals two. This class of equations allows another approach originated in the papers [3], [2], where the local estimates for gradients of solutions of the minimal surface equations (8) with $\mathcal{H} \equiv 0$ or, what is the same, Euler equation for integral (10) with $\mathcal{F} = |\mathcal{D}|$. We succeeded in developing this approach for the class of non-uniformly elliptic equations included the equation (8) and Euler equations for regular parametric variational problems. Namely, we consider the classical solutions of equations

$$(11) \qquad \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, u, u_x) + a(x, u, u_x) = 0$$

under the following assumptions about $a_i(x, u, p)$, $a(x, u, p)$

$$(12) \qquad \sum_{i=1}^{n} |a_i(x, u, p) + |a(x, u, p)| \leqslant \mu_1 ,$$

$$(13) \qquad \sum_{i=1}^{n} p_i a_i(x, u, p) \geqslant \mu_2 \sqrt{1+p^2} - \mu_3 , \quad \mu_i = \text{const} > 0 .$$

In addition we assume that the quadratic form

$$(A\xi, \xi) \equiv \sum_{i,j=1}^{n} \frac{\partial a_i(x, u, u_x)}{\partial u_{x_j}} \xi_i \xi_j , \qquad \xi \in R^n ,$$

is positive on the solution $u = u(x)$ and has a degeneracy of the following type:

$$(14) \qquad \mu_4 \frac{|\xi'|^2}{\sqrt{1+u_x^2}} \leqslant (A\xi, \xi) \leqslant \mu_5 \frac{|\xi'|^2}{\sqrt{1+u_x^2}} , \qquad \mu_4, \mu_5 > 0 ,$$

where the vector $\xi' \in R^{n+1}$ is a projection of $\tilde{\xi} = (\xi, 0)$ on the tangent plane to the hypersurface S defined by $u(x)$:

$$S = \left\{ X = (x_1, \ldots, x_{n+1}) : x = (x_1, \ldots, x_n) \in \Omega , \quad x_{n+1} = u(x) \right\} .$$

In other words, we require the quadratic form $(A\xi, \xi)$ to be non-degenerating on the vectors ξ for which $\tilde{\xi}$ are orthogonal to

$$\nu = (1 + u_x^2)^{-1/2} (-u_{x_1}, \ldots, -u_{x_n}, 1)$$

(ν is the unit normal to S).

Equation (8) and Euler equation for (10) near the points where $|\mathcal{D}| \neq 0$ have this kind of degeneracy. The latter can be expressed in the form

$$(15) \qquad \sum_{i=1}^{n} \frac{d}{dx_i} F_{u_{x_i}}(x, u, u_x) - F_u(x, u, u_x) = 0 ,$$

if we select the n components of the vector $X = (x_1, \ldots, x_{n+1})$ as the independent parameters $x = (x_1, \ldots, x_n)$ and denote the remaining one (let it be x_{n+1})

by u. The surface S is given be the equation $X_{n+1} = u(x)$, and the function $F(x, u, u_x)$ is given in terms of $\mathcal{F}(X, \omega)$ by

$$(16) \qquad F(x, u, u_x) = \mathcal{F}(X, \omega), \quad \omega = (-u_{x_1}, \dots, -u_{x_n}, 1).$$

As it is well known the condition of independence of the functional (10) in the parametric representation of the surface S has the form

$$\mathcal{F}(X, \lambda\omega) = \lambda \mathcal{F}(X, \omega), \quad \forall \lambda > 0,$$

and hence $(\mathcal{F}''\omega, \omega) \equiv \sum_{i,j=1}^{n+1} \mathcal{F}_{\omega_i \omega_j}(X, \omega)\, \omega_i\, \omega_j = 0$, i.e. the form $(\mathcal{F}''\xi, \xi)$, $\xi \in R^{n+1}$, is not positive definite. However in regular variational problems the degeneracy can occur only in the direction of the vector ω. In particular, the inequalities

$$(17) \qquad \mu_4 \frac{|\xi'|^2}{|\omega|} \leqslant (\mathcal{F}''\xi', \xi') \leqslant \mu_5 \frac{|\xi'|^2}{|\omega|}, \quad \mu_4, \mu_5 > 0,$$

hold for any $\xi' \in R^{n+1}$ which is orthogonal to ω.

Now it is clear that (14) for the equation (15) is a consequence of the regularity condition (17).

Conditions (12) and (13) are satisfied if $|\mathcal{F}_u(X, \omega)| \leqslant \mu_1$ and

$$\mathcal{F}(X, \omega) \geqslant \mu_0 |\omega|, \quad \mu_0 > 0.$$

In order to formulate the assumptions about the partial derivatives of the functions $a_i(x, u, p)$ and $a(x, u, p)$ consider the equation for the function $v = u_x^2$:

$$(18) \qquad -\frac{1}{2} \sum_{i,j} \frac{d}{dx_i}(a_{ij} v_{x_j}) + \sum a_{ij} u_{x_e x_i} u_{x_e x_j} = B,$$

where $a_{ij} = \dfrac{\partial a_i(x, u, u_x)}{\partial u_{x_j}}$ and

$$B = \frac{1}{2} \frac{\partial a_i}{\partial u} v_{x_i} + v \frac{d}{dx_i} \frac{\partial a_i}{\partial u} + u_{x_k} \frac{d}{dx_i} \frac{\partial a_i}{\partial x_k} + \frac{1}{2} \frac{\partial a}{\partial u_{x_i}} v_{x_i} + \frac{\partial a}{\partial u} + u_{x_k} \frac{\partial a}{\partial x_k}.$$

Our assumptions consist in the fact that for $x \in \Omega$ and $|u| \leqslant M$ the estimate

$$(19) \qquad B \leqslant \mu_6 |\delta u_x| + \mu_7 \sqrt{1 + u_x^2}$$

holds for B. Here $\delta f(x)$ is a projection of the vector $f_x = (f_{x_1}, \dots, f_{x_n}, 0)$ on the tangent plane to S. For the equation (15) conditions (12) and (19), roughly speaking, mean that each differentiation of $\mathcal{F}$ with respect to ω_i and u lowers the order of growth in $|\omega|$ at least by one, and differentiation with respect to x_i does not increase it. For the equation (8) our restrictions on $\mathcal{H}$ are just the same as in the case of global estimates.

Under the conditions (12) - (14) and (19) we proved the following.

THEOREM. — *For any strictly interior subdomain* Ω' *of* Ω, $\max_{\Omega} |u_x|$ *is estimated by a constant which depends only on M, the distance of* Ω' *from* $\partial\Omega$, *and on the constants appearing in the conditions* (12) - (14) *and* (19).

In proving this theorem we make an essential use of the inequality

$$(20) \qquad \left(\int_S |f|^{\frac{n}{n-1}} dS\right)^{\frac{n-1}{n}} \leqslant \beta \int_S |\delta f| \, dS ,$$

which is a generalization of an analogous inequality by M. Miranda [12] for the minimal surface equation ; (20) is true for any smooth function $f(x)$ with compact support in Ω and β depends only on constant μ_i appearing in the conditions (12) - (14). The proof of (20) is based on the isoperimetric inequality for integral currents established in [4].

If $u(x)$ satisfies one of the classical boundary conditions it is possible to estimate $\max_{\Omega'} |u_x|$ for subdomains $\Omega' \subset \Omega$ adjoining to $\partial\Omega$. Particularly similary results were obtained for the case of nonlinear boundary value problem

$$(21) \qquad \sum_{i=1}^{n} \frac{d}{dx_i} F_{u_{x_i}}(u_x) = \varphi(x , u) , \quad x \in \Omega ,$$

$$(22) \qquad \sum_{i=1}^{n} F_{u_{x_i}}(u_x) \cos(n , x_i) = 0 , \quad x \in \partial\Omega ,$$

where $n = n(x)$ is the exterior normal to $d\Omega$ at a point x and $F(u_x)$ is connected with the parametric integrant (10) in the above sense. These estimates permited to prove the solvability in the large of the problem (21), (22).

REFERENCES

[1] Бернштейн С. Н. — Собрание сочинений, том III, 1960, издательство АН СССР.

[2] Bombieri E., De Giorgi E. and Miranda M. — Una maggiorazione a priori relative alle ipersuperfici minimali non parametriche, *Arch. Rational Mech. Anal.*, Vol. 32, 1969, p. 255-267.

[3] De Giorgi E. — Una maggiorazione a priori relativa alle ipersuperfici minimali, *Atti del Convegno di Analisi Funzionale*, Roma, 1968.

[4] Federer H. and Fleming W.H. — Normal and integral currents, *Ann. of Math.*, 72, 1960, p. 458-520.

[5] Ладыженская О. А. — Первая краевая задача для квазилинейных параболических уравнений, Докл. Акад. Наук СССР, том 107, № 5, 1956, стр. 636-639; Труды ММО, том 7, 1958, стр. 149-177.

[6] Ладыженская О. А. и Уральцева Н. Н. — Линейные и квазилинейные эллиптические уравнения, Наука, Москва, 1964.

[7] Ладыженская О. А. и Уральцева Н. Н. — О некоторых классах неравномерио эллиптических уравнений, Записки научных семинаров Ленингр. отд. Матем. Инст. Стеклова, ЛОМИ, том II, 1968, стр. 129-149.

[8] Ладыженская О. А. и Уральцева Н. Н. — Тотальные оценки первых производных для решений квазилинейных эллиптическых и параболических уравнений, Записки научных семинаров Ленингр. отделения Матем. инст. Стеклова, ЛОМИ, том 14, 1969, стр. 127-155.

[9] Ladyzhenskaya O.A. and Uraltseva N.N. — Local estimates for gradients of solutions of non-Uniformly elliptic and parabolic equations, *Comm. Pure and Appl. Math.*, Vol. XVIII, 1970.

[10] Ладыженская О. А., Солонников В. А. и Уральцева Н. Н. — Линейные и квазилинейние параболические урав-я, Наука, Москва, 1967.

[11] Leray J., Schauder J. — Topologie et équations fonctionnelles, *Ann. Sci. Ecole Norm. Sup.*, 51, 1934, p. 45-78.

[12] Miranda M. — Disequaglianze di Sobolev sulle ipersuperfici minimali, *Rend. Sem. Math. Univ. Padova*, Vol. 38, 1967, p. 69-79.

[13] Осколков А. П. — Априорные оценки первых производных решений задачи Дирихле для неравномерно эллиптических квазилинейных уравнений, Труды Матем. Инст. Стеклова (Акад. Наук СССР), т. 102, 1967, стр. 105-127.

[14] Serrin J. — Problem of Dirichlet for quasi-linear elliptic differential equations with many independent variables, *Phil. Trans. Roy. Soc.*, London, A 264, 1969, p. 413-496.

[15] Уральцева Н. Н. — Нелинейные краевые задачи для уранений типа минимальной поверхности, Труды матем. Инст. Стеклова (Акад. Наук СССР), том 166.

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D 12 - SYSTÈMES DYNAMIQUES ET ÉQUATIONS DIFFÉRENTIELLES ORDINAIRES

SUR L'ALLURE FINALE DU MOUVEMENT DANS LE PROBLÈME DES TROIS CORPS

par V. M. ALEXEYEV

Introduction.

Le "Problème des trois corps" est parmi les plus connus en mathématique, en mécanique et en astronomie. 1687 –l'année où parurent les "Principia" de Newton – doit être considérée comme la date de naissance de ce problème. Depuis lors, soit presque 300 ans, le problème des trois corps a servi de pierre de touche aux générations successives de mathématiciens, mettant à l'épreuve leurs nouvelles méthodes.

A. Wintner remarqua une fois que chaque génération pose à sa propre manière les "questions fondamentales du problème des trois corps", et cherche à les résoudre toujours de sa propre façon. Suivant ici G.D. Birkhoff [1], je crois que du point de vue mathématique la question fondamentale est aujourd'hui celle de la description topologique de la décomposition de l'espace des phases en trajectoires des divers types. La classification des variétés intégrales invariantes est un cas particulier de ce problème.

Le problème ainsi posé est, semble-t-il, encore loin de la solution définitive. C'est pourquoi nous nous limiterons à l'un de ses aspects plus particuliers et plus approximatif, à savoir l'étude de **l'allure finale du mouvement**, c'est-à-dire l'étude des solutions lorsque $t \to \infty$.

La recherche dans cette direction commence avec les Mémoires de J. Chazy [2]-[4]. C'est pour rendre hommage à cet éminent mathématicien et astronome français, dont les travaux ont stimulé en grande partie ce qui est exposé ci-dessous, et aussi pour souligner la continuité de l'effort des diverses générations de mathématiciens, que j'ai donné à cette conférence le titre même de deux de ses Mémoires.

Le Mémoire [2] contient la description de toutes les allures finales unilatérales (c'est-à-dire se rapportant seulement au cas $t \to +\infty$ ou seulement au cas $t \to -\infty$). Du point de vue cosmogonique, tout aussi bien que du point de vue mathématique, il serait particulièrement intéressant de décrire les divers types d'évolution du système, c'est-à-dire déterminer quelles allures finales (pour $t \to +\infty$ et $t \to -\infty$) peuvent appartenir à un même mouvement.

La première partie de la presente conférence est un exposé des résultats obtenus dans cette direction. Dans la deuxième partie je traite des méthodes à l'aide desquelles ces résultats ont été obtenus.

Je voudrais remarquer que c'est A.N. Kolmogorov, qui, en 1954, a attiré pour la première fois mon attention sur ces problèmes. Depuis, son interêt amical et ses précieux conseils m'ont aidé plus d'une fois dans mes recherches.

I. RESULTATS

2. Classification des allures finales selon Chazy

Il est bien connu que le problème des trois corps se rapporte à l'étude du mouvement des corps (points matériels) P_i, $i = 1, 2, 3$, soumis aux forces d'attraction gravitationelles de Newton. Notons m_i la masse du corps P_i ; r_1 la distance entre P_2 et P_3 ; r_2 et r_3 se définissent d'une manière analogue.

Ce problème a donc $3 \times 3 = 9$ degrés de liberté et par conséquant l'espace des phases est de dimension 18. Par un choix approprié du système de coordonnées galiléen on peut considérer le centre de gravité immobile a l'origine des coordonnées, ce qui réduit le problème à 6 degrés de liberté ; soit M^{12} l'espace de phases. Ceci posé, le problème a toujours 4 premières intégrales algébriques, ce qui permet de réduire la dimension jusqu'à 8. Il nous sera néanmoins plus commode de nous arrêter à M^{12}, considéré comme fibré par les hypersurfaces isoénergétiques $H = h =$ Const.

D'après Chazy [2] M^{12} se décompose en sous-ensembles comme suit ($t \to +\infty$) :

(1) H. Mouvements **hyperboliques.**

(2) HP_i. Mouvements **hyperboliques-paraboliques.**

(3) P. Mouvements **paraboliques.**

Ici quant $t \to \infty$ on a $r_j \to \infty$ et $\dot{r}_j \to c_j$. Tout les c_j sont > 0 pour H et $= 0$ pour P. Dans les cas HP_i $c_i = 0$ et $c_j > 0$ pour $j \neq i$.

(4) HE_i. Mouvements **hyperboliques-elliptiques.**

(5) PE_i. Mouvements **paraboliques-elliptiques.**

Ici $\sup\limits_{t>0} \{r_i\} < \infty$ et $r_j \to \infty$, $\dot{r}_j \to c_j$ pour $j \neq i$; c_j est positif pour HE_i et égal à 0 pour PE_i.

(6) L. Mouvements **stables selon Lagrange.**

$$\sup_{i,\, t>0} \{r_i\} < \infty$$

(7) OS. Mouvements **oscillatoires.**

$$\overline{\lim_{t \to +\infty}} \sup_i \{r_i\} = \infty, \quad \underline{\lim}_{t \to +\infty} \sup_i r_i < \infty$$

Pour le cas $t \to -\infty$ les définitions sont analogues.

La position relative de ces sous-ensembles est représentée sur la figure 1. H et HP_i sont situés dans la région où la constante d'énergie $h > 0$; P sur l'hypersurface $h = 0$; L, PE_i et OS dans la région $h < 0$; les mouvements HE_i sont possibles pour h positif, nul ou négatif. Chacun des ensembles H et HE_i est ouvert dans M^{12} ; les HP_i forment des sous-variétés analytiques de codimension 1 ; P se compose de trois sous-variétés de codimension 2 (représentées par des points sur la figure 1) et d'une sous-variété de codimension 3 (non représentée sur la figure 1) ; la structure topologique des autres classes de mouvements est peu connue.

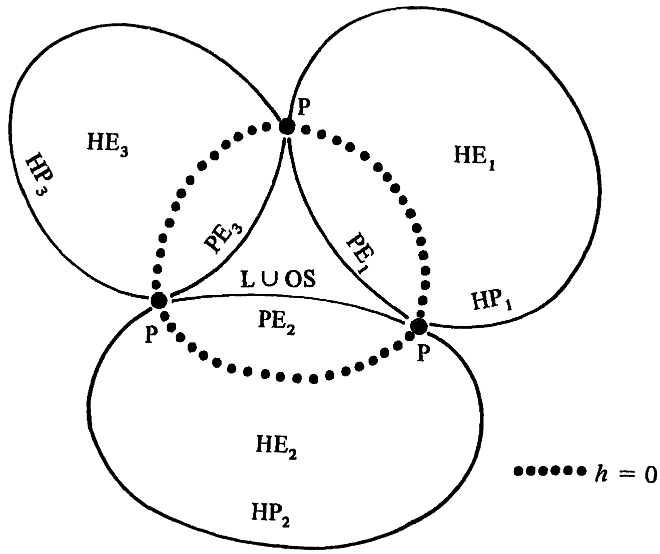

Fig. 1.

La classe *OS* fut introduite par Chazy à partir de considérations purement logiques, et longtemps l'existence de tels mouvements n'était pas établie. Enfin, en 1959, C.A. Sitnikov [11] a démontré, que $OS \neq \emptyset$. L'existence des autres types de mouvements était déjà connue. Dans ce qui va suivre nous nous limiterons aux types principaux : H, HE_i, L, OS, car les autres ont certainement une codimension positive. Pour différencier les types qui se rapportent à $t \to \pm\infty$ nous nous servirons des indices correspondants : H^+, L^- etc.

Il est bien connu qu'il n'existe pas dans M^{12} d'intégrales premières algébriques autres que les 4 classiques (Bruns) et même d'intégrales univoques analytiques, dépendant analytiquement des masses m_i (Poincaré).

HYPOTHESE. – Dans la région $H \cup \bigcup_i HP_i \cup HE_i$ il existe une famille complète d'intégrales premières univoques analytiques.

Les arguments tendant à confirmer cette hypothèse sont exposés dans [2] et [4].

3. Evolution du système ; la région $h > 0$ (table 1)

Les Mémoires [3] et [4] affirment que tout mouvement à les mêmes allures finales pour $t \to -\infty$ et $t \to +\infty$. Assez longtemps les mathématiciens et surtout les astronomes furent convaincus qu'une symétrie si remarquable avait effectivement lieu. Une certaine dissonance ne fut apportée que par les examples de L. Becker [5] qui appartenaient manifestement à $HE_1^- \cap HE_2^+$. Néanmoins Chazy les attribua à des erreurs d'intégration numérique et à l'impossibilité de fixer l'allure finale ($t \to \infty$) par intégration sur un intervalle fini du temps.

Table 1

		$t \to +\infty$	
		H^+	HE_i^+
$t \to -\infty$	H^-	Mesure >0 G.D. Birkhoff, 1927	**Capture partielle**, mesure >0 O. Yu. Schmidt (exemple numérique), 1947 C.A. Sitnikov (méthodes qualitatives), 1953
	HE_j^-	**Désintégration complète** Mesuré >0	$i = j$, Mesure >0 G.D. Birkhoff, 1927
			$i \neq j$, **échange**, mesure >0 L. Becker (exemples numériques), 1920 V.M. Alexeyev (méthodes qualitatives), 1956

En 1947 les conclusions de Chazy furent mises en doute par O.J. Schmidt. Son hypothèse cosmogonique fondée sur la capture gravitationelle à l'état pur, était en contradiction (tout au moins pour le problème des trois corps) avec le tableau symétrique de Chazy. Pour confirmer son hypothèse Schmidt obtint [12] par intégration numérique l'exemple d'une **capture partielle** ($H^- \cap HE^+$). Dans cet exemple trois étoiles indépendantes dans le passé formaient un sous-système stable (étoile double) tandis que la troisième étoile repartait à l'infini. Obtenu par intégration numérique l'exemple de Schmidt restait possible des mêmes critiques que les exemples [5]. Une des objections fut écartée par H.F. Hilmi [13], qui parvint à décrire par un système d'inégalités des sous-ensembles ouverts de H et HE_i ("critères des mouvements hyperboliques et hyperboliques-elliptiques"). En vérifiant ces inégalités aux extrémités de l'intervalle d'intégration on peut en tirer des conclusions valables sur tout l'axe du temps.

Mais les erreurs d'intégration numérique restaient très difficiles à évaluer. Ce n'est qu'avec l'exemple de C.A. Sitnikov (1953), obtenu par des méthodes purement qualitatives, que nous avons une démonstration rigoureuse de la possibilité d'une **capture partielle** ($H^- \cap HE^+$) et, par symétrie, de **désintégration complète** ($HE^- \cap H^+$).

L'exemple de Schmidt –sans doute un des premiers exemples d'une expérience de calcul numérique, entreprise pour vérifier l'hypothèse d'une théorie mathématique– servit de signal à toute une série de travaux consacrés tout aussi bien à l'étude critique des Mémoires de Chazy qu'au problème même de l'allure finale. Aujourd'hui le problème se trouve dans l'état résumé par les tables 1 et 2. Chaque rectangle correspond à une des possibilités logiques des allures finales du mouvement dans le passé et dans le futur, et correspond donc à un certain type d'évolution du système. Nous citons les auteurs et les dates des travaux où les types correspondants furent trouvés. Nous indiquons également la mesure de Lebesgue de l'ensemble correspondant dans M^{12}. Il faut remarquer que, vu la symétrie du temps, chaque résultat cité dans un des rectangles se rapporte tout aussi bien au rectangle symétrique par rapport à la diagonale principale de la table.

Dans le cas $h > 0$ (table 1) tous les 5 types d'évolution logiquement possibles se trouvent effectivement réalisés. Les ensembles H et HE étant ouverts, la probabilité positive (mesure > 0) de chaque type se trouve automatiquement vérifiée.

4. Evolution du système ; le cas $h < 0$ (table 2)

Table 2

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">$t \to +\infty$</td></tr>
<tr><td>HE_i^+</td><td>L^+</td><td>OS^+</td></tr>
<tr><td rowspan="4">$t \to -\infty$</td><td rowspan="2">HE_j^-</td><td>$i = j$, mesure > 0
G.D. Birkhoff, 1927</td><td rowspan="2">Capture complète
$\neq \emptyset$
V.M. Alexeyev, 1968

Mesure $= 0$
J. Chazy, 1929
G.A. Merman, 1954</td><td rowspan="2">$\neq \emptyset$
V.M. Alexeyev, 1968

Mesure $= 0$
J. Chazy, 1929
G.A. Merman, 1954</td></tr>
<tr><td>$i \neq j$, échange
Mesure > 0
L. Becker, 1920
(exemples numériq.)
V.M. Alexeyev, 1956
(méthodes qualit.)</td></tr>
<tr><td>L^-</td><td>Désintégration partielle
$\neq \emptyset$
Mesure $= 0$</td><td>L. Euler, J. Lagrange
H. Poincaré (exemples)
Mesure > 0
V.I. Arnold, 1963</td><td>$\neq \emptyset$
V.M. Alexeyev, 1968
Mesure ?</td></tr>
<tr><td>OS^-</td><td>$\neq \emptyset$, mesure $= ?$</td><td>$\neq \emptyset$, mesure $= ?$</td><td>$\neq \emptyset$, mesure $= ?$
C.A. Sitnikov, 1959</td></tr>
</table>

Dans la région $h < 0$ la situation est considérablement plus compliquée, que pour $h > 0$. Tout d'abord les ensembles $PE_i^{\pm}$, quoiqu'ils soient bien des sous-variétés de codimension 1, se trouvent certainement mal plongées dans M^{12}.

Il n'est pas clair qu'elles soient analytiques. Le point de vue de l'auteur est reflété par la suivante

HYPOTHESE. – Presque chaque point $p \in PE_i^+$ est accessible à partir de HE_i^+ et possède dans M^{12} un voisinage U, difféomorphe à $I \times D^{11}$ de sorte que

$$U \cap HE_j^+ \simeq A_j \times D^{11} \quad , \quad U \cap PE_j^+ \simeq B_j \times D^{11} \quad , \quad U \cap (L^+ \cup OS^+) \simeq C \times D^{11},$$

où D^{11} est un disque de dimension 11, $I = (0\,,\epsilon)$, $A_j\,, B_j\,, C \subset I$ et chacun de $A_j\,, j = 1,\ 2,\ 3$, consiste d'un ensemble dénombrable d'intervalles, B_j (dénombrable) est formé des extrémités des intervalles de l'A_j correspondant, et, enfin, C est zéro-dimensionel.

Probablement le difféomorphisme $U \to I \times D^{11}$ est analytique.

Les ensembles $HE_i^{\pm}$ sont ouverts et connexes, mais chacun d'entre eux a des ramifications compliquées dans M^{12} ; certaines branches s'entrelacent d'une manière très embrouillée. Birkhoff [1] imaginait HE_i^- comme trois courants venants de l'infini. En poursuivant cette analogie, on peut représenter chacun de ces courants ramifié en un ensemble dénombrable de ruisseaux, qui, tels les capillaires sanguins, transpersent l'espace des phases et se rassemblent en trois courants sortants HE_i^+.

L'analyse des raisonnements de Chazy a montré [15], que, contrairement à son opinion, $HE_i^- \neq HE_i^+$; toutefois on peut dire que les ensembles $HE^- = \bigcup_i HE_i^-$ et $HE^+ = \bigcup_i HE_i^+$ coincident dans la région $h < 0$ (à un ensemble de mesure nulle près) ("presque toute l'eau, apportée par les courants HE_i^-, sauf celle de certains rus de mesure totale nulle, est emportée par les courants HE_i^+").

Ainsi deux questions se posaient : existe-t-il dans la région $h < 0$ des mouvements des types :

a) $HE_i^- \cap HE_j^+$, $i \neq j$, ("échange") ; b) $HE_i^- \cap L^+$ (**"capture complète"**) et $HE_i^- \cap OS^+$.

L'auteur a obtenu ([16] — [18]) des réponses positives à ces deux questions. Il est particulièrement intéressant de constater la possibilité d'une capture complète. Contrairement aux exemples de Schmidt et Sitnikov, dans le cas d'une capture complète une étoile triple se forme sous l'effet de forces purement gravitationelles à partir d'une étoile double et d'une troisième étoile, venue de l'infini. La capture d'une comète par le système Soleil-Jupiter est de même nature.

La structure de l'ensemble $L^- \cap L^+$ est peu connue, quoique la plupart des publications sur le problème des trois corps se rapporte justement à cet ensemble. La théorie de Kolmogorov-Arnold-Moser a permis de démontrer l'existence de mouvements quasi-périodiques dans un grand nombre de problèmes non-intégrables en mécanique. En particulier, en 1963 V.I. Arnold [19] a démontré que $L^- \cap L^+$, pour une masse suffisament petite de deux des trois corps, contient un sous-ensemble de mesure positive constitué de tores pentadimensionels remplis de mouvements quasi-périodiques (cf. également [6], [20]).

Les mouvements quasi-périodiques remplissent une partie "régulière" de $L^- \cap L^+$, mais ne contiennent pas cet ensemble. L'ensemble $(L^- \cup OS^-) \cap (L^+ \cup OS^+)$ contient également une partie "quasi-aléatoire". Quelques considérations sur sa structure peuvent être obtenues en étudiant un cas particulier du problème des trois corps (§ 8 ci-dessous).

Le problème suivant reste ouvert :

PROBLEME. — L'ensemble OS a-t-il une mesure positive ou nulle dans M^{12} ? La même question peut être posée pour $OS^{\pm} \cap HE_i^{\mp}$, $OS^{\pm} \cap L^{\mp}$, $OS^- \cap OS^+$.

II. METHODES

5. Effet de la "couche adhérente" et solutions discontinues du problème idéal de Képler ([16], [17], [21]).

Soit M une variété riemanienne, Σ une sous-variété, W un fibré normal sur Σ, U un voisinage tubulaire de Σ dans M, difféomorphe au r-voisinage de la section nulle de W (de sorte que (x, y), $|x| < r$, $y \in \Sigma$, $x \perp T_y \Sigma$ peuvent être considérés comme coordonnées dans U). Tous les champs vectoriels considérés ci-dessous sont de la classe C^1 là où ils sont définis et dépendent d'une manière continue du paramètre $\alpha \geqslant 0$ au point $\alpha = 0$ pour la topologie de la convergence uniforme sur les compacts.

Considérons les systèmes d'équations différentielles suivants : le système **principal** (dans $M \backslash \Sigma$)

$$\dot{z} = Z_a (z) \tag{1}$$

ou (exprimés à l'aide des coordonnées $(x , y) \in U$) :

$$\begin{cases} x = X_a = X_a^1\left(\dfrac{x}{\mu}, y\right) + X_a^2 (x , y), \\ y = Y_a = \dfrac{1}{\mu} Y_a^1 \left(\dfrac{x}{\mu}, y\right) + Y_a^2 (x , y), \end{cases} \tag{2}$$

où $\mu = \mu(\alpha) \downarrow 0$ quand $\alpha \downarrow 0$; le système **auxiliaire** (dans U)

$$\begin{aligned} x &= X_a^2 (x , y) , \\ y &= Y_a^2 (x , y) ; \end{aligned} \tag{3}$$

le système **limite**

$$z = Z_0 (z) \qquad \text{ou} \qquad \begin{cases} x = X_0^2 (x , y), \\ y = Y_0^2 (x , y); \end{cases} \tag{4}$$

et enfin "le système **de la couche adhérente**"

$$\begin{aligned} \frac{d\xi}{d\tau} &= X_0^1 (\xi , \eta) + X_0^2 (0 , \eta), \\ \frac{d\eta}{d\tau} &= Y_0^1 (\xi , \eta). \end{aligned} \tag{5}$$

Ici les champs X_a^2 et Y_a^2 satisfont dans U aux conditions énumerées ci-dessus, leurs dérivées partielles sont uniformément (rel. à α) bornées sur chaque compact de U ; X_a^1 et Y_a^1 sont définies sur $W \backslash \Sigma$ (c'est-à-dire pour $\xi \neq 0$) et satisfont à la condition supplémentaire

$$\text{(A)} \qquad |X_a^1 (\xi , \eta)| \leqslant \varphi (|\xi|), \, |Y_a^1 (\xi , \eta)| \leqslant \psi (|\xi|),$$

où $\varphi(s)$, $\psi(s) \downarrow 0$ quand $s \uparrow +\infty$ et $\int \psi(s)\, ds < \infty$.

Sur un compact quelconque $K \subset U \backslash \Sigma$ d'après (A) les deuxièmes membres des systèmes (2) et (3) sont uniformément proches et les deux systèmes limites coïncident dans $U \backslash \Sigma$. Le deuxième de ces derniers systèmes est régulier dans toute la région U et par conséquant le premier (limite de (1) quand $\alpha \downarrow 0$) possède une extension régulière à toute la variété M.

Ainsi, le système limite n'a pas de singularité sur Σ (ses solutions "ne remarquent pas Σ"). Cependant on voit d'après (2) que, pour $\alpha \neq 0$, dans le μ-voisinage de Σ, les facteurs de perturbation X_a^1 et $\dfrac{1}{\mu} Y_a^1$ sont grands. Par conséquent dans la "couche adhérente" à Σ une variation rapide le long de Σ est possible et à la limite (quand $\alpha \downarrow 0$) la solution $Z_a(t)$ du système (1) peut converger vers de différentes solutions $Z_0^{\pm}(t)$ du système (4) (fig. 2).

Il est naturel, suivant (2), d'effectuer dans la couche adhérente le changement de variables $x = \mu\xi$, $y = \eta$, $t = \mu\tau$ qui donne, à la place de (2), le système suivant

$$\begin{cases} \dfrac{d\xi}{d\tau} = X_a^1(\xi, \eta) + X_a^2(\mu\xi, \eta), \\ \dfrac{d\eta}{d\tau} = Y_a^1(\xi, \eta) + \mu Y_a^2(\mu\xi, \eta). \end{cases} \tag{6}$$

A la limite, le système (6) dégénére, se transformant en "système de la couche adhérente" (5).

Supposons que l'on ait la condition

(B) La solution $(\xi(\tau), \eta(\tau))$ du système (5) issue de (ξ_0, η_0) est définie et contenue dans $W\backslash\Sigma$ pour $\tau \geqslant 0$ et satisfait en outre les égalités

$$\lim_{\tau \to +\infty} |\xi(\tau)| = \infty, \quad \lim_{\tau \to +\infty} \eta(\tau) = \eta_\infty^+ \tag{7}$$

THEOREME. — *Si* $z_a(t)$ *est une solution du système* (1) *telle que*

(1) $z_a(0) = (x_a(0), y_a(0)) \in U, \dfrac{x_a(0)}{\mu} \to \xi_0, y_a(0) \to \eta_0$ *pour* $\alpha \downarrow 0$.

(2) *Les conditions* **(A)** *et* **(B)** *sont satisfaites.*

(3) $X_0^2(0, \eta_\infty^+) \neq 0$.

(4) *La solution* $z^+(t)$ *du système* (4) *issue de* $z^+(0) = (0, \eta_\infty^+) \in \Sigma$ *est définie pour* $[0, T^+]$ *et se trouve dans* $M\backslash\Sigma$ *pour* $0 < t < T^+$;

alors pour tout $t \in (0, T^+)$ *on a*

$$\lim_{a \to 0} z_a(t) = z^+(t). \tag{8}$$

Remarquons, que $z_a(0) \to (0, \eta_0)$, qui, en général, différe de $(0, \eta_\infty^+)$. Si la condition **(B)** est renforcée comme suit :

$(\xi(\tau), \eta(\tau))$ est définie pour tout τ et

$$\lim_{\tau \to \pm\infty} |\xi(\tau)| = \infty, \quad \lim_{\tau \to \pm\infty} \eta(\tau) = \eta_\infty^{\pm} \tag{7a}$$

alors (8) est vérifié pour tout $t \in (0, T^+)$ et l'on a pour $t \in (T^-, 0)$,

$$\lim_{a \downarrow 0} z_a(t) = z^-(t), \tag{8a}$$

où $z^-(t)$ est la solution de (4) issue de $(0, \eta^-)$, définie pour $[T^-, 0]$ et contenue dans $M\backslash\Sigma$ pour $(T^-, 0)$.

Pour $\eta_\infty^- \neq \eta_\infty^+$ on a la situation représentée sur la figure 2.

Considérons maintenant le problème des trois corps du type "système planétaire" : $m_1 = 1 \gg m = m_2 = m_3$. Le mouvement est alors gouverné par un

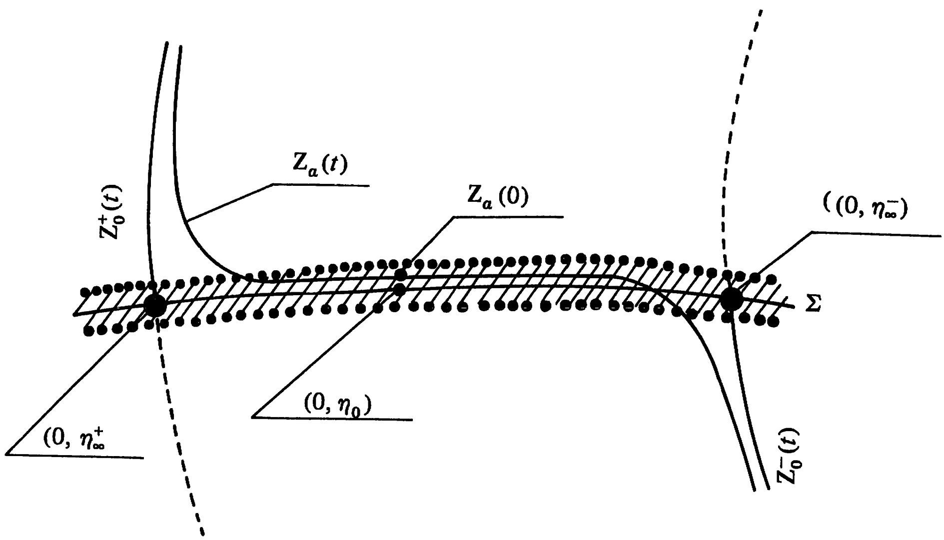

Fig. 2

système tel (2) où $\alpha = \mu = m$, le rôle de Σ étant joué par "la variété de choc" de corps P_2 et P_3, déterminée par l'équation $r_1 = 0$. Le système limite (4) correspond au "problème idéal de Képler" : le point P_1 est immobile à l'origine des coordonnées, P_2 et P_3 décrivent autour du P_1 des orbites Képleriennes. L'effet de la couche adhérente se manifeste quand P_2 et P_3 se rapprochent et le système (5) définit un mouvement Képlerien de ces deux corps autour du centre de gravité commun, l'interaction avec P_1 étant alors négligeable.

Le théorème ci-dessus permet de construire une famille de solutions du problème réel ($m > 0$) qui converge pour $m \to 0$ vers divers mouvements Kepleriens pour $t \gtrless 0$. Ainsi pour décrire valablement le passage à la limite il faut considérer les solutions "discontinues" du problème idéal de Kepler, solutions pour lesquelles les vitesses des corps P_2 et P_3 effectuent des bonds à l'instant du choc. C'est sur cette idée que se fondent les examples d'échange ($HE_i^- \cap HE_j^+$, $i \neq j$) dans la région $h < 0$, cités dans la table 2.

6. Systèmes dynamiques quasi-aléatoires

La théorie des systèmes dynamiques classiques se divise en deux grandes branches : topologique et ergodique. Cette dernière étudie les applications pour lesquelles une mesure reste invariante.

Limitons nous, pour simplifier, à une seule application $S : M \to M$ et une mesure normée ($\mu M = 1$). Si S conserve μ, on peut, suivant Kolmogorov, définir un invariant – **l'entropie métrique** $h_\mu(S)$. Un même S peut, en général, avoir plusieurs mesures invariantes, décrites par la théorie de Krylov-Bogolioubov.

Considérons un système mécanique, enregistrant successivement toutes les passages d'une trajectoire dans des ensembles $\{A_i\}$ formant une décomposition de M, c'est-à-dire considérons une fonction $x \to \{i_n(x)\}$ définie par la relation $S^n x \in A_{i_n}(x)$. Par exemple, chaque A_i correspond à une indication d'un appareil enregistreur, de sorte que $i_n(x)$ est la suite d'indications successives de cet appareil observant un même mouvement d'état initial x. Une mesure normée μ définit la distribution des probabilités sur M. Au choix aléatoire du point initial correspond une suite aléatoire $i_n(x)$. D'après un théorème de J.G. Sinaï [27], on peut, pour $h_\mu > 0$, choisir A_i de sorte que $i_n(x)$ soit une "suite de Bernoulli", au sens de la théorie des probabilités. Il est donc naturel de donner la définition suivante :

DEFINITION [22]. – Un système dynamique $S : M \to M$ s'appelle quasi-aléatoire s'il existe une mesure borélienne S-invariante μ, telle que $h_\mu(S) > 0$.

De même que l'entropie métrique, il est possible de définir [7] un invariant topologique de l'homéomorphisme – **l'entropie topologique** h_{top}.

THEOREME. – (J. Dynabourgue [23]). *Pour les homéomorphismes $S : M \to M$ d'un compact M de dimension finie on a $h_{top} = \sup\limits_{\mu} h_\mu(S)$ où μ décrit l'ensemble des mesures normées boreliennes S-invariantes.*

Remarquons, qu'il existe [24] un homéomorphisme d'un compact zéro-dimensionel et pour lequel le sup n'est pas atteint, c'est-à-dire que toujours $h_{top} > h_\mu$.

COROLLAIRE. – Un système dynamique $S : M \to M$ est quasi-aléatoire si et seulement si $h_{top} > 0$.

7. Chaines de Markov topologiques

Pour presque tous les exemples de systèmes dynamiques quasi-aléatoires que je connais, la propriété d'être quasi-aléatoire est liée à l'existence de sous-ensembles de Markov invariants.

Soit Ω^p le produit de Tychonov d'un ensemble dénombrable d'ensembles de p points, c'est-à-dire l'espace de suites infinies $\omega = \{a_n ; -\infty < n < +\infty\}$ où $a_n = 0, 1, \ldots p-1$; soit T l'homéomorphisme de déplacement vers la gauche. Chaque $(p \times p)$-matrice $\Pi = (\pi_{ij})$ formée de zéros et de uns détermine un sous-ensemble T-invariant $\Omega^\Pi \subset \Omega^p$ par la condition

$$a_n = \omega \in \Omega^\Pi \Leftrightarrow \pi_{a_n a_{n+1}} = 1$$

pour tout n. La restriction $T|\Omega^\Pi$ est appellée **chaine de Markov topologique** (CMT) à p états et à matrice de transformation Π.

Le graphe orienté Γ aux sommets $0, 1, \ldots, p-1$ et aux arêtes $\overrightarrow{ij}$ (pour celles des paires (i, j) qui satisfont $\pi_{ij} = 1$) détermine univoquement la CMT (Ω^Π, T). D'une manière analogue, à l'aide d'une matrice ou d'une graphe, on peut définir un CMT avec un ensemble dénombrable (on même arbitraire) d'états [22], [24].

THEOREME (PARRY [8] ; [24]). $h_{top}(T|\Omega^\Pi) = \log \lambda$, où λ *est la valeur propre positive maximale de la matrice Π $(p < \infty)$. Il existe une seule mesure borélienne μ T-invariante sur Ω^Π telle que $h_{top}(T) = h_\mu(T)$.*

Soit $S : M \to M$ un difféomorphisme de la variété M. Appelons le sous-ensemble $A \subset M$ **sous-ensemble invariant de Markov** du difféomorphisme S, s'il existe une CMT (Ω^Π, T) et une application continue $\varphi : \Omega^\Pi \to M$ telle que $A = \varphi(\Omega^\Pi)$, $\varphi \circ T = S \circ \varphi$ de sorte que $SA = A$, le diagramme

$$\begin{array}{ccc} \Omega^\Pi & \xrightarrow{T} & \Omega^\Pi \\ \varphi \downarrow & & \downarrow \varphi \\ A & \xrightarrow{S} & A \end{array}$$

étant commutatif. Si φ est un homéomorphisme, le restriction $S \mid A$ est topologiquement équivalent à la CMT $T \mid \Omega^\Pi$, ce qui permet de calculer h_{top} $(S \mid A)$ à l'aide du théorème de Parry. Malheureusement, ce n'est possible que pour $\dim A = 0$. Dans les exemples importants cités ci-dessous, φ cesse d'être injectif, mais seulement sur un ensemble de première catégorie dans Ω^Π ce qui rend possible la transformation de φ en isomorphisme métrique et permet de trouver la "mesure de l'entropie maximale" sur A.

Exemples.

(1) Le "fer à cheval de Smale" et ses modifications permettent pour tout p, Π et M ($\dim M \geqslant 2$) d'indiquer une C^1-ouvert $U_M \subset \text{Diff}(M)$ tel que chaque $S \in U_M$ a un sous-ensemble invariant de Markov A, sur lequel $S \mid A$ équivalent topologiquement à $T \mid \Omega^m$ [10], [22].

(2) Soient q_i, $1 \leqslant i \leqslant m$ des points hyperboliques périodiques du difféomorphisme S et r_j, $1 \leqslant j \leqslant s$ des points transversaux homo- et hétérocliniques, de sorte que $\lim\limits_{n \to \pm\infty} S^n r_j = q_{i\pm(j)}$.

Construisons le graphe Γ, dont la structure copie dans une certaine mesure cette situation. Ce graphe contient les cycles γ_i, $1 \leqslant i \leqslant m$, ne se coupant pas, (le nombre d'arêtes de γ_i étant égal à la période q_i) et les lignes brisées ouvertes π_j, $1 \leqslant j \leqslant s$ (à N_j arêtes) qui joignent les sommets $Q_j^- \in \gamma_{i-(j)}$ et $Q_j^+ \in \gamma_{i+(j)}$ (ces sommets peuvent être choisis arbitrairement sur les cycles correspondants (fig. 3)).

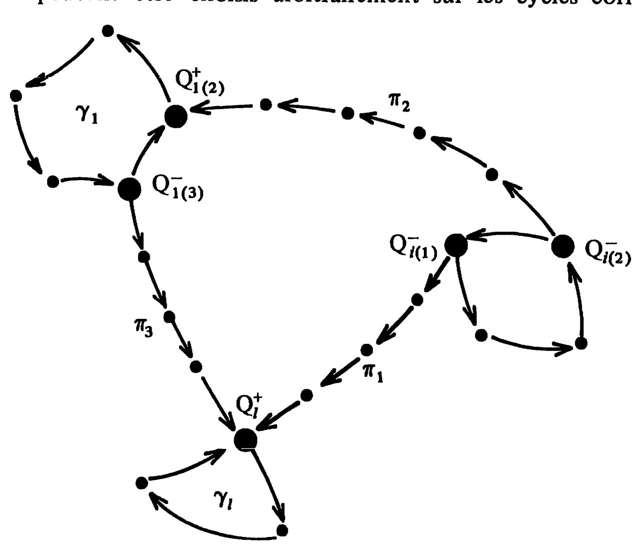

Fig. 3.

THEOREME. – *Pour chaque ouvert U contenant les trajectoires des points q_i, $1 \leqslant i \leqslant m$,* et r_j, $1 \leqslant j \leqslant s$ *il existe des N_j, $1 \leqslant j \leqslant s$, et un ouvert $V \subset U$ tels que la restriction de S, $S \mid A$, à l'ensemble A maximal S-invariant contenu dans V est topologiquement équivalente à la CMT définie par le graphe Γ ; $q_i, r_j \in A$.*

Remarquons que pour différents choix des points $Q_j^{\pm}$ on obtient des graphes Γ différents. Les ensembles V et A changent alors d'une manière correspondante.

(3) Soit $S : M \to M$ un C-difféomorphisme au sens de D.V. Anosov. J.G. Sinaï a démontré [25], [26] que M est un ensemble de Markov. La construction de la CMT (Ω^{Π}, T) et du "presque-homéomorphisme" $\varphi : \Omega^{\Pi} \to M$ peut être effectuée à l'aide d'une décomposition, dite **décomposition de Markov** de la variété M. L'application φ n'identifique que les points de l'ensemble $N \subset \Omega^{\Pi}$ de première catégorie et il est possible de construire sur M une "mesure d'entropie maximale" (décrite en plus de détail dans la communication de J.G. Sinaï à ce Congrés).

Fécemment, R. Bowen [9] à généralisé ce résultat à un "basic set" arbitraire du difféomorphisme S, à condition que celui-ci vérifie l'axiome A de S. Smale (cf. [10]). Pour un ensemble invariant hyperbolique (perronien) quelconque l'auteur a obtenu un résultat [28] plus faible (l'application φ est essentiellement non-univoque).

PROBLEME. – Supposons $h_{top}(S) > C > 0$. Existe-t-il alors pour S un sous-ensemble de Markov A sur lequel $h_{top}(S \mid A) = C$? Et si $C = h_{top}(S)$?

La méthode des "schémas d'itinéraire" proposée par l'auteur [22] est utile à la recherche des sous-ensembles de Markov des systèmes dynamiques correspondants à certains problèmes concrets. En particulier, c'est cette méthode qui a permi de remplir les "taches blanches" de la table 2.

8. Sur un cas particulier du problème des trois corps

Soit $m_1 = m_2$, les points P_1 et P_2 et leurs vitesses étant symétriques relativement à OZ à l'instant initial. Si le centre de gravité du système satisfait la condition usuelle (p. 2), P_3 se situe alors sur OZ et sa vitesse est verticale (fig. 4). Par raison de symétrie la configuration des corps P_1, P_2, P_3 restera analogue pour tout t, ce qui permet d'étudier le mouvement à l'aide d'un système à deux degrès de liberté.

Ce cas particulier a été considéré il y a bien longtemps (au plus tard en 1895), mais l'heureuse idée d'étudier la disposition des allures finales d'un tel mouvement dans l'espace des phases appartient à A.N. Kolmogorov. C'est justement pour cet exemple que C.A. Sitnikov [11] a démontré que $OS^{\pm} \neq \emptyset$ (vu la symétrie du temps on peut déduire du résultat de Sitnikov que $OS^{+} \cap OS^{-} \neq \emptyset$, ce que nous avons noté dans la table 2). Pour ce même exemple l'auteur a pu construire un ensemble invariant de Markov.

Supposons $m_3 = 0$ (la supposition $1 \geqslant m_3 > 0$ ne donne rien de réellement nouveau). Alors P_1 et P_2 décrivent dans XOY des orbites symétriques Képleriennes autour de 0, qui dans le cas $h < 0$ seront des ellipses. A l'instant où P_3 passe par 0, l'état de système est déterminé par la vitesse ν de ce corps et la phase τ (de l'anomalie réelle ou moyenne) du mouvement elliptique des corps P_1 et P_2. Considérons (ν, τ) comme coordonnées polaires dans un plan Φ (vu la symetrie relative à XOY nous pouvons négliger le signe de ν). Le déplacement

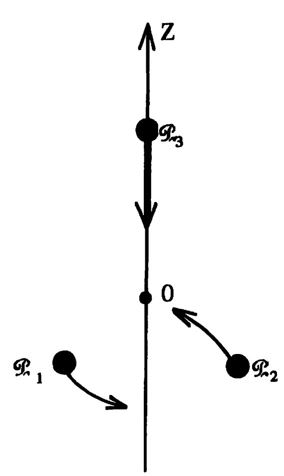

Fig. 4.

le long de la trajectoire dans l'espace des phases d'un passage de P_3 par 0 au passage suivant définit un difféomorphisme local $S : R^+ \to R^-$, $R^\pm \subset \Phi$. Il est possible d'indiquer un ouvert $V \subset \Phi$ tel que l'ensemble invariant maximal A contenu dans V est un ensemble de Markov (ou, en d'autres mots, admet une description en termes de "dynamique symbolique").

Soit Ω l'ensemble de toutes les suites

$[a_n \ ; n_1 \leqslant n \leqslant n_2]$, où $-\infty \leqslant n_1 \leqslant 0 < n_2 \leqslant +\infty$,

$a_n = (m_n, i_n)$ pour $n_1 < n < n_2$ où les entiers positifs $m_n \geqslant N$, $i_n = 0$ ou 1.

$a_{n_1} = (\nu^-, i_{n_2})$, $0 \leqslant \nu^- \leqslant \delta$, $i_n = 0$ ou 1, $n_1 \neq -\infty$.

$a_{n_2} = \nu^+$, $0 \leqslant \nu^+ \leqslant \delta$, $n_2 \neq +\infty$.

Ω admet une structure topologique naturelle, et sur le sous-ensemble Δ^+ (où $n_2 \neq 1$) on peut définir l'homéomorphisme local T (déplacement vers la gauche) ; $\Delta^- = T\Delta^+$.

THEOREME. – *Il existe des nombres N, δ, un homéomorphisme $\varphi : \Omega \to \Phi$ et un ouvert $V \subset \Phi$ tel que $\varphi(\Omega) = A$ soit un ensemble maximal invariant dans V tel que $\varphi(\Delta^\pm) \subset R^\pm$ et tel que le diagramme*

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{T} & \Delta^- \\ \varphi \downarrow & & \downarrow \varphi \\ R^+ & \xrightarrow{S} & R^- \end{array}$$

soit commutatif.

Le choix de $[a_n]$ influence de mouvement de P_3 comme suit. En choisissant le point $\varphi[a_n] \in \Phi$ comme condition initiale, nous obtenons une solution pour laquelle P_3 revient à l'origine, 0, n_2 fois pour $t > 0$ et n_1 fois pour $t < 0$. Entre le $(n-1)$-unième et le n-ième retour de P_3, les corps P_1 et P_2 effectuent m_n révolutions autour de 0 ; $i_n = 0 (= 1)$ indique que le n-ième retour a lieu près du moment du rapprochement (éloignement) maximal de P_1 et P_2.

Si $n_2 \neq +\infty$, après n_2 retours, P_3 s'éloigne à l'infini avec la vitesse limite $v^+ = a_{n_2}$. D'une manière analogue si $n_1 \neq -\infty$, v^- donne la vitesse l'éloignement à l'infini de P_3 lorsque $t \to -\infty$.

Lorsque m_n augmente, l'amplitude des oscillations du corps P_3 augmente aussi.

Le mouvement appartient à HE_3^+ et PE_3^+ pour $n_2 < +\infty$, et v^+ est supérieure ou égal à 0 respectivement; à L^+ si $n_2 = \infty$ et $\lim_{n\to+\infty} m_n < \infty$; à OS^+ si $n_2 = +\infty$, $\overline{\lim}_{n\to+\infty} m_n = +\infty$ et $\underline{\lim}_{n\to+\infty} m_n < \infty$. On détermine d'une manière analogue l'appartenance à HE_3^-, PE_3^-, L^- et OS^-. On vérifie aisément que $HE_3^\pm$, $PE_3^\pm$, $L^\pm$, $OS^\pm$ est l'ensemble complet des allures finales (pour $h < 0$) du mouvement considéré.

THEOREME [18]. – *Il existe un $\epsilon > 0$, tel que pour $\epsilon m_1 = \epsilon m_2 > m_3 \geqslant 0$ et pour $h < 0$, dans le cas particulier du problème des trois corps considéré, les 16 possibilités logiques des allures finales selon Chazy sont effectivement réalisables.*

Nous avons aussi le

COROLLAIRE. – *Dans ces mêmes conditions chaque intégrale première analytique est une fonction des intégrales d'énergie et de moment angulaire.*

Ce corollaire a un certain intérêt et ne découle pas du théorème de Poincaré cité au n. 2, puisque les masses sont fixées.

REFERENCES

[1] BIRKHOFF G.D. — Dynamical systems, *Amer. Math. Soc.*, Coll. Publ., Vol. 9, 1927.

[2] CHAZY J. — Sur l'allure finale du mouvement dans le problème des trois corps quand le temps croit indefiniment, *Annales de l'Ecole Norm. sup.*, 3^e sér., 39, 1922, p. 29-130.

[3] CHAZY J. — Sur l'allure finale du mouvement dans le problème des trois corps, *Journ. Math. pures et appl.*, 8, 1929, p. 353-380.

[4] CHAZY J. — Sur l'allure finale du mouvement dans le problème des trois corps, *Bull. Astron.*, 8, 1932, p. 403-436.

[5] BECKER L. — On capture orbits, *Monthly Notices Royal Astron. Soc.*, **80**, 1920, p. 590.

[6] MOSER J. — Quasi-periodic solutions of the threebody problem, « Coll. probl. gravit. des N corps, Paris, 1967 », *Bull. Astron.*, (3), **3**, 196, No. 1-2, p. 53-59.

[7] ADLER R.L., KONHEIM A.G. et MCANDREW M.H. — Topological entropy, *Trans. Amer. Math. Soc.*, **114**, No. 2, 1965, p. 309-319.

[8] PARRY W. — Intrinsic Markov chains, *Trans. Amer. Math. Soc.*, **112**, 1964, p. 55.

[9] BOWEN R. — *Markov partitions for Axiom A Diffeomorphism*. Preprint, 1970.

[10] SMALE S. — Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73**, 1967, p. 747-817.

[11] Ситников К. А. — Существование осциллирующих движений в задаче трех тел, Доклады АН СССР, 133, 2, 1960, p. 303-306.
[12] Шмидт О. Ю. — О возможности захвата в небесной механике, Доклады АН СССР, 58, 2, 1947, p. 213.
[13] Хильми Г. Ф. — Проблема n тел в небесной механике и космогонии, Изд. АН СССР, 1951.
[14] Ситников К. А. — О возможности захвата в задаче трех тел, Матем. Сборник, 32 (74), 3, 1953, p. 693-705.
[15] Мерман Г. А. — К вопросу об исследованиях Шази в задаче трех тел, Бюлл. инст. теор. астр. АН СССР, 5, 9 (72), 1954, p. 594-605.
[16] Алексеев В. М. — Обмен и захват в задаче трех тел, Доклады АН СССР, 108, 4, 1956, p. 599-602.
[17] Алексеев В. М. — Финальные движения в задаче трех тел, Сборник « Проблемы движения искусственных небесных тел », Изд. АН СССР, 1963.
[18] Алексеев В. М. — О возможности захвата в задаче трех тел при отрицательном значении константы полной энергии, Успехи мат. наук, 24, 1, 1969, p. 185-186.
[19] Арнольд В. И. — Малые знаменатели и проблемы устойчивости движения в классической и небесной механике, Успехи мат. наук, 18, 6, 1963, p. 91-191.
[20] Красинский Г. А. — Квазипериодигеские решения первого сорта в плоской задаче n тел, Труды институт. теор. астрон. АН СССР, 13. 1969, p. 105-168.
[21] Алексеев В. М. — Об одной задаче с малым параметром, Вестник Моск. ун-та 1962, № 4, 17-27.
[22] Алексеев В. М. — Квазислучайные динамические системы, I, II, III. Математич. Сборник, 76, 1, 1968, p. 72-134; 77, 4, 1968, p. 547-601; 78, 1, 1969, p. 3-50.
[23] Динабург Е. М. — Соотношение между топологической и метрической энтропией, Доклады АН СССР, 190, 1, 1970, p. 19-22.
[24] Гуревич Б. М. — Топологическая энтропия счетных марковских цепей, Доклады АН СССР, 187, 4, 1969, 715-718.
[25] Синай Я. Г. — Марковские разбиения и У-диффеоморфизмы, Функциональный анализ, 2, 1, 1968, p. 64-89.
[26] Синай Я. Г. — Построение марковских разбиений, Функциональный анализ, 2, 3, 1968, p. 70-80.
[27] Синай Я. Г. — О слабом изоморфизме преобразований с инвариантной мерой, Матем. Сборник, 63, 1, 1963, p. 23-42.
[28] Алексеев В. М. — Перроновские множества и топологические цепи Маркова, Успехи мат. наук, 24, 5, 1969, p. 227-228.

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ON THE CONTINUATION OF INVARIANT SETS OF A FLOW

by Charles C. CONLEY

1. Introduction.

One step towards understanding the structure of a flow involves describing the behavior of orbits near selected invariant sets ; particularly that behavior common to all nearby flows. The invariant set chosen is often an *isolated invariant set*, which means it is the largest invariant set in some (closed) neighborhood of itself. (cf. [1]) A typical example is a hyperbolic (elementary) rest point ; the behavior of nearby orbits is described in terms of the local stable and unstable manifolds – or, one could say, in terms of the Morse index.

This report describes a framework for the study of isolated invariant sets, beginning with some general remarks related to existence and stability.

2. Isolated Invariant Sets.

Let X be a compact connected metric space and let F be the space of flows f on X with the compact open topology. Let $C(X)$ be the space of non-empty closed subsets of X with the Hausdorff metric, and for each $f \in F$ let $\mathfrak{I}(f)$ be the collection of closed invariant sets of f. Then $C(X)$ is compact and connected, and $\mathfrak{I}(f)$ is a compact subset of $C(X)$; however, $\mathfrak{I}(f)$ is not generally connected.

Isolated invariant sets (defined in § 1) are characterized as those which occur as the maximal (under inclusion) element of some open and closed subset of $\mathfrak{I}(f)$. The (unique) maximal element of a component of $\mathfrak{I}(f)$ is called a quasi-isolated invariant set ; these are intersections of isolated ones. Thus the existence of isolated invariant sets corresponds to lack of connectivity of $\mathfrak{I}(f)$.

With regard to stability : suppose E is a subset of F and τ is a topology on E which is at least as strong as the (relative) compact-open topology and which makes E a Baire space. Then $\mathfrak{I}(f)$ is upper semi-continuous on the space E. Let $\mathfrak{I}\text{so}\ (f)$ be the closure in $C(X)$ of the set of isolated invariant sets. Then $\mathfrak{I}\text{so}(f)$ is lower semi-continuous at continuity points of $\mathfrak{I}$; i.e. on a dense G_δ of E.

3. Continuation.

The notion of "continuation" of an isolated invariant set to nearby flows is defined in terms of *isolating neighborhoods* : namely, a closed subset of X each of whose boundary points is sometimes carried into its complement by the flow. To each such neighborhood there corresponds a unique largest invariant set contained in it ; the latter is isolated provided it is not the empty set.

If N is an isolating neighborhood for f then it is also for nearby flows. Let $\mathfrak{S} \subset C(X) \times F$ consist of the pairs (I, f) such that $I \in \mathfrak{I}\text{so}\ (f)$. Suppose U is open in F and that N is an isolating neighborhood for each flow in U. Let $\sigma_N : U \to \mathfrak{S}$ assign to each f the pair (I, f) where I is the larget invariant set in N. Letting the range of these "sections" generate the topology of $\mathfrak{S}$, one obtains a sheaf over F.

If $s_1 = (I_1, f_1)$ and $s_2 = (I_2, f_2)$ are in the same component of $\mathfrak{S}$, we say s_1 and s_2 are related by continuation.

4. An Index.

Isolating neighborhoods do not generally reflect properties of the isolated invariant set inside, but certain special ones do. Namely, an isolating block is a closed subset $B \subset X$ such that for some $\epsilon > 0$, $x \in \partial B$ implies either $f(x, t) \notin B$ for $-\epsilon < t < 0$ or $f(x, t) \in B$ for $0 < t < \epsilon$; in particular, B is an isolating neighborhood for f. An example of an isolating block about a hyperbolic critical point in the plane is given in figure 1 ; reference to that figure makes the definitions to follow transparant.

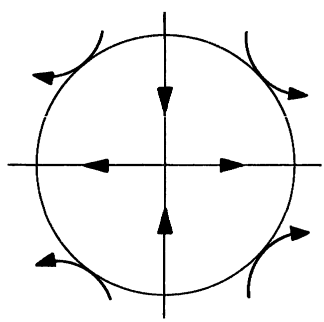

Figure 1

Subsets b^+ and b^- of $b = \partial B$ are defined to be the set of points which leave B in the negative direction and the set of points which leave B in the positive direction respectively. Subsets A^+ and A^- of B are defined to be the set of points whose positive orbit stays in B and the set of points whose negative orbit stays in B respectively. The isolated invariant set is just $A^+ \cap A^-$. The main lemma on blocks (cf. [2]) states that the flow defines in a natural way strong deformation retractions from $B - A^-$ to b^+ and from $B - A^+$ to b^-. Thus, for example, if b^+ is not a strong deformation retraction of B, then A^-, hence I, cannot be empty (cf. figure 1).

A fundamental result states that for $s = (I, f) \in \mathfrak{S}$, any neighborhood U of I contains one in the form of an isolating block. Furthermore if B_1 and B_2 are two blocks for s, the homotopy type of the pointed space B_1/b_1^+ is the same as that of B_2/b_2^+. Thus we can define the *homotopy index of s* to be the pair $h(s) \equiv [B/b^+]$, $[B/b^-]$ where B is any block for s. Finally a "perturbation theorem" for blocks shows that the homotopy index is constant on components of $\mathfrak{S}$; thus if s_1 and s_2 are related by continuation, $h(s_1) = h(s_2)$.

This homotopy index is something like the Morse index of a critical point of a gradient flow ; for such points it carries the same information as the Morse index. (cf. figure 1). The algebraic counterpart of the homotopy index is the pair of cohomology algebras $H^*(B, b^+)$ and $H^*(B, b^-)$ which again depend only on s and are constant on components of $\mathfrak{S}$. It should be noted that the homotopy type of the pair (B, b^+) is not independent of the block for fixed s ; the homotopy groups $\Pi_*(B, b^+)$ do give information about the invariant set, but do not play the role of an "index" in the above sense.

5. Bifurcation as revealed in terms of the algebraic index.

Cohomology properties of the invariant set I and the asymptotic sets A^+ and A^- are most easily described in terms of the sets $A = A^+ \cup A^-$ and

$$a^{\pm} \equiv A^{\pm} \cap b = A^{\pm} \cap b^{\pm} .$$

(cf. figure 1). In particular if we use Čech cohomology, then the inclusion $I \subset A$ induces an isomorphism from $H^*(A)$ to $H^*(I)$ as follows from the continuity of the Čech theory. In a similar way, one shows that the inclusions

$$(A, a^{\pm}) \subset (B, b^{\pm})$$

induce isomorphisms on cohomology.

It follows that cohomology classes in the index algebras $H^*(B, b^{\pm})$ correspond to cohomology either in $H^*(a^{\pm})$ or $H^*(I)$. However, this latter correspondence can switch under continuation ; the resulting "bifurcation" phenomena can be exemplified as follows.

Consider the diagram below (the rows are the exact sequences of triples, the columns are induced by inclusion) :

$$\begin{array}{ccccccc} \to H^*(b^+) & \xrightarrow{\delta} & H^*(B, b^+) & \xrightarrow{\beta} & H^*(B) & \to \\ \downarrow & & \downarrow & & \downarrow & \\ \to H^*(a^+) & \to & H^*(A, a^+) & \xrightarrow{\alpha} & H^*(A) & \to \end{array}$$

Define $K^*(s)$ to be ker α. From the diagram, ker β injects into $K^*(s)$ as well as into its correspondent for all smaller blocks. The fact that $H^*(b^+)$ converges to $H^*(a^+)$ as the blocks collapse to I together, say, with an ascending chain condition on the ideals of $H^*(A, a^+)$ implies that $K^*(s)$ is the isomorphic image of ker β for small enough blocks. Using the perturbation theorem for blocks, one then proves that the isomorphism from $H^*(A, a^+)$ to the corresponding algebra for nearby points of $\mathfrak{S}$ induces an injection from $K^*(s)$ to its correspondent. This is not always a surjection, but the "continuity points" form an open dense subset of the component of $\mathfrak{S}$. With an appropriate topology on the ideals of $H^*(A, a^+)$ one can dispense with the ascending chain condition and prove that K^* is lower semi-continuous on $\mathfrak{S}$. The discontinuity points are called bifurcation points (for K^*).

The above description depended on being able to recognize cohomology in $H^*(a^+)$ in the algebra $H^*(B, b^+)$ of some block, it is the blocks for which a perturbation theorem is available. In a similar way we can sometimes identify cohomology of $H^*(X, X - I)$ in the tensor product $H^*(B, b^+) \otimes H^*(B, b^-)$.

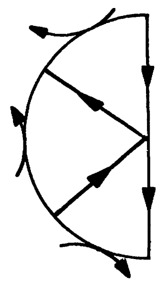

Figure 2.

Thus let $Q^*(B) \subset H^*(B, b)$ denote the range of the product under the homomorphism which sends $\alpha \otimes \beta$ to $\alpha \cup \beta$; and let $Q^*(s)$ denote the range of

$$H^*(B, B - A^-) \otimes H^*(B, B - A^*) \quad \text{in} \quad H^*(B, B - I) \approx H^*(X, X - I)$$

under the cup-product map. Since $b^{\pm}$ is a strong deformation retraction of $B - A^{\mp}$, the inclusion induced homomorphism from $H^*(X, X - I)$ to $H^*(B, b)$ is seen to induce a surjection from $Q^*(s)$ to $Q^*(B)$. Again using the perturbation theorem for blocks, one sees that the function Q^* defined on $\mathfrak{S}$ is lower semi-continuous with respect to an appropriate topology on the quotients of $H^*(A, a^+) \otimes H^*(A, a^-)$ and "bifuration points" can be defined as discontinuity points of Q.

In the case where X is a manifold, $H^*(X, X - I)$ is dual to $H^*(I)$ in the complementary dimensions. We use this to illustrate Q-bifurcation in the following example where $Q(s)$ is in fact dual to $H^*(I)$.

The example is that of a degenerate critical point for a flow in E^3 which is isolated by a 3-ball B. Figure two illustrates the flow provided one rotates the picture about the vertical axis which is a diameter of the 3-ball. It is easy to see that this critical point can bifurcate to either two non-degenerate critical points or to a periodic orbit. In either case $Q^*(s)$ (which is dual to $H^*(I)$ for either of these perturbations as for the unperturbed flow) increases, thus indicating the significance of the Q-bifurcation.

6. Conclusion.

We have stressed here the abstract aspects of our theory, however our belief is that the notions discussed will find applications to concrete problems. Steps in this direction are made in references [4], [5] and [7] as well as in work on various aspects of the three body problem by R. Easton, R. McGhee and the present author which is incomplete as of this writing.

REFERENCES

[1] URA T. — On the flow outside a closed invariant set; stability, relative stability, and saddle sets, MR 29 # 321, *Contributions To Differential Equations*, 3, 1964, p. 249-294.

[2] WAZEWSKI T. — Sur une méthode topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles, *Proc. of the International Congress of Mathematicians*, Vol. III, Amsterdam, 1954.

[3] CONLEY C.C. and EASTON R. — Isolated invariant sets and isolating blocks, *T.A.M.S.*, Sept. 1970.

[4] CONLEY C.C. — Invariant sets which carry a one-form, To appear in *J.D.E.*

[5] CONLEY C.C. — *Invariant sets in a monkey saddle. Proceedings United States Japan seminar on differential and functional equations* (William A., Harris Jr. and Yasutaha Sibuya, Ed.) W. A. Benjamin, Inc., New York.

[6] EATON R. — On the existence of invariant sets inside a submanifold convex to the flow, *J.D.E.*, Vol. 7, 1970.

[7] CONLEY C.C. and SMOLLER J.A. — Viscosity matrices for two dimensional nonlinear hyperbolic systems, To appear in *C.P.A.M.*

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LES SYSTÈMES LINÉAIRES D'ÉQUATIONS DIFFÉRENTIELLES ORDINAIRES

par V.M. MILLIONŠČIKOV

1. – Nous examinerons des systèmes linéaires d'équations différentielles

$$\dot{x} = \mathfrak{A}(t)\,x, \tag{1}$$

où x est un vecteur de l'espace euclidien E^n de dimension n, $\mathfrak{A}(t)$ une transformation linéaire $E^n \to E^n$ définie et dépendant continûment de t pour $t \geqslant 0$ ou pour tout t réel, avec de plus $\|\mathfrak{A}(t)\| \leqslant a_0$.

Rappelons tout d'abord la définition de Liapounov (1892). On appelle exposant caractéristique d'une solution $x(t)$ le nombre

$$\lambda_x = \overline{\lim_{t\to\infty}}\ \frac{1}{t}\operatorname{Log}\|x(t)\|.$$

Liapounov a démontré qu'il existe une base $x_1(t), \ldots, x_n(t)$ de l'espace des solutions du système (1) telle que, pour toute autre base $y_1(t), \ldots, y_n(t)$ du même espace telle que $\lambda_{y_1} \geqslant \ldots \geqslant \lambda_{y_n}$, on ait $\lambda_{y_i} \geqslant \lambda_{x_i}, i = 1, 2, \ldots, n$.

Les nombres $\lambda_1(\mathfrak{A}) \geqslant \ldots \geqslant \lambda_n(\mathfrak{A})$, où $\lambda_i(\mathfrak{A}) = \lambda_{x_i}$, s'appellent les exposants caractéristiques du système (1).

Par abus de langage, nous identifierons le système (1) à la fonction $\mathfrak{A}(t)$. Munissons l'ensemble des systèmes (1) d'une structure d'espace métrique M_n en introduisant la distance

$$\rho(\mathfrak{A}(t), \mathfrak{B}(t)) = \sup_t \|\mathfrak{A}(t) - \mathfrak{B}(t)\|$$

Perron, en 1930, a montré que les fonctions $\lambda_i(\mathfrak{A})$ ne sont pas partout continues dans M_n, puis a établi, en 1931, la continuité des fonctions $\lambda_i(\mathfrak{A})$ en tout point de l'ensemble $\mathfrak{J}_n$(1). Par définition, le système (1) appartient à l'ensemble $\mathfrak{J}_n$(2) si et seulement s'il existe une base $x_1(t), \ldots, x_n(t)$ de l'espace de ses solutions telle que

$$\frac{\|x_{i+1}(t)\|}{\|x_{i+1}(\tau)\|} : \frac{\|x_i(t)\|}{\|x_i(\tau)\|} \geqslant d e^{a(t-\tau)}$$

pour certains $a > 0$, $d > 0$ et tous les $t \geqslant \tau$, $i = 1, \ldots, n-1$.

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(1) cf. [2], pages 193-198, où le remplacement de la condition $p_{ik}(t) \to 0$, $t \to \infty$ $(i \neq k)$, par la condition $|p_{ik}(t)| < \delta$ $(i \neq k)$ n'introduit qu'un changement minime.

(2) B.F. Bylov et J.C. Lillo ont donné cette forme invariante à la définition de $\mathfrak{J}_n$. Dans [3], on montre que le système (1) appartient à $\mathfrak{J}_n$ si et seulement s'il se ramène par une transformation liapounovienne à un système $\dot{y}_i = p_{ii}(t) y_i$ $(i = 1, 2, \ldots, n)$ satisfaisant à la condition du théorème de Perron, [2]. p. 193.

J'ai obtenu les résultats suivants sur l'ensemble $\mathfrak{I}_n$.

THEOREME 1. [13]

$$\mathfrak{I}_n = \operatorname{Int} S_n ,$$

intérieur de l'ensemble S_n(1) *des points de l'espace* M_n *où toutes les fonctions* $\lambda_i(\mathfrak{A})$ *sont continues.*

THEOREME 2. [7]. – *Le système* (1) *appartient à l'ensemble* $\mathfrak{I}_n$ *si et seulement si, pour tout* $\epsilon > 0$, *il existe* $\delta > 0$ *tel que, pour toute solution* $y(t)$ *de tout système* $\dot{y} = \mathfrak{B}(t)\, y$ *satisfaisant à l'inégalité* $\rho(\mathfrak{B}(t), \mathfrak{A}(t)) < \delta$, *il existe une solution* $x(t)$ *du système* (1) *telle que* $\sphericalangle(x(t), y(t)) < \epsilon$ *pour tout* t.

THEOREME 3. [12].

$$\overline{\mathfrak{I}}_n = M_n .$$

2. – Pour l'étude des systèmes non autonomes $\dot{x} = g(x, t)$, et en particulier des systèmes (1), il est important d'introduire la notion suivante (cf. [4]).

DEFINITION 1. – On dit que $\widetilde{x}(t)$ est une solution généralisée du système

$$\dot{x} = g(x, t) \qquad \text{si} \qquad \widetilde{x}(t) = \lim_{k \to \infty} x_k(t_k + t),$$

où les t_k sont des nombres et les $x_k(t)$ des solutions du système, la limite étant uniforme sur tout segment.

Supposons $\mathfrak{A}(t)$ borné et uniformément continu sur la droite. Alors toute solution généralisée $\widetilde{x}(t)$ du système (1) est solution d'un système $\dot{x} = \widetilde{\mathfrak{A}}(t)x$, où

$$\widetilde{\mathfrak{A}}(t) = \lim_{k \to \infty} \mathfrak{A}(t_k + t),$$

la limite étant uniforme sur tout segment.

Utilisons la construction bien connue (cf. [2] p. 533-535) du système dynamique des translatées. L'ensemble des translatées de la fonction $\mathfrak{A}(t)$ est muni de la métrique de la convergence uniforme sur tout segment et complété pour cette métrique. Sur le compact $\mathcal{R}_{\mathfrak{a}}$ obtenu, considérons le système dynamique $\mathcal{D}_{\mathfrak{a}}$ défini par la formule

$$f(\widetilde{\mathfrak{A}}(t), \tau) = \widetilde{\mathfrak{A}}(t + \tau).$$

D'après le théorème de Bogolioubov et Krylov, le système $\mathcal{D}_{\mathfrak{a}}$ a des mesures normalisées invariantes(2).

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(1) Pour la description de l'ensemble S_n, voir [17] ou [13].

(2) L'étude des systèmes non autonomes a été faite par Favard en 1927, Stepanov et Tikhonov en 1934, Beboutov (1939-1941), Nemytski, qui m'a inspiré la note [4] celle-ci fut suivie d'un intéressant article de B.A. Chtcherbakov [18] et de communications de divers auteurs.

DEFINITION 2 [11]. — Nous dirons que le système (1) est absolument régulier s'il existe une base $x_1(t), \ldots, x_n(t)$ de l'espace de ses solutions possédant les propriétés suivantes :

(1) $$\lim_{|t|\to\infty} \frac{1}{t} \operatorname{Log} \|x_i(t)\| = \lambda_i \ (i = 1, 2, \ldots, n) ;$$

(2) Pour chaque $i = 1, 2, \ldots, n$ on a : pour tout $\epsilon > 0$, il existe T tel que l'ensemble des h pour lesquels il y a au moins une solution $x(t)$, satisfaisant à

$$\lim_{|t|\to\infty} \frac{1}{t} \operatorname{Log} \|x(t)\| = \lambda_i,$$

telle que, pour au moins un τ, $|\tau| \geqslant T$, on ait l'inégalité

$$\left| \frac{1}{\tau} \operatorname{Log} \frac{\|x(h+\tau)\|}{\|x(h)\|} - \lambda_i \right| \geqslant \epsilon$$

est de mesure relative $< \epsilon$ sur la droite.

THEOREME 4 [11](1). — *Pour presque tout $\widetilde{\mathcal{A}}(t) \in \mathcal{R}_{\mathcal{A}}$ (par rapport à une mesure normalisée invariante quelconque du système $\mathcal{D}_{\mathcal{A}}$), le système $\dot{x} = \widetilde{\mathcal{A}}(t)x$ est absolument régulier.*

DEFINITION 3 [15]. — Nous dirons que les exposants caractéristiques du système (1) sont presque sûrement stables si, pour $\sigma \to 0$, les exposants caractéristiques du système.

$$\dot{y} = \mathcal{A}(t)\, y + \sigma^2 C(t, \omega)\, y,$$

(où les éléments de la matrice $C(t, \omega)$ par rapport à une certaine base sont des bruits blancs, indépendants et non nuls), tendent avec une probabilité égale à 1 vers les exposants caractéristiques du système (1).

Le théorème suivant est un résultat fondamental de la théorie probabiliste des systèmes d'équations différentielles exposée dans ce paragraphe.

THEOREME 5 [15]. — *Les exposants caractéristiques de tout système absolument régulier sont presque sûrement stables.*

3. — Dans ce paragraphe, nous considérerons le système (1) pour une fonction $\mathcal{A}(t)$ presque périodique. Nous utiliserons ici les notions de système régulier de Liapounoff et de système presque réductible de B.F. Bylov que l'on peut trouver dans les livres [1] et [3]. Je ferai la remarque que tout système presque réductible est absolument régulier et que la régularité absolue entraine la régularité.

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(1) Ce théorème rassemble, sous une forme quelque peu améliorée, des résultats des articles [8] et [9] et équivaut, grosso modo, à l'ensemble des résultats de V.I. Osseledetz [16]. J'ai énoncé ces résultats des articles [8] et [9] sous forme de conjectures au séminaire de Nemytski l'hiver 1965-66 ; l'article [8] a été publié et les articles [9] et [11] ont été mis sous presse bien avant la sortie de l'article [16].

THEOREME 6 [14].

Pour tous $k > 1$, $n > 1$, il existe $\mathfrak{A}(t)$ k-périodique telle que le système (1) *ne soit pas régulier.*

(Ce théorème résout le problème de Erouguine)

Pour la démonstration de ce théorème j'ai utilisé le lemme suivant ($\mathfrak{A}(t)$ est dit récurrent, d'après Birkhoff, si chaque trajectoire du système $\mathcal{D}_{\mathfrak{A}}$ est partout dense dans $\mathcal{R}_{\mathfrak{A}}$) :

LEMME [6].

Si le système (1), pour $\mathfrak{A}(t)$ récurrent, n'est pas presque réductible, alors il existe $\tilde{\mathfrak{A}}(t) \in \tilde{\mathcal{R}}_{\mathfrak{A}}$ tel que le système $\dot{x} = \tilde{\mathfrak{A}}(t)\,x$ ne soit pas régulier.

On a le théorème suivant.

THEOREME 7 [11].

Pour que le système (1), *avec $\mathfrak{A}(t)$ presque périodique, soit presque réductible, il faut et il suffit que $\mathfrak{A}(t)$ soit un point de continuité des fonctions $\lambda_i(\mathfrak{A})$ ($i = 1, 2, \ldots, n$).*

Pour démontrer ce théorème, j'ai utilisé des arguments appliqués par la suite à la démonstration des théorèmes 1-3, le théorème 4, et certains arguments de caractère métrique (probabiliste).

Les systèmes (1) apparaissent, en particulier, comme systèmes aux variations des systèmes dynamiques différentiables. A ce sujet, voir [10] et [15].

BIBLIOGRAPHIE

[1] Ляпунов А. М. — Общая задача об устойчивости движения. Харьков. 1892 (2-е издание : ГТТИ, Москва-Ленинград, 1935).
[2] Немыцкий В. В., Степанов В. В. — Качественная теория дифференциальных уравнений. 2-е издание. ГТТИ, Москва-Ленинград, 1949.
[3] Былов Б. Ф., Виноград Р. Э., Гробман Д. М., Немыцкий В. В. — Теория показателей Ляпунова. Издательство « Наука », Москва, 1966.
[4] Миллионщиков В. М. — ДАН СССР, 161, № 1, 1965, 43-44.
[5] Миллионщиков В. М. — ДАН СССР, 166, № 1, 1966, 34-37.
[6] Миллионщиков В. М. — Дифференциальные уравнения, 3, № 12, 1967, 2127-2134.
[7] Миллионщиков В. М. — Математические заметки, 4, № 2, 1968, 173-180.
[8] Миллионщиков В. М. — Математический сборник, 75, № 1, 1968, 154-165.
[9] Миллионщиков В. М. — Математический сборник, 77, № 2, 1968, 163-173.
[10] Миллионщиков В. М. — Математические заметки, 5, № 1, 1969, 49-54.
[11] Миллионщиков В. М. — Математический сборник, 78, № 2, 1969, 179-201.
[12] Миллионщиков В. М. — Дифференциальные уравнения, 5, № 7, 1969, 1167-1170.
[13] Миллионщиков В. М. — Дифференциальние уравнения, 5, № 10, 1969, 1775-1784.

[14] Миллионщиков В. М. — Дифференциальные уравнения, 5, № 11, 1969, 1979-1983.
[15] Миллионщиков В. М. — Математические заметки, 7, № 4, 1970, 503-513.
[16] Оселедец В. И. — Труды Московского математического общества, 19, 1968, 179-210.
[17] Былов Б. Ф., Изобов Н. А. — Дифференциальные уравнения, 5, № 10, 1969, 1796-1805.
[18] Щербаков Б. А. — ДАН СССР, 167, № 5, 1966, 1004-1007.

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ERGODIC THEORY OF G-SPACES

by Willam PARRY

Introduction

The title of this talk might well have been "Ergodic theory of nilmanifolds" since most of the general theorems I shall mention developed out of a study of nilmanifolds and have direct applications in this area.

Let D be a uniform discrete subgroup of a simply connected, connected nilpotent Lie group (i.e. $N \supset N^1 \supset \ldots \supset N^k \supset N^{k+1} = e$, $N^{i+1} = [N, N^i]$ and N/D is compact). There is a unique normalised "Haar" measure m on N/D which is preserved by left translations. Let $A : N \to N$, $AD \subset D$ be an automorphism and let $a \in N$. Then $T(xD) = aAxD$ is called an *affine transformation* of the *nilmanifold* N/D. If $e_t \in N$ is a one-parameter group then $T_t(xD) = e_t xD$ is called a *nilflow*. The latter flows were studied by Auslander, Green and Hahn [1]. I shall be speaking about affines and nilflows but mainly the former. Obviously affines and nilflows preserve m. N/D can be viewed as a manifold resulting from a finite number of G_i extensions starting from the trivial one-point space where G_i are torii :

$$N/D \to N/N^k.D \to N/N^{k-1}.D \to \ldots \to N/N_1 D \to N/N\ ,\ G_i = N^i D/N^{i+1} D.$$

Moreover at each stage T induces an affine T_i on $X_i = N/N^{i+1}\ .\ D$ and denoting the action of G_i on X_i by $(g, x) \to gx$ we have $T_i(gx) = \tau_i(g)Tx$ for some endomorphism τ_i of G_i.

Quite generally let X be a compact metric space with normalised Borel measure m and let T be a measure preserving continuous map of X onto itself. Let G be a compact abelian group acting continuously on X and preserving m. If $T(gx) = \tau(g)Tx$ where τ is an endomorphism of G we say that T is a (τ, G) *extension* of the map T' induced on $X' = X/G$.

T is said to be *ergodic* if the induced unitary operator on $L^2(X)$., $U_T f = f \circ T$, has 1 as a simple eigenvalve i.e. $U_T f = kf$ implies f is constant, and *weak-mixing* if the only eigenfuntions are constant i.e. $U_T f = kf$ implies f is constant.

Without defining entropy we can say that a doubly invariant sub-σ-algebra $\mathcal{A} \subset \mathcal{B}$, $T^{-1}\mathcal{A} = \mathcal{A}$ ($\mathcal{B}$ is the σ-algebra of all measurable sets) has *zero-entropy* if it contains no decreasing sub-σ-algebras i.e. $\mathcal{C} \subset \mathcal{A}$, $T^{-1}\mathcal{C} \subset \mathcal{C}$ implies $T^{-1}\mathcal{C} = \mathcal{C}$. There is a maximum such σ-algebra according to a result of Pinsker [2], which we denote by $\mathcal{P}(T)$. If $\mathcal{P}(T) = \mathcal{N}$ the trivial σ-algebra of sets of measure zero or one, then T is said to have *completely positive entropy*. If T is invertible then a necessary and sufficient condition for T to have completely positive entropy is that there exists a sub-σ-algebra $\mathcal{A}$ with $T\mathcal{A} \supset \mathcal{A}$, $T^n\mathcal{A} \uparrow \mathcal{B}$ and $T^{-n}\mathcal{A} \downarrow \mathcal{N}$ (Rohlin, Sinai [3]) thus imitating the zero-one law for independent processes. Such transformations are called *Kolmogorov* or *K-automorphisms.*

1. Dynamical properties.

We have the following theorems, the first of which was proved jointly with R. Jones :

THEOREM 1. – *If T is a (τ, G) extension of T' : $X/G \to X/G$ where G acts freely and if T' is weak-mixing then for a residual set of $\phi : X \to G$ $(\phi(gx) = \phi(x))$ the maps $x \to \phi(x)Tx$ are also weak-mixing. (Note that such maps exhaust all possible (τ, G) extensions of T').*

THEOREM 2. – *If T is a weak-mixing (τ, G) extension of T' and if T' has completely positive entropy then T has completely positive entropy.*

This latter theorem has been significantly generalised by K. Thomas to arbitrary compact groups G, along the lines of Yuzvinskii's generalisation of Rohlin's theorem on the completely positive entropy of ergodic automorphisms of compact groups [4] [5]. In any case the latter theorem is relevant to nilmanifolds especially in view of :

THEOREM 3. – *If T is an affine transformation of $X_k = N/D$ then T is ergodic if and only if T_0 is ergodic on the torus $N/N^1 . D = X_0$. If T_0 is ergodic then every eigenfunction for T factors through $N/N^1 . D$ and in particular if T_0 is weak-mixing $(dA_e : N_e/N_e^1 \to N/N_e^1$ is without roots of 1 as eigenvalues) then T is weak-mixing (and in fact has completely positive entropy).*

If ergodicity is replaced by unique ergodicity or by minimality in the first part of this theorem then it remains valid. The corresponding statement concerning the minimality and ergodicity of nilflows was proved in [1].

If T has completely positive entropy then T is mixing of all orders and in particular is strong mixing. This latter fact also follows from the statement that U_T has countable Lebesgue spectrum in the orthogonal complement of the constant functions which in turn follows from the more general proposition that for any measure preserving transformation $U_T | L^2 (\mathfrak{P}(T))^{\perp}$ has countable Lebesgue spectrum if $\mathcal{B} \neq \mathfrak{P}(T)$ i.e. if $L^2(\mathfrak{P}(T))$ is non-trivial. Conze [6] has identified $\mathfrak{P}(T)$ as follows :

THEOREM 4. – *If T is an affine transformation of a nilmanifold X then there exists a unipotent affine $S = bB$ of a nilmanifold Y (unipotent means dB_e is unipotent) and a surjective homomorphism $\phi : X \to Y$ such that $\phi T = S\phi$ and $\mathfrak{P}(T) = \phi^{-1}\mathfrak{A}$ where $\mathfrak{A}$ is the Borel σ-algebra of Y. (This is a consequence of a general theorem of Conze's on homogeneous spaces).*

In view of the above, the complete spectral analysis of affines reduces to the analysis of unipotent affines. Once again this can be done via a general theorem on G-spaces. The final result is :

THEOREM 5. – *If T is an affine transformation of a nilmanifold then U_T has countable Lebesgue spectrum in the orthogonal complement of the space spanned by eigenfunctions. Each eigenfunction is of the form $\gamma \circ \pi$ where $\pi : N/D \to N/N^1 . D$ and where γ is a character of $N/N^1 . D$ such that $\gamma \circ \pi \circ A = \gamma \circ \pi$.*

To arrive at this result via unipotent affines we use the notion of G-stability of G-maps. Let T be a G-extension of T' i.e. T is a (τ, G) extension where τ is the identity. We say that T is *G-stable* if $gT \cong T$ for every $g \in G$ where the conjugacy is a G-map i.e. commutes with G.

THEOREM 6. – *If T is a G-extension of T' where G acts non-trivially and if G is connected then U_T has countable Lebesgue spectrum in the orthocomplement of the G invariant functions.*

The proof of this theorem is very simple and uses only the definitions, basic spectral theory and the fact that the Lebesgue measure class is the only one (on the circle) which is invariant under translations. Nevertheless it applies very well to unipotent affines, (and an analogous theorem applies also to nilflows) to yield the desired result.

2. Conjugacy.

We come next to the problem of conjugacy between two affine transformations. $S = aA$, $T = bB$ acting on nilmanifolds X, Y. When are S, T conjugate (ϕ maps X onto Y and $\phi S = T\phi$)

(i) topologically (ϕ a homeomorphism) ?

(ii) measure theoretically (ϕ invertible and measure preserving) ?

In line with a result of Adler and Palais [7] for automorphisms of a torus and Arov [8] for automorphisms of finite dimensional compact connected abelian groups Walters [9] has proved :

THEOREM 7. – *If $\phi S = T\phi$ where ϕ is a continuous map then ϕ is affine if U_S has no eigenvalue in common with $dB_e (S = aA, T = bB)$ other than perhaps 1. In other cases the exact form of ϕ can be specified.*

Essentially Walters has pushed the topological conjugacy problem into algebra.

The corresponding measure theoretic problem seems to be more difficult. However, there are substantial partial results. If S, T are Anosov diffeomorphisms (within the class we are considering this means dA_e, dB_e have no eigen values of unit absolute value) then Sinai [10] and Bowen [11] have shown that S, T are measure theoretically conjugate to Markov shifts which (in view of a result of Friedman and Ornstein [12]) are classified by their entropy h. (Using results for the torus it can be shown that

$$h(S) = \sum_{|\lambda|>1} \log |\lambda|$$

where summation is-over the eigen values of dA_e). Consequently such affines are classified by their entropies.

Starting from this point it seems there are three questions in this area which need to be answered.

(1) If T has completely positive entropy is it conjugate to a Markov shift ?

(2) Is an ergodic affine conjugate to the direct product of its maximum factor with zero entropy and a Markov shift ?

(3) Can the zero entropy affines be classified ?

The third question is answered by :

THEOREM 8. – *If $\phi S = T\phi$ where ϕ maps X onto Y and T is a unipotent (zero-entropy) affine and S is ergodic then ϕ is affine.*

This means that two unipotent affines are conjugate measure theoretically if and only if they are algebraically. This theorem is analogous to Abramov's classification theorem for quasi-discrete spectrum transformations [13], and also generalises the fact that eigenfunctions of affines must have a continuous version. (c.f. theorem 5).

An analogous result to theorem 8 also holds for nilflows (in fact it is a corollary of this theorem) and hence two nilflows are measure theoretically conjugate if and only if they are algebraically conjugate. Thus nilflows are completely classified. This is a problem which received little attention in [1], although partial classification according to discrete spectral type was achieved.

REFERENCES

[1] AUSLANDER L., GREEN L. and HAHN F. — Flows on homogeneous spaces. *Annals of Math. Studies,* 53, Princeton, 1963.
[2] PINSKER M.S. — Dynamical systems with zero or completely positive entropy *D.A.N. S.S.S.R.,* 133, 1960, p. 1025-1026; *Sov. Math.,* 1, 1960, p. 937-938.
[3] ROHLIN V.A. and SINAI Ya. G. — Construction and properties of invariant measurable partitions. *D.A.N.S.S.S.R.,* 141, 1961, p. 1038-1041; *Sov. Math.,* 2, 1961, p. 1611-1614.
[4] YUZVINSKII S.A. — Metric properties of endomorphisms of a compact group. *Izv. Akad. Nauk. S.S.S.R.,* Ser. Mat., 29, 1965, p. 1295-1328. (In Russian).
[5] ROHLIN V.A. — Metric properties of endomorphisms of compact commutative groups. *Izv.Akad. Nauk. S.S.S.R.,* Ser. Mat., 28, 1964, p. 867-874, (In Russian).
[6] CONZE J.P. — Entropie des transformations affines et des flots sur les espaces homogenes compacts. *C.R. Ac. Sc. Paris,* 270, 1970, p. 547-548.
[7] ADLER R. and PALAIS R. — Homeomorphic conjugacy of automorphisms on the torus. *Proc. A.M.S.,* 16, 1965, p. 1222-1225.
[8] AROV D.Z. — Topological similitude of automorphisms and translations of compact commutative groups. *Usp. Mat. Nauk. S.S.S.R.,* 18, 1963, p. 133-138. (In Russian).
[9] WALTERS P. — Topological conjugacy of affine transformations of nilmanifolds. *Math. Systems theory.* (To appear).
[10] SINAI Ja. G. — Markov partitions and *U-diffeomorphisms. Funk. Anal.,* 2, 1968, p. 64-89; Functional Analysis and its applications, 2, 1968, p. 61-82.
[11] BOWEN R. — *Markov partitions for Axiom A. diffeomorphisms.* (To appear).
[12] FRIEDMAN N.A. and ORNSTEIN D. — On isomorphism of weak Benoulli transformations. *Advances in mathematics.* (To appear).
[13] ABRAMOV L.M. — Metric automorphisms with quasidiscrete spectrum. *Izv. Akad. Nauk. S.S.S.R.,* Ser. Mat., 26, 1962, p. 513-530; *A.M.S. Transl.,* Ser 2, 39, 1964, p. 37-56.

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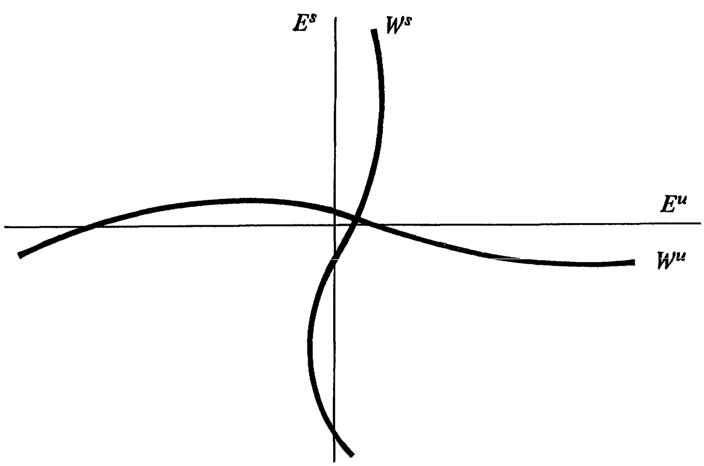
INVARIANT MANIFOLDS

par C. C. PUGH

I chose this title last spring because most topics I might discuss were related to it ; somehow. This talk will be in two parts. First I shall indicate a natural geometric way to prove smoothness of unstable and stable manifolds. Elaborations of this appear in [1] and [2]. Classical treatments occur in [3] and [4]. Second I shall pose several unsolved problems in the field of differential dynamics.

Let $H : R^m \to R^m$ be a linear hyperbolic isomorphism respecting $E^u \oplus E^s = R^m$; H leaves E^u, E^s invariant, expanding E^u and contracting E^s . $h : R^m \to R^m$ is a C^1 small perturbation and $f = H + h$. There are unique f-invariant manifolds W^u , W^s near E^u , E^s ; f expands W^u and contracts W^s.



THEOREM. – *If h is C^1 small and of class C^r , $r \geqslant 1$, then W^u, W^s are C^r.*

We shall deal only with $r = 1$ since $r \geqslant 2$ is a formal consequence of $r = 1$. Since W^s for f is W^u for f^{-1} we may concentrate entirely on W^u. We first prove that W^u is the graph of a Lipschitz function $g : E^u \to E^s$ with Lip $(g) \leqslant 1$, $\|g\|_0 \leqslant 1$. This is not hard to see by letting f act in the natural way on the space $\mathfrak{G}$ of all such functions g. Then $\Gamma_f : \mathfrak{G} \to \mathfrak{G}$ where $\Gamma_f g : E^u \to E^s$, the graph transform of g by f, satisfies

$$f(\text{graph } g) = \text{graph}\,(\Gamma_f g) \qquad g \in \mathfrak{G}$$

$$\Gamma_f g(x) = f^s \circ (1\,, g) \circ [f^u \circ (1\,, g)]^{-1}(x) \quad f = (f^u\,, f^s) \quad x \in E^u$$

Γ_f contracts $\mathcal{G}$ – the C^0 metric makes $\mathcal{G}$ complete because Lip(g) is uniformly bounded. (In general this natural action Γ_f does *not* seem to contract $\mathcal{G} \cap C^1$ furnished with a C^1 metric).

The fixed point of Γ_f gives W^u ; call it g_f, so W^u = graph g_f. So far this is all natural and straight-forward – even the estimates necessary to prove Γ_f contracts $\mathcal{G}$ are simple.

Classically one then shows that if $g_0 \in \mathcal{G}$ is of class C^1, say $g_0 \equiv 0$, then the sequence of first derivations of $\Gamma_{f^n}\, g_0$ is equicontinuous. (f^n means $f \circ \ldots \circ f$, n times).

By the Arzela-Ascoli Theorem, a subsequence converges uniformly and so $g_f = \lim \Gamma_{f^n}\, g$ has this subsequential limit as its first derivative. (Actually, the subsequence is a red herring because the whole sequence ($D\,\Gamma_{f^n}\, g_0$ converges uniformly to Dg_f). The estimates proving $\{D\,\Gamma_{f^n}\, g_0\}_{n \geqslant 0}$ is equicontinuous are laborious, in my opinion, and little feeling of why W^u is C^1 emerges.

To prove that W^u is C^1 it is not sufficient to *find* its tangent bundle. This much could be done by letting Tf act naturally on all u-plane bundles over W^u whose u-planes have inclination $\leqslant 1$. The fixed point of the resulting contraction would give TW^u – if we knew TW^u existed ! Instead of u-plane bundles over W^u we must consider "1-jet u-bundles over W^u". We shall need to speak of 1-jets of Lipschitz functions.

DEFINITION. – Two maps $\gamma_1, \gamma_2 : (E^u, 0) \to (E^s, 0)$ are 1-jet equivalent iff

$$\lim_{x \to 0} \frac{\gamma_1(x) - \gamma_2(x)}{|x|} = 0.$$

1-jets are the equivalence classes. $\mathcal{J} = \mathcal{J}(E^u, 0\,; E^s, 0)$ is the linear space of 1-jets of maps, Lipschitz at 0.

The norm $|j| = \lim_{x \to 0} \sup\, |\gamma x| / |x|$, $\gamma \in j$, Banachs $\mathcal{J}$ and makes the space of jets having differentiable representatives, $\mathcal{J}^{\text{diff}}$, closed. Having a differentiable representative is equivalent to having a linear one.

Let $\mathcal{M}$ = all maps $J : W^u \to \mathcal{J}$, $x \mapsto J_x$, such that $|J_x| \leqslant 1$, $x \in W^u$. With the sup norm, $\mathcal{M}$ is a complete metric space. Let f act on $\mathcal{M}$ in the natural way : If γ_x represents J_x, $J \in \mathcal{M}$, first translate graph (γ_x) to x, then apply f, and then translate back to the origin. This defines $\Gamma_f\,\gamma_x : (E^u, 0) \to (E^s, 0)$ by

$$\text{graph}(\Gamma_f\,\gamma_x) + fx = f(\text{graph}(\gamma_x) + x).$$

Then set

$$(\Gamma_f\, J)_{fx} = \text{the 1-jet of } \Gamma_f\,\gamma_x .$$

As one would expect, $\Gamma_f\, J$ is determined by Tf :

$$(\Gamma_f\, J)_{fx} = [K_x + C_x\, J_x] \circ [A_x + B_x\, J_x]^{-1}$$

where $T_x f = \begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix}$ respecting $R^m = E^u \oplus E^s$. This formula lets us prove that $\Gamma_f : \mathfrak{M} \to \mathfrak{M}$ is a contraction. This is even easier than proving that $\Gamma_f : \mathcal{G} \to \mathcal{G}$ is a contraction.

Let $J_f \in \mathfrak{M}$ be the fixed point of Γ_f. Since $fW^u = W^u$ and $W^u = \text{graph}\, g_f$, Γ_f leaves fixed the map in $\mathfrak{M}$

$$x \mapsto \text{the (translated) 1-jet of } g_f \text{ at } x$$

and so J_f = the (translated) 1-jet of g_f. On the other hand, it is clear that Γ_f carries the subspace $\mathfrak{M}^1$ into itself

$$\mathfrak{M}^1 = \{J \in \mathfrak{M} : x \mapsto J_x \text{ is continuous and } J_x \in \mathcal{J}^{\text{diff}}, x \in W^u\}$$

As remarked before, $\mathfrak{M}^1$ is closed in $\mathfrak{M}$ so that the unique fixed point of Γ_f lies in $\mathfrak{M}^1$. Therefore the 1-jet of g_f at x has a linear representative depending continuously on x which means that W^u is C^1.

Assorted Questions. — Most terms are explained in [5].

1. Is there a homotopy class of diffeomorphisms on some manifold M containing

(a) no structurally stable system

(b) no Axiom A system

(c) no "topologically-transitive-on- Ω_i" system ?

2. If $A : T^n \to T^n$ is linear and its eigenvalues are not roots of unity then A is ergodic ; if f is C^1 near A then is f ergodic or topologically transitive ?

3. If $f : M \to M$ is a diffeomorphism leaving a foliation $\mathfrak{F}$ invariant and the leaves of $\mathfrak{F}$ are smooth, $\cup T_x \mathfrak{F}_x = T\mathfrak{F}$ is continuous, $TM = N^u \oplus T\mathfrak{F} \oplus N^s$ is Tf invariant, and N^u is expanded more sharply that $T\mathfrak{F}$ by Tf while N^s is contracted more sharply that $T\mathfrak{F}$ by Tf (f is "normally hyperbolic") then is f *plaque-expansive* ? This means there exist $\epsilon > 0$, $\delta > 0$ such that for any two sequences of points $\{x_n\}$, $\{y_n\}$, $n \in Z$, with

$$x_n \in \mathfrak{F}(\delta, fx_{n-1}) \qquad y_n \in \mathfrak{F}(\delta, fy_{n-1})$$

$d(x_n, y_n) \geqslant \epsilon$ for some n . $\mathfrak{F}(\delta, x)$ is the plaque of radius δ in the $\mathfrak{F}$-leaf through x. When $\mathfrak{F}$ is smooth, the answer is "yes". Also "yes" when $f|T\mathfrak{F}$ is an isometry [6]. If f is plaque expansive and normally hyperbolic f mod $\mathfrak{F}$ is structurally stable [2].

4. If α is a G-action on M define $x \in \Omega$ iff $x \in$ Closure $[\alpha(G\text{-}K)y]$ for every compact K and some y in every neighborhood U of x. Is it generic for C^1 actions that

Ω = Closure of orbits of points with non compact isotropy group ?

This is a Closing Lemma for group actions. It might be provable for

G = Compact $\times$ R.

Hirsch [7] showed that it is foolish to expect

Ω = Closure of compact orbits

as a Closing Lemma for actions.

5. Do the unstable and stable foliations through the Ω_i of an Axiom A diffeomorphism f extend to locally f-invariant foliations of a neighborhood of Ω ? When there are no cycle among the Ω_i, Robbin [8] showed that f is structurally stable, circumventing the apparent need for this extension.

REFERENCES

[1] HIRSCH, PUGH. — Stable Manifolds for Hyperbolic sets. *Proc. A.M.S.* Global Analysis Conference.
[2] HIRSCH, PUGH and SHUB. — *Invariant Manifolds,* to appear.
[3] CODDINGTON LEVINSON. — *Theory of Ordinary Differential Equations.*
[4] HARTMAN. — *Ordinary Differential Equations.*
[5] SMALE. — Differential Dynamical Systems, *Bull. A. M. S.,* 1967.
[6] CHILLINGWORTH. — unpublished.
[7] HIRSCH. — *Invariant Foliations.*
[8] ROBBIN. — preprint *A Structural Stability Theorem.*

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MESURES INVARIANTES DES Y-SYSTÈMES

par Ia.G. SINAI

Nous désignons par *Y*-systèmes, *Y*-difféomorphismes et *Y*-flots, les systèmes, difféomorphismes et flots d'Anosov.

1. Introduction. – La distribution de Gibbs en théorie ergodique

A la base d'une grande partie des résultats exposés ci-dessous se trouve une idée empruntée à la physique statistique. Nous allons définir, dans le cadre de la théorie topologique générale des systèmes dynamiques, l'analogue de la distribution limite de Gibbs de la mécanique statistique (voir [1], [2], [3]).

Soient M un espace métrique compact et complet, $\{T_t\}$ un groupe à un paramètre d'homéomorphismes de M. Le paramètre t parcourt ou bien tous les nombres entiers, et dans ce cas les T_t représentent toutes les puissances d'une transformation $T_1 = T$, ou bien tous les nombres réels : $-\infty < t < \infty$. Soit μ une mesure normée invariante pour le groupe $\{T_t\}$. Construisons pour une fonction arbitraire bornée et mesurable $h(x)$, une famille de mesures normées $\mu_{t_1,t_2}(\cdot\,|h)$, $t_1 > 0$, $t_2 > 0$ absolument continues par rapport à la mesure μ de façon que soient vérifiées les égalités suivantes :

$$\frac{d\mu_{t_1,t_2}(x|h)}{d\mu} = \frac{\exp\left[\sum_{k=-t_2}^{t_1} h(T_x^k)\right]}{\Xi_{t_1,t_2}(h)} \quad , \quad \Xi_{t_1,t_2}(h) = \int_M \exp\left[\sum_{k=-t_2}^{t_1} h(T_x^k)\right] d\mu(x)$$

(pour le cas du temps discret)

$$\frac{d\mu_{t_1,t_2}(x|h)}{d\mu} = \frac{\exp\left[\int_{-t_2}^{t_1} h(T_u x)\,du\right]}{\Xi_{t_1,t_2}(h)} \quad , \quad \Xi_{t_1,t_2}(h) = \int_M \exp\left[\int_{-t_2}^{t_1} h(T_u x)\,du\right] d\mu(x)$$

(pour le cas du temps continu)

DEFINITION 1. – Toute mesure limite (au sens de la convergence faible) pour la famille de mesures $\mu_{t_1,t_2}(h)$ pour t_1, $t_2 \to \infty$ s'appelle distribution limite de Gibbs. Nous désignons par $\mu(h)$ la distribution limite de Gibbs.

Nous soulignons que $\mu(h)$ dépend aussi bien de la mesure initiale μ que de la fonction h.

Soient : M l'ensemble des mots périodiques $x = \{\ldots, x_{-n}, \ldots, x_0, \ldots, x_n, \ldots\}$, $x_i = x_{i+n}$ égaux à 0 ou 1, avec la période n ; $U(j)$ une fonction que l'on peut appeler de manière naturelle le potentiel d'interaction; μ_0 une mesure uniforme;

T un déplacement et $h(x) = \sum_{j=1}^{n-1} U(j)\, x_j\, x_0$; alors la distribution de Gibbs, construite selon nos définitions, coïncide avec la distribution de Gibbs habituelle pour un gaz sur un réseau de dimension un et de longueur n. La notion de distribution limite de Gibbs pour les systèmes de la mécanique-statistique a été énoncée pour la première fois dans l'exposé de N. N. Bogolubov et B.I. Hatzet. [1]. Dernièrement, des résultats importants dans cette direction ont été obtenus par D. Ruelle [2] et P.L. Dobrouchine [3]. Le cas considéré par nous correspond aux systèmes à une dimension de la physique statistique.

Il semble qu'il existe des cas où $\mu(h)$ n'est pas unique, mais nous ne les connaissons pas. D'autre part, la recherche des conditions générales d'unicité représente probablement des difficultés aussi grandes que le problème de l'unicité de la mesure invariante pour les chaînes de Markov dans le cas d'un espace de phases continu pour les états. Dans le cas où $\mu(h)$ est unique et $\mu(h) = \lim_{t_1,t_2} \mu_{t_1,t_2}(h)$, $\mu(h)$ sera une mesure invariante normée pour le groupe $\{T_t\}$. Ainsi, dans le cas d'unicité, nous obtenons une méthode assez efficace pour construire et étudier différentes mesures invariantes pour $\{T_t\}$. Plus loin, nous considérons des exemples de problèmes dans lesquels interviennent ces mesures.

2. Y-difféomorphismes transitifs et leurs mesures invariantes.

Nous supposons connues les définitions de Y-difféomorphismes et leurs propriétés principales. Rappellons que l'on appelle Y-difféomorphisme transitif un difféomorphisme dont chaque feuille d'une foliation transversale est partout dense.

Pour les Y-difféomorphismes transitifs, mon article [4] a établi en fait le résultat suivant : Soit μ une mesure différentielle normée quelconque sur la variété M, dans laquelle agit le Y-difféomorphisme T. Considérons la suite de mesures $\mu_n = \mu(T^*)^n$. Alors il existe des limites (au sens de la convergence faible)

$$\mu^{(p)} = \lim_{n \to \infty} \mu_n \quad , \quad \mu^{(c)} = \lim_{n \to -\infty} \mu_n ,$$

telles que

(1) $\mu^{(p)}$, $\mu^{(c)}$ sont des mesures invariantes pour T, et T , comme automorphisme métrique de l'espace avec une quelconque de ces mesures, est un K-automorphisme.

(2) pour chaque partition mesurable des feuilles locales de la foliation dilatante (contractante), les mesures conditionnelles induites $\mu^{(p)}$, $(\mu^{(c)})$ sur les éléments d'une telle partition sont données par des densités relatives à la mesure différentiable. Chaque mesure possédant les propriétés (1) et (2) est unique. De là on déduit que si T possède une mesure invariante compatible avec la structure différentiable, alors $\mu^{(c)} = \mu^{(p)} = \mu$ et la mesure μ est justement celle-ci. Inversement, si $\mu^{(c)} \neq \mu^{(p)}$, T ne possède pas de mesure invariante compatible avec la structure différentiable.

THEOREME 1. – *Soit $h \in C(M)$ satisfaisant à une condition de Hölder d'ordre positif. Alors la distribution limite de Gibbs est*

$$\mu^{(c)}(h) = \lim_{t_1,t_2 \to \infty} \mu^{(c)}_{t_1,t_2}(h) ;$$

elle est donc unique. T, comme automorphisme de l'espace avec la mesure $\mu^{(c)}(h)$, est un K-automorphisme. Les propositions analogues sont valables pour la mesure $\mu^{(p)}$.

Les mesures invariantes $\mu^{(c)}(h)$, $\mu^{(p)}(h)$ ne constituent pas toutes les mesures invariantes du Y-difféomorphisme T. A.M. Stepin a démontré que T possède aussi des mesures continues ergodiques invariantes, par rapport auxquelles il possède des propriétés très modérées de mélange, et, en particulier, a une entropie nulle.

THEOREME 2. – *Soient h_1 et h_2 vérifiant les conditions du Théorème 1. Alors l'égalité*

$$\mu^{(c)}(h_1) = \mu^{(c)}(h_2) \quad (\mu^{(p)}(h_1) = \mu^{(p)}(h_2))$$

a lieu si et seulement s'il existe une fonction $u(x)$, qui vérifie une condition de Hölder d'ordre positif, et une constante C telles que

$$h_1(x) = h_2(x) + u(Tx) - u(x) + C. \tag{1}$$

3. Mesure invariante d'entropie maximale. Récemment, E.I. Dinabourg [5] a démontré que l'entropie topologique d'un homéomorphisme d'un espace métrique compact de dimension finie est égale à la borne supérieure des entropies métriques, calculées selon toutes les mesures boréliennes invariantes. En tant qu'hypothèse, cette proposition a été formulée dans l'ouvrage connu de Adler, Konheim, McAndrew [6] dans lequel a été pour la première fois introduite l'entropie topologique. A la même époque, B.M. Gourewitch [7] a construit un exemple qui a montré que, dans ce théorème, la borne supérieure ne peut être remplacée par le max. M. Chtilman a construit un exemple d'homéomorphisme qui est topologiquement transitif et dans lequel l'entropie topologique est obtenue selon plusieurs mesures invariantes. L'exemple de M. Chtilman rappelle le phénomène connu de transition de phases de première espèce en physique statistique.

Pour les Y-difféomorphismes, nous avons construit, dans [4], une mesure invariante selon laquelle l'entropie métrique est égale à l'entropie topologique. B.M. Gourevitch et R. Bowen [8], indépendamment l'un de l'autre et presque simultanément, ont établi qu'une telle mesure est unique. Du point de vue métrique, T, en tant qu'automorphisme de l'espace avec une telle mesure, peut être réalisé par une chaîne ergodique de Markov (une classe, une sous-classe) avec un nombre fini d'états. D'ailleurs, R. Bowen a obtenu ses résultats dans des conditions plus générales : pour les difféomorphismes vérifiant l'axiome de S. Smale, Bowen a démontré aussi que l'entropie topologique détermine asymptotiquement l'exposant de l'exponentielle comme le nombre de points périodiques de l'homéomorphisme (quand la période tend vers l'infini).

Soit $\mu^{(\max)}$ la mesure invariante du Y-difféomorphisme transitif T par laquelle on obtient l'entropie topologique. On constate que les mesures invariantes $\mu^{(c)}$ et $\mu^{(p)}$ décrites ci-dessus, peuvent être obtenues en tant que distributions limites de Gibbs, calculées à l'aide de la mesure $\mu^{(\max)}$ à savoir

$$\mu^{(c)} = \mu^{(\max)}(\ln \lambda_c(x)) \quad , \quad \mu^{(p)} = \mu^{(\max)}(-\ln \lambda_p(x))$$

où $\lambda_c(x)$, $\lambda_p(x)$ sont les coefficients de dilatation et de contraction pour les feuilletages dilatant et contractant respectivement. Pour la mesure $\mu^{(\max)}$ le théorème 2 est aussi valable.

4. Problème de l'existence de la mesure invariante compatible avec la différentiabilité pour les *Y*-difféomorphismes.

Le problème a été posé dans l'exposé célèbre de S. Smale [9]. Il représente aussi une partie du problème plus général suivant : quelle est la catégorie de l'ensemble des C^∞-difféomorphismes d'une C^∞-variété compacte qui ont une mesure invariante compatible avec la différentiabilité. Le théorème suivant vaut :

THEOREME 3. – *L'ensemble des Y-difféomorphismes transitifs de classe C^∞ qui n'ont pas de mesure invariante compatible avec la différentiabilité contient un ensemble ouvert et partout dense.*

L'idée de la démonstration de ce théorème est la suivante : il suffit de considérer les Y-difféomorphismes T qui se trouvent dans un voisinage suffisamment petit d'un Y-difféomorphisme transitif T_0 de sorte que tous les T soient conjugués topologiquement avec T_0. Si U est un homéomorphisme pour lequel $T_0 U = UT$ alors $\mu^{(p)}U^* = \overline{\mu}^{(p)}$, $\mu^{(c)}U^* = \overline{\mu}^{(c)}$ sont des mesures invariantes pour le Y-difféomorphisme T_0. Il se révèle que ces mesures peuvent être obtenues comme les distributions limites de Gibbs en les construisant à l'aide de la mesure $\mu^{(\max)}$ avec l'entropie maximale pour T_0, à savoir de la façon suivante :

$$\overline{\mu}^{(p)} = \mu^{(\max)}(-\ln \lambda_p (U^{-1}x)) \quad , \quad \overline{\mu}^{(c)} = \mu^{(\max)}(\ln \lambda_c(U^{-1}x))$$

où λ_p, λ_c sont les coefficients de dilatation de contraction pour le Y-difféomorphisme T. De ce qui a été dit au § 2 découle que le Y-difféomorphisme T a une mesure invariante compatible avec la différentiabilité si et seulement si $\overline{\mu}^{(p)} = \overline{\mu}^{(c)}$.

Etant donné que U vérifie la condition de Hölder, $h_1(x) = \ln \lambda_p(U^{-1}x)$ et $h_2(x) = \ln \lambda_c(U^{-1}x)$ vérifient aussi cette condition. Et nous pouvons donc utiliser le théorème 2, dont on déduit que si $\overline{\mu}^{(p)} = \overline{\mu}^{(c)}$ ces deux fonctions vérifient la relation (1).

Utilisons maintenant un théorème de A.N. Lifchitz, selon lequel il faut et il suffit, pour que la relation (1) soit vérifiée, que, pour une constante C et pour toute trajectoire périodique $x, Tx, \ldots, T^{K-1}x, T^K x = x$, ait lieu l'égalité suivante

$$\sum_{i=0}^{K-1} h_1(T^i x) = \sum_{i=0}^{K-1} h_2(T^i x) + KC$$

c'est-à-dire que

$$\frac{1}{K} \sum_{i=0}^{K-1} [h_1(T^i x) - h_2(T^i x)] \tag{2}$$

soit le même pour toutes les trajectoires périodiques.

De là on déduit que l'ensemble des Y-difféomorphismes, pour lesquels h_1 et h_2 ne satisfont pas la condition (1), est ouvert. Pour démontrer que cet ensemble est partout dense, il suffit de considérer, pour un Y-difféomorphisme pour lequel se vérifie la condition (1), une perturbation arbitrairement petite dans le voisinage de deux trajectoires périodiques différentes et de vérifier que, pour une perturbation de forme générale, l'expression (2) devient différente pour ces deux trajectoires.

5. **Petite perturbation stochastique des Y-difféomorphismes.**

Le problème dont il s'agit maintenant est plus intéressant dans le cas du temps continu, et nous considérerons ce cas plus loin. Mais le cas du temps discret est un peu plus simple.

Considérons une famille de distributions de probabilités

$$q\,(\cdot \mid x \quad \epsilon) \qquad (x \in M)$$

où ϵ est un paramètre. Supposons que $q\,(\cdot \mid x\,,\epsilon)$ est une fonction continue du point x et que pour chaque ρ fixé, on a

$$\min q\,(0_\rho(x) | x\,,\epsilon) \to 1 \quad \text{si} \quad \epsilon \to 0 \tag{3}$$

où $0_\rho(x)$ est une boule de rayon ρ de centre x. Soit T un homéomorphisme quelconque de M. Construisons une famille de chaînes de Markov Π_ϵ dans laquelle le mouvement du point aléatoire x se déroule de la façon suivante : d'abord, x passe au point Tx et ensuite au point aléatoire z, choisi suivant la distribution $q\,(\cdot \mid Tx\,,\epsilon)$.

En raison de la condition (3), il est naturel d'appeler la famille de chaînes de Markov Π_ϵ une petite perturbation stochastique de l'homéomorphisme. Il est facile de démontrer la proposition suivante (voir aussi [10]) :

Si $\mathfrak{P}_\epsilon = \{\pi_\epsilon\}$ est une collection de mesures invariantes pour la chaîne de Markov Π_ϵ, alors toute mesure limite (au sens de la convergence faible) pour la famille $\mathfrak{P}_\epsilon$ (pour $\epsilon \to 0$) est une mesure invariante pour T.

Il est facile de construire des exemples de $\mathfrak{P}_\epsilon$ qui contiennent pour chaque ϵ plusieurs mesures. En même temps, l'étude de la limite des $\mathfrak{P}_\epsilon$, quand $\epsilon \to 0$, est intéressante dans le cas des homéomorphismes T qui possèdent une réserve suffisamment riche de mesures invariantes. Il suit de ce qui a été dit que les Y-difféomorphismes représentent l'un des moyens les plus intéressants d'étude de ce problème. L'étude des petites perturbations stochastiques du Y-difféomorphisme transitif T a été effectuée par E.I. Dinabourg et par nous. Nous avons considéré le cas où, pour des ϵ suffisamment petits, il existe des constantes positives $\delta_1(\epsilon), \delta_2(\epsilon)$ telles que

$$\delta_1(\epsilon) < \delta_2(\epsilon) \qquad \text{et} \qquad \delta_2(\epsilon) \to 0$$

pour

$$\epsilon \to 0 \qquad \text{et} \qquad q\,(0_{\delta_2(\epsilon)}(x) | x\,,\epsilon) \equiv 1$$

pour tout $x \in M$. La distribution $q\,(\cdot | x\,,\epsilon)$ est donnée par la densité $g(y | x\,,\epsilon)$ suivant la mesure induite par le volume Riemanien et

$$\frac{\max_{y \in O_{\delta_2(x)}} g(y|x, \epsilon)}{\min_{y \in O_{\delta_1(x)}} g(y|x, \epsilon)} \leqslant \text{const.}$$

Il est naturel d'appeler ces perturbations stochastiques "localisées". Sous nos hypothèses, il n'est pas difficile de démontrer que la mesure invariante π_ϵ est unique et dans ce cas le problème consiste à étudier le comportement asymptotique de π_ϵ quand $\epsilon \to 0$.

Nous pouvons considérer que la chaîne de Markov Π_ϵ est stationnaire et que son temps varie dans l'intervalle $(-\infty, +\infty)$.

Soient $\omega = \{\dots x_{-n}, \dots, x_0, \dots, x_n, \dots\}$ une réalisation aléatoire de la chaîne Π_ϵ, P_ϵ une mesure, dans l'espace de ces réalisations, invariante par rapport au déplacement T_m dans $\Omega = \{\omega\}_0$.

LEMME. – *Pour tout $\delta > 0$ et pour tout ϵ suffisamment petit, il existe, pour chaque réalisation ω, un et seulement un point $z_0 = z_0(\omega)$ tel que*

$$d(T^i z_0, x_i) < \delta \quad , \quad -\infty < i < +\infty.$$

Ce lemme, en fait, représente, avec de légères modifications, un raisonnement utilisé dans la démonstration du théorème d'Anossov sur la stabilité structurelle des Y-difféomorphismes.

En utilisant ce lemme, on construit une application $F : \Omega \to M$ en supposant que $F(\omega) = z_0$.

De l'égalité $FT_m = TF$, découle que la mesure μ_ϵ induite par cette application sur M est invariante par rapport à T. On démontre alors, que μ_ϵ est une distribution limite de Gibbs, construite à l'aide de la mesure $\mu^{(\max)}$ et d'une certaine fonction h_ϵ. La forme de cette fonction est assez complexe, et nous l'omettons ici. L'important est que la fonction h_ϵ satisfasse la condition de Hölder.

En fixant le point z_0, considérons l'ensemble $F^{-1}(z_0)$. Pour presque tout point z_0, sur $F^{-1}(z_0)$ on obtient une distribution conditionnelle de probabilité. Il est facile de démontrer que le processus conditionnel correspondant est une chaîne ergodique Markovienne non-homogène.

Soit $\tau_\epsilon(\cdot|z_0)$ la densité de la distribution pour la coordonnée z_0 calculée suivant la mesure conditionnelle dans l'espace $F^{-1}(z_0)$.

Alors

$$\pi_\epsilon(A) = \int_M d\mu_\epsilon(z_0) \int_A \tau_\epsilon(y|z_0)\,dy \quad , \quad A \subset M. \tag{4}$$

La densité τ_ϵ est différente de zéro dans un voisinage dont le diamètre n'est pas supérieur à $C\delta_2(\epsilon)$ où C dépend uniquement de T.

De (4) on déduit facilement que $\lim_{\epsilon \to 0} \pi_\epsilon = \lim_{\epsilon \to 0} \mu_\epsilon$ si au moins une de ces limites existe. Pour certaines hypothèses naturelles sur la dépendance de $q(\cdot|x, \epsilon)$ nous avons démontré que $\lim_{\epsilon \to 0} \mu_\epsilon$ existe et qu'elle est égale à $\mu^{(c)}$.

6. Mesures invariantes pour un Y-flot.

On n'a pas encore réussi à étendre complètement la théorie, exposée ci-dessus pour les Y-difféomorphismes, au cas du temps continu, bien qu'il existe des raisons sérieuses de croire que les théorèmes correspondants restent valables dans ce cas également.

G.A. Margouliss (11), (12), a construit une mesure invariante $\mu^{(\max)}$ avec une entropie maximale pour des Y-flots transitifs. A notre connaissance, le théorème d'unicité d'une telle mesure n'a pas été démontré. La mesure construite par G.A. Margouliss possède une propriété remarquable. Les coefficients de dilatation et de contraction, calculés suivant cette mesure, sont constants (ne dépendent pas du point) et la constante correspondante h est égale à l'entropie topologique. En outre, comme l'a démontré Margouliss, cette même constante h joue un rôle dans la valeur asymptotique du nombre des trajectoires fermées du Y-flot : si $\nu(t)$ est le nombre des trajectoires fermées de multiplicité 1 d'un Y-flot transitif dont période n'est pas supérieure à t, alors :

$$\lim_{t\to\infty} \frac{\ln \nu(t)}{t} = h.$$

Dans les articles de Margouliss se trouvent aussi des formules donnant les termes suivants du developpement asymptotique de la fonction $\nu(t)$.

M.E. Ratner (13), (14), en utilisant les méthodes de l'article [4] pour des Y-flots transitifs de dimension 3, a construit des mesures invariantes $\mu^{(c)}$ et $\mu^{(p)}$ liées à la structure différentiable de la variété de la même façon que dans le cas du temps discret, à savoir :

1) Le Y-flot $\{T_t\}$, comme groupe à un paramètre de transformations de l'espace M, conservant la mesure $\mu^{(c)}$ ou $\mu^{(p)}$, est un K-flot.

2) pour chaque partition mesurable des feuilles locales contractantes (dilatantes), les mesures conditionnelles, induites sur les éléments de la partition par la mesure $\mu^{(c)}$ $(\mu^{(p)})$, sont données par une densité suivant la mesure différentiable. Chaque mesure ayant les propriétés 1) et 2) est unique.

Pour les Y-flots à trois dimensions on obtient un théorème analogue au théorème 1, à savoir :

THEOREME 4. – *Pour chaque fonction $h \in C(M)$ vérifiant la condition de Hölder et chacune des mesures $\mu^{(c)}$ et $\mu^{(p)}$ il existe une seule distribution de Gibbs $\mu^{(c)}(h)$, $\mu^{(p)}(h)$. Le Y-flot $\{T_t\}$ avec une telle mesure est un K-flot.*

Les mesures $\mu^{(c)}$ et $\mu^{(p)}$ peuvent être obtenues comme les distributions limites de Gibbs, construites à l'aide de $\mu^{(\max)}$

7. Les petites perturbations stochastiques des champs vectoriels par des processus de diffusion.

Soit M une C^∞-variété Riemanienne sans bord. Pour les problèmes considérés ci-dessous, il est nécessaire d'introduire des processus Markoviens de diffusion sur la variété différentiable M. Pour la première fois, de tels processus sont apparus dans l'exposé de A.N. Kolmogoroff [15]. K. Ito [16] a construit, pour ces processus, des équations différentielles stochastiques. Notre approche est voisine de celle de R. Gangolli [17] décrite dans un article plus récent. Considérons l'espace tangent $\mathcal{C}_x$ au point x et une distribution régulière Gaussienne dans $\mathcal{C}_x$ centrée à l'origine et de matrice de dispersion $\sigma(x)$.

DEFINITION. – Soit $\sigma = \{\sigma(x)\}$ un champ de matrices définies positives sur M. On appelle "*bruit blanc*" sur la variété M la mesure définie dans le fibré tangent $\mathcal{C}(M)$ par

$$\nu_\sigma = \prod_{x \in M} g_{x,\sigma}$$

où $g_{x,\sigma}$ est la distribution Gaussienne dans $\mathcal{C}_x$ avec la matrice de dispersion $\sigma(x)$ $\left(\prod_{x \in M} \text{ désigne le produit direct des mesures}\right)$.

Soient α un C^∞-champs vectoriel sur M et ν_σ le bruit blanc sur M avec la C^∞-fonction $\sigma(x)$. Fixons $x_0 \in M$ et considérons l'espace $C_{x_0}([0\,,T])$ des applications continues $x(t)$ du segment $[0\,,T]$ dans M avec la condition $x(0) = x_0$. Nous allons construire une mesure $P_{x_0,T}$ dans C_{x_0} $([0\,,T])$. Divisons le segment $[0\,,T]$ en 2^n parties égales et pour chaque n définissons "la trajectoire aléatoire" sur M de la façon suivante : si la trajectoire aléatoire $x(t)$ est définie pour $0 \leqslant t \leqslant T\frac{m}{2^n}$, $m < 2^n$ alors nous choisissons un vecteur aléatoire b_m avec la distribution $g_{x(T\frac{m}{2^n})\sigma}$

et nous supposons que

$$x(t) = \exp_{x(T\frac{m}{2^n})} \left\{ \left(t - T\frac{m}{2^n}\right) \left[\alpha\left(x\left(T\frac{m}{2^n}\right)\right) + b_m \sqrt{\frac{2^n}{T}}\right] \right\},$$

$$T\frac{m}{2^n} \leqslant t \leqslant T\frac{m+1}{2^n}$$

où $\exp_z$ est une application exponentielle de $\mathcal{C}_z$ dans M.

Pour un n donné nous avons ainsi une distribution de probabilités $\mathcal{P}_{x_0,T}$ dans l'espace $C_{x_0}([0\,,T])$. A l'aide des théorèmes de Prokhorov [18] on démontre que $\mathcal{P}^{(n)}_{x_0,T}$ converge, pour $n \to \infty$, faiblement vers une mesure dans $C_{x_0}([0\,,T])$ que nous désignons par $\mathcal{P}_{x_0,T}$. Pour des T différents, les mesures qui sont construites de cette façon se raccordent naturellement, et leur limite, pour $T \to \infty$, est désignée par $\mathcal{P}_{x_0}$. Il est facile de voir que l'ensemble de toutes les mesures $\mathcal{P}_{x_0}$ définit un processus de diffusion de Markov. La mesure $\mathcal{P}_{x_0}$ définit une famille des probabilités de transition pour le processus Markovien au point x_0. Donc, le processus de diffusion Markovien sur la variété M est défini par le champ vectoriel α et le champ tensoriel σ des matrices définies positives du 2ème ordre. Nous désignons ce processus par $\mathcal{M}_{\alpha,\sigma}$.

Cette construction correspond à l'idée physique d'un processus de diffusion ou comme la limite d'un "random walk".

DEFINITION. – Le processus de diffusion Markovien pour lequel

$$\alpha \equiv 0 \quad , \quad \sigma(x) \equiv I$$

(où I est la matrice unité) s'appelle le mouvement brownien de la variété.

DEFINITION. – Soit, sur la variété M, un champ vectoriel α. On appelle "petite perturbation stochastique du champ vectoriel α" la famille des processus Markoviens de diffusion $\mathfrak{M}_{\alpha,\epsilon\sigma}$.

P.F. Khasminsky [10] a démontré en fait le théorème suivant :

THEOREME. – *Soient M une variété compacte, Π_ϵ un ensemble de mesures invariantes pour le processus Markovien $\mathfrak{M}_{\alpha,\epsilon\sigma}$. Alors, toute mesure qui est une limite faible des mesures de Π_ϵ (si $\epsilon \to 0$), est une mesure invariante pour le champ vectoriel α.*

Le problème d'étude de ces mesures μ a été posé à plusieurs reprises par A.N. Kolmogorov. Du point de vue analytique il s'agit d'étudier l'allure asymptotique des solutions positives des équations differentielles linéaires elliptiques du $2^{\text{ème}}$ ordre sur les variétés quand le coefficient des dérivées du degré supérieur est un paramètre ϵ tendant vers 0. Le cas d'une dimension a été considéré par Andronov, Vitt, Pontryaguin [19]. Certains champs vectoriels ont été étudiés, de ce point de vue, par R.Z. Kchasminsky sur le tore de 2 dimensions [10]. Récemment, des résultats importants ont été obtenus par A.D. Ventzel et M.I. Frildman [20]. Ils ont étudié en fait le cas où l'ensemble de points non-errants (no-wandering) du champ vectoriel est composé d'un nombre fini de points et de courbes fermées. Dans le paragraphe suivant, nous allons étudier les petites perturbations stochastiques des Y-flots et les problèmes qui s'y rattachent.

8. Les petites perturbations stochastiques des flux géodésiques dans les espaces à courbure négative et les questions connexes.

Le lemme du paragraphe 5 indique que, dans le cas des perturbations stochastiques localisées des Y-difféomorphismes, chaque réalisation de la chaîne de Markov se trouve dans un petit voisinage de la trajectoire du Y-difféomorphisme. Pour les perturbations non-localisées, ceci n'est plus exact. Cela l'est encore moins dans le cas des petites perturbations stochastiques des Y-flots. Cependant, il existe dans ce cas un analogue au lemme du paragraphe 5. Arrétons-nous plus en détail sur ces résultats.

Soient M le fibré tangent unitaire d'une surface compacte Q, ayant une courbure négative, et $\{T_t\}$ un flot géodésique dans M. On sait que $\{T_t\}$ représente un exemple classique de Y-flot. Nous désignons par α le champ vectoriel qui lui correspond et nous introduisons une petite perturbation stochastique $\mathfrak{M}_{\alpha,\epsilon\sigma}$. Considérons la surface $\widetilde{Q}$ de courbure négative, qui est le revêtement universel de Q ; soit $\widetilde{M}$ le fibré tangent unitaire de M. Alors $\widetilde{M}$ est le revêtement de M ; $\widetilde{M}$ est muni d'un champ vectoriel $\widetilde{\alpha}$ et d'une petite perturbation stochastique $\mathfrak{M}_{\tilde{a},\epsilon\tilde{\sigma}}$. Le théorème suivant s'applique aux réalisations du processus Markovien $\mathfrak{M}_{\tilde{a},\epsilon\tilde{\sigma}}$.

THEOREME. – *Pour chaque* ϵ *suffisamment petit, presque chaque réalisation* $\omega = \omega(t)$ *du processus* $\mathfrak{M}_{\tilde{a},\epsilon\tilde{\sigma}}$ *tend vers l'infini si* $t \to \infty$. *De plus, si* $q_0 \in \widetilde{Q}$ *est un point fixé, il existe une demi-géodésique unique* $q(t)$ *issu du point* q_0 *telle que* $d(\omega(t), q(t)) \leqslant$ const t, *où la constante dépend de* ω *mais non de* t.

Si $\widetilde{Q}$ est équivalent conformément à la surface $\widetilde{Q}_0$, de courbure négative constante, on peut alors réaliser les éléments linéaires sur $\widetilde{\mathfrak{M}}$ par des matrices unimodulaires du deuxième ordre. Le processus de diffusion Markovien dans l'espace de telles matrices représente une généralisation naturelle du produit des matrices aléatoires au cas du temps continu. Pour les produits des matrices aléatoires, on connaît le théorème de G. Furstenberg [21], montrant que le comportement asymptotique des produits des matrices aléatoires est semblable au comportement décrit par le théorème mentionné ci-dessus. C'est pourquoi on peut considérer notre théorème comme apparenté au Théorème de Furstenberg.

Il faut noter aussi le fait que pour les sommes des grandeurs aléatoires indépendantes, on connaît le principe d'invariance de Donsker, qui permet de ramener ces sommes au mouvement Brownien.

A notre connaissance un tel principe n'existe pas, jusqu'à présent, pour le produit des matrices aléatoires.

Un théorème analogue au précédent est le suivant qui a été démontré par F.I. Karpelevitch et par nous, mais qui est vraisemblablement connu de beaucoup d'autres mathématiciens.

THEOREME. – *Soit* Q *un espace à* n *dimensions, de courbure négative, et homéomorphe à* R^n. *On suppose que la courbure suivant chaque direction plane est comprise entre deux constantes négatives. Alors, pour presque toute trajectoire* $\widetilde{\omega}(t)$, $\widetilde{\omega}(0) = q_0$, *du mouvement brownien sur* Q, *il existe une semi-géodésique* $\widetilde{q}(t), \widetilde{q}(0) = q_0$, *telle que* $d(\widetilde{\omega}(t), \widetilde{q}(t)) \leqslant$ const. t, *où, comme dans le cas précédant la constante dépend de* $\widetilde{\omega}$ *et, non de* t.

Ces deux théorèmes sont en relation avec l'étude de la frontière de Martin des processus de diffusion considérés ici, mais les questions correspondantes n'ont pas encore été tirées au clair.

Revenons au problème de l'étude du comportement asymptotique (quand $\epsilon \to 0$) des mesures invariantes des processus de Markov $\mathfrak{M}_{a,\epsilon\sigma}$ où α est un champ vectoriel engendrant le Y-flot sur une variété compacte tridimensionnelle. Pour le problème analogue dans le cas du temps discret, nous construisons l'application $\mathfrak{F}$. Ici il n'existe pas d'application de ce type. Mais on peut tout de même construire une application $\mathfrak{F}_\epsilon$, dépendant de ϵ, qui possède un grand nombre de propriétés de l'application $\mathfrak{F}$. En particulier si μ_ϵ est l'image de la distribution Markovienne de probabilités par l'application $\mathfrak{F}_\epsilon$, la valeur asymptotique de la mesure

μ_ϵ (si $\epsilon \to 0$) coïncide avec celle de la distribution invariante pour notre processus de Markov quand $\epsilon \to 0$. A l'aide de cette idée, on peut démontrer le théorème suivant :

THEOREME. – *Pour* $\epsilon \to 0$ *les distributions invariantes du processus Markovien* $\mathfrak{M}_{a,\epsilon\sigma}$ *convergent faiblement vers la mesure* $\mu^{(c)}$. *En particulier, si le Y-flot* $\{T_t\}$ *a une mesure invariante* μ *compatible avec la différentiabilité, c'est-à-dire si* $\mu^{(c)} = \mu^{(p)} = \mu$ *alors les distributions ci-dessus convergent faiblement vers* μ.

BIBLIOGRAPHIE

[1] Боголюбов Н. Н. и Хацет Б. И. — О некоторых математических вопросах теории статистического равновесия. ДАН СССР, т. 66, н 3, 1949, 321-324.
[2] RUELLE D. — Correlation Functions, *Ann. of Phys.*, 25, 1963.
[3] Добрушин Р. Л. — Гиббсовские случайные поля для решетчатых систем. Фуикциональный анализ и его приложения, т. 2, вып 4, 1968, 31-43.
[4] Синай Я. Г. — Марковские разбиения и У-Диффеоморфизмы. Функциональный анализ и его приложения, т. 2, вып. 4, 1968, 64-89.
[5] Динабург Е. И. — Соотношение между топологической энтропией и метрической эитропией. ДАН СССР, т. 19О, и 1, 1970.
[6] ADLER R. L., KONHEIM A. G., MCANDREW M. M. — Topological entropy, *Trans. Amer. Math. Soc.*, 114, 1965, p. 309-319.
[7] Гуревич Б. М. — Топологическая энтропия счетной цепи Маркова. ДАН СССР, т. 187, и 4, 1969, 715-718.
[8] BOWEN R. — Markov Partitions for Axion A Diffeomorphisms. Preprint.
[9] SMALE S. — Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.*, 73, 1967, p. 747-817.
[10] Хасьминский Р. З. — О принципе усреднения для параболических и эллиптических дифферециальных уравнений и марковских процессов с малой диффузией. Теор. вероятн. и её прил., 8, 1, 1963, 3-25.
[11] Маргулис Г. А. — О некоторых применениях эргодической теории к изучению многообразий отрицательной кривизны. Функциональный анализ и его приложсния, т. 3, вып. 4, 1969, 89-90.
[12] Маргулис Г. А. — О некоторых мерах, связанных с У-потоками на компактных многообразиях. Функциональный анализ и его прилож, т. 4, вып. 1, 1970, 62-77.
[13] Ратнер М. Е. — Марковское разбиение для У-потока на трехмерном многообразий. Математические заметки, т. 6, н. 6, 1969, 693-704.
[14] Ратнер М. Е. — Об инварианной мере для У-потока на трехмерном многообразии. ДАН СССР, т. 186, н. 2, 1969, 261-263.
[15] Колмогоров А. Н. — Zum Umkehrbarkeit der statistische Naturgesetze, *Math. Ann.*, 113, 1937, p. 766-772.
[16] ITO K. — On stochastic differential equation in a differentiable manifold, *Nagoya Math. Y.*, 1, 1950, p. 35-47.
[17] GANGOLLI R. — On the construction of certain diffusion on a differentiable manifold, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, 1964, Bd. 2, N. 5, 406-419.

[18] Прохоров Ю. В. — Сходимость случайных процессов и предельние теоремы теории вероятностей. Теория вероят. и её приложения, 1, 1956, 177-238.
[19] Андронов А. А. Витт А. А., Понтрягин Л. С. — О статистическом рассмотрении динамических систем. Ж Э Т Ф, 3 : 3, 1933, 165-180.
[29] Венцель А. Д., Фрейдлин М. И. — О малых случайных возмущениях динамических систем. У М Н, т. 25, вып. 1, 1970, 3-55.
[21] Furstenberg H. — Non commuting random products, *Trans. Amer. Math. Soc.*, t. 108, 3, 1963, pp. 377-428.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 941 à 946.

LES SPECTRES DES SYSTÈMES DYNAMIQUES

par A.M. STEPIN

Une transformation qui conserve la mesure peut agir dans un espace muni d'une structure supplémentaire (topologie, différentiabilité, etc.) tout en conservant cette structure. Le plus grand intérêt est suscité par trois aspects de la théorie des transformations conservant la mesure, à savoir : la théorie des automorphismes de l'espace de Lebesgue, l'étude des homéomorphismes d'un espace métrique compact, conservant la mesure régulière,et, enfin, la théorie des difféomorphismes des variétés différentiables conservant la mesure absolument continue donnée par une densité différentiable.

Dans chacune des trois classes de transformations mentionnées, conservant la mesure, on peut introduire une topologie qui tienne compte de la présence d'une structure invariante supplémentaire et transformer la classe donnée en un groupe topologique. La question qui nous intéresse ici est l'étude spectrale de ces classes, c'est-à-dire la question des propriétés topologiques (catégorie, densité, etc.) des sous-ensembles du groupe topologique correspondant, composés de transformations dont les propriétés spectrales sont données.

A ces problèmes se rattache celui de la description des spectres des systèmes dynamiques.

1. Automorphismes de l'espace de Lebesgue.

(1) Dans la théorie métrique (et, en particulier, spectrale) des systèmes dynamiques s'est révélée utile la méthode des approximations périodiques [1]. Soit $f(p)$ une suite de nombres positifs. On dira que l'automorphisme T de l'espace de Lebesgue (X, μ) admet une approximation par des transformations périodiques $(a.t.p.)$ avec la vitesse $f(p)$, si l'on peut trouver une suite d'automorphismes T_n, convergeant vers T, et une suite de partitions mesurables finies ξ_n de l'espace X, telles que (*)

$$\xi_n \to \epsilon \quad , \quad T_n \xi_n = \xi_n \quad \text{et} \quad \sum_{C \in \xi_n} \mu(TC \,\Delta T_n C) \leqslant f(p_n),$$

où p_n est la période de la permutation T_n des éléments de la partition ξ_n, induite par la transformation T_n. Si, pour chaque n, la permutation T_n est composée d'un seul cycle, nous dirons que T admet une $(a.t.p.)$ cyclique avec la vitesse $f(p)$. Du fait que l'automorphisme T admet une $(a.t.p.)$ avec des propriétés déterminées (par exemple : cyclique) et avec "une vitesse suffisamment élevée" on

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(*) La notation $\xi_n \to \epsilon$ désigne la convergence de la suite des partitions ξ_n vers la partition mesurable de l'espace X, formée par les points isolés.

peut déduire des propriétés spectrales de l'automorphisme T, comme l'ergodicité ou *le mélange* (voir [1], [2]). En outre, si l'automorphisme T admet une $(\bar{a}.\bar{t}.\bar{p}.)$ cyclique avec la vitesse θ/p, où $\theta < 2 - \frac{2}{(m+1)}$, la multiplicité de son spectre n'est pas supérieure à m.

Un autre groupe de résultats, sur la relation entre la possibilité d'approximation et le spectre, est constitué par les théorèmes qui permettent, en partant de l'existence d'une approximation, d'obtenir des propriétés structurales du type spectral maximal de l'automorphisme (voir [1], [3]). Par exemple, si un automorphisme admet une $(a.t.p.)$ cyclique avec la vitesse θ/p $(\theta < 1)$, son type spectral maximal est singulier.

La méthode des approximations périodiques permet non seulement d'étudier les transformations individuelles (voir, par exemple, [4], [5], [6]), mais aussi de démontrer des théorèmes catégoriels d'un point de vue unitaire, ce qui est fait, pour l'essentiel, dans [1]. En outre, l'approche par approximation s'est révélée utile dans l'étude des transformations qui sont des *produits gauches* (skew-products), ainsi que dans l'étude des équations fonctionnelles de la forme

$$f(Tx) = W(x)\ f(x)$$

(voir, par exemple [1], [7], [8]).

(2) A.N. Kolmogoroff a émis une hypothèse selon laquelle le type spectral maximal d'un automorphisme ergodique subordonnerait son carré de convolution. Cette propriété du type spectral maximal, qui est l'analogue naturelle, dans le cas continu, la propriété de groupe du spectre d'un automorphisme ergodique possédant un spectre discret, a été établie par S.V. Fomine [9] pour les automorphismes engendrés par des processus gaussiens stationnaires, et par Ya.G. Sinaï [10] pour les automorphismes vérifiant la condition A.

Il s'est révélé, toutefois, que, dans le cas général, l'hypothèse mentionnée sur la structure de groupe du spectre n'est pas juste. Initialement, cela a été démontré pour un groupe d'automorphismes isomorphe au groupe de tous les nombres complexes qui sont des racines de l'unité d'ordre 2^n, $n = 1, 2, \ldots$, [11] et ensuite pour les automorphismes de [1]. Dans ces exemples, le type spectral maximal σ du groupe des transformations se présente sous la forme $\sigma_1 + \sigma_{-1}$, où σ_1 est de type discret et σ_{-1} de type continu, disjoint de son carré $\sigma_{-1} * \sigma_{-1}$. De là découle que le type σ ne subordonne pas son carré $\sigma * \sigma$. V.I. Osselebetz [12] a construit des exemples d'automorphismes avec spectre continu, dont le type spectral maximal ne subordonne pas son carré.

Récemment, on a réussi à démontrer que les spectres des automorphismes ergodiques ne possèdent pas en général, la propriété de groupe. Plus précisément, l'ensemble des automorphismes, dont le type spectral maximal dans le sous-espace invariant de $L^2(X, m)$, orthogonal au sous-espace des fonctions constantes, est disjoint de son carré, contient un sous-ensemble, partout dense, de type G_δ. La question, de savoir dans quel sens le type maximal spectral d'un automorphisme ergodique possède la propriété du groupe, reste ouverte.

2. Homéomorphismes conservant la mesure.

Oxtoby et Ulam ont étudié, dans leur article [13], les propriétés ergodiques des homéomorphismes conservant une mesure régulière μ pour le polyèdre simplicial M. Les conditions imposées au polyèdre M et à la mesure μ étaient les suivantes :

(a) M est régulièrement connexe ;

(b) l'ensemble des points non-réguliers est de mesure nulle ;

(c) la mesure d'un voisinage d'un point régulier est positive.

Sous ces hypothèses, on obtient dans [13] le résultat suivant :

Dans le groupe $H(M, \mu)$ de tous les homéomorphismes de M, qui conservent la mesure μ, muni de la topologie uniforme, les homéomorphismes ergodiques constituent un ensemble partout dense du type G_δ. A.B. Katok et l'auteur du présent exposé ont modifié dans leur article [2] la principale construction de l'approximation utilisée par Oxtoby et Ulam et l'ont reliée à la méthode des approximations périodiques. En suivant cette voie, on a réussi, pour une vaste classe d'espaces M (par exemple pour des polyèdres cellulaires régulièrement connexes de dimensions finies) à démontrer, pour le groupe $H(M, \mu)$ des homéomorphismes conservant la mesure μ de l'espace M, tous les théorèmes catégoriels obtenus auparavant pour les automorphismes de l'espace de Lesbegue.

Si l'on se limite à l'étude des propriétés ergodiques des homéomorphismes de polyèdres cellulaires, on peut alors remarquer une analogie presque complète entre le cas purement métrique et le cas continu.

A la base de cette analogie se trouve le fait suivant : les automorphismes périodiques (dans le cas métrique) et les homéomorphismes périodiques en dehors d'ensembles de mesure arbitrairement petite (dans le cas topologique) sont partout denses dans les groupes correspondants.

Notons que la démonstration du théorème sur la *massivité* de l'ensemble des automorphismes à spectre continu, démonstration due à Halmos, ne se transporte pas au cas topologique. Ceci s'explique par l'absence d'un lemme topologique analogue au lemme d'Halmos sur la conjugaison. Un problème qui reste pour le moment sans solution et qui vraisemblablement présente un intérêt peut être formulé de la façon suivante :

L'ensemble des homéomorphismes conjugués métriquement avec un homéomorphisme fixé, dont l'ensemble des points périodiques est de mesure nulle, est-il partout dense ?

L'article [2] comporte les deux principaux résultats suivants.

THEOREME A. — *L'ensemble des automorphismes de $H(M, \mu)$, qui admettent une approximation par une transformation périodique cyclique avec la vitesse $f(p)$, contient un sous-ensemble de type G_δ partout dense dans $H(M, \mu)$.*

THEOREME B. — *L'ensemble des automorphismes de $H(M, \mu)$, qui ont un spectre continu est un sous-ensemble partout dense dans $H(M, \mu)$, de type G_δ.*

De ces théorèmes et des résultats exposés dans [1], résulte :

COROLLAIRE – *Il existe un sous-ensemble de type G_δ qui est partout dense dans $H(M, \mu)$, dont les éléments sont des automorphismes ergodiques mais qui ne mélangent pas, et qui ont un spectre singulier simple et un spectre continu.*

Les théorèmes A et B sont valables pour un espace métrique compact M, dont l'ensemble M_r^n des points qui possèdent un voisinage homéomorphe à une sphère de dimension n, est connexe et tel que $\mu(M \setminus M_r^n) = 0$. On peut démontrer que sur un tel espace, il existe un groupe à un paramètre d'homéomorphismes qui conserve la mesure et qui a un spectre continu.

Notons que, de même que dans le cas métrique, l'ensemble des automorphismes de $H(M, \mu)$, dont les spectres ne possèdent pas la propriété de groupe, est de deuxième catégorie.

3. Les systèmes dynamiques différentiables.

(1) L'analogie entre le cas métrique et le cas continu disparaît dans le cas de différentiabilité, ce qui découle, par exemple, de l'existence de Y-difféomorphismes. Cependant, la méthode des approximations périodiques se révèle utile également pour l'étude des difféomorphismes à mesure invariante. Récemment, D.V. Anossov et A.B. Katok [14] ont proposé une construction qui donne des exemples de C^∞-difféomorphismes ergodiques conservant la mesure, agissant sur certaines variétés différentiables et possédant des propriétés métriques diverses et parfois inattendues. Ce sont des variétés différentielles connexes et compactes (avec ou sans bord) qui admettent une action différentielle non-triviale du groupe de rotations du cercle (le flot périodique). La mesure invariante dans ces exemples est donné par une densité régulière positive. Les propriétés métriques des difféomorphismes construits dans [14], en fonction du choix des paramètres, peuvent être les suivantes :

(a) spectre discret avec un nombre quelconque, fixé à l'avance (fini ou infini), de fréquences de base (sur l'anneau des nombres entiers).

(b) spectre simple singulier continu avec absence de mélange.

(c) (a) et (b) ensemble.

Les difféomorphismes construits dans [14] forment un ensemble, nulle part dense, dans le groupe Diff (M, μ) des difféomorphismes d'une variété donné M (qui conservent une mesure donnée μ) ce groupe étant muni de la topologie C^∞ ou C^n $(n \geqslant 11)$. En effet, ces difféomorphismes appartiennent à la fermeture de l'ensemble, nulle part dense dans Diff (M, μ), des difféomorphismes périodiques.

Il est possible que l'ensemble de tous les difféomorphismes ergodiques ne soit pas partout dense dans Diff (M, μ), du moins pour certains M (en dimension 2, cela découle des résultats de J. Moser). Sur la fermeture de l'ensemble des difféomorphismes qui appartiennent à des flots périodiques, la situation métrique rappelle beaucoup les cas topologique et métrique. Les théorèmes catégoriels correspondants ont été démontrés dans [14]. A.B. Katok a attiré notre attention sur le fait que la construction de [14] permet de construire, sur toute variété différentielle, admettant un flot périodique, un difféomorphisme ergodique à mesure différentiable invariante, dont le spectre ne possède pas la propriété de groupe.

En utilisant les méthodes exposées dans [14], D.V. Anossov et A.B. Katok ont montré [15], que, sur chaque variété de dimension $\geqslant 3$, il existe un flot ergodique différentiable (et donc un difféomorphisme) à mesure différentiable invariante.

(2) Comme il résulte de ce qui précède, il est possible, dans les cas métrique, topologique et différentiable, d'avoir un large éventail de spectres. Si l'on se limite à la classe des systèmes dynamiques sur des variétés homogènes des groupes Lie, il est naturel de supposer que les spectres de ces systèmes sont des combinaisons d'un spectre discret et d'un spectre de Lesbegue de multiplicité dénombrable. Tous les résultats obtenus jusqu'à présent dans l'analyse spectrale des systèmes dynamiques sur des espaces homogènes confirment cette hypothèse.

Dans l'article [16] cette hypothèse a été vérifiée pour les flots, sur l'espace-quotient $D\backslash G$ du groupe résoluble G de type exponentiel, engendrés par les éléments réguliers de l'algèbre de Lie $\mathfrak{g}$. Décomposons la complexifiée $\mathfrak{g}^{\mathbf{C}}$ de l'algèbre $\mathfrak{g}$ en somme des sous-espaces radiciels $\mathfrak{g}^{\mathbf{C}}_{\lambda}$ de l'élément régulier $X \in \mathfrak{g}$. Désignons par $\mathfrak{g}$ le sous-espaces constitué par les éléments réels de $\mathfrak{g}^{\mathbf{C}}_{\lambda} + \mathfrak{g}^{\mathbf{C}}_{\bar{\lambda}}$. Soient G_+ le sous-groupe de G, correspondant à la sous-algèbre $\sum_{Re\,\lambda \geqslant 0} \mathfrak{g}_{\lambda}$ et $\widehat{G}$ le sous-groupe dans G qui correspond à la sous-algèbre minimale dans $\mathfrak{g}$ contenant le sous-espace $\sum_{\lambda \neq 0} \mathfrak{g}_{\lambda}$. On constate que l'enveloppe mesurable de la partition de l'espace $D\backslash G$ en orbites du groupe G_+ est une partition en orbites du groupe $\widehat{G}$. De là et de la théorie de Ia. G. Sinaï (voir [17]), résulte notre proposition. Une telle approche, dans le cas des groupes semi-simples, permet de démontrer que le flot sur l'espace quotient d'un groupe de Lie semi-simple sans quotients compacts, engendré par un élément régulier réel de l'algèbre de Lie, est un K-flot; il possède donc un spectre de Lebesgue de multiplicité dénombrable.

BIBLIOGRAPHIE

[1] Каток А. Б. и Степин А. М. — Аппроксимации в эргодической теории, УМН, 22 : 5, 1967, 81-106.

[2] Каток А. Б. и Степин А. М. — Метрические свойства гомеоморфизмов, сохраняющих меру, УМН, 25 : 2, 1970, 193-220.

[3] Степин А. М. — Спектр и аппроксимации метрических автоморфизмов периодическими преобразованиями, Функ. анализ 1, № 2, 1967, 77-80.

[4] Каток А. Б., Степин А. М. — Об аппроксимациях эргодических динамических систем периодическими преобразованиями, ДАН, 171, № 6, 1966, 1268-1271.

[5] Каток А. Б. — Спектральные свойства динамических систем с интегральным инвариантом на торе, Функ. анализ, 1, № 4, 1967, 46-56.

[6] Шкловер М. Д. — О классических динамических системах на торе с непрерывным спектром. Известия вузов, № 10 (65), 1967, 113-124.

[7] Степин А. М. — О квадратных корнях из метрических автоморфизмов, ДАН, 176, № 5, 1967, 1023-1026.
[8] Малкин С. А. — Пример двух метрически неизоморфных эргодических автоморфизмов с простым спектром, Известия вузов, № 6, 73, 1968, 69-74.
[9] Фомин С. В. — Укр. матем. ж., 2, № 2, 1950, 25-47.
[19] Синай Я. Г. — Несколько замечаний о спектральных свойствах эргодических динамических систем, УМН 18, вып. 5, 1963, 41-54.
[11] Степин А. М. — О свойствах спектров эргодических динамических систем с локально компактным временем, ДАН 169, № 4, 1966, 773-776.
[12] Оселедец В. И. — Автоморфизм с простым и непрерывным спектром без группового свойства, Матем. заметки, 5, вып. 3, 1969, 323-326.
[13] Oxtoby J., Ulam S. — Measure-preserving homeomorphisms and metric transitivity, Ann. of Math., 49 : 4, 1941, p. 874-920.
[14] Аносов Д. В. и Каток А. Б. — Новые примеры в гладкой эргодической теории 1, Эргодические диффеоморфизмы, Труди Моск. матем. о-ва 23, 1970.
[15] Аносов Д. В., Каток А. Б. — Новые примеры эргодических диффеоморфизмов гладких многообразий, 25 : 4, 1970.
[16] Степин А. М. — Потоки на разрешимых многообразиях, УМН 24, вып. 5, 1969, 241-242.
[17] Синай Я. Г. — Классические динамические системы со счетнократным лебеговским спектром П, Изв. АН, сер. матем. 30, № 1, 1966, 15-68.

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Actes, Congrès intern. Math., 1970. Tome 2, p. 947 à 951.

THE STRUCTURE OF ATTRACTORS

by R. F. WILLIAMS

Our goal is to describe some recent work done toward classifying attractors (and other basic sets) of differentiable dynamical systems. As most of our results concern diffeomorphisms as opposed to vector fields, we take this point of view throughout. A basic reference is S. Smale's important survey paper [6]. Many appropriate papers are in the Proceedings of the Berkeley Conference on Global Analysis.

Let M be a compact C^r manifold and f a diffeomorphism of class C^r, $r \geqslant 1$. A closed set $\Lambda \subset M$ is an attractor of f provided

(a) Λ has a neighborhood N such that $f(N) \subset N$ and

$$\bigcap_{i=1}^{\infty} f^i(N) = \Lambda$$

(b) Λ is minimal relative to (a) ;

(c) each point of Λ is non-wandering.

This is so general that very little can be said. There is, however, an important deep,

Question (R. Thom) : Are attractors generically

(a) topologically transitive ?

(b) countable up to homeomorphism ?

(c) countable up to topologically conjugacy ?

One says $f : X \to X$ and $g : Y \to Y$ are topologically conjugate provided there is a homeomorphism $h : X \to Y$ such that $g = hfh^{-1}$.

1. Hyperbolic attractors.

Basic assumption. Λ has a hyperbolic structure. That is, there is a splitting of the tangent bundle of M restricted to Λ, $T_M \Lambda = E^u + E^s$ such that the derivative Tf of f leaves the splitting invariant, is an expansion on E^u and a contraction on E^s. We also use u, s to denote the dimensions of the fibers of E^u, E^s.

Basic problem. Characterize and classify hyperbolic attractors up to topological conjugacy.

THEOREM A (S. Smale [7]). – *There are only countably many conjugacy classes of hyperbolic attrators (even "basic sets").*

THEOREM B [9]. – *Hyperbolic attractors have rational zeta functions.*

This is now subsumed by the important

THEOREM (J. Guckenheimer [3]). *Hyperbolic basic sets have rational zeta functions.*

Here the zeta function for $f|\Lambda$ is invariant under conjugacy and is defined by $\zeta(t) = \exp\left(\sum_{i=1}^{\infty} N_i t^i/i\right)$ where N_i is the number of points of Λ left fixed under f^i. We next recall a part of the generalized stable manifold theorem of Smale [6] (formulation) and Hirsch-Pugh [4] (proof).

Roughly, this says that a neighborhood of Λ is foliated by s-planes, $\{W^s(x) | x \in \Lambda\} = \mathfrak{W}^s(\Lambda)$, that this foliation behaves well under f and has the fibers of E^s as tangents. Though its leaves are C^r, as a foliation it may not be smooth. However, this is an "eigen value difficulty" and it is our belief that it can be overcome by something like the

Conjecture. Given f, Λ, there is a topologically conjugate f', Λ' such that $\mathfrak{W}(\Lambda')$ is C^r.

THEOREM (Hirsch-Pugh [4]). – *In the special case that dim* $\Lambda = 1$, $\mathfrak{W}(\Lambda)$ *is of class* C^1.

Second assumption. $\mathfrak{W}(\Lambda)$ is of class C^r, $r \geqslant 1$.

Remark and Definition. As a consequence of the basic assumption it follows that dim $\Lambda \geqslant u$, as $\Lambda \supset W^u(x)$ for each $x \in \Lambda$. We will treat the extrême case: Λ is called an *expanding attractor* provided it has a hyperbolic structure and dim $\Lambda = u$, the dimension of the fiber of E^u.

Third assumption. Λ is an expanding attractor.

2. Branched manifolds.

THEOREM C [11]. – *One can choose the closed neighborhood N so that* $N/\sim$ *is a branched manifold.*

Here for $x, x' \in N$, $x \sim x'$ provided they lie together in the some component of $N \cap W$ for some leaf W of $\mathfrak{W}(\Lambda)$.

DEFINITION. – By a C^r *branched* manifold of dimension n is meant a metric space K together with

(i) a collection $\{U_i\}$ of closed subsets of K ;

(ii) a finite collection of $\{D_{ij}\}$ of closed C^r n-disks for each i ; and

(iii) for each i a map $\pi_i : U_i \to \mathbf{R}^n$;

subject to the following :

(a) $\cup_j D_{ij} = U_i$ and $\cup_i \operatorname{Int} U_i = K$;

(b) $\pi_i | D_{ij}$ is a C^r diffeomorphism into $\mathbf{R}^n$;

(c) There are C^r diffeomorphisms $\alpha_{i'i}$ defined an open subsets of $\mathbf{R}^n$; such that $\pi_{i'} = \alpha_{i'i} \circ \pi_i$ on $\pi_i(U_i \cap U_{i'})$;

(d) $\alpha_{i''i'} \circ \alpha_{i'i} = \alpha_{i''i}$.

Branched manifolds K have tangent bundles TK, admit smooth maps f which have derivatives Tf, etc. One says that a smooth map $f : K \to L$ is an *immersion* provided Tf is a monomorphism on each fiber. For example, each π_i is an immersion, $U_i \to \mathbf{R}^n$.

A point $x \in K$ is a *regular* point provided it has a neighborhood which is the union of open, smooth, n-dimensional sub-disks each containing x. Otherwise x is *singular.*

Conjecture. If K is compact and has empty singular set then there is an immersion $M^n \to K^n$ where M is a closed manifold of the same dimension of K.

3. **n-solenoids.**

Returning to our attractor there is the commutative diagram

$$(x) \qquad \begin{array}{ccc} f(N) & \xleftarrow{\;f\;} & N \\ \cap & & \\ N & & \Big\downarrow q \\ q \Big\downarrow & & \\ K & \xleftarrow[\;g\;]{} & K \end{array}$$

in which q is the quotient map $N \to N/\!\sim\, = K$. Then K is C^r as $\mathcal{W}^s(\Lambda)$ is C^r. g exists and is an immersion as $\mathcal{W}^s(\Lambda)$ is well behaved under f. Then g satisfies the following axioms :

(1) The nonwandering set of g is all of K.

(2) For each $x \in K$ there is a neighborhood V of x and an integer i such that $g^i(V)$ is a subset of a smooth n-cell.

(3^+) g is an expansion.

Then define Σ to be the inverse limit of the sequence

$$K \underset{g}{\leftarrow} K \underset{g}{\leftarrow} K \underset{g}{\leftarrow} \ldots$$

and $h : \Sigma \to \Sigma$ to be the coordinate shift,

$$h(x_0, x_1, \ldots) = (gx_0, gx_1, gx_2, \ldots) = (gx_0, x_0, x_1, \ldots).$$

Σ is called an *n-solenoid* and h the *shift map presented* by g.

Proposition. – If $g : K \to K$ satisfies axioms 1, 2, 3^+ then K has no singular points.

Theorem D [11]. – *Two presentations yield topologically conjugate shift maps iff they are shift equivalent (*[10, p. 349]).

Theorem E [11]. – *Under our assumptions each attractor* $f : \Lambda \to \Lambda$ *is topologically conjugate to the shift map of a n-solenoid and vice versa.*

Half of this is proved via the diagram (x). To prove the other half we take a presentation $g_0 : K_0 \to K_0$, find a nice one $g : K \to K$ which is shift equivalent, embed K in some $\mathbf{R}^m$ with a "tubular neighborhood" N and approximate the map g by a diffeomorphism $f : N \to N$.

4. Structure

THEOREM F [11]. – *The periodic points of an n-solenoid are dense* (also proved via the "Anasov closing lemma").

THEOREM G [11]. – *Each point of an n-solenoid has a neighborhood of the form (Cantor set)* $\times$ *(n-disk).*

Question. Does $g_* : \pi_1(K) \to \pi_1(K)$ determine $h : \Sigma \to \Sigma$ up to topological conjugacy (as in [4.5] and [10]) ?

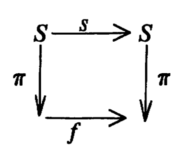
THEOREM H [11]. $H^*(\Sigma : \mathbf{Z}) \neq H^*$*(point) (Cech Theory).*

Conjecture I. If Σ is orientable it can be immersed into a closed manifold B of the same dimension. (note $I \Rightarrow H$. Also $I \Rightarrow J$).

Conjecture J. If Σ is orientable then there is a smooth fiber bundle $C \to \Sigma \to B$ where C is a Cantor set and B is a manifold covered by $\mathbf{R}^n$. In the special case dim $\Lambda = 1$ this theory is essentially complete ([8], [10]). However, in higher dimensions even expanding maps on manifolds (these satisfy axioms 1, 2, 3^+) are not yet classified (see M. Shub [4.5], [4.6]).

5. Sinai-Bowen theory.

The above theory can quite likely be improved, eg. by dropping the second and third assumptions. In particular one would need to replace axiom 3^+ with 3) $g : K \to K$ has a hyperbolic structure. There would be some difficulties. There is an entirely different appreach which would apply toward classifying all basic sets : the Sinai-Bowen theory of Markov partitions. R. Bowen [2], generalizing Ya. Sinai [5], proves that all basic sets (Λ, f) fit into a commutative diagram



in which (S, s) is a "subshift of finite type" (see below) and π is very well behaved. π is a measure theoretic isomorphism and nice enough topologically that one can hope to use it to classify all basic sets.

6. Subshifts of finite type.

If one is to carry out this approach one must classify all subshifts of finite type. This is done in [12] :

DEFINITION. – Let n be a positive interger and $A = [A(i, j)]$ be an $n \times n$ matrix of $0's$ and $1's$. Let $S(A)$ be the set of all doubly infinite sequences $\{x_i\}_{i \in \mathbf{Z}}$ such that $A(x_i, x_{i+1}) = 1$ for all $i \in \mathbf{Z}$. $S(A)$ become a 0-dimensional metric space with an obvious metric. The *shift map* $s : S(A) \to S(A)$ simply moves the index of each point over by one. Then $(S(A), s)$ is a *subshift of finite type.*

One says that two square matrics A, B with non-negative integral entries are *shift equivalent* provided there are rectangular non-negative integral matrices R, S and an integer m such that $RA = BR$, $SB = AS$, $SR = A^m$ and $RS = B^m$.

CLASSIFICATION THEOREM [12]. – *$S(A)$ and $S(B)$ are topologically conjugate iff A and B are shift equivalent.*

BIBLIOGRAPHY

[1] BOWEN R. — Topological entropy and axiom A. Global analysis, Vol. 14, *Proceeding of symposia in pure mathematics,* 1970, A.M.S. Providence R.I.

[2] BOWEN R. — Markor partitions for axiom A difeomorphisms, *Amer. J. of Math.,* Vol. 42, 1970, p. 725-747.

[3] GUCKENHEIMER J. — Axiom A + no cycle condition implies ζ is rational, *Bull. Amer. Math. Soc.,* Vol. 76, 1970, p. 592-594.

[4] HIRSCH M. and PUGH C. — The stable manifold theorem, Global analysis, Vol. (4), *op. cit.*

[4, 5] SHUB M. — Endomorphisms of compact differentiable manifolds, *Amer. J. of Math.,* 91, 1969, p. 175-199.

[4. 6] SHUB M. — Expanding maps, *Global analysis,* Vol. 14, p. 273-6.

[5] SINAI Ya. — Markov partitions and C-diffeomorphisms, *Func. Anal. and its Appl.,* 2, 1968, n° 1, p. 64-64.

[6] SMALE S. — Differentiable dynamical systems, *Bull. Amer. Math. Soc.,* 13, 1967, p. 747-817.

[7] SMALE S. — The Ω-stability theorem for basic sets, *Global Analysis,* Vol. 14.

[8] WILLIAMS R. — One dimensional non-wandering sets, *Topology,* 6, 1967, p. 473-487.

[9] WILLIAMS R. — The zeta function of an attractor, *Michigan state meeting on Algebraic Topology,* Prindle, Weber & Smith, Boston, p. 155-161.

[10] WILLIAMS R. — Classification of 1-dimensional attractors, *Global analysis,* Vol. 14, p. 341-361.

[11] WILLIAMS R. — *Expanding attractors* (to appear). Preprint, Northwestern Univ., 1971.

[12] WILLIAMS R. — *Classification of symbol spaces of finite type* (to appear). Preprint, Northwestern Univ., 1970.

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