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Algebraic K-Theory and Zeta Functions of Elliptic Curves

S. Bloch*

The classical regulator formula [6, Chapter 5]

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}Rh}{\sqrt{|D|}\cdot w}$$

computes the residue of the zeta function of a number field at $s=1$. Various generalizations have been proposed [19], [2], [20]. Lichtenbaum, noting that $h=\#K_0(\mathcal{O}_K)$ and $m=\#K_1(\mathcal{O}_K)_{\text{tors}}$ suggested a formula relating $\zeta_K(m+1)$ to $\#K_{2m}(\mathcal{O}_K)$, $\#K_{2m+1}(\mathcal{O}_K)$ and a higher regulator R_m. Borel [4], [5], studied a regulator map

$$r_{2m+1}\colon K_{2m+1}(\mathcal{O}_K) \to \mathbf{R}^{d_m}$$

where

$$d_m = \left\{\begin{array}{ll} r_2, & m=2n+1>0 \\ r_1+r_2, & m=2n>0 \\ r_1+r_2-1, & m=0 \end{array}\right\} = \text{order of zero of } \zeta_K \text{ at } s=-m.$$

He showed that r_{2m+1} embedded $K_{2m+1}(\mathcal{O}_K)/\text{torsion}$ as a lattice of maximal rank with volume a rational multiple of

$$\pi^{-d_m}\lim_{s\to -m}\zeta_K(s)(s+m)^{-d_m} \in \pi^{-d(m+1)}|D|^{1/2}\zeta_K(m+1)\cdot Q.$$

In another direction, let E be an elliptic curve defined over a number field K, and let $L(E/K, s)$ be the "Hasse–Weil zeta function". Birch and Swinnerton–Dyer

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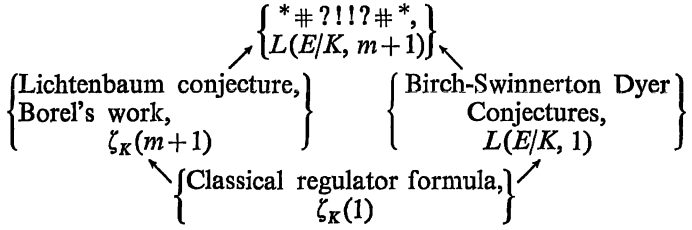
have conjectured [2], [20], [21] that $L(E/K, s)$ should vanish to order $\mathrm{rk}\, E(K)$ at $s=1$ and they have proposed a regulator formula for the first nonvanishing coefficient in the Taylor expansion.

One can envision an amalgated sum of conjectures:



Oddly enough, the fireworks at the top may be easier to deal with than the right hand side. I will sketch the construction of a regulator map $R_q\colon K_2(E)\to \boldsymbol{C}$ ($q=e^{2\pi i\tau}$), and relate in special cases the values of R_q with $L(E/K, 2)$. As an example, consider $E/\boldsymbol{Q}$ with complex multiplication by the full ring of integers $\mathcal{O}_k=\boldsymbol{Z}+\boldsymbol{Z}\tau$, k imaginary quadratic. Assume for simplicity the conductor N of E lies in $\boldsymbol{Z}$. Let χ^{Gross} denote the Grössencharakter associated to E so $L(E/\boldsymbol{Q}, s)=L(\chi^{\mathrm{Gross}}, s)$, and let χ be the corresponding Dirichlet character on $\mathcal{O}_k$ with conductor N, so $\chi^{\mathrm{Gross}}(\mathfrak{p})=\bar{h}\chi(h)$ if $\mathfrak{p}=(h)$. Write

$$\hat{\chi}=\frac{1}{N}\sum_{a,b=0}^{N-1}\chi(a+b\tau)\,e^{2\pi i b/N}.$$

THEOREM. *There exists* $U\in K_2(E)$ *such that*

$$L(\chi^{\mathrm{Gross}}, 2)=\frac{\pi|\mu_k|\hat{\chi}}{i(\mathrm{Im}\,\tau)^2N^4}R_q(U).$$

(More generally, if $N\in k$, $\hat{\chi}$ *is replaced by a more complicated character sum.)*

For any E over a number field K, one might

CONJECTURE. $\mathrm{rk}\, K_2(E)=$order of zero of $L(E/K, s)$ at $s=0$.

The conjecture, of course, presumes analytic continuation of $L(E/K, s)$.

REMARK. There is a basic analogy between $K_{i+1}(k)$ and $K_i(E)$ where k is a field and E is an elliptic curve. In fact, $K_{i+1}(k)$ is contained in the relative K-group $K_i(\boldsymbol{P}^1_k, \{0, \infty\})$ which can be thought of (perhaps not too literally as I do not know if excision holds in this case) either as K_i of the degenerate nodal elliptic curve or as "K_i with compact supports" of $\boldsymbol{G}_m$. To use this dictionary to formulate a Birch–Lichtenbaum–Swinnerton-Dyer conjecture is perhaps premature. A more accessible problem might be to formulate and prove an analogue for $K_1(E)$ of the exact sequence due to Moore

$$K_2(k)\to \coprod_{\text{places of } k}\mu_{k_v}\to\mu_k\to 0.$$

Let E be an elliptic curve over C, A an abelian group. Given divisors $\delta=\sum n_i(a_i)$ and $\delta'=\sum m_j(b_j)$ on E, define

$$\delta^- * \delta' = \sum n_i m_j (b_j - a_i),$$

$$C(E)^* \otimes C(E)^* \to \coprod_{x \in E} Z,\ f \otimes g \mapsto (f)^- * (g).$$

A set theoretic function $D: E \to A$ extends to $D: \coprod_{x\in E} Z \to A$. D is a *Steinberg function* if $D((f)^- * (1-f))=0$ for any $f \in C(E)^*$, $f \neq 0, 1$. A Steinberg function induces a map $K_2(C(E)) \to A$.

Let R be the semilocal ring at $\{0, \infty\}$ on P^1_C, $I \subset R$ the radical. Using the group structure on $P^1 - \{0, \infty\}$ one defines $(f)^- * (g) \in \coprod_{x \in C^*} Z$ for $f \in 1+I, g \in C(P^1)^*$. A function $D: C^* \to A$ is a relative Steinberg function if $D((f)^- * (1-f))=0$ for $f \in 1+I$. Using work of Keune [**10**] one shows that a relative Steinberg symbol induces a map $K_2(R, I) \to A$.

The key transcendental object is the single valued(!) function

$$D(x) = \log|x| \cdot \arg(1-x) - \operatorname{Im} \int_0^x \log(1-t) \frac{dt}{t}$$

first discovered by D. Wigner. The functional properties of $D(x)$ seem unbelievably rich: ((i) and (ii) below are joint work with Wigner)

THEOREM. (i) $D(x) = -D(x^{-1}) = -D(1-x) = -D(\bar{x})$. $D(0)=D(1)=D(\infty)=0$.

(ii) *For* $g \in \mathrm{SL}_2(C)$, *let* $\bar{g} \in \mathrm{SL}_2(C)/B \cong P^1_C$. *Let* $\{\bar{g}_1, \ldots, \bar{g}_4\}$ *denote the cross ratio. Then* $D(\{\bar{g}_1, \ldots, \bar{g}_4\})$ *is a measurable* 3*-cocycle on* $\mathrm{SL}_2(C)$. *If* $V(\bar{g}_1, \ldots, \bar{g}_4)$ *denotes the volume of the geodesic tetrahedron in the Poincaré upper half space with vertices* $\bar{g}_i \in P^1$ *lying at* ∞, *then* $D(\{\bar{g}_1, \ldots, \bar{g}_4\}) = \pm \frac{2}{3} V(\bar{g}_1, \ldots, \bar{g}_4)$.

(iii) $D(x)$ *is a relative Steinberg function and so induces*

$$D: K_3(C) \to K_2(P^1_C, \{0, \infty\}) \to K_2(R, I) \to R.$$

(iv) *Write* $E = C^*/q^Z$ *with* $|q|<1$. *The series*

$$D_q(x) = \sum_{n=-\infty}^{\infty} D(xq^n)$$

converges. D_q *is a continuous Steinberg function on* E *and induces a map* $K_2(E) \to K_2(C(E)) \to R$.

Given a number field k and an embedding $\sigma: k \to C$ one gets $D_\sigma: K_3(k) \to R$. One builds in this way the Borel regulator for K_3. The function $J(x) = \log|x| \cdot \log|1-x|$ is also a relative Steinberg function, although the map on $K_2(R, I)$ factors

$$K_2(R, I) \xrightarrow{\text{tame}} \coprod_{x \in C^*} C^* \xrightarrow{\log||\cdot\log||} R$$

and hence is trivial on $K_3(C)$.

In the elliptic case define

$$J_q(x) = \sum_{n=0}^{\infty} J(xq^n) - \sum_{n=1}^{\infty} J(x^{-1}q^n).$$

Given divisors $(f)=\sum n_i(a_i)$, $(g)=\sum m_j(b_j)$ on E we can choose lifting $(\tilde{f})=\sum n_i(\alpha_i)$, $(\tilde{g})=\sum m_j(\beta_j)$ to divisors on $\boldsymbol{C}^*$ such that $\sum n_i=\sum m_j=0$, $\sum^{n_i}\alpha_i=\sum^{m_j}\beta_j=1$.

THEOREM. *The expression*

$$J_q\{f, g\} = J_q((\tilde{f})^- * (\tilde{g})) = \sum n_i m_j J_q(\alpha_i^{-1}\beta_j)$$

is well defined independent of the choices. Moreover,

$$J_q\{f, 1-f\} = 0$$

so there is an induced map $J_q: K_2(\boldsymbol{C}(E)) \to \boldsymbol{R}$.

Define, finally

$$R_q = J_q + iD_q: K_2(E) \to K_2(\boldsymbol{C}(E)) \to \boldsymbol{C}.$$

Assume now E defined over an arbitrary field k of characteristic 0. Recall that the sequence

$$K_2(E) \to K_2(k(E)) \xrightarrow[\text{symbol}]{\text{tame}} \coprod_{x\in E} k(x)^*$$

is exact. To study the image of R_q we construct elements in Ker (tame) as follows: let f, g be functions on E and assume the divisors (f), (g) are supported on the points of order N of E. Assume for simplicity these points of order N are defined over k. Then there exist $c_i\in k^*$, $f_i\in k(E)^*$ such that

$$S_{f,g} = \{f, g\}^N \cdot \prod_i \{f_i, c_i\} \in \text{Ker (tame)}.$$

R_q is trivial on symbols with one entry constant, so when $k \hookrightarrow \boldsymbol{C}$, $R_q(S_{f,g})$ is well defined. Let ϱ have a pole of order 1 at every $x\in E_N$, $x\neq 0$, and a zero of order N^2-1 at 0. Let $x\in E_N$ and let f_x have a zero of order N at x and a pole of order N at 0. Define $S_x = S_{\varrho, f_x}$. If, for example, $E=\boldsymbol{C}/\boldsymbol{Z}+\boldsymbol{Z}\tau$, one finds

$$R_q(S_{(a+b\tau)/N}) = \frac{(\operatorname{Im}\tau)^2 N^3}{\pi} \sum_{m,n=-\infty}^{\infty}{}' \frac{\sin\left(2\pi((an-bm)/N)\right)}{(m+n\tau)^2(m+n\bar{\tau})}.$$

REMARKS. (i) The $S_{f,g}$ are analogous to cyclotomic units. They are available when certain torsion points of the curve are rational over k. I do not expect they generate Ker (tame) in general.

(ii) The techniques discussed in this report are *ad hoc*. One could try to give a general regulator construction by interpreting the higher K-groups of a variety

as relative algebraic cycles, e.g. $K_1(C) \cong$ picard variety of $\boldsymbol{P}^1_C$ relative to $\{0, \infty\}$. The *Akel–Jacoby* map would associate to these cycle points in a relative *Griffiths intermediate jacobian.* Factoring out by the maximal compact subgroup of this torus yields invariants in a real vector space which frequently inherits a complex structure from Hodge theory.

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Unstable Vector Bundles and Curves on Surfaces

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In this article I will describe some results on the structure of vector bundles over projective varieties. All of these are obtained by investigating the sections of tensor bundles associated to the given bundle. For simplicity we work over the complex field.

1. Consider a projective variety V and a left principal fibre bundle G_γ over V with reductive algebraic group G as fibre. For R an algebraic variety with right G-action we set $R_\gamma = R \times_G G$ (the *associated* fibre bundle). If $h: R \to Q$ is a regular G-morphism of G-spaces then there is a corresponding bundle morphism $h: R_\gamma \to Q_\gamma$. Every linear representation ϱ of G defines a vector bundle E_ϱ over V.

There is a natural correspondence between irreducible representations of the group G and characters χ of a fixed maximal torus T of G. Namely, for each nonzero χ in the character group $\hat{T}$ of T there exists a unique parabolic subgroup $P \subset G$ containing T and such that χ extends to P, with the induced line bundle E on G/P positive. So for every $\chi \in \hat{T}$ we define a bundle $(G/P, E_\chi)$ and a representation ϱ_χ of G on $H^0(G/P, E_\chi) = W_\chi$. Consider in W_χ the subvariety A_χ defined as the closure of the orbit of a vector of highest weight. This is isomorphic to the line bundle $(G/P, E_{\chi^*})$ with the zero-section blown down. As a G-space A_χ consists of two orbits, 0 and $A_\chi \setminus 0$. For each parabolic subgroup $P \subset G$ we define the associated fibre bundle $G/P_\gamma = G_\gamma \times_G G/P$. The principal bundle $\tilde{G}_\gamma$ over G/P_γ induced by the projection $\pi: G/P_\gamma \to V$ has P as a structure group. Every positive character χ on P defines a line bundle L_χ on G/P_γ and a vector bundle $R^0_\pi(L_\chi)$ on V, with $R^0_\pi(L_\chi) = E_\varrho$. Since $R^i_\pi(L_\chi) = 0$ for $i > 0$ we have $H^q(V, E_\varrho) = H^q(G/P_\gamma, L_\chi)$.

EXAMPLE. If S^iE^* is the ith symmetric power of the vector bundle E^* then $H^q(V, S^iE^*)=H^q(P(E), O(i))$, where $P(E)\to V$ is the projectivisation of E.

2. Assume that $G=\mathrm{GL}(n)$. Parabolic subgroups of G are in 1-to-1 correspondence with flags in n-dimensional vector space C^n. For F: $E_{a_1}\subset E_{a_2}\subset\ldots\subset E_{a_k}\subseteq C^n$ the parabolic group P_F is the stabiliser of F. Every positive character of the subgroup $P_F\subset G$ can be written as $\chi=\sum_{i=1}^{n} n_i\chi_i$, where $\chi_i(A)=\det A|_{E_i}$, $\dim E_i=i$ and $n_i>0$ for all $i<n$, $i\in\{a_1, a_2, \ldots, a_k\}$.

If $G=\mathrm{PGL}(n)$ then the parabolic subgroups correspond to proper flags in C^n. The bundles E_ϱ with $\det E_\varrho=0$, where ϱ is an irreducible representation of $\mathrm{GL}(n)$ are associated to representations induced from $\mathrm{PGL}(n)$ via the natural homomorphism $p\colon \mathrm{GL}(n)\to\mathrm{PGL}(n)$. The corresponding characters can be written as $\chi_\varrho=\sum_{i=1}^{n-1} n_i(\chi_i-i\chi_n/n)$ with $n_i>0$ for all i.

Now fix a principal bundle G_γ over V with group $G=\mathrm{GL}(n)$, and let E be the associated vector bundle, corresponding to $-\chi_1$. Then we call a bundle E_ϱ a *cotensor bundle* for E if $\chi_\varrho=\sum n_i\chi_i$ with $n_i>0$. Every cotensor bundle is a direct summand of the tensor power $E^{*\otimes k}$ for some k. We denote by $h(\varrho)$ the number $h(\varrho)=\max\{i, n_i>0\}$, where the cotensor representation ϱ corresponds to $\chi=\sum n_i\chi_i$.

3. If $i\colon C^k\subset C^n$ is a vector subspace we can define for every flag F in C^k the flag iF in C^n. So we have a natural morphism $i\colon G/P_F\to G'/P_{iF}$, where $G=\mathrm{GL}(k)$, $G'=\mathrm{GL}(n)$. The character χ_{a_i} on P_F is induced from the character χ'_{a_i} on P_{iF}, and the line bundles $L_{\chi_{a_i}}$ on G/P_F are also induced. If the character $\chi=\sum n_i\chi_i$ with $n_i>0$ for all $i\leqslant k$ then the bundles L_χ on G/P_F and G'/P_{iF} are positive and the induced homomorphism $i\colon H^0(G/P_{iF}, L_{\chi'})\to H^0(G/P_F, L_\chi)$ is onto.

For a subbundle $i\colon E_1\subset E_2$ we define an imbedding of the flag bundles $i\colon G/P_{E_1F,\gamma}\to G'/P_{E_2F,\gamma'}$ by a similar procedure. For any character χ of P_F which is induced from a χ' on P_{iF} we have $L_{1\chi'}=i^*L_{2\chi'}$. So if $E_{1,\varrho}$ is a cotensor bundle for E_1 then i induces a linear map $i\colon H^0(V, E_{2,\varrho'})\to H^0(V, E_{1,\varrho})$.

Note that for a proper flag $E_{a_1}\subset E_{a_2}\subset\ldots\subset E_{a_m}\subset C^k$ we can define two different proper flags in C^n. The first is iF and the second is $iF\subset C^k\subset C^n$.

4. DEFINITION. A vector bundle E over a projective variety V is called *improper* if E is induced from a bundle E' on some variety V' of smaller dimension by a regular morphism. In the opposite case we say that E is *proper*.

THEOREM 1. *Let V be a projective variety of dimension n, and let E be a vector bundle of rank k over V, with E proper and not containing a trivial subbundle. Suppose that E is a subbundle of a trivial bundle C^n_V, and not contained in any smaller trivial bundle. Then*

(i) *if $N-k<n$ then $H^0(V, S^iE^*)=H^0(V, S^iC^N_V)$;*

(ii) *if $N=k+1$ then for every representation ϱ of G with $h(\varrho)<\min(k, n)$ we have*

$$H^0(V, E_\varrho) = H^0(V, (C^N)_\varrho).$$

PROOF. Set $G=\mathrm{GL}(k)$, $G'=\mathrm{GL}(N)$; for every parabolic subgroup $P_F\subset G$ we have a bundle morphism $G/P_{F,\gamma}\to G'/P_{iF,\gamma'}$ and the projection of the trivial bundle $G'/P_{iF,\gamma}$ onto its fibre G'/P_{iF} so that any line bundle L_χ over $G/P_{F,\gamma}$ is induced by the composite pi from a line bundle on $G'/P_{F,\gamma}$. If pi is surjective with connected fibres then the result follows.

We can interpret the projection of the general fibre of pi onto V as a pull-back of some complete intersection in the Grassmanian $\mathrm{Gr}(N,k)$ under the canonical morphism $h\colon V\to\mathrm{Gr}(N,k)$, and the result follows from the conditions of the theorem.

COROLLARY 1. *Suppose that $i\colon V\to T^n$ is an immersion of a smooth projective variety V into an Abelian variety T^n; assume that $i(V)$ does not lie in any proper Abelian subvariety of T^n, and that it is not a fibre space with Abelian subvariety $T^k\subset T^n$ as fibre. Then*

(i) *if* $\dim V>n/2$, $H^0(V, S^i\Omega^1_V)=H^0(T^n, S^i\Omega^1_{T^n})$ [3];

(ii) *if* $\dim V=n-1$ *then* $H^0(V, T_{V,\varrho})=H^0(T^n, T_{T^n,\varrho})$ *for every cotensor bundle* T_V, *such that* $h(\varrho)<n-1$.

Here the imbedding of T_V into the trivial bundle is induced by the differential *di*.

COROLLARY 2. *Suppose that $i\colon V\to P^n$ is an immersion of a smooth V into P^n. Define the bundle E on V as the extension $0\to O(-1)\to E\to T_V\otimes O(-1)\to 0$ by the class $c_1(O_V(1))\in H^1(V,\Omega^1_V)$. The differential di defines an imbedding of E into a trivial vector bundle. Assume that V is not isomorphic to a projectivised variety $P(\mathscr{F})$, for a coherent sheaf $\mathscr{F}$ of rank $\geqslant 2$ over some V', and assume that $i(V)$ does not lie in a hyperplane of P^n. Then E satisfies the conditions of the theorem, and we obtain:*

(i) *if* $\dim V>n/2$ *then* $H^0(V, S^iE^*)=H^0(V, S^iC^{n+1})$;

(ii) *if* $\dim V=n-1$ *then* $H^0(V, E^*_\varrho)=H^0(V, (C^{n+1})_\varrho)$ *for every cotensor representation* ϱ *with* $h(\varrho)<n-1$.

From (ii) we get the theorem of Brückmann for smooth hypersurfaces of P^n: $H^0(V, T_{V,\varrho})=0$ if $h(\varrho)<n-1$ [4].

5. Suppose that $s\in H^0(V, T_{V,\varrho})$ is a cotensor section. For a surjective morphism $p\colon V'\to V$ we have the pull-back p^*s on V'. If $H^0(V', \Omega^1_{V'})\neq 0$ then V' has some special cotensors which are constructed from holomorphic 1-forms on V'.

CONJECTURE. For every cotensor s on a smooth projective variety V there exists a projective variety V' and a surjective morphism $p\colon V'\to V$ such that the cotensor p^*s on V' is a polynomial expression in holomorphic 1-forms of V'. In other words there should exist a regular morphism $h\colon V'\to T$ to an Abelian variety, and a cotensor s' on T such that $h^*s'=p^*s$.

It can easily be seen that for each cotensor s there exists a k such that the symmetric power s^n of s satisfies this condition for all $n>k$; however, the most interesting problem is to investigate the case for $s\in H^0(V,\Omega^i_V)$ for $i>0$.

6. Assume now that V is a smooth projective surface. As usual K denotes the canonical class of V, and $\chi=c_2(T_V)$ the second Chern class of V. Consider a vector bundle E on V of rank n, and let $c_i=c_i(E)$ be its Chern classes. The following proposition is a sort of converse to a theorem of Kleiman [7].

LEMMA 1. *Assume that* $c_1^2-c_2>0$ *then* $h^0(V, S^iE)+h^0(V, S^iE^*)>ai^{n+1}+b$ *for some constants* $a>0$, b, *and all* i.

The proof is by computing the Euler characteristic of S^iE [1].

REMARK. If $\operatorname{rk} E=2$ and, for some $n>0$, $H^0(V, n\det E)\neq 0$ then $h^0(V, S^iE)>a'i^3+b'$ with $a'>0$.

For $c_1^2-c_2>0$ some positive multiple of the tautological bundle $O(1)$ on the projectivisation $P(E)$ defines a birational map, [6].

7. The group $NS_QV=NS(V)\otimes_Z Q$ has a natural intersection form with signature $(1, n)$. So the subset $\{x^2>0\}$ is divided into two components; we denote by K^+ the component which contains positive elements of $\operatorname{Pic} V$, and $K^-=-K^+$. Every element $L\in\operatorname{Pic} V$ whose class $\chi\in K^+$ has $H^0(V, nL)>an^2+b$ for some $a>0$ and all $n>0$.

For a vector bundle E of rank n define the *reduced* c_2 as the rational number $c_2^0(E)=c_2(E)-(n-1)\,c_1^2(E)/2n$; this invariant is unchanged on multiplying E by some line bundle L.

THEOREM 2. *Assume that* E *is a vector bundle over a smooth surface with* $c_2^0(E)<0$. *Then there exists a bundle* F *and an injective homomorphism* $h: F\to E$ *for which*

$$f=\left(\det F-\frac{\operatorname{rk} F}{n}\det E\right)\in NS_QV$$

lies in K^+, *and* $f^2\geqslant -c_2^0(E)$.

We describe the main steps in the proof.

8. The reductive group G and the standard models A_χ are as in § 1.

THEOREM 3. *Let* ϱ *be any representation of* G *in* $\mathbf{C}^n$, *and let* X *be an affine* G*-invariant subvariety of unstable points in* $\mathbf{C}^n$, *with* $\dim X\geqslant 1$.

Then there exists a regular geometrically surjective G*-morphism* $f: X\to A_\chi$ *for some* χ.

REMARK. The proof for $G=\mathrm{SL}(2)$ follows from Mumford's description of the unstable forms in 2 variables. The general case is contained in [2].

9. DEFINITION. A vector bundle E over V of rank n is *T-unstable* if there exists a representation ϱ of $\mathrm{PGL}(n)$ and a nonzero section $s\in H^0(V, E_\varrho)$ which vanishes at some point of V [2], [12].

LEMMA 2. *If E is T-unstable then there exists a flag of coherent subsheaves $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots \subset O(E)$, locally direct summands outside a finite subset of V, and a vector $(a_1, \ldots, a_k)$ of positive integers such that the line bundle*

$$F(a_1, \ldots, a_k) = \sum_{i=1}^{k} a_i(n \det F_i - n_i \det E)$$

has a nonzero section.

The proof of this lemma follows from Theorem 3 applied by means of the transfer morphisms indicated in § 1 to the models A_χ, together with an explicit description of these models for $G = \mathrm{PGL}(n)$. Remark that every $\mathrm{PGL}(n)$-invariant polynomial on the representation space transfers s into the 0-section of O_V, so that s is unstable at every point.

10. From the inequality $c_2^0(E) < 0$ and Lemma 1 we get that $h^0(V, S^{in}E \otimes i \det E) + h^0(V, S^{in}E \otimes i \det E) > ai^{n+1} + b$. Since both these bundles are constructed from representations of $\mathrm{PGL}(n)$ we obtain the T-instability of E.

We prove Theorem 2 starting from the flag constructed in Lemma 2 using an inductive procedure based on the formula

$$c_2(E) = c_2(F) + c_2(F') + c_1(F)(c_1(E) - c_1(F)) + N$$

where

$$0 \to \mathscr{F} \to E \to \mathscr{F}' \to 0$$

is an exact sequence of sheaves, locally split outside a finite number of points, and $N \geqslant 0$ is a measure of the extent to which $\mathscr{F}'$ is not locally free.

REMARK. Theorem 2 was proved for $V = P^2$ and vector bundles of rank 2 in [**14**]; the general proof can be found in [**2**].

11. DEFINITION (TAKEMOTO–MARUYAMA). A vector bundle E on a surface V is H-unstable for $H \in K^+$ if there exists a subsheaf $\mathscr{F} \subset O(E)$ such that

$$c_1(\mathscr{F}) \cdot H > \frac{\mathrm{rk}\, F}{\mathrm{rk}\, E} c_1(E) \cdot H.$$

Using Lemma 2 and the Nakai–Moishezon ampleness criterion we prove the following proposition:

LEMMA 3. *If a vector bundle E on a surface V is H-unstable for each $H \in NS_R V = NS(V) \otimes_Z \boldsymbol{R}$ in the closure of the cone generated by polarisations then E is T-unstable.*

12. Let X be a smooth curve on the surface V, and let $h\colon E|_X \to F$ be a surjective bundle morphism on X. The sheaf $\mathscr{L} = \ker(hi\colon \mathscr{E} - \mathscr{F})$ is locally free, so corresponds to a vector bundle L [**13**]. The Chern classes $c_i(L)$ are easily computed in terms of those of E, F and the class of X:

$$c_1(L) = c_1(E) - rX, \quad \text{where} \quad r = \mathrm{rk}F, \; n = \mathrm{rk}E,$$

$$c_2^0(L) = c_2^0(E) - \frac{r(n-r)}{2n} X^2 + r\left(\frac{\deg F}{r} - \frac{c_1(E)X}{n}\right).$$

If $E|_X$ is unstable and

$$\frac{\deg F}{r} < \frac{c_1(E)\cdot X}{n}$$

then

$$c_2^0(L) < c_2^0(E) - \frac{r(n-r)}{2n} X^2,$$

and L is unstable if X^2 is large enough. As follows from Theorem 2 L contains a special subsheaf F with corresponding $f \in K^+$ (see § 7). If we consider F as a subsheaf of E we get some information about E.

For simplicity consider the case $\operatorname{rk} E=2$.

LEMMA 4. *For V a smooth surface with $\dim_Q NS_Q V=1$ suppose that the restriction of a rank 2 vector bundle E to a curve X with $X^2>(2c_2^0(E))^2$ is unstable; then E is unstable on V.*

This result generalises to vector bundles of arbitrary rank over any surface.

13. In this section we describe some results on subsheaves of cotensor bundles for the tangent bundle of projective varieties.

THEOREM 4. *Let V be a smooth projective variety, and consider a rank 1 subsheaf $\mathscr{F} \subset \Omega_V^i$; then there exist constants a and b such that for all n*

$$h^0(V, nF) < an^i + b.$$

Furthermore, if for all $n>0$

$$h^0(V, nF) > a_0 n^{i'} + b_0$$

for some $i' \leqslant i$ and constants $a_0>0$ and b_0 then there is a dominant rational map $p: V \to V'$ for some variety V' with $\dim V'=i'$ such that $\mathscr{F}$ lies in the ideal of the exterior algebra $\bigwedge \Omega_V^1$ generated by a rank 1 subsheaf $\mathscr{L} \subset \Omega_V^{i'}$ containing $p^*(\Omega_{V'}^{i'})$.

The proof can easily be reduced to the case $\dim V=i+1$ when F defines a 1-dimensional foliation F_s on V. If s_1 and s_2 are sections of $\mathscr{F} \subset \Omega_V^i$ then the meromorphic function s_1/s_2 is constant on the leaves of F_s, as follows from the fact that $ds_l=0$. By an argument using branched covers of V the same is true for sections of $\mathscr{F}^{\otimes n}$. The theorem follows.

The same result holds for rank 1 subsheaves of the sheaf of logarithmic forms $\Omega^i(\log D)$. In fact by a theorem of Deligne [5] we have that sections of $\Omega^i(\log D)$ are closed forms.

COROLLARY 1. *For V a surface of general type T_V is stable, so that $c_1^2 \leqslant 4c_2$. Moreover T_V is H-semistable for some nonempty cone of polarisations in $NS_R V$.*

COROLLARY 2. *If V is a smooth projective variety with $\operatorname{Pic} V=\mathbf{Z}$ and $K_V \geqslant 0$ then the tangent bundle T_V is stable. All tensors $s \in H^0(V, T_\varrho)$ are locally constant for ϱ any representation of $\operatorname{PGL}(n)$.*

The same is also true if $K<0$ and $-K$ is a generator of Pic V. In both cases we have the inequality

$$\left(c_1^2-\frac{2n}{n-1}c_2\right)\cdot H^{n-2}\leqslant 0,$$

where H is an ample generator of Pic V.

THEOREM 5 (Y. MIYAOKA [8]). *For a surface of general type we have* $c_1^2\leqslant 3c_2$.

The theorem follows from the following assertion, which strengthens Theorem 4 for the case of surfaces:

Let V be a minimal surface of general type, and let $\mathscr{F}\subset S^m\Omega_V^1$ be a rank 1 subsheaf; then $c_1(F)\cdot K_V\leqslant\max(mc_2(V),0)$.

Corollaries 1 and 2 above, and Miyaoka's Theorem 5 can be proved in stronger forms by the differential-geometric methods of S. T. Yau [9], [10].

14. We consider some applications of these methods to the study of curves on smooth projective surfaces.

We define a subset $R\subset A_2(V)$ of the set of algebraic homology classes by the condition that $z\in R$ if there exists a smooth curve X and a birational imbedding $f: X\to V$ such that $f(X)$ realizes z. On R we define the function $g(z)=\inf_X g(X)$, where $g(X)$ is the geometric genus of X. There is an obvious upper bound for $g(X)$, $g'(z)=(z^2+Kz)/2+1$; we are interested in giving a lower bound.

For every nontrivial morphism $f: X\to V$ we define the regular tangent morphism $t_f: X\to P(T_V)$ into the projectivised tangent bundle.

LEMMA 5. *The degree of the restriction of* $O(1)$ *to* $t_f(X)$ *is bounded by* $2g(X)-2$.

Note that $H^0(P(T_V),O(i))=H^0(V,S^i\Omega_V^1)$, so that if $h^0(V,S^i\Omega_V^1)>ai^3+b$ for $a>0$ then for some i $O(i)$ induces a birational imbedding $h_i: P(T)\to P^N$. If B is the proper subset of $P(T_V)$ where h_i is not a regular imbedding then all curves X in V for which $t_f(X)\not\subset B$ have degree $h_i(t_f(X))\leqslant 2g(X)-2$. So all curves such that $g(X)\leqslant g_0$ lie in an algebraic family. The curves with $t_f(X)\subset B$ are tangent to a 1-dimensional foliation on B if B projects surjectively on V; all these curves lie in an algebraic family (of dimension $\leqslant 1$), as follows from the following result:

LEMMA 6. *If* F_s *is a 1-dimensional foliation on a surface* V' *then either*

(i) *there are only a finite number of algebraic curves tangent to* F_s; *or*

(ii) *there is a rational map* $p: V'\to Y$ *of* V' *to a curve* Y *such that* F_s *is tangent to the fibres of* p.

We conclude:

THEOREM 6. *Suppose that for some* $a>0, b$ *and every* i *we have* $h^0(V,S^i\Omega_V^1)>ai^3+b$; *then for every ample class* $h\in NS(V)$ *there exists a constant* $\alpha=\alpha(h)$ *such that*

$$g(z)>\alpha h\cdot z$$

for all but a finite number of $z\in R$.

REMARK. The condition of the theorem is automatically satisfied by any surface of general type with $c_1^2 > c_2$.

Using symmetric tensors with coefficients in a line bundle we can prove the following analogous result for any surface of general type:

THEOREM 7. *Suppose that V is a surface of general type with $\varrho = \operatorname{rank} NS(V) \geqslant 2$. Define the subcone $K_{D,\varepsilon}$ in $NS_R V$ by the equation*

$$x^2 > \alpha_\varepsilon (Kx)^2, \quad \text{where } \alpha_\varepsilon = \left(\frac{3}{4c_2 - c_1^2} - \varepsilon\right).$$

Then the subset of $R \setminus K_{D,\varepsilon}$ defined by $g(z) < \alpha(h) h \cdot z$ is finite, for $h \in K^+$ and suitable $\alpha(h) > 0$ [1], [11].

COROLLARY. *There are only a finite number of curves of fixed geometric genus and negative self-intersection on any surface of general type.*

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Some Applications of Geometric Invariant Theory to Moduli Problems

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The geometric invariant theory of Hilbert and Mumford provides a general method for constructing quasi-projective moduli spaces for certain types of objects in algebraic geometry. After reviewing geometric invariant theory, we will discuss the construction of moduli spaces for surfaces of general type, and for stable bundles on a smooth projective variety.

1. Let G be a reductive algebraic group over an algebraically closed field k, and let V be an n-dimensional representation of G. Assume G acts through $\mathrm{SL}(V)$. Let $x \in V^*$.

DEFINITION 1.1 (Mumford [6]). x is stable (resp. semistable) if the stabilizer of x is finite and the orbit $O^G(x)$ is closed in V^* (resp. $0 \notin \overline{O^G(x)}$).

Unstable is the opposite of stable. Let $\boldsymbol{P}(V)_s$ (resp. $\boldsymbol{P}(V)_{ss}$) denote the image of the stable (resp. semi-stable) points in $\boldsymbol{P}(V)$, the set of hyperplanes in V.

PROPOSITION 1.2. *Let* $X = \mathrm{Proj}\, k[V]^G$. *Then there is a diagram*

$$\begin{array}{ccccc} \boldsymbol{P}(V) & \supseteq & \boldsymbol{P}(V)_{ss} & \supseteq & \boldsymbol{P}(V)_s \\ & & \pi\downarrow & & \downarrow \\ & & X & \supseteq & X_s \end{array}$$

For each $x \in X$, $\pi^{-1}(x)$ *contains a unique orbit of* G *closed in* $\boldsymbol{P}(V)_{ss}$. *If* $x \in X_s$, $\pi^{-1}(x)$ *is exactly an orbit.*

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Let $\lambda: k^* \to G$ be a one-parameter subgroup.

DEFINITION 1.3. $x \in V^*$ is unstable (resp. not stable) with respect to λ if $\lim_{\alpha \to 0} x^{\lambda(\alpha)} = 0$ (resp. $\lim_{\alpha \to 0} x^{\lambda(\alpha)}$ exists in V^*).

PROPOSITION 1.4. *x is unstable (resp. not stable) if and only if x is unstable (resp. not stable) with respect to some nontrivial one-parameter subgroup of G.*

Propositions 1.2 and 1.4 are due to Mumford in characteristic zero, and in general to the efforts of Nagata, Haboush, Seshadri [8], and for $G = \mathrm{SL}(n)$, Procesi and Formanek.

Kempf [3] has shown that if x is unstable, there is a one-parameter subgroup λ so that $x^{\lambda(\alpha)}$ approaches 0 faster than $x^{\mu(\alpha)}$ for any other suitably normalized subgroup μ and that λ is unique up to conjugation. (Rousseau has established a similar result.) This result gives a proof of the stability of the Hilbert point of abelian varieties embedded by complete linear systems.

2. Let X be a subscheme of $\boldsymbol{P}(V)$ with Hilbert polynomial $P_X(m) = \chi(\mathcal{O}_X(m))$. For each $m \in \boldsymbol{Z}^+$, there is a natural map

$$\varphi_{X,m}: S^m(V) = H^0\big(\boldsymbol{P}(V), \mathcal{O}(m)\big) \to H^0\big(X, \mathcal{O}_X(m)\big).$$

Thus we have a map for $m \gg 0$,

$$\psi_{X,m}: \bigwedge^{P(m)} S^m(V) \to \bigwedge^{P(m)} H^0\big(X, \mathcal{O}_X(m)\big) \cong k.$$

Thus $\psi_{X,m}$ can be considered as an element of $\boldsymbol{P}\big(\bigwedge^{P(m)} S^m(V)\big)$, and may be called the mth Hilbert point of X. Let $G = \mathrm{SL}(V)$ which acts on $\bigwedge^{P(m)} S^m(V)$.

Let $\lambda: k^* \to G$ be a one-parameter subgroup. There is a basis $v_1, \ldots, v_n$ of V^* so that $v_i^{\lambda(\alpha)} = \alpha^{r_i} v_i$. If $M = v_1^{j_1} \ldots v_n^{j_n} \in S^m(V)$ is a monomial in the v's, we define $w_\lambda(M) = \sum j_i r_i$. The following is an easy consequence of Proposition 1.4:

PROPOSITION 2.1. *$\psi_{X,m}$ is stable with respect to λ if and only if there are monomials $M_1, \ldots, M_{P(m)}$ in $S^m(V)$ so that their images form a basis of $H^0\big(X, \mathcal{O}(m)\big)$ and $\sum \omega_\lambda(M_i) < 0$.*

The following is the key step constructing a quasi-projective moduli space for surfaces of general type modulo birational equivalence with fixed K^2 and $\chi(\mathcal{O}_X)$ [1].

THEOREM 2.2. *Let X be a surface of general type. For any sufficiently large n, there is an m so that the mth Hilbert point of the n canonical image of X is stable. Further, m and n depend only on K^2 and $\chi(\mathcal{O}_X)$.*

Let v_i and r_i be as above. Let F_i be the subsheaf of $\mathcal{O}(nK)$ generated by $v_1, \ldots, v_i$. We may blow up X, $\pi: \tilde{X} \to X$, so that $\pi^*(F_i) = \mathcal{O}_{\tilde{X}}(D_i)$ is invertible. We define

$$D(i,j) = \tfrac{1}{3}(D_i^2 + D_i \cdot D_j + D_j^2).$$

PROPOSITION 2.3. *For $n \gg 0$ and for each λ, there is a sequence $1=i_1<\ldots<i_l=h^0(X, \mathcal{O}(nK))=L$ so that*

$$\sum_{k=1}^{l} D(i_k, i_{k+1})(r_{i_{k+1}}-r_{i_k}) > r_L n^2 K^2.$$

Propositions 2.1 and 2.3 imply Theorem 2.2 without difficulty. Lower bounds for $D(i,j)$ used to prove Proposition 2.3 involve estimates for $h^1(\tilde{X}, \mathcal{O}(D_i))$. Chow points may be used instead of Hilbert points.

3. Let X be a smooth projective variety of dimension d and let $\mathcal{O}(H)$ be a very ample divisor on X. Let E be a coherent sheaf on X, and let r be the rank of E at a generic point of X. We define $P_E(n)=\chi(E(n))/r$.

DEFINITION 3.1. E is H-stable (resp. H-semistable) if E is coherent torsion free and whenever F is a proper subsheaf of E, then $P_F(n)<P_E(n)$ (resp. $P_F(n)\leqslant P_E(n)$) for $n \gg 0$.

There is an alternate definition which compares the coefficients of n^{d-1} in P_E and P_F. The above is more closely tied to geometric invariant theory than the alternate definition as is seen in:

PROPOSITION 3.2. *E is H-stable (resp. H-semistable) if and only if there is an N so that if $n \geqslant N$, then the map*

$$T_{E,n}\colon \bigwedge^{r} H^0(E(n)) \to H^0(\det E(n))$$

is stable (resp. semistable) with respect to the action of $\mathrm{SL}(H^0(E(n)))$ *on* $\mathrm{Hom}(\bigwedge^r H^0(E(n)), H^0(\det E(n)))$.

Here $\det(E(n))$ is the unique line bundle which agrees with $\bigwedge^r E(n)$ wherever it is locally free.

One can show that any H semi-stable sheaf has a filtration $0=E_0 \subseteq E_1 \subseteq \ldots \subseteq E_k=E$ so that $F_i=E_i/E_{i-1}$ is H stable with $P_{F_i}=P_E$. Set $\mathrm{Gr}\, E=\oplus F_i$. $\mathrm{Gr}\, E$ depends only on E, and E and E' are said to be S equivalent if $\mathrm{Gr}\, E \cong \mathrm{Gr}\, E'$.

THEOREM 3.3. *A coarse moduli space U_s for H stable sheaves with fixed numerical characters exists. For $d \leqslant 2$, U_s is quasiprojective and its compactification consists of H semistable sheaves modulo S equivalence.*

Theorem 3.3 is due to Mumford, Narashiman and Seshadri for $d=1$. For $d=2$, Proposition 3.2 was established and used to prove Theorem 3.3 [2]. For $d>2$, both are due to Maruyama [4]. The main obstacle to establishing the quasi-projectivity of U_s in higher dimensions is that it is unknown if the N in Proposition 3.2 can be chosen to depend only on the numerical characters of E.

The following theorem of Morrison [5] indicates an interesting connection between two different notions of stability.

THEOREM 3.4 *Let X be a curve of genus >1, and E be a bundle on X of large degree. Then E is stable (resp. semistable) if and only if $P(E)$ embedded by $\mathcal{O}_{P(E)}(1)$ has stable (resp. semistable) mth Hilbert point for $m \gg 0$.*

4. One of the interesting questions in geometric invariant theory is the nature of the compactification by semi-stable elements of the moduli space. Theorem 3.3 answers this question for bundles. It has also been answered for the case of curves. A projective curve is stable in the sense of Mumford–Deligne if it has only nodes as singularities and has ample dualizing bundle. For $n \geq 5$, Mumford [7] has shown that the n-canonical image of a curve stable in the sense of Mumford–Deligne has stable Chow point. This result is used to construct a projective moduli space $\overline{\mathfrak{M}}_g$ for stable curves. (F. Knutsen has given a different construction of $\overline{\mathfrak{M}}_g$.)

In general, we have the following definition:

DEFINITION 4.1. $X \subseteq P^n$ is asymptotically stable if the Chow point of X re-embedded by $\mathcal{O}_X(m)$ is stable for $m \gg 0$.

Thus the n-canonical images of Mumford–Deligne stable curves are asymptotically stable, as are n-canonical images of surfaces of general type. An asymptotically stable singular surface has many good properties. For instance, X is reduced and has normal crossings in codimension one. Shah [9] has classified the Cohen–Macaulay singularities which can occur on such an X if the embedding dimension e of the singularity is equal to its multiplicity, or if $e \leq 3$ (provided char $k=0$).

The nature of the singular schemes arising from the compactification of surfaces of general type is unknown. It is hoped that they will be asymptotically stable, or at least have good local and global properties.

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Homogeneous Spaces Associated to Certain Semi-Universal Deformations

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It is believed that the discriminant variety of a semiuniversal deformation of an isolated hypersurface singularity has nice properties, both from the algebro-geometric and homotopy theoretic point of view. Until now, complete results have only been obtained for rational double points. The aim of this lecture is to report on recent progress concerning this question in the case of a unimodular singularity, which supports this belief.

1. The discriminant variety and related notions. We begin with recalling the definition of a semi-universal deformation.

Let X_0 be a complex space with an isolated singular point $x_0 \in X_0$. A *deformation* of the germ (X_0, x_0) is a flat morphism $(\mathscr{X}, x_0) \to (S, s_0)$ of complex spaces together with an isomorphism of (X_0, x_0) with the fibre over s_0. We say that a deformation $(\mathscr{X}, x_0) \to (S, s_0)$ is *semi-universal,* if for any deformation $(\mathscr{X}', x_0) \to (S', s_0')$ of (X_0, x_0) there exist a morphism $\Phi: (\mathscr{X}', x_0) \to (\mathscr{X}, x_0)$ respecting the identification of (X_0, x_0) with the special fibres and a morphism $\varphi: (S', s_0') \to (S, s_0)$ such that the resulting square is cartesian and $d\varphi(s_0')$ is unique with respect to these properties.

Following Grauert [9], semi-universal deformations exist and are unique. In the case we will consider, namely, when X_0 is a complete intersection at $x_0 \in X_0$, this was proved earlier (but in a somewhat weaker form) by Kas–Schlessinger and independently by Tjurina. Then $(\mathscr{X}, x_0)$ and (S, s_0) are smooth and $(\mathscr{X}, x_0) \to (S, s_0)$ is a stable map-germ in the sense of Mather.

For a *good* representative $F: \mathscr{X} \to S$ of the semi-universal deformation, any fibre X_0 has at most isolated singular points, the *discriminant* $D := \{s \in S: X_s$ is

singular} is an irreducible hypersurface in S and the restriction of F to the union of smooth fibres: $\mathcal{X}-F^{-1}(D)\to S-D$ is a C^∞-locally trivial fibre bundle whose typical fibre has the homotopy type of a finite bouquet of n-spheres, where $n:=\dim_C(X_0, x_0)$. We put $S':=S-D$, fix a base point $*\in S'$ and write V for the lattice $H_n(X_*; \mathbf{Z})$ endowed with its intersection form $\langle\ ,\ \rangle$. Then $\pi_1(S', *)$ acts on V via monodromy transformations; the subgroup $\Gamma\subset \mathrm{Aut}(V)$ so obtained is called the *monodromy group*. The group $\pi_1(S', *)$ is generated by simple loops around simple points of the discriminant and the monodromy transformation along such a loop is given by the *Picard–Lefschetz pseudo-reflection:* $s_\delta(x)=x-(-1)^{n(n-1)/2}\langle x, \delta\rangle\delta$ where $\delta\in V$ is a so called *vanishing cycle* which satisfies $\langle\delta, \delta\rangle=(-1)^{n(n+1)/2}+(-1)^{n(n-1)/2}$. So Γ is generated by the s_δ's. As the non-singular part of D is connected, the s_δ's are all in one Γ-conjugacy class. Moreover, the vanishing cycles generate V [4]. Note that for even n, $\langle\ ,\ \rangle$ is symmetric and s_δ is a reflection with respect to $\langle\ ,\ \rangle$ mapping δ to $-\delta$.

Arnol'd introduced an effective way to measure the complexity of the germ (X_0, x_0); define an equivalence relation on $\mathcal{X}$ by putting $x\sim x'$ iff the germs $(X_{F(x),x})$ and $(X_{F(x'),x'})$ are analytically equivalent. Then the *modality* of (X, x_0) is the smallest dimension of an analytic subvariety of $\mathcal{X}$ meeting each equivalence class. The classification of the 0-modular (or *simple*) and 1-modular (or *unimodal*) hypersurface singularities is due to Arnol'd [2]. We are going to discuss the discriminants of these germs. For technical reasons it is convenient to take $n=\dim_C(X_0, x_0)=2$.

2. Rational double points. The two-dimensional simple germs are the rational double points [3], [2]; they are characterized by the fact that $\langle\ ,\ \rangle$ is negative definite. Then Γ must be finite. Since Γ is generated by reflections which are all conjugate in Γ and stabilizes a lattice, Γ has to be a Weyl group of type A, D or E. Each of these occurs [1]. The discriminant admits a very neat description in terms of (V, Γ) which we proceed to describe. First recall that the Γ-invariant expressions in the symmetric algebra of V_Q form a polynomial algebra (this is true for any finite reflection group). So the orbit space V_C^*/Γ is as an affine algebraic variety isomorphic to $C^{\mathrm{rk}(V)}$. The canonical map $V_C^*\to V_C^*/\Gamma$ is a Γ-covering, whose branch locus D_Γ is the image of the union of reflection hyperplanes in V_C^*. The following result was conjectured by Grothendieck and proved by Brieskorn [5]:

THEOREM. *The germs* (S, D, s_0) *and* $(V_C^*/\Gamma, D_\Gamma, 0)$ *are isomorphic.*

A generalization of this theorem to the other rational singularities is due to Lipman [14]. There is also an analogue in the real case [18]. Brieskorn's proof is very computational and has been greatly simplified by Slodowy [24].

Both proofs depend on a detailed analysis of the unipotent variety of a simple Lie group/C. As this has little chance to generalize; we sketch an alternative proof

which generalizes at least to the unimodal singularities. It involves a period mapping which, loosely speaking is part of the period map for $K3$ surfaces.

Choose nowhere vanishing holomorphic forms of maximal rank on both $\mathscr{X}$ and S. Then their "quotient" determines on the nonsingular part of each fibre X_s a holomorphic 2-form $\omega(s)$. So for any $s', s \in S'$ we get a cohomology class $[\omega(s)] \in H^2(X_s; \boldsymbol{C})$. There is a well-defined map $P'\colon S' \to V_{\boldsymbol{C}}^*/\Gamma$ which assigns to $s \in S'$ the Γ-orbit of the class $[\omega(s)]$ displaced along a path in S' connecting s' with $*$. It can be proved [15] that P' extends to a map $P\colon S \to V_{\boldsymbol{C}}^*/\Gamma$ which is a local isomorphism at s_0 and maps (D, s_0) to $(D_\Gamma, 0)$.

Alternatively, we could pass to the unramified Γ-covering $\tilde{S}' \to S'$ of $(S', *)$. The above result comes down to that near s_0, (i) $\tilde{S}' \to S'$ extends in a unique way to a normal Γ-covering $\tilde{S} \to S$ ramifying over D and (ii) the canonical lift $\tilde{S}' \to V_{\boldsymbol{C}}^*$ of $D'P'$ extends to a local isomorphism $\tilde{S} \to V_{\boldsymbol{C}}^*$. This suggests a similar construction in the general case. Let $S_f \subset S$ denote the Zariski open set of $s \in S$ such that X_s has at most rational double points. Then we may form the Γ-covering $\tilde{S}_f \to S_f$ of $(S_f, *)$ which ramifies over $D \cap S_f$. It has the property that $\tilde{S}_f$ is smooth, Γ acts on $\tilde{S}_f$ as a reflection group and the period mapping, defined in the same way as above, extends to a Γ-equivariant mapping $\tilde{S}_f \to V_{\boldsymbol{C}}^*$. A period mapping, thus defined is not unique. But the philosophy is that its "limit behaviour" is independent of the choices involved.

We now consider the unimodal singularities. They fall into three classes: the *simple elliptic,* the *hyperbolic* and the *exceptional* singularities. All are minimally elliptic in the sense of Laufer [13].

3. Simple-elliptic singularities. A *simple-elliptic singularity* is obtained by taking an elliptic curve C, a line bundle l over C of negative degree and collapsing the zero section of l to a point. The result is an affine surface X_0 with an isolated singular point x_0. Following Saito [23]; X_0 is a hypersurface iff $\deg(l) \in \{-1, -2, -3\}$. The form $\langle\ ,\ \rangle$ is then negative on V with kernel $V^\perp$ of rank two. It follows from work of Gabrielov [8] that the set of vanishing cycles in V map onto a root system $R \subset V/V^\perp$ of type $E_{9+\deg(l)}$ and that Γ acts on $V/V^\perp$ through the Weyl group $W(R)$ of R. This yields a semi-direct product decomposition for Γ:

$$0 \to V^\perp \otimes V/V^\perp \xrightarrow{\exp} \Gamma \to W(R) \to 1,$$

where $\exp(a \otimes b)(x) = x + a\langle b, x\rangle$.

In order to describe the discriminant of (X_0, x_0) we construct a certain homogeneous space out of the pair (V, Γ), but where R is allowed to be any irreducible root system. Fix an ordered basis (a, b) of $V^\perp$ and put

$$\Omega = \{\omega \in \operatorname{Hom}(V, \boldsymbol{C})\colon \operatorname{Im}[\omega(b)/\omega(a)] > 0\}/\boldsymbol{C}^*.$$

There is a canonical projection $\pi\colon \Omega \to \mathscr{H}$ (=upper half plane) given by $\omega \mapsto \omega(b)/\omega(a)$. The algebraic group $\Gamma_R := \{g \in \operatorname{Aut}(V_R)\colon g|V^\perp = \mathrm{id}\}$ acts transitively on each fibre of π and gives it the structure of a riemannian affine space.

THEOREM [16]. *There is an (essentially unique) Γ-line bundle $\mathscr{L}$ over Ω such that $\oplus_{k=0}^{\infty}(\pi_*\mathscr{L}^{\otimes k})^{\Gamma}$ is a polynomial $\mathcal{O}_{\mathscr{H}}$-algebra on $\mathrm{rk}\,(V)-1$ generators.*

Spec of this algebra may be described as follows: identify two points in the total space of the dual $\mathscr{L}^*$ if they lie in the same Γ-orbit or when both lie in the zero section and have the same image in $\mathscr{H}$. The resulting space S_Γ is analytic and the canonical map $\Pi\colon S_\Gamma \to \mathscr{H}$ is smooth. Mumford pointed out in a letter to Brieskorn (May '76) that $\mathscr{L}$ naturally turns up along certain rational boundary components of a symmetric domain of type IV. In particular, he observed that S_Γ may be viewed as a partial compactification of Baily–Borel type of the arithmetic quotient ($\mathscr{L}^*$-zero section)/Γ.

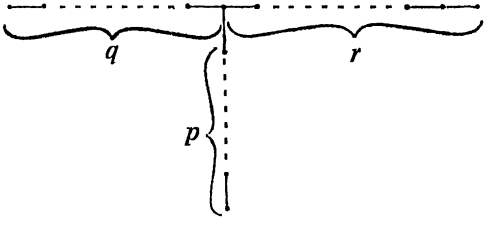
How this pertains to our problem is expressed by the following result, most of which was independently obtained by Pinkham [21].

THEOREM [17]. *Let (X_0, x_0) be a simple elliptic singularity coming from the elliptic curve $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ with $\tau\in\mathscr{H}$ and line bundle l with $\deg(l)\in\{-1,-2,-3\}$. Then for a suitable representative of its semi-universal deformation the fibres of F are affine surfaces (generically del Pezzo surfaces) and there is a Γ-equivariant isomorphism of $\tilde{S}_f$ onto ($\mathscr{L}^*$-zero section) $|\pi^{-1}(U)$ where U is a neighbourhood of τ in $\mathscr{H}$. Moreover the induced map $S_f \to \Pi^{-1}(U)$ is defined by the period mapping and extends to an isomorphism $S\to\Pi^{-1}(U)$. Finally, the induced map $S'\to S_\Gamma$ classifies the mixed Hodge structure on H^2 (fibre).*

In view of Mumford's interpretation, we may think of S_f as an arithmetic quotient and S as a partial compactification of it. This point of view turns out to be also very fruitful for the study of the discriminant of the other unimodal singularities.

The above theorem also enables us to classify the topological types of the fibres of F.

4. Hyperbolic singularities. The hyperbolic germs are characterized by the property that $\langle\ ,\ \rangle$ has only one positive eigenvalue; the kernel $V^\perp$ of $\langle\ ,\ \rangle$ is then of rank one. According to Karras [10], they are just the two-dimensional cusp singularities of embedding dimension three. More explicitly, a hyperbolic germ is given by an equation $x^p+y^q+z^r+xyz$ with $1/p+1/q+1/r<1$ (we take $p\leqslant q\leqslant r$), which we denote by $T_{p,q,r}$. The structure of Γ in this case is also due to Gabrielov [8] and is best described as follows. There exists a basis B of $V/V^\perp$ consisting of projections of vanishing cycles such that the intersection diagram of B is given by



(Here each element B is represented by a vertex and two vertices are (not) connected if the corresponding elements have inner product equal to one (zero).) The reflections $s_\delta: x \mapsto x + \langle x, \delta\rangle \delta$ $(\delta \in B)$ generate a Coxeter group W for which $C := \{x \in V_R / V_R^\perp : \langle x, \delta\rangle \geqslant 0 \text{ for all } \delta \in B\}$ is a chamber. This Coxeter group is hyperbolic iff (p, q, r) is one of the triples $(3, 3, 4)$, $(2, 4, 5)$ or $(2, 3, 7)$. Now Γ acts on $V/V^\perp$ via W, yielding a semi-direct product decomposition

$$0 \to V^\perp \otimes V/V^\perp \xrightarrow{\exp} \Gamma \to W \to 1.$$

The following construction and the subsequent theorem will be in terms of the pair (V, Γ). (For this purpose the assumption that the intersection diagram of B is of type $T_{p,q,r}$ may be weakened to $\langle \delta, \delta'\rangle \geqslant 0$ if $\delta, \delta' \in B$, unless $\delta = \delta'$ for then $\langle \delta, \delta\rangle = -2$ and the group W generated by the s_δ's is irreducible.)

The set $\{x \in C: x = \sum_{\delta \in B} x_\delta \delta \text{ with } x_\delta > 0\}$ is convex nonvacuous and contained in precisely one of the two components of $\{x \in V_R / V_R^\perp : \langle x, x\rangle > 0\}$. We denote this component by H_+. Choose a generator a for $V^\perp$ and put

$$\Omega := \{\omega \in \mathrm{Hom}(V, \boldsymbol{C}): \omega/\omega(a) \text{ maps } H_+ + V^\perp \text{ onto the upper half plane}\}/\boldsymbol{C}^*.$$

We may (and in fact, will) identify Ω through the assignment $\omega \mapsto \omega/\omega(a)$ with the set of $\omega \in \mathrm{Hom}(V, \boldsymbol{C})$ with $\omega(a) = 1$ and $\mathrm{Im}(\omega)$ positive on $H_+ + V^\perp$. In this case, $\mathrm{Im}(\omega)$ factorizes over $V/V^\perp$ and we shall therefore think of $\mathrm{Im}(\omega)$ as an element of $\mathrm{Hom}(V/V^\perp, \boldsymbol{R})$ (which is positive on H_+). It is not hard to see that Ω is a Siegel domain of the first kind. The group Γ acts properly and discretely on Ω. We shall describe a partial compactification of Ω/Γ. Let $B^\vee \subset V_Q / V_Q^\perp$ denote the dual basis of B and put for any $N \in \boldsymbol{R}$

$$\Omega(N) := \{\omega \in \Omega: \mathrm{Im}(\omega)(\delta) \geqq 0 \text{ for all } \delta \in B \text{ and } \mathrm{Im}(\omega)(\varepsilon) > N \text{ for all } \varepsilon \in B^\vee\}.$$

A result of the following kind has been independently conjectured by Mumford.

THEOREM. *The set of Γ-invariant holomorphic functions which converge on (the Γ-orbit of) some $\Omega(N)$ form a convergent power series algebra in* $\mathrm{rk}(V) - 1$ *indeterminates. This algebra defines a smooth partial compactification $\Omega/\Gamma \to (\Omega/\Gamma)^\wedge$, which as a set is obtained by adjoining for each subdiagram B' of B, having no connected component isomorphic to a Dynkin diagram, a connected manifold $S_{B'}$ of dimension $|B - B'|$.*

This partial compactification describes the semi-universal deformation of (some of) the $T_{p,q,r}$ singularities:

THEOREM. *If (p, q, r) is a Dolgachev triple* [2], *then for a suitable representative $F: \mathcal{X} \to S$ of a semi-universal deformation of a $T_{p,q,r}$ singularity the period mapping determines a Γ-equivariant isomorphism of $\tilde{S}_f$ onto the Γ-orbit of some $\Omega(N)$. The induced map $S_f \to \Omega/\Gamma$ extends to a map $S \to \Omega/\Gamma$ which yields a local isomorphism $(S, s_0) \to (\Omega/\Gamma, S_B)$.*

Most likely, this theorem holds for all $T_{p,q,r}$ singularities but we haven't been able to remove the first condition yet. The result also permits us to determine which singularities may occur on a degenerate fibre of F.

5. Exceptional singularities. A normal surface singularity is called exceptional with Dolgachev triple (p, q, r) $(p \leqslant q \leqslant r)$ if it admits a resolution whose exceptional divisor consists of four smooth rational curves E, E_1, E_2, E_3 such that $E \cdot E_i = 1$, $E_i \cdot E_j = 0$ if $i \neq j$, $E^2 = -1$, $E_1^2 = -p$, $E_2^2 = -q$ and $E_3^2 = -r$. In order to satisfy Grauert's criterion we must have $1/p + 1/q + 1/r < 1$. With this condition there are, according to Laufer [**12**], precisely two exceptional germs with Dolgachev triple (p, q, r) (up to analytic isomorphism): one with $\boldsymbol{C}^*$-action and one without. Following Dolgachev [7], for only fourteen of these triples we get hypersurface singularities. We only consider the fourteen hypersurface singularities with $\boldsymbol{C}^*$-action.

For a suitable representative $F: \mathscr{X} \to S$ of the semiuniversal deformation of such a singularity the $\boldsymbol{C}^*$-action on X_0 extends naturally to one on $\mathscr{X}$ and S such that F becomes equivariant. The union $\mathscr{Y}$ of the $\boldsymbol{C}^*$-orbits in $\mathscr{X}$ having x_0 in their closure is of the form $F^{-1}(T)$ where $\mathscr{Y}$ and T are smooth and of codimension one in $\mathscr{X}$ resp. S. Following Pinkham [**21**], the smooth fibres of $F: \mathscr{Y} \to T$ are then affine $K3$ surfaces which admit a common compactification by adjoining at infinity a divisor with normal crossings which consists of nodal curves intersecting each other according to a $T_{p,q,r}$ diagram. In fact, Pinkham shows in [**22**] that $T \cap S'$ is a moduli space for marked $K3$ surfaces endowed with a $T_{p,q,r}$ configuration of nodal curves as described above. Using the global Torelli theorem for $K3$ surfaces, he then obtains a good description for Γ. In Brieskorn's reformulation it amounts to the following: if $V^\vee \subset V_{\boldsymbol{Q}}$ denotes the dual of V, then Γ consists of the $g \in \mathrm{Aut}\,(V)$ which act trivially on $V^\vee/V$ and leave each of the two components of the space of oriented positive 2-planes in $V_{\boldsymbol{R}}$ invariant.

Pinkham's compactification of X_* determines a primitive embedding of V in the 2nd homology lattice L of a $K3$ surface (following Nikulin [**19**] such an embedding is unique) and a distinguished base B of the orthogonal complement of V in L (the elements of B are the classes of the nodal curves). We say that a set of linearly independent elements, $B' \subset L$ *disturbs the polarization* if for any $\delta, \delta' \in B$ we have $\langle \delta, \delta' \rangle \geqslant 0$ unless $\delta = \delta'$: $\langle \delta, \delta \rangle = -2$, each element of B is a positive linear combination of the elements of B' and $B \nsubseteq B'$. (If $B' \subset L$ disturbs the polarization and consists of algebraic cycles, then at least one of the elements of B is necessarily represented by a reducible curve.)

We now let Ω denote one of the (two) components of $\{\omega \in \mathrm{Hom}\,(V, \boldsymbol{C}): \langle \omega, \omega \rangle = 0$ and $\langle \omega, \bar{\omega} \rangle > 0\}$. This is the total space of the canonical $\boldsymbol{C}^*$-bundle of a symmetric domain of type IV. We further put $\Omega' = \{\omega \in \Omega: \omega | V \cap \boldsymbol{Z} \cdot B'$ is nonzero for any subset $B' \subset L$ which disturbs the polarization$\}$. The difference $\Omega - \Omega'$ is a union of hyperplanes which is locally finite.

THEOREM. *The period mapping for* $K3$ *surfaces determines an isomorphism* $\tilde{T}_f \to \Omega'$.

The main ingredients of the proof of this theorem are Laufer's characterization of rational double points [11] and the local Torelli theorem for $K3$ surfaces. On the whole it is fairly elementary (we do not use the global Torelli theorem). Still lacking is a description of the partial compactification $\Omega'/\Gamma \cong T_f \subset T$. There is a canonical extension $\Omega/\Gamma \to T$, but this map blows up over the set of $t \in T$ where X_t contains an exceptional singularity. It is almost certain that we must add the classifying space of the mixed Hodge structure on H^2 (fibre less its singular points) for each occurring fibre type. For the three cases of lowest dimension (where $\Omega = \Omega'$), Brieskorn [6] verified that $\Omega/\Gamma \subset T$ is topologically of Baily–Borel type.

Added in proof. The theorems in sections 3 and 4 describing partial compactifications admit a very natural common generalization in the framework of generalized root systems (in the sense of Kac and Moody). Details are forthcoming.

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Young Diagrams, Standard Monomials and Invariant Theory

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The topics I want to treat have various sources: (1) the fundamental theorems of invariant theory of the classical groups, (2) the representation theory of classical groups, (3) the geometry of the Grassmann and flag varieties.

The initial interest in these questions came for me from the solution given by Formanek and myself, [8], of Mumford's conjecture for the linear group, [25], obtained independently from the general solution due to Haboush [9].

Our approach consisted in relating the geometric reductivity of the group $\mathrm{Gl}(V)$ with a property of the symmetric group S_m acting on the tensor space $V^{\otimes m}$.

In fact, in characteristic zero, one classical approach to the representation theory of $\mathrm{Gl}(V)$, and in particular to the proof that it is linearly reductive, is the following [27], [29]:

The algebras of operators spanned by the symmetric group and by $\mathrm{Gl}(V)$ on $V^{\otimes m}$ are each the full centralizer of the other. In characteristic zero then one can apply the theory of semisimple algebras.

In positive characteristic, if we consider the algebra Σ_m spanned by S_m on $V^{\otimes m}$, we have that Σ_m is no more semisimple but nevertheless the limit $\Sigma = \lim_{m\to\infty} \Sigma_m$ is semisimple in the sense of Jacobson and this suffices to prove the geometric reductivity of $\mathrm{Gl}(V)$.

This analysis led naturally to the question of extending the classical relations between $\mathrm{Gl}(V)$ and S_m to all characteristics; in particular to prove that $\Sigma_m = \mathrm{End}_{\mathrm{Gl}(V)}(V^{\otimes m})$.

It was well known that this equality is a simple consequence of a statement in invariant theory, the so-called first fundamental theorem, to which it is equivalent

in characteristic zero. Similar questions arise for the other classical groups which require a similar (or sometimes more complicated) analysis [29].

The relation with invariant theory comes from the identification $\mathrm{End}(V^{\otimes m}) \cong (V^{\otimes m} \otimes V^{*\otimes m})^*$ and hence the identification of $\mathrm{End}_{\mathrm{Gl}(V)}(V^{\otimes m})$ with the multilinear invariant polynomials in m vector and m covector variables.

It turns out, in positive characteristic, that it is possible to study first all the invariant polynomials and then deduce the form of the multilinear ones.

1. Fundamental theorems of classical invariant theory. Let us state, in a form slightly different from the usual one, the fundamental theorems of invariant theory of the classical groups, (see [29] for the theory in char 0).

(i) $G = \mathrm{Gl}(n, k)$. Consider multiplication of matrices $\pi: M_{k,n} \times M_{n,h} \to M_{k,h}\{(A, B) \to AB\}$, with image the variety V_n of $k \times h$ matrices of rank $\leqslant n$.

FIRST FUNDAMENTAL THEOREM. *The coordinate ring of V_n is the ring of invariants of G acting on $M_{k,n} \times M_{n,h}$ by $X \cdot (A, B) = (AX^{-1}, XB)$.*

SECOND FUNDAMENTAL THEOREM. *The ideal of functions on $M_{k,h}$ vanishing on V_n is generated by the $n+1 \times n+1$ minors.*

We should remark that there exists now in characteristic 0 a beautiful theory of the higher syzygies due to A. Lascoux [15], [22], [23]. Similar theorems hold for the other classical groups.

These theorems, with the exception of the very last, are all classical in characteristic zero and can be found in H. Weyl's book [29].

The second fundamental theorems and various qualitative results on the determinantal varieties have been the object of intensive study by many authors both in positive and zero characteristics [10], [11], [21], [26].

In a joint paper with De Concini [5] we attacked all these problems in the spirit of Igusa's proof of the projective normality for the Grassmann variety [14], again a classical theorem of invariant theory for $\mathrm{Sl}(n, K)$. His approach was through a careful analysis of the projective coordinate ring of the Grassmann variety. Such ring, as the one for the flag variety, were always fundamental objects in invariant theory [1] (the classical primary covariants of the Capelli–Deruyts expansion), the explicit bases (standard bases) for such rings were popularized by Hodge [12], [13] (although they seem to have been known to Young [30]).

They give, in algebra, very explicit descriptions of the irreducible representations of the linear groups and the symmetric group (in characteristic zero), in geometry a very thorough understanding of the cellular decomposition of the Grassmann and flag varieties (by Schubert cells).

In char 0 the theory is based on the fact that the irreducible polynomial representations of $\mathrm{Gl}(V)$ can be indexed, as $L_\sigma(V)$, by partitions σ. One has the classical

plethysm formulas:

$$S[V\otimes W] = \bigoplus_{\sigma} L_\sigma(V)\otimes L_\sigma(W),$$

$$S[S^2V] = \bigoplus L_\sigma(V),\ \sigma \text{ has even columns},$$

$$S[\Lambda^2 V] = \bigoplus L_\sigma(V),\ \sigma \text{ has even rows}$$

(nevertheless the space $L_\sigma(V)$ is defined over Z!).

2. Standard bases. If $Y=(y_{ij})$ is an $n\times m$ matrix, we indicate by $p=(i_k\dots i_2 i_1|j_1 j_2\dots j_k)$ the determinant of the minor with rows i_t's, columns j_u's. If $p_1, p_2, \dots, p_k$ are minors of size $\sigma: h_1\geqslant h_2\geqslant\dots\geqslant h_k$, we display their product M as a double tableau with rows the p_i's. We say that M is a standard monomial (of shape σ) if the indices are strictly increasing on each row, non decreasing on each column, (separately on the right and on the left) [4], [7]. A similar definition holds for Y symmetric or antisymmetric.

THEOREM. *The standard monomials (in each case) are a Z basis of $Z[y_{ij}]$.* (Cf. [4], [7], [12].)

The theorems stated are very convenient to study the coordinate rings of the various determinantal varieties appearing in the fundamental theorems of invariant theory, in fact such theorems can be proved in a characteristic free way [5], precisely by using such standard bases.

In fact the previous form of the theorems contains explicit algorithms, by use of quadratic equations, to express a non standard product in terms of standard ones.

To a partition $\sigma: k_1\geqslant k_2\geqslant\dots\geqslant k_r$ associate the dual partition $\check{\sigma}: h_1\geqslant h_2\geqslant\dots\geqslant h_l$ with $h_i=\{\#j|k_j\geqslant i\}$ and a sequence $\gamma_i(\sigma)=\sum_{j\geqslant i} h_j$; then set $\sigma\geqslant\tau$ if $\gamma_i(\sigma)\geqslant\gamma_i(\tau)$ for all i.

A filtration of $Z[y_{ij}]$ is defined setting A_σ=span of all standard monomials of shape $\geqslant\sigma$. It has the following geometric interpretation [3]: A_0 is exactly the ideal of functions vanishing, for each i, on the variety P_i of matrices of rank i to order $\geqslant\gamma_{i+1}(\sigma)$.

The graded space of this G invariant filtration has again the direct sum decomposition as in the Plethysm formulas.

3. Admissible pairs. More or less at the same time that these standard bases were studied for the invariant theory of classical groups, Musili [26] and Seshadri [28] analyzed the relationship between standard bases and Schubert cells having in mind a better understanding of the vanishing theorems (proved in general by Kempf [16]; see also [17]) for the cohomology of line bundles in the positive chamber on a variety G/P, G a reductive group, P a parabolic subgroup.

If the parabolic P is associated to a fundamental weight ω, with L_ω the corresponding line bundle, the purpose of the analysis is first of all to understand $H^0(G/P, L_\omega^m)$. This was done, first of all, when ω is minuscule. In this case one has

a basis of $H^0(G/P, L_\omega)$ by extremal weight vectors which index the Bruhat cells of G/P.

The face ordering of Bruhat cells induces a partial ordering on these generalized Plücker coordinates which also satisfy quadratic equations, so that:

THEOREM. *The standard monomials of degree m in such coordinates are a basis of $H^0(G/P, L_\omega^m)$.*

This theorem includes, by suitable interpretation, the standard basis theory for $S[V\otimes W]$ and $S[\Lambda^2 V]$.

The case $S[S^2 V]$ can be interpreted as a theorem on $H^0(G/P, L_\omega^m)$ (G the symplectic group) but it is related to a non minuscule weight.

From this example Lakshmibai, Musili and Seshadri have been able to formulate [20] and prove [19] a general theorem. The hypothesis is that ω is a "classical weight", this means that intersecting a Schubert variety (closure of a Bruhat cell) with the hyperplane class one obtains the Schubert varieties σ_i faces of σ with multiplicity $\leqslant 2$. We can draw a diagram of the ordered set of Bruhat cells, with a double bond each time that the intersection multiplicity is 2.

One defines an admissible pair of cells as one which can be joined by a sequence of double bonds. The pairs (τ, τ) are considered admissible and called trivial pairs.

THEOREM. (LAKSHMIBAI, MUSILI, SESHADRI [19].)

(a) *There is a basis $P_{\tau,\sigma}$ of $H^0(G/P, L_\omega)$ indexed by admissible pairs. $P_{\tau,\sigma}$ is a weight vector of weight $-\frac{1}{2}(\tau(\omega)+\sigma(\omega))$.*

(b) *Define a product $P_{\tau_1\sigma_1}P_{\tau_2\sigma_2}\dots P_{\tau_k\sigma_k}$ to be standard if $\tau_1 \geqslant \sigma_1 \geqslant \tau_2 \geqslant \sigma_2 \geqslant \dots \geqslant \tau_k \geqslant \sigma_k$. The standard monomials of degree m are a basis of $H^0(G/P, L_\omega^m)$.*

A similar but more complicated analysis gives, for classical groups, standard bases of $H^0(G/B, L)$, B a Borel subgroup and L any positive line bundle.

4. Combinatorial theory of invariant ideals. We go back to the decomposition $A = S[V\otimes W] = \oplus L_\sigma(V)\otimes L_\sigma(W)$. The module $M_\sigma = L_\sigma(V)\otimes L_\sigma(W)$ is irreducible under $G = \mathrm{Gl}(V)\times \mathrm{Gl}(W)$. Any invariant subspace I of A is thus of the form $\oplus_{\sigma\in\mathscr{I}} M_\sigma$ for some set $\mathscr{I}$ of partitions. In particular one may study G invariant ideals; this has been accomplished in joint work with De Concini and Eisenbud [3]. We summarize the results.

DEFINITION. A set $\mathscr{I}$ of partitions is an "ideal" if: $\sigma\in\mathscr{I}$ and $\tau \supseteq \sigma$ implies $\tau\in\mathscr{I}$. (If $\sigma: k_1 \geqslant k_2 \geqslant \dots \geqslant k_r$, $\tau: m_1 \geqslant \dots \geqslant m_r$ we say $\tau \supseteq \sigma$ if $m_i \geqslant k_i$ for all i.)

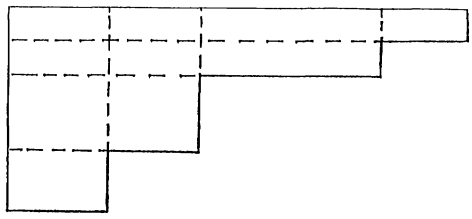
THEOREM. *The decomposition $I = \oplus_{\sigma\in\mathscr{I}} M_\sigma$ establishes a 1–1 correspondence between G-invariant ideals of A and "ideals" of partitions.*

DEFINITION. $\sigma\cdot\tau$ is the partition having as columns the sum of the corresponding columns.

DEFINITION. (a) An "ideal" $\mathscr{I}$ of partitions is prime if $\sigma\tau\in\mathscr{I}$, $\sigma\notin\mathscr{I}$ implies $\tau\in\mathscr{I}$.
(b) $\sqrt{\mathscr{I}}=\{\sigma|\sigma^k\in\mathscr{I} \text{ for some } k\}$.
(c) $\mathscr{I}$ is primary if $\sigma\tau\in\mathscr{I}$, $\sigma\notin\sqrt{\mathscr{I}}$ implies $\tau\in\mathscr{I}$.

THEOREM. *The* 1–1 *correspondence previously given preserves the notions of prime, primary and radical.*

As an example let us define I_σ as the (G-invariant) ideal generated by M_σ. I_σ corresponds to the "principal ideal" of partitions $(\sigma)=\{\tau|\tau\supseteq\sigma\}$. Then I_σ has a nice primary decomposition: e.g.

$\sigma =$ 

$\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the diagrams of maximal rectangles in σ, then: $I_\sigma=I_{\sigma_1}\cap I_{\sigma_2}\cap I_{\sigma_3}\cap I_{\sigma_4}$.

THEOREM. (i) $A_\sigma=I_{k_1}I_{k_2}\ldots I_{k_r}$ $(\sigma: k_1\geqslant k_2\geqslant\ldots\geqslant k_r)$.
(ii) $A_\sigma=I_1^{(\gamma_1(\sigma))}\cap I_2^{(\gamma_2(\sigma))}\cap\ldots\cap I_n^{(\gamma_n(\sigma))}$
(where $P^{(m)}$ *means the symbolic power) is a primary decomposition.*
(iii) A_σ *is the integral closure of* I_σ.

THEOREM. *An integrally closed G-invariant ideal is of the form* ΣA_{σ_i} *with the restriction:*
If $\gamma(\tau)\geqslant$ *convex combination of* $\gamma(\sigma_i)$, *then* $\gamma(\tau)\geqslant$ *some* $\gamma(\sigma_i)$.

We want to mention a last result which can be obtained by using all the ingredients of the theory [4]: Let $I_k^{(n)}$ denote the nth symbolic power of I_k. The algebra $B=\oplus_n I_k^{(n)}/I_k^{(n+1)}$ has a very explicit theory of standard monomials and:
(i) B is a finitely generated algebra;
(ii) B is normal and Cohen–Macaulay;
(iii) B is the ring of global functions on the normal bundle (in the affine space of matrices) of the variety of matrices of rank $k-1$.

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Vector Bundles over Algebraic Curves

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1. The subject of vector bundles over curves goes back, as much else in algebraic geometry, to A. Weil. In his famous *Generalization des fonctions abeliennes* [**18**] which appeared in 1938, he was principally motivated in his study by the possibility of creating a "non-abelian class field theory" for function fields. From this point of view, Drinfeld's recent solution of the Langlands conjecture and the Ramanujan conjecture is one of the high points. In [**18**], Weil realized the importance of vector bundles obtained from *unitary representations* of the fundamental group. He was no doubt puzzled by the fact that the equivalence classes of unitary representations of the fundamental group have the same number of parameters as a generic vector bundle, but is, on the other hand, a compact variety. However, he remarks that every vector bundle ("without restriction") does not necessarily come from a unitary representation.

2. Despite Atiyah's results on vector bundles over elliptic curves [**2**] and a slight dent on the genus 2 case [**1**], matters stood still more or less until Geometric Invariant Theory gave a clue to Mumford [7] why all bundles are not classifiable and more significantly, what bundles can be classified, in the sense of providing a variety whose points correspond to isomorphism-classes of vector bundles. These were called *stable vector bundles* by him after his concept of stability of points under action by reductive groups. (See Gieseker's talk in these Proceedings.) If E is a vector bundle of rank $n>0$, let us denote by $\det E$ the nth exterior power of E. Now $\det E$ is a line bundle and hence has the notion of a degree, namely the sum of multiplicities of zeros or poles, of any of its nonzero meromorphic sections (which exist). We will write $\deg E=\deg(\det E)$ and $\mu(E)=\deg E/\mathrm{rk}\, E$. A *stable*

bundle E is a vector bundle such that for every proper subbundle F, we have $\mu(F)<\mu(E)$.

3. Mumford [7] showed that the equivalence classes of stable vector bundles of a given topological type form a quasiprojective variety with reasonable universal properties. These results were completed by Seshadri [15] in an essential detail which we will now describe. A vector bundle E on X is semistable if every subbundle F satisfies $\mu(F)\leqslant\mu(E)$. If E is any semistable vector bundle with $\mu(E)=\mu$, then it is easy to see that there exists a flag

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E \tag{3.1}$$

of E with $\mu(E_i)=\mu$ and E_i/E_{i-1} stable for $i=1, \ldots, r$. Although such a flag is not necessarily unique, we nevertheless have, as in Jordan–Hölder

THEOREM. *$\sum E_i/E_{i-1}$ is uniquely determined by E.*

Let us call this bundle Gr E. We say

(3.2) E is *S-equivalent* to E' if Gr $E\sim$Gr E'. Notice that if E is stable, S-equivalance is the same as isomorphism. Then Seshadri proved that the S-equivalence classes of semistable vector bundles of fixed rank n and degree d form a projective variety $M(n, d)$. When $n=1$, this is the classical Jacobian.

4. I must point out here that a stable bundle E is *indecomposable* in the sense that E is not a direct sum of proper subbundles F_1 and F_2, for, we cannot have $\mu(F_1)<\mu(E)$ and $\mu(F_2)<\mu(E)$ since $\deg E=\deg F_1+\deg F_2$. On the other hand, a semistable vector bundle could well be decomposable. Secondly, if the rank n and degree d are coprime, then there is no difference between the requirements $\mu(F)<\mu(E)$ and $\mu(F)\leqslant\mu(E)$ so that the Mumford variety is itself projective.

5. We will now discuss in what sense the structures on $M(n, d)$ are natural. If T is any variety and E is a vector bundle on $T\times X$ with $E/t\times X$ semistable of rank n and degree d for every $t\in T$, then the map $t\mapsto$(S-equivalence class of $E/tx X$) of T into $M(n, d)$, is a morphism. Actually, one might also like to get a vector bundle $P/T\times X$ with $P/t\times X$ a semistable bundle in the S-equivalence class of t, for every $t\in M(n, d)$. In the case of Jacobians, such a bundle does exist and is known as the Poincaré bundle. However, in general this turns out to be too optimistic. I showed [13] that actually the existence of such a bundle is equivalent to the assumption that n and d are coprime. Thus if n and d are coprime, $M(n, d)$ is a projective, nonsingular variety which is the solution of an obvious universal problem. Incidentally, except for the case $n=2$, $d\equiv 0 \pmod 2$ and the curve is of genus 2, $M(n, d)$ is nonsingular if and only if n and d are coprime [8]. Since bundles given by unitary (resp. irreducible unitary) representations of the fundamental group (or other Fuchsian groups) can be easily proved to be semistable (resp. stable), it can be deduced [11] that $M(n, d)$ is the same as the space Weil had in mind and one may say that the puzzle is satisfactorily solved.

6. Regarding a study of these varieties, consider the morphism $\det: M(n, d) \to M(1, d)$, the latter being the classical Jacobian variety. This is actually a fibration and hence one might restrict oneself to a study of the fibre $S(n, \xi)$ over $\xi \in M(1, d)$. Since the dependence of $S(n, \xi)$ on ξ is minimal one might also write $S(n, d)$ for this fibre.

7. A crucial advantage in the point of view that $M(n, d)$ is a space of semistable bundles rather than unitary representations of a Fuchsian group is that the construction works over fields of any characteristic. In particular, if X is defined over a finite field F_q, one might expect (at least after replacing F_q by a finite extension and when n and d are coprime) that $M(n, d)$ is also defined over F_q and that the F_q-rational points of $M(n, d)$ are the same as F_q-rational vector bundles over X. In this regard, there is an interesting melange which uses the Weil conjectures for $S(n, \xi)$, the computation of Tamagawa measure for $\mathrm{SL}(n)$ over function fields and the topology of $S(n, \xi)$. This was pointed out to us several years ago by Weil and has been exploited by G. Harder, M. S. Narasimhan, U. V. Desale and myself [5], [6]. Indeed, the fact that the Tamagawa measure is 1 can be interpreted to mean that the expression

$$\sum_{\det E = \xi} \frac{q-1}{\#\operatorname{Aut} E}$$

where the summation is taken over all F_q-rational bundles over X with fixed determinant, is actually $q^{\dim S(n,d)} \zeta_X(2) \ldots \zeta_X(n)$, where ζ_X is the zeta function of X. Note that in the above expression, bundles E with no nontrivial automorphisms are counted only once, so that the contribution from the stable bundles alone would give the zeta function of $M(n, d)$. On the other hand, the nonsemistable bundles can be written uniquely as extensions of semistable bundles of lower rank of a definite type and this enables one to get a recurrence formula for the zeta function of $M(n, d)$.

8. If $S(n, \xi)$ is considered the nonabelian Jacobian–Weil calls it the *hyper Jacobian*—then one might ask for analogues for the classical theorems on Jacobians. For example, is the Torelli theorem valid? In other words, if the hyper Jacobians of two curves are isomorphic, are the curves themselves isomorphic? (By the way, since $S(n, \xi)$ can be proved to have a unique polarisation, one need not talk of polarised isomorphisms). The answer is yes, as first noticed by Tjurin [17]. But the surprising fact found by M. S. Narasimhan and myself [9] is that any deformation of $S(n, \xi)$ arises from that of the curve. This is in marked contrast with the classical case, where the moduli $\mathcal{M}$ of curves interpreted as principally polarised Abelian varieties satisfying some conditions (such as the Matsusaka criterion) is a subvariety of the Siegel Half space $\mathcal{H}$ modulo the modular group Γ, of high codimension.

9. As far as the generalisation of theta functions to $S(n, \xi)$ is concerned (if, by that one means a study of linear systems on $S(n, \xi)$), not much is known. Unlike

the case of the Jacobian (indeed, any abelian variety) where the space of theta functions is a representation space for a rather simple nilpotent group (a Heisenberg group), the only group that operates on linear systems of $S(n, \xi)$ is the group of n-division points of the Jacobian. The only other "structure" of the variety $S(n, \xi)$ that has been exploited is the Hecke transformation at a place of the curve. This consists in changing the fibre of a vector bundle at a given point to get a new vector bundle. Although the correspondence is "non-flat", this has proved to be a good tool in our proof of the theorem mentioned in § 8 above, and also in the construction of a desingularised model for $S(2, 0)$ [**10**]. As a matter of fact, the latter is a rather difficult exercise in the flattening of the Hecke correspondence mentioned above. For an alternate description, see Seshadri [**16**].

10. The varieties $S(2, \xi)$ have been explicitly determined [**5**] in the case when X is hyperelliptic. Since this is a very simple description, I shall give it in full. Consider in P^{2g+1} the equations

$$\sum X_i^2 = 0, \quad \sum \lambda_i X_i^2 = 0$$

where λ_i's are distinct scalars determined by X. The linear subspaces of dim $(g-2)$ contained in both the quadrics above, form a variety isomorphic to $S(2, 1)$. I have obtained further results recently [**14**] interpreting linear subspaces of other dimensions contained in these quadrics as i-invariant "spin bundles of odd type" over X. In particular, the intersection of quadrics is itself a moduli space of some kind over X. This was proved by Newstead [**12**] and Narasimhan and Ramanan [**8**] in the case of a curve of genus 2. Curiously, in a short note, Weil [**19**] while verifying his famous conjectures for the intersection of quadrics, raises the question of the relationship between the pencil of quadrics and the hyperelliptic curve.

11. Besides the intrinsic interest of the study of vector bundles, it may be pointed out that a deeper understanding of these varieties have thrown light on apparently unrelated questions. One such has been mentioned in § 10 above. The Schottky relations defining $\mathcal{M}$ (at least as a component) in $\mathcal{H}/\Gamma$ with the notation in § 8 and the behaviour of the 2θ-linear system on the Jacobian where θ is the principal polarisation are examples of this kind. A particularly interesting result due to Atiyah and Bott [**3**] ties the moduli of vector bundles with the Yang–Mills fields and actually yields topological results, making fascinating interconnection between algebraic and differential geometry.

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Classification of Algebraic Manifolds

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0. Introduction. By the classification theory, we mean the birational classification of algebraic manifolds which is the generalization of the Enriques classification of algebraic surfaces. Namely, we classify all algebraic manifolds of a fixed dimension into a finite number of classes and study general properties of manifolds in each class. It also includes the numerical characterization of special algebraic manifolds. For example, if S is a surface with $\varkappa(S)=-\infty$, $q(S)=0$, then S is birationally equivalent to $\boldsymbol{P}^2$ and if $\varkappa(S)=0$, $q(S)=2$, then S is birationally equivalent to an abelian surface.

The classification of algebraic surfaces was done by Italian algebraic geometers, especially Enriques and Castelnuovo more than sixty years ago. For the classification of higher dimensional manifolds, in spite of several efforts, except few results, there was no essential progress until the middle of the 1960s. In 1964 Hironaka [2] proved the resolution theorem of singularities and rational mappings and gave us the foundation of the birational geometry. Kodaira [5] studied the structure of analytic surfaces and succeeded to generalize the classification of algebraic surfaces to that of analytic surfaces. Inspired by Kodaira's work, in 1965 Kawai [4] succeeded to prove the theorem that a compact complex manifold of dimension 3 whose meromorphic function field is of transcendence degree 2 has a structure of fibre space over an algebraic surface whose general fibres are elliptic curves. The result of Kawai is of nonalgebraic nature. But this was the first general statement concerning the structure of higher dimensional manifolds and his success encouraged us to build the classification theory.

The first definite step toward the classification theory was taken by Iitaka [3] in 1971. He introduced the notion of Kodaira dimension and proved the funda-

mental theorem for pluricanonical fibrations. The main purpose of the present paper is to report the recent development in this direction.

In the present paper, all algebraic varieties are assumed to be complete, irreducible and defined over the complex number field $\boldsymbol{C}$. A non-singular algebraic variety is called an algebraic manifold. By a fibre space $\varphi: V \to W$ of algebraic varieties, we mean that φ is a surjective morphism between algebraic varieties and general fibres of φ are connected.

1. Kodaira dimension. Let V be an algebraic manifold and K_V the canonical line bundle (or a canonical divisor). For a positive integer m, we put $P_m(V) = h^0(V, \mathcal{O}(mK_V))$ and call it the m-genus of V. $P_1(V)$ is often written as $p_g(V)$ and called the geometric genus of V. Put $N(V) = \{m \geqslant 1 | P_m(V) \geqslant 1\}$. Assume $N(V) \neq \emptyset$. Then, for each integer $m \in N(V)$, we define a rational mapping (the mth *canonical mapping*) as

$$\begin{array}{ccc} \Phi_{mK}: V & \to & \boldsymbol{P}^N \\ \cup & & \cup \\ z & \mapsto & (\varphi_0(z): \varphi_1(z): \ldots : \varphi_N(z)) \end{array}$$

where $\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$ is a basis of $H^0(V, \mathcal{O}(mK_V))$ and $\boldsymbol{P}^0$ is, by definition, a point. The *Kodaira dimension* $\varkappa(V)$ of V is defined by

$$\varkappa(V) = \begin{cases} \max\limits_{m \in N(V)} \dim \Phi_{mK}(V), & \text{if } N(V) \neq \emptyset, \\ -\infty, & \text{if } N(V) = \emptyset. \end{cases}$$

Thus $\varkappa(V)$ takes one of the values $-\infty, 0, 1, \ldots, \dim V$. $P_m(V)$ and $\varkappa(V)$ are birational invariants. Therefore, for a singular algebraic variety V, the m-genus $P_m(V)$ and the Kodaira dimension $\varkappa(V)$ of V are defined by

$$P_m(V) = P_m(V^*), \quad \varkappa(V) = \varkappa(V^*)$$

where V^* is a nonsingular model of V.

The Kodaira dimension has the following important properties.

(1.1) For an algebraic manifold V, with $\varkappa(V) \geqslant 0$, there exist positive numbers α, β such that for any integer $m \in N(V)$ we have

$$\alpha m^{\varkappa(V)} \leqslant P_m(V) \leqslant \beta m^{\varkappa(V)}.$$

(1.2) If $\varkappa(V) \geqslant 0$ for an algebraic manifold V, we have

$$\varkappa(V) = \operatorname{tr.deg}_C R[V] - 1^1$$

where $R[V] = \oplus_{m=0}^{+\infty} H^0(V, \mathcal{O}(mK_V))$ is the canonical ring of V.

(1.3) If $\varphi: V \to W$ is a surjective rational mapping with $\dim V = \dim W$, we have $\varkappa(V) \geqslant \varkappa(W)$. Moreover, if φ is étale, we have $\varkappa(V) = \varkappa(W)$.

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[1] Because of this fact, some mathematicians use $\varkappa = -1$ instead of $\varkappa = -\infty$. But this is quite inconvenient to write the formulas (1.4) (1.5), Conjecture C_n below, etc.

(1.4) $\varkappa(V\times W)=\varkappa(V)+\varkappa(W)$.

(1.5) Let $\varphi: V\to W$ be a fibre space. Then we have

$$\varkappa(V)\leqq \dim W+\varkappa(V_w)$$

where V_w is a general fibre of φ.

EXAMPLE 1. (a) For a nonsingular curve C we have the following table.

$\varkappa(C)$	genus	structure
$-\infty$	0	P^1
0	1	elliptic curve
1	$\geqq 2$	K_C is ample

(b) Let V_d^n be a nonsingular hypersurface of degree d in P^{n+1}.

$$\varkappa(V_d^n)=\begin{cases}-\infty, & \text{if } d<n+2,\\ 0, & \text{if } d=n+2,\\ n, & \text{if } d>n+2.\end{cases}$$

(c) Suppose that there is a surjective rational mapping from an abelian variety A to an algebraic variety V. Then we have

$$\varkappa(V)\leqq 0.$$

Another important birational invariance of an algebraic manifold V is the *first irregularity* $q(V)$ defined by $q(V)=h^1(V,\mathcal{O}_V)$. Since V is algebraic, we have $q(V)=g_1(V)=\dim A(V)$, where $g_1(V)=h^0(V,\Omega_V^1)$ and $A(V)$ is the Albanese variety of V.

2. Pluricanonical mappings and Albanese mappings. The following theorem is important for our classification theory.

THEOREM 1 (IITAKA [3]). *Assume* $\varkappa(V)\geqq 0$. *Then there exist algebraic manifolds* V^*, W *and a surjective morphism* $\varphi: V^*\to W$ *which satisfy the following conditions.*

(1) $\dim W=\varkappa(V)$.

(2) φ *has connected fibres.*

(3) *There exists a dense subset* U *of* W *in the complex topology such that for each* $u\in U$, $V_u=\varphi^{-1}(u)$ *is smooth and* $\varkappa(V_u)=0$.

(4) $\varphi: V^*\to W$ *is birationally equivalent to* $\Phi_{mK}: V\to W_m=\Phi_{mK}(V)$ *for a sufficiently large* $m\in N(V)$.

EXAMPLE 2. Let V be a subvariety of an abelian variety. Then there exists an abelian variety A and an algebraic variety Y such that V has a structure $\pi: V\to Y$ of a fibre bundle over Y in the sense of étale topology whose fibre is an abelian variety A and that $\dim Y=\varkappa(V)=\varkappa(Y)$. $\pi: V\to Y$ is birationally equivalent to the above fibration $\varphi: V^*\to W$.

An algebraic variety V is said to be of *elliptic type* (resp. *parabolic type, hyperbolic type* **(general type)**) if $\varkappa(V)=-\infty$ (resp. $\varkappa(V)=0, \varkappa(V)=\dim V$). The above theorem shows that the study of the birational structures of algebraic manifolds is reduced to the following problems.

Problem A. Study the structures of algebraic manifolds of elliptic, parabolic and hyperbolic types.

Problem B. Study the fibre spaces whose general fibres are of parabolic type.

From the above theorem we infer that if V is of parabolic type, then Φ_{mK} is birational for sufficiently large m.

Let us recall the classification of algebraic surfaces.

$\varkappa$	q	Structure
2		Φ_{mK} is birational for $m\geqslant 5$. (surface of general type)
1		general fibres of Φ_{mK} are elliptic curves for $m\gg 0$. (elliptic surface)
0	2	abelian surface
	1	hyperelliptic surface
	0	$K3$ surface, Enriques surface
$-\infty$	$\geqslant 1$	ruled surface
	0	rational surface

In this table, the cases $\varkappa=2, 0, -\infty$ give the solutions to Problem A and the study of elliptic surfaces due to Kodaira [5] gives the solution to Problem B. Surfaces of general type are being studied by several mathematicians.

The classification theory of surfaces of parabolic or elliptic type shows us the importance of Albanese mappings The following theorem plays an important role for the classification of higher dimensional algebraic manifolds.

THEOREM 2 (UENO [8, I]). *Let* $\alpha: V\to A(V)$ *be the Albanese mapping of* V. *Then we have* $\varkappa(\alpha(V))\geqslant 0$. *Moreover, the equality holds if and only if* α *is surjective.*

The fibres of the Albanese mapping $\alpha: V\to\alpha(V)$ is not necessarily connected. Hence we take the Stein factorization $\beta: V\to W$, $\gamma: W\to\alpha(V)$ such that $\alpha=\gamma\cdot\beta$, β has connected fibres and γ is finite. Then we have $\varkappa(W)\geqslant\varkappa(\alpha(V))\geqslant 0$. $\beta: V\to W$ is called the *fibre space associated with the Albanese mapping.*

Concerning the Albanese mapping of algebraic manifolds of parabolic type, we have the following conjectures.

CONJECTURE A_n. *Let V be an n-dimensional algebraic manifold of parabolic type. Then the Albanese mapping is surjective.*

CONJECTURE B_n. *If $\varkappa(V)=0$, $q(V)=\dim V=n$, then the Albanese mapping of V is birational.*

More generally we have

CONJECTURE K_n. *Let V be an n-dimensional algebraic manifold of parabolic type. Then the Albanese mapping $\alpha: V\to A(V)$ is birationally equivalent to an étale fibre bundle over $A(V)$ whose fibre is an algebraic manifold of parabolic type.*

The converse is true by virtue of Theorem 3 below.

3. Conjecture $C_{n,m}$ and algebraic threefolds. To attack the above conjectures, it is important to consider the following

CONJECTURE $C_{n,m}$. *Let $\varphi: V\to W$ be a fibre space with $\dim V=n$, $\dim W=m$. We have the inequality*

$$\varkappa(V)\geqq\varkappa(W)+\varkappa(V_w)$$

where V_w is a general fibre of φ.

For several applications, the following weaker conjecture is sufficient.

CONJECTURE $C'_{n,m}$. *Let $\varphi: V\to W$ be the same as above. Assume, furthermore, that W is of hyperbolic type and $\varkappa(V_w)\geqq 0$, for a general fibre V_w of φ. Then we have $\varkappa(V)>0$.*

Combining Theorem 2 and Example 2, we can prove the following

PROPOSITION 1. *Assume that $C'_{n,m}$ holds for any $m<n$. Then the Conjecture A_n holds.*

Similarly we show the following

PROPOSITION 2. *Let V be an n-dimensional algebraic manifold of elliptic type. Assume that $C_{n,m}$ holds for any $m<n$. Then general fibres of the fibre space $\beta: V\to W$ associated with the Albanese mapping of V are of elliptic type.*

We have the following affirmative answers to the above conjecture $C_{n,m}$.

THEOREM 3 (NAKAMURA–UENO [6]). *Let $\varphi: V\to W$ be an analytic fibre bundle over a compact complex manifold W whose fibre is an algebraic manifold F with structure group* Aut (F). *Then we have*

$$\varkappa(V)=\varkappa(W)+\varkappa(F).$$

Note that for a compact complex manifold we can define the Kodaira dimension similarly. The proof is based on the following interesting observation.

As Aut (F) operates on $H^0(V, \mathcal{O}(mK_F))$, we have a natural representation

$$\varrho_m: \text{Aut}\,(F)\to \text{Aut}\,(H^0(V, \mathcal{O}(mK_F))).$$

Then the image of ϱ_m is a finite group, provided F is *algebraic*. This fact does *not* necessarily hold for a *nonalgebraic* manifold. Hence Conjecture $C_{n,m}$ does *not* necessarily hold for an analytic fibre bundle whose fibre is a non-algebraic compact complex manifold [6].

THEOREM 4 (VIEHWEG [11]). *$C_{n,n-1}$ holds.*

The proof is based on the moduli theory of stable curves. That is, we need a good compactification of the moduli space of curves. The next theorem is also proved by using the moduli theory of polarized abelian varieties. Though there is no good theory of stable abelian varieties, we can use the toroidal relative compactifications of families of polarized abelian varieties due to Namikawa [7].

THEOREM 5 (UENO [10]). *If general fibres of a fibre space $\varphi: V \to W$ of algebraic manifolds are birationally equivalent to abelian varieties, then $C_{n,m}$ holds.*

THEOREM 6 (FUJITA [1], UENO [8, II]). *Assume that W is a nonsingular curve of genus $g \geqq 2$. Assume, moreover, $p_g(V_w) \geqq 1$ for a general fibre of $\varphi: V \to W$, or $\varkappa(V) \geqq 0$. Then $C_{n,1}$ holds.*

For the proof we use the theory of variations of Hodge structures and the duality theorem. We also have partial results when $g(W)=1$.

The above theorems show that the conjecture $C_{n,m}$ is deeply related to the moduli theory of polarized algebraic manifolds.

From these theorems and a generalization of Theorem 6, we infer the following

THEOREM 7. *If $\dim V \leqq 3$, $\varkappa(V)=0$, then the Albanese mapping is surjective and has connected fibres. Hence, a fortiori we have $q(V) \leqq \dim V$.*

Moreover, we can show

THEOREM 8. *Let V be an algebraic threefold of parabolic type.*

(1) *$q(V)=3$, if and only if the Albanese mapping is birational.*

(2) *$q(V)=2$, if and only if $p_g(V)=0$ and the Albanese mapping $\alpha: V \to A(V)$ is birationally equivalent to an étale fibre bundle over $A(V)$ whose fibre is an elliptic curve.*

(3) *If $q(V)=1$, then $\alpha: V \to A(V)$ is birationally equivalent to an étale fibre bundle over $A(V)$ whose fibre is a surface of parabolic type, or general fibres of α are surfaces of general type with $p_g \leqq 1$.*

If $C_{3,1}$ is true, in (3) we can exclude the last case.

For algebraic surfaces the similar theorem is a part of the Enriques classification. In this case the proof relies heavily on the theory of relatively minimal models of surfaces. To prove Theorem 8, we need not use the theory of relatively minimal models of algebraic threefolds, which is rather complicated and incomplete at the moment. We remark that there are several examples of algebraic threefolds of parabolic type with $q=0$ whose properties are different from those of surfaces of parabolic type.

As a generalization of Theorem 8 (2), we have the following

PROPOSITION 3. *Let $\pi: V \to A$ be a fibre space over an abelian variety A whose general fibres are birationally equivalent to abelian varieties. If $\varkappa(V)=0$, then $\pi: V \to A$ is birationally equivalent to an étale fibre bundle over A whose fibre is an abelian variety.*

As for algebraic threefolds of elliptic type, we have the following

PROPOSITION 4. *Let V be an algebraic threefold of elliptic type with $q(V)>0$ and $\beta: V \to W$ the fibre space associated with the Albanese mapping. Then,* $\dim W \leqq 2$.

(1) *If* $\dim W=2$, *then general fibres of β are $\boldsymbol{P}^1$.*

(2) *If* $\dim W=1$, *then general fibres of β are rational or ruled surfaces, or surfaces of general type with $p_g=0$ and $\pi_1^{\mathrm{alg}}=\{e\}$.*

If $C_{3,1}$ holds, in (2) the last case does not occur. If this is the case, then all algebraic threefolds of elliptic type with $q>0$ are uniruled, where an algebraic variety V is called to be uniruled if there is a finite ramified covering of V which is birationally equivalent to $\boldsymbol{P}^1 \times W$. It is interesting to know whether all algebraic threefolds of elliptic type with $q=0$ are uniruled or not.

Finally we rewrite the above results in the following tables.

$\dim V=3$, $\varkappa(V)>0$.

$\varkappa(V)$	Birational structure of V
3	Φ_{mK} is birational for $m \gg 0$
2	fibre spaces of elliptic curves over a surface
1	fibre space of surfaces of parabolic type over a curve

$\dim V=3$, $\varkappa(V) \leqq 0$.

$\varkappa(V)$	$q(V)$	Birational structure of V
0	3	abelian variety
	2	étale fibre bundle over $A(V)$ whose fibre is an elliptic curve
	1	étale fibre bundle over $A(V)$ whose fibre is a surface of parabolic type (or fibre space of surfaces of general type with $p_g<1$)
	0	?
$-\infty$	1	uniruled (or fibre space of surfaces of general type with $p_g=0$, $\pi_1^{\mathrm{alg}}=\{e\}$)
	0	uniruled?

Added in proof. Recently Viehweg succeeded to prove $C_{3,1}$. Hence, in the above table, $\dim V = 3$, $k(V) \leqq 0$; the exceptional cases do not occur.

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Jacquet Modules for Real Reductive Groups

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In the past few years work of several people has established a precise relationship between the behaviour at infinity of matrix coefficients of admissible representations—of both real and p-adic reductive groups—and embeddings into representations induced from parabolic subgroups, as well as the relationship of both to characters. Historically, these ideas were already implicit in Harish-Chandra's theory of the "constant term", and one might consider the new results as an algebraic elaboration of this theory. One crucial observation was made by Jacquet [**10**] concerning the p-adic case, and indeed this case has had a strong influence on the development of the other. For p-adic groups the relationships established are fundamental and perhaps indispensable to the whole theory of admissible representations (see [**1**], [**3**], [**8**]). For real groups they seem to me somewhat more technical, and I must admit that it is not clear to me how important they will turn out to be in the long run. At any rate their present interest lies mainly in the light they shed on the nature and significance of the series expansions of matrix coefficients, a notoriously obscure matter up to now. They also offer a new approach to old results on the algebra of representations; have been applied to a problem in cohomology of arithmetic groups; have been used to formulate and prove some striking analytical properties of intertwining operators among principal series.

The p-adic case is simpler than the other in both statements and proofs, and if only for that reason I would have liked to say something about it here. It was not possible because of limitations on time and space but I would like to call attention — as Harish-Chandra is fond of doing—to the continuing suggestive parallels in the two theories.

Most of what I will say should appear soon in joint papers with Dragan Milicic

and Nolan Wallach. It represents to some extent independent work on their part and to some extent joint work with them. I also include some recent and not yet published results of Henryk Hecht and Wilfried Schmid.

1. Let

G=$\boldsymbol{R}$-valued points on a Zariski-connected reductive group defined over $\boldsymbol{R}$,
K=a maximal compact subgroup of G,
P=a minimal parabolic subgroup of G,
N=the unipotent radical of P,
M=a reductive component of P stable under the Cartan involution associated to K,
A=the (topological) neutral component of the maximal split torus of G contained in M.

Let $\mathfrak{g}, \mathfrak{k}$, etc. be their complexified Lie algebras.

I call a *Harish–Chandra module* of the pair $(\mathfrak{g}, K)$ a representation (π, V) of $\mathfrak{g}$ and K simultaneously on the same space such that (1) the representation of K is the direct sum of continuous finite-dimensional irreducibles; (2) the representation of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$ agrees with the differential of the representation of K; (3) for all $X\in\mathfrak{g}$, $k\in K$, $\pi(\mathrm{Ad}(k)X)=\pi(k)\pi(X)\pi(k)^{-1}$; (4) it is $Z(\mathfrak{g})$-finite—i.e. every $v\in V$ is contained in a finite-dimensional subspace stable under the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$. It is well known that condition (4) is superfluous if the representation has finite length and that K-representations occur with finite multiplicity if it is finitely generated.

If (π, V) is a continuous representation of G (on a reasonable topological vector space) then $\mathfrak{g}$ and K act on the dense subspace of smooth K-finite vectors. Frequently—for example if π is unitary and irreducible—this yields a Harish–Chandra module. In fact, almost all interest in such modules arises in the intimate relationship with G-spaces. It is only recently, however, that one knows:

LEMMA (PRISCHEPIONOK). *Every finitely generated Harish–Chandra module is the canonical representation on the K-finite vectors in some smooth representation of G.*

As will be seen, one may even choose this to be on a nuclear Fréchet space.

There are available two rather different proofs of this—see [5] and [15]. Prischepionok even constructs a canonical extension to G.

Of course when π is irreducible the Lemma follows from an old and famous result of Harish–Chandra (which is proved nicely in [13]).

If (π, V) is a finitely generated Harish–Chandra module, its contragredient is the natural representation on the K-finite linear functionals on V. It is again a Harish–Chandra module—although not obviously finitely generated, but at least with finite K-multiplicity. The importance of the above Lemma for my purposes is that to each pair $v\in V$, $\tilde{v}\in\tilde{V}$, one may associate the function $\langle\pi(g)v, \tilde{v}\rangle$ on G, the *matrix coefficient* of the pair. It is smooth and annihilated by some ideal of

differential operators in $Z(\mathfrak{g})$ of finite codimension, hence even real-analytic. (In [5] I construct the matrix coefficients directly as solutions of a system of differential equations on G, and deduce the Lemma from that.)

2. Let (π, V) be a finitely generated Harish–Chandra module. For each $n \in \boldsymbol{N}$ define $\mathfrak{n}^n V$ to be the subspace of V spanned by $\{\pi(v_1)\ldots\pi(v_n)v | v_i \in \mathfrak{n}, v \in V\}$. For $n \geqslant m$, $\mathfrak{n}^n V \subseteq \mathfrak{n}^m V$ so that there is a canonical projection $V/\mathfrak{n}^n V \to V/\mathfrak{n}^m V$. Define the Jacquet module $V_{[\mathfrak{n}]}$ of V associated to P to be the projective limit of these spaces. Each $V/\mathfrak{n}^n V$ is naturally a $(\mathfrak{g}, K \cap P)$-module, and so is the limit. In fact, since V is finitely generated over $U(\mathfrak{n})$ (see [6]) each $V/\mathfrak{n}^n V$ is finite-dimensional; this implies that each $V/\mathfrak{n}^n V$, hence $V_{[\mathfrak{n}]}$, is a P-space. There is a canonical map from V to $V_{[\mathfrak{n}]}$.

The construction of $V_{[\mathfrak{n}]}$ is motivated by an analogous p-adic definition. The reason that things here are more complicated is that in the p-adic case all finite-dimensional P-spaces are trivial on N, so that one need only consider the analogue of $V/\mathfrak{n}V$, while here one only knows that a finite-dimensional P-space is nilpotent on $\mathfrak{n}$.

The most immediate result about $V_{[\mathfrak{n}]}$ is a version of Frobenius reciprocity. Recall that the Harish–Chandra module $\mathrm{Ind}(\sigma|P, G)$ induced from the finite-dimensional representation (σ, U) of P is the right regular representation on the K-finite $f: G \to U$ such that $f(pg) = \sigma\delta^{1/2}(p)f(g)$ for all $p \in P, g \in G$. (Here δ is the modulus character of P.) Let $\Omega: \mathrm{Ind}(\sigma) \to U$ be the $(\mathfrak{g}, K \cap P)$-morphism from $\mathrm{Ind}(\sigma)$ to the P-module $\sigma\delta^{1/2}, f \to f(1)$. Composition with Ω gives a map

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(V, \mathrm{Ind}(\sigma)) \to \mathrm{Hom}_{(\mathfrak{g}, K \cap P)}(V, \sigma\delta^{1/2}).$$

Since $\mathfrak{n}$ acts nilpotently on U, any $\mathfrak{p}$-map from V to U must factor through $V/\mathfrak{n}^n V$ for $n \gg 0$, hence through $V_{[\mathfrak{n}]}$. It is elementary to see:

PROPOSITION (FROBENIUS RECIPROCITY). *Composition with Ω induces an isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(V, \mathrm{Ind}(\sigma)) \cong \mathrm{Hom}_P(V_{[\mathfrak{n}]}, \sigma\delta^{1/2}).$$

In other words, $\mathfrak{p}$-morphisms from V into induced representations are determined by the structure of $V_{[\mathfrak{n}]}$ as a P-space. In particular, maps into principal series are determined by the M-module $V/\mathfrak{n}V$: if V is irreducible, any non-zero M-morphism $V/\mathfrak{n}V \to \sigma\delta^{1/2}$ gives an embedding of V into $\mathrm{Ind}(\sigma)$. (Incidentally, it is important to realize that $V/\mathfrak{n}V$ may not be M-semisimple.)

Although each finite quotient $V/\mathfrak{n}^n V$ is only a P-space, something more can be said about the limit.

LEMMA. *There exists $d \geqslant 0$ such that for every $n \geqslant d$ and $X \in \mathfrak{g}$*

$$X\mathfrak{n}^n U(\mathfrak{g}) \subseteq \mathfrak{n}^{n-d} U(\mathfrak{g}).$$

As a consequence, the element X induces a sequence of maps $V/\mathfrak{n}^n V \to V/\mathfrak{n}^{n-d} V$, hence an endomorphism of $V_{[\mathfrak{n}]}$. Thus $V_{[\mathfrak{n}]}$ *becomes a module over* $(\mathfrak{g}, P)$.

3. A $\mathfrak{g}$-module is $\mathfrak{p}$-finite if every element in it is contained in a finite-dimensional $\mathfrak{p}$-stable subspace. If it is in addition finitely generated over $U(\mathfrak{g})$ then it is a quotient of some $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U$, U a finite-dimensional $\mathfrak{p}$-module (this is called a *generalized Verma module*). Recall that if U is irreducible then $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U$ has a unique irreducible $\mathfrak{g}$-quotient $X(U)$.

PROPOSITION. *Let X be a $\mathfrak{g}$-module. Then it is finitely generated and $\mathfrak{p}$-finite if and only if it satisfies these two conditions:* (a) *every $x \in X$ is annihilated by some $\mathfrak{n}^n$;* (b) *the subspace of $x \in X$ annihilated by $\mathfrak{n}$ is finite-dimensional. In this case it has finite $\mathfrak{g}$-length and every composition factor is of the form $X(U)$ for certain U.*

Continue to let (π, V) be a finitely generated Harish-Chandra module. Define $\hat{V}^{[\mathfrak{n}]}$ to be the subspace of the linear dual of V consisting of functionals annihilated by some $\mathfrak{n}^n$—i.e. trivial on some $\mathfrak{n}^n V$. The Lemma in § 2 implies that it is a $\mathfrak{g}$-module. It is in fact simply a sort of dual of $V_{[\mathfrak{n}]}$, the topological dual if $V_{[\mathfrak{n}]}$ is assigned the $\mathfrak{n}$-adic topology. And it is not hard to see that $V_{[\mathfrak{n}]}$ is the entire linear dual of $\hat{V}^{[\mathfrak{n}]}$. In other words, $V \rightsquigarrow V_{[\mathfrak{n}]}$ is a functor from the category of finitely generated Harish–Chandra modules to that of linear duals of finitely generated $\mathfrak{p}$-finite $\mathfrak{g}$-modules.

4. Let R for the moment be the ring $U(\mathfrak{n})$, I the two-sided ideal of R generated by $\mathfrak{n}$. The module $V_{[\mathfrak{n}]}$ is nothing but the completion of the finite R-module V with respect to powers of I. When $\mathfrak{n}$ is abelian this construction is standard, but it turns out that even for general nilpotent $\mathfrak{n}$ something can be said about it.

LEMMA. (a) *The completion $R_I = \varprojlim R/I^n$ is a Noetherian ring.*

(b) (ARTIN–REES). *If $A \subseteq B$ are two finite R-modules then there exists $d \geqslant 0$ such that for $n \gg 0$*

$$I^n B \cap A \subseteq I^{n-d} A.$$

In fact this is true for any two-sided ideal of R. (There is a large literature attempting to extend results of commutative algebra to rings like R—see [7], [14] for example.)

PROPOSITION. *The functor $V \rightsquigarrow V_{[\mathfrak{n}]}$ is exact.*

Again, the motivation for expecting such a result comes from something similar for p-adic groups.

What one would really like to do is apply general results about modules over R to obtain something about the structure of V as an R-module. For example, if $\mathfrak{n}$ is abelian then in [6] Osborne and I were able to show that the associated prime ideals of V are rather restricted, and obtain (implicitly, I am afraid) a useful filtration of V as a $\mathfrak{p}$-module. But for general $\mathfrak{n}$ the known results on associated primes are not sufficient to say anything. This is reasonable in light of the fact that pathologies can occur even for finite R-modules which are at the same time $\mathfrak{p}$-modules (as Wallach has described to me).

5. I have given no reason so far to expect that $V_{[\mathfrak{n}]} \neq 0$. An easy non-commutative version of Nakayama's Lemma shows that $V_{[\mathfrak{n}]}=0$ if and only if $V/\mathfrak{n}V=0$, but even this apparently simple possibility is hard to rule out. The first proof of this fact involved looking at the asymptotic expansions of matrix coefficients at infinity; this is no accident, and in fact the deepest results about $V_{[\mathfrak{n}]}$ are related to these expansions.

For simplicity, assume G for the moment to be semisimple. Let Δ be the set of simple roots of $\mathfrak{g}$ with respect to A determined by the choice of P (so that $\mathfrak{n}$ is the sum of positive root spaces). Embed A in $\mathbf{C}^{\Delta}$: $a \to (\alpha(a))$. For each $s \in \mathbf{C}^{\Delta}$ define functions which are single-valued on A, multivalued on the complement of coordinate hyperplanes in $\mathbf{C}^{\Delta}$:

$$a^s \log^m a = \prod \alpha(a)^{s_\alpha} \log^{m_\alpha} \alpha(a) \quad (\alpha \in \Delta).$$

LEMMA. *There exist finite sets* $S \subseteq \mathbf{C}^{\Delta}$, $\mathcal{M} \subseteq \mathbf{N}^{\Delta}$ *such that for every* $v \in V$, $\tilde{v} \in \tilde{V}$, *there exist functions* $f_{s,m}$ $(s \in S, m \in \mathcal{M})$ *holomorphic in the region* $|\alpha| < 1$ *with*

$$\langle \pi(a)v, \tilde{v} \rangle = \sum f_{s,m}(a) a^s \log^m a$$

on $\{a \in A \mid \alpha(a) < 1$ *for all* $\alpha \in \Delta\}$.

The possible sets $S, \mathcal{M}$ depend on V, but are not unique.

This is a restatement of results of Harish-Chandra (see the Appendix to **[16]**, also the forthcoming paper with Milicic).

Let

$$f_{s,m} = \sum f_{s,m,n} a^n \quad (n \in \mathbf{N}^{\Delta})$$

be the Taylor's series expansion of f at the origin.

LEMMA. *For* $v \in \mathfrak{n}^n V$ *or* $v \in (\mathfrak{n}^-)^n \tilde{V}$, $f_{s,m,n}=0$ *whenever* $\sum n_i < n$.

Here $\mathfrak{n}^-$ is the opposite of $\mathfrak{n}$.

Neither is this much different from what Harish-Chandra has shown; it depends on the expression of elements of $\mathfrak{n}^n U(\mathfrak{g})$ in so-called radial coordinates in terms of the Cartan decomposition $G=KAK$ (see Chapter 9 of **[16]**, also again the paper with Milicic).

Let $v \in V$ be given, $v \neq 0$. Choose $\tilde{v} \in \tilde{V}$ with $\langle v, \tilde{v} \rangle \neq 0$. Then the restriction of the corresponding matrix coefficient to A is not trivial. Since it is analytic, its expansion around the origin (as in the above Lemma) must not be trivial. The second Lemma implies the existence of some n with $v \notin \mathfrak{n}^n V$.

THEOREM. *The canonical map from* V *to* $V_{[\mathfrak{n}]}$ *is an injection.*

The kernel of this map, after all, is the intersection of the $\mathfrak{n}^n V$.

As one consequence, $V/\mathfrak{n}V \neq 0$. If V is irreducible, then, earlier remarks show that V occurs as the subspace of at least one principal series representation. As another, since $\tilde{V}^{[\mathfrak{n}]}$ has finite length one sees immediately that V does too.

For $n \gg 0$, these two facts imply that V embeds into Ind $(V/\mathfrak{n}^n V \cdot \delta^{-1/2})$. Wallach has been able to show recently that the closure of the image of V in the corresponding space of smooth functions is as a topological vector space independent of n. One defines in this way a canonical extension of V to G which should play an important role.

The statement that V embeds into $V_{[\mathfrak{n}]}$ is purely algebraic, and one might like to see a purely algebraic proof. For abelian $\mathfrak{n}$ this follows from the result in [6] mentioned earlier, and indeed from a very much easier argument. Recently Wallach and Stafford have constructed a satisfactory algebraic proof of this whenever G has real rank one, and also when $G = \mathrm{SL}_n$. Still, one should keep in mind that part of the charm of $V_{[\mathfrak{n}]}$ lies in the link to matrix coefficients. This link is even stronger that I have so far said; Hecht has been able to show that not only do matrix coefficients imply something about $V_{[\mathfrak{n}]}$, but that one has a converse as well. (He uses a result of Milicic [19] which is a special case of what I have in mind.) Thus one has criteria for whether V is square-integrable, or tempered, or has matrix coefficients vanishing at infinity, all purely in terms of the M-module $V/\mathfrak{n}V$.

If one considers parabolic subgroups which are not necessarily minimal, one obtains results about the asymptotic behaviour of matrix coefficients in directions other than the origin in $\boldsymbol{C}^A$ (in Harish–Chandra's terminology, "along the walls").

6. A few other applications and related ideas: (1) The $\mathfrak{n}$-homology of V is the same as that of $V_{[\mathfrak{n}]}$, and dual to the cohomology of $\hat{V}^{[\mathfrak{n}]}$. What my remarks in §5 amount to is that in some sense the M-module $V/\mathfrak{n}V = H_0(\mathfrak{n}, V)$ determines the leading terms of the symptotics matrix coefficients. These observations have been joined to an ingenious sequence of arguments by Schmid to obtain a new proof of the cohomology vanishing theorem of Borel–Wallach [2] and Zuckerman [17], which in fact relates the extent of cohomology vanishing to the rapidity of matrix coefficient decrease of a unitary representation.

(2) One can obtain a new proof of some of Zuckerman's results in [18] concerning tensoring Harish–Chandra modules with finite-dimensional ones, and perhaps even to recover some information about certain extensions which arise in his construction (he is able to say something only about semisimplifications of the modules occurring, but one may be able to combine the two techniques to say more).

(3) What may seem most amazing, one obtains a proof that *any* $\mathfrak{g}$*-morphism between K-finite induced representations extends continuously to the corresponding smooth representations of* G. (In certain cases, Kashiwara has also proved this.) The proofs involved in these results are to some extent a generalization of those in Bruhat's thesis, and illuminate Langlands' arguments in [12] as well. (In my talk I asserted that all intertwining operators have closed images but I was overconfident. Some delicate points remain to be shown.)

(4) Hecht and Schmid have proven Osborne's conjecture concerning the restriction of characters to noncompact tori. (Refer to [4] for a statement of the conjecture as well as a proof of the p-adic analogue.) This involves $\mathfrak{n}$-homology, and in view

of the relationship between that and matrix coefficients gives what seems to me the most satisfactory proof of the relationship between the asymptotics of matrix coefficients and those of characters (see [9]).

(5) One may define the completion of V with respect to other maximal ideals of $U(\mathfrak{n})$, and this is related to results of Kostant (and, in weaker form, of Zuckerman and myself) on Whittaker models for Harish-Chandra modules (see [11]).

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Langlands' Conjecture for GL(2) over Functional Fields

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Introduction

0.1. Let k be a global field of characteristic $p>0$, $\mathfrak{A}$ its adele ring, W the Weil group of k (W consists of elements of $\mathrm{Gal}(\bar{k}/k)$ whose image in $\hat{\boldsymbol{Z}}$ belongs to $\boldsymbol{Z}$, the topology on W being induced by the embedding $W \subset \mathrm{Gal}(\bar{k}/k)\times\boldsymbol{Z}$). If v is a place of k, k_v will denote the completion of k at v and W_v will denote the Weil group of k_v.

For each number field E denote by $\Sigma_1(E)$ the set of isomorphism classes of compatible systems of absolutely irreducible 2-dimensional λ-adic representations of W (λ belongs to the set of places of E which do not divide ∞ and p). Compatibility of a system $\{\varrho_\lambda\}$ means, by definition, that for every place v of k and every $h\in W_v$ $\mathrm{Tr}\,\varrho_\lambda(h)$ belongs to E and does not depend on λ. Put

$$\Sigma_1 = \varinjlim_{E\subset\bar{\boldsymbol{Q}}} \Sigma_1(E).$$

Denote by Σ_2 the set of isomorphism classes of irreducible representations of $\mathrm{GL}(2,\mathfrak{A})$ over $\bar{\boldsymbol{Q}}$ which occur in the space of cusp forms.

Let $E\subset\bar{\boldsymbol{Q}}$, $[E:\boldsymbol{Q}]<\infty$, $\varrho=\{\varrho_\lambda\}\in\Sigma_1(E)$, $\pi\in\Sigma_2$. π is said to be compatible with ϱ if for some λ and almost all places v of k $L(s-\frac{1}{2},\pi_v)=[\det(1-F_v\,q_v^{-s},\varrho_\lambda)]^{-1}$ where L denotes Jacquet–Langlands' L-function, $F_v\in W$ is a geometric Frobenius element of v, q_v is the order of the residue field of v (of course, the words "for some λ", can be replaced by "for all λ"). Now let $\varrho\in\Sigma_1$, $\pi\in\Sigma_2$. π is said to be compatible with ϱ if there exists a subfield $E\subset\bar{\boldsymbol{Q}}$ of finite degree over $\boldsymbol{Q}$ and an element $\varrho_E\in\Sigma_1(E)$ such that ϱ is the image of ϱ_E and π is compatible with ϱ_E. Denote by Γ the set of pairs $(\varrho,\pi)\in\Sigma_1\times\Sigma_2$ such that π is compatible with ϱ.

0.2. In this report I shall sketch a proof of the following theorem.

THEOREM A (LANGLANDS' CONJECTURE). *Γ is the graph of a bijection $\Sigma_1 \simeq \Sigma_2$.*

The following theorems are easy consequences of the proof of Theorem A.

THEOREM B (PETERSSON'S CONJECTURE). *Let π be an irreducible unitary representation of* GL$(2, \mathfrak{A})$ *which occurs in the space of cusp forms. Then for every place v of k π_v does not belong to the complementary series.*

Recall that the complementary series for GL$(2, k_v)$ consists of unitary representations of GL$(2, k_v)$ having the form $\pi(\mu, \nu)$, where μ and ν are quasicharacters of k_v^* and not characters.

THEOREM C. *Let π be an irreducible unitary representation of* GL$(2, \mathfrak{A})$ *which occurs in the space of cusp forms. Suppose that either one of the local components of π belongs to the discrete series, or every field finitely generated and of transcendence degree* 3 *over $\bar{F}_p$ has a smooth projective model. Then all the zeroes of $L(s, \pi)$ are on the line* $\operatorname{Re} s = \frac{1}{2}$.

Recall that the discrete series for GL$(2, k_v)$ consists of supercuspidal and special representations.

0.3. Let us show that theorem A is implied by the following proposition.

THEOREM A′. *For each $\pi \in \Sigma_2$ there exist a number field E containing the field of definition of π and 2-dimensional λ-adic representations ϱ_λ of W (for all places λ of E except those dividing ∞ and p) such that for almost all places v of k and all λ $L\left(s-\frac{1}{2}\right)$, $\pi_v = [\det(1 - \boldsymbol{F}_v q_v^{-s}, \varrho_\lambda)]^{-1}$.*

First of all, the representations ϱ_λ in Theorem A′ are absolutely irreducible (because for every quasicharacter ω of $\mathfrak{A}^*/k^*$ $L(s, \pi \otimes \omega)$ is holomorphic). In order to prove that $\{\varrho_\lambda\} \in \Sigma_1(E)$ it remains to show that for every place v of k the restriction of the character of ϱ_λ to W_v belongs to E and does not depend on λ. In fact the results of [1] imply that it is enough to show this for almost all v, which is trivial. We have shown (using Theorem A′) that the projection $\Gamma \to \Sigma_2$ is surjective.

The surjectivity of the projection $\Gamma \to \Sigma_1$ was proved by P. Deligne [1] using the results of [5] (he also proved that if $\varrho \in \Sigma_1(E)$ is compatible with $\pi \in \Sigma_2$ then for all places of k the local L- and ε-functions of ϱ and π coincide).

The projection $\Gamma \to \Sigma_1$ is injective because an irreducible representation of GL$(2, \mathfrak{A})$ which occurs in the space of cusp forms is uniquely determined by almost all of its local components. The injectivity of the projection $\Gamma \to \Sigma_2$ is an easy consequence of Chebotarev's density theorem.

0.4. The proof of Theorem A′ can be considered as a noncommutative generalization of Lang's theory [10]. On the other hand it is closely connected to Eichler–Shimura's theory.

1. *F*-sheaves and their moduli schemes

1.1. Let $\boldsymbol{F}_q$ be the field of constants of k, denote by X the smooth projective curve over $\boldsymbol{F}_q$ corresponding to k. We shall write $Y \otimes Z$, $Y \times Z$ instead of $Y \otimes_{\boldsymbol{F}_q} Z$, $Y \times_{\boldsymbol{F}_q} Z$. If S is a scheme over $\boldsymbol{F}_q$, Fr_S will denote the Frobenius endomorphism of S (over $\boldsymbol{F}_q$). Instead of $\mathrm{Fr}_{\mathrm{Spec}\, B}$ we shall write Fr_B.

DEFINITION. Let S be a scheme over $\boldsymbol{F}_q$. A *left (right) F-sheaf over S of rank d* is, by definition, a diagram

$$\mathscr{F} \xrightarrow{j} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L} \quad \mathscr{F} \xrightarrow{i} \mathscr{L} \qquad \text{(respectively} \qquad (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L} \xrightarrow{i} \mathscr{F} \xleftarrow{j} \mathscr{L}\text{)}$$

where $\mathscr{L}$ and $\mathscr{F}$ are locally free sheaves of $\mathcal{O}_{X \times S}$-modules of rank d, Coker i is the direct image of an invertible sheaf on the graph of a morphism $\alpha\colon S \to X$, Coker j is the direct image of an invertible sheaf on the graph of a morphism $\beta\colon S \to X$. α is called the *zero* of the F-sheaf, β is called the *pole* of the F-sheaf. An *(α, β)-F-sheaf* is an F-sheaf whose zero equals α and whose pole equals β.

The role of rank 2 F-sheaves in the proof of Theorem A′ is similar to that of elliptic curves in Eichler–Shimura's theory.

REMARK. Suppose that α and β are *disjoint* (this means, by definition, that the graph of α doesn't intersect the graph of β). Then a left (α, β)-F-sheaf is, in fact, the same as a right (α, β)-F-sheaf. In this situation the word "left" or "right" will be omitted. When α and β are disjoint an (α, β)-F-sheaf can be imagined as an "isomorphism" from $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L}$ to $\mathscr{L}$ which has "singularities" (a "simple zero" at α and a "simple pole" at β).

REMARK. Instead of "F-sheaf $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L} \to \mathscr{F} \leftarrow \mathscr{L}$" we shall often say "$F$-sheaf $\mathscr{L}$".

Let $\mathscr{L}$ be an F-sheaf (left or right) over S of rank d. Suppose that the zero and pole of $\mathscr{L}$ are *disjoint of D* (this means, by definition, that the zero and pole are morphisms from S to $X - D$). Denote by $\mathscr{L}_D$ the restriction of $\mathscr{L}$ to $D \times S$. The F-structure on $\mathscr{L}$ induces an isomorphism $f\colon (\mathrm{id}_D \times \mathrm{Fr}_S)^* \mathscr{L}_D \xrightarrow{\sim} \mathscr{L}_D$. In this situation the notion of level D structure on $\mathscr{L}$ is introduced.

DEFINITION. A level D structure on $\mathscr{L}$ is an isomorphism $h\colon \mathscr{L}_D \xrightarrow{\sim} \mathcal{O}_{D \times S}^d$ such that the diagram

$$\begin{array}{ccc} (\mathrm{id}_D \times \mathrm{Fr}_S)^* \mathscr{L}_D & \xrightarrow{(\mathrm{id}_D \times \mathrm{Fr}_S)^* h} & \mathcal{O}_{D \times S}^d \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle h} & \\ & \mathscr{L}_D & \end{array}$$

is commutative.

Let us agree that there are no level D structures if the zero or the pole is not disjoint of D. It is easy to show that a level D structure always exists if S is replaced by an étale covering of S (provided the zero and pole are disjoint of D).

It is quite natural that 2-dimensional vector bundles appear in the proof of Theorem A': the set of isomorphism classes of 2-dimensional vector bundles over X can be identified with $U\backslash \mathrm{GL}(2, \mathfrak{A})/\mathrm{GL}(2, k)$ (where U is the standard maximal compact subgroup of $\mathrm{GL}(2, \mathfrak{A})$), while $\mathrm{GL}(2, \mathfrak{A})/\mathrm{GL}(2, k)$ can be identified with the set of isomorphism classes of 2-dimensional vector bundles over X whose restrictions to all finite subschemes of X are compatibly trivialized.

1.2. THEOREM 1.1. *For every finite subscheme $D \subset X$ and positive integer d, the coarse moduli scheme of left (right) F-sheaves with level D structure exists.*

These schemes will be denoted by ${}^d_D\mathscr{M}$ (for left F-sheaves) and $\mathscr{M}^d_D$ (for right F-sheaves). ${}^d_D\mathscr{M}$ and $\mathscr{M}^d_D$ are schemes over $(X-D)^2=(X-D)\times(X-D)$; the structure morphism $\mathscr{M}^d_D \to (X-D)^2$ or ${}^d_D\mathscr{M} \to (X-D)^2$ maps an (α, β)-F-sheaf to (α, β).

THEOREM 1.2. *There exist open subschemes $U_i \subset \mathscr{M}^d_D$ $(i=1, 2, 3, \ldots)$ such that $U_i \subset U_{i+1}$, $\bigcup_i U_i = \mathscr{M}^d_D$ and for every i U_i is a quotient of a quasiprojective smooth scheme over $(X-D)^2$ of relative dimension $2d-2$ by a finite group. This is also true for ${}^d_D\mathscr{M}$.*

1.3. We shall define some natural morphisms between the moduli schemes constructed above.

(1) Let $\pi: {}^d_D\mathscr{M} \to X^2$, $\bar{\pi}: \mathscr{M}^d_D \to X^2$ be the structure morphisms, denote by Δ the diagonal in X^2. Then $\pi^{-1}(X^2-\Delta)$ can be identified with $\bar{\pi}^{-1}(X^2-\Delta)$; this follows from the remark in 1.2 (after the definition of F-sheaf). Moreover ${}^1_D\mathscr{M}$ can be identified with $\mathscr{M}^1_D$.

(2) If $\mathscr{L}$ is an F-sheaf with level D structure then $\det \mathscr{L}$ is an F-sheaf with level D structure. Therefore we have morphisms $\det: \mathscr{M}^d_D \to \mathscr{M}^1_D$, $\det: {}^d_D\mathscr{M} \to {}^1_D\mathscr{M} = \mathscr{M}^1_D$.

(3) If $\mathscr{L}$ is a left (right) (α, β)-F-sheaf then $\mathscr{L}^*$ is a right (left) (β, α)-F-sheaf; level D structure on $\mathscr{L}$ induces level D structure on $\mathscr{L}^*$. Therefore we have an isomorphism $*: \mathscr{M}^d_D \simeq {}^d_D\mathscr{M}$ such that the diagram

$$\begin{array}{ccc} \mathscr{M}^d_D & \xrightarrow{\ *\ } & {}^d_D\mathscr{M} \\ \downarrow & & \downarrow \\ (X-D)^2 & \xrightarrow{(a,b)\mapsto(b,a)} & (X-D)^2 \end{array}$$

is commutative.

(4) If $\mathscr{L} \xleftarrow{i} \mathscr{F} \xrightarrow{j} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L}$ is a left (α, β)-F-sheaf then

$$(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{F} \xrightarrow{(\mathrm{id}_X \times \mathrm{Fr}_S)^* i} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L} \xleftarrow{j} \mathscr{F}$$

is a right $(\mathrm{Fr}_X \circ \alpha, \beta)$-$F$-sheaf. If $(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L} \xrightarrow{i} \mathscr{F} \xleftarrow{j} \mathscr{L}$ is a right (α, β)-F-sheaf then

$$(\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{F} \xleftarrow{(\mathrm{id}_X \times \mathrm{Fr}_S)^* j} (\mathrm{id}_X \times \mathrm{Fr}_S)^* \mathscr{L} \xrightarrow{i} \mathscr{F}$$

is a left $(\alpha, \mathrm{Fr}_X \circ \beta)$ F-sheaf. In both cases level D structure on $\mathscr{L}$ induces level D structure on $\mathscr{F}$. Therefore we obtain morphisms $F_1: {}^d_D\mathscr{M} \to \mathscr{M}^d_D$ and $F_2: \mathscr{M}^d_D \to {}^d_D\mathscr{M}$ such that the diagrams

$$\begin{array}{ccc} {}^d_D\mathscr{M} & \xrightarrow{F_1} & \mathscr{M}^d_D \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{\mathrm{Fr}_X \times \mathrm{id}_X} & X^2 \end{array} \qquad \begin{array}{ccc} \mathscr{M}^d_D & \xrightarrow{F_2} & {}^d_D\mathscr{M} \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{\mathrm{id}_X \times \mathrm{Fr}_X} & X^2 \end{array}$$

are commutative. Note that $F_1 F_2 = \mathrm{Fr}_{(\mathscr{M}^d_D)}$, $F_2 F_1 = \mathrm{Fr}_{({}^d_D\mathscr{M})}$.

Moral. $\mathscr{M}^d_D$ looks like a product of two schemes over X (in fact it is not a product).

(5) If $D' \supset D$ there are obvious morphisms ${}^d_{D'}\mathscr{M} \to {}^d_D\mathscr{M}$, $\mathscr{M}^d_{D'} \to \mathscr{M}^d_D$.

(6) $\mathrm{GL}(d, \mathfrak{A})$ acts on $\varprojlim_D {}^d_D\mathscr{M}$ and $\varprojlim_D \mathscr{M}^d_D$. We shall not give the precise definition. The definition is based on the following construction. Let $\mathscr{L}$ be an F-sheaf over S of rank d with level D structure, let P be a quotient module of $\mathscr{O}^d_D$. Put $\tilde{P} = i_* \pi^* P$ where π is the projection $D \times S \to D$ and i is the embedding $D \times S \hookrightarrow X \times S$. The level D structure on $\mathscr{L}$ induces an epimorphism $\mathscr{L} \to \tilde{P}$ whose kernel has a natural F-structure.

REMARK. The action of $\mathrm{GL}(d, \mathfrak{A})$ on the set of isomorphism classes of d-dimensional vector bundles over X compatibly trivialized over finite subschemes (recall that this set can be identified with $\mathrm{GL}(d, \mathfrak{A})/\mathrm{GL}(d, k)$) is described by a similar construction.

2. Lang's theory

2.1. Let us consider the case $d=1$. Denote by $\underline{\mathrm{Pic}}_D X$ the moduli scheme of invertible sheaves on X whose restriction to D is trivialized. Recall that (1) $\underline{\mathrm{Pic}}_D X$ is a group scheme over $\boldsymbol{F}_q$; (2) there is an exact sequence $0 \to \underline{\mathrm{Pic}}^0_D X \to \underline{\mathrm{Pic}}_D X \to^{\deg} \boldsymbol{Z} \to 0$, where $\underline{\mathrm{Pic}}^0_D X$ is the connected component of $\mathrm{Pic}_D X$; (3) $\underline{\mathrm{Pic}}^0_D X$ is an algebraic variety (i.e. of finite type); (4) $\underline{\mathrm{Pic}}^0_D X$ is called the generalized Jacobian variety of X of conductor D. There is a natural mapping $X - D \to \underline{\mathrm{Pic}}_D X$ (the image of $u \in (X-D)(\bar{\boldsymbol{F}}_q)$ in $\underline{\mathrm{Pic}}_D X(\bar{\boldsymbol{F}}_q)$ will be denoted by $\bar{u}$).

There is a Cartesian square

$$\begin{array}{ccc} \mathscr{M}^1_D = {}^1_D\mathscr{M} & \longrightarrow & \underline{\mathrm{Pic}}_D X \\ \downarrow & & \downarrow \mathrm{Fr}-1 \\ (X-D)^2 & \xrightarrow{(\alpha,\beta) \mapsto \beta - \bar{\alpha}} & \underline{\mathrm{Pic}}_D X \end{array}$$

Let $y \in \mathscr{M}^1_D(\bar{\boldsymbol{F}}_q)$; $y = (\alpha, \beta, l)$, where $\alpha, \beta \in (X-D)(\bar{\boldsymbol{F}}_q)$, $l \in (\underline{\mathrm{Pic}}_D X)(\bar{\boldsymbol{F}}_q)$, $\mathrm{Fr}(l) - l = \bar{\beta} - \bar{\alpha}$. Then $*y = (\beta, \alpha, -l)$, $F_1(y) = (\mathrm{Fr}(\alpha), \beta, l - \bar{\alpha})$, $F_2(y) = (\alpha, \mathrm{Fr}(\beta), l + \bar{\beta})$. Put $\mathrm{Pic}_D X = (\underline{\mathrm{Pic}}_D X)(\boldsymbol{F}_q)$. $\mathrm{Pic}_D X$ acts on $\mathscr{M}^1_D$: if $(\alpha, \beta, l) \in \mathscr{M}^1_D(\bar{\boldsymbol{F}}_q)$, $m \in \mathrm{Pic}_D X$, then $m \cdot (\alpha, \beta, l) = {}^{\mathrm{def}} = (\alpha, \beta, l + m)$. Therefore $\mathfrak{A}^*/k^* = \varprojlim_D \mathrm{Pic}_D X$ acts on $\varprojlim_D \mathscr{M}^1_D$. We have described the action of $\mathfrak{A}^*$ on $\varprojlim_D \mathscr{M}^1_D$ mentioned in 1.3.

2.2. Theorem A′ follows from a more precise statement (Theorem 4.1) describing the cohomology of the compactified moduli space of F-sheaves of rank 2. We are

going to discuss a similar theorem for $d=1$ (Theorem 2.1), which is a rather awkward reformulation of a part of Lang's theory.

Let $D \subset X$ be a finite subscheme. Recall that $\mathrm{Pic}_D X$ acts on $\mathscr{M}_D^1$ (see 2.1). Choose a splitting of the exact sequence $0 \to \mathrm{Pic}_D^0 X \to \mathrm{Pic}_D X \to^{\deg} \mathbf{Z} \to 0$: $\mathrm{Pic}_D X = \mathrm{Pic}_D^0 X \times J$, $J \simeq \mathbf{Z}$. Denote by π the natural morphism $\mathscr{M}_D^1/J \to (X-D)^2$. Consider $\pi_* \mathbf{Q}_l$ (in the étale sense). The action of $\mathrm{Pic}_D X/J$ on $\mathscr{M}_D^1/J$ induces an action of $\mathrm{Pic}_D X/J$ on $\pi_* \mathbf{Q}_l$. The action of F_1 on $\mathscr{M}_D^1/J$ induces an isomorphism $(\mathrm{Fr}_{X-D} \times \mathrm{id}_{X-D})^* \pi_* \mathbf{Q}_l \simeq \pi_* \mathbf{Q}_l$.

2.3. Definition. A *PF-sheaf* of sets on $(X-D)^2$ is an étale sheaf of sets $\mathscr{F}$ on $(X-D)^2$ with an isomorphism $(\mathrm{Fr}_{X-D} \times \mathrm{id}_{X-D})^* \mathscr{F} \xrightarrow{\sim} \mathscr{F}$.

Remark. Note that $(\mathrm{Fr}_{X-D} \times \mathrm{Fr}_{X-D})^* \mathscr{F} = \mathscr{F}$ for any étale sheaf $\mathscr{F}$ on $(X-D)^2$.

Theorem 2.1. *The category of constructible locally constant PF-sheaves of sets on $(X-D)^2$ is equivalent to the category of finite $\pi_1(X-D) \times \pi_1(X-D)$-sets.*

Remark. The category of constructible locally constant étale sheaves of sets on $(X-D)^2$ is equivalent to the category of finite $\pi_1((X-D)^2)$-sets. But the natural morphism $\to \pi_1((X-D)^2) \to \pi_1(X-D) \times \pi_1(X-D)$ is not an isomorphism: if $D=\emptyset$ it is injective but not surjective (for instance, $\pi_1(\boldsymbol{P}^1 \times \boldsymbol{P}^1) = \hat{\boldsymbol{Z}} \neq \hat{\boldsymbol{Z}} \times \hat{\boldsymbol{Z}}$); if $D \neq \emptyset$ it is even not injective.

2.4. Let us return to the situation of 2.2. $\pi_* \mathbf{Q}_l = (\pi_* \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$, $\pi_* \mathbf{Z}_l = \varprojlim_n \pi_*(\mathbf{Z}/l^n \mathbf{Z})$ and $\pi_*(\mathbf{Z}/l^n \mathbf{Z})$ is a constructive *PF*-sheaf of sets. Therefore $\pi_* \mathbf{Q}_l$ corresponds to an l-adic representation of $\pi_1(X-D) \times \pi_1(X-D)$. $\mathrm{Pic}_D X/J$ acts on $\pi_* \mathbf{Q}_l$, therefore we have, in fact, a representation of

$$\pi_1(X-D) \times \pi_1(X-D) \times (\mathrm{Pic}_D X/J);$$

denote it by V_l.

Theorem 2.2.

$$V_l \underset{\mathbf{Q}_l}{\otimes} \bar{\mathbf{Q}}_l = \bigoplus_{\chi \in \mathrm{Hom}(\mathrm{Pic}_D X/J,\, \bar{\mathbf{Q}}_l^*)} \varrho_\chi \otimes \varrho_\chi^{-1} \otimes \chi$$

where $\varrho_\chi \in \mathrm{Hom}(\pi_1(X-D), \bar{\mathbf{Q}}_l^*)$ is compatible with χ (this means that for every closed point $v \in X-D$ $\varrho_\chi(F_v) = \chi(\bar{v})$, where $F_v \in \pi_1(X-D)$ is the geometric Frobenius element of v and $\bar{v}$ is the canonical image of v in $\mathrm{Pic}_D X/J$).

3. Compactification

3.1. Denote by K the fraction field of $k \otimes k$, put $\mathscr{N}_D = \mathscr{M}_D^2 \otimes_{X \times X} K = {}^2_D\mathscr{M} \otimes_{X \times X} K$. We are going to describe the nature of nonquasicompactness of $\mathscr{N}_D$.

For every $n \in \mathbf{Z}$ denote by $\mathscr{N}_D^n$ the open subscheme of $\mathscr{N}_D$ corresponding to sheaves of degree n; then $\mathscr{N}_D = \coprod_n \mathscr{N}_D^n$. For every $m, n \in \mathbf{Z}$ denote by $\mathscr{N}_D^{m,n}$ the open subscheme of $\mathscr{N}_D^n$, corresponding to F-sheaves $\mathscr{L}$ which do not contain invertible subsheaves of degree more than m; $\mathscr{N}_D^{m,n} \subset \mathscr{N}_D^{m+1,n}$ and $\mathscr{N}_D^n = \bigcup_m \mathscr{N}_D^{m,n}$.

THEOREM 3.1. *$\mathcal{N}_D^{m,n}$ is a quasiprojective surface. If $2m-n\geqslant 2$ then $(\mathcal{N}_D^n\otimes_K\bar{K})-(\mathcal{N}_D^{m,n}\otimes_K\bar{K})$ is isomorphic to the countable disjoint sum of affine lines.*

3.2. If $2m-n\geqslant 2$, $\mathcal{N}_D^{m,n}$ has a canonical compactification $\bar{\mathcal{N}}_D^{m,n}$. We are going to describe its properties.

(1) $\bar{\mathcal{N}}_D^{m,n}$ is a proper algebraic space over K (I don't know whether it is a scheme), $\mathcal{N}^{m,n}$ is an open dense subspace of $\bar{\mathcal{N}}_D^{m,n}$.

(2) $\bar{\mathcal{N}}_D^{m,n}$ is normal. If Y is a fibre of a singularity resolution of $\bar{\mathcal{N}}_D^{m,n}\otimes_K\bar{K}$ then $H^0(Y,\mathbf{Q}_l)=\mathbf{Q}_l$, $H^1_{et}(Y,\mathbf{Q}_l)=0$ (in other words, Y_{red} is either a point or a curve whose irreducible components are rational and whose intersection graph is a tree).

(3) If $m'\geqslant m$ there is a commutative diagram

$$\begin{array}{ccc}\bar{\mathcal{N}}_D^{m',n} & \longrightarrow & \bar{\mathcal{N}}_D^{m,n}\\ \uparrow & & \uparrow\\ \mathcal{N}_D^{m',n} & \hookleftarrow & \mathcal{N}_D^{m,n}\end{array}$$

(4) Put $\Gamma_D^{m,n}=\bar{\mathcal{N}}_D^{m,n}-\mathcal{N}_D^{m,n}$ ($\Gamma_D^{m,n}$ is considered as a reduced subspace). The morphism $\Gamma_D^{m+1,n}\to\Gamma_D^{m,n}$ induced by $\bar{\mathcal{N}}_D^{m+1,n}\to\bar{\mathcal{N}}_D^{m,n}$ is radical; this means that it is finite and its geometric fibres contain exactly one point (but they may not be reduced).

(5) Let $m'\geqslant m$. The morphism $\bar{\mathcal{N}}_D^{m',n}\otimes_K\bar{K}\to\bar{\mathcal{N}}_D^{m,n}\otimes_K\bar{K}$ maps each irreducible component of $\mathcal{N}_D^{m',n}\otimes_K\bar{K}-\mathcal{N}_D^{m,n}\otimes_K\bar{K}$ onto a point of $\Gamma_D^{m,n}\otimes_K\bar{K}$ (different components correspond to different points).

(6) If $D'\supset D$ the morphism $\mathcal{N}_{D'}^{m,n}\to\mathcal{N}_D^{m,n}$ is the restriction of a finite morphism $\bar{\mathcal{N}}_{D'}^{m,n}\to\bar{\mathcal{N}}_D^{m,n}$.

(7) The action of $\mathrm{GL}(2,\mathfrak{A})$, F_1, F_2 and $*$ on $\varprojlim_D\mathcal{N}_D$ is the restriction of an action of $\mathrm{GL}(2,\mathfrak{A})$, F_1, F_2 and $*$ on $\Pi_n\varprojlim_{D,m}\bar{\mathcal{N}}_D^{m,n}$ (the projective limit is understood formally).

3.3. DEFINITION. The R-category is the category of schemes localized by radical morphisms.

$\Gamma_D^{m,n}$ considered as an object of the R-category will be denoted by Γ_D^n (because it doesn't depend on m). We are going to describe $\Gamma_\varnothing^n$.

Denote by $Y^{i,j}$ the fibred product of $X\times X\times X$ by $\underline{\mathrm{Pic}}^iX\times\underline{\mathrm{Pic}}^jX$ over $\underline{\mathrm{Pic}}^0X\times\underline{\mathrm{Pic}}^0X$, the morphisms $X\times X\times X\to\underline{\mathrm{Pic}}^0X\times\underline{\mathrm{Pic}}^0X$ and $\underline{\mathrm{Pic}}^iX\times\underline{\mathrm{Pic}}^jX\to\underline{\mathrm{Pic}}^0X\times\underline{\mathrm{Pic}}^0X$ being given respectively by $(\alpha,\xi,\beta)\mapsto(\xi-\bar{\alpha},\beta-\bar{\xi})$ and $\mathrm{Fr}-1$. The morphism $Y^{i,j}\to Y^{i+1,j-1}$ given by $(\alpha,\xi,\beta,a,b)\to(\alpha,\mathrm{Fr}(\xi),\beta,a+\bar{\xi},b-\bar{\xi})$ (where $\alpha,\ \xi,\ \beta\in X(\bar{F}_q)$, $a\in\underline{\mathrm{Pic}}^iX(\bar{F}_q)$, $b\in\underline{\mathrm{Pic}}^jX(\bar{F}_q)$) is radical, therefore in the R-category $Y^{i,n-i}$ doesn't depend on i. The result is: $\Gamma_\varnothing^n=Y^{i,n-i}\otimes_{X\times X}K$, the morphism $Y^{i,n-i}\to X\times X$ being given by $(\alpha,\xi,\beta)\mapsto(\alpha,\beta)$.

The description of Γ_D^n for $D\neq\varnothing$ is more complicated; the main difference from the case $D=\varnothing$ is that Γ_D^n is not R-isomorphic to a smooth curve.

3.4. The construction of $\bar{\mathcal{N}}_D^{m,n}$ is based on the notion of *degenerate F-sheaf*. Let E be a field, $\alpha\in X(E)$, $\beta\in X(E)$, $\alpha\neq\beta$.

DEFINITION. Let $\mathcal{L}$ be a 2-dimensional locally free sheaf on $X\otimes E$. An (α,β)-*transform of* $\mathcal{L}$ is a subsheaf $\mathcal{L}'\subset\mathcal{L}(\beta)$ such that $\operatorname{Supp}\mathcal{L}(\beta)/\mathcal{L}'=\{\alpha,\beta\}$ and $\dim_E H^0(\mathcal{L}(\beta)/\mathcal{L}')=2$ (we identify α and β with the corresponding points of $X\otimes E$).

DEFINITION. A *degenerate* (α,β)-*F-sheaf over* E *of rank* 2 is a triple $(\mathcal{L},\mathcal{L}',\varphi)$ consisting of a 2-dimensional locally free sheaf $\mathcal{L}$ on $X\otimes E$, its (α,β)-transform $\mathcal{L}'$ and a morphism $\varphi\colon(\mathrm{id}_X\times\mathrm{Fr}_E)^*\mathcal{L}\to\mathcal{L}'$ such that (1) φ has rank 1 at each point of $X\otimes E$, (2) $\operatorname{Im}\varphi\not\subset\mathcal{L}$, (3) $\varphi|_{(\mathrm{id}_X\times\mathrm{Fr}_E)^*(\mathcal{L}\cap\operatorname{Im}\varphi)}\neq 0$.

In this situation put $A=\mathcal{L}\cap\operatorname{Im}\varphi$. φ induces an injection $\varphi'\colon(\mathrm{id}_X\times\mathrm{Fr}_K)^*A\to\operatorname{Im}\varphi=A(\beta)$. $\operatorname{Supp}\operatorname{Coker}\varphi'$ consists of a single point ξ.

DEFINITION. ξ is called the *degenerator of* $\mathcal{L}$.

REMARK. The third projection $\Gamma_D^{m,n}\to X$ is given by the degenerator.

4. Proof of Theorems A, B and C

4.1. Choose a subgroup $J\subset\mathfrak{A}^*$ such that the composition $J\to\mathfrak{A}^*/k^*\xrightarrow{\deg}\mathbf{Z}$ is an isomorphism. Let $l\neq p$ be a prime number. The action of $\mathrm{GL}(2,\mathfrak{A})$ on $\Pi_{n\in\mathbf{Z}}\varinjlim_{D,m}\bar{\mathcal{N}}_D^{m,n}$ induces an action of $\mathrm{GL}(2,\mathfrak{A})$ on $\Pi_n\varinjlim_{D,m}H^*_{et}(\bar{\mathcal{N}}_D^{m,n}\otimes_K\bar{K},\mathbf{Q}_l)$; put $V_l^i=[\Pi_n\varinjlim_{D,m}H^i_{et}(\bar{\mathcal{N}}_D^{m,n}\otimes_K\bar{K},\mathbf{Q}_l)]^J$. Note that the representation of $\mathrm{GL}(2,\mathfrak{A})$ on V_l^2 is not admissible.

Put $F\,\mathrm{Gal}(\bar{K}/K)=\{\gamma\in\mathrm{Aut}\,\bar{K}\mid\exists m,n\in\mathbf{Z}\ \gamma|_{k\otimes k}=\mathrm{Fr}_k^m\otimes\mathrm{Fr}_k^n\}/\{\mathrm{Fr}_K^n\mid n\in\mathbf{Z}\}$. The action of F_1 and F_2 on $\Pi_n\varinjlim_{D,m}\bar{\mathcal{N}}_D^{m,n}$ and the action of $\mathrm{Gal}(\bar{K}/K)$ on $\bar{K}$ induce an action of $F\,\mathrm{Gal}(\bar{K}/K)$ on V_l^i. There is a canonical homomorphism $f\colon F\,\mathrm{Gal}(\bar{K}/K)\to\mathrm{Gal}(\bar{k}/k)\times\mathrm{Gal}(\bar{k}/k)$: an element of $F\,\mathrm{Gal}(\bar{K}/K)$ represented by $\gamma\in\mathrm{Aut}\,\bar{K}$ is mapped to $(\gamma\,\mathrm{Fr}_K^{-m}|_{\bar{k}\otimes 1},\ \gamma\,\mathrm{Fr}_K^{-n}|_{1\otimes\bar{K}})$, where $\gamma|_{k\otimes k}=\mathrm{Fr}_k^m\otimes\mathrm{Fr}_k^n$. The image of f contains $W\times W$. One can deduce from theorem 2.1. that the action of $F\,\mathrm{Gal}(\bar{K}/K)$ on V_l^i is trivial on $\operatorname{Ker}f$. Therefore we have a representation of $W\times W\times\mathrm{GL}(2,\mathfrak{A})$ on V_l^i.

Denote by V_l^{cusp} the greatest $\mathrm{GL}(2,\mathfrak{A})$-invariant subspace of V_l^2 contained in $[\Pi_n\varinjlim_D H^2_{et}(\bar{\mathcal{N}}_D^{[(n+3)/2],n},\mathbf{Q}_l)]^J$. V_l^{cusp} is $W\times W$-invariant. The representation of $\mathrm{GL}(2,\mathfrak{A})$ on V_l^{cusp} is already admissible.

THEOREM 4.1. $V_l^{\mathrm{cusp}}\otimes_{\mathbf{Q}_l}\bar{\mathbf{Q}}_l=\oplus_{\pi\in\Sigma_2^J}\varrho_\pi\otimes\varrho_{\hat{\pi}}\otimes\pi$, *where* $\Sigma_2^J=\{\pi\in\Sigma_2\mid$ *the restriction of* π *to* J *is trivial*$\}$, $\hat{\pi}$ *denotes the representation contragradient to* π, ϱ_π *is a 2-dimensional representation of* W *over* $\mathbf{Q}_l$ *compatible with* π.

Theorem A′ follows from Theorem 4.1.

REMARK. The proof of Theorem 4.1. gives in fact a complete description of V_l for all i. It turns out that $V_l^1 = V_l^3 = 0$, V_l^2 is the direct sum of V_l^{cusp} and its orthogonal complement V_l^{Eis}, V_l^{Eis} has an explicit description, the description of V_l^0 and V_l^4 follows from the fact that the fibres of det: $\mathcal{N}_D \otimes_K \bar{K} \to \mathcal{M}_D^1 \otimes_{X \times X} \bar{K}$ are connected.

4.2. The proof of Theorem 4.1. is based on Lefschetz formula, Selberg trace formula and an explicit description of $\mathcal{M}_D^2(\bar{F}_q)$ similar to Deuring's results on elliptic curves over F_p. This method was used in [6] and [8]. It turns out that the structure of the spaces $\mathcal{N}_D^{m,n}$ is closely related to the structure of Selberg trace formula. For instance, the fact that Γ_D^n is a scheme over $X \times X \times X$ (not only $X \times X$) corresponds to the fact that Selberg trace formula involves integrals of the form $\int f(s) d \ln L(s)$, where $L(s)$ is an abelian L-function (so that in order to compute the trace of the Hecke operator corresponding to a place of k one has to use L-functions containing information on *all* places of k).

4.3. Let v and w be closed points of X, $v \neq w$. The proof of Theorem 4.1 uses the computation of the ζ-function of an open subscheme of $\mathcal{M}_D^2 \times_{X^2} (v \times w)$ (this ζ-function is expressed in terms of cusp forms). Theorem B follows from the fact that the poles of the ζ-function of a surface over F_q are on the lines $\operatorname{Re} s = 0, \frac{1}{2}, 1, 2$. This fact is a consequence of Weil's conjecture proved by Deligne and Abhyankar's theorem (resolution of singularities for surfaces).

If every field finitely generated and of transcendence degree 3 over $\bar{F}_p$ has a smooth projective model, the zeroes and poles of the ζ-function of a 3-dimensional scheme over F_q are on the lines $\operatorname{Re} s = n/2$, $n \in \mathbf{Z}$. Applying this to a suitable open subscheme of $\mathcal{M}_D^2 \times_{X^2} (X \times v)$ for some closed point $v \in X - D$, we obtain the second part of Theorem C.

In [2], [3] Theorem A' was proved assuming that π_∞ is in the discrete series. In [3] the representation of W corresponding to π appears as a part of $H_{et}^1(F \otimes_X \bar{k}, \mathbf{Q}_l)$ for some scheme F over X of relative dimension 1. The zeroes and poles of $\zeta(F, s)$ are on the lines $\operatorname{Re} s = n/2$, $n \in \mathbf{Z}$, because F is a surface. This enables us to prove the first part of theorem C.

4.4. The surface F mentioned in 4.3 is the moduli scheme of *elliptic modules* (see [2]) with some additional structures. Elliptic modules are functional analogues of elliptic curves, and the result of [3] mentioned in 4.3 is analogous to Eichler–Shimura's Theorem.

On the other hand, elliptic modules are, in some sense, F-sheaves of special type (this is not an obvious consequence of the definitions of elliptic module and F-sheaf). The precise statement and proof can be found in [4], [9]. The method use 4] is analogous to the method used by Burchnall, Chaundy, Baker and Kričever in order to describe the commutative rings of differential operators (see [7], [9]).

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Algebraic Representations of Reductive Groups

George R. Kempf

My original interest in this area arose from an effort to prove the following statement.

THEOREM. *Let X be a complete homogeneous space under a reductive algebraic group G. If $\mathscr{L}$ is an invertible sheaf on X with a nonzero section, then the cohomology groups $H^i(X, \mathscr{L})$ are zero whenever i is positive.*

When the characteristic of the ground field is zero, this theorem was originally proven by A. Borel and A. Weil in the early 1950s. They deduced it from Kodaira's vanishing theorem. In positive characteristics, the result seems more difficult to prove. Under the assumption (A) that the stabilizer group scheme of a point of X is reduced, I was able to prove the theorem in the paper [**8**] (with an argument which made no reference to the characteristic). It would be interesting to know to what extent this theorem remains true when the assumption (A) is dropped.

For the reader who is not interested in cohomological vanishing theorems, I will explain some of the significance of the theorem. The theorem implies that the vector space $\Gamma(X, \mathscr{L})$ has the right dimension, i.e., $\dim \Gamma(X, \mathscr{L})$ equals $\sum(-1)^i \dim H^i(X, \mathscr{L})$, which is the easily computed Euler characteristic of the sheaf $\mathscr{L}$. Also it implies that $\dim \Gamma(X, \mathscr{L})$ is locally constant (better yet, $\Gamma(X, \mathscr{L})$ varies continuously with the parameters) when X and $\mathscr{L}$ are varied in an algebraic family (see [**9**, § 13]). The most important case of these variational consequences is when one is actually changing the characteristic of the ground field.

One main reason for looking at invertible sheaves $\mathscr{L}$ on the homogeneous space X is that they provide standard representations of the group G. If we are given a G-linearization of $\mathscr{L}$ [**10**], then $\Gamma(X, \mathscr{L})$ has a natural action of G and is called an induced representation of G. Another classical result of Borel and Weil is that,

in characteristic zero, the induced representations are irreducible. This irreducibility fails in characteristic p. A current topic of interest is to determine to what extent it fails. Some information on this problem may be found in [4], [5], [6] and [3].

In some sense the truth of the theorem seems to be a miracle. In characteristic zero, Bott generalized the theorem as follows. He proved that, for an arbitrary $\mathscr{L}$, there is only one level i for which $H^i(X, \mathscr{L})$ is not zero. Unfortunately, Bott's form of the theorem no longer holds when the characteristic is finite. In his thesis [2] and, also [12], Mumford's student W. L. Griffith systematically determined when Bott's formulation fails in the case where X is the variety of flags in a three dimensional vector space.

One wants to know as much as possible about the representations of G on the cohomology $H^i(X, \mathscr{L})$ of a G-linearized $\mathscr{L}$. In characteristic zero, Bott settled this over twenty years ago. In characteristic p, this difficult problem is currently being attacked by Griffith and H. H. Anderson [1].

When I was working on proving the theorem, I first tried to prove it G-equivariantly. This approach never achieved any results. I only made progress with a B-equivariant approach where B is a Borel subgroup of G. One may filter the vector space $\Gamma(X, \mathscr{L})$ in a B-equivariant way and get better control on what is, or is not, in $\Gamma(X, \mathscr{L})$. One would like to have some standard ways of writing a T-eigenbasis for $\Gamma(X, \mathscr{L})$, which are compatible with the natural B-filtrations of $\Gamma(X, \mathscr{L})$, where T is a maximal torus contained in B.

This problem about the internal structure of the induced representations is quite interesting. One approach to it, which has been developed by a multitude of authors, attempts to generalize Hodge's standard monomial theorem for the Plücker coordinate ring over the Grassmann varieties. Currently this approach is being developed by Seshadri and his associates [11]. Also see Processi's talk in these PROCEEDINGS for his work on standard monomials and the group ring $\Gamma(G, \mathscr{O}_G)$ for classical groups.

In closing, I want to mention two topics which I would like to see better developed. One involves the singularities of Schubert varieties, i.e., the closure of a B-orbit in X. For an initial problem, determine the singular locus of a Schubert variety. The other problem asks for a different proof of the theorem. Can anyone prove directly with Lie algebra methods that the Grothendieck–Cousin complex of $\mathscr{L}$ [7] and [9] is exact when G is a Chevalley group over $\mathbf{Z}$?

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Lie Algebras and Combinatorics

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One of the most intriguing of combinatorial formulas is the Rogers–Ramanujan identity

$$(1) \qquad \prod_{j\geq 1}(1-q^{5j-1})^{-1}(1-q^{5j-4})^{-1} = 1+\sum_{j\geq 1} q^{j^2}/(1-q)(1-q^2)\dots(1-q^j),$$

where q is an indeterminate. See [1] for the history, proof, combinatorial interpretation and generalization of (1) and its companion identity. I shall describe the evolution of a program, just completed, for understanding such identities as (1) in terms of Kac–Moody Lie algebras. This program is related to topology, and probably physics,[1] as well as combinatorics.[2]

A Kac–Moody Lie algebra ([9], [16]) is defined by a generalized Cartan matrix (GCM)—an $n\times n$ integral matrix A such that the diagonal entries are 2, the other entries are ≤ 0, and there is an $n\times n$ positive definite rational diagonal matrix D such that DA is symmetric. Consider the complex Lie algebra defined by $3n$ generators h_i, e_i, f_i $(i=1, \dots, n)$ subject to the relations $[h_i, h_j]=0$, $[h_i, e_j]=A_{ij}e_j$, $[h_i, f_j]=-A_{ij}f_j$, $[e_i, f_j]=\delta_{ij}h_i$ and for $i\neq j$, $(\operatorname{ad} e_i)^{-A_{ij}+1}e_j=0=(\operatorname{ad} f_i)^{-A_{ij}+1}f_j$.[3] The Kac–Moody (or GCM) Lie algebra $I(A)$ is the quotient of this algebra by a certain radical, conjectured to be zero. If A is the Cartan matrix of a finite-dimen-

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[1] and finite group theory (see footnote 4).

[2] Also, surprising connections have been found between certain non-Euclidean Kac–Moody Lie algebras and certain cusps on Hilbert modular surfaces [19].

[3] Cf. N. Jacobson, *Lie Algebras*, Interscience, New York, 1962, Chapter VII and J.-P. Serre, *Algèbres de Lie semi-simples complexes*, Benjamin, New York, 1966, Chapter VI, Appendice.

sional semisimple Lie algebra $\mathfrak{g}$, then $I(A)=\mathfrak{g}$.[3] Otherwise, $I(A)$ is infinite-dimensional. See also [9 (e)].

Let A correspond to a simple $\mathfrak{g}$ of rank l. Then $A_{ij}=2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, $i, j=1, \ldots, l$, where the α_i are the simple roots and $(\cdot, \cdot)$ is the usual inner product. Let α_0 be minus the highest root, and define $\tilde{A}_{ij}=2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ with $i, j=0, \ldots, l$. Then $\tilde{A}$ is an $(l+1)\times(l+1)$ GCM. Denote $I(\tilde{A})$ by $\hat{\mathfrak{g}}$. The algebras $\hat{\mathfrak{g}}$ are "most" of the "Euclidean" Kac–Moody Lie algebras ([9 (a)], [16 (a)], [16 (b)]). The center $\mathfrak{c}$ of $\hat{\mathfrak{g}}$ is 1-dimensional, and $\hat{\mathfrak{g}}/\mathfrak{c}=\tilde{\mathfrak{g}}$, which is defined as $\mathfrak{g}\otimes_C C[t, t^{-1}]$. ($C[t, t^{-1}]$ is the algebra of Laurent polynomials in t, the $\otimes$ is ring extension, and $\tilde{\mathfrak{g}}$ is the corresponding complex Lie algebra.)

Let $\varphi(q)=\prod_{j\geq 1}(1-q^j)$ and let $\eta(q)=q^{1/24}\varphi(q)$ (Dedekind's η-function). Euler's formula

$$\eta(q)=\sum_{j\in Z}(-1)^j q^{(1/24)(6j+1)^2}, \tag{2}$$

Gauss' formula $\eta(q)^2/\eta(q^2)=\sum_{j\in Z}(-1)^j q^{j^2}$ and Jacobi's formula $\eta(q)^3=\sum_{j\geq 0}(-1)^j(2j+1)q^{(1/8)(2j+1)^2}$ follow by appropriate specializations of Jacobi's 2-variable "triple product identity" (cf. [1]). Also, (2) is equivalent to Euler's recursion for the classical partition function (cf. [1]). For each simple $\mathfrak{g}$, Macdonald found a remarkable multivariable power series identity [15 (a)] generalizing Jacobi's triple product identity, the case $\mathfrak{g}=\mathfrak{sl}(2, C)$. Suitable specialization gave a formula for $\eta(q)^{\dim \mathfrak{g}}$ [15 (a)], generalizing Jacobi's formula for $\eta(q)^3$. Dyson [unpublished] independently found these multivariable and 1-variable formulas for $\mathfrak{g}$ classical (cf. [3]). Macdonald in fact found a somewhat broader family of identities, corresponding to the "affine root systems" [15 (a)].

Kac [9 (c)] and Moody [16 (c)] independently recognized Macdonald's multivariable identities as precisely the assertions of Weyl's denominator formula (the product expansion of the denominator in Weyl's character formula) for the Euclidean Lie algebras. Kac [9 (c)] at the same time reproved and considerably generalized these identities by proving the assertions of Weyl's character *and* denominator formulas for *all* Kac–Moody Lie algebras using a refinement of the method of Bernstein–Gelfand–Gelfand [2 (a)]. (For an exposition, see [12 (f)].) Kac's generalizations of finite-dimensional irreducible $\mathfrak{g}$-modules are called *standard* modules.

Now $\mathfrak{g}=\mathfrak{g}\otimes t^0$ behaves like the reductive part of a parabolic subalgebra of $\tilde{\mathfrak{g}}$, with the subalgebra $\mathfrak{u}\equiv\mathfrak{g}\otimes\{f(t)\in C[t] \mid f(0)=0\}$ playing the role of the nilpotent part. By determining $H^*(\mathfrak{u})$ as a graded $\mathfrak{g}$-module, Garland [5] reinterpreted Macdonald's identity for $\mathfrak{g}$ via the Euler–Poincaré principle. This determination and Kostant's classical analogue for parabolics of semisimple Lie algebras [11 (b)] are together generalized to Kac–Moody Lie algebras, and in fact the "weak Bernstein–Gelfand–Gelfand resolution" [2 (b)] is proved in the same generality, in a joint paper with Garland [6]. This gives a new proof of the Dyson–Macdonald–Kac formulas. The possibility of using a resolution to prove the Dyson–Macdonald identities was conjectured by Verma [18].

Let G be the compact simply connected Lie group associated with $\mathfrak{g}$. Garland empirically observed [5] that the number of irreducible $\mathfrak{g}$-module components in $H^j(\mathfrak{u})$ equals $\dim H^{2j}(\Omega(G), C)$ (which had been determined by Bott). Now $\tilde{\mathfrak{g}}$ may clearly be viewed as the Lie algebra of algebraic functions from the circle into $\mathfrak{g}$. In a formal sense, this is the complex Lie algebra of the free loop group $\Lambda(G)$ of G. Since $\Omega(G)=\Lambda(G)/G$ (G acting by translations on $\Lambda(G)$), $\tilde{\mathfrak{g}}/\mathfrak{g}$ is formally the tangent space at the origin of $\Omega(G)$. It is thus plausible that $H^*(\tilde{\mathfrak{g}}, \mathfrak{g}) \simeq H^*(\Omega(G), C)$ as graded vector spaces, and this is proved in [**12** (d)], by generalization to Kac–Moody Lie algebras of Kostant's work [**11** (c)] on generalized flag manifold cohomology. The general results on relative Lie algebra cohomology in [**12** (d)] uniformly "explain" both Garland's empirical equality and Bott's analogous classical "strange equality", which Kostant [**11** (c)] had previously explained a different way. For Euclidean Lie algebras, these results are combined in [**12** (d)] with Kac's classification [**9** (b)] of the automorphisms of finite order of semisimple Lie algebras to illuminate additional work of Bott on path space cohomology. In particular, $\Omega(G)$ and certain more general path spaces studied by Bott turn out to be good analogues for Euclidean Lie algebras of generalized flag manifolds [**12** (d)]. (Cf. also [**7**].)

The method of combining relative homological algebra with generalized Verma modules in [**12** (d)] applies also to a couple of aspects [**12** (a)], [**12** (b)] of the broad problem of constructing a purely algebraic theory of Harish-Chandra modules.

It is natural to choose as power series variables in the Weyl–Macdonald–Kac formulas the exponentials of minus the simple roots of the corresponding Kac–Moody Lie algebra [**12** (d)], and this leads to the interpretation of Dyson's and Macdonald's identities as multivariable vector partition theorems [**13**]. Combining this choice of variables with ideas from [**9** (b)] and [**15** (a)], one can naturally associate a 1-variable specialization [**12** (c)], [**12** (d)] of a multivariable Macdonald identity to every automorphism θ of finite order of $\mathfrak{g}$, in such a way that the Dyson–Macdonald identities for $\eta(q)^{\dim \mathfrak{g}}$ come from $\theta=1$ (cf. also [**9** (d)], [**15** (b)]). Choosing θ to be regular instead, we get interesting "regular specializations": The "principal" automorphism [**11** (a)] of order equal to the Coxeter number h of $\mathfrak{g}$ produces the *principal specialization* [**12** (d)]—the setting of the exponentials of minus the simple roots of $\hat{\mathfrak{g}}$ (or by extension of the terminology, any Kac–Moody Lie algebra) all equal to q. For example, principal specialization for $\mathfrak{sl}(n, C)\hat{}$ gives

$$\eta(q)^n/\eta(q^n) = \sum (\operatorname{sgn}\sigma) q^{(1/2)\sum_{i=1}^n (\sigma(i)-i+\mu_i)^2},$$

the sum being over σ in the symmetric group of $\{1, \ldots, n\}$ and over $\mu_1, \ldots, \mu_n \in n\boldsymbol{Z}$ subject to $\sum \mu_i = 0$. This generalizes and "explains" Gauss' formula, and by (2), it gives a formula for $\eta(q)^n$. The (essentially unique) inner regular automorphism of $\mathfrak{g}$ of order $h+1$ gives

$$\eta(q)^{\operatorname{rank}\mathfrak{g}} = \sum (\operatorname{sgn} w) q^{(1/4)(h+1)\|w(2\varrho)-\varrho'/(h+1)\|^2},$$

where the sum is over w in the affine Weyl group of $\mathfrak{g}$, ϱ (resp., ϱ') is half the sum

of the positive roots (resp., co-roots), and $\|\cdot\|^2$ is the canonical quadratic form ([**12** (c)], [**12** (d)]; the same specialization had already been used in [**15** (a)]). This includes and "explains" (2) as the case $\mathfrak{g}=\mathfrak{sl}(2, C)$. One thus gets formulas for arbitrary positive powers of $\eta(q)$, and hence several new formulas for Ramanujan's τ-function different from Dyson's [**3**].

To pass from formulas for $\varphi(q)^{\dim \mathfrak{g}}$ to formulas for $\eta(q)^{\dim \mathfrak{g}}$, Dyson and Macdonald had invoked the Freudenthal–de Vries "strange formula" $\|\varrho\|^2=(\dim \mathfrak{g})/24$. The analogous transition from $\varphi(q)^{\operatorname{rank} \mathfrak{g}}$ to $\eta(q)^{\operatorname{rank} \mathfrak{g}}$ uses a new type of strange formula [**12** (c)], [**12** (d)]:

$$(1/4)(h+1)\|2\varrho-\varrho'/(h+1)\|^2 = (\operatorname{rank} \mathfrak{g})/24.$$

There is a similar new strange formula associated with principal specialization [**12** (c)], [**12** (d)]. Kac (using ideas of Deligne and Kazhdan) and Macdonald have independently generalized these strange formulas and η-function identities ([**9** (d)], [**15** (b)]).

These identities all come from specializations of the denominator formula for Euclidean Lie algebras. In joint work with Milne [**13**], a link was discovered between the Weyl–Kac *character* formula and the Rogers–Ramanujan identities: Principal specialization of the character of a certain standard module, say V, for $\mathfrak{sl}(2, C)\hat{}$ equals F times the left-hand side of (1), where the "fudge factor" F is $\prod_{j\geq 1}(1-q^{2j-1})^{-1}$. This surprising finding was illuminated by the recognition [**12** (e)] that principal specialization of the numerator in the Weyl–Kac character formula for any standard module of any Kac–Moody Lie algebra has a certain product expansion, the "numerator formula"; see also [**9** (d)]. Stanley [**17**] has observed that for finite-dimensional simple Lie algebras, this numerator formula, which was essentially well known, simplifies Hughes' proof [**8**] that the polynomial $\prod_{i=1}^{n}(1+q^i)$, and certain other polynomials, are unimodal. Using the numerator formula, one also obtains from the Dyson–Macdonald identity for $\mathfrak{sl}(n, C)$ a formula [**12** (e)] for the reciprocal of the generating function for n-rowed plane partitions, generalizing Euler's recursion (2), the case $n=1$.

In a first attempt to prove (1) by Lie-algebraic methods, the characters of V and of other standard modules were "directly" computed in a joint paper with Feingold [**4**], leading not to (1), but to two new formulas equating the left-hand side of (1) with expressions involving interlocking recursions. It was also unexpectedly found [**4**] that the weight multiplicities of the "basic" $\mathfrak{sl}(2, C)\hat{}$-module V_0 are precisely the values of the classical partition function; this led (using the numerator formula) to a Lie-algebraic interpretation of a second Gauss formula, $\varphi(q^2)^2/\varphi(q)=\sum_{j\in Z} q^{j(2j+1)}$. These first character computations for Euclidean Lie algebras have been extended by Kac, giving new multivariable identities [**9** (d)].[4]

[4] These turn out to be related to the modular function j [**20**], suggesting the striking possibility of a connection between the "monster" sporadic simple group and Euclidean Lie algebras, in view of recent empirical discoveries of McKay, Thompson, Conway and Norton.

Let $\mathfrak{g}=\mathfrak{sl}(2, \boldsymbol{C})$. A "construction" of $\hat{\mathfrak{g}}$ is accomplished in joint work with Wilson [14]: The occurrence of the partition function in the character formula for V_0 suggested trying to realize V_0 concretely as a space of polynomials in infinitely many variables and to construct $\hat{\mathfrak{g}}$ as an algebra of differential operators on V_0. The result is that $\hat{\mathfrak{g}}$ acts faithfully on $\boldsymbol{C}[x_1, x_3, x_5, \ldots]$, this algebra being graded by setting $\deg x_j=-j$, and $\hat{\mathfrak{g}}$ is spanned by the identity, the multiplications and differentiations by the x_j, and the homogeneous components of

$$\exp\left(\sum 2x_j/\sqrt{j}\right)\exp\left(-\sum 2(\partial/\partial x_j)/\sqrt{j}\right),$$

where the sums are over all positive odd integers j [14]. This construction gives the central extension $\hat{\mathfrak{g}}$ the appearance of a "quantization" of the function space $\tilde{\mathfrak{g}}$. The "exponential generating function" for $\hat{\mathfrak{g}}$ is strikingly similar to the vertex operator in the string model in quantum field theory, as Garland has observed. In [10], this construction is generalized to all Euclidean Lie algebras (using at one point the numerator formula together with, coincidentally, exactly the two "regular specializations" discussed above).

The realization of $\hat{\mathfrak{g}}$ depends on the presence of a "principal" infinite-dimensional Heisenberg subalgebra of $\hat{\mathfrak{g}}$ [14], which at the same time turns out to "explain" the fudge factor F. It has now been shown that (1) "counts" the dimensions of certain spaces of polynomials associated with the concrete realization [14] of $\hat{\mathfrak{g}}$ and that (1) has a Lie-algebraic interpretation and proof (current joint work with Wilson and Kac).

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Tate Theory for Reductive Groups and Distinguished Representations

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Dedicated to the Baire family for their generous support which made possible the establishment of the Baire chair at Tel-Aviv University

J. Tate in his famous thesis introduced L and ε factors and showed that the classical global L and ε are products of local ones. R. P. Langlands [2] conjectured that a similar theory is true for an arbitrary reductive group G. More precisely, he conjectured that it is possible to attach to each (or almost each) irreducible representation π_p of G_{k_p} L and ε factors in such a way that for each irreducible automorphic representation $\pi = \otimes \pi_p$ of G_A the function

$$L(\pi, s) = \prod_p L_p(\pi_p, s) \tag{1}$$

has a meromorphic continuation (with a finite number of poles) to the whole complex plane and a functional equation

$$L(\pi, s) = L(\pi, 1-s)\varepsilon(\pi, s), \tag{2}$$

where $\varepsilon(\pi, s) = \prod_p \varepsilon(\pi_p, s, \psi_p)$, ψ_p being the local components of a character ψ of $k \backslash A$, and $\tilde{\pi}$ is the contragredient representation. He also conjectured that for a given reductive group G there are many constructions of L and ε, which can be labelled by irreducible representations of some complex reductive group ${}^L G$. We are still very far from a solution of this problem.

For classical modular forms of one variable there are three different constructions of L-functions due to (I) E. Hecke, (II) R. A. Rankin and A. Selberg (independently and about the same time), (III) G. Shimura. These constructions give different L-functions. The aim of this talk is to describe generalizations of these constructions to the case of automorphic forms on adeles of a reductive group G.

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(I) Hecke construction. Assume that there exists a subgroup $H \subset G$ and a homomorphism $H \to^{\mu} k^x$ and that $\lambda(h)$ is an automorphic form on H_A such that

$$J(\varphi, s) = \int_{H_k \backslash H_A} \varphi(h) |\mu(h)|^s \lambda(h)\, dh \tag{3}$$

has the following properties for every cuspidal automorphic form φ on G_A: (1) it converges in some half-plane, (2) it has a meromorphic continuation to the whole complex plane, and (3) for a decomposable φ lying in an irreducible representation π, $J(\varphi, s)$ can be written as an Euler product. The simplest example of such a situation is for $G = \mathrm{GL}(2)$ [1]. In this case we can take

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in k^* \right\}, \quad \mu \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a, \quad \lambda(h) \equiv 1.$$

The second known example is $G = \mathrm{GL}(n) \times \mathrm{GL}(m)$, $n > m$. Here we take

$$H = \left\{ \left(\begin{bmatrix} x & 0 & y \\ 0 & 1 & u \\ 0 & 0 & z \end{bmatrix}, x \right), \; x \in \mathrm{GL}(m), \; y \in M^{m \times l}, \; u \in M^{1 \times l}, \; z \in Z_l \right\}, \tag{4}$$

$$\mu(x) = \det x, \quad \lambda(h) = \psi(z_{12} + \ldots + z_{l-1\,l} + u_1),$$

where $l = n - m - 1$, Z_l is the unipotent upper triangular group in $\mathrm{GL}(l)$, and ψ is a nondegenerate character on k. The case $n = 3$, $m = 1$ was considered in a paper by Jacquet, Shalika and myself (submitted in Annals). The general case has not yet been worked out in all details; we are working on it now (see also [8]). Some examples of subgroups H in the orthogonal and symplectic groups were obtained by M. Novodvorsky [9]. But in Novodvorsky's examples there exist irreducible cuspidal automorphic representations π of G_A such that the integral (3) is zero identically for all $\varphi \in \pi$. It is plausible that the only examples of G and H such that for every π the integral (3) is non-zero for some $\varphi \in \pi$ (and then in fact equals $L(\pi, s)$ for suitable φ) are the groups $\mathrm{GL}(n)$ itself and $\mathrm{GL}(n) \times \mathrm{GL}(m)$, $n > m$, with H as above.

It is well known that for $G = \mathrm{GL}(n) \times \mathrm{GL}(m)$, ${}^L G = \mathrm{GL}(n, C) \times \mathrm{GL}(m, C)$. It is possible to show that the L-functions constructed using the subgroup (4) correspond to the representation of ${}^L G$ which is the tensor product of the standard representations of $\mathrm{GL}(n, C)$ and $\mathrm{GL}(m, C)$. How to construct L-functions for other representations of ${}^L G$ is almost unknown.

We now describe the local L and ε factors, following the pattern given by J. Tate. We consider a set of integrals and we claim that all these integrals converge in some half-plane and can be meromorphically continued to the whole complex plane. We define $L(\pi, s)$ in such a way that $L(\pi, s)$ times any of these integrals is an entire function. For the definition of ε we should consider another set of integrals. There is a natural one-one correspondence between these sets such that the corresponding integrals are proportional, and the coefficient of proportionality is the ε-factor.

For GL (2), for instance, we put

$$L(W, s) = \int W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|^{s-1/2}\, d^x x,$$

$$\tilde{L}(W, s) = \int W\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} |x|^{1/2-s}\, d^x x,$$

where W belongs to the Whittaker model. (Recall that a *Whittaker model* for an irreducible representation π of $\mathrm{GL}(n, k_p)$ is a right invariant space $W(\pi, \psi)$ of smooth functions on $\mathrm{GL}(n, k_p)$ satisfying $W(zg)=\psi(z_{12}+\ldots+z_{n-1n})W(g)$, $\forall z \in Z_n$, such that the representation of $\mathrm{GL}(n, k_p)$ on $W(\pi, \psi)$ is equivalent to π.) Then we have the following theorem:

(1) Both integrals converge absolutely in some half-plane $\mathrm{Re}\, s > c_0$.

(2) They are rational functions of q^s.

(3) There exist two Euler factors $L(\pi, s)$ and $\tilde{L}(\pi, s)$ such that $L^{-1}(\pi, s)L(W, s)$ and $\tilde{L}^{-1}(\pi, s)\tilde{L}(W, s)$ are polynomials in q^s, q^{-s}.

(4) There is a function $\varepsilon(\pi, s, \psi)$ which is of the form aq^{ns} and such that

$$\tilde{L}^{-1}(\pi, 1-s)\tilde{L}(W, 1-s) = \varepsilon(\pi, s, \psi)L^{-1}(\pi, s)L(W, s)$$

for all $W \in W(\pi, \psi)$.

(5) If π and ψ are unramified, W_0 is invariant under the corresponding maximal compact group and $W_0\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)=1$, then

$$L(W_0, s) = L(\pi, s).$$

For $\mathrm{GL}(n)\times\mathrm{GL}(m)$ we have to consider the following integrals. Let π and τ be irreducible representations of $\mathrm{GL}(n, k_p)$ and $\mathrm{GL}(m, k_p)$. We put for $W \in W(\pi, \psi)$ and $W^1 \in W(\tau, \psi^{-1})$

$$L(W, W^1, s) = \int W\begin{pmatrix} x & 0 & 0 \\ y & 1_l & 0 \\ 0 & 0 & 1 \end{pmatrix} W^1(x) |\det x|^{s-1/2}\, d(x, y),$$

$$\tilde{L}(W, W^1, s) = \int W\begin{pmatrix} 0 & 1_{l+1} \\ x & 0 \end{pmatrix} W^1(x) |\det x|^{-s+l+1/2}\, d^x x.$$

Then for the integrals L and $\tilde{L}$ the same theorem as in the case GL (2) holds.

(II) Rankin-Selberg construction. R. Rankin and A. Selberg gave a method of analytic continuation for the convolution of Dirichlet series relating to classical modular forms. The reformulation of their construction in the language of representation theory was considered by H. Jacquet ([**10**] for $\mathrm{GL}(2)\times\mathrm{GL}(2)$ with explicit computation) and Piatetski-Shapiro [7] in generality, but without explicit computation). We will outline [7]. Assume that H is a reductive subgroup of G and P a maximal parabolic subgroup of H. For simplicity we assume that the center of G is trivial. We put $P=MN$, where M is a reductive group and N its unipotent radical. Let Φ be a nice function on $N_A \backslash H_A$ and μ a homomorphism $P \to k^\times$ and set

$$f(g, \Phi, s) = \int_{M_A} \Phi(mg) |\mu(m)|^{-s}\, dm.$$

It is easy to show that this integral converges in some half-plane and has meromorphic continuation to the whole plane. Now we introduce our main hero "Eisenstein series"

$$E(g, \Phi, s) = \sum_{\gamma \in P_k \backslash H_k} f(\gamma g, \Phi, s).$$

It is known that this Eisenstein series converges in some half-plane and we know due to R. Langlands [5] that this series can be meromorphically continued to the whole complex plane. In this case the global L function can be written in the form $J(\varphi, s)$, where for any cuspidal automorphic form φ on G we put

$$J(\varphi, s) = \int_{P_k \backslash H_A} \varphi(g) f(g, \Phi, s)\, dg = \int_{H_k \backslash H_A} \varphi(g) E(g, \Phi, s)\, dg.$$

It is known that $\varphi(g)$ rapidly decreasing and $E(g, \Phi, s)$ slowly increasing implies that $J(\varphi, s)$ is a meromorphic function. We will give now a partial list of cases when we can use the Rankin–Selberg construction. In all details it was done only for $G = \mathrm{GL}(2) \times \mathrm{GL}(2)$ by H. Jacquet [10]. In this case $H = \mathrm{GL}(2)$, diagonally embedded. H. Jacquet also considered the case $\mathrm{GL}(2, K)$, where K is a quadratic extension of a global field k, and $H = \mathrm{GL}(2, k)$, embedding induced by the natural embedding $k \to K$. The cases $\mathrm{GL}(n) \times \mathrm{GL}(n)$ and $\mathrm{GL}(n, K)$ can be considered similarly, but many difficult problems have not been solved yet. The case $G = \mathrm{GSp}(4, k)$, $H = \{g \in \mathrm{GL}(2, K), \det g \in k^*\}$ can be considered [7]. Some cases where G is an orthogonal or sympletic group were considered by M. Novodvorsky [9].

(III) Shimura's method. This was developed for modular forms of one variable by Shimura. Here we will give a general construction. Let G be a reductive group over k. For simplicity, we assume that the center of G is trivial. We will call an irreducible automorphic representation σ *distinguished* if for a maximal parabolic subgroup P, we have

(1) For any automorphic form $\theta \in \pi$ we have that

$$\theta_0(g) = \int_{Z_k \backslash Z_A} \theta(zg)\, dz = \int_{N_k \backslash N_A} \theta(ng)\, dn$$

where N is the unipotent radical of P and Z the center of N.

(2) All nontrivial characters ψ of $Z_k \backslash Z_A$ for which the Fourier coefficient

$$\theta_\psi(g) = \int_{Z_k \backslash Z_A} \theta(zg) \psi^{-1}(z)\, dz$$

is not identically zero have the form $z \to \psi_0(pzp^{-1})$ for some $p \in P_k$ and some fixed ψ_0.

(3) Denote by R the stationary subgroup of ψ_0 in P, viewed as an algebraic subgroup over k. It is clear that $\theta_\psi(rg) = \theta_\psi(g)$, $\forall r \in R_k$. Our third assumption says that for every $\theta \in \sigma$ the restriction of θ_ψ to R_A belongs to a fixed irreducible automorphic representation τ of R_A.

We define Eisenstein series $E(g, \Phi, s)$ as above, with the additional assumption that the homomorphism μ is trivial on R. Then for any cuspidal automorphic

form φ, lying in any irreducible representation π of G, the integral

$$J(\varphi, s) = \int_{G_k \backslash G_A} \varphi(g)\overline{\theta(g)}E(g, \Phi, s)\, dg$$

has an analytic continuation. Moreover, if $\Phi = \prod_p \Phi_p$ and if we assume that for all p there is at most one map from π_p to τ_p (this assumption is similar to the uniqueness of Whittaker models in the classical case), then $J(\varphi, s)$ has an Euler product. Since the proof is short and has not been published, we give it here.

First, replacing E by the series defining it, we have

$$J(\varphi, s) = \int_{P_k \backslash G_A} \varphi(g)\overline{\theta(g)}f(g, \Phi, s)\, dg \quad (\mathrm{Re}\,(s) \gg 0).$$

From the definition of a distinguished representation it follows that

$$\theta(g) = \theta_0(g) + \sum_{p \in R_k \backslash P_k} \theta_{\psi_0}(pg).$$

Hence

$$J(\varphi, s) = \int_{P_k \backslash G_A} \varphi(g)\overline{\theta_0(g)}f(g, \Phi, s)\, dg + \int_{R_k \backslash G_A} \varphi(g)\overline{\theta_{\psi_0}(g)}f(g, \Phi, s)\, dg.$$

Since θ_0 is left N_A-invariant and φ is cuspidal, the first term is zero. Since μ is trivial on R we have $f(rg, \Phi, s) = f(g, \Phi, s)$, $\forall r \in R_A$, and therefore

$$J(\varphi, s) = \int_{R_A \backslash G_A} \langle \varphi(g), \theta(g) \rangle f(g, \Phi, s)\, dg,$$

where

$$\langle \varphi(g), \theta(g) \rangle = \int_{R_k \backslash R_A} \varphi(rg)\overline{\theta_{\psi_0}(rg)}\, dr.$$

Denote by L a map from π to τ. By our assumption this map is unique and equals the tensor product of the corresponding local maps L_p. This immediately implies that

$$\langle \varphi(g), \theta(g) \rangle = \prod_p \langle L_p(\pi(g)\varphi_p), L_p(\sigma(g)\theta_p) \rangle$$

and hence that $J(\varphi, s)$ has the Euler product

$$J(\varphi, s) = \prod_p \int \langle L_p(\pi(g_p)\varphi_p), L_p(\sigma(g_p)d_p) \rangle f(g_p, \Phi_p, s)\, dg.$$

In many situations Weil representations are distinguished. For instance, this is the case when G is the split three-dimensional unitary group. Here we can also prove the local assumption made above (uniqueness of $\pi_p \to \tau_p$). Details were given by the author in lectures in Yale University (1977–1978). [1]

[1] During the congress I have learned that T. Shintani has considered independently afterwards (1978) a similar construction in the special case of the holomorphic modular forms in the two dimensional ball. He did not give a definition for local L and ε factors. He found remarkable necessary and sufficient conditions for holomorphic modular form in the two-dimensional ball to be an eigen function of Hecke operators in terms of its Fourier—Jacobi series.

We can also apply this method to a group $\bar{G}$ which is a central extension of G, for instance the metaplectic covering of GL (2). Here again Weil representations are distinguished (Gelbart) and the local uniqueness assumption is a theorem (joint work with S. Gelbart and R. Howe). The main result which was obtained in this case is the construction of L-functions; it will be published in a forthcoming paper of S. Gelbart and the author (see also [11]). The main application is the proof of the existence of a Shimura correspondence. This says that there exists an injective map S_p from the set of irreducible admissible representations of $\bar{G}_p$ into the set of irreducible admissible representations of G_p which preserves L and ε and has the following property: if $\pi = \otimes \pi_p$ is an irreducible automorphic cuspidal representation, then $S(\pi) = \otimes S_p(\pi_p)$ is an irreducible automorphic representation and is cuspidal iff π does not come from a Weil representation.

Unfortunately, the image of S or S_p is unknown. Despite lack of evidence I dare to conjecture that the analogous map exists for any central extension of any reductive group.

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On Special Values of Zeta Functions of Totally Real Algebraic Number Fields

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1. Hecke presented two different approaches to the Hilbert twelfth problem. One is the study of higher dimensional complex multiplication which is now developed to a magnificent theory by Shimura. Another is the study of Kronecker limit formula for general algebraic number fields [4]. Namely he proposed that the study of analytic expressions for values at $s=1$ of the Hecke L-series for a given field k could lead to a discovery of analytic functions suitable (natural) special values of which generate abelian extensions of k (the author is indebted for the present view on Hecke's works to Siegel [15] and Honda [7]). In the present talk we review our papers [11]–[14] in the context of Hecke's second program.

Notation. For an algebraic number field k, let h_k, d_k and $\mathfrak{O}_k$ be, respectively, the class-number, the discriminant and the ring of integers of k. For each ideal a of k, $N(a)$ is the absolute norm of a. The group of units (resp. totally positive units) of k is denoted by $E(k)$ (resp. $E(k)_+$). The mth Bernoulli polynomial in x is $B_m(x)$.

2. Let k be an algebraic number field of degree n. For an integral ideal $\mathfrak{f}$ of k, we denote by $H_k(\mathfrak{f})$ the group of narrow ideal classes modulo $\mathfrak{f}$ of k. For each $c\in H_k(\mathfrak{f})$, $\zeta_k(s, c)$ is the ray class zeta function corresponding to c. For any character χ of $H_k(\mathfrak{f})$, $L_k(s, \chi)=\sum\chi(c)\zeta_k(s, c)$ is the Hecke L-series associated with χ. For the arithmetic nature of $L_k(1, \chi)$, Stark [20] conjectured that, if the number of infinite primes of k where χ splits is $a(\chi)$, $L_k(1, \chi)\pi^{a(\chi)-n}$ would be a homogeneous form of degree $a(\chi)$ with algebraic coefficients in logarithms of units of a certain abelian extension of k. In view of the conjecture, one may say that an analytic expression for $L_k(1, \chi)$ of "desirable nature" should have a natural

interpretation as a homogeneous form of degree $a(\chi)$ in logarithms of special values of certain analytic functions. In particular, if $a(\chi)=0$, "good formula" for $L_k(1, \chi)$ should be an elementary arithmetic one. Recall, when k is either rational or imaginary quadratic, classical formulas for $L_k(1, \chi)$ are all of "desirable nature" and that special values of exponential or elliptic modular functions are involved in these formulas. We will show that for a *totally real* field k, an analytic formula of "desirable nature" is available for $L_k(1, \chi)$ if $a(\chi)=0$ or 1.

3. Let A (resp. x) be an $n\times r$ (resp. $1\times n$) matrix *with positive entries*. Denote by $L_m(z)=\sum a_{m\alpha} z_\alpha$ $(\alpha=1, \ldots, r)$ the linear form in r variables corresponding to the mth row of A and set

$$\zeta(s, A, x) = \sum_{z} \prod_{m=1}^{n} \{L_m(z)+x_m\}^{-s}, \tag{1}$$

where the summation with respect to z is over all the r-tuples of *nonnegative* integers. The Dirichlet series (1) is a natural generalization of the Riemann–Hurwitz zeta function. It is convergent if $\mathrm{Re}\, s>r/n$ and is extended to a meromorphic function on C. Our starting point is that, *if k is totally real, $\zeta_k(s, c)$ is a finite linear combination* of Dirichlet series of type (1). In the following we always assume k to be totally real. Let $x\mapsto(x^{(1)}, \ldots, x^{(n)})$, where $x^{(1)}, \ldots, x^{(n)}$ are n conjugates of x, be a natural embedding of k into $\boldsymbol{R}^n$. By componentwise multiplication, the group $E(k)_+$ acts on $\boldsymbol{R}^n_+$. It is shown that a fundamental domain for $\boldsymbol{R}^n_+$ with respect to the action of $E(k)_+$ is realized as a disjoint union of *finite number* of open simplicial cones *with generators in* $\mathfrak{O}_k$. In more detail, for r linearly independent vectors $v_1, \ldots, v_r$ of $\boldsymbol{R}^n$, we call the set of all positive linear combinations of $v_1, \ldots, v_r$ the r-dimensional open simplicial cone with generators $v_1, \ldots, v_r$. It is proved (cf. Proposition 4 of [11]) that there is a *finite* system of open simplicial cones $\{C_j; j\in J\}$ *with generators in* $\mathfrak{O}_k\cap\boldsymbol{R}^n_+$ such that

$$\boldsymbol{R}^n_+ = \bigcup_{j\in J} \bigcup_{u\in E(k)_+} uC_j \quad \text{(disjoint union)}. \tag{2}$$

For each $j\in J$, we denote by $r(j)$ the dimension of C_j and choose and fix a system of generators $v_{j1}, \ldots, v_{jr(j)}\in\mathfrak{O}_k$ of C_j once and for all. For each $x\in C_j$ we set

$$x = \sum_{\alpha=1}^{r(j)} z(x)_\alpha v_{j\alpha} \quad (z(x)_\alpha\in\boldsymbol{R}_+).$$

Note that if $x\in C_j\cap k$, $z(x)_1, \ldots, z(x)_{r(j)}$ are all *rational*. Choose and fix integral ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_0}$ of k so that they form a complete set of representatives for *narrow* ideal classes of k. For each $c\in H_k(\mathfrak{f})$, there uniquely exists an index i $(1\leqq i\leqq h_0)$ such that c and $\mathfrak{a}_i\mathfrak{f}$ are in the same narrow ideal class of k. For each $j\in J$, put

$$R(c, C_j) = \{x\in C_j\cap(\mathfrak{a}_i\mathfrak{f})^{-1};\ (x)\mathfrak{a}_i\mathfrak{f}\in c,\ 0<z(x)_\alpha\leqq 1\ (\alpha=1, \ldots, r(j))\}. \tag{3}$$

Note that the set $R(c, C_j)$ is always *finite*. Let A_j be the $n\times r(j)$ matrix whose (α, β)-entry is given by $v_{j\beta}^{(\alpha)}$. Then (2) and (3) imply the following formula for

$\zeta_k(s, c)$ (cf. [11]):

$$N(\mathfrak{a}_i\mathfrak{f})^s\zeta_k(s, c) = \sum_{j\in J}\sum_{x\in R(c, C_j)}\zeta(s, A_j, x). \tag{4}$$

Note that a formula of type (4) was first introduced by Zagier [24] in the case of real quadratic fields.

Hence, evaluations of special values of $\zeta_k(s, c)$ are reduced to those of $\zeta(s, A_j, x)$. Now, *at non-positive integers,* $\zeta(s, A_j, x)$ is evaluated in an elementary manner. In more detail, let $L_i^*(t)=\sum a_{\alpha i}t_\alpha$ $(\alpha=1, \ldots, n)$ be the linear form in n variables corresponding to the ith column of A and let $B_m(A, x)^{(\beta)}/(m!)^n$ be the coefficient of $u^{n(m-1)}(t_1\ldots t_{\beta-1}t_{\beta+1}\ldots t_n)^{m-1}$ in the Laurent expansion at the origin of the following function in t and u:

$$\left[\exp\left(-u\sum_{\alpha=1}^{n}t_\alpha x_\alpha\right)\prod_{i=1}^{r}\{1-\exp(-uL_i^*(t))\}^{-1}\right]_{t_\beta=1}.$$

Furthermore, set $B_m(A, x)=\sum_{\beta=1}^{n}B_m(A, x)^{(\beta)}/n$. We may regard $B_m(A, x)$ as a generalization of Bernoulli polynomials. Applying a modification of the classical method of contour integrals due to Riemann, Hurwitz and Barnes [2], we can prove (cf. Proposition 1 of [11]) that

$$\zeta(1-m, A, x) = (-1)^{n(m-1)}m^{-n}B_m(A, x) \quad (m = 1, 2, \ldots). \tag{5}$$

Combining (4) and (5), we obtain an *explicit formula* for $\zeta_k(1-m, c)$ $(m=1, 2, \ldots)$ (for real quadratic fields, such a formula was given in [17] by a different method) which yields a simple proof of the Klingen–Siegel theorem [19] that $\zeta_k(s, c)$ is rational valued at non-positive integers (cf. Theorem 1 of [11]). When $\mathfrak{f}=\mathfrak{O}_k$, an explicit formula for $\zeta_k(1-m, c)$ of entirely different nature was established in [18] (cf. [6]). Note that, when k is real quadratic, Cassou-Noguès [3] and Zagier [25] also calculated $\zeta_k(1-m, c)$ starting from the formula (4).

Now let χ be a *primitive* character of the group $H_k(\mathfrak{f})$ which *ramifies at all infinite primes of* k. Then the functional equation for $L_k(s, \chi)$ implies that

$$L_k(1, \chi) = L_k(0, \chi^{-1})w(\chi)\pi^n/\sqrt{d_kN(\mathfrak{f})},$$

where $w(\chi)$ is the root number. Applying our formula for $\zeta_k(0, c)$, we obtain the following (cf. Theorem 3 of [11]):

THEOREM 1. *The notation and assumptions being as above,*

$$L_k(1, \chi)\sqrt{d_kN(\mathfrak{f})}/(\pi^n w(\chi)) = \sum_c \chi^{-1}(c)\sum_j\sum_x B_1(C_j, x) \tag{6}$$

$$(c\in H_k(\mathfrak{f}),\ j\in J,\ x\in R(C_j, c)),$$

where

$$B_1(C_j, x) = \frac{(-1)^{r(j)}}{n}\sum_l \mathrm{tr}_{k/Q}\left(\prod_{\alpha=1}^{r(j)}v_{j\alpha}^{l_\alpha-1}\right)\prod_{\alpha=1}^{r(j)}\frac{B_{l_\alpha}(z(x)_\alpha)}{l_\alpha!},$$

where l *ranges over all the* $r(j)$*-tuples of non-negative integers such that* $l_1+\ldots+l_{r(j)}=r(j)$.

In particular, assume χ to be of order 2. Then χ corresponds, in class field theory, to a totally imaginary quadratic extension K of k with relative discriminant $\mathfrak{f}$ and the left side of (6) is equal to $2^{n-1}h_K/h_k[E(K), E(k)]$. Thus, the formula (6) gives *an affirmative answer to the Hecke conjecture* [5] that the relative class number of K with respect to k would admit an elementary arithmetic expression. For real quadratic fields k, a formula equivalent to (6) was given by Hecke [5] and was further studied by Meyer [8] and Siegel [16].

4. Next, we assume that χ is a *primitive* character of $H_k(\mathfrak{f})$ *which splits at only one* of n *infinite primes of* k. Then the functional equation implies that

$$L_k(1, \chi) = w(\chi)L_k'(0, \chi^{-1})(2\pi^{n-1})/\sqrt{d_k N(\mathfrak{f})}.$$

Thus, in the present case, the evaluation of $L_k(1, \chi)$ is reduced to that of $\zeta'(0, A, x)$. For a $1\times r$ matrix A and a variable x, Barnes ([1], [2]) introduced *multiple gamma function* $\Gamma_r(x, A)$ with modulus A by setting

$$\zeta'(0, A, x) = \log\{\Gamma_r(x, A)/\varrho_r(A)\}, \tag{7}$$

where $\varrho_r(A)^{-1}$ is the residue at $x=0$ of $\exp\zeta'(0, A, x)$. He established various properties of Γ_r analogous to the ordinary gamma function. For a general $n\times r$ matrix A with positive entries, $\zeta'(0, A, x)$ is also evaluated in terms of multiple gamma functions.

In more detail, let A_m be the mth row of A and set, for each r-tuple l of non-negative integers,

$$C_l(A) = \sum_{(\alpha,\beta)} \int_0^1 \left\{ \prod_{m=1}^{r} (a_{\alpha m}+a_{\beta m}u)^{l_m-1} - \prod_{m=1}^{r} a_{\alpha m}^{l_m-1} \right\} \frac{du}{u},$$

where (α, β) ranges over all pairs of positive integers such that $1\leqq\alpha, \beta\leqq n$, $\alpha\neq\beta$. Furthermore, for ${}^t x=Az$ $(z\in \boldsymbol{R}_+^r)$, set

$$T(x, A) = \prod_{m=1}^{n} \left\{ \frac{\Gamma_r(x_m, A_m)}{\varrho_r(A_m)} \right\} \exp\left\{ \frac{(-1)^r}{n} \sum_l C_l(A) \prod_{\alpha=1}^{r} \frac{B_{l_\alpha}(z_\alpha)}{l_\alpha!} \right\},$$

where l ranges over all r-tuples of nonnegative integers such that $l_1+\ldots+l_r=r$. Then it is proved (cf. Proposition 3 of [12] and Proposition 1 of [13]) that if ${}^t x=Az$ $(z\in \boldsymbol{R}_+^r)$, $\zeta'(0, A, x)=\log T(x, A)$. Thus, we are led to the following expression for $L_k(1, \chi)$ as a linear combination of logarithms of special values of multiple gamma functions (cf. Theorem 1 of [13]):

THEOREM 2. *The notation and assumptions being as above,*

$$w(\chi)^{-1}\sqrt{d_k N(\mathfrak{f})}L_k(1, \chi)/(2\pi^{n-1}) = \sum_{c\in H_k(\mathfrak{f})} \chi^{-1}(c)\log T(c), \tag{8}$$

where

$$T(c) = \prod_{j\in J} \prod_{x\in R(C_j, c)} T(x, A_j).$$

Note that $T(c)$ does depend upon the choice of representatives $\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_0}$ for narrow ideal classes of k. For χ of order 2, (8) yields a generalization of Dirichlet class number formula for real quadratic fields.

Formula (8), together with conjectures of Stark, seems to suggest that multiple gamma functions may play a significant role in construction of class fields over totally real fields. In more detail let $S(\mathfrak{f})$ be the subgroup of $H_k(\mathfrak{f})$ generated by those principal ideals (x) with $x\equiv 1 \bmod (\mathfrak{f})$. Assume that there exists a character λ of $S(\mathfrak{f})$ given by $\lambda((x))=\prod_{i=2}^{n}\operatorname{sgn} x^{(i)}$. In [22], Stark introduced invariants $\varepsilon_m(c)$ $(m\in\boldsymbol{Z},\ c\in H_k(\mathfrak{f}))$ by setting $\varepsilon_m(c)=\exp\{m\sum_{s\in S(\mathfrak{f})}\lambda(s)\zeta'(0, cs)\}$. He conjectured ([21], [22], [23]) that $\varepsilon_m(c)$ would be a unit of the abelian extension of k corresponding to the kernel of λ and that the mapping $c\mapsto\varepsilon_m(c)$ would be compatible with the Artin canonical isomorphism. We can show

$$\varepsilon_m(c)=\prod_{s\in S(\mathfrak{f})}T(cs)^{\lambda(s)m}. \tag{9}$$

Thus, the Stark invariants are described in terms of values at conjugates of k of multiple gamma functions with modulus belonging to conjugates of k.

Stark's conjectures, which are strongly supported by his numerical experiments, suggest also that Hecke's second program (mentioned in the introduction) would ultimately work for totally real fields if a *really good* analytic formula for $\varepsilon_m(c)$ is available. At present one can only say that (9) may be a candidate for such a formula.

5. For a real quadratic field k, Theorem 2 is described in a simpler manner. Take a fundamental totally positive unit ε of k. Then we may put (cf. (2)),

$$\bigcup_{j\in J}C_j=\{x+y\varepsilon;\ x>0,\ y\geqslant 0\}.$$

Hence we may put (cf. (3))

$$\bigcup_{j\in J}R(c, C_j)=R(\varepsilon, c)$$
$$=\{z=x+y\varepsilon\in(\mathfrak{a}_i\mathfrak{f})^{-1},\ (z)\mathfrak{a}_i\mathfrak{f}\in c,\ 0<x\leqslant 1,\ 0\leqslant y<1,\ x, y\in\boldsymbol{Q}\}.$$

Set

$$A=\begin{pmatrix}1 & \varepsilon^{(1)}\\ 1 & \varepsilon^{(2)}\end{pmatrix}.$$

Then (4) is simplified to

$$N(\mathfrak{a}_i\mathfrak{f})^s\zeta_k(s, c)=\sum_{z\in R(\varepsilon, c)}\zeta(s, A, z).$$

Assume that the group $H_k(\mathfrak{f})$ has a character χ which splits at only one of two infinite primes of k. Take $v\in k$ such that $v^{(1)}, v^{(2)}>0$ and $v\equiv -1 \pmod{\mathfrak{f}}$. Denote by the same letter v the element of $H_k(\mathfrak{f})$ represented by (v). Then $\chi(v)=-1$. Set

$$F(z, (1, \varepsilon))=\Gamma_2(z, (1, \varepsilon))/\Gamma_2(1+\varepsilon-z, (1, \varepsilon))$$

(cf. (7)). The function F is, up to a constant factor, the unique meromorphic simultaneous solution of the difference equations $F(z+1)=F(z)2\sin(\pi z/\varepsilon)$, $F(z+\varepsilon)=F(z)2\sin\pi z$. It is shown (cf. Theorem 1 of [14])

$$\zeta'(0, c)-\zeta'(0, cv)=\log X_{\mathfrak{f}}(c),$$

where

$$X_{\mathfrak{f}}(c)=\prod_{z\in R(\varepsilon,c)}F(z^{(1)},(1,\varepsilon^{(1)}))F(z^{(2)},(1,\varepsilon^{(2)})).$$

Let G be the kernel of χ and put $X_{\mathfrak{f}}(c,G)=\prod_{g\in G}X_{\mathfrak{f}}(cg)$. Then we have, if χ is primitive, the following simplified version of (8):

$$L(1,\chi)w(\chi)^{-1}\sqrt{d_kN(\mathfrak{f})}/(2\pi)=\sum_{c\in H_k(\mathfrak{f})/\langle G,\nu\rangle}\chi^{-1}(c)\log X_{\mathfrak{f}}(c,G),$$

where $\langle G,\nu\rangle$ is the group generated by G and ν.

Let K_G be the class field over k corresponding to the subgroup G of $H_k(\mathfrak{f})$. *If K_G is a quadratic extension of its maximally absolutely abelian subfield, $L_k(s,\chi)$* coincides with an L-function of a suitable imaginary quadratic field. Applying results of Ramachandra [9] on the arithmetic nature of $L_\varkappa(1,\chi)$ for $\varkappa$ imaginary quadratic, we obtain the following result (cf. Theorem 2 of [14]) which is consistent with the Stark conjecture.

THEOREM 3. *If K_G is quadratic over its maximally absolutely abelian subfield, then for a suitable positive integer m, $X_{\mathfrak{f}}(c,G)^m$ is a unit of K_G and the mapping $c\mapsto X_{\mathfrak{f}}(c,G)^m$ is compatible with the Artin canonical isomorphism.*

Theorem 3 implies that suitable multiplicative combinations of special values of the function $F(z,(1,\varepsilon))$ at k do generate certain non-trivial abelian extensions of k. However it remains quite mysterious why they do. It seems that most significant properties of the double (or multiple) gamma functions remain to be discovered. We wonder if our results are related to Shimura's theory [10] on construction of class fields over real quadratic fields.

Finally, we present a numerical example for Theorem 3. Set $k=\boldsymbol{Q}(\sqrt{21})\subset\boldsymbol{R}$. The fundamental totally positive unit is given by $\varepsilon=(5+\sqrt{21})/2$. Put $\mathfrak{f}=(\varepsilon-1)$. Then the group $H_k(\mathfrak{f})$ is isomorphic to the direct product of two copies of a cyclic group of order 2. There uniquely exists the character χ of $H_k(\mathfrak{f})$ with $a(\chi)=1$ which splits at the infinite prime corresponding to the prescribed embedding of k into $\boldsymbol{R}$. Then $K_G=k(\sqrt{\varepsilon-1})$. We may put $\mathfrak{a}_1=\mathfrak{O}_k$, $\mathfrak{a}_2=\mathfrak{f}$. For this example, Theorem 3 is valid for $m=1$. We have

$$\begin{aligned}(10)\qquad X_{\mathfrak{f}}(1)&=\{F(1/3)F(1+\varepsilon/3)F((2+2\varepsilon)/3)\}^2\\&=(\varepsilon-2+\sqrt{\varepsilon-1})/2,\end{aligned}$$

where we put $F(z)=F(z,(1,\varepsilon))$ (cf. example 3 of § 3 of [12]). It would be quite interesting if one could prove the equality (10) *directly*.

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Метод Усреднений в Теории Ортогональных Рядов

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Метод усреднений в вопросах расходимости рядов Фурье по общим ортогональным системам состоит в том, что каждому виду сходимости сопоставляется некоторый способ усреднений, т. е. строится случайный объект, один и тот же для всех ортогональных систем. Этот случайный объект должен давать нужное явление расходимости в среднем, независимо от системы. Тогда для каждой ортогональной системы найдется реализация случайного объекта, на которой для данной системы достигается требуемая расходимость. Таким образом, конструкция случайной функции определяется видом сходимости, а траектории, на которых достигается расходимость, зависят от ортогональной системы.

Метод усреднений позволяет установить, что все основные факты, относящиеся к расходимости тригонометрических рядов Фурье, по своей природе не являются тригонометрическими, а могут быть единообразно получены для всех ортогональных систем.

Кроме того, указанный метод усреднений устанавливает своего рода двойственность между прямыми и обратными утверждениями в вопросах сходимости рядов Фурье. Действительно, для конкретных ортонормированных систем прямое утверждение о сходимости ряда Фурье было общим фактом, относящимся к определенному классу функций, а обратное утверждение о расходимости носило характер контрпримера. При указанном подходе и прямое и обратное утверждения являются общими фактами, и оба эти утверждения связаны определенным отношением двойственности. Так, если прямое утверждение относится к произвольной функции из определенного класса и к конкретной ортонормированной системе, то обратное утверждение,

полученное методом усреднений, относится к конкретной случайной функции из определенного класса и к произвольной ортонормированной системе.

Рассмотрим вопрос об абсолютной сходимости рядов Фурье. Оказывается, что для абсолютной сходимости рядов Фурье существует общая теорема переноса, относящаяся к двум произвольным ортогональным системам [1].

Теорема 1. *Пусть* $\{f_n\}$*—ортонормированная система и* $\{\varphi_n\}$*—ортонормированная, полная в* $L_2(0, 1)$ *система. Если*

$$\sum_{n=1}^{\infty} a_n^2 < \infty, \quad \sum_{n=1}^{\infty} |a_n|^p = \infty \quad \textit{при некотором} \quad 0 < p < 2,$$

то для почти всех $x \in [0, 1]$ *функция*

$$F_x(t) = \sum_{n=1}^{\infty} \tau_n(x) \cdot a_n \cdot f_n(t),$$

где $\{\tau_n(x)\}$ — *система Радемахера, удовлетворяет соотношению*

$$\sum_{n=1}^{\infty} |b_n(F_x)| = \infty, \text{ где } b_n(F_x) = \int_0^1 F_x(t) \cdot \varphi_n(t)\, dt.$$

Возьмем в качестве системы $\{f_n\}$ систему Франклина. Тогда функция

$$F_x(t) = \sum_{n=1}^{\infty} \frac{\tau_n(x) \cdot f_n(t)}{n^{\alpha+1/2}}$$

при всех $x \in [0, 1]$ принадлежит классу $\operatorname{Lip} \alpha$ и в силу теоремы переноса при почти всех $x \in [0, 1]$

$$\sum_{n=1}^{\infty} |b_n(F_x)|^{2/(2\alpha+1)} = \infty,$$

т. е. одна и та же случайная функция $F_x(t)$ дает обратное утверждение к теореме Бернштейна—Саса об абсолютной сходимости тригонометрических рядов Фурье одновременно для всех полных ортонормированных систем.

Другие факты такого же типа, а также исторические сведения, относящиеся к теореме Бернштейна—Саса, содержатся в [1].

Система Франклина играет важную роль в теории базисов. Система Франклина используется при построении базиса в пространстве функций, аналитических в круге и непрерывных на границе [2]. Кроме того, для рядов по системе Франклина справедливы неравенства [1], [2]

$$\operatorname{mes}\left\{\left(\sum_{k=1}^{\infty} a_k^2 f_k^2(t)\right)^{1/2} > y\right\} \leqslant \frac{B}{y} \int_0^1 \left|\sum_{k=1}^{\infty} a_k f_k(t)\right| dt,$$

$$\operatorname{mes}\left\{\left|\sum_{k=1}^{\infty} \pm a_k f_k(t)\right| > y\right\} \leqslant \frac{B}{y} \int_0^1 \left|\sum_{k=1}^{\infty} a_k f_k(t)\right| dt,$$

из которых, в частности, следует, что система Франклина есть безусловный базис в L_p, $1<p<\infty$.

Вопрос об абсолютной сходимости рядов Фурье функций ограниченной вариации оставался открытым и в теории тригонометрических рядов. А. Зигмунд [3] установил, что условие

$$\sum_{n=1}^{\infty}\frac{\sqrt{\omega(1/n)}}{n}<\infty$$

достаточно для абсолютной сходимости рядов Фурье функций $f\in H^{\omega}\cap V$. Однако получить необходимое и достаточное условие до последнего времени не удавалось. Различные необходимые условия были получены Р. Салемом [4] $\left(\sum_{n=1}^{\infty}(1/n)[\omega(1/n)]^{1/2+\varepsilon}<\infty,\ \varepsilon>0\right)$ Ж-П. Каханом [5] $\left(\overline{\lim}_{j\to\infty} j^2\cdot\omega(2^{-j})<\infty\right)$, И. Виком [6]

$$\left(\sum_{n=1}^{\infty}\omega(1/n)\log n/n<\infty\right).$$

Для решения этого вопроса следует произвести усреднение по некоторому континууму сингулярных функций [1], [7].

Теорема 2. *Пусть* $\{f_n\}$ *— полная ортонормированная система, удовлетворяющая условию* $\|f_n\|_{\infty}\leqslant M$, $n=1, 2, \ldots$. *Тогда для любого модуля непрерывности* $\omega(\delta)$ *существует такое семейство функций* $F_x(t)$, $F_x(t)\in H^{\omega}\cap V$ *при всех* $x\in[0, 1]$, *что*

$$\int_0^1\sum_{l=1}^{\infty}|a_l(F_x)|dx\geqslant B(M)\cdot\sum_{n=1}^{\infty}\frac{\sqrt{\omega(1/n)}}{n},$$

где $a_l(F_x)=(F_x, \varphi_l)$.

Из Теоремы 2 следует, что обратное утверждение к теореме Зигмунда действительно справедливо, и не только для тригонометрической системы, но и для любой ограниченной полной ортонормированной системы.

Семейство функций $F_x(t)$ имеет следующий вид:

$$F_x(t)=\varphi_2(t)+\sum_{m=0}^{\infty}\frac{1}{2^{m+1}}\sum_{n=N_m+1}^{N_{m+1}}\sum_{k=2^n+1}^{2^{n+1}}\mu_k(x)\cdot\varphi_k(t),$$

где $\{\varphi_k\}$ — система Шаудера, $2^{-m-1}\leqslant\omega(2^{-n})<2^{-m}$ при $N_m\leqslant n<N_{m+1}$ и $\{\mu_k(x)\}$ — некоторые полиномы по системе Хаара, которые определяются по индукции.

Отметим, что вариация внутренней суммы этого ряда удовлетворяет соотношению

$$\frac{1}{2^{m+1}}\bigvee_0^1\sum_{k=2^n+1}^{2^{n+1}}\mu_k(x)\cdot\varphi_k(t)=2$$

при $m=0, 1, \ldots, N_m\leqslant n<N_{m+1}$, $x\in[0, 1]$, но благодаря специальному сложению $\bigvee_0^1 F_x=2$ при всех $x\in[0, 1]$, т. е. общая вариация остается постоянной.

Аналитическим аппаратом при усреднении по сингулярным функциям являются новые неравенства, дающие оценку снизу нормы в L_1 [1], [7].

Пусть

$$\delta_n(x) = \sum_{k=2^n+1}^{2^{n+1}} a_k \chi_k(x) = \sum_{k=2^n+1}^{2^{n+1}} c_k \cdot \psi_k(x),$$

где $\{\chi_k\}$ и $\{\psi_k\}$ — системы Хаара и Уолша.

Теорема 3. *Если существует последовательность номеров $\{R_m\}$ и последовательность чисел $\{q_m\}$, для которых*

$$\left\| \left(\sum_{n=R_m+1}^{R_{m+1}} \delta_n^2(x) \right)^{1/2} \right\|_\infty \leqslant q_m, \quad \sum_{n=m}^{\infty} q_n \leqslant B \cdot q_m,$$

то справедливо неравенство

$$\int_0^1 \sum_{m=0}^{\infty} \frac{1}{q_m} \sum_{n=R_m+1}^{R_{m+1}} \delta_n^2(x)\, dx \leqslant B \int_0^1 \left| \sum_{n=0}^{\infty} \delta_n(x) \right| dx.$$

Это есть некоторое новое неравенство о мультипликаторах. Отметим, что условия теоремы 3 выполняются, если например, $\|\delta_n(x)\|_\infty \leqslant d_n$ и $\sum_{n=1}^{\infty} d_n^2 < \infty$. Если же $\sum_{n=1}^{\infty} d_n^2 = \infty$, то может существовать нуль-ряд и тогда нельзя получить оценку снизу для L_1-нормы.

Частным случаем усреднений по сингулярным функциям является усреднение по сдвигам [8]. При этом используется следующий частный случай общей оценки снизу нормы в L_1:

$$\int_0^1 \sum_{n=0}^{\infty} 2^n \cdot \delta_n^2(x)\, dx \leqslant B \int_0^1 \left| \sum_{n=0}^{\infty} \delta_n(x) \right| dx,$$

если $\|\delta(x)\|_\infty \leqslant 1/2^n$. Здесь возникает мультипликатор степенного вида $\{n^{1/2}\}$. Доказательство оценки снизу в этом весьма частном случае значительно проще, чем доказательство общей оценки.

Если сумма ряда $\sum_{n=0}^{\infty} \delta_n(x) = f(x)$ — абсолютно непрерывная функция, то из предыдущего неравенства, ввиду оценки

$$\|\delta_n(x)\|_\infty \leqslant B \cdot \|f'\|_\infty / 2^n,$$

следует неравенство

$$\sum_{k=0}^{\infty} (f', \varphi_k)^2 \leqslant B \|f'\|_\infty \cdot \|f\|_1.$$

Если $a_1, a_2, \ldots$ — произвольная последовательность чисел, то получаем неравенство

$$\max_{1 \leqslant k \leqslant 2^n} |a_k| \left(\sum_{p=1}^{2^n} \left| \sum_{k=1}^{p} a_k \right| \right) \geqslant B \sum (\Delta_k)^2,$$

где Δ_k — двоичные порции последовательности $a_1, \ldots, a_{2^n}$, взятые с линейными срезками (функциями Шаудера), и суммирование справа произ-

водится по всем двоичным порциям. Интегрируя числовое неравенство, получаем, что справедлива

Теорема 4. *Пусть* $\{a_k\}$ *— произвольная последовательность чисел и* $\{f_k\}$ *— произвольная ортонормированная система. Тогда*

$$\max_{1\leqq k\leqq n}\|a_k f_k\|_\infty\left(\sum_{p=1}^{n}\int_0^1\left|\sum_{k=1}^{p}a_k f_k(t)\right|dt+n\int_0^1\left|\sum_{k=1}^{n}a_k f_k(t)\right|dt\right)$$

$$\geqslant B\cdot\log n\cdot\sum_{k=1}^{n}a_k^2.$$

Таким образом, частный случай оценки снизу L_1-нормы дает новое числовое неравенство, из которого следует важное неравенство для ортогональных систем. Отметим, что более слабое неравенство для ортогональных систем, с заменой средних арифметических на максимум, было без использования числового неравенства доказано А. М. Олевским [9]. Из Теоремы 4 следуют логарифмический рост средних арифметических от функций Лебега равномерно ограниченных ортонормированных систем и справедливость в среднем гипотезы Литтлвуда.

Чтобы получить для произвольных ограниченных ортонормированных систем обобщение известной теоремы А. Н. Колмогорова [10] о существовании расходящегося почти всюду тригонометрического ряда Фурье следует произвести усреднение по носителям δ-функций в n-мерном пространстве. Здесь необходимо было устранить связанное с распределением знаков взаимодействие ядер Дирихле, соответствующих различным δ-функциям. Казалось, что это невозможно сделать, не зная явного вида ядер Дирихле. Тем не менее удалось найти весьма общий способ преодалеть указанную трудность [1], [11].

Теорема 5. *Пусть* $\{f_k\}$ *— ортонормированная, ограниченная в совокупности система функций. Тогда для любого* $N=1,2,\ldots$ *существуют такое множество* Ω *точек* $(t,\theta_1,\ldots,\theta_N)$ *единичного куба в* R^{N+1}, $\operatorname{mes}\Omega\geqslant\gamma>0$, *и такая последовательность натуральных чисел* $\{m_p(t)\}$, $Np\leqslant m_p(t)<N(p+1)$, *зависящая от* t, *что для всякой точки* $(t,\theta_1,\ldots,\theta_N)\in\Omega$ *выполняется соотношение*

$$\overline{\lim_{p\to\infty}}\sum_{i=1}^{N}\sum_{k=Np}^{m_p(t)}f_k(t)\cdot f_k(\theta_i)\geqslant B\cdot N\cdot\log N.$$

Из Теоремы 4 следует существование для произвольной ограниченной ортонормированной системы ряда Фурье, расходящегося на множестве положительной меры. Для тригонометрической системы и системы Уолша отсюда автоматически может быть получено существование расходящихся почти всюду рядов Фурье.

В заключение я приведу новую оценку снизу мажоранты частных сумм для переставленной системы Уолша [12].

Теорема 6. *Для любого* $n=1, 2, \ldots$, *существуют полином* $P_n(t)=\sum_{k=1}^{m} a_k\psi_k(t)$, $\|P_n\|_2=1$, *и перестановка* $\sigma(k)$, *для которых имеет место неравенство*

$$\operatorname{mes}\left\{\max_{m\leqq n}\left|\sum_{k=1}^{m} a_{\sigma(k)}\cdot\psi_{\sigma(k)}(t)\right|>\alpha\sqrt{\log n}\right\}\geqslant\gamma,$$

где α *и* γ — *положительные постоянные.*

Из Теоремы 6 следует, что последовательность $\varrho(n)=\bar{o}(\log n)$ не является множителем Вейля для безусловной сходимости почти всюду рядов Фурье-Уолша. В наилучшем предыдущем результате, принадлежащем С. Наката [13], последовательность

$$\varrho(n)=\sqrt{\log n}\cdot\log\log n\ldots\underbrace{\log\ldots\log}_{N\text{ раз}}n.$$

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Contractive Intertwining Dilations and Waves in Layered Media

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1. The aim of this talk is to present some old and new developments in the dilation theory of contractions on Hilbert spaces, which, because of their abundance in open questions and of their potential connection with other branches of science, seem promising for further research.[1]

2. The beginning of our story is the following well known dilation theorem of B. Sz.-Nagy [**31**]: For any contraction T on $\mathscr{H}$ there exists an isometric operator U on some $\mathscr{K}$ containing $\mathscr{H}$ as a subspace, such that

$$P_{\mathscr{H}} U = T P_{\mathscr{H}}, \quad \mathscr{K} = \bigvee_{n=0}^{\infty} U^n \mathscr{H}.$$

The operator U, uniquely determined up to an "isomorphism", is called a *minimal isometric dilation* of T. Minimal isometric dilations though their construction is now very simple (see e.g. [**33**, p. 11]), were connected with fruitful developments in Operator theory (see e.g. [**23**], [**33**], [**17**], [**32**], etc.); essentially, because they provide an adequate geometrical frame (actually meaningful also in Systems theory; see [**8**], [**20**], [**21**])[2] for studying and understanding problems involving

[1] In the sequel, all spaces will be complex Hilbert spaces, all operators are linear and bounded Hilbert space mappings, all contractions are operators of norm ≤ 1 and finally the orthogonal projection of any space on its subspace $\mathscr{H}$ will be denoted by $P_{\mathscr{H}}$; also for a contraction T, we shall denote $D_T = (I - T^*T)^{1/2}$, $\mathscr{D}_T = (\text{Range } D_T)^-$, where I denotes the identity operator (on any Hilbert space). Otherwise the terminology is that of [**33**].

[2] For the connection of operator theory to systems theory see also the pioneering work of M. S. Livšic [**25**].

contractions on Hilbert spaces. This will be illustrated by our discussion on the following dilation theorem for intertwining contractions:

(I) *Let T on $\mathcal{H}$ and T' on $\mathcal{H}'$ be two contractions and let U on $\mathcal{K}$ and U' on $\mathcal{K}'$ be minimal isometric dilations of T and T', respectively. Let A be a contraction from $\mathcal{H}$ to $\mathcal{H}'$ intertwining T' and T (i.e. $T'A=AT$). Then there exists a contraction B (called, in the sequel, a contractive intertwining dilation of A) from $\mathcal{K}$ to $\mathcal{K}'$ such that*

$$U'B = BU, \quad P_{\mathcal{H}'}B = AP_{\mathcal{H}}.$$

This theorem was established early in 1968 [34]. Although it had, as paradigm, an illuminating discovery made, shortly before that date, by D. Sarason [29] (namely, its particular functional case: $U'=U=$ the canonical multiplication unilateral shift on H^2, $T=T'$; see also [35]) and could be also inferred (see [27], [37]) from an older theorem of T. Ando [6] (on the existence of isometric commuting dilations of two commuting contractions; see also [33, p. 20]), its proof in [34], recurrently exploiting the geometry of minimal isometric dilations, was genuinely new (see also [16] and [13] for variants of this proof).[3]

3. Theorem I proved to be a useful tool in Operator theory, particularly in the study of the commutant of a contraction, of the generalized Jordan models for C_0-contractions and of the similarity of certain operators (see [36], [11], [38], [39], etc.). Moreover it is an abstract generalization of several extremal problem theorems in classical and modern interpolation theory. Such theorems are, for instance, those answering the classical interpolation problems of Nehari, of Caratheodory–Fejér and of Nevanlinna–Pick, as well as their generalized operator versions ([29], [3], [4], [5], [26]). Indeed if any of these problems is solvable, then the set of its solutions can be identified with the set of the contractive intertwining dilations of a contraction A suitably given together with the operators T, T', U and U' ([29], [5], [26]). The first complete description of the set of all solutions of an interpolating problem (namely the Caratheodory–Fejér one, in the case

$$\|A\| < 1) \tag{3.1}$$

was given by I. Schur [30]. The operator generalization of Schur's description was obtained much later by V. M. Adamjan, D. Z. Arov and M. G. Krein [2], [3], [4], [5]. These descriptions label the solutions by arbitrary contractive analytic functions $\{\mathcal{E}, \mathcal{E}, \sigma(z)\}$, where $\mathcal{E}$ and $\mathcal{E}'$ are some suitable fixed spaces (see footnote one). (In Schur's case: $\dim \mathcal{E}=\dim \mathcal{E}'=1$.) Therefore the labelling by contractive analytic functions of the contractive intertwining dilations of a contraction A (given together with T, T', U and U' as in (I)) constitutes a natural generalization

[3] Let us also note that in case $T=T'$ the theorem, formulated for the adjoint operators $U_1=U^*$ and $A_1=A^*$, is currently referred to as the "Commutant Lifting Theorem" (see e.g. [16], [26], [19], [12], [22], etc.).

of all these descriptions. In order to give such a general labelling (valid, in the former particular cases, even if $\|A\|=1$ (see (4.1)) let us introduce the spaces

$$\mathscr{L}=((U-T)\mathscr{H})^-,\quad \mathscr{L}'=((U'-T')\mathscr{H}')^-,\quad \mathscr{L}_*=((I-UT^*)\mathscr{H})^-,$$

$$\mathscr{F}=\{DTh+(U-T)h:\ h\in\mathscr{H}\}^-,\quad \mathscr{G}=(\mathscr{D}+\mathscr{L})\ominus\mathscr{F}, \tag{3.2}$$

$$\mathscr{F}'=\{Dh\oplus(U'-T')Ah:\ h\in\mathscr{H}\}^-,\quad \mathscr{G}'=(\mathscr{D}\oplus\mathscr{L}')\ominus\mathscr{F}'$$

(where $\mathscr{D}=\mathscr{D}_A$, $D=D_A$) and the unitary operator ω from $\mathscr{F}$ to $\mathscr{F}'$ defined by

$$\omega(DTh+(U-T)h)=Dh\oplus(U'-T')Th\ \ (h\in\mathscr{H}).$$

Also we shall denote by j and j' the canonical identifications of $\mathscr{D}$ and $\mathscr{L}'$ with $\mathscr{D}\oplus\{0\}$ and $\{0\}\oplus\mathscr{L}'$, respectively (where the last two spaces are viewed as subspaces of $\mathscr{D}\oplus\mathscr{L}'$). The existence of the general labelling is given by the following theorem [13]:

(II) *Let T, T', U, U' and A be as in* (I). *Then there exists a one-to-one correspondence between the set of all contractive intertwining dilations B of A and the set of all contractive analytic functions $\{\mathscr{G},\mathscr{G}',\sigma(z)\}$.*

Consequently there exists exactly one contractive intertwining dilation of A if and only if at least one of the space $\mathscr{G}$ and $\mathscr{G}'$ reduces to $\{0\}$; thus (II) contains the uniqueness theorem given in [7]. The correspondence yielded by Theorem II is made explicit in [9], in the following way:

(III) *Let T, T', U, U' and A be as in* (I). *For any contractive intertwining dilation B of A, set*

$$B(z)=B_0+zB_1+z^2B_2+\ldots\quad (|z|<1)$$

where

$$B_n=U'^{*n}P_{\mathscr{M}}B|\mathscr{L}_*,\quad \mathscr{M}=U'^n\mathscr{L}'\quad (n=0,1,2,\ldots). \tag{3.3}$$

Then B is uniquely determined by the function $B(z)$ (and also can be easily recovered from $B(z)$) and

$$B(z)=j'^*(\omega P_{\mathscr{F}}+\sigma_B(z)P_{\mathscr{G}})\cdot(I-zj^*(\omega P_{\mathscr{F}}+\sigma_B(z)P_{\mathscr{G}}))^{-1} \tag{3.4}$$

$$\cdot(DP_{\mathscr{H}}+I-P_{\mathscr{H}})|\mathscr{L}_*\ \ (|z|<1),$$

where $\{\mathscr{G},\mathscr{G}',\sigma_B(z)\}$ is the contractive analytic function corresponding to B by virtue of (II).

The involvement of these results in problems of electrical engineering (as those of V. M. Adamjan, D. Z. Arov and M. G. Krein were in J. W. Helton's exciting paper [20]) as well as the connections to the Sarason type representations given in [29], [35] and to the Schur type descriptions given in [3], [4], [5] will be reported elsewhere.

4. The general Schur type labelling given by (II)–(III) was obtained (see [13], [9]) by means of another labelling (actually new, since it is only implicitly present in the

Schur–Adamjan–Arov–Krein theory). This new label, called *choice sequence* (see [13, Definition 3.1]) is any sequence of contraction R_n from $\mathscr{G}_n$ to $\mathscr{G}'_n$ $(n=0, 1, 2, \ldots)$ where

$$\text{(4.1)} \quad \mathscr{G}_0 = \mathscr{G},\ \mathscr{G}'_0 = \mathscr{G}' \text{ (see (4.2))},\ \mathscr{G}_n = \mathscr{D}_{R_{n-1}},\ \mathscr{G}'_n = \mathscr{D}_{R^*_{n-1}} \quad (n = 1, 2, \ldots).$$

We have (see [13, Propositions 2.2 and 3.1]):

(IV) *Let T, T', U, U' and A be as in* (I). *Then there exists a one-to-one correspondence between the set of all contractive intertwining dilations of A and the set of all choice sequences.*

The explicit formulae for the correspondence yielded by (IV) were established in [9], though they are already implicitly used in [14]. The next theorems exhibit them:

(V) *Let T, T', U, U' and A be as in* (I), *let B be any contractive intertwining dilation of A and let B_n $(n=0, 1, 2, \ldots)$ be the operators defined by* (3.3). *Let also $\{R_n\}_{n=0}^{\infty}$ be the choice sequence corresponding to B, by virtue of* (IV). *Then*

$$\text{(4.2)} \quad B_n = B_n^0 + C'_n R_n C_n^* \quad (n = 0, 1, 2, \ldots),$$

where

$$\text{(4.3)} \quad \begin{aligned} C'_0 &= j'^*|\mathscr{G}'_0, \quad C'_n = C'_0 D_{R_1^*} \ldots D_{R^*_{n-1}}|\mathscr{G}'_n \ (n = 1, 2, \ldots), \\ C_0 &= P_{\mathscr{L}_*}(DP_{\mathscr{H}} + I - P_{\mathscr{H}})|\mathscr{G}_0, \quad C_n = C_0 D_{R_1} \ldots D_{R_{n-1}}|\mathscr{G}_n \ (n = 1, 2, \ldots) \end{aligned}$$

and B_n^0 depends only on T, T', U, U', A and $R_0, R_1, \ldots, R_{n-1}$ $(n=0, 1, 2, \ldots)$.

For the computation of B_n^0 we introduce the following definition: For $N=0, 1, 2, \ldots$, the Nth interface operator V_N of any choice sequence $\{R_n\}_{n=0}^{\infty}$ is defined as the operator from

$$\mathscr{G}_{-1} \oplus \ldots \oplus (\mathscr{G}_{N-1} \oplus \mathscr{G}'_N) \oplus \mathscr{G}_{N+1} \oplus \ldots$$

to

$$\mathscr{G}_{-1} \oplus \ldots \oplus (\mathscr{G}'_{N-1} \oplus \mathscr{G}_N) \oplus \mathscr{G}_{N+1} \oplus \ldots$$

(where $\mathscr{G}_{-1} = \mathscr{G}'_{-1} = \mathscr{F}$; see (3.2), (4.1)) acting like the identity on all components except those in brackets on which it acts as the matrix

$$\begin{bmatrix} P_{\mathscr{F}} j^* \omega & -P_{\mathscr{F}} j^*|\mathscr{G}' \\ P_{\mathscr{G}} j^* \omega & -P_{\mathscr{G}} j^*|\mathscr{G}' \end{bmatrix} \quad \text{for } N = 0,$$

$$\begin{bmatrix} -R_{N-1} & D^*_{R_{N-1}} \\ D_{R_{N-1}} & R^*_{N-1} \end{bmatrix} \quad \text{for } N = 1, 2, \ldots.$$

(VI) *Under the same conditions as in* (V), *let X_n^j $(j=0, 1, \ldots, n;\ n=0, 1, 2, \ldots)$ be the operator from $\mathscr{L}_*$ to $\mathscr{G}_{j-1}$ defined by the following recurrent procedure:*

$$\text{(4.4)} \quad X_0^0 = P_{\mathscr{F}}(DP_{\mathscr{H}} + I - P_{\mathscr{H}})|\mathscr{L}_*$$

and, for $n=1, 2, \ldots,$

$$X_n^0 1_* \oplus X_n^1 1_* \oplus \ldots \oplus X_n^n 1_* \oplus C_n^* 1_* \oplus 0 \oplus 0 \oplus \ldots$$
$$= V_0 V_1 \ldots V_n (X_{n-1}^0 1_* \oplus X_{n-1_*}^1 \oplus \ldots$$
$$\oplus X_{n-1}^{n-1} 1_* \oplus C_{n-1}^* 1_* \oplus 0 \oplus 0 \oplus 0 \oplus \ldots) \quad (1_* \in \mathscr{L}_*) \tag{4.5}$$

where $\{V_n\}_{n=0}^{\infty}$ *is the sequence of the interface operators of* $\{R_n\}_{n=0}^{\infty}$. *Then*

$$B_n^0 = \begin{cases} C_0' \omega X_0^0 & \text{for } n = 0, \\ C_0' \omega X_0^0 + C_0' R_0 X_n^1 + \ldots + C_{n-1}' R_{n-1} X_n^n & \text{for } n = 1, 2, \ldots. \end{cases} \tag{4.6}$$

In spite of their complicated appearance, the formulae (5.2)–(5.5) provide an algorithm for the computation of either the contractive intertwining dilation or the corresponding choice sequence if the other is given. The algorithm can, at least in some interesting particular cases, be efficiently programmed on the computer. Such programs can be used for the synthesis of artificial seismic traces and, endowed with some stabilizing tricks, even to the analysis of real seismic traces. Our further discussion will also offer an explanation of this possibility.

5. We shall give now the simplest general model occurring in the discretization of the scattering of plane waves by layered media and of their multiple reflection through these media (see [**15**, Chapter 8]). For $\lambda, \lambda' \in \Lambda(R^d)$, let $A(\lambda, \lambda')$, and $E(\lambda, \lambda') \geqslant 0$ be some operators on $\mathscr{E}$ such that

$$E(\lambda, \lambda')\mathscr{E} = \mathscr{E},$$
$$\left((A(\lambda, \lambda')E(\lambda, \lambda') + E(\lambda', \lambda'')B(\lambda', \lambda''))\mathscr{E}\right)^- = \mathscr{E} \; (\lambda, \lambda', \lambda'' \in \Lambda) \tag{5.1}$$

and that for any sequence $\lambda_\cdot = \{\lambda\}_{j=-\infty}^{\infty} \subset \Lambda$, the scheme

$$u_j^{n+1} = B(\lambda_j, \lambda_{j+1}) u_{j-1}^n + A(\lambda_j, \lambda_{j+1}) u_{j+1}^n \tag{5.2}$$

where j, $n=0, \pm 1, \pm 2, \ldots$, explicit for $n \geqslant 0$ and implicit for $n \leqslant 0$, enjoys the property $\|u_\cdot^{n\pm 1}\| = \|u_\cdot^n\|$ $(n=0, \pm 1, \ldots)$, where for $u_\cdot = \{u_j\}_{j=-\infty}^{\infty}$ we set

$$\|u_\cdot\|^2 = \sum_{j=-\infty}^{\infty} (E(\lambda_j, \lambda_{j+1}) u_j, u_j).$$

Let $\mathscr{H}(\lambda_\cdot)$ be the space of all sequences $u_\cdot$ enjoying $\|u_\cdot\| < \infty$, and let $U(\lambda_\cdot)$ be the unitary operator on $\mathscr{H}(\lambda_\cdot)$ defined by (5.2) (i.e. $U(\lambda_\cdot)u_\cdot^n = u_\cdot^{n+1}$). We take now two sequences $\lambda_\cdot = \{\lambda_j\}$, $\lambda_\cdot^0 = \{\lambda_j^0\}$ such that $\lambda_j^0 \equiv \lambda^0$ $(j=0, \pm 1, \pm 2, \ldots)$ and $\lambda_j = \lambda^0$ for $j \leqslant s$ and set $U = U(\lambda_\cdot)$, $\mathscr{H} = \mathscr{H}(\lambda_\cdot)$ and $U^0 = U(\lambda_\cdot^0)$, $\mathscr{H}^0 = \mathscr{H}(\lambda_\cdot^0)$. Moreover set $\mathscr{E}_+(\lambda^0) = (A(\lambda^0, \lambda^0)\mathscr{E})^-$ $(= A(\lambda^0, \lambda^0)\mathscr{E})$, $\mathscr{E}_-(\lambda^0) = \mathscr{E} \ominus \mathscr{E}_+(\lambda^0)$, and define

$$\mathscr{D}_\pm^0 = \{u_\cdot = (u_j) \in \mathscr{H}^0, \; u_j \in \mathscr{E}_\pm(\lambda^0) \; (j < s), \; u_j = 0 \; (j \geqslant s)\}. \tag{5.3}$$

The operators U, U^0 and the spaces $\mathscr{H}, \mathscr{H}^0, \mathscr{D}_\pm^0$ constitute an Adamjan–Arov–Lax–Phillips type scattering model with discrete time [**24**], [**1**] (see also [**8**], [**18**]) and therefore we can consider its Lax–Phillips scattering operator S. Taking T and T' as the null operators on $\mathscr{D}_-^0 \ominus (U^0)^{-1}\mathscr{D}_-^0$ and on $\mathscr{D}_+^0 \ominus U^0\mathscr{D}_+^0$ and U and

U' as the restrictions of $(U^0)^{-1}$ to $\mathscr{D}_-^0$ and to $(\bigvee_{n=-\infty}^{\infty} U^{0n}\mathscr{D}_+^0)\ominus U^0\mathscr{D}_+^0$, respectively, the operator S is a contractive intertwining dilation of the null operator from $\mathscr{D}_-^0\ominus(U^0)^{-1}\mathscr{D}_-^0$ to $\mathscr{D}_+^0\ominus U^0\mathscr{D}_+^0$. Therefore, by virtue of (IV), (V) and (VI) we obtain a unique *computable* choice sequence $\{\Gamma_n(S)\}_{n=0}^{\infty}$.

(VII) *We have*

$$
\begin{aligned}
&\Gamma_{2j}(S) = 0 \quad \textit{for } j = 0, 1, 2, \ldots \\
&\Gamma_{2j+1}(S) = \omega_*(\lambda_{s-1}, \lambda_s) \ldots \omega_*(\lambda_{s+j-1}, \lambda_{s+j}) \\
&\qquad \cdot R(\lambda_{s+j}, \lambda_{s+j+1}) \\
&\qquad \cdot \omega(\lambda_{s+j-1}, \lambda_{s+j})^* \ldots \omega(\lambda_{s-1}, \lambda_s)^* \quad \textit{for } j = 0, 1, 2, \ldots,
\end{aligned} \tag{5.3}
$$

where for any $\lambda, \lambda' \in \Lambda$, $R(\lambda, \lambda')$ *is the operator* $A(\lambda, \lambda)B(\lambda, \lambda')$ *regarded as operator from* $\mathscr{E}_-(\lambda^0)$ *to* $\mathscr{E}_+(\lambda^0)$ *and* $\omega(\lambda, \lambda')$ *and* $\omega_*(\lambda, \lambda')$ *are some adequate canonical unitary operators from* $\mathscr{D}_{R(\lambda,\lambda')}$ *and* $\mathscr{D}_{R(\lambda,\lambda')^*}$ *to* $\mathscr{E}_-(\lambda')$ *and* $\mathscr{E}_+(\lambda')$, *respectively.*

Since if $\lambda_{s+1}=\lambda_{s+2}=\ldots=\lambda^1$ we have

$$\Gamma_1 = \omega_*(\lambda^0, \lambda^0)R(\lambda^0, \lambda^1)\omega(\lambda^0, \lambda^0)^*, \quad \Gamma_j = 0 \quad \text{for } j \neq 1,$$

it is natural to consider that the parameter λ is scattering discernible if

$$R(\lambda^0, \lambda^1) = R(\lambda^0, \lambda^2) \quad \text{implies} \quad \lambda^1 = \lambda^2 \ (\lambda^0, \lambda^1, \lambda^2 \in \Lambda). \tag{5.9}$$

In this case from the choice sequence $\{\Gamma_j(R)\}_{j=0}^{\infty}$ we obtain the parameters λ_{s+1}, $\lambda_{s+2}, \ldots$.

6. Here is an illustration of the above approach: Consider the system

$$\varrho(x)\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \quad \frac{1}{E(x)}\cdot\frac{\partial \sigma}{\partial t} = \frac{\partial v}{\partial x}, \tag{6.1}$$

where $\varrho(x)$ and $E(x)$ are piecewise constant, left continuous, functions >0, and constant for $x\leqslant 0$.

We replace in (6.1), the differential operators $\gamma(x)\partial/\partial t$ and $\partial/\partial x$ by the following "difference operators" (with $\gamma(x)$ representing some adequate "slight modifications" of $\varrho(x)$ and $1/E(x)$)

$$
\begin{aligned}
D_t\omega(x, t) = &\frac{1}{\tau}\frac{C(x)\gamma(x)+C(x+0)\gamma\big(x+C(x+0)\tau\big)}{C(x)+C(x+0)}\omega(x, t+\tau) \\
&-\frac{C(x)\gamma(x)}{C(x)+C(x+0)}\omega\big(x-C(x)\tau, t\big) \\
&+\frac{C(x+0)\gamma\big(x+C(x+0)\tau\big)}{C(x)+C(x+0)}\omega\big(x+C(x+0)\tau, x\big)
\end{aligned}
$$

and

$$D_x\omega(x, t) = \frac{\omega\big(x+C(x+0)\tau, t\big)-\omega\big(x-C(x)\tau, t\big)}{\big(C(x)+C(x+0)\big)\tau}$$

respectively, where $\tau>0$ and $C(x)\,(=\sqrt{E(x)/\varrho(x)})$ denotes the speed. The "difference" equations (with variable space mesh) established in this way are of the form (5.2) with $\mathscr{E}=C^2$ and

$$A(\lambda,\lambda')=(\lambda+\lambda')^{-1}\begin{bmatrix}\lambda' & 1\\ \lambda'\lambda & \lambda\end{bmatrix},\quad B(\lambda,\lambda')=(\lambda+\lambda')^{-1}\begin{bmatrix}\lambda & -1\\ -\lambda\lambda' & \lambda'\end{bmatrix},$$

where the parameter λ is represented by the impedance $I(x)\,(=\varrho(x)C(x))$.[4] Setting also

$$E(\lambda,\lambda')=\frac{\lambda+\lambda'}{2}\begin{bmatrix}1 & 0\\ 0 & 1/\lambda\lambda'\end{bmatrix}$$

all the conditions required in the preceding section are fulfilled. This yields that $\Gamma_{2j+1}(R)$ is the multiplication by r_j from the space $C\{1,-I(0)\}$ $(\subset H^2(C^2))$ to the space $Cz\{1,I(0)\}$ $(\subset H^2(C^2))$, where

$$r_j=[I(x_j)-I(x_{j+1})]/[I(x_j)+I(x_{j+1})]$$

with $x_0=0$ and $x_{j+1}=x_j+C(x_j+0)\tau$ $(j\geqslant 0)$. Thus, the impedance $I(x)$ is a scattering discernible parameter. Recalling that

$$r(x)=[I(x)-I(x+0)]/[I(x)+I(x+0)]$$

represents the reflexion coefficient at point x (see [**15**, p. 173]), it follows that if τ is small enough, our procedure[5] "yields correctly" all the "layers".

7. Unhappily, in a very important practical case involving the preceding particular example, the "experimental" values $R_2^0, R_4^0, \ldots$ are with respect to the "real" values $R_2, R_4, \ldots$ in the following relation

$$\sum_{j=0}^{k}|R_{2j}-R_{2j}^0|\approx 1,\quad \sum_{j=k+1}^{\infty}|R_{2j}-R_{2j}^0|=\delta\ll 1. \tag{7.1}$$

In order to overcome the difficulties implied by (7.1), one has to provide answers to a number of questions, which have natural formulations in the general framework discussed in §§ 4 and 5. For instance: with the same notations as in Theorem III, find a way to construct all A compatible with fixed $B_0, B_1, \ldots, B_{K-k}$. (This will yield a solution to the idealized case when $\delta=0$ in (7.1).)

We conclude with the hope that the interplay between the dilation theory of contractions and the search for effective formulae in the scattering inverse problem for layered media, we sketched above, will be fruitful for both sides.

---

[4] Thus $\Lambda=(0,\infty)$.

[5] Which is different from that of **[10]**, **[15]**, or **[28]**.

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Some Combinatorial Methods in Real Analysis

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Introduction. I shall speak of some work carried out over a period of some years (1970–1976) by a number of people including myself, E. Rodemich, H. Rumsey, Jr., H. Taylor, S. Milne, P. Deland and T. W. Park. A fairly complete account of the results obtained appears in [2].

The setting can be roughly described as follows. We are given that a generalized modulus of continuity of some function f (of one or more variables) has certain integrability properties and we conclude that f has a certain degree of smoothness.

These conclusions are expressed in the form of analytic inequalities between a "norm" of the function and a "norm" of its modulus of continuity.

A number of applications of these inequalities have been obtained in several areas of real analysis. For instance results have been obtained on

(1) Continuity of sample paths for certain stochastic processes.
(2) Sobolev type inequalities.
(3) Convergence of Fourier series.
(4) Continuity of local times for certain Markov processes.

All these results are consequences of a single purely "combinatorial" inequality whose significance may very well transcend the applications that so far have been made of it.

In [2] a brief description may be found of the path that led us to the discovery of this inequality; a proof of it may be found in [5]. Today we shall limit ourselves to stating it.

Time permitting we shall also state some of the analytic inequalities that follow from it and give a description of the applications.

1. The combinatorial inequality. We shall describe this result in the simplest possible setting. For each integer $n>1$ let G_n denote the set of pairs $\{(i,j)\colon i,j=1,2,\ldots,n;$

$i<j\}$. We introduce a partial ordering on G_n by setting

$$a=(i,j)\leqslant a'=(i',j') \tag{1.1}$$

if and only if

$$i'\leqslant i<j\leqslant j'. \tag{1.2}$$

In Figure 1 we illustrate the Hasse diagram of G_6.

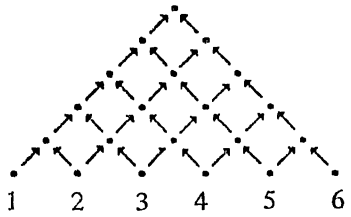

FIGURE 1

This given we shall say that a certain subset $A\subset G_n$ is "below" another subset $B\subset G_n$ and write

$$A\leqslant B \tag{1.3}$$

if and only if we can find an injection θ of A *into* B which satisfies the condition

$$a\leqslant\theta(a)\quad\forall a\in A. \tag{1.4}$$

In other words $A\leqslant B$ means that each point $a\in A$ can be "matched" to a point $a'\in B$ that is above it in a one-to-one manner.

Now for each permutation $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ of $1,2,\ldots,n$ and each $M\leqslant n$ let

$$\mathscr{D}_M(\sigma)=\{(i,j)\colon|\sigma_i-\sigma_j|\leqslant M\}.$$

For instance in Figure 2 we illustrate the set $\mathscr{D}_2(2\ 4\ 6\ 1\ 3\ 5)$.

Let us also denote by R_M the set of elements in the first M rows of G_n. In Figure 3 we have R_2 (when $n=6$).

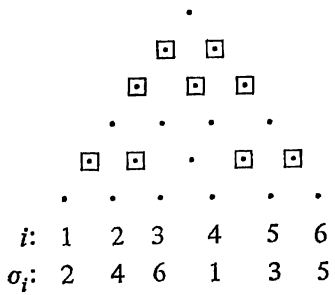

FIGURE 2

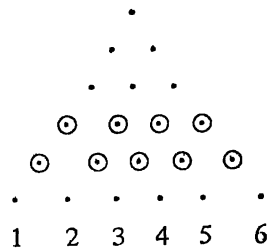

FIGURE 3

Clearly $R_M = \mathscr{D}_M(\varepsilon)$ where ε denotes the identity permutation.

This given, the combinatorial inequality can be stated as follows

THEOREM 1.1. *For all $M \leqslant n$ and all permutations $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ we have*

$$R_M \leqslant \mathscr{D}_M(\sigma). \tag{1.5}$$

In Figure 4 we have illustrated an instance of this result by superimposing the diagrams $\mathscr{D}_2(2\ 4\ 6\ 1\ 3\ 5)$ (squares) and R_2 (circles) and indicating a possible map θ (arrows) satisfying the condition (1.4).

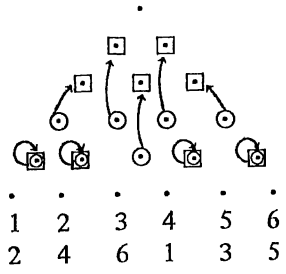

FIGURE 4

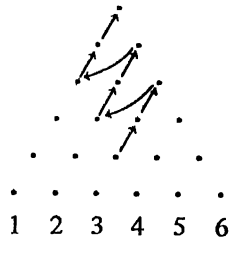

FIGURE 5

It can be shown that given $\mathscr{D}_M(\sigma)$ a map θ of R_M onto $\mathscr{D}_M(\sigma)$ satisfying 1.4 can always be constructed by matching each $a \in R_M$ in a "lexicographic" manner. That is we go from "left to right" in R_M following the lexicographic order of the pairs (i, j) and match each $a \in R_M$ with the first available pair in $\mathscr{D}_M(\sigma)$ that is in its "right diagonal", or in the right diagonal immediately to the left of $\mathscr{D}$, or in any of the right diagonals that are found progressively further and further left.

This procedure is best illustrated by an example. In Figure 5 we give the search paths for finding the match to the point (3,4) in G_6.

Theorem 1.2 has a d-dimensional analogue which may be stated as follows. Let $I_d(n)$ be the set of d-tuples of integers $(i_1, i_2, \ldots, i_d)$ taking the values $1, 2, \ldots, n$. For each one-to-one map σ of the set of integers $[n^d] = [1, 2, \ldots, n^d]$ into $I_d(n)$ let

$$\mathscr{D}_M(\sigma) = \{(i, j) \in G_{n^d} : \overline{\sigma_i \sigma_j} \leqslant M\}$$

where for $X = (i_1, \ldots, i_d)$, $Y = (j_1, \ldots, j_d)$ we set

$$\overline{XY} = \sqrt{\sum_{\nu=1}^{d} (i_\nu - j_\nu)^2}.$$

This given it can be shown that there is a constant c_d depending only on the dimension d such that

THEOREM 1.2. *For all $M \leqslant n^d$ and all one-to-one maps σ of $[n^d]$ onto $I_d[n]$ we have*

$$R_{M^d/c_d} \leqslant \mathscr{D}_M(\sigma). \tag{1.6}$$

We do not know what is the best possible c_d but we can show that we can take

$$c_d = 3^d(d+3)^{d/2}.$$

2. The basic analytic inequalities. Let now φ be a function on G_n that is nonnegative and nondecreasing with respect to the partial order (1.1) of G_n. We see that if A is below B then we must have

$$\sum_{a\in A}\varphi(a) \leqslant \sum_{b\in B}\varphi(b). \tag{2.1}$$

Indeed, using the map $\theta: A\to B$ we can write

$$\sum_{b\in B}\varphi(b) = \sum_{a\in A}\varphi(\theta(a))$$

but then the monotonicity of φ and (1.4) give $\varphi(\theta(a))\geqslant\varphi(a)$. Thus (2.1) is immediate.

It is not difficult to show (using the *marriage theorem*) that the validity of (2.1) for all such φ is actually equivalent to the combinatorial inequality $A\leqslant B$.

Let now $\Phi(u)$ be a real valued nondecreasing function of the absolute value of u. That is

$$\Phi(u) = \Phi(-u)\uparrow \quad \text{as} \quad |u|\uparrow. \tag{2.2}$$

Let $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n$ be reals and set for $a=(i,j)$

$$\varphi(a) = \Phi(\lambda_i-\lambda_j). \tag{2.3}$$

Note that for $i'\leqslant i\leqslant j\leqslant j'$ we have

$$\Phi(\lambda_i-\lambda_j) \leqslant \Phi(\lambda_{i'}-\lambda_{j'}).$$

In other words, the function φ in (2.3) is nondecreasing with respect of the partial ordering of G_n. Thus by Theorem 1.1 we must necessarily have (2.1) with $A=R_M$ and $B=\mathscr{D}_M(\sigma)$.

This gives

$$\sum_{|i-j|\leqq M}\Phi(\lambda_i-\lambda_j) \leqslant \sum_{|\sigma_i-\sigma_j|\leqq M}\Phi(\lambda_i-\lambda_j). \tag{2.4}$$

Note now that if $\pi=(\pi_1,\pi_2,\ldots,\pi_n)$ is the inverse of the permutation σ then (2.3) can also be written in the form

$$\sum_{|i-j|\leqq M}\Phi(\lambda_i-\lambda_j) \leqslant \sum_{|i-j|\leqq M}\Phi(\lambda_{\pi_i}-\lambda_{\pi_j}).$$

We have therefore established the following result.

THEOREM 2.1. *Let $f_1,f_2,\ldots,f_n$ be arbitrary reals and $f_1^*\geqslant f_2^*\geqslant\ldots\geqslant f_n^*$ denote their nonincreasing rearrangement. Then for all Φ satisfying* (2.2) *and for all $M\leqslant n$ we have*

$$\sum_{|i-j|\leqq M}\Phi(f_i^*-f_j^*) \leqslant \sum_{|i-j|\leqq M}\Phi(f_i-f_j). \tag{2.5}$$

This inequality has a continuous analogue which can be obtained by a suitable passage to the limit (see [5]). To state it we need some further notation.

Given a measurable function f on $[0, 1]$ let us denote by f^* the non-increasing *rearrangement* of f. That is f^* is the unique right continuous non-increasing function such that

$$m\{x: f^*(x) \geqslant \lambda\} = m\{x: f(x) \geqslant \lambda\} \quad \forall \lambda. \tag{2.6}$$

We then have

THEOREM 2.2. *For any measurable function f on $[0, 1]$ and any Φ satisfying* (2.2) *we have for* $0 \leqslant \delta \leqslant 1$

$$\iint\limits_{|x-y| \leqq \delta} \Phi(f^*(x)-f^*(y))\,dx\,dy \leqslant \iint\limits_{|x-y| \leqq \delta} \Phi(f(x)-f(y))\,dx\,dy. \tag{2.7}$$

It can be shown that conversely (2.7) implies (2.5) and that in turn (2.5) implies the combinatorial inequality in (1.5). We shall refer to it as the *basic analytic inequality.*

Theorem 2.2 has a d-dimensional analogue which can be stated as follows. For a given $f(x_1, x_2, \ldots, x_d)$, measurable on the d-dimensional cube

$$I_d = [0, 1] \times [0, 1] \times \ldots \times [0, 1] \ (d \text{ times}),$$

we let again $f^*(x)$ denote the unique non-increasing right continuous function on $[0, 1]$ which satisfies

$$m\{x \in [0, 1]: f^*(x) \leqslant \lambda\} = m\{(x_1, \ldots, x_d) \in I_d: f(x_1, \ldots, x_d) \geqslant \lambda\}.$$

Then we can show that for some constant c_d depending only on d we have

THEOREM 2.3. *For any measurable f on I_d and any Φ satisfying* 2.2 *and for* $0 \leqslant \delta \leqslant 1$

$$\iint\limits_{|x-y| \leqq \delta} \Phi(f^*(x)-f^*(y))\,dx\,dy \leqslant \iint\limits_{\overline{XY} \leqq c_d \delta^{1/d}} \Phi(f(X)-f(Y))\,dX\,dY \tag{2.8}$$

where $X=(x_1, \ldots, x_d)$, $Y=(y_1, \ldots, y_d)$ and $\overline{XY}$ denotes the Euclidean distance between X and Y.

Of course, we could obtain (2.8) by passing to the limit in (1.6) in the same way (2.7) is obtained from (1.5). However, it develops that both (1.6) and (2.8) are immediate consequences of (2.7) via the existence of *Peano curves* satisfying certain special conditions.

To see how this comes about we shall derive (2.8) from (2.7).

It can be shown (see [8]) that there exists a map $\mathscr{F}: [0, 1] \to I_d$ satisfying the following conditions

(a) $\mathscr{F}(x)$ is measure preserving.

(b) $\mathscr{F}(x)$ is Lipschitzian of order $1/d$. That is for some constant c_d we have

$$\overline{\mathscr{F}(x)\mathscr{F}(y)} \leqq c_d |x-y|^{1/d} \quad \forall x, y \in [0, 1]. \tag{2.9}$$

It is interesting to point out that unbeknown to Peano himself the original Peano curves do satisfy both these conditions.

Now, given f as in Theorem 2.3, let us apply (2.7) to the function $g(x)=f(\mathscr{F}(x))$. Observe that since $\mathscr{F}$ is measure preserving we must necessarily have

$$g^*(x)=f^*(x),$$

and (2.7) then gives

$$\iint_{|x-y|\leqq\delta}\Phi(f^*(x)-f^*(y))\,dx\,dy\leqslant\iint_{|x-y|\leqq\delta}\Phi(f(\mathscr{F}(x))-f(\mathscr{F}(y)))\,dx\,dy.$$

Now, using (2.9) (b) we see that the right hand side is no bigger than

$$\iint_{\overline{\mathscr{F}(x)\mathscr{F}(y)}\leqq c_d\delta^{1/d}}\Phi(f(\mathscr{F}(x))-f(\mathscr{F}(y)))\,dx\,dy.$$

Making the substitutions $X=\mathscr{F}(x)$, $Y=\mathscr{F}(y)$ and using again the measure preserving property of $\mathscr{F}$ we see that this integral is equal to

$$\iint_{\overline{XY}\leqq c_d\delta^{1/d}}\Phi(f(X)-f(Y))\,dX\,dY.$$

Thus (2.8) follows as asserted.

3. Applications. To give a flavor of the uses that the basic inequality (2.7) can be put to we shall state some of its consequences.

Let Ψ and p be restricted as follows

(3.1)
(a) $\Psi(u)$ is defined and continuous in $(-\infty,+\infty)$ and $\Psi(u)=\psi(-u)\uparrow$ as $|u|\uparrow$.
(b) $\Psi(e^x)$ is convex in $(-\infty,+\infty)$.
(c) $p(u)$ is defined and continuous in (0, 1) and $p(u)=p(-u)\downarrow 0$ as $|u|\downarrow 0$.

Then it can be shown [5] that (2.7) implies that

THEOREM 3.1. *For any f measurable in* [0, 1] *and any pair p, Ψ satisfying the conditions in* (3.1)

$$\int_0^1\int_0^1\Psi\left(\frac{f^*(x)-f^*(y)}{p(x-y)}\right)dx\,dy\leqslant\int_0^1\int_0^1\Psi\left(\frac{f(x)-f(y)}{p(x-y)}\right)dx\,dy.$$

This inequality has several remarkable consequences. For instance by a suitable choice of Ψ and p we get [5] that

$$\int_0^1\int_0^1\frac{|f(x)-f(y)|^p}{(x-y)^2}\,dx\,dy\leqslant B^p\quad(p>1)$$

implies

$$\int_0^1\exp\left[\left(\frac{\log 2}{2}\right)\frac{|f(x)-m_f|}{B}\right]^q dx\leqslant 2\sqrt{2}$$

where m_f denotes the *median* of f and $q=p/(p-1)$.

Set now for $f\in L_p(0,1)$

$$Q_p(\delta,f)=\left[\frac{1}{\delta}\iint_{|x-y|\leqq\delta}|f(x)-f(y)|^p\,dx\,dy\right]^{1/p}.$$

This is perhaps the most natural way to define the L_p modulus of continuity of a function on [0, 1]. Indeed we see that (2.7) with $p(u)=|u|^p$ gives

$$Q_p(\delta, f^*) \leqslant Q_p(\delta, f) \quad \forall d \leqslant 1.$$

Now, upon a suitable integration of this inequality we get the following remarkable result [4]:

THEOREM 3.3. *If $Q_p(\delta, f)/\delta^{1+1/p}$ is integrable then f is essentially continuous and at all points of the Lebesgue set of f we have*

$$|f(x)-f(y)| \leqslant \frac{2^{1+2/p}}{\log 3/2} \int_0^{|x-y|} Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}. \tag{3.2}$$

In general we always have

$$\left.\begin{matrix} f^*(x)-f^*(1/2) \\ f^*(1/2)-f^*(1-x) \end{matrix}\right\} \leqslant \frac{4^{1/p}}{\log 3/2} \int_x^1 Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}} \quad (0 < x \leqslant 1/2). \tag{3.3}$$

This inequality can be used to derive the convergence of certain Fourier series. To see how this comes about observe that (3.2) need not be restricted to functions defined on [0,1]. By a change of scale argument it can be established [4] for periodic function of period 2π.

More precisely if $\omega_p(\delta, f)$ denotes the customary L_p modulus of continuity of such a function, that is

$$\omega_p(\delta, f) = \left[\int_0^{2\pi} |f(x+\delta)-f(x)|^p \, dx\right]^{1/p}.$$

Then from 3.2 we can derive

$$|f(x)-f(y)| \leqslant c_p \int_0^{|x-y|} \omega_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}$$

where c_p is a constant depending only on p. In particular if $S_n(x, f)$ denotes the nth partial sum of the Fourier series of f, we must also have (using the theorem of M. Riesz)

$$|S_n(x, f)-S_n(y, f)| \leqslant c_p' \int_0^{|x-y|} \omega_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}$$

where c_p' is another such constant.

We see then that if $\omega_p(\delta, f)/\delta^{1+1/p}$ is integrable the partial sums $S_n(x, f)$ are necessarily equicontinuous and therefore uniformly convergent.

Now, this result for $1 \leqslant p \leqslant 2$ can be considerably improved [4], (the Fourier series of f converges absolutely then). However, for $p > 2$ it does not appear to be a consequence of any of the well-known criteria assuring uniform convergence.

As a last application we shall show how a combination of (2.8) and (3.3) can be used to derive Sobolev-type inequalities. To this end set for $f \in L_p(I_d)$

$$Q_p(\delta, f) = \left[\frac{1}{\delta^d} \int_{I_d} \int_{I_d} |f(X)-f(Y)|^p \, dX \, dY\right]^{1/p}$$

$$\overline{XY} \leqq \delta$$

and note that if the gradient ∇f of f is in $L_p(I_d)$ then we must have

$$Q_p(\delta, f) \leqslant c\delta \tag{3.4}$$

where c is a constant depending on p, d and the L_p norm of ∇f. Now (2.8) for $\Phi(u)=|u|^p$ yields

$$Q_p(\delta, f^*) \leqslant c_d^{d/p} Q(c_d \delta^{1/d}, f). \tag{3.5}$$

Combining with (3.4) and substituting in (3.3) we get then

$$\left.\begin{matrix} f^*(x)-f^*(1/2) \\ f^*(1/2)-f^*(1-x) \end{matrix}\right\} \leqq c' \int_x^1 \delta^{1/d} \frac{d\delta}{\delta^{1+1/p}}$$

and that shows that f is in "weak" L_q where

$$q = \frac{1}{1/p-1/d}.$$

Of course for $p=d$ we get that f is exponentially integrable. It can also be shown, using the d-dimensional analogue of (3.2) that (3.4) when $p>d$ forces f to be Hölder continuous with exponent $1-d/p$.

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The Pade Approximants

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In the last few years a new interest has developed towards Pade approximants. This subject is fairly old but is not still sufficiently studied. The object of the present paper is to outline the problems with which the Pade approximants are connected and applied to. We shall restrict ourselves to diagonal Pade approximants.

1. Definitions and some elementary properties

1°. The main definition. Let $f(z)$ be a function, analytic at infinity and $f(\infty)=0$. Find a polynomial $A(z)\not\equiv 0$, $\deg A(z)\leqslant n$ such that

$$A(z)f(z) = B(z)+\frac{c_{1,n}}{z^{n+1}}+\frac{c_{2,n}}{z^{n+2}}+\ldots \tag{1}$$

where $B(z)$ is another polynomial. The condition (1) gives us n linear equations in $n+1$ unknown coefficients of the polynomial $A(z)$. Thus $A(z)$ exists but is not necessarily unique. The rational fraction

$$\pi_n(z) = B(z)/A(z)$$

however, is determined uniquely. It is called the nth Pade fraction. All such indexes $\Lambda=\{n_1, n_2, \ldots\}$ that degree of $A(z)$ is exactly n, are called normal. If $n_k\leqslant n<n_{k+1}$ then

$$\pi_n(z) = \pi_{n_k}(z)$$

Let

$$f(z) = \frac{c_0}{z}+\frac{c_1}{z^2}+\ldots$$

index n is normal if and only if the so-called Hankel determinant

$$H_n = \begin{vmatrix} c_0 & c_1 \dots c_{n-1} \\ c_1 & c_2 \dots c_n \\ \dots\dots\dots\dots \\ c_{n-1} & c_n \dots c_{2n-2} \end{vmatrix}$$

does not vanish.

2°. Relations with continued fractions. For a function

$$f(z) = Q(z) + \frac{c_0}{z} + \frac{c_1}{z^2} + \dots \tag{2}$$

where $Q(z)$ is a polynomial we denote $[f(z)] = Q(z)$ by analogy with real numbers and call it a whole part of the function $f(z)$. By the same analogy with continued fractions for real numbers, we may relate each function with a pole at infinity to a functional continued fraction

$$f(z) \sim Q_1(z) + \frac{1}{Q_2(z)} + \frac{1}{Q_3(z)} + \dots \tag{3}$$

where $Q_1, Q_2, \dots$ are polynomials. We may note here that all the calculations are purely formal and so we put each formal power series (2) into correspondence with the Pade approximants, as well as with a continued fraction (3).

It turns out that if $\deg Q_k = d_k$ then the set of normal indices for f is $\Lambda = \{d_1, d_1+d_2, d_1+d_2+d_3, \dots\}$ and

$$\pi_{n_k}(z) = Q_1 + \frac{1}{Q_2} + \dots + \frac{1}{Q_k}.$$

The Pade approximants could be constructed at any point. We choose infinity merely for the convenience reasons.

2. Calculations in physics and mechanics. In some problems of physics and mechanics the investigator has often to deal with the following question. Given a power series expansion of a function, find the location of poles and branch points. The Pade approximants happen to be of a great use here. Although the general theorems embrace but some special classes of function, it became clear nevertheless that the poles of Pade fractions $\pi_n(z)$ in some sense well imitate the poles of the given function and some cuts joining the branch points. We shall not dwell at these applications but direct the interested reader to the works [1] and [2] containing an extensive bibliography.

3. Number theory applications. This is one of the most important fields of applications of Pade approximants. Besides the definition given earlier the notion of approximation forms is of a great importance here. Let $f_1(z), \dots, f_m(z)$ be functions, analytic in a neighbourhood of zero. For a set of nonnegative integers $(n_1, \dots, n_m)$

there exist polynomials $P_1(z), \ldots, P_m(z)$ not all identically zero of degree no greater than n_k respectively such that

$$\sum_{k=1}^{m} P_k(z) f_k(z) = cz^{\sigma} + \ldots$$

where $\sigma = \sum_{k=1}^{m} (n_k + 1) - 1$.

These approximation forms turn out to be very useful in the proofs of transcendency of certain numbers. The first to apply these to the proof of transcendency of e was Hermite [3]. See also a work by Siegel [4] providing the further development of the method. Baker [5] shows by the help of Pade approximants of the function $(1+x)^{1/3}$ that

$$\left| 2^{1/3} - \frac{p}{q} \right| \geqslant \frac{10^{-6}}{q^{2,955}}$$

for each pair of integers p and q. The entire lists could be compiled of the works on diophantine approximations which employ the approximation forms Pade fractions. We also point out a work by Stepanoff [6], who by employing Pade approximants ideas in analytic number theory obtained some strong results.

4. Algebraic aspects. Let $f(z) = \sum_{k=0}^{\infty} c_k / z^{k+1}$ be an algebraic function. Uchiyama [7] showed that for every $\varepsilon > 0$ there exists a constant c_ε such that if $A(z) \not\equiv 0$ and $B(z)$ are polynomials of degree not greater than n and

$$A(z) f(z) - B(z) = dz^{-\lambda} + \ldots \quad (d \neq 0)$$

then $\lambda \leqslant (1+\varepsilon) n + c_\varepsilon$. This provides in fact some characteristic of the set $\Lambda = \{n_k\}$ of normal indices for algebraic functions. We see in particular that

$$\lim_{k \to \infty} \frac{n_{k+1}}{n_k} = 1.$$

This theorem is a functional analogue of Thue–Siegel–Roth theorem on rational approximations of algebraic numbers and enjoys as well as the latter a prolonged prehistory. The analogy becomes obvious by the respective formalization of the problem with its strongly algebraic features (see [7]). The proof of Uchiyama is also somewhat analogous to the proof of Thue–Siegel–Roth theorem. However in recent years some new considerations have been proposed here apart from the algebraic number theory ideas (see [8], [9]).

The similar questions, on the structure of Λ as a matter of fact, were not only for algebraic functions (see [10], [11], [12]).

5. Moment problem. The origins of the theory dating as far back as Chebyshev, the relations between Pade approximants and moment problem are as follows.

Let μ be a measure on $(-\infty, \infty)$ with infinitely many growth points. Consider the function

$$f(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x - z}$$

which will be called hence forth Markoff function. To this function a power series is put into correspondence, namely

$$f(z) \sim \sum \frac{c_k}{z^{k+1}}, \quad c_k = \int_{-\infty}^{\infty} x^k \, d\mu.$$

This is the formal expansion of $f(z)$ in the powers of z^{-1} which ceases to be formal if μ vanishes outside some compactum. Construct Pade approximants for $f(z)$. The Hankel determinants (see §1) are distinct from zero as they are the main determinants of positively determined quadratic forms

$$\sum_{i,j=0}^{m} c_{i+j}\xi_i\xi_j = \int_{-\infty}^{\infty} \left[\sum_{0}^{m} \xi_k x^k\right]^2 d\mu.$$

Thus all the indices are normal: $\Lambda=\{0, 1, \ldots\}$, and we see that $\pi_n=P_n/Q_n$ where $\deg Q_n=n$ and formal continued fraction for f is of the type

$$f \sim \frac{1}{A_1 z+B_1} + \frac{1}{A_2 z+B_2} + \ldots.$$

Here, a remarkable fact is that the polynomials $Q_n(x)$ coincide with the sequence of orthogonal polynomials for the measure μ. Indeed,

$$Q_n f - P_n = \int_{-\infty}^{\infty} \frac{Q_n(x)}{x-z} \, d\mu - \int_{-\infty}^{\infty} \frac{Q_n(x)-Q_n(z)}{x-z} \, d\mu = \frac{h_n}{z^{n+1}} + \ldots.$$

Hence

$$P_n = \int_{-\infty}^{\infty} \frac{Q_n(x)-Q_n(z)}{x-z} \, d\mu(x) \quad \text{and} \quad \int_{-\infty}^{\infty} x^k Q_n(x) \, d\mu = 0,$$

$k=0, 1, \ldots, n-1$.

We owe to Markoff the following theorem. If the measure μ vanishes outside some finite segment Δ then $\pi_n \rightrightarrows f$ uniformly on every closed set K disjoint from Δ. The study of Pade approximants for Markoff functions is reduced in essence to the study of moment problem and orthogonal polynomials. The main contributors here are Stieltjes and Carleman. Their works illuminate the connections between the problems of convergence of Pade approximants and the problem of determinateness of moment problem. We indicate here the work [13] which, apart from formulation of the classical results contains some other interesting reports.

2°. A concrete moment problem could be given by either of the three data: the measure, the moments, the $A_k z+B_k$ in the expansion (2). One of the parts in moment problem is to define the rest of the elements given one of the three. The most serious part is probably to reconstruct the measure given the moments. More specifically, it remains unknown which conditions on moments could guarantee the measure to be of one or another functional class. The problems of interest are perhaps as well those of reconstruction of (A_k, B_k) by the given moments $\{C_k\}$. Although there are some formulas connecting A_k, B_k and C_k, employing some expressions via

determinants similar to H_n, the connection is a very obscure one and does not yield any satisfactory qualitative picture. The same is true for the reverse problem, i.e. reconstruction of C_k by (A_k, B_k).

One of the important problems in the theory of orthogonal polynomials is the problem of Stekloff which is conceived in a broader sense recently than it was originally. Given the properties of the measure μ define the behaviour of the orthogonal system of polynomials $\{Q_n\}$. Even in its original version the problem still stands unsolved. Namely, let $w(x) \geqslant c > 0$ on $[a, b]$. Prove that the system $\{Q_n\}$ of orthogonal polynomials on $[a, b]$ with respect to measure $w(x)dx$ satisfies the inequality

$$|Q_n(x)| \leqslant C_\varepsilon, \quad x \in [a+\varepsilon, b-\varepsilon].$$

We will not detain the reader on the other aspects of the problem here. See in this connection **[14]**.

The similar matters are dealt with in the recent theorem of Rakhmanoff **[15]** which states that if

$$d\mu/dx > 0 \tag{4}$$

almost everywhere on $[-1, 1]$ then for a system of orthogonal polynomials $Q_n(z)$ we have

$$\frac{Q_{n+1}(z)}{Q_n(z)} \to \frac{1}{2}\left(z + \sqrt{z^2-1}\right) \tag{5}$$

uniformly inside the $C \setminus [-1, 1]$ (the branch of the root is chosen in such a way that $\sqrt{z^2-1} > 0$ while $z > 1$). The formula (5) follows from Szegő asymptotics but the latter is true by the stronger suppositions than (4).

3°. *Special weights.* The problem of finding the properties of orthogonal polynomials given the specific weights and measures is in the domain of classical analysis. We will make so bold as to observe that in the recent times the investigations in the field of special systems of orthogonal polynomials are unduly underestimated. It could prove to be of a great use to expand the list of those weights which permit concrete formulas for orthogonal polynomials. The main reason for this is that as a rule the applications of concrete systems in physics as well as in other branches of mathematics itself pretty often predominates over the applications of the general results. As a matter of example we point out the following interesting problem. Obtain explicit formulas if possible for orthogonal polynomials to the weight

$$d\mu = dt/\left(e^{2\pi\sqrt{t}} - 1\right), \quad t \in [0, \infty).$$

Although this problem does not seem to promise any immediate applications right away, nevertheless it would be very useful to have a clearer notion of the properties of the said weight in detail. Its Mellin transformation can be expressed through the Riemann ζ-function, its moments are closely connected with Bernoulli numbers and as pointed out in **[16]** the function $(e^t - 1)^{-1}$ is employed in Planck formula for equilibrium emission, the same function appears in the solutions of some dif-

ference equations. At the first glance this function seems to be very like the function e^{-t} but the analytic properties of the former however, are much richer than those of the latter.

4°. Some results on multidimensional moment problem are contained in [17] and [18] where further references are to be found. It should be noted that in multidimensional moment problem a vast number of uncertainties occur, even in such a simple for one-dimensional case question as the solvability of moment problem. We formulate here one problem of the multidimensional moment problem. Let $k=(k_1, \ldots, k_N), k_j \geqslant 0, x^k=x_1^{k_1} \ldots x_N^{k_N}$. For the given number $\{c_k\}$, $|k|=k_1+\ldots+k_N \leqslant n$, let the contracted moment problem be solvable, i.e. there exists a measure μ such that $c_k=\int_{R^N} x^k d\mu$ for each k, $|k|<n$. The set $\mathfrak{M}$ of these measures is convex and closed in the corresponding space of charges. The problem is to give a description of the set of extreme points of $\mathfrak{M}$. Is it possible as in the analogical problem for $N=1$ to state that there exists for every extreme point μ a polynomial P such that

$$P(\operatorname{supp} \mu) = 0?$$

It seems that the greatest nuisance in the multidimensional case is the absence of any representation of positive polynomials through the squares of polynomials.

5°. We omit here the well-known applications of orthogonal polynomials in the theory of probabilities. Of certain interest are the relations of orthogonal polynomials with difference method which immediately leads us to recurrence relations for orthogonal polynomials. In this connection see [19, p. 219].

6. Complex analysis. This part deals mainly with the recent works on Pade approximants carried out in the U.S.S.R. The present survey could be supplemented by [20].

One of the most difficult problems in Pade approximants theory is the convergence problem. As far back as at the beginning of the century different examples of entire functions were constructed of which the Pade approximants were diverging in every point of the plane. Perhaps precisely these counterexamples coupled with the general difficulties of the object of investigation caused the decay of interest on the part of complex analysis towards the Pade approximants. For a pretty long time the only positive result on the convergence of Pade fractions had been the above formulated Markoff theorem. In 1970 Nutall [21] proved that if $f(z)$ is meromorphic in C then its Pade approximants constructed in the zero point converge to f in measure. Pommerenke substituted the convergence in measure in the result of Nutall by convergence in capacity. Gončar noted that if we have also the corresponding information on the poles of Pade fractions and the poles of $f(z)$ the convergence in this theorem could be proved to be uniform in the corresponding domains. Gončar [23] considered the functions of the type

$$f(z) = \int_{\Delta} \frac{d\mu(x)}{z-x} + r(z), \quad \Delta \subset R,$$

where $r(z)$ is the rational function with poles outside Δ and $r(\infty)=0$. He showed that if the measure μ satisfies the condition (4) then $\pi_n(f, z)\to f(z)$ uniformly inside $\bar{C}\setminus\Delta\setminus P_f$ where P_f is the set of poles of $r(z)$ inside $C\setminus\Delta$.

The next result in the same direction is due to Rakhmanoff [24]. Let μ be restricted to two intervals Δ_1 and Δ_2 and let it satisfy on each of these the Szegő condition, i.e.

$$\int_{\alpha}^{\beta} \frac{\ln \mu'(x)}{\sqrt{(x-\alpha)(\beta-x)}}\, dx > -\infty$$

where α, β are the extreme points of either of Δ. Let $r(z)$ be a rational function with poles outside $\Delta_1 \cup \Delta_2$ and $r(\infty)=0$. Then there exists an analytic curve $\Gamma \subset C\setminus(\Delta_1 \cup \Delta_2)$ whose extreme points belong respectively to Δ_1 and Δ_2, and such that the Pade approximants for the function

$$f(z) = \int_{\Delta_1 \cup \Delta_2} \frac{d\mu}{z-x} + r(z)$$

converge uniformly to $f(z)$ inside

$$\bar{C}\setminus\Delta_1\setminus\Delta_2\setminus\Gamma\setminus P_f$$

where P_f is the set of poles of $r(z)$. The curve Γ converts into a finite set of points if the harmonic measure of the interval Δ_1 with respect to $C\setminus(\Delta_1 \cup \Delta_2)$ in infinity is a rational number. Some negative statement is also given with respect to the behaviour of π_n in the neighbourhood of any point of Γ. The interesting fact is that Γ depends on the set $\Delta_1 \cup \Delta_2$ but not on the measure μ. It should be noted that the work of Rakhmanoff employs some delicate questions of complex analysis extensively studied by Widom [25].

It would be very interesting to investigate the convergence of Pade approximants for algebraic functions. The results available at the present are scarce. The work [26] shows that even for the simplest algebraic functions convergence or divergence of its Pade fractions depends essentially on the point where these are constructed. It is note worthy that the conditions emerging here as well as those of Rakhmanoff theorem are of arithmetic character. The following conjecture seems to be reasonable. For every algebraic function there exists a point converging outside some cuts of the complex plane which permit to pick up a single-valued branch of $f(z)$.

In the present survey of restricted space we were bound to omit a vast number of investigations on Pade approximants. At all events it is clear that the Pade ractions are a very important and interesting object of mathematical analysis.

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What Problems Do Spectral Theory and Complex Analysis Solve for Each Other?

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0. Very old and close relations join the branches of mathematics mentioned in the title. Every sufficiently complete description of these relations must contain the full history and perspective of mathematical analysis as a whole. The aim of this report is much more definite: I shall try to trace the interplay between spectral theory and complex analysis during the last, say, ten years and only in the following two domains:

I. Generalized spectral decompositions and the Corona Theorem, the functional model of Sz.-Nagy and Foiaş and the free interpolation by analytic functions.

II. Spectral synthesis, divisor ideals and the factorization of analytic functions.

The rapid progress in these directions during the last years is based upon:

A. The Beurling Theorem (and its generalizations) on invariant subspaces of the shift operator $f \to zf$ in the Hardy space H^2, a fundamental fact connecting geometric (spectral) objects with the multiplicative structure of analytic functions;

B. The clear realization of the method of dilations of linear operators making the shift operator a universal operator and giving rise to the model mentioned in I;

C. The Carleson Corona Theorem, its generalizations and the methods connected with it.

Similar ideas have played a role earlier (before the last 10–15 years); the same features appear already in the theory of elementary divisors in Linear Algebra (compare with A) and in the Hilbert–von Neumann spectral theorem (compare with B). A new feature is the essential widening of modern tools (Nevanlinna–Hardy classes and the whole corresponding analytic machinery) and of the circle of problems under consideration including now many of the principal problems

of Function Theory. Our bibliography is not exhaustive but we hope it can help in finding many other publications concerning topics we are discussing below.

1. Model, interpolation, corona. These comprise the right ordering of the three remarkable domains of analysis inside spectral theory. I am going to explain the relations between these matters in their simplest manifestation, namely in the context of the one-dimensional model of Sz.-Nagy and Foiaş [1]. This model covers, however, certain operators occurring in applications (having a "physical meaning"), and its investigation leads to the separation of technical difficulties inherent in spectral decompositions in their purest form, without any influence of the non-commutative matrix structure inevitable in the study of operators of more general form. It is important also that all assertions concerning the one-dimensional functional model can be interpreted as theorems in the spectral function theory which are of independent interest.

Thus, let H be a Hilbert space, T a linear continuous operator in H of class C_{00}, i.e. $\lim_n T^n x=0$, $\lim_n T^{*n}x=0$, $\forall x\in H$. Then T is unitarily equivalent to its model, namely to the operator $P_K S|K$, where $K=H_E^2\ominus\theta H_E^2$, H_E^2 being the Hardy class of all E-valued and holomorphic functions f in $\boldsymbol{D}=\{\zeta\colon|\zeta|<1\}$ such that

$$\|f\|_2^2 \stackrel{\text{def}}{=} \sup_{0\leq r<1}\int_0^{2\pi}\|f(re^{it})\|_E^2\,dt<\infty,$$

S the shift-operator in H_E^2 ($Sf=zf$) and θ an inner operator function (holomorphic in $\boldsymbol{D}$, $\theta(\zeta)\colon E\to E$, $\|\theta(\zeta)\|\leq 1$ for $|\zeta|<1$ and $\theta(\zeta)$ a unitary operator in E for almost all ζ, $|\zeta|=1$); P_K an orthogonal projector onto the subspace K. The "one-dimensionality" discussed above is the case $\dim E=1$, all functions being the usual complex functions in the disc $\boldsymbol{D}$.

One of the aims of the spectral theory is the construction of tools enabling us to answer questions traditionally put to operators in the widest possible set of situations. One such question is the computation and the estimation of functions of operators (especially of characteristic functions), the description of the spectrum, the computation of its multiplicity, of the ind, the spur, etc. The answer to the above question can yield the Spectral Theorem if it remains true (in a sense) for the operator under consideration. We are going to consider a variant of such a Spectral Theorem, generalizing the Dunford spectrality condition: $\mathscr{B}_0$ will be a subalgebra of the Borel σ-algebra of $\boldsymbol{C}$, $\mathscr{E}$ a projector-valued measure on $\mathscr{B}_0$ such that $\mathscr{E}(\sigma)H$ ($\forall\sigma\in\mathscr{B}_0$) is a spectral subspace (in the sense of Bishop and Foiaş) of the operator T (with the spectrum $\operatorname{clos}\sigma$). We shall discuss the case when $\mathscr{B}_0$ is spanned by a countable decomposition $\{\sigma_i\}_{i\in I}$ of the spectrum, i.e. the case when the $\mathscr{B}_0$-spectrality means the family of spectral subspaces $\{\mathscr{E}(\sigma_i)H\}_{i\in I}$ forms an unconditional base. The $\mathscr{B}_0$-spectrality coincides with the Dunford spectrality if all σ_i are points and $\bigvee(\mathscr{E}(\sigma_i)H\colon i\in I)=H$.

The spectral subspaces corresponding to the scalar functional model ($T=PS|K$, $K=H^2\ominus\theta H^2$) are $K_\vartheta=\vartheta'(H^2\ominus\vartheta H^2)$, where $\vartheta'=\theta\bar{\vartheta}$, ϑ being inner relatively

prime (disjoint) factors of θ. Let Ξ be a family of such (disjoint) factors of θ. Since we are interested in whether the family $\{K_\vartheta\}_{\vartheta\in\Xi}$ forms an unconditional base, we are led to spectral projectors $\mathscr{E}(\sigma)$: $\mathscr{E}(\sigma)|K_\sigma=\mathbf{0}$, $\mathscr{E}(\sigma)|K_{\sigma'}=I$, σ denoting a finite product of elements of Ξ, $\sigma'=\theta\sigma^{-1}$. A key to the problem of the criteria of generalized spectrality gives the following remarkable

Theorem of Sarason. Let T be a model operator, $A\in\{T\}'$ (i.e. $AT=TA$). Then there is a function $f, f\in H^\infty$, satisfying $f(T)=A, \|f\|_\infty=\|A\|$.

Here H^∞ denotes the class of all functions analytic and bounded in $\boldsymbol{D}$ endowed with the sup-norm $\|\cdot\|_\infty$. Simple computations show that the functions f_σ corresponding to $\mathscr{E}(\sigma)$ have the following properties:

$$f_\sigma\in\sigma H^\infty,\quad \mathbf{1}-f_\sigma\in\sigma' H^\infty\quad(\sigma'\stackrel{\text{def}}{=}\theta/\sigma). \tag{1}$$

These properties admit a clear interpolatory interpretation and are closely connected with Carleson's Corona Theorem. Indeed, the existence of f_σ is equivalent with the solvability (with respect to $g_i, g_i\in H^\infty$) of the equation

$$g_1\sigma+g_2\sigma'=\mathbf{1}.$$

Further, if θ is a Blaschke product,

$$\theta=\prod_\lambda b_\lambda^{k(\lambda)},$$

$$b_\lambda\stackrel{\text{def}}{=}\frac{|\lambda|}{\lambda}\frac{\lambda-z}{1-\bar\lambda z}\quad\text{and}\quad \Xi=\{b_\lambda^{k(\lambda)}\}_\lambda,$$

the corresponding subspaces $K_\lambda=^{\text{def}}K_{b_\lambda^{k(\lambda)}}$ are the root subspaces corresponding to the eigenvalue λ, and (1) leads to the interpolation with multiplicities: $f_\sigma/b_\lambda^{k(\lambda)}\in H^\infty$, $\lambda\in\sigma$; $(\mathbf{1}-f_\sigma)/b_\lambda^{k(\lambda)}\in H^\infty$, $\lambda\notin\sigma$ (here σ is a subset of the point spectrum of T). It is convenient to remember now the Lorch–Grinbljum theorem (an unconditional base $\equiv$ any bounded family $\{c_\vartheta\}_{\vartheta\in\Xi}$ of numbers defines a continuous operator C with $C|K_\vartheta=c_\vartheta I$, $\forall\vartheta\in\Xi$), and to introduce the following turn of speech: the family of numbers $c=\{c_\vartheta\}_{\vartheta\in\Xi}$ is interpolated (by definition) by a function $f, f\in H^\infty$, with respect to the family Ξ of inner functions if $(f-c_\vartheta)/\vartheta\in H^\infty$, $\vartheta\in\Xi$.

THEOREM. *Let θ be an inner function, and Ξ a family of its inner factors such that the least common multiple of Ξ is θ. The following assertions are equivalent:*

1. $\{K_\vartheta\}_{\vartheta\in\Xi}$ *forms an unconditional base.*
2. $\forall\sigma, \sigma\subset\Xi, \exists f_\sigma\in H^\infty: f_\sigma$ *interpolates the characteristic function* χ_σ *with respect to* Ξ.
3. $\forall c, c\in l^\infty(\Xi), \exists f_c\in H^\infty: f_c$ *interpolates* c *with respect to* Ξ.
4. $\forall f_\vartheta, f_\vartheta\in H^\infty, \sup_\vartheta\|f_\vartheta\|_\infty<\infty, \exists f\in H^\infty: (f-f_\vartheta)/\vartheta\in H^\infty, \forall\vartheta\in\Xi$.
5. $\inf_{\vartheta\in\Xi}\inf_{|\zeta|<1}\big(|\vartheta(\zeta)|+|(\theta/\vartheta)(\zeta)|\big)>0$.
6. $|\theta(\zeta)|\geqslant$ const. $\inf_{\vartheta\in\Xi}|\vartheta(\zeta)|$, $|\zeta|<1$.

The connections between 1, 2, 3, 4 can be understood by means of Sarason's Theorem and other considerations mentioned above. The equivalence of 6 and

of the first four assertions is a deep result of V. I. Vasjunin [2], [3]. It means in particular that a family of T-invariant subspaces forms an unconditional base iff $\sup_{\vartheta\in\Xi}\|\mathscr{E}(\{\vartheta\})\|<\infty$, i.e. iff it is uniformly minimal. It is worth noting that for general families of subspaces (or vectors) these two properties are quite far apart. In the particular case $\theta=\prod_\lambda b_\lambda$ (a Blaschke product with simple zeros) the condition 6 turns out to be the well-known Carleson condition

$$\inf_\mu \prod_{\lambda\neq\mu} |b_\lambda(\mu)| > 0.$$

The same is still true if the multiplicities of zeros are bounded, otherwise there arises a new condition imposed on the zeros of the Blaschke product θ (and on their multiplicities) and not reducible to other known conditions occurring in interpolation theory (see [3]). The most general interpolatory corollaries can be deduced from assertion 4: if $\theta=\prod_\lambda b_\lambda^{k(\lambda)}$ is a Blaschke product, the theorem asserts the equivalence of the generalized Carleson condition 6 and of the existence of a bounded analytic function f with prescribed germs of height $k(\lambda)$ ($\sup_\lambda\|f_\lambda\|_\infty<\infty$; $(f-f_\lambda)/b_\lambda^{k(\lambda)}\in H^\infty$, $\forall\lambda$).

The well-known theorem of N. K. Bari enables us to use also assertion 1 of the theorem to obtain interpolatory corollaries.

COROLLARY. *Let $\theta=\prod_\lambda b_\lambda^{k(\lambda)}$ be a Blaschke product. The following assertions are equivalent:*

1. $|\theta(\zeta)|\geqslant \text{const.}\inf_\lambda |b_\lambda(\zeta)|^{k(\lambda)}$, $|\zeta|<1$.
2. *If* $f_\lambda\in H^2$,

$$\sum_\lambda \|P_\lambda f_\lambda\|_2^2<\infty, \tag{2}$$

where P_λ is the orthogonal projector onto the subspace $H^2\ominus b_\lambda^{k(\lambda)}H^2$, then there is a function f, $f\in H^2$, such that $(f-f_\lambda)/b_\lambda^{k(\lambda)}\in H^2$, $\forall\lambda$.

Taking into account the magnitudes of the germs f_λ in the form (2) we can dispense with the construction of special combinations of $|f^{(k)}(\lambda)|$, $0\leqslant k<k(\lambda)$ (as is very often the case in Hermitian interpolation). Moreover, it is possible to show that in the case $\sup_\lambda k(\lambda)=\infty$ it is impossible in principle to give a condition of this kind equivalent to the true condition (2) imposed on the interpolation data (see [4]).

2. The nonfree interpolation can differ not too much from the free one considered in § 1 (as is the case, for instance, with the Nevanlinna–Pick or Carathéodory interpolation when the space of data is essentially an ideal space, and the freedom of interpolation is lost only because of the location of the interpolating function within a distinguished convex set), but can be also connected with some new singularities of the family $\{K_\vartheta\}_{\vartheta\in\Xi}$ of subspaces. We could discuss here the bases with respect to a matrix summation method by which it is possible to express (as in § 1) in terms of multipliers ([4], [5]) of the family $\{K_\vartheta\}_{\vartheta\in\Xi}$ (compare with the assertion 3 of the theorem). Avoiding precise statements, let us remark only that the answer has

not been found even for the simplest summation methods. For example, in the case of Abel's method we have to describe the sequences $\{\lambda_n\}_{n\geq 1}$ of points of the disc $\boldsymbol{D}$ serving as knots of interpolation for the family of powers:

$$\overline{\lim_{t\to 1-0}} \inf\{\|f_t\|_\infty : f_t(\lambda_n) = t^n,\ n \geq 1\} < \infty;$$

moreover, $\sup_n t^n |B_n(\lambda_n)|^{-1} < \infty$, $\forall t$ $(0 \leq t < 1)$, where $B_n = \prod_{k\neq n} b_{\lambda_k}$. It seems natural to apply to this problem the recent interesting theorem of Garnett [6]; unfortunately computations show it gives only obvious results in this problem.

A more detailed discussion of the relations existing between the summability of spectral decompositions and interpolatory problems can be found in [4].

3. Methods of summing and the hereditary completeness. Any topologically free family of subspaces $\{X_n\}_{n\geq 1}$ of the space X has non-orthogonal ("spectral") projectors $\mathscr{E}_n, \mathscr{E}_n : X \to X$, $\mathscr{E}_n | X_k = \delta_{nk} I$. This fact enables us to construct formal Fourier series $\sum_{n\geq 1} \mathscr{E}_n x$ corresponding to an arbitrary element x of X: $x \sim \sum \mathscr{E}_n x$. This representation is a useful tool in problems connected with the family $\{\mathscr{E}_n\}_{n\geq 1}$ if it is complete $(\bigvee(\mathscr{E}_n X : n \geq 1) = X)$ and total $(\mathscr{E}_n x = 0,\ \forall n \geq 1 \Rightarrow x = 0)$. These properties are however very far from the real possibility of reconstituting vectors x, $x \in X$, from their Fourier series, i.e. from the inclusion $x \in \bigvee(\mathscr{E}_n x : n \geq 1)$, $\forall x \in X$. If the last property holds, the family $\{X_n\}_{n\geq 1}$ is called hereditarily complete. It is known [7] that the hereditary completeness is equivalent to the possibility of spectral synthesis (see below) for every operator with eigen subspaces X_n and with sufficiently "thin" spectrum (for example, compact operators). It is clear that the hereditary completeness is necessary for the existence of a general method of summing of Fourier series (i.e. of a sequence $\{P_n\}_{n\geq 1}$ of operators such that $P_n x \in \bigvee(\mathscr{E}_k x : 1 \leq k \leq n)$, $P_n \in_k = 0$ $(k > n)$, $\lim_k P_k x = x$, $\forall x$). If the X_n's are proper or root subspaces of an operator T, the summing aggregates are usually constructed as functions of the operator T $(P_n = f_n(T))$, and such a construction gives rise (as in § 2) to interpolatory problems with knots in the eigenvalues of T. We shall return to this discussion in the next paragraph and shall now note only one unsolved problem: is the hereditary completeness sufficient for the existence of the above-mentioned method of summing? There are three concrete families of interest in connection with this question: (1) the family $\{e^{int} : n \in \boldsymbol{Z}\}$ in the space $L^2(T, \mu)$, μ being a finite measure on the circumference $T = \{\zeta \in \boldsymbol{C} : |\zeta| = 1\}$; (2) the family $\{k(\cdot, \lambda) : \lambda \in \sigma\}$ of vectors of a Hilbert space of functions analytic in $\boldsymbol{D}$ with the reproducing kernel k, where σ is a countable subset of $\boldsymbol{D}$; (3) the family of trigonometric binomials spanning the subspace of $L^p(T)$, $p \neq 2$, without the approximation property (as in examples by Enflo, Fiegel and others). There are a number of facts concerning methods of summing for the first family (Helson, Szegö, M. Rosenblum, Muckenhoupt, Hunt and others, see [8]).

4. Rational expansions (expansions in eigen- and root-vectors of the left shift operator $f \to S^*(f - f(0))/z$) give the dual form of generalized canonical Weierstrass

products. The last term is understood for example in the sense of Rubel [9] and thus the above-mentioned duality can be stated as follows. The operator S^* is considered in a space Y of functions analytic in $\boldsymbol{D}$, Y being in a natural duality with another space X of the same kind ($\langle x, y\rangle = \sum_{n\geq 0} \hat{x}(n)\hat{y}(n)$ on polynomials x, $x = \sum_{n\geq 0} \hat{x}(n) z^n$). For a nontrivial S^*-invariant subspace E of Y the principal question about the spectral synthesis for S^* is whether E is spanned by the root-vectors of S^* contained in E. The answer is yes if the formal Fourier series (as in §3) is summed (as in §3) to its defining element of E. Just in this way the possibility of spectral synthesis is proved in the works of L. Schwartz, A. F. Leontjev and their successors, the Summation method being chosen after the investigation of "the characteristic (or minimal annihilating) function" of $S^*|E$, i.e. of the function θ satisfying (in an appropriate sense) the equation $\theta(S^*|E) = \mathbf{0}$, see [8], [10]. Analogous considerations (lower estimates of θ on corresponding contours) were used by V. B. Lidskii [11] in a different situation connected with the summing of spectral expansions of abstract nonselfadjoint operators.

The dual form (in the space X) of the problem of synthesis is the problem of divisorial character of S-invariant subspaces ($Sf = zf$) i.e. the problem of finding conditions ensuring the implication

$$SE \subset E (E \subset X) \Rightarrow E = X_k \stackrel{\text{def}}{=} \{f \in X\colon k_f \geq k\},$$

$k_f(\lambda)$ being the multiplicity of the zero of the function f at the point λ, k an integervalued function in $\boldsymbol{D}$. It is clear that the summability of root-vector series of S^* corresponds to the representability of functions of X as generalized Weierstrass products which is sufficient for the divisorial character of S-invariant subspaces [9], [12]. All that needs, of course, more precise formulations. They can be found in works already cited and in [5], [10].

5. The divisorial character of S-invariant subspaces, arising in §4, needs for its proof a variety of tools in modern function theory, from the factorization theorems (à la Weierstrass, Nevanlinna, etc.), interpolation theorems, to the theory "of the common spectrum with estimates" for the elements of a topological algebra (Corona theorems). This theme is too large and difficult to be presented in just a few pages, so I shall outline briefly some directions related to my own work.

The main idea, described in detail in [13], can be expressed as follows: the divisorial character of all S-invariant subspaces implies a considerable "softness" of the topology of the space X under consideration (for every continuous seminorm it is possible to find another one which is much stronger), and without such a property the only subspaces defined by a much smoother element than the general one are of divisorial character. Thus we remove from our consideration all Banach spaces of analytic functions in advance (conjecture: any such space contains an S-invariant subspace of nondivisorial character) together with certain Banach-like spaces, although by using an appropriate modification of the notion of a divisor (taking into account the information concerning boundary zeros), one could possibly also obtain

a positive answer in this case. See the bibliography in [8] about this very interesting case, as well as [14], [15].

The softness of the topology of the space X must (first of all) ensure the stability of X with respect to the division $(f\in X,\ f(\lambda)=0\Rightarrow f/(z-\lambda)\in X)$ and even the following uniformity of this property: $(\{f_\lambda\}$ is a bounded family in $X, f_\lambda(\lambda)=0)\Rightarrow$ $(\{f_\lambda/(z-\lambda)\}$ is a bounded family in $X)$. This condition is essentially weaker than the representability by means of generalized Weierstrass products from §4 and does not imply the divisorial character of S-invariant subspaces of X. The following two principal methods need certain new assumptions about X (still weaker than those needed by the method of decomposition into elementary factors) but they give the desired divisorial character of invariant subspaces: (1) the method of compensated division by Schwartz; (2) the method of the factor-operator.

The first is an improvement of the method of elementary factors mentioned in §4 and can be described more precisely as follows.

Let I be an ideal in the algebra X of analytic functions, $g\in X, f\in I$, $k_g\geqslant k_I$. To prove the divisorial character of I let us approximate g by reconstructing f into g: $g=\lim_n g\cdot f/\pi_n$, π_n being finite Weierstrass-type products for f (so $fg/\pi_n\in I$, $\forall n$, because of the assumed stability). The weakness of this method is just the consequence of the universal character of this "reconstruction". The method of the compensated division is to pick out (possibly infinite) subproducts for f but in a manner imposed by f itself. One of the possible variants is to prove that $f^{(n)}g/f\in X$, $n\geqslant 1$ (then $f^{(n)}g\in I$, $n\geqslant 1$), and then to approximate $\mathbf{1}$ by linear combinations $\sum a_n f^{(n)}$. The first part of this reasoning (i.e. the inclusion $f^{(n)}g/f\in X$) is proved by induction (with respect to n) using the convergence of the series (for the case $k_g\geqslant k_f$)

$$\frac{f'g}{f}=\sum_{k(\lambda)>_f 0} a_\lambda \frac{g}{z-\lambda}$$

in the topology of X (under certain assumptions concerning X). The second part (i.e. the approximation of $\mathbf{1}$ by linear combinations of derivatives $f^{(n)}$, or, using the z-invariance of I, by linear combinations of Weyl derivatives $\sum_{k\geqq 0} k^n \hat{f}(k)z^k$) is reduced to the weighted approximation (in the sense of S. N. Bernstein) by polynomials on the set of integers.

The method of the factor-operator stems from Waelbroeck (and even from Carleman) and also splits the problem of divisorial character of the ideal I (or of the submodule I if X is not an algebra) into two parts: to construct (for a given g, $g\in X$, $k_g\geqslant k_I$) a bounded (in X) subset B, $B\subset I$ with the upper envelope $\theta_B(\lambda)=^{\text{def}}\sup_{b\in B}|b(\lambda)|$ majorizing g ($|g|\leqslant\theta_B$) and then to deduce the inclusion $g\in I$. This last deduction provides the name of the method: it consists in proving "the emptiness of the spectrum" of the factor-operator z/I in the space X_k/I, $k=k_I$; see the details in [10], [16], [17], [18]. The construction of the desired set B needs some delicate lower estimates of modules of analytic functions [5], [19], [20]. The implication $|g|\leqslant\theta_B\Rightarrow g\in I$ is a topological variant of the Carleson Corona Theorem (consider finite sets B and $g=\mathbf{1}$).

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How the *-Function Solves Extremal Problems*

Albert Baernstein II

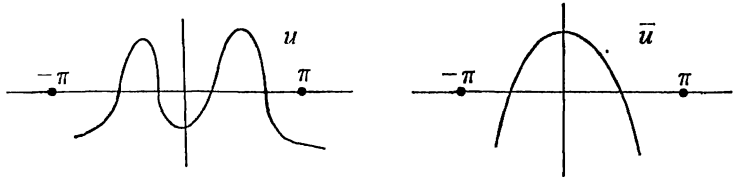
Let $u(z)$ be a real valued integrable function defined in the plane annulus $A: r_1<|z|<r_2$. We define a new function u^*, whose domain is the upper half A^+ of A, by the formula

$$u^*(re^{i\theta}) = \sup_E \int_E u(re^{i\varphi})\, d\varphi, \quad 0 \leqslant \theta \leqslant \pi,$$

where the supremum is taken over all sets $E \subset [0, 2\pi]$ with Lebesgue measure $|E| = 2\theta$. An alternative formula for u^* is

$$u^*(re^{i\theta}) = \int_{-\theta}^{\theta} \bar{u}(re^{it})\, dt,$$

where we use $\bar{u}$ to denote the *symmetric decreasing rearrangement* of u on circles $|z| = r$ [4, p. 149].



The passage from u to u^* has been the crucial step in the recent solution of some extremal problems from various areas of complex function theory. The prob-

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lems have in common the circumstance that subharmonic functions play a role. The solutions have in common reliance on the following basic result, proved in [4], or on some variation of it.

THEOREM A. *If u is subharmonic in A, then u^* is subharmonic in A^+.*

In this article I will discuss three situations in which *-functions have been of use, and hope to convey the main idea of how they are used in proofs. As we shall see, although the problems originated in different parts of classical function theory, the principal feature of their solution is pretty much the same: The contending functions all have subharmonic *'s, whereas the extremal function has a symmetry which imparts to it a *harmonic* *.

I will consider here only problems in R^2. There are also *-functions in R^n. One version is studied in [9], where further references may be found. Also in [9] is a proof of Theorem A which is based on an analogue for circles of F. Riesz's triple convolution inequality

$$\int_{-\infty}^{\infty} h(y) \int_{-\infty}^{\infty} f(y-x)g(x)\,dx \leqslant \int_{-\infty}^{\infty} \bar{h}(y) \int_{-\infty}^{\infty} \bar{f}(y-x)\bar{g}(x)\,dx.$$

1. A simple form of the spread relation. The *-function was first used in [2], in the solution of a problem in Nevanlinna theory, about functions f *meromorphic in the plane,* which had been posed by Edrei. He needed to obtain a certain sharp lower bound, depending on the lower order λ of f and the defect $\delta(\tau,f)$, on the size of the set where f is close to the value τ. This bound, which had been conjectured independently by Teichmüller [17], is now called the spread relation. Edrei's application of it to obtain a sharpening of Nevanlinna's defect relation is in [12]. Hayman's book [15] contains a good general introduction to Nevanlinna theory.

Here we will prove what amounts to an idealized version of a special case ($\lambda=1$, $\tau=\infty$) of the spread relation. Let $\mathscr{E}$ denote the set of all *entire functions* f which satisfy the conditions

$$T(r,f) \leqslant r \quad \text{for} \quad 0 < r < \infty \quad \text{and} \quad T(1,f) = 1,$$

where

$$T(r,f) = (2\pi)^{-1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\,d\theta.$$

Let $\sigma(f)=\{\theta\in(0,2\pi): |f(e^{i\theta})|\geqslant 1\}$. Following Edrei, we call $\sigma(f)$ the "spread of f over ∞". The problem is to *minimize* $|\sigma(f)|$ as f runs through the class $\mathscr{E}$.

Note that $g(z)=e^{\pi z}\in\mathscr{E}$, and that $T(r,g)=r$ for every r. Also, $|g(re^{i\theta})|$ is a *symmetric decreasing function* of θ, so that its spreads are all in one piece and pushed up toward the front. This leads us to suspect that g might be extremal, and, in fact, it is.

SPREAD RELATION *(Simple form). If $f\in\mathscr{E}$, then $|\sigma(f)|\geqslant\pi$.*

PROOF. Let's first assume that the set $\sigma(f)$ is a *single interval*, $\sigma(f)=[-\theta_0, \theta_0]$, say. Form a new function $F(z)$ in the upper half plane $\boldsymbol{C}^+$ by

$$F(re^{i\theta}) = \int_{-\theta}^{\theta} \log|f(re^{i\varphi})|\, d\varphi, \quad 0 \leqq \theta \leqq \pi. \tag{1}$$

Then

$$F(e^{i\theta_0}) = 2\pi T(1,f) = 2\pi. \tag{2}$$

Now, if h is a harmonic function in $\boldsymbol{C}$, a simple computation shows that the indefinite integral $\int_{-\theta}^{\theta} h(re^{i\varphi})\, d\varphi$ is harmonic in $\boldsymbol{C}^+$. The function $\log|f|$ is subharmonic in $\boldsymbol{C}$, and with just a little more trouble, it can be shown that F is *subharmonic* in $\boldsymbol{C}^+$.

Let $G(z)$ be the function obtained in (1) when $f=g$. Then G is harmonic in $\boldsymbol{C}^+$. For $r>0$, we have $F(r)=G(r)=0$, and also $F(ir)\leqslant 2\pi T(r,f)\leqslant 2\pi r = G(ir)$. Hence $F\leqslant G$ on the *boundary* of the first quadrant Q. The functions do not grow too fast at ∞, so, using the *maximum principle*, we conclude that $F\leqslant G$ throughout Q. In particular, if $\theta\in(0, \frac{1}{2}\pi)$ then $F(e^{i\theta})\leqslant G(e^{i\theta})<2\pi$. Comparing with (2), we deduce $\frac{1}{2}\pi\leqslant|\theta_0|=\frac{1}{2}|\sigma(f)|$. Q.E.D.

How can we adapt the proof to take care of the general case when $\sigma(f)$ is no longer a single interval? The idea occurs that we should replace the "fixed" integral defining F by a "maximal" integral over the set where $\log|f|$ is large. This leads us to form $(\log|f|)^*$. The argument with $(\log|f|)^*$ in place of F works exactly as above, the only questionable point being the subharmonicity of $(\log|f|)^*$. But this turns out to be true, as asserted by Theorem A, and the spread relation is now completely proved.

In the proof just given, the extremal function $v=\log|G|$ is both symmetric decreasing on circles and harmonic in $\boldsymbol{C}$. As a result of the former, its *-function is given by

$$v^*(re^{i\theta}) = \int_{-\theta}^{\theta} v(re^{i\varphi})\, d\varphi,$$

which, combined with the latter, shows that v^* is harmonic in $\boldsymbol{C}^+$. This harmonicity of the extremal *, sometimes on just a subset of $\boldsymbol{C}^+$, is typical of *-function proofs. It is quite to be expected, since harmonicity is an extreme form of subharmonicity.

Extensions and analogues of the spread relation appear in [1], [3], [6], and [13]. The idea of integrating a subharmonic function and using the subharmonicity of the result appears implicitly in the proof of the Edrei–Fuchs "ellipse theorem" [15, p. 109] and also in Petrenko's proof of an extended form of Paley's conjecture (See, e.g., [14]). Conversely, these latter two results are quite easy to prove using subharmonicity of the *-function together with Edrei's notion of "Pólya peaks".

2. Univalent functions. In this section f will denote a function analytic and *univalent* in the unit disk $|z|<1$, and S will denote the class of all such f which satisfy the normalizations $f(0)=0$, $f'(0)=1$.

An important role in S is played by the Koebe function $k(z)=z(1-z)^{-2}$, which maps the disk onto the plane with a single slit along the negative axis from $-\frac{1}{4}$ to $-\infty$ removed. In some sense k has the largest possible image of any function in S, and this suggests it should be extremal for many problems about S. In [4] I showed that this is the case for a large class of problems about integral means.

THEOREM. *If $f\in S$ and Φ is a convex function, then*

$$\int_{-\pi}^{\pi}\Phi\left(\log|f(re^{i\theta})|\right)d\theta\leqq\int_{-\pi}^{\pi}\Phi\left(\log|k(re^{i\theta})|\right)d\theta,\quad 0<r<1. \tag{3}$$

In particular, the L^p norm for $0<p\leqslant\infty$ on circles $|z|=r$ is largest for k. Previously this sharp bound was known only for $p=\infty$. The paper [4] also contains results for some other classes of univalent functions, as well as results about what happens to Green's functions and harmonic measures under circular symmetrization. More related results appear in [5], [8], and [16].

PROOF OF THEOREM (OUTLINE). First of all, the validity of (3) for all convex Φ is equivalent to the validity of the inequalities $T(r,\varrho^{-1}f)\leqq T(r,\varrho^{-1}k)$ for every $\varrho>0$ [4, p. 150]. Here T denotes the Nevanlinna characteristic, as in the preceding section. Next, we move the problem from the z-plane to the $\zeta=f(z)$ plane, where we can see the geometry. This is accomplished by *Cartan's formula* [15, p. 8], which in the special case of functions in S asserts that

$$T(r,\varrho^{-1}f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}[u(\varrho e^{i\varphi})+\log r]^{+}\,d\varphi, \tag{4}$$

where $u(\zeta)$ denotes the Green's function of the range of f with pole at $\zeta=0$, $u=0$ outside the range.

Thus, our theorem is equivalent to the assertion that the right hand side of (4) does not decrease when u is replaced by v, the Green's function of the slit plane, which, in turn [4, p. 150] is equivalent to the majorization $u^*\leqslant v^*$ throughout $\boldsymbol{C}^+$.

Now u is subharmonic in $\boldsymbol{C}-\{0\}$, so, by Theorem A, u^* is subharmonic in $\boldsymbol{C}^+$. On the other hand, the symmetry of the slit plane causes $v(\varrho e^{i\theta})$ to be a symmetric decreasing function of θ, so that $v^*(\varrho e^{i\theta})=\int_{-\theta}^{\theta}v(\varrho e^{i\varphi})\,d\varphi$, which is *harmonic* in $\boldsymbol{C}^+$. Thus to deduce $u^*\leqslant v^*$ throughout $\boldsymbol{C}^+$ it suffices to prove it on the real axis. This final step in the proof is not difficult, but it takes a bit of writing, and so we refer the interested reader to [4, p. 155].

Write $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. The most famous unsolved problem about univalent functions is *Bieberbach's conjecture,* which asserts that $|a_n|$ is largest when $f=k$, i.e., that $|a_n|\leqslant n$. So far it had been verified for $n\leqslant 6$, and the best known bound in general is $|a_n|<1.07n$, due to D. Horowitz. The survey article [11] contains an extensive discussion and bibliography.

The integral means theorem with $p=2$ shows that $\sum|a_n|^2r^n\leqslant\sum n^2r^n$ for $0<r<1$, while from $p=1$ together with Cauchy's estimates one obtains $|a_n|<\frac{1}{2}en$. Using the full strength of the theorem it is also possible to deduce Loewner's theorem that

the coefficients of the *inverse function* are largest when $f=k$. However, to obtain sharper bounds for the $|a_n|$ themselves I think it will be necessary to combine the *-function techniques, which are essentially potential theoretic, with some kind of Fourier analytic ideas which make effective use of orthogonality.

3. **Conjugate functions.** In this section, f will denote a *real valued* function on the unit circle T, and $\tilde{f}$ will denote its conjugate function. Thus, $f+i\tilde{f}$ is the boundary function of a function analytic in the unit disk, which we also denote by $f+i\tilde{f}$.

If $f\in L^1(T)$ it is not necessarily true that $\tilde{f}\in L^1(T)$. However, Kolmogorov proved in 1925 that $\tilde{f}\in L^p(T)$ for $0<p<1$, and that $\|\tilde{f}\|_p\leq C_p\|f\|_1$. The problem of finding the *smallest* possible C_p has been solved only very recently, by Burgess Davis **[10]**. Davis's proof is probabilistic. He manages, by means of a beautifully imaginative argument, to deduce the function theoretic result from the solution to a certain optimal stopping problem for sums of independent random variables.

The extremal "function" in Davis's theorem is actually a singular measure, the two point measure on T with mass $\frac{1}{2}$ at $z=1$ and mass $-\frac{1}{2}$ at $z=-1$. The associated analytic function conformally maps the disk onto the plane with two symmetric vertical slits removed. This picture suggests that there might also be a way to prove Davis's theorem with *-functions. In [7], I succeeded in doing this, and also proved some related results which do not seem accessible by Davis's methods. It is not clear what the connection is, if any, between the probabilistic and *-function techniques.

Here is one result from [7]. Let g denote the symmetric decreasing rearrangement of f.

THEOREM. $\|\tilde{f}\|_p\leq\|\tilde{g}\|_p$ *for* $1\leq p\leq 2$, *and* $\|\tilde{f}\|_p\geq\|\tilde{g}\|_p$ *for* $2\leq p\leq\infty$.

The second inequality is not stated in [7], but was pointed out to me later by M. Essén and D. Shea, who have studied uniqueness questions associated with these results.

To prove the theorem, we once again start by moving the problem to the ζ-plane. For $1<p<\infty$ the formula is

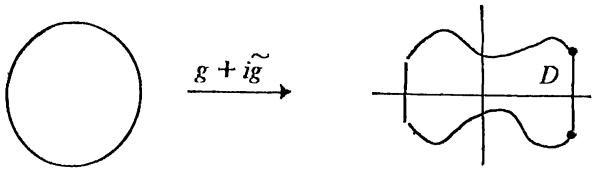
$$\int_{-\pi}^{\pi}|\tilde{f}(e^{i\theta})|^p\,d\theta = p(p-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}N(\xi+i\eta)\,|\eta|^{p-2}\,d\xi\,d\eta \tag{5}$$

where $N(\zeta)=N(1,\zeta,f+i\tilde{f})$ is Nevanlinna's weighted counting function. Letting $\bar{N}$ denote the counting function for $g+i\tilde{g}$, one proves that

$$\int_{-\eta}^{\eta}N(\xi+is)\,ds \leqq \int_{-\eta}^{\eta}\bar{N}(\xi+is)\,ds,\ \xi\in(-\infty,\infty),\quad \eta>0, \tag{6}$$

with equality for $\eta=\infty$. The theorem then follows from (5) by an integration by parts argument. The main ideas in the proof of (6) are to consider this time "vertical" *-functions of N and $\bar{N}$,

$$N^*(\xi+i\eta)=\sup_{|E|=2\eta}\int_E N(\xi+is)\,ds,\ E\subset(-\infty,\infty)$$



and to make use of the fact that $g+i\tilde{g}$ maps the disk *univalently* onto a domain D with strong vertical symmetry. A theorem of Lehto's asserts that N is subharmonic in $C-\{f(0)\}$, and so, by a variant of Theorem A, N^* is subharmonic in C^+, except for a certain correction term. The symmetry of D causes $\overline{N}^*$ to be harmonic in D^+, except for the same correction term. Now one uses the maximum principle, together with some other facts, to prove the majorization $N^* \leqslant \overline{N}^*$, which implies (6).

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Some Problems in Complex Analytic Geometry with Growth Conditions

Phillip A. Griffiths*

The purpose of this talk is to raise a few questions in the general area of complex-analytic geometry with growth-conditions.

1. Notations. We shall be concerned with a complex manifold M of the form

$$M = \overline{M} - N$$

where $\overline{M}$ is an n-dimensional compact complex manifold and N is a k-dimensional complex submanifold. We assume given a metric on $\overline{M}$ and an exhaustion function

$$\tau\colon\ M \to \boldsymbol{R}$$

such that near N we have approximately

$$\tau(p) \sim -\log \delta(p, N)$$

where $\delta(p, N)$ is the distance from M to N. If we set $M[r] = \{p M : \tau(p) \leqslant \log r\}$ then the Levi form

$$L(\tau) = \sqrt{-1}\, \partial\bar{\partial}\tau/2$$

in the holomorphic tangent spaces to $\partial M[r]$ will, for large r, have $\geqslant n-k-1$ negative eigenvalues in the directions normal to N. The sign of the remaining eigenvalues will depend on the curvature in the normal bundle to N.

A prototypical example is when

$$\overline{M} = \boldsymbol{P}^n, \quad N = \boldsymbol{P}^k \quad \text{and} \quad M = \boldsymbol{P}^n - \boldsymbol{P}^k.$$

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Setting $n=m+k+1$ we may consider $\boldsymbol{P}^n$ as the join

$$\boldsymbol{P}^n = \boldsymbol{P}^m + \boldsymbol{P}^k \tag{1}$$

corresponding to the direct sum decomposition $\boldsymbol{C}^{n+1}=\boldsymbol{C}^{m+1}\oplus\boldsymbol{C}^{k+1}$, and use homogeneous coordinates

$$[z, w] = [z_0, \ldots, z_m;\ w_0, \ldots, w_k]$$

on $\boldsymbol{P}^n$. Then we may take

$$\tau([z, w]) = \log \|w/z\|$$

for an exhaustion function. For $k=n-1$, M is $\boldsymbol{C}^n$ with Euclidean coordinates $(\zeta_1, \ldots, \zeta_n)=(w_0/z_0, \ldots, w_{n-1}/z_0)$ and $\tau(\zeta)=\log\|\zeta\|$.

We will be interested in the asymptotic growth properties of analytic and meromorphic functions, holomorphic vector bundles and their sections, analytic subvarieties, etc. as we go to infinity in M, cf. [4] and [12]. Thus we are studying the behavior of essential singularities of analytic objects along an analytic subvariety, as opposed to the rather different and more difficult questions of singularities along the real $(2n-1)$-dimensional boundary of a domain.

2. The Bezout problem. The growth of an analytic subvariety $V\subset M$ will be measured by

$$\mu(V, r) = \text{vol}\,(V[r])$$

where $V[r]=V\cap M[r]$ and $\text{vol}\,(V[r])$ is the volume of $V[r]$ relative to the given metric on $\overline{M}$. We recall that

$$\text{vol}\,(V[r]) = \frac{1}{k!}\int_{V[r]} \varphi^k$$

where φ is the (1,1) form associated to the metric (Wirtinger theorem). It is a basic theorem due to Bishop and Stoll (cf. [10]) that V has a removable singularity along N; i.e., $\overline{V}$ is an analytic subvariety of $\overline{M}$, if and only if $\mu(V, r)$ is bounded. The transcendental Bezout problem is to estimate the growth of the intersection $V\cap W$ in terms of the growth of the analytic subvarieties V and W of M.

The problem arises already when M is $\boldsymbol{C}^n$. By the diagonal construction we may reduce to the case where W is a linear space (cf. [6]). Then the Bezout estimate holds in case V is a hypersurface, but fails when codim $V\geqslant 2$. Thus, for an analytic curve in $\boldsymbol{C}^2$ the growth of the number of points of intersection with a line is estimated by the growth of the area of the curve, but Cornalba and Shiffman [5] gave an analytic curve V in $\boldsymbol{C}^3$ where the corresponding statement is false. If we let

$$\Gamma = \overline{V}\cap\boldsymbol{P}^2$$

be the limits of the asymptotic directions $\overline{0p}$ as $p\in V$ tends to infinity, then the intuitive reason for the failure of Bezout seems to be the somewhat arbitrary character of Γ; in any case, it certainly need not be evenly distributed. The Bezout estimate is concerned with the intersection properties of a neighborhood of Γ with lines in

the P^2 at infinity in C^3, and there have been estimates on this intersection in terms of $\mu(V, r)$ for almost all lines (cf. Carlson [3] and Gruman [8]), and in terms of $\mu(V, r)$ together with the growth of the osculating spaces associated to V (cf. [6] and Stoll [11]).

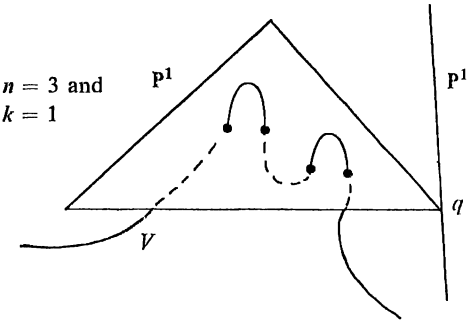
In his Harvard thesis [2], Moshe Breiner has clarified the Bezout problem and to some extent shed light on the general character of essential singularities of analytic varieties. To explain what he did we take

$$M = P^n - P^l$$

and recall that for a k-dimensional analytic subvariety $V \subset M$ the Remmert–Stein theorem [9] implies that V is algebraic if either

$$k > l, \quad \text{or} \quad k = l \quad \text{and} \quad \overline{V} \cap P^k \quad \text{omits an open set.}$$

We take the case $k=l$ and set $n=m+k+1$ so that the decomposition (1) holds.



For each point $q \in P^k$ the linear span $\overline{P^m; q}$ of P^m and q is a $P^{n-k}(q)$ and a special case of the main theorem in [2] is: *Given* $\alpha > 1$ *there exists* $C_\alpha > 0$ *such that for any* $V_k \subset P^n - P^k$ *and any* $q \in P^k$

$$\mu(V \cap P^{n-k}(q), r) \leqq C_\alpha \mu(V, \alpha r). \tag{2}$$

For $k=1$ we recover the aforementioned Bezout theorem for analytic hypersurfaces in C^n. We may informally paraphrase his result by saying that the analytic Bezout theorem holds in the first dimension in which the Remmert–Stein theorem allows an essential singularity.

Breiner's proof uses integral formulas from Nevanlinna theory and we would like to discuss intuitively what he does. For each point q we define the attraction of V to q to be measured by the area of $V[r] \cap U$ where U is a neighborhood of q in $\overline{M}$. Then Breiner's argument gives that the attraction to any particular q is bounded by the average attraction to all points of P^k, and by integral geometry this average attraction turns out to essentially be the area of $V[r]$. In particular it follows that the attraction of V to q is the same for almost all points of P^k, and so the erratic behavior encountered in the set of asymptotic directions to a curve in C^3 is precluded (this erratic behavior is even more evident for points in C^2).

Now his result about the attraction of V to points $q \in N$ is undoubtedly true for general situations $V \subset M = \overline{M} - N$ provided that $\dim V \geqslant \dim N$. Still likely but somewhat less evident is the

Question. Is the attraction of V_k to $q \in N$ equidistributed provided that for large r the Levi form $L(\tau)$ has $\geqslant n-k-1$ negative eigenvalues in the holomorphic tangent spaces to $\partial M[r]$?

For example, if we let M be the usual quadratic transformation of $\boldsymbol{P}^n$ along $\boldsymbol{P}^k$ and N the total transform of $\boldsymbol{P}^k$, then N is the projectivized normal bundle to $\boldsymbol{P}^k$ in $\boldsymbol{P}^n$, and the question asks not only about the attraction of V to points of $\boldsymbol{P}^k$ but also about the normal component of the tangent spaces to V.

Another question we should like to discuss briefly arises from the use of curvature integrals to measure growth. For an entire analytic set $V_k \subset \boldsymbol{C}^n$ we consider the Gauss map

$$\gamma\colon V \to G(k, n)$$

that assigns to each smooth point $p \in V$ the complex tangent plane $T_p(V)$ viewed as lying in the Grassmannian $G(k, n)$ of k-planes through the origin in $\boldsymbol{C}^n$. Clearly γ extends to a meromorphic mapping on all of V, and at the smooth points the usual curvature matrix Ω_V of V is the negative transpose of the pullback under γ of the curvature in the universal subbundle over the Grassmannian. The Chern forms $c_k(\Omega_V)$ are defined by

$$\det\left(tI + \frac{\sqrt{-1}}{2\pi}\,\Omega_V\right) = \sum_{l=0}^{k} (-1)^l t^{k-l} c_l(\Omega_V).$$

We denote by φ the standard Kähler form on $\boldsymbol{C}^n$, and recall that $(1/k!)\int_{V[r]} \varphi^k$ is the Euclidean area $\operatorname{vol}(V[r])$ of $V[r]$ and

$$\mu_0(V, r) = \frac{1}{k!\, r^{2k}} \int_{V[r]} \varphi^k = \frac{\operatorname{vol}(V[r])}{r^{2k}}$$

is an increasing function of r with

$$\lim_{r \to 0} \mu_0(V, r) = \operatorname{mult}_0(V)$$

being the multiplicity of V at the origin. In [7] it is proved that the expressions

$$\mu_l(V, r) = \frac{1}{(k-l)!\, r^{2(k-l)}} \int_{V[r]} c_l(\Omega_V) \wedge \varphi^{k-l}$$

are well defined and increasing in r, and whose limits as $r \to 0$ have to do with the singularity structure of V at the origin. On the other hand, the quantities

$$r^{2(k-l)} \mu_l(V, r) = \frac{1}{(k-l)!} \int_{V[r]} c_k(\Omega_V) \wedge \varphi^{k-l}$$

are the coefficients in the expansion in powers of ε of the volume of the ε-tube

$\tau_\varepsilon(V, r)=\{q\in \boldsymbol{C}^n\colon \delta(q, V[r])\leqslant\varepsilon\}$ around $V[r]$. These integrals may be thought of as measuring the growth of the currents obtained by the standard smoothing of the current defined by integration over V, and as such may be expected to play a role in such questions as extending functions from V to $\boldsymbol{C}^n$ preserving growth conditions. Here we should like to pose the

Question. Is there a Bezout estimate for the refined growth indicator

$$\mu(V, r)=\sum_{l\geqq 0}\mu_l(V, r)?$$

We remark that the analogue of Crofton's formula

$$\mu_0(V, r)=\int_{A\in G(n-k+l,\,n)}\mu_0(A\cap V, r)\,dA$$

is provided by the kinematic formula given in [7], so that an affirmative answer to this question would follow from plurisubharmonic properties of the elementary symmetric functions of the 2nd fundamental form of V in $\boldsymbol{C}^n$.

3. Representing homology classes by analytic cycles. Recall that an analytic cycle Z on a complex manifold is a locally finite formal sum $\sum_i n_i Z_i$ of irreducible analytic varieties with integer coefficients. The growth of Z will mean that of the analytic variety $|Z|=\sum_i |n_i| Z_i$. If Z has pure dimension $n-k$ there is the fundamental class ($\boldsymbol{Q}$-coefficients)

$$\eta_Z\in H_{2n-2k}(M)\cong H^{2k}(M).$$

A long-standing general problem is how much of $H^{2k}(M)$ is represented by such fundamental classes? When M is a compact algebraic variety there is the famous Hodge conjecture. At the opposite extreme, when M is Stein a theorem of Grauert implies that all of $H^{2k}(M)$ is represented by analytic cycles. Here the natural analogue of the Hodge conjecture is to impose growth conditions on the cycles.

In general we may look for restrictions on η_Z imposed by Hodge theory. Suppose we denote by $H^{2k}_{\mathrm{DR}}(M)$ the complex deRham cohomology and recall the Hodge filtration $F^pH^{2k}_{\mathrm{DR}}(M)$ that may be defined as follows:

We consider the usual double complex

$$A^*(M)=\bigoplus_{p,q}A^{p,q}(M)$$

obtained by decomposing the C^∞ forms into (p, q) type and writing $d=\partial+\bar\partial$. The associated total complex is the deRham complex, and the Fröhlicher spectral sequence has

$$E_1^{p,q}\cong H^q_{\bar\partial}(M, \Omega^p),\quad E_\infty\Rightarrow H^*_{\mathrm{DR}}(M).$$

The Hodge filtration is that induced on the abutment of the E_∞ term. When M is a compact Kähler manifold, $E_1=E_\infty$ and the Hodge filtration is

$$F^pH^{2k}(M)=H^{2k,0}(M)\oplus\ldots\oplus H^{p,2k-p}(M)$$

where $H^l_{\mathrm{DR}}(M)=\oplus_{r+s=l}H^{r,s}(M)$ is the Hodge decomposition on cohomology. When M is Stein $E_1^{p,q}=0$ for $q>0$ and all $F^pH^{2k}_{\mathrm{DR}}(M)=H^{2k}_{\mathrm{DR}}(M)$.

Alternatively, we consider the holomorphic deRham complex

$$0 \to \boldsymbol{C} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0.$$

By the holomorphic Poincaré lemma this complex separates into short exact sequences

$$\begin{aligned}
&0 \to \boldsymbol{C} \to \Omega^0 \xrightarrow{d} \Omega^1_c \to 0,\\
&0 \to \Omega^1_c \to \Omega^1 \xrightarrow{d} \Omega^2_c \to 0,\\
&\qquad\vdots\\
&0 \to \Omega^{n-1}_c \to \Omega^{n-1} \xrightarrow{d} \Omega^n \to 0,
\end{aligned}$$

where Ω^q_c is the sheaf of closed holomorphic q-forms. In cohomology we have

$$\begin{array}{ccccc}
H^{2k-1}(M,\Omega^1_c) & \to & H^{2k}(M,\boldsymbol{C}) & \to & H^{2k}(M,\Omega^0)\\
& \searrow & & & \\
H^{2k-2}(M,\Omega^2_c) & \to & H^{2k-1}(M,\Omega^1_c) & \to & H^{2k-1}(M,\Omega^1)\\
& \searrow & & & \\
& & \vdots & & \\
& \searrow & & & \\
H^0(M,\Omega^{2k}_c) & \to & H^1(M,\Omega^{2k-1}_c) & \to & H^1(M,\Omega^{2k-1})
\end{array} \tag{3}$$

A class $\eta\in H^{2k}_{\mathrm{DR}}(M)$ is in $F^pH^{2k}_{\mathrm{DR}}(M)$ if it is in the image of $H^{2k-p}(M,\Omega^p_c)$ in (3). A basic fact is that $\eta_Z\in F^kH^{2k}_{\mathrm{DR}}(M)$ for any analytic cycle Z. When M is Stein this imposes no conditions and Grauert's theorem provides the existence theorem.

Suppose now that $\overline{M}\subset\boldsymbol{P}^N$ is a projective algebraic manifold, $N=\overline{M}\cdot\boldsymbol{P}^{N-k}$ is a linear section of $\overline{M}$, and $M=\overline{M}-N$. For example, when $k=1$, M is an affine algebraic variety and hence a Stein manifold. If $\eta\in H^{2k}_{\mathrm{DR}}(\overline{M})$ restricts to $\eta\in H^{2k}_{\mathrm{DR}}(M)$ then we have seen that $\eta=\eta_Z$ for a generally transcendental analytic cycle Z in M. In fact, it is possible to provide a lower bound on the transcendence level of Z as follows [4]: If $\bar{\eta}$ is primitive and $\bar{\eta}^{k+l,k-l}\neq 0$ in the Hodge decomposition, then

$$\mu(|Z|,r)\geqslant C^{te}r^l \tag{4}$$

where C^{te} is a positive constant. In other words $|Z|$ must be of finite order $\geqslant l$. It was also proved in [4] that the estimate (4) is sharp in case $k=1$. Intuitively the reason that we were able to establish this had to do with the fact that the analytic Bezout theorem is valid in the codimension one case, and consequently the analytic formalism goes well.

Now, recalling that for general k,

$$M=\overline{M}-\overline{M}\cdot\boldsymbol{P}^{N-k}$$

a theorem of Andreotti–Grauert [1] gives

$$H^p(M,\Omega^{2k-p})=0 \quad \text{for} \quad p>k,$$

so that

$$H^k(M, \Omega_c^k) \to H^{2k}_{\mathrm{DR}}(M)$$

is surjective; i.e., $F^k H^{2k}_{\mathrm{DR}}(M) = H^{2k}_{\mathrm{DR}}(M)$ in this case. Consequently there are no Hodge-theoretic objections to representing all of $H^{2k}(M)$ by analytic cycles in this particular dimension; note that this is exactly the dimension where the Remmert–Stein theorem first allows transcendental analytic varieties. We we may ask the

Question 3. For $M = \overline{M} - \overline{M} \cdot \boldsymbol{P}^{N-k}$ as above, is all of $H^{2k}(M)$ represented by analytic cycles? Can we choose these cycles to have finite order $\leqslant k$?

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Analytic Functions of Unbounded Characteristic and Beurling Algebras

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1. Introduction. The classical representation and factorization theory due to R. Nevanlinna [1] permits a very satisfactory understanding of functions regular and bounded in $D=\{z\in C|\ |z|<1\}$ (these functions constitute the algebra H^∞) and of meromorphic functions belonging to the quotient field N of H^∞. It is well known that the Nevanlinna representation theorem is based on the application of the Poisson–Jensen formula to smaller disks $|z|\leqslant r<1$ and on a subsequent transition to the limit as $r\uparrow 1$ involving use of the Riesz–Herglotz formula. In fact, the definition of the Nevanlinna characteristic itself is based on the following simplest version of the Poisson–Jensen formula:

$$\log|f(0)| = -\sum_{|\alpha_\nu|<r}\log\frac{r}{|\alpha_\nu|}+\sum_{|\beta_\nu|<r}\log\frac{r}{|\beta_\nu|}+\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta, \tag{1.1}$$

where α_ν and β_ν are the zeros and the poles of meromorphic function $f(z)$.

All this shows that there are some inherent limitations to the described classical approach:

(i) Since (1.1) involves only the *moduli* of α_ν and β_ν, it hardly can be the basis for an understanding of those classes of analytic functions in D whose possible zero sets are characterized not just by the moduli of the zeros, but depend on the distribution of their arguments as well. This observation also applies to some generalizations of the Nevanlinna theory (see [2] and [3]).

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(ii) The Riesz–Herglotz theorem is indispensable for the theory. Until quite recently no such theorem involving more general classes of harmonic functions was available.

(iii) The real-analytical tool of Lebesgue–Stieltjes integration is good for N, but it may be inadequate for larger classes of meromorphic functions.

In what follows we consider algebras $A^{\langle k\rangle}$ of regular functions $f(z)$ in $\boldsymbol{D}$ that satisfy the relation

$$|f(z)| \leqslant C_f \exp\{a_f k(|z|)\}, \tag{1.2}$$

where $k(r)$ is some nondecreasing continuous function on $[0; 1)$, $k(0)=0$, $k(r)\to\infty$ $(r\uparrow 1)$. We also consider the quotient field $N^{\langle k\rangle}$ of $A^{\langle k\rangle}$. It is known [4] that $A^{\langle k\rangle}$ always contains some functions whose zeros do not satisfy the Blaschke condition; therefore $A^{\langle k\rangle} \not\subset N$. Given some regularity conditions and growth restrictions on $k(r)$, the $A^{\langle k\rangle}$ coincide with (topological) algebras introduced by A. Beurling [5]. The results of A. Beurling imply that, as regards the structure of closed ideals of $A^{\langle k\rangle}$, there exists a critical rate of growth for $k(r)$: if

$$\int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr = \infty \tag{1.3}$$

then the (closed) ideals of $A^{\langle k\rangle}$ are completely characterized by the set of common zeros of their elements, whereas for

$$\int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr < \infty \tag{1.4}$$

there exist some proper ideals in $A^{\langle k\rangle}$ whose elements have no common zeros in $\boldsymbol{D}$. The structure of ideals in $A^{\langle k\rangle}$ (in the latter case) is probably similar to the structure of invariant subspaces in H^2, as described by a classical theorem of A. Beurling [6], but there is little hope to solve the ideal problem for $A^{\langle k\rangle}$ without a meaningful extension of the Nevanlinna theory to the classes $A^{\langle k\rangle}$.

The case

$$k(r) = \log\frac{1}{1-r} \tag{1.5}$$

may be of some special interest, because the corresponding algebra, which we denote by $A^{-\infty}$, consists of analytic functions having power rate of growth:

$$|f(z)| \leqslant A_f/(1-|z|)^{n_f}. \tag{1.6}$$

The class $A^{-\infty}$ is the smallest extension ring of H^∞, which is invariant under differentiation; $A^{-\infty}$ contains (in fact, is the union of) the Bergman classes B^p $(0<p<\infty)$, where B^p consists of those analytic functions $f(z)$ in $\boldsymbol{D}$ for which

$$\iint_D |f(x+iy)|^p\, dx\, dy < \infty. \tag{1.7}$$

Now, the problem of describing the so-called "weakly invertible", or "cyclic",

elements f of B^2 (for which clos $\{f\mathscr{P}\}=B^2$, $\mathscr{P}$ being the set of all algebraic polynomials) has been outstanding (and almost completely at a standstill) for over 40 years. Clearly, any progress in the closed-ideal problem for $A^{-\infty}$ would mark at the same time some progress in the above-mentioned problem for B^2.

In what follows we describe some recent results in the study of $A^{-\infty}$, of its quotient field $\mathscr{N}$ and of the corresponding class $\mathscr{H}$ of harmonic functions. These results include a complete characterization of possible zero sets Z of functions in $A^{-\infty}$, representation of harmonic functions of the class $\mathscr{H}$ by means of a generalized Poisson integral involving so-called "premeasures" (distributions of the first class), a factorization for the class $\mathscr{N}$, and a complete description of closed ideals in $A^{-\infty}$. Finally we discuss some results indicating possible extensions of the theory to more general Beurling algebras.

2. Zero sets of $A^{-\infty}$ [7].

DEFINITION 2.1. The *entropy* $\varkappa(F)$ of a finite set $F\subset\partial\boldsymbol{D}$ is

$$\varkappa(F)=\frac{1}{2\pi}\sum_{\nu}\left(|I_\nu|+\log\frac{2\pi e}{|I_\nu|}\right), \tag{2.1}$$

where I_ν are the complementary arcs of F and $|I_\nu|$ are their (angular) lengths.

DEFINITION 2.2. For each $\zeta\in\partial\boldsymbol{D}$ the Stoltz domain S_ζ is the interior of the smallest convex set containing the disc $\{|z|\leqslant\frac{1}{2}\}$ and the point ζ. For each finite set $F\subset\partial\boldsymbol{D}$

$$S_F=\bigcup_{\zeta\in F}S_\zeta. \tag{2.2}$$

DEFINITION 2.3. Let $\alpha=\{\alpha_\nu\}_1^\infty$ be a sequence of complex numbers, $0\leqslant|\alpha_1|\leqslant|\alpha_2|\leqslant\ldots<1$, $\lim_{\nu\to\infty}|\alpha_\nu|=1$. For each finite set $F\subset\partial\boldsymbol{D}$ the corresponding *partial Blaschke sum* is

$$\sigma(\alpha;F)=\sum_{\alpha_\nu\in S_F}(1-|\alpha_\nu|). \tag{2.3}$$

THEOREM. *For an* $\alpha=\{\alpha_\nu\}_1^\infty$ *to be the zero set of a function* $f\in A^{-\infty}$ *it is necessary and sufficient that*

$$\sup_F\frac{\sigma(\alpha;F)}{\varkappa(F)}<\infty, \tag{2.4}$$

where sup *is taken over all finite* $F\subset\partial\boldsymbol{D}$.

3. Harmonic functions and premeasures [7], [8].

DEFINITION 3.1. (i) $\mathscr{H}^+$ is the class of (real) harmonic functions $u(z)$ in $\boldsymbol{D}$ satisfying

$$u(z)\leqslant A_u\log\frac{1}{1-|z|}+B_u. \tag{3.1}$$

(ii) $\mathscr{H}=\mathscr{H}^+-\mathscr{H}^+$, i.e. $u\in\mathscr{H}$ means that

$$u=u_1-u_2 \tag{3.2}$$

for some $u_1, u_2\in\mathscr{H}^+$.

DEFINITION 3.2. Let $\mathscr{K}$ be the set of all open, closed and halfclosed arcs of ∂D, including all one-point sets, ∂D itself and $\emptyset$. A function $\mu\colon \mathscr{K}\to R$ is called a *premeasure* if

(i) $\mu(I_1\cup I_2)=\mu(I_1)+\mu(I_2)$ for all $I_1, I_2\in\mathscr{K}$ such that

$$I_1\cup I_2\in\mathscr{K},\ I_1\cap I_2=\emptyset;$$

(ii) $\lim_{\nu\to\infty}\mu(I_\nu)=0$ whenever $I_\nu\in\mathscr{K}$, $I_1\supset I_2\supset\ldots$, $\cap I_\nu=\emptyset$.

DEFINITION 3.3. (i) A premeasure μ is called *$\varkappa$-bounded* if there exists a positive constant C such that

$$\mu(I)\leqq C|I|\log\frac{2\pi e}{|I|}\quad(\forall I\in\mathscr{K}) \tag{3.3}$$

(for $|I|=0$ that means $\mu(I)\leqslant 0$). The set of all $\varkappa$-bounded premeasures will be denoted $\varkappa B^+$.

(ii) The $\varkappa$-variation of a premeasure μ is

$$\varkappa\operatorname{Var}\mu=\sup_F\frac{\sum_\nu|\mu(I_\nu)|}{\varkappa(F)}, \tag{3.4}$$

where sup is taken over all finite sets $F\subset\partial D$ (I_ν are the complementary arcs of F). The set of all premeasures of bounded $\varkappa$-variation will be denoted $\varkappa V$.

THEOREM. (i) *Each premeasure $\mu\in\varkappa V$ defines a harmonic function $u\in\mathscr{H}$ by means of the generalized Poisson integral*

$$u(z)=\frac{1}{2\pi}\int_{\partial D}\frac{1-|z|^2}{|\zeta-z|^2}\mu(|d\zeta|). \tag{3.5}$$

If $\mu\in\varkappa B^+$, then $u\in\mathscr{H}^+$.

(ii) *Conversely, for each $u\in\mathscr{H}$ there is a $\mu\in\varkappa V$ such that* (3.5) *holds; μ is (uniquely) determined by*

$$\frac{1}{2}[\mu(I)+\mu(\bar{I})]=\lim_{r\uparrow 1}\int_I u(r\zeta)|\,d\zeta|, \tag{3.6}$$

where $I\subset\partial D$ is an arbitrary open arc.

If $u\in\mathscr{H}^+$, then $\mu\in\varkappa B^+$.

4. Factorization [7]. If $f\in\mathscr{N}$ and $\alpha=\{\alpha_\nu\}$, $\beta=\{\beta_\nu\}$ are the sets of zeros and poles of $f(z)$, then α and β satisfy condition (2.4). There are many ways of constructing quasi-Blaschke products $B(z,\alpha)$ and $B(z,\beta)$ which belong to $A^{-\infty}$ and have zeros precisely at α and β. Then $f(z)B(z,\beta)/B(z,\alpha)=g(z)$ also belongs to $\mathscr{N}$

and has no zeros or poles. Using Theorem 2 we get the following factorization

$$f(z) = \lambda \frac{B(z, \alpha)}{B(z, \beta)} \exp\left\{\int_{\partial D} \frac{\zeta + z}{\zeta - z} \mu(|d\zeta|)\right\}, \tag{4.1}$$

where $|\lambda| = 1$ and μ is a premeasure of bounded $\varkappa$-variation.

5. Closed ideals in $A^{-\infty}$ [8]. Each premeasure μ of bounded $\varkappa$-variation generates a finite Borel measure on Carleson sets, i.e. on closed sets $F \subset \partial D$, which have measure zero, and whose complementary arcs I_ν satisfy the relation $\sum |I_\nu| \log(2\pi e/|I_\nu|) < \infty$. This Borel measure (defined on every Carleson set) is called the *$\varkappa$-singular part* of μ and denoted by σ_μ. Although μ in (4.1) may depend upon the particular way of constructing quasi-Blaschke products $B(z, \alpha)$ and $B(z, \beta)$, its $\varkappa$-singular part σ_μ does not and is intrinsically linked to the function f. If $f \in A^{-\infty}$, then μ in (4.1) is $\varkappa$-bounded and its $\varkappa$-singular part σ_μ is nonpositive.

A description of all closed ideals in $A^{-\infty}$ is given in [8]. Each such ideal is uniquely characterized by its zero set and its $\varkappa$-singular measure which is completely analogous to the classical result of A. Beurling [6]. For the sake of brevity we confine ourselves here to the case of weakly invertible elements.

THEOREM. *An element $f \in A^{-\infty}$ is weakly invertible if and only if*

(i) *$f(z)$ has no zeros in $\boldsymbol{D}$;*

(ii) *The $\varkappa$-singular measure associated with f is zero. Clearly, these conditions remain necessary for an $f \in B^2$ to be weakly invertible in B^2.*

CONJECTURE. The same conditions are necessary and sufficient for an $f \in B^2$ to be weakly invertible in B^2.

6. Possible generalizations. Should we try to generalize the above approach to other Beurling algebras $A^{\langle k \rangle}$, then the following class $\mathscr{H}^+_{\langle k \rangle}$ of harmonic functions:

$$u(z) \leqslant a_u + b_u k(|z|) \quad (z \in \boldsymbol{D}) \tag{6.1}$$

whould have to be considered with the view to extending the Riesz–Herglotz formula to $\mathscr{H}^+_{\langle k \rangle}$ using premeasures. The first step in this direction is to find out what are the conditions on $k(r)$ under which the limit

$$\lim_{r \uparrow 1} \int_I u(r\zeta) |d\zeta| \tag{6.2}$$

exists for each $u \in \mathscr{H}_{\langle k \rangle}$ and for each arc $I \subset \partial \boldsymbol{D}$.

THEOREM [9]. *The limit* (6.2) *exists for each $u \in \mathscr{H}^+_{\langle k \rangle}$ and for each $I \subset \partial \boldsymbol{D}$ if and only if $k(r)$ satisfies condition* (1.4).

In view of the cited result of A. Beurling [5], the above theorem shows that the method of premeasures may be available exactly when it is needed in the closed-ideal problem, i.e. when the "boundary singularities" should play a role in the description of ideals in $A^{\langle k \rangle}$.

Turning now to the problem of zero sets for $A^{\langle k\rangle}$, let us observe that under (1.4) each nonzero $f(z)$ belonging to $A^{\langle k\rangle}$ has the following property:

$$\sum(1-\varrho_\nu)<\infty, \tag{6.3}$$

where ϱ_n are the positive zeros of $f(z)$; moreover, if (1.3) holds, then there exists functions in the class $A^{\langle k\rangle}$ for which (6.3) fails [**10**].

It was in a paper by H. S. Shapiro and A. L. Shields [**11**] that the property (6.3) was first noted and proved for some special classes of functions.

Finally, let us note that condition (1.4) seems to be of crucial importance in many other (but unmistakably related) problems (see [**12**] and [**13**]).

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The Holomorphic Equivalence of Real Hypersurfaces

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1. We are concerned with the mapping problem in several complex variables. In its simplest form it amounts to deciding which domains $D_1, D_2 \subset C^{n+1}$ $(n \geqslant 1)$ [1] can be mapped biholomorphically into each other and to determine the group Aut (D) of biholomorphic selfmappings of such a domain. More generally we seek invariant quantities or geometrical objects of such domains. As a rule we will consider domains of holomorphy and even domains with smooth strictly pseudoconvex boundary. The answers are in stark contrast to the results in one complex variable where any two bounded simply connected domains are biholomorphically equivalent.

In several complex variables one has much more rigidity. For example it was known to Poincaré that the polydisc D_1: $|z_\alpha|<1$ $(\alpha=1, 2, \ldots, n+1)$ is not biholomorphically equivalent to the ball D_2: $\sum_{\alpha=1}^{n+1}|z_\alpha|^2<1$; this is most easily seen by showing that the groups Aut (D_j) $(j=1, 2)$ have different dimensions.

However, it is only quite recent that it was possible to decide which ellipsoids

$$(E) \qquad \sum_{\alpha=1}^{n+1} a_\alpha x_\alpha^2 + b_\alpha y_\alpha^2 = 1, \quad a_\alpha \geqslant b_\alpha > 0, \quad n \geqslant 1,$$

with $z_\sigma = x_\alpha + iy_\alpha$ are biholomorphically equivalent. S. Webster [20], [22] showed with interesting arguments

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[1] The choice of C^{n+1} in place of C^n is a habit which was adhered to in order to conform with notation in §§ 2—3.

THEOREM 1. *Any two such ellipsoids are biholomorphically equivalent if and only if the ratios* $(a_j-b_j)/(a_j+b_j)$ *(up to ordering) agree. Moreover, in that case the biholomorphic mapping between such ellipsoids is necessarily birational. In particular, for distinct* $(a_j-b_j)/(a_j+b_j)$ *the group* Aut (E) *of the above ellipse is given by the* 2^{n+1} *trivial maps*

$$(z_1, \dots, z_{n+1}) \to (\pm z_1, \pm z_2, \dots, \pm z_{n+1}).$$

Another interesting result characterizing holomorphic balls (i.e. domains which are biholomorphically equivalent to the ball $\sum |z_\alpha|^2<1$) was proven (see [5]) by Burns and Shnider, Klembeck and Wong; the latter gave a very simple proof in [26].

THEOREM 2. *If for a strictly pseudoconvex domain* $D\subset C^{n+1}$ $(n\geqslant 1)$ *with smooth boundary the group* Aut (D) *is not compact then* D *is holomorphically equivalent to a ball.*

There are only few isolated results of this specific nature. It is the purpose of this lecture to report on some recent work attempting to attack this equivalence problem more systematically. We will assume in the following that the boundary is smooth (C^∞) and the Levi form nondegenerate.

The first observation is that this equivalence problem—at least for strictly pseudoconvex domains—can be reduced to an equivalence problem of the boundaries $\partial D_j=M_j$. For this purpose we note that the complex structure of C^{n+1} induces a so-called CR-structure (Cauchy Riemann structure) on M. If J: $z \to iz$ defines the complex structure in C^{n+1} we denote the holomorphic tangent space by $H_p(M)=T_p(M)\cap JT_p(M)$ which is of real codimension 1 in $T_p(M)$.[2] Thus $H(M)$ is the maximal complex subspace of $T_p(M)$. A CR-structure on M is given by $H(M)$ together with $J|_{H(M)}$.

For two CR hypersurfaces $(M_j, H(M_j), J_j)$, $j=1, 2$, one calls a mapping $f: M_1\to M_2$ a CR-mapping if $df H(M_1)=H(M_2)$ and $df\circ J_1=J_2\circ df$. A CR equivalence is a diffeomorphism for which f and f^{-1} are CR mappings.

The basic theorem in this connection is due to Fefferman [12]: A biholomorphic mapping $\varphi: D_1\to D_2$, where D_1, D_2 are bounded strictly pseudoconvex domains with smooth boundary, can be extended to C^∞—map $\bar\varphi$ of $\bar D_1\to\bar D_2$.* According to an argument of H. Lewy [16] the extended map is real analytic if ∂D_1, ∂D_2 are real analytic.

Clearly the induced mapping $f=\bar\varphi|_{\partial D_1}$ of $M_1=\partial D_1$ into $M_2=\partial D_2$ is a CR-equivalence.

THEOREM 3 (SEE [4]). *Let* D_1, D_2 *be bounded, strictly pseudoconvex domains in* C^{n+1} $(n\geqslant 1)$ *with* C^∞ *(or* C^ω*)*[3] *boundaries. Then* D_1, D_2 *are biholomorphically*

[2] We will only consider CR structures of codimension 1. For more general CR structures — which are irrelevant in our context — see R. O. Wells, Jr. [25].

* A simplified proof of Fefferman's theorem is contained in Webster [31].

[3] C^ω refers to real analytic mappings.

equivalent if and only if $\partial D_1=M_1$, $\partial D_2=M_2$ *are* CR*-equivalent via a* C^∞ *(or* C^ω*) diffeomorphism.*

Having reduced the problem to an equivalence of the boundary ∂D we turn to the *local problem.* Given two real hypersurfaces M_1, M_2 with distinguished points $p_j \in M_j$ we seek conditions for the existence of a holomorphic mapping f taking $p_1 \to p_2$, $M_1 \cap U_1 \to M_2 \cap U_2$ where U_j are neighborhoods of p_j.

This problem turns out to be of differential geometric nature. We describe briefly the main results of the paper [8] which contains two different approaches: One is *extrinsic* and refers to real hypersurfaces in C^{n+1}—or in any $n+1$ dimensional complex manifold—and the other *intrinsic,* and refers to abstract CR manifolds. In the following two sections we outline the main results of these two approaches. For a fuller description we refer to the excellent survey article [3]. In § 4 we collect a number of related studies which have been published recently. We supply references primarily of the last four years; the earlier literature is carefully cited in [3] and in the quoted references.

I want to express my appreciation to D. Burns, S. Webster, S. T. Yau for discussions on this subject and, in particular, to K. Diederichs who made his unpublished lecture notes on the invariants for CR hypersurfaces available.

2. Normalform. (a) We consider a real analytic hypersurface M in C^{n+1} with a distinguished point $p \in M$. Without loss of generality we assume that p is given by $z_1=\ldots=z_{n+1}=0$ and T_pM by $\operatorname{Im} z_{n+1}=0$. Writing $z_{n+1}=w=u+iv$ and $z=(z_1, z_2, \ldots, z_n)$ we can represent M locally by $v=F(z, \bar{z}, u)$ where F is real analytic and $F(0)=0$, $dF(0)=0$.

We subject M to the group $\mathcal{G}$ of biholomorphic transformations $z^*=f(z, w)$, $w^*=g(z, w)$ with $f(0, 0)=0$, $g(0, 0)=0$, $g_z(0, 0)=0$ preserving 0 and $T_p(M)$. The aim is to find a simple normalform for the transformed hypersurface $v^*=F^*(z^*, \bar{z}^*, u^*)$ and to study the isotropy group of these normalforms.

(b) We assume that the hermitian form

$$\sum_1^n h_{\alpha\bar{\beta}}(0)\zeta_\alpha\bar{\zeta}_\beta = \langle\zeta, \zeta\rangle, \quad h_{\alpha\bar{\beta}} = \frac{\partial^2 F}{\partial z_\alpha \partial \bar{z}_\beta}$$

is nondegenerate, having p positive and q negative eigenvalues, hence $p+q=n$. Strict pseudoconvexity amounts to $p=0$ or $q=0$. The construction of a normalform can be viewed as approximating M to high order by the holomorphic image of the hyperquadric Q: $v=\langle z, z\rangle$. In projective coordinates

$$\zeta_\alpha\zeta_0^{-1} = z_\alpha, \quad \zeta_{n+1}\zeta_0^{-1} = w;$$

this equation becomes

$$\sum_1^n h_{\alpha\bar{\beta}}(0)\zeta_\alpha\bar{\zeta}_\beta + \frac{i}{2}(\bar{\zeta}_0\zeta_{n+1} - \zeta_0\bar{\zeta}_{n+1}) = 0.$$

Here the left-hand side is a hermitian form of type $p+1, q+1$. Thus Q admits

SU $(p+1, q+1)$ as its automorphism group. To obtain an effective action we consider the group,

$$G = \mathrm{SU}(p+1, q+1)/K$$

where K, the center of SU $(p+1, q+1)$, is given by the mappings $\zeta \to \varepsilon\zeta$ where $\varepsilon^{n+2}=1$. The isotropy group H of G consists of all those transformations in G leaving $\zeta=(\zeta_0, 0, \ldots, 0)$ fixed. Thus $Q=G/H$ is a homogeneous space.

(c) We subject $v=F(z, \bar{z}, u)$ to a transformation $\varphi=(f, g)\in\mathcal{G}$ and characterize a normalform in the orbit $\{\varphi^* F|\varphi\in\mathcal{G}\}$. We say that $v=N(z, \bar{z}, u)$ is in normalform if

$$N(z, \bar{z}, u) = \langle z, z\rangle + \sum_{k,l} N_{kl}(z, \bar{z}, u)$$

where $N_{kl}(tz, s\bar{z}, u)=t^k s^l N_{kl}(z, \bar{z}, u)$ and

$$N_{kl} = 0 \quad \text{for} \quad \min(k, l) \leqslant 1,$$

$$\mathrm{tr}\, N_{22} = 0, \quad (\mathrm{tr})^2 N_{23} = 0, \quad (\mathrm{tr})^3 N_{33} = 0;$$

here "tr" denotes the contraction with respect to the form $\langle z, z\rangle$.

THEOREM 4. *The real analytic manifold M can be transformed into a normalform $v=N(z, \bar{z}, u)$ by a transformation $\varphi\in\mathcal{G}$. This normalizing transformation φ is not unique; but the most general such transformation $\varphi=\varphi_h$ depends on $h\in H$; and the map $\varphi_h: M\to\mathcal{G}$ is injective.*

The above result (see [15], [8]) holds also for formal series expansions and can be used to construct holomorphic invariants for C^∞-manifolds.

The above normalform still depends on $h\in H$ as φ_h does. On account of this action of H the coefficients of N are not holomorphic invariants.[4] Moreover, it is a nontrivial problem to decide when two normalforms are equivalent.**

(d) S. Webster [21] as well as Burns, Shnider and Wells [4] used this extra freedom to construct a "restricted normalform" under the assumption $N_{22}\neq 0$ at the origin (hence $n\geqslant 2$). They showed that one can achieve that the function

$$\psi = \sum_{\alpha,\beta,\gamma,\delta} \left| \frac{\partial^4 N}{\partial z_\alpha \partial z_\beta \partial \bar{z}_\gamma \partial \bar{z}_\delta} \right|^2$$

satisfies

$$\psi = 1, \quad \mathrm{grad}\,\psi = 0 \quad \text{at} \quad z = \bar{z} = u = 0.$$

The transformations into this restricted normalform are—in the pseudoconvex case—unique up to $(z, w)\to(Uz, w)$, $\langle Uz, Uz\rangle=\langle z, z\rangle$. This remark can be used to construct infinitely many invariant functions on M by forming unitarily invariant polynomials from the coefficients of the restricted normalform.

[4] The construction of such invariants is the object of a forthcoming paper by C. Fefferman, *Parabolic invariant theory in complex analysis.*

** A normalform for the differential equations describing an abstract CR-structure was found by Luk [28].

In this restricted normalform $\partial/\partial u$ defines a distinguished invariant vectorfield on M for which Webster [18] gave an elegant description.

(e) The explicit nature of the normal form shows that along the u-axis the manifold $v=N(z,\bar{z},u)$ and the hyperquadric Q osculate to third order (and for $n=1$ even to fifth order). This means that on the given manifold M there is a curve through p along which M osculates the holomorphic image of Q to third or fifth order. The action of H on the normal form gives rise to a family of curves through p transversal to the complex tangent space of M at p. These distinguished curves are governed by a second order system of differential equations which is equivalent to $(\mathrm{tr})^2 N_{32}=0$ provided $N_{kl}=0$ for $\min(k,l)\leqslant 1$. These curves are called the chains of M. For example, for the hyperquadric Q the chains are simply the intersection of complex lines in C^{n+1} with Q.

Similarly the conditions $\mathrm{tr}\, N_{22}=0$, $(\mathrm{tr})^3 N_{33}=0$ have geometrical interpretations.

If the osculation of the hyperquadric is one order higher than generic, i.e. if $N_{22}(0)=0$ for $n\geqslant 2$ or $N_{42}(0)=0$ for $n=1$, one calls the origin an umbilical point—in a geometrical analogy.

3. Chern's connection. (a) The construction of a normalform referred to an *embedded* real hypersurface. Instead one may consider an abstract CR manifold and study CR equivalences between them. This approach can be viewed as an analogue to the intrinsic description of Riemannian manifolds using Cartan's formalism of frame bundles, connections, structure equations and curvature forms. This program was carried out by S. S. Chern [8] after E. Cartan had settled the case $n=1$ and after further studies by Tanaka (see [3] for references).

An abstract CR manifold (of codimension1) is a $(2n+1)$-dimensional real manifold M with a subbundle $H(M)$ of codimension 1 of the tangent bundle carrying a complex structure J, i.e. $J^2=-I$. This CR structure is required,to satisfy an integrability condition: If $\theta\in T^*M$ annihilates $H(M)$ and $\theta^1\,\theta^2,\ldots,\theta^n$ are chosen so that θ, $\mathrm{Re}\,\theta^\alpha$, $\mathrm{Im}\,\theta^\alpha$ form a basis for T^*M and satisfy $J\theta^\alpha=i\theta^\alpha$ then it is required that

$$d\theta, d\theta^\alpha = 0 \mod(\theta,\theta^1,\ldots,\theta^n).$$

The CR structure is called nondegenerate if the hermitian matrix $(h_{\alpha\bar{\beta}})$ defined by

$$d\theta = i h_{\alpha\bar{\beta}}\theta^\alpha\wedge\bar{\theta}^{\bar{\beta}} \pmod{\theta}$$

is nondegenerate. It is called strictly pseudoconvex if $(h_{\alpha\bar{\beta}})$ is positive or negative definite. We define the real form θ^{n+1} such that

$$d\theta = i h_{\alpha\bar{\beta}}\theta^\alpha\wedge\bar{\theta}^{\bar{\beta}}+\theta\wedge\theta^{n+1}.$$

Clearly every strictly pseudoconvex real hypersurface M of C^{n+1} carries a CR structure which is strictly pseudoconvex. However, it is not known whether every strictly pseudoconvex CR manifold can be locally embedded in C^{n+1} in this way. While this is true for real analytic CR structures it is false in the C^∞ case, at least

if $n=1$. The relevant counterexample was found by L. Nirenberg [16], [17]. However, for $n\geqslant 2$, such counterexamples are not available since the integrability condition imposes severe restrictions. Therefore the embedding problem of CR manifolds of $2n+1\geqslant 5$ dimensions remains open (except for the result of Boutet de Monvel [6], for *compact* strictly pseudoconvex CR manifolds).

Therefore a CR manifold is to be considered as a more general object than a real hypersurface of C^{n+1}. The advantage of Chern's construction is that it refers to such abstract CR manifolds and that it proceeds intrinsically. It provides intrinsic invariants for the CR equivalence problem which in the analytic case are also sufficient to distinguish all different CR structures.

(b) In Chern's approach a principal bundle Y is constructed over M which has as fiber group the isotropy group H of G, defined in the previous section. We recall that the hyperquadric Q is given as homogeneous space G/M. With $\mathfrak{g}, \mathfrak{h}$ we denote the Lie algebras of G, H. Moreover, a connection of this principal bundle which is a $\mathfrak{g}$-valued one-form ω is defined in an invariant manner.

The principal bundle Y is of the same dimension as G, and the fiber dimension agrees with $\dim H$. For an H-principal bundle $Y\to M$ a Cartan connection is a $\mathfrak{g}$-valued one-form defined on Y which defines an isomorphism $\omega: T_y Y\simeq\mathfrak{g}$ for all $y\in Y$, satisfies $R_h^*\omega=(\mathrm{Ad}\, h^{-1})\omega$ for all $h\in H$, where R_a is right action of H on Y. Finally, if ξ^* is the vertical vectorfield on Y induced by $\xi\in\mathfrak{h}$ then $\omega(\xi^*)=\xi$.

The curvature of this connection is defined as usual as

$$\Omega = d\omega+(\omega)\wedge(\omega).$$

For the hyperquadric case $M=Q$ one has $\Omega=0$, the flat case. Conversely, if for the above connection ω on Y one has $\Omega=0$ then M is locally biholomorphically equivalent to the hyperquadric Q.

The basic result concerning this connection is the following which shows that at least for real analytic CR-manifolds the equivalence problem is solved:

THEOREM 5. *Two real analytic manifolds M_1, M_2 are* CR *equivalent if and only if the corresponding bundles Y_1, Y_2 with their connections ω_1, ω_2 are locally isomorphic.*

The terms of the $\mathfrak{g}$-valued curvature tensor can be related to the lower order terms of the normalform $N_{2,2}, N_{3,2}, N_{3,3}$ of the previous section and in this way provides a geometrical interpretation of the latter [8], [3]. See also Jacobowitz [27].

4. Applications. We summarize briefly various applications of this theory.

(a) *Rigidity results* [4]: It is to be expected that for most pseudoconvex domains $\mathrm{Aut}(D)$ consists of the identity only. The proof of such statements has become feasible via the above theory. Burns, Shnider and Wells [4] proved the following result: Let D_0 be a compact, strictly pseudoconvex domain in a complex manifold X with smooth boundary M. If $U=U(M)$ is some neighborhood of M let

$M=D_0$ be defined by the equation $r(z,\bar{z})=0$ where r is a real smooth function on U. Given any positive integer k and a closed ball $B\subset R^k$ one can construct C^∞-functions $\varrho_t(z,\bar{z}): B\times U\to R$ near r such that the domains $D_{t,\delta}$ bounded by

$$M_{t,\delta}=\{z\in U,\ \varrho_t=\delta\}$$

are biholomorphically distinct, i.e.

$$D_{t_1,\delta_1}\sim D_{t_2,\delta_2} \quad \text{only if} \quad t_1=t_2,\quad \delta_1=\delta_2.$$

Moreover, $\operatorname{Aut} D_{t,\delta}=(\mathrm{id})$. In other words the number of "moduli" for this equivalence problem is infinite.

The proof is based on the local invariants provided by the previous sections and a transversality argument which allows one to distinguish two boundaries at any two points. For $n\geqslant 2$ the invariants used in this connection are the ones mentioned in § 2 (d).

(b) *Variational principle for the chains* [13]. One of the most interesting geometrical objects holomorphically invariantly attached to a hypersurface M is the family of chains defined above. In § 2 they appeared as curves along which one can osculate M by holomorphic spheres and in the intrinsic theory of § 3 they appear as curves along which the tangents are parallel in the sense of the connection.

In 1976 Fefferman [13] gave an entirely different description of these curves and showed that they are governed by Hamiltonian systems of differential equations. Surprisingly the relevant variational principle is not defined on the boundary M but on a circle bundle over M.

If M is locally given by $r(z,\bar{z})=0$ then the pseudoconvexity amounts to the condition

$$\Delta(r)=(-1)^{n+1}\det\begin{pmatrix} r_k & r_j \\ r_{\bar{k}} & r_{j\bar{k}}\end{pmatrix}>0$$

where $r_j=\partial r/\partial z^j$ etc., if $r>0$ refers to the interior of D.

Motivated by the boundary behavior of the Bergman kernel Fefferman required the function r to satisfy the Monge–Ampère equation

$$\Delta(r)=1 \quad \text{and} \quad r=0,\quad dr\neq 0 \quad \text{on} \quad \partial M.^{5}$$

In terms of such a defining function (which is needed only locally) Fefferman introduced the metric ds^2 on $M\times S^1$:

$$ds^2=\frac{i}{n+1}(r_j\,dz^j-r_k\,dz^k)\,d\gamma+r_{j\bar{k}}\,dz^j\,dz^k)$$

for $(z,e^{i\gamma})\in M\times S^1$. If one extends a biholomorphic map

$$z\to\varphi(z)$$

[5] Recently S. Y. Cheng and S. T. Yau established the existence of smooth solutions of the corresponding global problem.

to

$$(z, e^{i\gamma}) \to (\varphi(z), |\det \varphi_z|(\det \varphi_z)^{-1} e^{i\gamma}) = (z^*, e^{i\gamma^*})$$

then

$$(ds^*) = |\det \varphi_z|^{-2/(n+1)} ds^2,$$

i.e. both metrics are conformally equivalent. The metric so defined is indefinite and has type $(n+1, 1)$. The nullgeodesics of this problem are clearly invariant under holomorphic transformations. The chains are the projections of these null geodesics into M, which provides a third characterization of the chains.

One shortcoming of this approach is the need of a solution of the complicated Monge–Ampère equation. However, obviously it suffices to satisfy the above equation only to second order, i.e. solve

$$\Delta(r) = 1 + O(r^2)$$

and Fefferman gave a simple algebraic procedure to obtain such a function.

(c) *Intrinsic description of chains* [1], [19]. While Fefferman's metric refers to embedded hypersurfaces one may seek an intrinsic description of such a metric on a circle bundle of a CR-manifold. This was carried out in [1], [19]. The circle bundle is obtained as a subbundle of the frame bundle Y of Chern.

It is interesting that this approach [1] relates the equivalence problems for CR-structures to that for conformal structures on the above mentioned circle bundle.

(d) In § 1 we mentioned Theorem 1. It is an example of a result which did not require the theory of §§ 2, 3; its proof [26] is simpler and more direct. An earlier result due to Webster refers to abstract CR manifolds [18]: Let M be a compact, connected strictly pseudoconvex CR manifold. If the group of CR-equivalences of M is connected and noncompact then M is locally equivalent to the hyperquadric Q.

The proof of this theorem uses the connection of § 3, though in a rudimentary way. For a study of spherical hypersurfaces see also [2].

(e) We mention some indirectly related work. In connection with Theorem 1 Webster was led to study CR immersion of a $2n-1$ dimensional CR manifold not in C^n but in S^{2n+1}, the unit sphere in C^{n+1}. A $2n-1$ submanifold M of S^{2n+1} is called a CR-hypersurface if the complex tangent space $T(M) \cap JT(M)$ has complex dimension $n-1$. An example of such a CR hypersurface is the intersection of S^{2n+1} with a complex hypersurface in C^{n+1} transverse to S^{2n+1}.

Let $f: M \to M'$ be a CR mapping of one such CR hypersurface of S^{2n+1} into another. M is called *rigid* in S^{2n+1} if any such mapping is the restriction of a fractional linear map of S^{2n+1} into itself.

Webster [23], [24] established the following interesting results:

(i) If $n \geqslant 3$ the image of a local CR immersion of $M = S^{2n-1}$ in S^{2n+1} is rigid, i.e. it is contained in a hyperplane section of S^{2n+1}.

(ii) If $n \geqslant 4$ any CR hypersurface in S^{2n+1} is rigid.

For $n=2$ the statement (i) is false as the example (due to H. Alexander) shows: The holomorphic mapping

$$(z_1, z_2) \to (z_1^2, \sqrt{2}z_1z_2, z_2^2)$$

induces a CR immersion of S^3: $|z_1|^2+|z_2|^2=1$ into S^5 which clearly does not lie in a hyperplane section.

(f) *Segre families.* If a real analytic hypersurface is locally described by an equation $r(z, \bar{z})=0$, $z\in C^{n+1}$, where r is real analytic one can extend this manifold into the complex and define the complex manifold

$$\tilde{M} = \{(z, \zeta)\in C^{2n+2},\ r(z, \zeta) = 0\}.$$

Alternately, one can view $\tilde{M}$ as constituted of the n-dimensional complex manifolds

$$Q_\zeta = \{z\in C^{n+1},\ r(z, \zeta) = 0\}$$

the so-called Segre family. In generalization of the above one can seek the equivalence problem for such Segre families, where z and ζ are separately subject to biholomorphic transformation.

This problem was solved by Faran [**11**] extending earlier work by Chern [7]. Faran also derived a normalform for Segre families and studied their relation to the normalform for real hypersurfaces.

(g) Concerning the boundary behavior of proper holomorphic mappings of pseudoconvex domains with real analytic boundaries, see Pinčuk [**29**], [**30**].

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Extension Problems in Several Complex Variables

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One of the features that distinguish the theory of several complex variables from that of a single variable is the extension property of analytic objects in the case of several variables. We give here a survey on some of the important, especially recent, known results and unsolved problems concerning such extension properties.

1. Holomorphic and meromorphic functions. Let Δ be the open unit disc in C. For $k, n \geq 1$ and $0<c<1$ let H be the "Hartogs figure" consisting of all points $z=(z_1, \ldots, z_{k+n}) \in \Delta^{k+n}$ such that $\max(|z_1|, \ldots, |z_k|)>1-c$ or $\max(|z_{k+1}|, \ldots, |z_{k+n}|)<c$. We say that an analytic object is *Hartogs k-extendible* if it can always be extended from H to Δ^{k+n}. Hartogs [**6**] (respectively Levi [**11**]) proved that holomorphic (respectively meromorphic) functions are Hartogs 1-extendible.

2. Analytic subsets. Rothstein [**16**] obtained the Hartogs k-extendibility for analytic subsets whose every branch has codimension $<k$.

By the *Thullen k-extendibility* of an analytic object we mean its extendibility across any analytic subset A of codimension k if it is assumed to be extendible across some point of every k-codimensional branch of A. Hartogs k-extendibility implies Thullen k-extendibility but not Thullen $(k-1)$-extendibility. Analytic subsets of pure codimension k are Thullen k-extendible. The special case of 1-dimensional subsets in C^2 was proved by Thullen [**34**] and the general case by Remmert–Stein [**14**]. One corollary is that an irreducible k-dimensional analytic

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subset can be extended across an analytic subset of dimension $<k$. As a generalization of this corollary Shiffman [17] obtained the extendibility of irreducible k-dimensional analytic subsets across closed subsets of zero Hausdorff $(2k-1)$-measure.

Bishop [2] proved that a pure-dimensional analytic subset of finite volume can be extended across another analytic subset. A special case was earlier obtained by Stoll [31]. We call this property the *Bishop extendibility*.

For analytic objects on domains in $\boldsymbol{C}^n$, Hartogs $(n-1)$-extendibility implies extension across $\boldsymbol{R}^n$ (see e.g. [27]). Hence irreducible analytic subsets of dimension >1 can be extended across $\boldsymbol{R}^n$. Alexander [1] proved that pure 1-dimensional analytic subsets which are stabilized by the conjugation of the coordinates are also extendible across $\boldsymbol{R}^n$.

3. Coherent analytic sheaves and subsheaves. For a coherent analytic sheaf F, the kth gap-sheaf $F^{[k]}$ of F is the sheaf defined by the presheaf which assigns to an open subset U the inductive limit of $\Gamma(U-A, F)$ as A goes through all analytic subsets of dimension $\leqslant k$ in U. A coherent analytic sheaf F on a domain in $\boldsymbol{C}^n$ is Hartogs $(n-k)$-extendible if $F=F^{[k]}$. A coherent analytic subsheaf F of a coherent analytic sheaf G on a domain in $\boldsymbol{C}^n$ is Hartogs $(n-k)$-extendible if $F=F^{[k]}\cap G$. The sheaf extension result yields a new proof of the theorem of Grauert–Remmert that every α-space is a β-space [22]. The subsheaf extension result specializes to Rothstein's result on analytic subset extension when the subsheaf is the ideal-sheaf of the analytic subset. The theory of coherent analytic sheaf and subsheaf extension is due to Thimm [32], [33], Trautmann [35], Frisch–Guenot [4], Siu [21]–[24], and Siu–Trautmann [29].

Positive holomorphic vector bundles, when regarded as coherent analytic sheaves, are Thullen 1-extendible [25]. The line bundle case was earlier proved by Shiffman [19].

4. Complex spaces. A strictly k-pseudoconvex function f on an open subset U of $\boldsymbol{C}^n$ means a smooth real-valued function whose complex Hessian has at least $n-k+1$ positive eigenvalues at every point of U. A boundary point x of a domain D is said to be strictly k-pseudoconcave if for some open neighborhood U of x, $D\cap U=U\cap\{f>0\}$. For analytic objects, extendibility across strictly k-pseudoconcave boundary points is equivalent to Hartogs $(n-k)$-extendibility (see e.g. [27]). Rossi [15] obtained the following result on extending complex spaces across pseudoconcave boundaries. If X is a complex manifold (or even a connected normal complex space) of dimension $\geqslant 3$ and g is a strictly 1-pseudoconvex function on X such that $g: X\to(a, b)$ is a proper map for some $-\infty\leqslant a<b\leqslant\infty$, then X is an open subset of a normal complex space $\tilde{X}$ such that $(\tilde{X}-X)\cup\{g\leqslant c\}$ is compact for $a<c<b$. This extension result was generalized by Ling [12] to the case with holomorphic parameters. It is conjectured that Rossi's result remains valid if $\dim X\geqslant k+2$ and g is strictly k-pseudoconvex.

5. Holomorphic and meromorphic maps. Griffiths [5] and Shiffman [18] independently obtained the Hartogs 1-extendibility of holomorphic maps into complete

Hermitian manifolds with nonpositive holomorphic sectional curvature. Recently Shiffman [20] proved the Hartogs k-extendibility of holomorphic maps into compact complex manifolds whose bundles of holomorphic k-vectors in a certain sense carry metrics with nonpositive curvature.

Meromorphic maps (in the sense of Remmert) into compact Kähler manifolds are Thullen 1-extendible [28]. Certain special cases were proved earlier by Griffiths [5] and Shiffman [19]. This result was used by Sommese [30] to obtain extension theorems for reductive group actions on compact Kähler manifolds. For target manifolds without curvature conditions meromorphic maps and Kähler conditions are needed respectively because of the examples $C^{n+1}-0\to P_n$ and $C^n-0\to$ the quotient of C^n-0 by the action of the group generated by multiplication of the coordinates by 2.

Meromorphic maps from complex manifolds of dimension n into compact Kähler manifolds can be extended across closed subsets of zero Hausdorff $(2n-3)$-measure [28]. It is conjectured that meromorphic maps into compact Kähler manifolds are Hartogs 1-extendible. There are pieces of evidence that support such a conjecture. For example, if E is a nowhere dense closed subset of a Stein manifold D which does not locally disconnect D and if f is a holomorphic (or even meromorphic) map from $D-E$ to a compact Kähler manifold M with Kähler form ω, then f can be extended to a meromorphic map from D to M when D is the envelope of holomorphy of $D-E$ and when the pullback of ω by f admits a closed positive (1,1)-current on D as an extension (or even only a superextension). A second example is that meromorphic (and also holomorphic) maps into compact Kähler manifolds with zero second homotopy group are Hartogs 1-extendible. The second example can rather easily be proved by using Bishop's theorem on limits of analytic subsets. According to Yau it can also be proved by using results from the theory of minimal surfaces.

6. Closed positive currents. The meromorphic map extension obtained in [28] depends heavily on two results on closed positive currents proved in [26]. The first one is that the set of points where the Lelong number of a closed positive (k,k)-current does not exceed any fixed positive number is an analytic subset of codimension $\geqslant k$. A shorter proof of this result was later given by Lelong [10]. The second one is the Thullen k-extendibility of closed positive (k,k)-currents. A closed positive (k,k)-current can be regarded as the generalization of an analytic subset of pure codimension k, because integration over the regular points of such an analytic subset defines such a current. Generalizing Shiffman's result on analytic subset extension, Harvey [7] proved that closed positive (k,k)-currents on open subsets of C^n can be extended across closed subsets of zero Hausdorff $(2n-2k-1)$-measure. It is conjectured that closed positive (k,k)-currents are Bishop extendible, with the total variations of currents taking the place of volumes. The special case of the conjecture where the analytic subset across which the (k,k)-current is to be extended is of codimension k has been proved [9], [28].

7. Extension from boundaries. There is a theory dealing with the extension of analytic objects defined only on boundaries of domains. Bochner [3] proved that a smooth function on the smooth connected boundary of a compact subset K of C^n with $n \geqslant 2$ which satisfies the tangential Cauchy–Riemann equation can be extended to a smooth function on K holomorphic on the interior of K. Polking–Wells [13] generalized Bochner's result to the case of hyperfunction boundary values.

Recently Harvey–Lawson [8] obtained the following important generalization of Bochner's result. For $k>1$, a compact oriented smooth submanifold M of real dimension $2k-1$ in a Stein manifold X bounds, in the sense of currents, a positive integral linear combination of analytic subsets of pure dimension k in $X-M$ if the complex subspace of the tangent space T_xM of M at every x is of real codimension 1 in T_xM. This yields Bochner's result when one considers the graph of the function there. It is conjectured that the result of Harvey–Lawson remains valid if instead of the Steinness of X one assumes that X is complete Kähler with negative holomorphic sectional curvature. This is related to the conjecture that a complete simply connected Kähler manifold with negative holomorphic sectional curvature is Stein. In conjunction with this conjecture it should be mentioned that the result of Griffiths–Shiffman can easily be shown to reamin valid in a Bochner type formulation.

In Bochner's result, since holomorphic functions are harmonic, it is clear that the extension function must be the solution of the Dirichlet problem for the given boundary values. Therefore a consequence of Bochner's result is that the solution of the Dirichlet problem must be holomorphic. From this point of view, Bochner's result is related to the problem of affirming the following conjecture of Yau on rigidity by proving that certain harmonic maps are automatically holomorphic. Two compact Kähler manifolds of equal dimension $\geqslant 2$ with negative sectional curvature are biholomorphic or conjugate biholomorphic if they are of the same homotopy type.

Added in proof. Meanwhile I succeeded in proving a modified version of Yau's conjecture, which was announced in Proc. Nat. Acad. Sci. U.S.A. **76** (1979), 2107—2108.

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Integral Methods and Zeros of Holomorphic Functions

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Introduction. The entire functions are so closely related with the theory of systems of partial differential equations with constant coefficients and of convolution equations (cf. [**1**], [**27**], [**29**]), that about 1950 a deep study of their properties seemed necessary to many mathematicians. Especially, the connection between the growth of entire functions and the growth of the sets of zeros appeared of great interest. For one variable, it is easy to deal with the Jensen formula and with the Weierstrass canonical products, in a very classical way. For several variables there is no trivial extension of the Weierstrass products but we are able to build entire functions with a given set of zeros and whose growth is connected with the growth of zeros. Working on these problems, the mathematicians were led to use more systematically tools of complex analysis as positive closed currents, plurisubharmonic functions, L^2 estimates or integral methods for the $\bar{\partial}$ operator.

1. The current of integration and the Lelong—Poincaré equation. Let X be an analytic subset of dimension p in a complex manifold of dimension n. In 1953, P. Lelong [**23**] has proved that it is possible to define a current $[X]$ using integration over the regular part $\operatorname{Reg} X$ of X:

$$\langle [X], \varphi \rangle = \int_{\operatorname{Reg} X} \varphi, \tag{1}$$

where φ is a form of total degree $2p$. Besides $[X]$ is closed, of bidegree $(n-p, n-p)$ and positive; a current θ of bidegree $(n-p, n-p)$ is said to be positive if:

$$i^p \theta \wedge \alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge \alpha_p \wedge \bar{\alpha}_p \geqslant 0,$$

for all $(1, 0)$ forms α_j (the orientation of the manifold is defined by the (n, n) form $i^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$).

Let in $\boldsymbol{C}^n$ be: $\alpha=i\partial\bar\partial\log|z|^2/2$, $\beta=i\partial\bar\partial|z|^2/2$, let σ and ν be the positive measures associated to θ by:

$$\sigma=\frac{1}{p!}\theta\wedge\beta^p,\quad \nu=\pi^{-p}\theta\wedge\alpha^p. \tag{2}$$

Let $\sigma(r)$ be the measure σ supported by the closed euclidean ball of radius r. If $\theta=[X]$, σ is the $2p$-dimensional area of X. The significant growth of θ (resp. $[X]$) is measured by the projective indicator: $\nu(r)=\pi^{-p}p!\,r^{-2p}\sigma(r)$.

The growth of an entire application F of $\boldsymbol{C}^n$ in $\boldsymbol{C}^k$ will be described by:

$$M_F(r)=\sup_{|z|\leqq r}\|F(z)\|.$$

When X is a hypersurface which is defined by only one equation F, then X and F are connected by the Lelong–Poincaré equation of currents:

$$\frac{i}{\pi}\partial\bar\partial\log|F|=\sum_j n_j[X_j]=\theta \tag{3}$$

(where n_j is the multiplicity of F on the irreducible branch X_j of X).

With the Poisson–Jensen formula, it is possible exactly as in the case of one variable, to obtain a bound for $\nu(r)$:

$$\nu(r)\leqslant C(\varepsilon,n)\log M_F(r+\varepsilon r),$$

for all $\varepsilon>0$ (and supposing $|F(0)|=1$ for the simplicity). Conversely, if V is a solution of the equation:

$$i\partial\bar\partial V/\pi=\theta, \tag{4}$$

then P. Lelong observed in [**24**] that necessarily $V=\log|F|$, for some entire function F such that $F^{-1}\{0\}=X$.

When θ is of finite order (i.e. $\limsup_{r\to+\infty} r^{-\varrho}\nu(r)<+\infty$ for some $\varrho\geqslant 0$), P. Lelong [**24**] has built in 1953 an explicit canonical potential V, using a modification of the kernel $-|z-x|^{-2n+2}$ by harmonic terms, which is a solution of (4). This potential exactly generalizes the canonical Weierstrass product. The difficulty is to prove that V, which is already a solution of $4\Delta V=\sigma$, is in fact a solution of (4).

Therefore, using V we can define a given X by an entire function of the same order as X. W. Stoll, using H. Kneser's work [**21**], reaches the same result in [**39**], but does not obtain a globally convergent representation for $\log|F|$.

In 1970, in [**33**], [**34**], we dealed with the general case, without restriction about X. We resolved the equation (4), using a regularization of θ, the classical Cartan–Poincaré homotopy formula for $d=\partial+\bar\partial$ and the Hörmander's L^2 estimates for $\bar\partial$. We dealt also with the case where F verifies the weaker condition $F^{-1}\{0\}\supset X$, but not necessarily $F^{-1}\{0\}=X$.

E. Bombieri in [2] (1970) has given a beautiful application of Lelong's results to the arithmetical properties of entire functions. This work will be pursued by M. Waldschmidt [**43**].

2. The case of analytic sets of codimension >1. When the analytic set X is a complete intersection, defined as the set of zeros of a holomorphic map $F=(F_1, F_2, \ldots, F_{n-p})$ of C^n (or of a manifold) in C^{n-p}, P. A. Griffiths and J. R. King [10] proved in 1973 the following "Poincaré–Martinelli" equation of currents:

$$(2\pi)^{-k} i\,\partial\bar\partial\,[\log\|F\|^2 (i\partial\bar\partial \log\|F\|^2)^{k-1}] = [X], \tag{5}$$

where $k=n-p$, where the form in the brackets has locally summable coefficients, and where each irreducible branch of X is counted with the appropriate multiplicity of F (this multiplicity is defined for example in [7]). But the relation (5) is less useful than the Lelong–Poincaré equation, as well to obtain a bound of $\nu(r)$ when a bound of $M_F(r)$ is given, as well to try to build the functions F_j when X is given.

A counterexample of M. Cornalba and B. Shiffman, in 1972 [5], shows that it is not possible in general to obtain a bound of $\nu(r)$ when a bound of $M_F(r)$ is given. Precisely, they proved:

THEOREM 1. *Let $s: \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing function. There exists a holomorphic map F of C^2 in C^2 such that $X=F^{-1}\{0\}$ has dimension 0, such that $M_F(r)$ is of order zero, but such that $\nu(r)$ grows faster than $s(r)$, that is:*

$$\lim_{r\to+\infty} \frac{\nu(r)}{s(r)} = +\infty,$$

$$\limsup_{r\to+\infty} r^{-\varrho} \operatorname{Log} M_F(r) = 0, \quad \textit{for all} \quad \varrho > 0.$$

This means that the Bezout theorem over the algebraic curve has no transcendental equivalent. Nevertheless, different works have been done by W. Stoll [41], P. A. Griffiths [11] and L. Gruman [13] in order to obtain weaker forms of bounds of "transcendental Bezout theorem" type.

It is therefore surprising that the inverse problem, which seems at first more difficult, received a positive general answer, given by the author in 1972 in [35]. We have:

THEOREM 2. *Let X be an analytic subset of C^n, such that $0 \notin X$, let ε and $\delta \in \mathbf{R}^+$. Then there exist $n+1$ entire functions $F=(F_1, F_2, \ldots, F_{n+1})$ such that $F^{-1}\{0\}=X$ and such that for all $r>0$, F verifies one of the following bounds:*

1. $\log M_F(r) \leqslant C(\varepsilon)\,\nu(r+\varepsilon r)\cdot \log^2 r$,
2. $\log M_F(r) \leqslant C(\varepsilon,\delta)\, r^{\delta} \int_0^r t^{-\delta-1}\nu(t+\varepsilon t)\,dt$,

where $C(\varepsilon)$ and $C(\varepsilon,\delta)$ are independent from r.

Particularly, if $\nu(r) \leqslant C_1 r^{\varrho}$, then we are able to find F such that $\log M_F(r) \leqslant C_2 r^{\varrho}$, for some constant C_1 and C_2 (choose $\delta<\varrho$). We obtained in [35] different technical bounds which we do not reproduce for simplicity and which especially give the

following result of W. Stoll [40] and E. Bishop [1] (1966) about algebraic subset of C^n.

THEOREM 3. *X is algebraic if and only if $r \mapsto v(r)$ is a bounded function.*

We shall now briefly talk about the methods of the proof. We do not use (5) which supposes that X is a complete intersection. We begin to build a plurisubharmonic function U which behaves intuitively as $\log \|F\|$, where F is the requested map. At first, we locally build U as an explicit negative potential, by integration of the kernel $-|z-x|^{-2p}$ over X. Then using a partition of unity, we obtain a global $\leqslant 0$ potential U_0 which is nearly plurisubharmonic (i.e. modulo C^∞ strictly plurisubharmonic functions). We estimate the lack of plurisubharmonicity of U_0 and we add to U_0 a strictly plurisubharmonic function with controlled growth to get U. When X has a low growth, such that $\int_1^{+\infty} t^{-2} v(t)\, dt < +\infty$ and when $0 \notin X$, we can take very simply (and globally):

$$U(z) = \int_{C^n} [-|z-x|^{-2p} + |x|^{-2p}] \beta^p \wedge \theta(x), \tag{6}$$

where $\theta = [X]$. The difficulty is to prove that U is plurisubharmonic. To be sucessful we represent U as a direct image of a current on $C^n \times C^n$. Let π_1 and π_2 be the projections of $C^n \times C^n$ over C^n, τ the diagonal map $(x, z) \to z - x$, and $K = -|x|^{-2p} \beta^p$. We prove that choosing a convenient C^∞ function $\chi \geqslant 0$, which is equal to 1 on a neighbourhood of the diagonal of $C^n \times C^n$, we have:

$$U = \pi_{2*}(\chi \tau^* K \wedge \pi_1^* \theta) + \text{technical terms},$$
$$i\partial\bar{\partial} U = p \pi_{2*}(\chi \tau^* \alpha^{p+1} \wedge \pi_1^* \theta) + \text{technical terms},$$

such that the positivity of $i\partial\bar{\partial} U$ is a consequence of the positivity of α and θ and of the fact that positivity is invariant through the direct and inverse image (modulo precise estimates for the technical terms, cf. [34] and [35]). Now we build the functions $F_1, F_2, \ldots, F_{n+1}$ using a theorem of Hörmander–Bombieri ([2], [1]) which gives (because of the plurisubharmonicity of U) the existence of a non trivial entire function f such that

$$\int_{C^n} |f|^2 \exp(-lU)(1+|z|^2)^{-n-2}\, d\lambda(z) < +\infty, \tag{7}$$

where $l > 0$ and where $d\lambda$ is the Lebesgue measure.

U were built such that for some $l > 0$, $\exp(-lU)$ is not locally summable in all points of X. Therefore the estimates (7) imply that f is null over X. Besides the estimates (7) give bounds for the growth of f. We repeat a classical argument of H. Grauert in order to obtain $n+1$ functions satisfying (7) such that $F^{-1}\{0\} = X$.

The results of Theorem 2 seem the best from the point of view of the comparison of growth of F and X. There is still an open problem, that is to reduce the number $n+1$ of functions defining X in theorem 2 without loss of growth. According to O. Forster and K. J. Ramspott [9] n functions are always sufficient to define X (with-

out controlled growth). When X is a submanifold of C^n, topological conditions which were shown in [10] gives the insurance that X is a complete intersection (for instance in codimension 2, the necessary and sufficient condition is the vanishing of the first Chern class of X). New ideas are requested to obtain the equivalent of Theorem 2 in these cases.

The construction of the potential U associated to X in the proof of the Theorem 2 is much generally valid for an arbitrary positive, closed current θ. Therefore to each such $(n-p, n-p)$ current is associated a plurisubharmonic function U and a positive (1, 1) current $i\partial\bar{\partial}U$, whose properties are intimately connected with the properties of θ. The density or Lelongs number of θ at z is defined by:

$$\nu(z) = \lim_{r \to 0} \pi^{-p} p!\, r^{-2p} \int_{B(z,r)} \beta^p \wedge \theta$$

($B(z, r)$ is the euclidean closed ball of radius r centered at z). $\nu(z)=0$ if θ is C^0 in a neighbourhood of z. If $\theta=[X]$, $\nu(z)$ is an integer at each point $z \in X$ and $\nu(z)=1$ if $z \in \operatorname{Reg} X$. Conversely, the following result of Y. T. Siu [32] (1974), whose particular case is solved by Bombieri [2], King [15], Harvey [14], Skoda [38] says how much of a given θ is an analytic set.

THEOREM 4. *Let θ be a positive, closed, (k, k) current on a complex manifold Ω. For all $c>0$, the set $E_c=\{z \in \Omega | \nu(z) \geqslant c\}$ is an analytic subset of Ω (of smaller dimension than $n-k$).*

The proof heavily depends on the Hörmander–Bombieri result (7).

Recently, P. Lelong [25] has proved that the (1, 1) current $i\partial\bar{\partial}U$, locally associated to θ as in (6), has the same density as θ in all points. Therefore, it is sufficient to prove Theorem 4 for a (1, 1) current. In this last case, the proof of Y. T. Siu in [32] is particularly elegant.

The result of Theorem 2 has easy extensions to an open pseudoconvex subset of C^n and to Stein manifolds (cf. H. Skoda [35, Proposition 9.1]) but the precision of the bounds is limited by the L^2 estimates and is not always the best. Therefore new methods were necessary.

3. Fine results for strictly pseudoconvex open sets in C^n. Let Ω be a bounded, strictly pseudoconvex, open set in C^n, of class C^2, that is: $\Omega=\{z|\varrho(z)<0\}$, where ϱ is a real function, defined, of class C^2, strictly plurisubharmonic in a neighbourhood of $\bar{\Omega}$ and verifying $d\varrho \neq 0$ on $\partial\Omega$. Let Ω_ε be the set $\{z|\varrho(z)<-\varepsilon\}$.

The Nevanlinna class $N(\Omega)$ (resp. the space $H^p(\Omega)$, $0<p<+\infty$) is the set of holomorphic functions f on Ω such that:

$$\limsup_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\partial\Omega_\varepsilon} \log^+ |f|\, dS_\varepsilon < +\infty, \tag{8}$$

resp.

$$\limsup_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\partial\Omega_\varepsilon} |f|^p\, dS_\varepsilon < +\infty,$$

where dS_ε is the euclidean area of $\partial\Omega_\varepsilon$. We trivially have

$$H^\infty(\Omega)\subset H^p(\Omega)\subset N(\Omega),$$

for all p, where $H^\infty(\Omega)$ is the space of bounded functions. An hypersurface X verifies the Blaschke condition if by definition:

$$\int_X |\varrho(z)|\, d\sigma(z) < +\infty, \tag{9}$$

where $d\sigma$ is the area element on X. For one variable, it is classical that the Blaschke condition (9) characterizes the set of zeros of functions of $N(\Omega)$ and of all $H^p(\Omega)$ $0<p\leqslant+\infty$. For several variables, it is easy to prove using the Poisson–Jensen formula (cf. [4]) that the zero set of $f\in N(\Omega)$ verifies (9). But W. Rudin [31] has proved [11] that the characterization of zeros of functions in $H^p(\Omega)$ for the euclidean ball must necessarily depend on p, so that such a characterization is probably much more complicated as in the case $n=1$. Nevertheless, in 1975 G. M. Henkin [17] and the author [37], [38] have independently but by very similar methods proved the following:

THEOREM 5. *Let Ω be a strictly pseudoconvex open set such that $H^2(\Omega, \mathbf{Z})=0$ and let X be an hypersurface of Ω verifying the Blaschke condition, then X is the zero set of some $f\in N(\Omega)$.*

More generally, we solved the Lelong–Poincaré equation $i\partial\bar\partial V=\theta$, where θ is a given (1, 1) positive, closed, current verifying:

$$\int_\Omega |\varrho|\theta\wedge\beta^{n-1} < +\infty, \tag{10}$$

and where V is built so that

$$\limsup_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} V^+\, dS_\varepsilon < +\infty.$$

Partial results were obtained by L. Gruman [13] and G. Laville [22]. In fact, Theorem 5 is a consequence of an existence theorem for the $\bar\partial$ which is especially conceived for the theory of H^p spaces. For simplicity, we only consider the (0, 1) form.

THEOREM 6. *If f is a (0, 1) current on $\Omega, \bar\partial$ closed and if the coefficients of f and of the current $|\varrho|^{-1/2}d''\varrho\wedge f$ are bounded measures on Ω, then there exists $u\in L^1(\Omega)$ such that*

$$\bar\partial u = f \quad \text{in } \Omega,$$

and such that u has a boundary value in $L^1(\partial\Omega)$ in the sense of Stoke's formula:

$$\int_{\partial\Omega} u\wedge\varphi = \int_\Omega f\wedge\varphi + \int_\Omega u\wedge\bar\partial\varphi,$$

for all $(n, n-1)$ forms φ of class C^1 in $\bar\Omega$.

Theorem 6 was obtained by explicit integral formula for differential form of Cauchy–Leray–Poisson–Szegö type, which are closely related with the kernels for $\bar{\partial}$ already built in order to obtain L^∞ estimates for $\bar{\partial}$ (cf. [26], [15], [20], [30]).

Theorem 5 follows from Theorem 6 using classical splitting of $\partial\bar{\partial}$ in d and $\bar{\partial}$, homotopy formula for d. The new argument is that (10) implies very strong restriction on the *tangential* coefficients of θ, for instance:

$$\int_\Omega \theta \wedge \partial_\varrho \wedge \bar{\partial}_\varrho \wedge (\partial\bar{\partial}_\varrho)^{n-2} < +\infty$$

(found in another more restrictive form by P. Malliavin [28]).

In another direction, Theorem 6 is closely connected with the "corona problem" for $H^\infty(\Omega)$ (i.e. the determination of the spectrum of $H^\infty(\Omega)$) and with Carleson–Hörmander measures on Ω (cf. N. Varopoulos [42] and [4], [38], [18]).

Other results for hypersurfaces with polynomial growth:

$$\int_X |\varrho(z)|^{\alpha+1}\, d\sigma(z) < +\infty \quad (\alpha > 0)$$

where recently in 1977 obtained by similar methods by G. M. Henkin [6] (cf. also [44]). All results of this section are in fact a consequence of an existence theorem for $\bar{\partial}$.

Perhaps, methods which will be more specific to the real $i\partial\bar{\partial}$ operator, will permit us to reach better results concerning for instance zeros of $H^p(\Omega)$. Besides, there is no result similar to Theorem 5 for analytic sets of codimension >1.

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A Survey of Quasiregular Maps in R^n

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1. Introduction. Classical complex analysis is concerned with analytic functions $f\colon G\to C$, where G is an open set in the complex plane C. Since C is equal to the two-dimensional euclidean space R^2, it is natural to ask whether we can get anything similar if we consider maps $f\colon G\to R^n$, G open in R^n. It is fairly obvious that we cannot get everything, and the purpose of my lecture is to give some idea of what we can get.

We start at the following well-known characterization of analyticity: A map $f\colon G\to R^2$ is (complex) analytic if and only if it satisfies the conditions

(1) f is C^1 in the real sense,

(2) $|f'(x)|^2=J_f(x)$ for all x in G.

Here the derivative $f'(x)$ is the linear map $R^2\to R^2$ satisfying $f(x+h)=f(x)+f'(x)h+o(h)$, and $J_f(x)=\det f'(x)$ is the jacobian of f. The partial derivatives of f are given by

$$\partial_j f(x)=f'(x)e_j \tag{1.1}$$

where e_1, e_2 are the standard basis vectors of R^2. Geometrically, (2) means that $f'(x)$ maps every circle onto a circle or to a point.

This definition can immediately be extended to higher dimensions. However, we obtain a surprisingly small class of maps. Indeed, if $n\geq 3$ and G is open in R^n, then a map $f\colon G\to R^n$ satisfies the conditions

(1) f is C^1,

(2) $|f'(x)|^n=J_f(x)$ for all $x\in G$,

if and only if f is either a constant or a sense-preserving Möbius transformation. For C^3-maps this was already proved by Liouville in 1850. It implies that the maps satisfying (1) and (2) cannot have any branch points. In order to get a more interesting theory we relax these conditions. First observe that (2) is equivalent to the inequality $|f'(x)|^n \leq J_f(x)$, since the opposite inequality is automatically true. We replace this by the weaker inequality $|f'(x)|^n \leq KJ_f(x)$, where $K \geq 1$ is a constant. Geometrically, this means that the image of a ball under $f'(x)$ is an ellipsoid E such that if B is the ball circumscribing E, then $m(B) \leq Km(E)$. Thus E cannot be arbitrarily flat or thin. We must also allow some non-differentiability to get interesting maps. In fact, a C^3-map $f: G \to R^n$, $n \geq 3$, satisfying $|f'(x)|^n \leq KJ_f(x)$ is always a local homeomorphism. It has turned out that the most convenient class is the Sobolev space W_n^1, also denoted by ACL^n. This consists of all functions which have generalized first order partial derivatives which are locally L^n-integrable. Such a map $f: G \to R^n$ has almost everywhere the ordinary partial derivatives $\partial_i f(x)$, and we use (1.1) to define the linear map $f'(x): R^n \to R^n$ for almost every $x \in G$. We have now obtained the class of quasiregular maps:

DEFINITION. Let G be open in R^n, let $f: G \to R^n$ be continuous, and let $K \geq 1$. Then f is *K-quasiregular* if

(1) $f \in W_n^1$,

(2) $|f'(x)|^n \leqq KJ_f(x)$ a.e. in G.

The smallest constant K for which (2) is true is called the *outer dilatation* of f. In the literature it is customary to use a slightly different definition of K-quasiregularity which also involves the so-called inner dilatation, but in order to keep the presentation simple I ignore that. A map is *quasiregular* (abbreviated qr) if it is K-qr for some $K \geq 1$; it is *quasiconformal* if it is a qr homeomorphism onto some open set.

The definition of qr maps has nothing to do with complex numbers, and in fact, this theory does not belong to complex analysis at all, but it is usually classified in the section "complex analysis" in the review journals and in this congress, because its roots are in complex analysis, and no better place has been found. The theory has nothing in common with the analytic functions of several complex variables, except that both classes include the analytic functions of one complex variable.

Historical remarks. The two-dimensional qr maps were introduced by H. Grötzsch [3] in 1928. Higher dimensional maps were first mentioned by M. A. Lavrentiev [4] in 1938, but a systematic study only began considerably later, in 1966, by Ju. G. Rešetnjak [11].

2. Example. Let (r, φ, z) be the cylindrical coordinates of R^3; thus $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. Let $f: R^3 \to R^3$ be the map defined by $f(r, \varphi, z) = (r, 2\varphi, z)$. It keeps all points of the axis $Z = \{(r, \varphi, z) | r = 0\}$ fixed and maps every circle

$\{(r, \varphi, z)|r=r_0>0, z=z_0\}$ onto itself by a two-to-one covering map. As a Lipschitz map, f is in W_n^1. It is differentiable outside Z, and an easy computation shows that $|f'(x)|^3=4J_f(x)$ for all x in $R^3\setminus Z$. Thus f is 4-qr.

This map is a local homeomorphism outside Z, but not at any point of Z. Thus Z is the *branch set* of f. The branch set B_f of a map $f: G\to R^n$ is defined as the set of all $x\in G$ such that f is not a local homeomorphism at x. It is clearly always closed in G. If $f: G\to R^2$ is a nonconstant analytic function, B_f is the set of all zeros of $f'(x)$, and is discrete. In higher dimensions B_f never contains isolated points.

3. Properties. I shall give a survey of some of the most interesting properties of qr maps. Before that, I remark that the following properties of analytic functions cannot be extended to qr maps: (1) power series; (2) algebraic structure (the sum of two qr maps is not usually qr); (3) unique continuation.

In the rest of this section we assume that $f: G\to R^n$ is a given *nonconstant K-quasiregular map*.

(3.1) f is open, that is, f maps open sets onto open sets. Hence f satisfies the maximum principle.

(3.2) f is discrete, that is, $f^{-1}(y)$ is a discrete set for every $y\in R^n$.

(3.3) $\dim B_f=\dim fB_f\leq n-2$, where dim is the topological dimension. This result is purely topological, since it is true for every discrete open map $f: G\to R^n$. It has been conjectured that $\dim B_f=n-2$ whenever $B_f=\emptyset$, but in the topological case, this has recently been disproved by P. T. Church and J. G. Timourian [2, 5.6] with the aid of the double suspension theorem of J. W. Cannon and R. D. Edwards. However, I do not know whether it is true for qr maps.

(3.4) If $B_f\neq\emptyset$, then $\dim_H fB_f\geq n-2$, where $\dim_H$ is the Hausdorff dimension.

(3.5) $m(B_f)=m(fB_f)=0$. In fact, $\dim_H B_f<n$, $\dim_H fB_f<n$.

(3.6) f is differentiable a.e.

(3.7) f maps every set of measure zero onto a set of measure zero.

(3.8) (Liouville's theorem in R^n.) If $G=R^n$, then f is unbounded.

All these results are at least seven years old, except the Hausdorff dimension result of (3.5), which was proved by J. Sarvas [16] in 1975. For proofs and some other results, see [5], [6], [7], [12], [13].

For analytic functions $f: R^2\to R^2$ we have the Picard theorem, which is much stronger than (3.8). It states that $\mathbf{C}fR^2$ contains at most one point, where $\mathbf{C}$ denotes complement in R^2. This is known to be true also for twodimensional qr maps, but for $n\geq 3$ it is an important open problem. However, there are intermediate results, which lie between Liouville and Picard. For a long time, the best result was that $\mathbf{C}fR^n$ is of n-capacity zero. However, a few months ago, S. Rickman proved that this set is in fact finite:

(3.9) If $G=R^n$, then $\operatorname{card}\mathbf{C}fR^n\leq q(n, K)<\infty$.

An *isolated singularity* of a map $f: G \to R^n$ is an isolated boundary point b of G. We say that b is a removable singularity, a pole, or an essential singularity according as f has a finite limit, an infinite limit, or no limit at all at b. Corresponding to the big Picard theorem, the result (3.9) has also a local version:

(3.10) If b is an essential isolated singularity of f, then $\operatorname{card} \mathbf{C}f[U \setminus \{b\}] \leq q(n, K)$ for every neighborhood U of b.

(3.11) (Iversen's theorem in R^n.) If b is an essential isolated singularity of f and if f omits z, then z is an asymptotic value of f.

Here z can also be the point at infinity. In fact, one can easily introduce quasimeromorphic maps, but for simplicity, I consider only finitevalued maps.

This result means that there is a path $\alpha: (0, 1] \to G$ such that $\alpha(t) \to b$ and $f(\alpha(t)) \to z$ as $t \to 0$. I shall outline its proof in § 4.

If $f: B^2 \to R^2$ is a bounded analytic function in the unit disc B^2 and if f has an asymptotic value z at a boundary point b, then, by a theorem of Lindelöf, f has also the angular limit z at b. The result is also true for two-dimensional qr maps, but S. Rickman has recently proved that the corresponding result is false in higher dimensions:

(3.12) For every $n \geq 3$, there exists a bounded quasiregular $f: B^n \to R^n$ and a point $b \in \partial B^n$ such that f has an asymptotic value but no angular limit at b.

(3.13) If $G = R^n$, $B_f = \emptyset$ and $n \geq 3$, then f is a homeomorphism onto R^n.

This is due to V. A. Zorič [19]. For related results, see [1], [8], and [20]. The result is not true for $n=2$, as is seen from the counterexample $f(z) = e^z$. We see from (3.12) and (3.13) that there are essential differences between the cases $n=2$ and $n \geq 3$.

I stop the list here, because I also want to say a few words on the methods and tools used in this theory. For some further properties of qr maps, especially for value distribution theory, we refer to the expository article of Rickman [15] and to the references given there.

4. Tool. The most important tool in the theory of qr maps is the *modulus* of a path family, introduced by Ahlfors and Beurling in 1950. By a path in R^n we mean a continuous map $\alpha: \Delta \to R^n$ where Δ is a real interval, which may be open, half open, or closed. If Γ is a family of paths in R^n, its modulus is defined by

$$M(\Gamma) = \inf_{\varrho} \int_{R^n} \varrho^n \, dm$$

where the infimum is taken over all nonnegative Borel functions $\varrho: R^n \to R^1 \cup \{\infty\}$ such that

$$\int_{\alpha} \varrho \, ds \geq 1$$

for all $\alpha \in \Gamma$. Here the line integral is defined using the Hausdorff linear measure for all paths α, but we obtain the same modulus if we only consider the rectifiable paths in Γ.

The modulus M is an outer measure in the space of all paths in R^n. For the basic theory of the modulus, we refer to [17, Chapter 1]. As an important example, consider the family Γ of all paths joining the boundary components of the annulus $A=\{x\in R^n \mid a<|x|<b\}$ in A. Then

$$M(\Gamma)=\omega_{n-1}\left(\log\frac{b}{a}\right)^{1-n}$$

where ω_{n-1} is the area of the unit sphere S^{n-1}. The formula is also true in the limiting case $a=0$, from which it follows that given a point x_0 in R^n, the family of all paths through x_0 or converging to x_0 is of modulus zero.

Suppose that $f: G\to R^n$ is a nonconstant K-qr map, and let Γ be a family of paths in G. Then f maps Γ onto the path family $f\Gamma=\{f\alpha \mid \alpha\in\Gamma\}$. The moduli of these families satisfy the important inequality

$$M(f\Gamma)\leqslant K^{n-1}M(\Gamma), \tag{4.1}$$

proved by E. A. Poleckii [10] in 1970. In the literature, the factor K^{n-1} is usually replaced by K, because of a different definition of K-qrty. Conversely, a sense-preserving discrete open map $f: G\to R^n$ satisfying (4.1) for all path families Γ in G is qr.

This inequality and some related inequalities [5, 3.2], [18] have turned out to be extremely useful. I shall illustrate the use of (4.1) by giving a sketch of the proof of (3.11), the n-dimensional Iversen's theorem.

So let b be an essential isolated singularity of a qr map f, and let f omit a point z. Choose a ball U centered at b such that f is defined in $\overline{U}\setminus\{b\}$. Choose another ball $B(z,r)$ such that (1) $\overline{B}(z,r)\cap f\partial U=\emptyset$ and (2) $U\cap f^{-1}(y)\neq\emptyset$ for every $y\in\partial B(z,r)$. The condition (2) holds for almost every r, because of (3.10) or the older capacity result. We may assume $r=1$. For every $y\in\partial B(z,r)$, we define a path $\beta_y: (0,1]\to R^n$ by $\beta_y(t)=z+t(y-z)$. Furthermore, we choose a point $x_y\in U\cap f^{-1}(y)$ and a maximal lifting $\alpha_y: (r_y,1]\to U$ of β_y, terminating at x_y. This means that $\alpha_y(1)=x_y$, $f\alpha_y=\beta_y\mid(r_y,1]$, and α_y has no extension to r_y. For topological reasons, $\alpha_y(t)\to b$ as $t\to r_y$. If $r_y=0$, then α_y is the sought-for path for which $f(\alpha_y(t))\to z$ as $t\to 0$. We show that $r_y=0$ for almost every $y\in\partial B(z,r)$. It suffices to show that for every j, the set $E_j=\{y\mid r_y>1/j\}$ is of $(n-1)$-measure zero. Since all paths of the family $\Gamma=\{\alpha_y \mid y\in E_j\}$ converge to b, $M(\Gamma)=0$. By (4.1), this implies $M(f\Gamma)=0$. On the other hand, easy modulus estimates yield $M(f\Gamma)\geqslant m_{n-1}(E_j)\,(\log j)^{1-n}$, which implies $m_{n-1}(E_j)=0$.

This proof is also valid for complex analytic functions. In fact, it gives the stronger result due to W. Gross that almost every line segment from z can be chosen to be the image of an asymptotic path. In general, the path family method gives new proofs for many classical theorems. The new proofs are often more complicated but sometimes simpler than the old ones, and they usually give new geometric points of view to the questions. There are also some results in the non-integrated value distribution theory of qr maps [14] which are new also for analytic functions.

In the above proof, (4.1) was not used in its full strength. In fact, with some extra care to verify the property (2), the proof is also valid for *weakly quasiregular* maps. A map $f: G \to R^n$ is weakly qr if it is sensepreserving, discrete and open, and maps every path family of modulus zero onto a family of modulus zero.

5. Open problems.

(5.1) Is the Picard theorem true for $n \geqslant 3$?

(5.2) Is it possible that $0 \leqslant \dim B_f \leqslant n-3$ for some qr map?

(5.3) Let f be qr and C^1, and let $n \geqslant 3$. Is $B_f = \emptyset$? The answer is yes if f is C^3 or C^2 for $n \geqslant 4$.

(5.4) In the definition of qrty, can one replace W_n^1 by W_1^1?

(5.5) Let $f: G \to R^n$ be sense-preserving, discrete and open, and let $f|G \setminus B_f$ be qr. Is f qr?

(5.6) Can every analytic function $f: R^2 \to R^2$ be extended to a qr map $g: R^3 \to R^3$?

(5.7) Let $f: B^n \to R^n$ be bounded and qr. Does f have a radial limit at some boundary point? It is known that f has an asymptotic value at every point of a dense set in ∂B^n.

(5.8) What are the analytic properties of weakly qr maps? For example, is a weakly qr map differentiable at some point?

(5.9) Given a discrete group G of Möbius transformations of B^n, does there exist a quasimeromorphic map $f: B^n \to R^n \cup \{\infty\}$ which is automorphic with respect to G? Cf. [8].

(5.10) Let G be the half space $x_n > 0$, and let $f: G \to R^n$ be qr such that the cluster set $C(f, x) \subset R^{n-1}$ for all $x \in B^{n-1}$. Can f be extended by reflection to a qr map of $R^n \setminus (R^{n-1} \setminus B^{n-1})$?

(5.11) Let E be closed in an open set $G \subset R^n$, and let $m_{n-2}(E) < \infty$. Does every bounded qr map of $G \setminus E$ have a qr extension to G?

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Algèbres Enveloppantes

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1. Introduction. 1.1. Soit G un groupe de Lie réel connexe d'élément neutre e. Pour la convolution, les distributions sur G concentrées en e forment une algèbre U sur C. L'ensemble des vecteurs tangents complexes à G en e est une sous-algèbre de Lie $\mathfrak{g}$ de U, et $\mathfrak{g}$ est l'algèbre de Lie complexifiée de G.

1.2. On peut construire U à partir de $\mathfrak{g}$ de manière algébrique: soient T l'algèbre tensorielle de $\mathfrak{g}$, I l'idéal bilatère de T engendré par les $x\otimes y-y\otimes x-[x,y]$ où $x,y\in\mathfrak{g}$; l'algèbre T/I, notée $U(\mathfrak{g})$, s'appelle *l'algèbre enveloppante* de $\mathfrak{g}$. Le plongement naturel de $\mathfrak{g}$ dans $U(\mathfrak{g})$ est le plongement *universel* de $\mathfrak{g}$ dans une algèbre associative, d'où la notation $U(\mathfrak{g})$; et U s'identifie canoniquement à $U(\mathfrak{g})$. Si $(x_1,\ldots,x_n)$ est une base de $\mathfrak{g}$, les $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ où $\alpha_1,\ldots,\alpha_n\in N$ forment une base de $U(\mathfrak{g})$, d'où une vue assez concrète de $U(\mathfrak{g})$. Beaucoup de problèmes relatifs à $\mathfrak{g}$ ne se comprennent bien qu'après passage à $U(\mathfrak{g})$. On note $Z(\mathfrak{g})$ le centre de $U(\mathfrak{g})$.

1.3. Toute représentation (=rep.) linéaire π de $\mathfrak{g}$ se prolonge de manière unique en une rep. π' de $U(\mathfrak{g})$. L'application $\pi\mapsto\pi'$ est une bijection entre rep. de $\mathfrak{g}$ et rep. de $U(\mathfrak{g})$, qui conserve l'équivalence et la simplicité.

1.4. Les *démonstrations* des résultats ci-dessous font appel à l'algèbre non commutative, à la théorie des représentations, à la géométrie algébrique, et à des méthodes analytiques (groupes de Lie).

1.5. L'algèbre $U(\mathfrak{g})$ est utile dans d'autres contextes: corps de base quelconque, algèbres de Moody–Kac, superalgèbres de Lie. Nous n'en parlerons pas.

2. Représentations de G, de $\mathfrak{g}$, de $U(\mathfrak{g})$. 2.1. Soit $U(\mathfrak{g})\hat{}$ l'ensemble des classes de représentations simples de $U(\mathfrak{g})$, ou de $\mathfrak{g}$. La recherche de $U(\mathfrak{g})\hat{}$ a longtemps paru impraticable. Pourtant, $U(\mathfrak{g})\hat{}$ vient d'être calculé quand par exemple $\mathfrak{g}$ est

l'algèbre de Heisenberg de dimension 3 [2], [2 bis]. L'ensemble $U(\mathfrak{g})\hat{}$ est très gros, et a peu de liens avec les rep. de G.

2.2. N. Jacobson a associé à tout anneau, par exemple à $U(\mathfrak{g})$, l'espace topologique Prim $U(\mathfrak{g})$ formé de tous les idéaux primitifs de $U(\mathfrak{g})$. (Un idéal primitif est le noyau d'une représentation simple). On a une surjection évidente $U(\mathfrak{g})\hat{} \to$ Prim $U(\mathfrak{g})$; bien que $U(\mathfrak{g})\hat{}$ soit énorme, Prim $U(\mathfrak{g})$ est raisonnable.

2.3. Soit π une rep. continue de G dans un espace de Banach H. On lui associe une rep. π' de $\mathfrak{g}$, donc de $U(\mathfrak{g})$: soit H' l'ensemble des $\xi \in H$ tels que la fonction $g \mapsto \pi(g)\xi$ sur G soit C^∞; H' est un sousespace vectoriel dense de H; tout $x \in \mathfrak{g}$ (réel) définit, par dérivation de $t \mapsto \pi(\exp tx)$, un endomorphisme $\pi'(x)$ de H', d'où une rep. π' de $\mathfrak{g}$ dans H'. Supposons π simple, au seul sens intéressant, i.e. *topologiquement:* tout endomorphisme continu de H est limite forte de combinaisons linéaires des $\pi(g)$, $g \in G$. Alors, malheureusement, π' n'est pas *algébriquement* simple. Mais Ker $\pi' \in$ Prim $U(\mathfrak{g})$ [10].

En particulier, notant $G\hat{}$ l'ensemble des classes de rep. unitaires simples de G, on a une *application canonique* $\theta: G\hat{} \to$ Prim $U(\mathfrak{g})$. Soit Prim$_a U(\mathfrak{g})$ l'ensemble des idéaux primitifs de $U(\mathfrak{g})$ qui sont autoadjoints, i.e. invariants pour l'involution canonique de $U(\mathfrak{g})$. Alors $\theta(G\hat{}) \subset$ Prim$_a U(\mathfrak{g})$. Si G est nilpotent simplement connexe, l'application $G\hat{} \to$ Prim$_a U(\mathfrak{g})$ est *bijective* [8]; en général, elle n'est ni injective, ni surjective, mais les espaces $G\hat{}$ et Prim$_a U(\mathfrak{g})$ ont quelque ressemblance. Conjecture: si G est algébrique, les fibres de θ sont finies.

2.4. Pour prouver que π simple $\Rightarrow$ Ker π' primitif, il faut certaines *caractérisations des idéaux primitifs.* Soit dans $U(\mathfrak{g})$ un idéal bilatère I, premier (i.e. tel que $u, v \in U(\mathfrak{g})$, $uU(\mathfrak{g})v \subset I \Rightarrow u \in I$ ou $v \in I$; un idéal primitif est premier), et considérons les conditions suivantes: (a) I est primitif; (b) soit A l'anneau de fractions de $U(\mathfrak{g})/I$ (qui existe et est un anneau de matrices d'après Goldie); alors le centre de A est $\boldsymbol{C}$; (c) l'intersection des idéaux primitifs qui contiennent strictement I est distincte de I. Alors (c) $\Rightarrow$ (a) $\Leftrightarrow$ (b) [10]. Si $\mathfrak{g}$ est résoluble (et probablement en général), (c) $\Leftrightarrow$ (a) [9]. Ces propriétés cessent d'être vraies en algèbre non commutative générale [16], [24].

3. Recherche de Prim $U(\mathfrak{g})$. 3.1. Pour calculer Prim $U(\mathfrak{g})$, on cherche à imiter pour $U(\mathfrak{g})$ les théorèmes de Mackey concernant les rep. induites [10], et la méthode des orbites de Kirillov.

3.2. La méthode des orbites réussit pour $\mathfrak{g}$ *résoluble.* Soit $\mathfrak{g}^*$ l'espace dual de $\mathfrak{g}$. Soit $f \in \mathfrak{g}^*$. On appelle polarisation de $\mathfrak{g}$ en f une sous-algèbre de $\mathfrak{g}$ qui est un sous-espace vectoriel totalement isotrope maximal pour la forme alternée $(x, y) \mapsto f([x, y])$ sur $\mathfrak{g} \times \mathfrak{g}$. *Il existe* des polarisations de $\mathfrak{g}$ en f. Soit $\mathfrak{h}$ l'une d'elles. La forme $f|\mathfrak{h}$ est une rep. de dimension 1 de $\mathfrak{h}$. Soit ϱ la rep. "induite tordue" de $\mathfrak{g}$, i.e. associée au $U(\mathfrak{g})$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \boldsymbol{C}$, où $\boldsymbol{C}$ est considéré comme $\mathfrak{h}$-module via la forme $x \mapsto f(x) + (1/2) \operatorname{trad}_{\mathfrak{g}/\mathfrak{h}} x$ sur $\mathfrak{h}$. Alors: (1) Ker $\varrho \in$ Prim $U(\mathfrak{g})$; (2) Ker ϱ ne dépend que de f et non du choix de $\mathfrak{h}$; posons Ker $\varrho = I(f)$; (3) l'application $I: \mathfrak{g}^* \to$ Prim $U(\mathfrak{g})$ est surjective; (4) soit Γ le

groupe adjoint algébrique de $\mathfrak{g}$, i.e. le plus petit groupe algébrique d'automorphismes de $\mathfrak{g}$ dont l'algèbre de Lie contient ad $\mathfrak{g}$; alors $I(f)=I(f')\Leftrightarrow f'\in If$. *On a donc défini une bijection* $\mathfrak{g}^*/I\to\mathrm{Prim}\,U(\mathfrak{g})$ [9]. Cette bijection est même un homéomorphisme pour $\mathfrak{g}$ nilpotente [5]; on ignore s'il est de même pour $\mathfrak{g}$ résoluble.

3.3. Pour $\mathfrak{g}$ *non résoluble,* il y a de sérieuses difficultés. (a) Certains éléments de $\mathfrak{g}^*$ sont non polarisables. Toutefois, disons que $f\in\mathfrak{g}^*$ est *régulier* si son stabilisateur dans $\mathfrak{g}$ est de dimension minimale; l'ensemble $\mathfrak{r}$ des éléments réguliers est ouvert dans $\mathfrak{g}^*$ pour la topologie de Zariski. Si $f\in\mathfrak{r}$, f admet une polarisation résoluble. (b) Si $f\in\mathfrak{g}^*$ admet 2 polarisations $\mathfrak{p}_1,\mathfrak{p}_2$, les idéaux obtenus comme en 3.2 par induction à partir de $\mathfrak{p}_1$ et $\mathfrak{p}_2$ peuvent être distincts. Toutefois, si $\mathfrak{p}_1$ et $\mathfrak{p}_2$ sont résolubles, l'idéal obtenu est le même pour $\mathfrak{p}_1$ et $\mathfrak{p}_2$, et est primitif. *On a donc défini une application I:* $\mathfrak{r}\to\mathrm{Prim}\,U(\mathfrak{g})$, constante sur les orbites du groupe adjoint algébrique Γ [9]. (c) Les idéaux $I(f)$ précédents sont complètement premiers [6] (un idéal I de $U(\mathfrak{g})$ est complètement premier si $u,v\in U(\mathfrak{g})$, $uv\in I\Rightarrow u\in I$ ou $v\in I$). Un idéal premier de $U(\mathfrak{g})$ est complètement premier si $\mathfrak{g}$ est résoluble mais pas pour $\mathfrak{g}$ quelconque. Soit $\mathrm{Primc}\,U(\mathfrak{g})$ l'ensemble des idéaux primitifs complètement premiers de $U(\mathfrak{g})$. La méthode des orbites, sous la forme précédente, ne peut donner au mieux que $\mathrm{Primc}\,U(\mathfrak{g})$.

3.4. Supposons $\mathfrak{g}=\mathfrak{sl}(n)$. Alors tout $f\in\mathfrak{g}^*$ est polarisable [25]. L'idéal obtenu comme en 3.2 est indépendant du choix de la polarisation, et est primitif; notons — le $I(f)$. L'application $I\colon\mathfrak{g}^*\to\mathrm{Primc}\,U(\mathfrak{g})$ est constante sur les Γ-orbites, *d'où une application de* $\mathfrak{g}^*/\Gamma$ *dans* $\mathrm{Primc}\,U(\mathfrak{g})$ qui est *injective* [3], [4]. On ignore si elle est surjective. Elle l'est pour $n\leqslant 5$ [20]. La situation est moins bonne pour $\mathfrak{g}$ semi-simple quelconque.

3.5. Après ce qu'on a dit en 3.3 (c), on peut tenter d'induire à $\mathfrak{g}$ des rep. simples de dimension finie >1. Cela ne suffit pas, hélas, pour obtenir $\mathrm{Prim}\,U(\mathfrak{g})$ tout entier: même pour $\mathfrak{g}=\mathfrak{sl}(n)$, les idéaux primitifs «non induits» sont assez nombreux [4]. Nous reviendrons sur le calcul de $U(\mathfrak{g})$ au § 6.

4. Le centre de $U(\mathfrak{g})$. 4.1. Soit $Z(\mathfrak{g})\hat{}$ l'ensemble des caractères de $Z(\mathfrak{g})$ (=homomorphismes de $Z(\mathfrak{g})$ dans $\boldsymbol{C}$). Si π est une rep. simple de $U(\mathfrak{g})$, $\pi|Z(\mathfrak{g})$ appartient à $Z(\mathfrak{g})\hat{}$ et ne dépend que de $\mathrm{Ker}\,\pi$, d'où *une application canonique* φ: $\mathrm{Prim}\,U(\mathfrak{g})\to Z(\mathfrak{g})\hat{}$. Il importe donc de bien connaître $Z(\mathfrak{g})$.

4.2. Soit $S(\mathfrak{g})$ l'algèbre symétrique de $\mathfrak{g}$. Il existe une bijection linéaire β de $S(\mathfrak{g})$ sur $U(\mathfrak{g})$, la *symétrisation,* telle que $n!\,\beta(x_1x_2\cdots x_n)=\sum_{\sigma\in S_n}x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$ quels que soient $x_1,\ldots,x_n\in\mathfrak{g}$ (S_n: groupe symétrique); et β est un isomorphisme de $\mathfrak{g}$-modules pour les représentations adjointes. En particulier, notant $Y(\mathfrak{g})$ l'ensemble des *invariants* de $S(\mathfrak{g})$, on a $\beta(Y(\mathfrak{g}))=Z(\mathfrak{g})$. Si $\mathfrak{g}$ est nilpotente, $\beta|Y(\mathfrak{g})$ est un isomorphisme d'algèbres de $Y(\mathfrak{g})$ sur $Z(\mathfrak{g})$. Pour $\mathfrak{g}$ quelconque, considérons la fonction $x\mapsto(\text{dét}\,(\mathrm{sh}\,\frac{1}{2}\,\mathrm{ad}\,x/\frac{1}{2}\,\mathrm{ad}\,x))^{-1/2}$ sur $\mathfrak{g}$ au voisinage de 0; elle définit un opérateur de multiplication dans l'algèbre de séries formelles $S\hat{}(\mathfrak{g}^*)$ donc par transposition un opérateur différentiel d d'ordre infini dans $S(\mathfrak{g})$. L'application

$u \mapsto d\beta^{-1}(u)$, où u parcourt $Z(\mathfrak{g})$, *est un isomorphisme d'algèbres* de $Z(\mathfrak{g})$ *sur* $Y(\mathfrak{g})$ [12]. Cela ramène l'étude de $Z(\mathfrak{g})$ à celle de $Y(\mathfrak{g})$.

4.3. Il peut arriver que $Z(\mathfrak{g})$ ne soit pas de type fini. Mais supposons $\mathfrak{g}$ semi-simple. Soient $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{g}$, $r = \dim \mathfrak{h}$, W le groupe de Weyl, $S(\mathfrak{h})^W$ l'algèbre des éléments W-invariants de $S(\mathfrak{h})$. Il y a des isomorphismes connus $Y(\mathfrak{g}) \to S(\mathfrak{h})^W$ (Chevalley) et $Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ (Harish-Chandra); ils sont liés à l'isomorphisme $Z(\mathfrak{g}) \to Y(\mathfrak{g})$ de 4.2 par un triangle commutatif. On sait que $S(\mathfrak{h})^W$ est une algèbre de polynômes à r générateurs, donc il en est de même de $Z(\mathfrak{g})$; on sait même décrire dans chaque cas des générateurs explicites de $Z(\mathfrak{g})$.

Les caractères de $S(\mathfrak{h})^W$ s'identifient aux W-orbites dans $\mathfrak{h}^*$. Compte tenu de l'isomorphisme de Harish-Chandra, on obtient une *surjection* $\lambda \mapsto \chi_\lambda$ *de* $\mathfrak{h}^*$ *sur* $Z(\mathfrak{g})\hat{}$, et *une bijection de* $\mathfrak{h}^*/W$ *sur* $Z(\mathfrak{g})\hat{}$.

4.4. Revenons au cas général. Soit $C^\infty(G)$ l'ensemble des fonctions complexes C^∞ sur G. Si $u \in U(\mathfrak{g})$ et $f \in C^\infty(G)$, on a $u*f$, $f*u \in C^\infty(G)$, et $f \mapsto u*f$ (resp. $f*u$) est un opérateur différentiel (OD) invariant à droite D_u (resp. à gauche D'_u) sur G; l'application $u \mapsto D_u$ (resp. D'_u) est un isomorphisme (resp. antiisomorphisme) de l'algèbre $U(\mathfrak{g})$ sur l'algèbre des OD invariants à droite (resp. à gauche) sur G. Si $u \in Z(\mathfrak{g})$, on a $D_u = D'_u$ et $u \mapsto D_u$ est un isomorphisme de $Z(\mathfrak{g})$ sur l'algèbre des OD *biinvariants* sur G (exemple: les OD à coefficients constants sur $\boldsymbol{R}^n$). Grâce à 4.2, on montre que *si* Δ *est un OD biinvariant sur* G, *et si* $f \in C^\infty(G)$, *il existe* $g \in C^\infty(G)$ *telle que* $\Delta g = f$ *au voisinage de* e [12].

5. Modules de Verma. 5.1. On a dit que $U(\mathfrak{g})\hat{}$ est énorme. Mais pour $\mathfrak{g}$ semi-simple, on sait construire des sous-ensembles intéressants de $U(\mathfrak{g})\hat{}$: on définit des «séries» importantes $(\varrho_\lambda)_{\lambda \in \Lambda}$ de rep. de $\mathfrak{g}$, qui sont simples pour les valeurs génériques de λ, et seulement de longueur finie pour certaines valeurs exceptionnelles de λ; même pour ces valeurs, la considération des sous-quotients de ϱ_λ fournit bien entendu des éléments de $U(\mathfrak{g})\hat{}$. Parmi ces séries, citons *les modules de Verma* [26], *les modules de Verma généralisés* [23], *les modules de Whittaker* [21], les modules de Harish-Chandra et parmi eux les *modules de la série principale algébrique* [22] (liés à la série principale de G), les *modules d'Enright-Varadarajan* [13] (liés à la série discrète de G), généralisés par Enright-Wallach [14]. Cf. [15] pour les modules de Harish-Chandra indécomposables sur certaines algèbres $\mathfrak{g}$.

5.2. Parlons seulement des modules de Verma (qui servent d'ailleurs à étudier les autres modules de 5.1). Soient $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{g}$, $\mathfrak{b} \supset \mathfrak{h}$ une sous-algèbre de Borel, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, $\lambda \in \mathfrak{h}^*$, ϱ la demi-somme des racines >0. On prolonge λ par 0 sur $\mathfrak{n}$, d'où une rep. de dimension 1 de $\mathfrak{b}$. Par induction tordue à $\mathfrak{g}$, on obtient le $\mathfrak{g}$-*module de Verma* $M(\lambda)$, qui admet le caractère infinitésimal χ_λ. En tant que $\mathfrak{h}$-module, $M(\lambda)$ est somme directe de sous-espace poids de dimension finie; c'est le module universel parmi les modules engendrés par un vecteur de plus grand poids $\lambda - \varrho$. Avec les notations classiques, $M(\lambda)$ simple $\Leftrightarrow \lambda(H_\alpha) \notin \{1, 2, 3, \ldots\}$ pour toute racine positive α. Donc $M(\lambda)$ est en général simple. Pour les valeurs

exceptionnelles de λ, les multiplicités des sous-quotients simples de $M(\lambda)$ ne sont pas entièrement connues. [7], [17].

5.3. Pour tout λ, $M(\lambda)$ admet un plus grand sous-module distinct de $M(\lambda)$; soit $L(\lambda)$ le module quotient. Quand λ est dominant entier, $L(\lambda+\varrho)$ est le module simple de dimension finie bien connu de plus grand poids λ. La théorie des modules de Verma permet de retrouver bien des résultats classiques: formule des caractères de Weyl, formule de Kostant pour la multiplicité des poids, théorème de Bott-Kostant sur la cohomologie $H^*(\mathfrak{n}, L(\lambda))$, homomorphisme de Harish-Chandra, y compris dans le cas sphérique, etc. [1], [22].

6. Recherche de Prim $U(\mathfrak{g})$ **(suite).** 6.1. Conservons les notations de 4.3. et 5.3. Pour tout $\lambda\in\mathfrak{h}^*$, l'annulateur $J(\lambda)$ de $L(\lambda)$ appartient à Prim $U(\mathfrak{g})$. L'application composée

$$\mathfrak{h}^* \xrightarrow{J} \operatorname{Prim} U(\mathfrak{g}) \xrightarrow{\varphi} Z(\mathfrak{g})\hat{} \xrightarrow{\sim} \mathfrak{h}^*/W$$

est la surjection canonique. *L'application J est surjective* [11].

6.2. L'ensemble Prim $U(\mathfrak{g})$ est donc «intermédiaire» entre $\mathfrak{h}^*$ et $\mathfrak{h}^*/W$. Soient $\lambda\in\mathfrak{h}^*$, et $\lambda\hat{}$ la W-orbite de λ ou le caractère correspondant de $Z(\mathfrak{g})$. Si $\lambda(H_\alpha)\notin\mathbf{Z}$ pour toute racine α, on a $\#\varphi^{-1}(\lambda\hat{})=1$. Mais la situation est beaucoup plus subtile quand par exemple λ est *entier régulier*; on a alors $2^r\leqslant\#\varphi^{-1}(\lambda\hat{})\leqslant i$, où i est le nombre d'involutions de W; ces inégalités sont strictes en général, mais $\#\varphi^{-1}(\lambda\hat{})=i$ quand $\mathfrak{g}=\mathfrak{sl}(n)$ [19]; cf. [19] pour des conjectures quand $\mathfrak{g}\neq\mathfrak{sl}(n)$.

6.3. Fixons λ entier dominant. Quand w parcourt W, l'annulateur de $L(w\lambda)$ parcourt $\varphi^{-1}(\lambda\hat{})$, d'où une surjection $W\to\varphi^{-1}(\lambda\hat{})$. L'inclusion dans $\varphi^{-1}(\lambda\hat{})$ définit par image réciproque un préordre et par suite une relation d'équivalence intéressants sur W [18].

6.4. Supposons $\mathfrak{g}$ simple. L'ensemble Prim $U(\mathfrak{g})$ n'est complètement connu que pour $\mathfrak{g}=\mathfrak{sl}(n)$ ou $\mathfrak{g}$ de rang$\leqslant 2$. Il est presque complètement connu pour $\mathfrak{g}$ de rang 3.

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(Très incomplète; on trouvera dans [9] des références aux mémoires originaux.)

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Extensions of C^*-Algebras and Algebraic Topology

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Ever since the creation of K-theory and its appearance in index theory, one has been aware of the close connection between certain aspects of operator theory and of algebraic topology. In particular, Atiyah [5] and Janich [14] showed that the set of Fredholm operators on a complex separable Hilbert space is a classifying space for complex K-theory. As we know K-theory is a generalized cohomology theory. In this talk I want to report on some results proved over the last five years by myself and others concerning operator theory and K-homology. I do not have time to develop all the motivations but will allow the results to speak for themselves.

We shall consider extensions of C^*-algebras, that is, short exact sequences of the form

$$0 \to \mathscr{A} \to \mathscr{B} \xrightarrow{\varphi} \mathscr{C} \to 0.$$

One says that $\mathscr{B}$ is an extension of $\mathscr{A}$ by $\mathscr{C}$. Now there is not much that one can say in this generality. We shall restrict attention to cases where $\mathscr{A}$ is the C^*-algebra $\mathscr{K}(\mathscr{H})$ of compact operators on the complex separable Hilbert space $\mathscr{H}$. Also we shall require $\mathscr{C}$, and hence also $\mathscr{B}$, to be separable. Two extensions $(\mathscr{B}_1, \varphi_1)$ and $(\mathscr{B}_2, \varphi_2)$ are *strongly equivalent* if there exists a *-isomorphism $\alpha: \mathscr{B}_1 \to \mathscr{B}_2$ such that $\alpha \circ \varphi_1 = \varphi_2$.

There is an equivalent definition of extension obtained by considering the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{K} & \to & \mathscr{B} & \xrightarrow{\varphi} & \mathscr{C} & \to & 0 \\ & & \| & & \cap & & \downarrow \tau & & \\ 0 & \to & \mathscr{K} & \to & \mathscr{L} & \xrightarrow{\pi} & \mathscr{Q} & \to & 0 \end{array}$$

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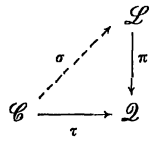
where the Calkin algebra $\mathcal{Q}(\mathcal{H})$ is defined to be the quotient of all bounded linear operators $\mathcal{L}(\mathcal{H})$ by $\mathcal{K}(\mathcal{H})$ and π is the quotient map. By diagram chasing one can show the existence of the *-monomorphism $\tau: \mathcal{C} \to \mathcal{Q}$ and conversely, given τ one can recapture the extension by defining $\mathcal{B} = \pi^{-1}[\tau(\mathcal{C})]$ and $\varphi = \tau^{-1} \circ \pi|_{\mathcal{B}}$. Two *-monomorphisms $\tau_1, \tau_2: \mathcal{C} \to \mathcal{Q}$ are strongly equivalent if there exists a unitary U on $\mathcal{H}$ such that $\tau_1 = \alpha_{\pi(U)} \circ \tau_2$, where $\alpha_{\pi(U)}$ is the automorphism defined in $\mathcal{Q}$ by conjugation by $\pi(U)$. In this context it might be more natural to allow inner automorphisms. We call the resulting notion simply *equivalence* and let $\mathrm{Ext}(\mathcal{C})$ denote the collection of equivalence classes. For $\mathcal{C}$ commutative the two notions of equivalence coincide. One could allow all automorphisms of $\mathcal{Q}$ in defining equivalence, but it is not known whether $\mathcal{Q}$ has outer automorphisms.

An extension $\tau: \mathcal{C} \to \mathcal{Q}$ is said to be trivial if it lifts, that is, if there exists $\sigma: \mathcal{C} \to \mathcal{L}$ such that the following diagram commutes:



We define addition on $\mathrm{Ext}(\mathcal{C})$ such that for τ_1, τ_2 in $\mathrm{Ext}(\mathcal{C})$, we let $[\tau] = [\tau_1] + [\tau_2$ where

$$\tau(f) = \tau_1(f) \oplus \tau_2(f) \in \mathcal{Q}(\mathcal{H}) \oplus \mathcal{Q}(\mathcal{H}) = \mathcal{Q}(\mathcal{H} \oplus \mathcal{H}).$$

One easily shows that $\mathrm{Ext}(\mathcal{C})$ is a commutative semigroup. Moreover, we have]

THEOREM. *For $\mathcal{C}$ a separable C^*-algebra, $\mathrm{Ext}(\mathcal{C})$ is a commutative semigroup in which trivial extensions define a unique element which acts as the additive identity.*

For $\mathcal{C} = C(X)$ and $X \subset \boldsymbol{R}$, this is essentially the Weyl–von Neumann theorem [30], [20], while for $X \subset \boldsymbol{C}$ it is Berg's generalization [8]. For $\mathcal{C} = C(X)$, X compact metrizable, this result was proved by Brown, Fillmore and myself in [10] and in [11], [12] we extended it to a class including the n-homogeneous C^*-algebras. A further extension was obtained by Pearcy and Salinas [22], [23]. Finally, in a surprising paper [29] Voiculescu proved this result for general separable C^*-algebras. A nice exposition of his proof was given by Arveson [4].

Voiculescu obtains many results in operator theory and the theory of C^*-algebras as a consequence of his work. We offer two examples. First, he answered a question of Halmos by showing that the reducible operators are norm-dense in $\mathcal{L}(\mathcal{H})$. Secondly, he showed that the double commutant theorem holds for separable C^*-subalgebras of $\mathcal{Q}$, where one means relative commutant.

On the basis of the examples of Ext which we were able to compute in our early work, Brown, Fillmore and I began to suspect $\mathrm{Ext}(C(X))$ might be a group for general compact metrizable X and finally by quite indirect means, we established this in [10]. A direct proof of this was given by Arveson [3] who observed by using a dilation theorem of Naimark that to show $\tau: \mathcal{C} \to \mathcal{Q}$ had an inverse it was sufficient to exhibit a completely positive lifting $\sigma: \mathcal{C} \to \mathcal{L}$ such that $\tau = \pi \circ \sigma$. (Recall a map $\alpha: \mathcal{A} \to \mathcal{B}$ is said to be completely positive if $\alpha \otimes 1: \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$ is

positive for each n.) For $\mathscr{C}=C(X)$ this lifting was supplied by results of Andersen [1], and Vesterstrøm [28]. More recently, Choi and Effros [13] established this lifting result for $\mathscr{C}$ a separable nuclear C^*-algebra. In [4] Arveson also gave an improved exposition of this result. Thus we have by virtue of these results

THEOREM. *For $\mathscr{C}$ a separable nuclear C^*-algebra,* Ext $(\mathscr{C})$ *is an abelian group.*

Unfortunately, Ext $(\mathscr{C})$ is not always a group. Anderson has produced an example [2] of a C^*-algebra $\mathscr{C}_0$ closely related to the free group on two generators such that Ext $(\mathscr{C}_0)$ is not a group. He exhibits a *-monomorphism $\tau: \mathscr{C}_0 \to \mathscr{Q}$ having no completely positive lifting.

In [12] Brown, Fillmore and I showed further that $X \mapsto \text{Ext}(C(X))$ defines a homotopy invariant covariant factor. Our proof involved considerable use of techniques from algebraic topology. More operator theoretic proofs of homotopy invariance have been given by O'Donovan [21] and Salinas [25] with the latter covering the case of Ext $(\mathscr{C})$ for $\mathscr{C}$ a separable quasi-diagonal nuclear C^*-algebra. Finally, in a recent paper by Pimsner, Popa, and Voiculescu [24] prove

THEOREM. *The correspondence $\mathscr{C} \mapsto$* Ext $(\mathscr{C})$ *is a contravariant homotopy functor rom the category of separable type* I C^**-algebras to the category of abelian groups.*

By homotopy invariance we mean the following: If $\varphi: [0, 1] \times \mathscr{C}_1 \to \mathscr{C}_2$ is continuous and a homomorphism for each t in $[0, 1]$, then the maps φ_0^*, φ_1^*: Ext $(\mathscr{C}_2) \to$ Ext $(\mathscr{C}_1)$ are equal. This agrees with the usual notion when $\mathscr{C}_1$ and $\mathscr{C}_2$ are commutative.

We now discuss one of the applications. An operator T on $\mathscr{H}$ is said to be essentially normal if its self-commutator $[T, T^*]=TT^*-T^*T$ is compact. One seeks to classify such operators up to unitary equivalence modulo the compacts and to indicate the possibilities. A necessary condition for a unitary U to exist such that $U^*T_1U-T_2$ is compact is for $\sigma_e(T_1)=\sigma_e(T_2)$, where the essential spectrum $\sigma_e(T)$ is defined $\sigma_e(T)=\sigma_{\mathscr{Q}}(\pi(T))$. The theorem of Weyl–von Neumann–Berg shows that for $T_i=N_i+K_i$, $i=1, 2$, N_i normal and K_i compact, that this condition is also sufficient. For general essentially normal operators the Fredholm index is also important. Recall that an operator T is said to be Fredholm if $\pi(T)$ is invertible in $\mathscr{Q}$ and the index of T is defined by ind $(T)=\dim \ker T-\dim \ker T^*$.

THEOREM. *For essentially normal operators T_1 and T_2 on $\mathscr{H}$, there exists a unitary operator U such that $U^*T_1U-T_2$ is compact if and only if* (1) $\sigma_e(T_1)=\sigma_e(T_2)$ *and* (2) ind $(T_1-\lambda)=$ ind $(T_2-\lambda)$ *for λ in $\mathbf{C}\setminus\sigma_e(T_i)$. Moreover, for each compact subset X of $\mathbf{C}$ and continuous function $m: \mathbf{C}\setminus X \to \mathbf{Z}$ there exists an essentially normal operator T such that $\sigma_e(T)=X$ and* ind $(T-\lambda)=m(\lambda)$ *for λ in $\mathbf{C}\setminus X$.*

This result was proved in [10] as a consequence of our characterization of Ext (X) for $X \subset \mathbf{C}$. The relation between Ext and essentially normal operators is established as follows.

If T is essentially normal, then the C^*-algebra $\mathscr{E}_T$ generated by T and $\mathscr{K}$

gives rise to an extension, since $\mathscr{E}_T/\mathscr{K} \cong C(\sigma_e(T))$ by the spectral theorem. The theorem on essentially normal operators follows from the fact that there exists an isomorphism γ_1: Ext $(X) \to$ Hom $(\pi^1(X), \boldsymbol{Z})$ for $X \subset \boldsymbol{C}$. Actually, this is one aspect of the relation between Ext and K-theory and for general X one considers γ_∞: Ext $(X) \to$ Hom $(K^1(X), \boldsymbol{Z})$ which is onto and has kernel equal to the "torsion subgroup". We also show in [12] that Ext (X) is a $K^0(X)$-module. If one defines higher Ext groups using suspension, then one obtains

THEOREM. Ext *defines a generalized Steenrod homology theory* E_* *of period two which is dual to K-theory.*

The fact that the homology theory has period two requires proving a periodicity theorem analogous to Bott periodicity for K-theory. This theorem extends to the category of separable type I C^*-algebras using the results of Pimsner, Popa, and Voiculescu [24].

The relation of Ext to K-theory can be used to solve a problem posed by Atiyah in [6], where he considered a generalization of the notion of elliptic operator. For X a finite complex he let Ell (X) denote the collection of triples (σ_1, σ_2, T), where σ_1, σ_2: $C(X) \to \mathscr{L}(\mathscr{H}_1)$, $\mathscr{L}(\mathscr{H}_2)$ are *-homorphisms and T: $\mathscr{H}_2 \to \mathscr{H}_1$ is a Fredholm operator for which $\sigma_1(f)T - T\sigma_2(f)$ is compact for each f in $C(X)$. Addition is defined by direct sum and after defining a pairing of Ell (X) with K-theory, Atiyah established an onto map Ell $(X) \to K_0(X)$, where $K_0(X)$ is defined using Spanier–Whitehead duality. Atiyah asked that the equivalence relation on Ell (X) defined by this map be determined.

Assuming $\sigma_1 = \sigma_2$ (which one can reduce to) let $T = UH$ be the polar decomposition; then setting $\mathscr{E}$ equal to the C^*-algebra generated by the range of σ and U, we obtain an element of Ext (Y), where $Y \subset X \times S^1$, which projects to 0 in Ext (X). Such an element lies in the zeroth homology group $E_0(X)$ and hence we have the commutative diagram

$$\begin{array}{ccc} \text{Ell}(X) & & \\ \downarrow & \searrow & \\ E_0(X) & \longrightarrow & K_0(X) \end{array}$$

We show in [12] that the bottom line is an isomorphism and therefore we can answer Atiyah's question using properties of equivalence on Ext.

In [18], [19] Kasparov also solved Atiyah's problem in a related way. In our terminology Kasparov considers for arbitrary $\mathscr{C}$ the maps τ: $\mathscr{C} \to \mathscr{Q}$ which have inverses and he allows homotopy as well as conjugation in his equivalence relation. He also obtains a generalized homology theory and establishes periodicity. He further claims applications to Novikov's higher homotopy conjectures. Kasparov considers the real and quaternionic case as well as the complex and allows the action of a compact group. He makes use of indexing by Clifford algebras in the manner of Karoubi [17].

More precise results on the relation of Ext to K-theory have been obtained by

Kahn, Kaminker and Schochet [16], [15]. They show that for every cohomology theory defined by spectra, there is associated a canonical Steenrod generalized homology theory on the category of compact metrizable spaces and that if one starts with K-theory, then the homology theory defined by Ext is obtained.

We conclude with an index theorem in the context of Ext first formulated by Singer [26] and a couple of specific examples. Recall that for X compact the Chern homomorphism is defined $\mathrm{ch}^*\colon K^1(X)\to H^{\mathrm{odd}}(X, Q)$. Analogously, there is a homomorphism defined $\mathrm{ch}_*\colon \mathrm{Ext}(X)\to H_{\mathrm{odd}}(X, Q)$ (cf. [16]). Thus for $(\mathscr{E}, \varphi)$ in $\mathrm{Ext}(X)$ there is an "index class" $\mathrm{ch}_*(\mathscr{E}, \varphi)$ in $H_{\mathrm{odd}}(X, Q)$ for which

$$\mathrm{ind}(T) = \gamma_\infty(\mathscr{E}, \varphi)(\pi(T)) = [\mathrm{ch}_*(\mathscr{E}, \varphi)\cap \mathrm{ch}^*(\pi(T))]$$

for T in $\mathscr{E}\otimes M_n(\boldsymbol{C})$, where [ ] denotes the natural pairing between homology and cohomology. If X is an oriented connected n-dimensional manifold, then we reformulate using Poincaire duality $D\colon H^i(X, Q)\to H_{n-i}(X, Q)$ to obtain:

THEOREM. *If X is an oriented connected compact manifold and $\mathscr{I}(\mathscr{E}, \varphi) = D^{-1}(\mathrm{ch}_*(\mathscr{E}, \varphi))$ is the index class in $H^*(X, Q)$, then*

$$\mathrm{ind}(T) = [\mathscr{I}(\mathscr{E}, \varphi)\cup \mathrm{ch}^*(\pi(T))][X]$$

for T in $\mathscr{E}\otimes M_n(\boldsymbol{C})$.

To use this result one must calculate $\mathscr{I}(\mathscr{E}, \varphi)$ concretely for explicit extensions. There are two general cases where this has been done. For M a smooth compact manifold, the pseudo differential operators of order 0 define bounded operators on the L^2 space of some fixed smooth measure. The C^*-algebra $\mathscr{P}_M$ generated by these operators and $\mathscr{K}$ together with the classical symbol map σ_M determine an extension in $\mathrm{Ext}(\mathrm{Sph}^*(M))$, where $\mathrm{Sph}^*(M)$ denotes the cosphere bundle of M. The Atiyah–Singer index theorem [7] yields that $\mathscr{I}(\mathscr{P}_M, \sigma_M)$ is the Todd class in $H^*(\mathrm{Sph}^*(M), Q)$. For Ω a strongly pseudo convex domain in a Stein manifold one considers the compression of the continuous multipliers on $\bar{\Omega}$ to the closure of the analytic functions to L^2 of volume measure. The resulting C^*-algebra $\mathscr{T}_\Omega$ generated by these operators and $\mathscr{K}$ determines an element of $\mathrm{Ext}(\partial\Omega)$, closely related to Toeplitz operators. Again the index class $\mathscr{I}(\mathscr{T}_\Omega)$ is the Todd class. This was proved for $\Omega = B^n\subset \boldsymbol{C}^n$ by Venugopalkrishna [27] and in full generality by Boutet de Monvel [9].

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Infinite Dimensional Groups, Their Representations, Orbits, Invariants

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1. Representation theory for infinite dimensional groups does not exist as a theory although such groups occur long ago in several branches of mathematics and its applications. Among the most important examples are:

(a) groups of automorphisms of infinite dimensional vector spaces with some additional structures (unitary, symplectic, Fredholm etc.);

(b) groups of diffeomorphisms of smooth manifolds (also with extra structures);

(c) continuous products of finite-dimensional groups (so called "current groups" in mathematical physics);

(d) gauge groups in field theory; usually they are constructed as semidirect products of groups of the two preceding types.

2. One gets a possible approach to representation theory of infinite-dimensional groups by applying the method of orbits. All groups mentioned above have a natural structure of infinite-dimensional Lie group G. Hence there is a corresponding Lie algebra $\mathfrak{g}$, the dual space $\mathfrak{g}^*$ and the coadjoint representation of G in $\mathfrak{g}^*$.

In the cases investigated until now (there are rather few of them) one observes a satisfactory correspondence between unitary irreducible representations of G and its orbits in $\mathfrak{g}^*$. This correspondence is especially clearly visible in the case of orbits of finite dimension of finite codimension.

3. Orbits of finite dimension in $\mathfrak{g}^*$ correspond to those unitary irreducible representations of G which have finite functional dimension, i.e. can be naturally realized in spaces of functions of a finite number of variables. (A variant of the exact definition of functional dimension can be done in terms of the image of $\mathfrak{g}$ and its enveloping algebra $U(\mathfrak{g})$ under the given representation.)

The classification of all finite-dimensional orbits for some groups of types (c) and (d) as well as the description of the corresponding representations were obtained in [1], [2]. We remark an interesting phenomenon which makes representation theory of infinite-dimensional groups somewhat similar to the classical character theory of abelian groups; the product of two irreducible representations is, as a rule, also irreducible.

4. Much more difficult and more interesting case is that of orbits of infinite dimension. The corresponding representations have infinite functional dimension. The analysis of known examples shows that in the construction of such representations the role of the "universal object" is played by the representation of the (inhomogeneous) isometry group of a Hilbert space H in the Vock space $\exp M$ (see [3], for example). In contrast, the method of induced representations—the main tool in the representation theory for locally compact groups—does not work in the infinite dimensional situation due to the absence of quasiinvariant measures (see, however [4]).

5. An interesting class of infinite-dimensional orbits which can be thoroughly investigated arises in representation theory of certain groups of type (a). Namely, let G be one of the Banach–Lie group $O(p, \infty)$, $U(p, \infty)$, $\mathrm{Sp}(p, \infty)$ (i.e. the group of all bounded invertible operators in a Hilbert space over $\boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$ which preserve an indefinite hermitian form with p negative squares and differ from 1 (unity operator) by a compact one).

Here $\mathfrak{g}^*$ is the space of all skew hermitian operators of trace class and the collection of all integral G-orbits in $\mathfrak{g}^*$ can be easily described.

In the representation theory this circumstance is reflected by the following unexpected fact: the group G is tame (belongs to the type I in the Dixmier–von Neumann sense) and the dual object $\hat{G}$ admits an explicit description. For the case $p=0$ this result was obtained in [5], the general case is treated in [6], where the corresponding motion groups are also considered.

6. The study of the relation between orbits and representations involves many interesting and difficult problems of different kinds. In particular, the definition of the natural symplectic structure on the orbits leads to the notion of local Lie algebra.

Let E be a vector bundle over the smooth manifold M. By a local Lie algebra we mean a Lie algebra L such that: (1) the underlying vector space is $\Gamma^\infty(E)$—the set of all smooth sections of E over M; (2) the Lie operation is bicontinuous in the C^∞-topology on $\Gamma^\infty(E)$ and (3) is local in the natural sense: $\mathrm{supp}\,[s_1, s_2] \subset \mathrm{supp}\, s_1 \cap \mathrm{supp}\, s_2$.

It turns out that for the linear bundles (=vector bundles with one-dimensional fiber) the complete classification of local Lie algebras can be obtained [7]. In fact, every local Lie algebra splits into the so called transitive ones and every transitive local Lie algebra is determined up to an automorphism of E in the neighbourhood

of the given point by dimension of M. For even dimensional manifolds this is the Poisson algebra and for odd dimensional manifolds—the Lie algebra of contact (Lagrange) brackets. This result explains the necessity of the appearance of the symplectic and contact geometry in the mathematical models of mechanics.

The decomposition of $\mathfrak{g}^*$ on G-orbits illustrates the general theorem about the decomposition of a local Lie algebra into transitive ones. Many interesting questions about local Lie algebras (for instance, their stability) are not answered yet.

7. The main result of the preceding section says that each Lie algebra structure on the function space is induced by the natural bracket operation on vector fields. This fact suggests to rise once more the old problem of the classification of natural operations on geometric objects (in classical terms—of finding of all differential concomitants). The algebraic formulation reduces this problem to looking for the invariants of some non-semisimple Lie groups (that of jets of diffeomorphisms) acting on certain category of modules.

All unary linear operations are now known ([9], [11]); there are essentially only two of them: the multiplication by a scalar and the exterior derivative of a differential form.

As for binary bilinear operation, their list was essentially replenished by J. Schouten and A. Nijenhuis (see [8]) and now contains five types of operations ([13]): (1) Lie operator and the "Lagrange concomitant" of Schouten, (2) symmetric concomitant of Schouten (=Poisson bracket for function on the cotangent bundle which is polynomial on each fibre), (3) generalized commutator of k-vector densities, (4) Nijenhuis' bracket for vector-valued differential forms, (5) binary operation arising from unary ones and the tensor product. Some recent results of A. N. Rudakov [10], P. Ja. Grosman [11] and the author [13] give a hope that the complete classification of bilinear invariant operations over tensor fields will be obtained in the near future. The list given above is complete in the case $n=2$ [11].

Rather interesting combinatorial problems arise in the study of multilinear invariant operations on the real line. For example, the invariant n-linear operations on functions correspond to the polynomial solutions of the following system of partial differential equation (which of course can be reduced to the linear algebraic system): $L_k P=0$, $k=1, 2, \ldots$, where $L_k=\sum_{\alpha=1}^{n} x_\alpha \partial^{k+1}/\partial x_\alpha^{k+1}$.

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Recent Developments in the Theory of Unbounded Derivations in C^*-Algebras

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1. Introduction. The study of derivations in C^*-algebras is making great strides in recent years. It is divided into two major steps. The first is for bounded derivations; the second for unbounded derivations with dense domain. At the early stage, mathematicians devoted their effort to the study of bounded derivations. This is understandable, because bounded derivations can be more easily handled than unbounded ones so that one can expect a beautiful theory as mathematics and also knowledge on bounded derivations may contribute to the study of unbounded derivations. In fact, the study of bounded derivations is now attaining maturity. On the other hand, the study of unbounded derivations occurred much later and was initially motivated by the problem of the construction of dynamics in statistical mechanics. Soon it became apparent that the work of Šilov [**26**] also has strong influence on the study of unbounded derivations in commutative C^*-algebras. For bounded derivations, the main theme is when they are inner. On the other hand, for unbounded ones, it is rich in variety, because they are closely related to dynamical systems in quantum physics and differentiations in manifolds.

In this, I would like to give a very brief survey of recent developments in the theory of unbounded derivations in C^*-algebras. Let $\mathfrak{A}$ be a C^*-algebra. A linear mapping δ in $\mathfrak{A}$ is said to be a *-derivation in $\mathfrak{A}$ if it satisfies the following conditions: (1) The domain $\mathscr{D}(\delta)$ of δ is a dense *-subalgebra of $\mathfrak{A}$; (2) $\delta(ab)=\delta(a)b+a\delta(b)$ $(a, b\in\mathscr{D}(\delta))$; (3) $\delta(a^*)=\delta(a)^*$ $(a\in\mathscr{D}(\delta))$. If $\mathscr{D}(\delta)=\mathfrak{A}$, then δ is closed so that by the closed graph theorem it is bounded [**20**]. Therefore the study of everywhere defined derivations on $\mathfrak{A}$ is equivalent to the study of bounded derivations.

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2. Closability. If $\mathscr{D}(\delta)\subsetneqq\mathfrak{A}$, then δ is not necessarily closable **[5]**. An element x of the self-adjoint portion $\mathscr{D}(\delta)^S$ of $\mathscr{D}(\delta)$ is said to be well-behaved if there is a state φ_x on $\mathfrak{A}$ such that $|\varphi_x(x)|=\|x\|$ and $\varphi_x(\delta(x))=0$. Let $W(\delta)$ be the set of all well-behaved elements; then it is dense in $\mathfrak{A}^S$. δ is said to be well-behaved (quasi well-behaved) if $W(\delta)=\mathscr{D}(\delta)^S$ (the interior $W(\delta)^0$ is dense in $\mathscr{D}(\delta)^S$). Any (quasi) well-behaved *-derivation is closable and its closure is again (quasi) well-behaved **[3]**, **[11]**, **[25]**. δ is well-behaved if the positive portion of $\mathscr{D}(\delta)$ is closed under the square root operation **[19]**. A closed *-derivation is bounded if the positive portion is closed under the square root operation **[15]**. δ is well-behaved if there is a sequence of self-adjoint elements (h_n) in $\mathfrak{A}$ such that $\delta(x)=\lim i\,[h_n, x]$ $(x\in\mathscr{D}(\delta))$ **[18]**. One can easily see that the (infinitesimal) generator δ of a strongly continuous one-parameter group of *-automorphisms on $\mathfrak{A}$ is always well-behaved and the differentiation d/dt on the unit interval is quasi well-behaved.

3. Domain of closed derivations. In mathematical physics, unbounded derivations are often defined by Hamiltonians. In those cases, it is easily seen that the derivations are closable. If $\mathfrak{A}$ is commutative, δ is closed and $f\in C^1(R)$ (continuously differentiable), then $f(a)\in\mathscr{D}(\delta)$ for $a\in\mathscr{D}(\delta)^S$. For a noncommutative $\mathfrak{A}$, one has to replace $C^1(R)$ by $C^2(R)$ **[6]**, **[14]**, **[17]**.

4. Differentiations. Let I be the unit interval and $\delta_0=d/dt$. If δ is a derivation in the commutative C^*-algebra $C(I)$ such that $\mathscr{D}(\delta)=C^n(I)$ (n-times continuously differentiable) for some n $(n=\infty, 1, 2, 3, \ldots)$, then there is a unique continuous function λ on I such that $\delta=\lambda\delta_0$. In particular, δ is closable **[25]**. Suppose that δ_0, δ are two *-derivations in $C(K)$ (K, a compact Hausdorff space), δ_0 is closed and $\mathscr{D}(\delta)=\mathscr{D}(\delta_0)$ or $\bigcap_{n=1}^{\infty}\mathscr{D}(\delta_0^n)$; then there is a unique continuous function λ on K such that $\delta=\lambda\delta_0$. In particular, δ is closable **[3]**. It is an open question whether the result can be extended to $n=2, 3, \ldots$.

CONJECTURE. Let δ_0 be a closed *-derivation in $\mathfrak{A}$ and let δ be a *-derivation in $\mathfrak{A}$ with $\mathscr{D}(\delta)=\mathscr{D}(\delta_0)$. Then δ is closable. It would be an interesting problem to study the relationship between closed *-derivations in $C(K)$ and differential structures in K.

5. Generators. Let $\{\varrho(t)\}$ $(t\in R)$ be a strongly continuous one-parameter group of *-automorphisms on $\mathfrak{A}$ with identity. The system $\{\mathfrak{A}, \varrho(t)\}$ is said to be a C^*-dynamics. It is said to be approximately inner if there is a sequence of uniformly continuous one-parameter groups $\{\varrho_n(t)\}$ of inner *-automorphisms on $\mathfrak{A}$ such that $\|\varrho_n(t)(a)-\varrho(t)(a)\|\to 0$ uniformly on every compact subset of R for each fixed $a\in\mathfrak{A}$. We shall denote this by strong $\lim \varrho_n(t)=\varrho(t)$. All C^*-dynamics appearing in quantum lattice systems and Fermion field theory are approximately inner. In mathematical physics, we are often concerned with a C^*-algebra $\mathfrak{A}$ containing an identity and an increasing sequence of C^*-subalgebras $\{\mathfrak{A}_n\}$ of $\mathfrak{A}$ such that $1\in\mathfrak{A}_n$ and the uniform closure of $\bigcup_{n=1}^{\infty}\mathfrak{A}_n$ is $\mathfrak{A}$. In addition, we are given a

*-derivation δ in $\mathfrak{A}$ satisfying the following conditions: (1) $\mathscr{D}(\delta)=\bigcup_{n=1}^{\infty}\mathfrak{A}_n$; (2) there is a sequence of selfadjoint elements (h_n) in $\mathfrak{A}$ such that $\delta(a)=i[h_n, a]$ $(a\in\mathfrak{A}_n)$. We shall call such a derivation a normal *-derivation in $\mathfrak{A}$. If a normal *-derivation δ satisfies the following conditions: (1) $h_n\in\mathfrak{A}_{n+p}$ for a fixed p; (2) there is an element k_n in $\mathfrak{A}_n$ such that $\|h_n-k_n\|=O(n)$, then the closure $\bar{\delta}$ of δ is a generator and strong $\lim \exp t\delta_{ih_n}=\exp t\bar{\delta}$, where $\delta_{ih_n}(x)=i[h_n, x]$ $(x\in\mathfrak{A})$ [4]. (This includes the case of two-dimensional quantum lattice systems with finite range interaction.) If a normal *-derivation δ satisfies $\|h_n-k_n\|=O(1)$ for some $k_n\in\mathfrak{A}_n$, then $\bar{\delta}$ is a generator and strong $\lim \exp t\delta_{ih_n}=\exp t\bar{\delta}$ [11]. (This includes the case of one-dimensional lattice systems with bounded surface energy.) A normal *-derivation δ is said to be commutative if one can choose the sequence (h_n) such that $h_nh_m=h_mh_n$ $(m, n=1, 2, 3, \ldots)$. If δ is commutative, then it has an extension δ_1 such that δ_1 is a generator, strong $\lim \exp t\delta_{ih_n}=\exp t\delta_1$, and moreover $\exp t\delta_1(a)=\exp t\delta_{ih_n}(a)$ $(a\in\mathfrak{A}_n)$ [21]. This result is applicable to all classical lattice systems. There are generalizations of these results to dissipative operators in Banach spaces [3], [4], [9].

6. Ground states. We introduce a class of states of some importance in quantum physics. Let $\{\mathfrak{A}, \varrho(t)\}$ be a C^*-dynamics and let δ be the generator of $\{\varrho(t)\}$. A state φ on $\mathfrak{A}$ is said to be a ground state for $\{\varrho(t)\}$ if $-i\varphi(a^*\delta(a))\geqslant 0$ for $a\in\mathscr{D}(\delta)$. A ground state is invariant under $\{\varrho(t)\}$. If a C^*-dynamics is approximately inner, then it has a ground state [18]. There is a nontrivial example of a C^*-dynamics without ground state [13].

7. KMS states. We introduce another class of states on a C^*-dynamics which is important in quantum physics. For a real number β, a state φ_β on $\mathfrak{A}$ is said to be a KMS state for $\{\mathfrak{A}, \varrho(t)\}$ at inverse temperature β if for $a, b\in\mathfrak{A}$, there is a bounded continuous function $F_{a,b}$ on the strip $0\leqslant \mathrm{Im}\,(z)\leqslant\beta$ $(0\geqslant \mathrm{Im}\,(z)\geqslant\beta)$ in the complex plane which is analytic on $0<\mathrm{Im}\,(z)<\beta$ $(0>\mathrm{Im}\,(z)>\beta)$ so that $F_{a,b}(t)=\varphi_\beta(a\varrho(t)(b))$ and $F_{a,b}(t+i\beta)=\varphi_\beta(\varrho(t)(b)a)$. The KMS condition gives every evidence of being the abstract formulation of the condition for equilibrium of a state [8]. A KMS state is invariant under $\{\varrho(t)\}$. If $\{\mathfrak{A}, \varrho(t)\}$ is approximately inner and $\mathfrak{A}$ has a tracial state, then it has a KMS state at each inverse temperature β $(-\infty<\beta<+\infty)$ [18]. If a C^*-dynamics $\{\mathfrak{A}, \exp t\delta\}$ has a KMS state φ_β at β and there is a sequence of bounded *-derivations (δ_n) on $\mathfrak{A}$ such that $\delta_n(a)\to\delta(a)$ for $a\in\mathscr{D}$, where $\mathscr{D}$ is a dense subset of $\mathscr{D}(\delta)$, then $\{\mathfrak{A}, \exp t\delta\}$ has a tracial state [10], [12]. There is a C^*-dynamics $\{\mathfrak{A}_n, \varrho_n(t)\}$ which has a KMS state at $\beta=\log n$ only $(n=2, 3, \ldots)$ [16]. Let φ_{β_n} be a KMS state at β_n for $\{\mathfrak{A}, \varrho(t)\}$ and $\beta_n\to\beta$; then any accumulation point of $\{\varphi_{\beta_n}\}$ is a KMS state for $\{\mathfrak{A}, \varrho(t)\}$ at β; if $\beta_n\to\infty$, then any accumulation point of $\{\varphi_{\beta_n}\}$ is a ground state for $\{\mathfrak{A}, \varrho(t)\}$.

8. UHF algebras. A C^*-algebra $\mathfrak{A}$ is said to be a UHF algebra if there is an increasing sequence $\{\mathfrak{A}_n\}$ of finite type I-subfactors such that $1\in\mathfrak{A}_n$ and the

uniform closure of $\bigcup_{n=1}^{\infty}\mathfrak{A}_n=\mathfrak{A}$. Such algebras are appearing in quantum lattice systems and Fermion field theory. Let δ be a closed *-derivation in a UHF algebra $\mathfrak{A}$; then there is an increasing sequence $\{\mathfrak{A}_n\}$ of finite type I-subfactors in $\mathscr{D}(\delta)$ such that $1\in\mathfrak{A}_n$ and $\bigcup_{n=1}^{\infty}\mathfrak{A}_n$ is dense in $\mathscr{D}(\delta)$ [5], [21]. We define a normal *-derivation in a UHF algebra more restrictively than general cases. A *-derivation δ in a UHF algebra $\mathfrak{A}$ is said to be normal if there is an increasing sequence $\{\mathfrak{A}_n\}$ of finite type I-subfactors such that $1\in\mathfrak{A}_n$ and $\mathscr{D}(\delta)=\bigcup_{n=1}^{\infty}\mathfrak{A}_n$. Let δ be a normal *-derivation and let $\{e_{ij}^n|i,j=1,2,\ldots,p_n\}$ be a matrix unit of $\mathfrak{A}_n$. Set $ih_n=\sum_{j=1}^{p_n}\delta(e_{j1}^n)e_{1j}$; then $\delta(a)=i[h_n,a]$ $(a\in\mathfrak{A}_n)$. It is easily seen that h_n is a self-adjoint element. All derivations appearing in quantum lattice systems and all quasi free derivations in the canonical anticommutation relation algebra have normal *-derivations as their cores. Let δ be a generator; then there is a normal *-derivation $\tilde{\delta}$ such that $\tilde{\delta}\subset\delta$ and $\mathscr{D}(\tilde{\delta})$ is contained in the *-subalgebra $A(\delta)$ of all analytic elements with respect to δ [21].

CONJECTURE. Any C^*-dynamics $\{\mathfrak{A},\varrho(t)\}$ with a UHF algebra $\mathfrak{A}$ is approximately inner.

PROBLEM. Let δ be a generator in a UHF algebra. Then can we find a normal *-derivation $\tilde{\delta}$ such that $\tilde{\delta}$ is the core of δ (i.e., the closure of $\tilde{\delta}=\delta$)?

9. Bounded perturbations. Let δ be a normal *-derivation in a UHF algebra $\mathfrak{A}$ with $\mathscr{D}(\delta)=\bigcup_{n=1}^{\infty}\mathfrak{A}_n$. Then for $\varepsilon>0$, there is a normal *-derivation δ_ε such that $\mathscr{D}(\delta)=\mathscr{D}(\delta_\varepsilon)$, $\delta_\varepsilon(\mathscr{D}(\delta))\subset\mathscr{D}(\delta)$ and $\delta-\delta_\varepsilon$ is a bounded *-derivation with $\|\delta-\delta_\varepsilon\|<\varepsilon$ [7], [25]. This implies that an infinite range interaction may move to a finite range interaction by bounded perturbations. Next suppose that $\delta(\mathscr{D}(\delta))\subset\mathscr{D}(\delta)$; then by choosing a suitable subsequence of $\{\mathfrak{A}_n\}$, we may assume that $\delta(\mathfrak{A}_n)\subset\mathfrak{A}_{n+1}$. Then we have the following decomposition: $\delta=\delta_1+\delta_2$, where δ_1,δ_2 are normal *-derivations with $\mathscr{D}(\delta)=\mathscr{D}(\delta_1)=\mathscr{D}(\delta_2)$, $\delta_1(\mathfrak{A}_{2n})\subset\mathfrak{A}_{2n}$ and $\delta_2(\mathfrak{A}_{2n+1})\subset\mathfrak{A}_{2n+1}$. In particular, δ_1 and δ_2 are commutative normal *-derivations [7], [25]. Generally, let $\{\mathfrak{A},\exp t\delta\}$ be a C^*-dynamics and let δ_0 be a bounded *-derivation on $\mathfrak{A}$; then a C^*-dynamics $\{\mathfrak{A},\exp t(\delta+\delta_0)\}$ has a ground state (a KMS state at β) if and only if $\{\mathfrak{A},\exp t\delta\}$ has a ground state (a KMS state at β) [1].

10. Phase transition. Let $\{\mathfrak{A},\varrho(t)\}$ be a C^*-dynamics. Suppose that it has a KMS state at every β $(-\infty<\beta<+\infty)$. If it has only one KMS state at β, then we say that the dynamics has no phase transition at β. If it has at least two KMS states at β, then we say that it has phase transition at β. If $\{\mathfrak{A},\exp t\delta\}$ has no phase transition at β, then $\{\mathfrak{A},\exp t(\delta+\delta_0)\}$ has no phase transition at β for a bounded *-derivation δ_0 [1]. Let δ be a normal *-derivation in a UHF algebra with $\mathscr{D}(\delta)=\bigcup_{n=1}^{\infty}\mathfrak{A}_n$. Let P_n be the canonical conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}_n$ such that $\tau(xa)=\tau(P_n(x)a)$ $(a\in\mathfrak{A}_n)$, where τ is the unique tracial state on $\mathfrak{A}$. Let (h_n) be a sequence of self-adjoint elements in $\mathfrak{A}$ such that $\delta(a)=i[h_n,a]$ $(a\in\mathfrak{A}_n)$.

Then, if $\|h_n - P_n(h_n)\| = O(1)$, then δ is a pregenerator and the approximately inner C^*-dynamics $\{\mathfrak{A}, \exp t\delta\}$ has no phase transition at every β $(-\infty < \beta < +\infty)$ [2], [11], [23], [24]. This implies that a quantum lattice system with bounded surface energy has no phase transition at every β and quasi free C^*-dynamics in the canonical anticommutation relation algebra has no phase transition at every β. To develop the theory of phase transition for normal *-derivations more deeply, we need to find how to construct all KMS states at each β. For commutative normal *-derivations, we have a fairly detailed description of the construction of all KMS states at β. Let δ be a commutative normal *-derivation in a UHF algebra $\mathfrak{A}$ such that $\delta(\mathscr{D}(\delta)) \subset \mathscr{D}(\delta)$. Then there is a sequence (h_n) of self-adjoint elements in $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ such that $\delta(a) = i[h_n, a]$ $(a \in \mathfrak{A}_n)$ and $h_n h_m = h_m h_n$ $(m, n = 1, 2, 3, \ldots)$. Let $\mathfrak{B}_n$ be a *-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_n, h_n$. Since $\mathfrak{B}_n \subset \mathfrak{A}_m$ for some m, $\mathfrak{B}_n$ is finite-dimensional. Set $\mathfrak{B}_n = \sum_{j=1}^{p_n} \mathfrak{B}_n z_{n,j}$, where $z_{n,j}$ is the minimal central projection of $\mathfrak{B}_n$. Now let φ_β be a KMS state at β for the C^*-dynamics $\{\mathfrak{A}, \varrho(t)\}$ with $\varrho(t)(a) = \exp t\, \delta_{ih_n}(a)$ $(a \in \mathfrak{A}_n)$; then there is a unique family $(\tilde{h}_n)$ of mutually commuting self-adjoint elements in $\mathfrak{A}$ such that $\tilde{h}_n \in \mathfrak{B}_n$, $\delta(a) = i[\tilde{h}_n, b]$ $(b \in \mathfrak{B}_n)$ and $\varphi_\beta(b) = \tau(b \exp(-\beta \tilde{h}_n))$ for $b \in \mathfrak{B}_n$ $(n = 1, 2, 3, \ldots)$ [24]. This result can be extended to commutative normal *-derivations with infinite range interactions [24].

PROBLEM. Can we find how to construct all KMS states at each β for normal *-derivations?

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The Spectrum of Compact Quotients of Semisimple Lie Groups

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1. Introduction. Let G be a Lie group and let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. One of the basic problems of harmonic analysis on G can be expressed (rather imprecisely) as follows:

If (π, H) is a unitary representation of G express π in terms of $\hat{G}$ using some sort of explicit formula.

Examples of the solution of this problem are:

(A) $G=\boldsymbol{R}$ with the additive group structure, $H=L^2(\boldsymbol{R})$, $\pi(x)$ translation for $x\in\boldsymbol{R}$. The solution to the problem is the L^2-Fourier transform.

(B) G compact, $H=L^2(G)$, $\pi(x)$ right translation for $x\in G$. The solution to the problem is the Peter–Weyl theorem combined with the theorem of the highest weight.

In this lecture I will be concerned with the following situation: G a Lie group, $\Gamma\subset G$ a co-compact discrete subgroup of G. $H=H_\Gamma=L^2(\Gamma\backslash G)$ and $\pi=\pi_\Gamma$ the right regular representation of G on H.

The problem has a qualitative answer (Gelfand, Graev, Piatetski–Shapiro [6])

$$\pi_\Gamma = \sum_{\omega\in\hat{G}} N(\Gamma,\omega)\omega,\quad N(\Gamma,\omega) \text{ a nonnegative integer.}$$

This means that $H_\Gamma=\oplus_{i=1}^{\infty} H_i$ a Hilbert space direct sum, H_i is $\pi_\Gamma(G)$-invariant and irreducible and precisely $N(\Gamma,\omega)$ of the H_i are in the class ω.

To complete the answer to the problem we need only compute the integers $N(\Gamma,\omega)$. To date there is no closed formula for these numbers unless ω is a fairly special type of element of $\hat{G}$.

In this lecture we will give a discussion of a few techniques that have been developed to study the numbers $N(\Gamma,\omega)$. We will also indicate some implications of the results

to the structure of Γ. For a systematic account of the relationship between the $N(\Gamma, \omega)$ and the cohomology of Γ the interested reader should consult [2].

2. The trace formula. Let G, Γ be as in the introduction. Fix dg a Haar measure on G. If $\gamma \in \Gamma$ let $[\gamma]$ denote its Γ-conjugacy class. Let $[\Gamma]$ denote the set of Γ-conjugacy classes in Γ.

If $\omega \in \hat{G}$ let θ_ω denote the character of ω. The trace formula says

(1) If $f \in C_c^\infty(G)$ then $\pi_\Gamma(f) = \int_G f(g)\pi_\Gamma(g)\, dg$ is of trace class and

$$\sum_{\omega \in \hat{G}} N(\Gamma, \omega)\theta_\omega(f) = \sum_{[\gamma] \in [\Gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, d(G_\gamma g)$$

here $G_\gamma = \{g \in G \mid g\gamma g^{-1} = \gamma\}$ and the Haar measure on G_γ and the G-invariant measure on $G_\gamma \backslash G$ are normalized so that Fubini's theorem is true.

(1) was originally pointed out by Selberg [13]. As Mackey (and others) have pointed out: if $G = \boldsymbol{R}^n$ and Γ is a lattice in G then (1) is just the Poisson summation formula. It is also clearly a special case of the induced character formula.

We now specialize to the case when G is a connected, linear, semi-simple Lie group.

(1) can be applied to a more general class of functions than $C_c^\infty(G)$. We denote the class by $\mathscr{C}^1(G)$. It is an analogue of the Schwartz space of $\boldsymbol{R}^n$, except that it is based on L^1 rather than L^2.

Let $K \subset G$ be a maximal compact subgroup of G. Let $(\pi, H) \in \omega \in \hat{G}$. A vector $v \in H$ is said to be *K-finite* if $\pi(K)v$ spans a finite dimensional subspace of H. ω is said to be of class L^1 if there is a nonzero K-finite vector $v \in H$ so that $f: g \to \langle \pi(g)v, v \rangle$ is in $\mathscr{C}^1(G)$. If ω is of class L^1 then ω is square integrable. We denote by $d(\omega)$ the formal degree of ω.

Harish–Chandra has shown that if $\gamma \in \Gamma$ and γ is not G-conjugate to an element of K then

$$\int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, d(G_\gamma g) = 0$$

for f as above [7], cf. [17, 9.3.17].

If Γ is torsion free then the only $\gamma \in \Gamma$, G-conjugate to an element of K, is 1. Thus for f as above the formula (1) becomes

$$N(\Gamma, \omega)\theta_\omega(f) = \operatorname{vol}(\Gamma \backslash G) f(1). \tag{2}$$

Now $f(1) = d(\omega)\theta_\omega(f)$ (cf. [17]). Hence we have

THEOREM 2.1 (LANGLANDS [9]). *If ω is of class L^1 and Γ is torsion free then*

$$N(\Gamma, \omega) = \operatorname{vol}(\Gamma \backslash G)\, d(\omega).$$

If G is $\mathrm{SL}(2, \boldsymbol{R})$ and $\Gamma \subset \mathrm{SL}(2, \boldsymbol{R})$ is a co-compact discrete subgroup of G then properly interpreted, Theorem 2.1 is the classical formula for the dimension of the space of automorphic forms of weight $k > 2$.

The formula in Theorem 2.1 for $\omega\in\hat{G}$ square integrable but not of class L^1 is false in general. For example, if $G=\mathrm{SL}(2,\boldsymbol{R})$ and $\omega\in\hat{G}$ is square integrable but not integrable the compact formula is

$$N(\Gamma,\omega)=d(\omega)\,\mathrm{vol}\,(\Gamma\backslash G)+1. \tag{3}$$

This corresponds to the fact that the dimension of the space of automorphic forms of weight 2 on $\Gamma\backslash H$ (H the upper half plane) is g, the genus of $\Gamma\backslash H$. Whereas $\mathrm{vol}\,(\Gamma\backslash G)d(\omega)=g-1=\frac{1}{2}\,\mathrm{vol}\,(\Gamma\backslash H)$ for the corresponding square integrable representation.

Langland's formula has been extended to a larger class of square integrable elements of $\hat{G}$ by Schmid [12] and Hotta–Parthasarathy [8] using fixed point theorems or index theorems and vanishing theorems.

When the vanishing theorems break down they do so for the following reason:

PROPOSITION 2.2 [3]. *Let $\omega\in\hat{G}$ be square integrable. Then there exists $\omega_1,\ldots,\omega_r\in\hat{G}$, ω_i nonsquare integrable and $a_j\in\boldsymbol{Z}$, $j=1,\ldots,r$, such that*

$$N(\Gamma,\omega)+\sum a_jN(\Gamma,\omega_j)=d(\omega)\,\mathrm{vol}\,(\Gamma\backslash G).$$

((3) above for $\mathrm{SL}(2,\boldsymbol{R})$ corresponds to $r=1$, $a_1=-1$ and ω_1 in the class of the trivial representation.) This formula has been computed explicitly for $G=\mathrm{SU}(2,1)$ in [16] and $G=\mathrm{SO}(4,1)$ in [11]. Also in [3] the a_j and ω_j are given group theoretic interpretations.

3. Uniform distribution theorems. Let $\Gamma\supset\Gamma_1\supset\Gamma_2\supset\ldots$, $\bigcap\Gamma_j=\{1\}$ and suppose that Γ_j is a normal subgroup of Γ with finite index. Such a family $\{\Gamma_j\}$ will be called a tower for Γ. Towers for Γ always exist (see [1]). Fix one.

If $\omega\in\hat{G}$ set $\lambda(\omega)$ equal to the eigenvalue of the Casimir operator on a representative of ω. We will say that $S\subset\hat{G}$ is bounded if the following two conditions are satisfied:

(a) $|\lambda(\omega)|\leqslant C<\infty$, $\omega\in S$.

(b) There is a finite subset $T\subset\hat{K}$ such that

$$[\omega|_K:\tau]\neq 0 \quad \text{for} \quad \tau\in T,\ \omega\in S.$$

Put for $S\subset\hat{G}$ a bounded subset

$$\mu_j(S)=\mathrm{vol}\,(\Gamma_j\backslash G)^{-1}\sum_{\omega\in S}N(\Gamma_j,\omega). \tag{1}$$

It is natural to ask:

Does $\lim_{j\to\infty}\mu_j(S)$ exist? If $\lim_{j\to\infty}\mu_j(S)$ exists what is its relationship with μ, the Plancherel measure of G relative to dg?

The simplest case of these equations is if $S=\{\omega\}$. Notice that if ω is of class L^1 then

$$\mu_j(\{\omega\})=d(\omega)=\mu(\{\omega\}).$$

THEOREM 3.1 [3]. *If* $\omega \in \hat{G}$ *then*

$$\lim_{j\to\infty} \mu_j(\{\omega\}) = \begin{cases} 0 & \text{if } \omega \text{ is not square integrable,} \\ d(\omega) & \text{if } \omega \text{ is square integrable.} \end{cases}$$

THEOREM 3.2 [4]. *Assume that* G *has* $\mathbf{R}$*-rank* 1. *If* $S \subset \hat{G}$ *then*

$$\lim_{j\to\infty} \mu_j(S) = \mu(S)$$

with μ *the Plancherel measure of* G.

Theorem 3.2 can be looked upon as a uniform distribution theorem for the $N(\Gamma_j, \omega)$ for ω contained in the continuous series for G.

These results do not compute $N(\Gamma, \omega)$; however they do give qualitative information about these numbers.

We give an application of Theorem 3.1 to the Eilenberg–Mac Lane cohomology of Γ with coefficients in an irreducible finite dimensional representation of G.

COROLLARY 3.3. *Let* F *be an irreducible finite dimensional representation of* G. *Suppose that* G *has a compact Cartan subgroup. Set* $2d = \dim(G/K)$. *Let* $\Gamma \subset G$ *be discrete and co-compact. Then*

(1) *There exists a normal subgroup* Γ_1 *of finite index in* Γ *so that*

$$H^d(\Gamma_1; F) \neq 0.$$

(2) *If* $\Gamma \supset \Gamma_1 \supset \Gamma_2, \ldots$ *is a tower for* Γ *then there is a constant* C, *depending only on* dg *and* G, *so that*

$$\lim_{j\to\infty} \operatorname{vol}(\Gamma_j \backslash G)^{-1} \dim H^q(\Gamma_j, F) = \begin{cases} C(\dim F) & \text{if } q = d, \\ 0 & \text{otherwise.} \end{cases}$$

4. The heat equation method. In this section we look at a circle of ideas initiated in [6] and [5], centering on the heat equation.

Fix $\gamma \in \hat{K}$ and Γ as above. Consider

$$g_{\gamma,\Gamma}(t) = \sum_{\omega\in\hat{G}} N(\Gamma, \omega)[\gamma : \omega|_K] e^{t\lambda(\omega)}; \tag{1}$$

here $\lambda(\omega)$ is the eigenvalue of the Casimir operator of $(\Pi, H) \in \omega$, and $[\gamma : \omega|_K]$ is the multiplicity of γ in ω as a K-representation.

The theory of the heat equation and a proportionality principle implies that

$$g_{\gamma,\Gamma}(t) \sim \operatorname{vol}(\Gamma \backslash G)(4\pi t)^{-n/2} \left(\sum_{k=0}^{\infty} a_{k,\gamma} t^k \right), \quad t \to 0+ \tag{2}$$

where $n = \dim G/K$. The numbers $a_{k,\gamma}$ depend only on γ and not on Γ.

The results of [6], [5], [15] imply that

$$a_{0,\gamma} = (\dim \gamma). \tag{3}$$

The main question is: How can we compute the $a_{k,\gamma}$?

In his thesis, Miatello computed the $a_{k,\gamma}$ for G of split rank 1 and γ satisfying a technical condition (that is automatically satisfied for G locally $\mathrm{SO}(n, 1)$ or

SU $(n, 1)$ but is not satisfied for Sp $(n, 1)$ or the R-rank 1 real form of F_4). His formula has as its first step

$$(4) \quad \left|g_{\gamma,\Gamma}(t) - \mathrm{vol}\,(\Gamma\backslash G) \int_G e^{t\lambda(\omega)}[\omega\colon \gamma]\, d\mu(\omega)\right| \leqslant C_{G,\Gamma} e^{-h_{G,\Gamma}/t} \quad \text{for} \quad 0 < t \leqslant T$$

with $C_{G,\Gamma}$ and $h_{G,\Gamma}$ positive constants.

He then computed the asymptotic expansion of

$$\int_G e^{t\lambda(\omega)}[\omega\colon \gamma]\, d\mu(\omega)$$

for G of R-rank 1.

Using an idea of Donally, De George and I have recently shown that (4) is true for all connected semi-simple, Lie groups with finite center. This implies that Miatello's formulas are true for all G of R-rank 1.

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Coherent Translation of Characters of Semisimple Lie Groups

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Dedicated to the memory of Rufus Bowen

By a character of a connected semisimple Lie group G we will here mean an element θ of the Grothendieck group $\mathrm{Ch}(\mathfrak{g}, K)$ of virtual Harish–Chandra modules (see [Z1]) over the pair $(\mathfrak{g}, K)$, $\mathfrak{g}$ the complexified Lie algebra of G, K a maximal compact subgroup of G. The ring $R(G)$ of virtual finite dimensional characters operates on $\mathrm{Ch}(\mathfrak{g}, K)$ via tensor product [Z1]: $\theta \to F \otimes \theta$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, C)$, then $\lambda \in \mathfrak{h}^*$ determines in a canonical fashion a C-valued character χ_λ of the center of the enveloping algebra of $\mathfrak{g}$. In turn, λ determines a projection operation $\theta \to \theta_\lambda$, where θ_λ is the χ_λ-primary component of θ. Tensoring and projecting in general do *not* commute.

Let $L \subseteq \mathfrak{h}^*$ be the lattice of weights occurring in finite dimensional representations of G. For any $F \in R(G)$ let $m(F: \nu)$ be the multiplicity of weight ν in F.

DEFINITION ([S1], [H–S1], [S2], [Sp–V]). A coherent family of characters is a function

$$\theta(\,): L \to \mathrm{Ch}(\mathfrak{g}, K)$$

such that

(a) $\theta(\lambda)_{\lambda_0+\lambda} = \theta(\lambda)$ for all $\lambda \in L$ and for some fixed base point $\lambda_0 \in \mathfrak{h}^*$ (but not necessarily in L).

(b) For any $F \in R(G)$, $F \otimes \theta(\lambda) = \sum_{\nu \in L} m(F, \nu)\theta(\lambda + \nu)$.

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EXAMPLE 1 [L–W]. Suppose G is real split, with split Cartan subgroup H and Borel subgroup $B=H$ nilradical (B). Each $\lambda\in L\subseteq\mathfrak{h}^*$ determines a continuous character $\xi(\lambda)$ of B into $C^\times$. Fix an arbitrary continuous character $\xi_0: B\to C^\times$. Let $\theta_{\xi_0}(\lambda)$ be the element of $\mathrm{Ch}(\mathfrak{g}, K)$ corresponding to the induced representation $\mathrm{Ind}_B^G(\xi_0\xi(\lambda))$. $\theta_{\xi_0}(\ )$ is a coherent family of characters with base point $\lambda_0=\log\xi_0$. In this example $\theta_{\xi_0}(\lambda)$ is always a true character.

EXAMPLE 2 (Any connected G). Fix a system Δ^+ of positive roots for $\mathfrak{h}$ in $\mathfrak{g}$. If $\lambda\in L$, let

$$\mathrm{sgn}(\lambda)=\prod_{\alpha\in\Delta^+}\mathrm{sgn}(\lambda,\alpha),$$

and let $\varrho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. For each $\lambda\in L$ let

$$\theta_{\Delta^+}(\lambda)=\mathrm{sgn}(\lambda+\varrho)F^{w(\lambda+\varrho)-\varrho},$$

where F^μ is the irreducible finite dimensional character with highest weight μ, and w is an element of the Weyl group such that $w(\lambda+\varrho)$ is dominant. Then $\theta_{\Delta^+}(\ )$ is a coherent family based at $\lambda_0=0$.

AN ALGEBRAIC LEMMA. *Suppose* $\lambda\in\mathfrak{h}^*$ *is regular* $((\lambda,\alpha)\neq 0\ \forall\alpha\in\Delta^+)$, $\theta\in\mathrm{Ch}(\mathfrak{g}, K)$, *and* $\theta=\theta_{\lambda_0}$. *Then there exists at most one coherent family* $\theta(\ ): L\to\mathrm{Ch}(\mathfrak{g}, K)$ *such that* $\theta(0)=\theta$.

THEOREM A. ([H–S1], [Sp–V]). *For* θ *with* $\theta=\theta_{\lambda_0}$ *and* λ_0 *regular, there always exists a coherent family* $\theta(\ ): L\to\mathrm{Ch}(\mathfrak{g}, K)$ *such that* $\theta(0)=\theta$.

DEFINITION. In Theorem A we call $\theta(\lambda)$ the (unique by the lemma) coherent translate (also called coherent continuation) of θ from λ_0 to $\lambda_0+\lambda$.

The original proof of Theorem A used Harish–Chandra's analytical theory of distributional characters [**HC1**]. Hecht and Schmid [**H–S1**] defined and studied the coherent families generated by the distributional discrete series characters [**HC2**] and they obtained a proof of a formula conjectured by Blattner for the K-multiplicities of these characters. Not long afterwards, Enright [**E**] proved Blattner's formula for a series of algebraically constructed Harish–Chandra modules which can be identified with the infinitesimal discrete series. However, Enright did not find an algebraic construction of the coherent families generated by the discrete series characters.

We have recently presented [**Z2**] a new homological algebraic method for simultaneously constructing the discrete series characters and *all* of their coherent translates. We suppose that G has a compact Cartan subgroup T, and we choose any Borel subalgebra $\mathfrak{b}$ in $\mathfrak{g}$ with $\mathfrak{b}=\mathfrak{t}+$nilradical $(\mathfrak{b})$, $\mathfrak{t}$ the complexified Lie algebra of T. Each character ξ of T determines a line bundle $\mathscr{L}_\xi$ over G/T;

$\mathfrak{b}$ gives rise to a Dolbeault complex of $(0, *)$ forms with values in $\mathscr{L}_\xi$. We pass to the *new* complex of K-finite ∞-jets of $(0, *)$ forms at the coset eT, and show that the cohomology H_ξ^* of this complex is a graded Harish–Chandra module.

THEOREM B. *Let* $\theta_{\mathfrak{b}}(\xi) = \sum (-1)^i [H_\xi^i]$.

(a) $\lambda \mapsto \theta_{\mathfrak{b}}(\xi \circ \xi(\lambda))$ *is the coherent family generated by* $\theta_{\mathfrak{b}}(\xi)$.

(b) *The K-multiplicities of* $\theta_{\mathfrak{b}}(\xi)$ *are given by Blattner's formula.*

(c) *Every discrete series character is equal to* $\pm\theta_{\mathfrak{b}}(\xi)$ *for some* $\mathfrak{b}$ *and some* ξ *positive relative to* $\mathfrak{b}$.

Combining a more general version of our homological methods with techniques from D. Vogan's [V1] homological algebraic classification of irreducible Harish–Chandra modules, we have, jointly with Vogan, obtained enough characters to span Ch $(\mathfrak{g}, K)$. As an immediate consequence we obtain an algebraic proof of the existence of coherent translation (Theorem A.)

We conclude by noting that additional interesting and powerful applications of the existence of coherent translation can be found in references [H–S2], [H1], [J], [K–Z], [M], [Sp–V], [V1], [V2], [V3], [V4], and [Z1].

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Rate of Convergence and Large Deviations in Invariance Principle

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The paper is devoted to the recent investigations connected with the well-known Donscer–Prokhorov's invariance principle. This principle is one of the main results in the theory of the summation of random variables.

Let $\xi_{1,n}, \ldots, \xi_{n,n}$, $n=1, 2, \ldots$, be the double sequence of independent random variables,

$$S_{k,n} = \sum_{j=1}^{k} \xi_{j,n}, \quad S_{0,n} = 0,$$

$$E\xi_{j,n} = 0, \quad D\xi_{j,n} = \sigma_{j,n}^2, \quad t_{k\,n} = \sum_{j=1}^{k} \sigma_{j,n}^2, \quad t_{n,n} = 1, \quad t_{0,n} = 0.$$

We denote $s_n = s_n(t)$ the random continuous broken line on the interval $[0, 1]$ with basic points $(t_{k,n}, S_{k,n})$, $k=0, \ldots, 1$.

Let P_n be the distribution of s_n in the space $C(0, 1)$ with σ-algebra $\mathscr{B}$ of Borel sets and W be the distribution of standard Wiener process $w=w(t)$ in the same space. If Ljapunov condition holds

$$\Lambda_s = \sum_{k=1}^{n} E|\xi_{k,n}|^s \to 0, \quad s > 2,$$

as $n \to \infty$ then $P_n \Rightarrow W$. It is the invariance principle. We can write it in the equivalent form

$$L(P_n, W) \to 0 \quad \text{as} \quad h \to \infty,$$

where $L(P, Q)$ is the Levy–Prokhorov's distance. The latter is defined by relations: $L(P, Q) < \varepsilon$ iff

$$P\big(G_\varepsilon(B)\big) > Q(B) - \varepsilon, \quad Q\big(G_\varepsilon(B)\big) > P(B) - \varepsilon$$

for any $B\in K$, where $G_\varepsilon(B)=\{y\in C(0,1)\colon \varrho(z,y)<\varepsilon,\ z\in B\}$, K and ϱ are a given class of sets and a metric.

For the sake of accuracy we would write $L_{K,\varrho}(P,Q)$, but we omit K and ϱ if $K=\mathscr{B}$, ϱ is a uniform metric.

I. Rate of convergence. What can we say about the difference

$$\Delta_n(B)=|P_n(B)-W(B)|$$

if we estimated the distance $L(P_n, W)$?

We call the set $B\in\mathscr{B}$ a Lipschitz set with respect to metric ϱ and measure W ((ϱ, W)-Lip. set), if

$$W(G_\varepsilon(\partial B))<c\varepsilon,$$

where $c=\text{const}$, ∂B is the boundary of B.

It is easy to see that for (ϱ, W)-Lip. sets

$$\Delta_n(B)\leqslant(c+1)L(P_n,W). \tag{1}$$

The similar question can arise about the nearness of the distributions of $f(s_n)$ and $f(w)$ for some functional f, that is, about the difference

$$\Delta_n(f,u)=\left|P(f(s_n)<u)-P(f(w)<u)\right|.$$

If $|f(y)-f(z)|<c_1\varrho(z,y)$ and the distribution of $f(w)$ has the density, then the set $B=\{y\colon f(y)<u\}$ will be (ϱ, W)-Lip. and, hence,

$$\Delta_n(f,u)\leqslant(c+1)L(P_n,W).$$

Thus, the estimates for $L(P_n, W)$ allow us to estimate the values important for us $\Delta_n(B)$ and $\Delta_n(f,u)$.

The first and rather exact (as it became clear later) estimate for $L(P_n, W)$ was obtained by Prokhorov (1956)

$$L(P_n,W)=o(\Lambda_3^{1/4}\ln^2\Lambda_3).$$

This estimate was improved in the i.i.d. case by Rosenkrantz (1967), Heyde (1969) and Dudley (1972). In 1973 Borovkov proved that

$$L(P_n,W)\leqslant c\Lambda_s^{1/(s+1)},\ 2<s\leqslant 3. \tag{2}$$

Sahanenko (1974) and Arak (1975) showed that this estimate is unimprovable.

At about the same time it was shown that $\Delta_n(B)$ and $\Delta_n(f,u)$ admit stronger estimates than (1): if B is (ϱ, W)-Lip. set, then

$$\Delta_n(B)\leqslant c\Lambda_s^{1/s}\ln^\beta\Lambda_s,\ 3<s\leqslant 4,\ \beta>0 \tag{3}$$

(Sahanenko, 1974).

There exist the examples constructed by Nagaev and Nevsorov (Borovkov 1973) for which the boundary $c\Lambda_s^{1/s}$ is achieved.

We consider now i.i.d.-case.

In this case $\xi_{k,n}=\xi_k/\sqrt{n}$, where ξ_k are i.i.d. random variables and we have

$$\Delta_s = E|\xi_1|^s n^{(2-s)/2}.$$

Our unimprovable estimates give us the boundaries

$$\Delta_s^{1/(s+1)} = cn^{(2-s)/(2s+2)}, \quad \Delta_s^{1/s} = cn^{(2-s)/2s}.$$

For instance, if $E|\xi_k|^3<\infty$ $(s=3)$, we obtain

$$L(P_n, W) \leqslant cn^{-1/8}, \quad \Delta_n(B) \leqslant cn^{-1/6}\ln^{\beta} n.$$

These results arouse suspicions that Δ_s is not the best value for the estimations in i.i.d.-case because we do not know a single example of functional f (even merely continuous) for which the rate of convergence would be worse than $cn^{-1/2}$ if $E|\xi_k|^3<\infty$.

We have here a large break between $n^{-1/6}$ and $n^{-1/2}$.

On the other hand, the use of the value Δ_s is connected first of all with the method of investigation and with the central limit theorem, where it is very natural. The second point is that all the examples of unimprovement of estimates (2), (3) are based on not i.i.d. random variables.

For some examples of sets B or functionals f there exist instead of $\Delta_s^{1/s}$ functions of summands (for estimation $\Delta_n(B)$ or $\Delta_n(f, u)$), which are transformed in i.i.d. case into $cn^{-1/2}$.

We have, for instance,

$$\Delta_n(f, u) \leqslant \frac{c}{1+u^2}\max_k \frac{E|\xi_{k,n}|^3}{E|\xi_{k,n}|^2}$$

for $f(y)=\sup_{t\in[0,1]}y(t)$ and $\Delta_n(B)\leqslant c\Delta_s$ for cylindrical sets B.

The whole problem on final estimates in i.i.d. case is not solved yet.

If $E|\xi_k|^s<\infty$ and s is large, then the break between the right hand side of (3) and $cn^{-1/2}$ becomes narrower. If the Cramer condition holds

$$Ee^{t|\xi_k|}<\infty \quad \text{for some } t>0 \tag{C}$$

then it follows from the recent papers of J. Komlos, P. Major and G. Tusnady (1975, 1976) that

$$L(P_n, W) \leqslant \frac{c\ln n}{\sqrt{n}}. \tag{4}$$

We remark, that the estimates (2), (4) were obtained for other metrics ϱ in (Borovkov 1973).

We have already mentioned that for some special classes of sets B or functionals f it is possible to find stronger estimates for $\Delta_n(B)$ and $\Delta_n(f, u)$.

For instance, for the band B between two smooth boundaries $g_1(t)$ and $g_2(t)$ we have

$$\Delta_n(B) \leqslant \frac{c(g_1, g_2)E|\xi_1|^3}{\sqrt{n}}$$

(Nagaev, (1970), Sahanenko, (1974)), where $c(g_1, g_2)$ depends only on g_1, g_2. The same estimate holds for linear functionals f (Borisov (1976)). It is easy to see that these results are final.

For functionals

$$f(y) = \int_0^1 \varphi(t, y(t))\, dt, \quad \varphi \in C_1$$

under condition $E|\xi_k|^s < \infty$, and some conditions on φ holds

$$\Delta_n(f, u) \leqslant \frac{c}{\sqrt{n}} \ln^\beta n, \quad \beta > 0$$

(Borisov (1976)). There are also some results about integral functionals in more complicated situations (see Borisov (1976), Sawyer (1972), O'Relly (1974), Nikitin (1974)).

We cited the principal results related to the estimation of the speed of convergence.

What are the further possibilities here?

The main method of investigation in this area is the one of common probability space (c.p.s.). At the moment there are three different ways to construct s_n and w on the c.p.s.

Prokhorov method. For special points $u_1, u_2, \ldots, u_k$ the increments $s_n(u_k) - s_n(u_{k-1})$ are defined as the function of $w(u_k) - w(u_{k-1})$ with the help of quantile transformations. The remaining increments of s_n are not essential.

Skorokhod method. The values $s_n(t_k)$ are defined as $w(\tau_k)$ for special random points $\tau_1, \tau_2, \ldots$.

Method of Major, Tusnady, Komlos. This method is more delicate and in some sense unimprovable. But it is developed so far only for i.i.d. case.

All these three methods were essential for the development and advance in invariance principle.

The third method allows us to construct the trajectories s_n and w with the minimal possible (in some sense) distance between them. The following assertion of Major, Tusnady, Komlos (1975) is true.

Let there exist increasing function $H(x) \geqslant x^{3+\delta}$, $\delta > 0$, $H(x) \leqslant c_1 e^{\alpha x}$, $\alpha > 0$, such that $EH(|\xi_k|) < \infty$. Then for the constructed trajectories and for $x \leqslant c_2 \sqrt{n \ln n}$, $H(x) \geqslant n$ holds

$$P\left(\varrho(s_n, w) > \frac{x}{\sqrt{n}}\right) \leqslant \frac{cn}{H(ax)}, \quad c > 0, \quad a > 0. \tag{5}$$

Assertions (2), (4) follow from this inequality almost immediately.

Inequality (5) almost completely expresses the possibilities of c.p.s. methods. It is unimprovable in the following sense

$$P(\varrho(s_n, w) > cx_n/\sqrt{n}) \to 1$$

for *any* c.p.s. method, where x_n is the solution of equation $cn/H(ax)=1$ which we obtain if we put the right hand side of (5) equal to 1 (Borovkov (1973), Komlos, Major, Tusnady (1975)).

Since $x_n \geqslant \ln n \to \infty$ it is impossible to obtain by c.p.s. methods the estimate for $L(P_n, W)$ and $\Delta_n(B)$ of the form $cn^{-1/2}$.

There is also another shortcoming of c.p.s. methods: all these three mentioned methods at the moment are very difficult to generalize for the case when ξ_k take values from space R^m and the more so for ξ_k from arbitrary Banach space Y.

It was rather unexpected to discover that there exists another, rather rough approach suitable for the spaces of general nature and giving results close to (2). For the sake of simplicity we consider only i.i.d. case.

Let Y be a Banach space with Borel σ-algebra $\mathscr{B}$, and let $\xi_j \in Y$, $E\xi_j=0$, $T_j=T$ be the corresponding to ξ_j covariation operator,

$$S_{k,n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \xi_j.$$

We denote Γ_T and p_n respectively the gaussian distribution in Y with the same operator T and the distribution of $S_{n,n}$.

As before, we can construct in the space $Y \times [0, 1]$ the broken line $s_n=s_n(t)$ and gaussian process $w=w(t)$ with independent increments and Γ_T as the distribution of $w(1)$. P_n and W as before will denote the distribution of s_n and w in the space $Y \times [0, 1]$.

The above-mentioned approach arises from the following statement of the problem: we suppose that we know the behaviour of $L(p_n, \Gamma_T) \to 0$ in Y. Then we can estimate the rate of convergence of distributions of vectors $(s_n(u_1), \ldots, s_n(u_k))$ and the large deviations of the processes s_n and w on $[u_j, u_{j+1}]$. This allows us to estimate by direct calculations the behaviour of $L(P_n, W)$ in $Y \times [0, 1]$.

Here is the result (Borovkov, Sahanenko): if $L(p_n, \Gamma_T)=\varphi(n) \to 0$ as $n \to \infty$, then

$$L(P_n, W) \leqslant cn\varphi(k)/k,$$

where k is defined from equation

$$(n/k)^3 \varphi^2(k) = \ln n.$$

For instance, if $\varphi(n)=cn^{-1/2}$ (that is true for $Y=R$, $E|\xi_j|^3<\infty$), then $L(P_n, W)< cn^{-1/8} \ln^{\beta} n$, $\beta>0$.

If $\varphi(n)=cn^{-1/8} \ln^{\alpha} n$ (that is true for separable Hilbert space Y, if $E(\sum |\xi_{1,l}|)^3<\infty$, where $\xi_{1,l}$ are coordinates of ξ_1 (Senatov (1977)) then

$$L(P_n, W) \leqslant cn^{-1/26} \ln^{\beta} n, \quad \beta > 0.$$

We note that instead of $L(p_n, \Gamma_T)$ we can consider $L_K(p_n, \Gamma_T)$ for some class K of sets in Y (K may be, for instance, class of all the convex sets) and investigate

the behaviour of $L_K(P_n, W)$ for class $\tilde{K}$ of sets B in $Y\times[0,1]$ with the cuts $B\{t=t_0\}\in K$.

II. Large deviations. We consider in this section only i.i.d. case.

The object of investigation here is the asymptotic behaviour of $P(S_n\in xB)$, where $B\subset Y\times[0,1]$, $x\to\infty$ as $n\to\infty$, $0\notin B$.

We shall say that invariance principle for large deviations (i.p.l.d.) for the set $B\subset Y\times[0,1]$ takes place if

$$P(s_n\in xB)\sim P(w\in xB). \tag{6}$$

For the case $Y=R$, $Ee^{t|\xi_1|^\alpha}<\infty$, $\alpha>0$, $t>0$ the following assertion holds. If B contains at least one interior point and for $y=o(x^{-1})$

$$W(G_y(\partial xB))=o(W(xB)) \tag{7}$$

then for $x=o(n^{\alpha/2(\alpha+2)})$ the i.p.l.d. is true.

If $\alpha=1$ (Cramer condition (C)), then the admissible region of deviations is $x=o(n^{1/6})$. For $\alpha<1$ the region $o(n^{\alpha/2(\alpha+2)})$ may be extended. Condition (7) is essential.

If B is the exterior of the band between two smooth functions $g_1(t)$ and $g_2(t)$, the asymptotic behaviour of $P(S_n\in xB)$ under condition (C) was found in an explicit form for all $x=O(n)$ (Borovkov (1964)). But i.p.l.d. is true only for $x=o(n^{1/6})$.

We also consider the so-called rough invariance principle for large deviations (r.i.p.l.d.):

$$\ln P(s_n\in xB)\sim \ln P(w\in xB) \tag{8}$$

or, which is the same,

$$\ln P(s_n\in xB)\sim -\frac{x^2}{2}\inf_{\varphi\in B\cap C_1}\int_0^1\left(\frac{d\varphi}{dt}\right)^2 dt \tag{9}$$

On the right-hand side here is the so-called action functional for Wiener process.

The r.i.p.l.d. is true under condition (C) for a broad class of sets B for all $x=o(n^{1/2})$ (Borovkov (1967), Mogulsky (1970)).

The generalization of i.p.l.d. (6) on the case $\xi_j\in Y$ for Banach space Y is apparently a difficult problem. At the same time the generalization of r.i.p.l.d. (8), (9) may be extended very far. I present only one recent assertion of Borovkov, Mogulsky which is connected with the recent results of R. R. Bahadur, S. L. Zabele.

Let Y be a Hilbert space and Cramer condition be fulfilled $Ee^{t(\xi_j,g)}<\infty$, $\forall g\in Y$ for sufficiently small $|t|$. Further, let T be the covariation operator generated by ξ_j: $(Tg,h)=E(\xi_j,g)(\xi_j,h)$. We denote by D the domain of values of $Tg, g\in Y$. We can formulate now the following assertion. For a broad class H of sets B in $Y\times[0,1]$ and for $x=o(n^{1/2})$

$$\ln P(s_n\in xB)\sim -x^2\Lambda(B)/2$$

where

$$\Lambda(B) = \inf_{\varphi \in B \cap C_1} \int_0^1 \lambda\left(\frac{d\varphi}{dt}\right) dt,$$

$$\lambda(\psi) = \begin{cases} -\infty & \text{if } \psi \notin D, \\ E(T^{-1}\psi, \xi_1)^2 & \text{if } \psi \in D. \end{cases}$$

In particular the convex sets B with property $\Lambda(B-\partial B)=\Lambda(B+\partial B)$ belong to H. This result can be extended on the arbitrary local convex space Y with tight distribution of ξ_1.

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Un Survol de la Théorie de l'Integrale Stochastique

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Je voudrais présenter ici, de manière compréhensible pour un lecteur n'ayant que quelques notions de théorie des processus (stochastiques), les principaux résultats d'une part importante du calcul différentiel stochastique en temps continu (plus précisément, de la part où la probabilité de base P ne joue que par la classe des ensembles négligeables qu'elle définit): après quelques préliminaires sur la théorie des processus, nous aborderons le problème de la définition de l'intégrale stochastique, puis celui du calcul d'une différentielle stochastique, et enfin celui de la résolution d'équations différentielles stochastiques. Vu l'espace qui m'est imparti, je ne parlerai pas de l'historie de la théorie, hors quelques indications sommaires à l'occasion de la bibliographie, elle-même rudimentaire, et je n'aborderai pas non plus des parts importantes de la théorie, comme tout ce qui touche à la notion, pourtant fondamentale de martingale locale.

Préliminaires. Nous fixons ici notre terminologie en rappelant quelques données de base de la théorie des processus, et introduisons la notion fondamentale de *semimartingale:* nous verrons, au paragraphe suivant, que les semimartingales forment la classe naturelle des processus «intégrants» dans la théorie de l'intégrale stochastique.

On se donne un espace probabilisé complet $(\Omega, \mathscr{F}, P)$, muni d'une *filtration* $(\mathscr{F}_t)_{t\in R_+}$ *vérifiant les conditions habituelles:* $(\mathscr{F}_t)$ est une famille croissante de sous-tribus de $\mathscr{F}$ (la tribu $\mathscr{F}_t$ étant la collection des événements connus à l'instant $t \geqslant 0$), continue à droite (on a $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$), telle que $\mathscr{F}_0$ contienne tous les ensembles négligeables. On suppose de plus, pour simplifier, que $\mathscr{F} = \mathscr{F}_\infty = \bigvee_t \mathscr{F}_t$.

Un *processus* $X = (X_t)_{t\in R_+}$ sera pour nous ce qu'on appelle en général un *processus mesurable et adapté,* i.e. une application mesurable $(t, \omega) \to X_t(\omega)$ de

$(R_+\times\Omega, \mathscr{B}(R_+)\times\mathscr{F})$ dans $(R, \mathscr{B}(R))$ telle que, pour t fixé, la v.a. X_t soit $\mathscr{F}_t$-mesurable. Deux processus X et Y sont dits *indistinguables* si l'ensemble $\{\omega: \exists t X_t(\omega)\neq Y_t(\omega)\}$ est négligeable. Dans toute la suite, nous travaillerons «à l'indistinguabilité près»: c'est en particulier ainsi qu'il faudra entendre les assertions d'unicité. On dira qu'un processus est *càdlàg* (resp. *càglàd*) si, pour tout $\omega\in\Omega$, la trajectoire $t\to X_t(\omega)$ est *c*ontinue *à d*roite, et pourvue de *l*imites *à g*auches (pour $t>0$) (resp. continue *à* gauche, pour $t>0$, et pourvue de limites *à* droite). Un processus càdlàg V est dit *à variation finie* si toute trajectoire de V est une fonction à variation bornée sur tout intervalle fini.

Un processus càdlàg M est une *martingale* (resp. *surmartingale, sousmartingale*) si M_t est une v.a. intégrable pour tout t et si l'on a, pour $s<t$, $M_s=E[M_t|\mathscr{F}_s]$ p.s. (resp. $\geqslant$, $\leqslant$).

En général, les trajectoires d'une martingale sont bien trop oscillantes pour être à variation finie (il en est en particulier ainsi pour le mouvement brownien unidimensionnel, qui est l'exemple fondamental de martingale continue): c'est ce qui fait tout le charme de l'intégrale stochastique, qui ne peut être réduite à une intégrale de Stieltjès trajectoire par trajectoire.

Voici maintenant la définition d'une semimartingale, «combinaison linéraire locale» d'une martingale et d'un processus à variation finie

DEFINITION 1. *Un processus càdlàg X est une* semimartingale *s'il existe une suite croissante (T_n) de v.a.$\geqslant 0$, tendant vers $+\infty$, vérifiant la condition: pour tout n, il existe une martingale M^n et un processus à variation finie V^n (non uniques!) tels que l'on ait $X_t(\omega)=M_t^n(\omega)+V_t^n(\omega)$ pour tout $t\in[0, T_n(\omega)[$.*

Si la filtration est triviale, i.e. si $\mathscr{F}_t=\mathscr{F}$ pour tout t, les semimartingales sont exactement les processus càdlàg à variation finie.

REMARQUE. Pour le développement de la théorie, il est important de savoir qu'on peut choisir les T_n, M^n et V^n de sorte que les T_n soient des «temps d'arrêt» de la filtration, et que M^n soit bornée et V^n à variation bornée (la borne ne dépendant pas de ω).

La classe $\mathscr{S}=\mathscr{S}(\Omega, (\mathscr{F}_t), P)$ des semimartingales vérifie de nombreuses propriétés de stabilité. En voici deux, très importantes (et non triviales!); nous en verrons d'autres, encore plus importantes pour le calcul différentiel, plus loin

THÉORÈME 1. (a) *$\mathscr{S}(\Omega, (\mathscr{F}_t), P)$ est une* algèbre, *et aussi une* lattice, *qui contient évidemment les martingales et les processus à variation finie, mais aussi les surmartingales et les sousmartingales.*

(b) *Si Q est une probabilité sur $(\Omega, \mathscr{F})$ équivalente à P, alors $\mathscr{S}(\Omega, (\mathscr{F}_t), Q)$ est égale à $\mathscr{S}(\Omega, (\mathscr{F}_t), P)$.*

Noter, en contraste avec (b), que la nation de martingale dépend fortement de la probabilité de base P, tant par la condition d'intégrabilité que par la prise d'espérances conditionnelles.

L'integrale stochastique. Nous dirons qu'un processus càglàd borné Y est *élémentaire* s'il existe des instants $0=t_0<t_1<\ldots<t_n$ et, pour $i=0, 1, \ldots, n$, une v.a. bornée U_i $\mathscr{F}_{t_i}$-mesurable tels que $Y_t=U_i$ pour $t\in]t_i, t_{i+1}]$ (avec $t_{n+1}=+\infty$). Si Y est un tel processus, on définit « l'intégrale triviale » de Y par rapport à X, processus càdlàg quelconque, en posant, si j est le plus grand entier tel que $t_j<t$,

$$\int_0^t Y_s\,dX_s = \sum_{i<j} U_i(X_{t_{i+1}}-X_{t_i})+U_j(X_t-X_{t_j})$$

l'intégration ayant lieu sur $]0, t]$. Mais, une intégrale « digne de ce nom » doit vérifier des propriétés de continuité par rapport à la chose intégrée. Or, on a le résultat suivant

THÉORÈME 2. *Munissons l'espace $\mathscr{E}$ des processus élémentaires de la topologie de la convergence uniforme et l'espace L^0 des (classes de) v.a. finies de la topologie de la convergence en probabilité. Alors X étant un processus càdlàg fixé, l'intégrale triviale $Y\to\int_0^t Y_s\,dX_s$ définit, pour chaque t, une application continue de $\mathscr{E}$ dans L^0 si et seulement si X est une semimartingale.*

si bien qu'on ne peut définir une «véritable» intégrale qu'en intégrant par rapport à une semimartingale.

Dégageons maintenant la classe des processus que l'on saura finalement intégrer par rapport à une semimartingale, en étendant de manière naturelle l'intégrale triviale. Soit $\mathscr{P}$ la sous-tribu de $\mathscr{B}(\boldsymbol{R}_+)\times\Omega$, sur $\boldsymbol{R}_+\times\Omega$, engendrée par les processus élémentaires: *les processus mesurables par rapport à cette tribu sont dits -prévisibles*; tels sont, par exemple, les processus càglàd (mais, en général, pas tous les processus càdlàg!). Nous désignerons par Π l'ensemble des processus prévisibles Y vérifiant la condition de « bornitude locale » suivante (satisfaite par tout processus càglàd): il existe une suite croissante (T_n) de v.a. $\geqslant 0$, tendant vers $+\infty$, et une suite de constantes (c_n) telles que l'on ait $|Y_t(\omega)|\leqslant c_n$ pour tout $t\in[0, T_n(\omega)[$. La construction de l'intégrale stochastique aboutit finalement au résultat suivant

THÉORÈME 3. *Soit X une semimartingale. Il existe, pour tout t, une unique application linéaire $Y\to\int_0^t Y_s\,dX_s$ de Π dans L^0 vérifiant:*

(a) *pour $Y\in\mathscr{E}$, $\int_0^t Y_s\,dX_s$ est égale à l'intégrale triviale;*

(b) *si (Y^n) est une suite d'éléments de Π, majorée en valeur absolue par un élément de Π, et convergeant simplement vers Y ($\in\Pi$), alors $\int_0^t Y^n_s\,dX_s$ converge en probabilité vers $\int_0^t Y_s\,dX_s$.*

De plus, pour tout $Y\in\Pi$, il existe une unique semimartingale Z telle que l'on ait $Z_t=\int_0^t Y_s\,dX_s$ p.s. pour tout t, si bien que l'ensemble des semimartingales est stable pour la formation d'intégrales stochastiques.

Quoique l'intégrale stochastique « $Z=\int Y\,dX$ » ne peut, en général, être définie trajectoire par trajectoire, elle se comporte souvent « comme si ». Par exemple,

(a) si T est une v.a. $\geqslant 0$, pour presque tout ω, le saut $\Delta Z_t(\omega)=Z_t(\omega)-Z_{t-}(\omega)$ en $t=T(\omega)>0$ est égal à $Y_t(\omega)\Delta X_t(\omega)$;

(b) si, pour $A\in\mathscr{F}$, la trajectoire $t\to X_t(\omega)$ est à variation finie pour tout $\omega\in A$, alors, pour (presque) tout $\omega\in A$, la trajectoire correspondante de Z est égale à l'intégrale de Stieltjès de la trajectoire de Y par rapport à celle de X;

(c) si $Z'=\int Y'dX'$ est une autre intégrale stochastique et si, pour presque tout $\omega\in A$ ($A\in\mathscr{F}$), les processus Y et Y' (resp. X et X') ont même trajectoire, (fonctionnelle) il en est de même de Z et Z'.

La formule de changement de variables. Notons d'abord que, si une semimartingale n'est pas, en général, à variation finie, elle a, *en un certain sens,* une 2-variation finie:

THÉORÈME 4. *Soit X une semimartingale. Il existe un unique processus càdlàg croissant, que l'on note $[X,X]$, vérifiant la condition: pour tout t, la v.a. $[X,X]_t$ est la limite en probabilité des sommes $X_0^2+\sum_i (X_{t_{i+1}}-X_{t_i})^2$, où (t_i) est une subdivion finie de $[0,t]$, lorsque le pas de la subdivision tend vers 0. De plus, pour tout $t>0$, on a $\Delta[X,X]_t=(\Delta X_t)^2$.*

Si X est à variation finie, on a $[X,X]_t=X_0^2+\sum_{0<s\leqq t}(\Delta X_s)^2$, qui est donc nul si $X_0=0$ et si X est de plus continu. Mais, par exemple, si X est un mouvement brownien unidimensionnel, issu de 0, on a $[X,X]_t=t$ pour tout t. Par ailleurs, le «en un certain sens» résulte du fait que les subdivisions que l'on prend de $[0,t]$ ne dépendent pas de ω: prise trajectoire par trajectoire, la «vraie» 2-variation du mouvement brownien est p.s. infinie!

Ceci dit, si V est un processus continu à variation finie et si f est une fonction de classe $\mathscr{C}^1$ sur $\boldsymbol{R}$, il est bien connu que $f(V)$ est encore à variation finie et que $f(V_t)-f(V_0)=\int_0^t f'(V_s)\,dV_s$; c'est, trajectoire par trajectoire, la formule classique de «changement de variable», qui se visualise bien en écrivant, pour une subdivision finie (t_i) de $[0,t]$, $f(V_t)-f(V_0)=\sum_i\big(f(V_{t_{i+1}})-f(V_{t_i})\big)$ et en développant chaque paranthèse par la formule de Taylor à l'ordre 1. Si V, à variation finie, est seulement càdlàg, on obtient la formule

$$f(V_t)-f(V_0)=\int_0^t f'(V_{s-})\,dV_s+\sum_{0<s\leqq t}\big(f(V_s)-f(V_{s-})-f'(V_{s-})\Delta V_s\big)$$

moins classique, où, dans la famille sommable qui tient compte des sauts, on reconnait un avatar du développement de Taylor à l'ordre 1. Dans le cas d'une semimartingale, on a une formule analogue — appelée souvent *formule d'Ito*, Ito ayant été le premier à établir une telle formule, dans le cas du mouvement brownien —, mais il y apparait un développement de Taylor à l'ordre 2 (cf. Théorème 4):

THÉORÈME 5. *Soient X une semimartingale et f une fonction de classe $\mathscr{C}^2$ sur $\boldsymbol{R}$. Alors $f(X)$ est encore une semimartingale et l'on a*

$$f(X_t)-f(X_0)=\int_0^t f'(X_{s-})\,dX_s+\frac{1}{2}\int_0^t f''(X_{s-})\,d[X,X]_s+\sum_{0<s\leqq t}U_s$$

où $U_s=\{f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s-\frac{1}{2}f''(X_{s-})^2\}$, *la première intégrale étant une intégrale stochastique et la famille* $(U_s)_{s\leqq t}$ *étant p.s. sommable.*

Bien entendu, si X est à variation finie, on retrouve la formule citée plus haut.

La formule de changement de variables qui, malgré son air un peu rébarbatif, est d'une grande efficacité dans le calcul différentiel stochastique, admet une extension aux fonctions de classe $\mathscr{C}^2$ à plusieurs variables. Mais, avant de formuler cette extension, il nous faut définir le «crochet mixte» $[X, Y]$ de deux semimartingales X et Y: on pose tout simplement, par polarisation,

$$[X, Y]=\frac{1}{2}([X+Y, X+Y]-[X, X]-[Y, Y])$$

et $[X, Y]_t$ est encore limite en probabilité de sommes du genre $X_0Y_0+\sum_i(X_{t_{i+1}}-X_{t_i})(Y_{t_{i+1}}-Y_{t_i})$; de plus, si l'une des semimartingales X ou Y est à variation finie, on a $[X, Y]_t=X_0Y_0+\sum_{0<s\leqq t}\Delta X_s\Delta Y_s$, et, de manière générale, on a $\Delta[X, Y]_t=\Delta X_t\Delta Y_t$ pour $t>0$. Soient maintenant $X^1, \ldots, X^n$ n semimartingales et f une fonction de classe $\mathscr{C}^2$ sur $\boldsymbol{R}^n$, admettant les dérivées partielles D_if au premier ordre et D_iD_jf au second ordre. Alors, posant $\vec{X}=(X^1, \ldots, X^n)$, $f(\vec{X})$ est encore une semimartingale et l'on a

$$f(\vec{X}_t)=f(\vec{X}_0)+\sum_i\int_0^t D_if(\vec{X}_{s-})\,dX_s^i+\frac{1}{2}\sum_{i,j}\int_0^t D_iD_jf(\vec{X}_{s-})\,d[X^i, X^j]_s$$

$$+\sum_{0<s\leqq t}\{f(\vec{X}_s)-f(\vec{X}_{s-})-\sum_i D_if(\vec{X}_{s-})\Delta X_s^i-\frac{1}{2}\sum_{i,j}D_iD_jf(\vec{X}_{s-})\Delta X_s^i\Delta X_s^j\}.$$

En appliquant cette formule au cas où $f(x, y)=xy$, on obtient la formule importante d'*intégration par parties:* si X, Y sont deux semimartingales, alors

$$X_tY_t=\int_0^t Y_{s-}\,dX_s+\int_0^t X_{s-}dY_s+[X, Y]_t$$

et, en particulier, pour $X=Y$, $X_t^2=2\int_0^t X_{s-}\,dX_s+[X, X]_t$.

Equations differentielles stochastiques. Nous nous bornerons à énoncer un théorème d'existence et d'unicité «global», sous des hypothèses analytiques analogues à celles du théorème classique de Cauchy—Lipschitz.

THÉORÈME 6. *Soient* X *une semimartingale et* H *un processus càdlàg. Alors l'équation intégrale stochastique*

$$Z_t(\omega)=H_t(\omega)+\int_0^t f(s, \omega, Z_{s-}(\omega))\,dX_s(\omega)$$

admet une solution unique Z, *processus càdlàg, lorsque la fonction* $f(s, \omega, x)$ *sur* $\boldsymbol{R}_+\times\Omega\times\boldsymbol{R}$ *satisfait aux conditions suivantes*

(1) *pour x fixé, $(s, \omega) \to f(s, \omega, x)$ est un processus càglàd,*

(2) *pour (s, ω) fixé, $x \to f(s, \omega, x)$ est une fonction lipschitzienne, avec une constante de Lipschitz ne dépendant pas de s.*

Trois remarques au sujet de l'énoncé du théorème:

(a) Pour Z càdlàg, le processus $(s, \omega) \to f(s, \omega, Z_{s-}(\omega))$ est càglàd, et peut donc bien être intégré «stochastiquement» par rapport à une semimartingale.

(b) L'équation considérée est plus générale que les usuelles en ce sens que le processus H, qui joue le rôle de «condition initiale», peut évoluer au cours du temps. Ce gain de généralité permet en fait de simplifier les démonstrations!

(c) Si H est une semimartingale, la solution Z en est aussi une.

L'énoncé que nous avons donné ne fait intervenir qu'une seule semimartingale mais, en fait, il est possible de remplacer l'unique intégrale dans l'équation intégrale par une somme finie d'intégrales du même type et, plus généralement, de considérer des «systèmes» d'équations.

Une équation particulièrement importante est celle définissant «l'exponentielle d'une semimartingale, au sens des semimartingales»

$$Z_t = 1 + \int_0^t Z_{s-}\, dX_s$$

Sa solution, notée $\varepsilon(X)$, admet une représentation explicite: si X est continu, on a

$$\varepsilon(X)_t = \exp\left(X_t - \frac{1}{2}[X, X]_t\right)$$

où «exp» est la fonction exponentielle ordinaire; dans le cas général, on a (avec un produit infini p.s. absolument convergent)

$$\varepsilon(X)_t = \exp\left(X_t - \frac{1}{2}[X, X]_t\right) \prod_{0<s\leqq t} (1+\Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right).$$

Par ailleurs, X et Y étant deux semimartingales, l'exponentielle ε vérifie l'équation fonctionnelle $\varepsilon(X)\varepsilon(Y) = \varepsilon(X+Y+[X, Y])$.

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L'intégrale stochastique par rapport au mouvement brownien doit ses débuts à Wiener (intég) ration de processus «déterministes»), mais c'est à Ito que revient la création d'un véritable calcul différentiel stochastique attaché au mouvement brownien, dans une série d'articles allant de 1944 à 1961. Une référence classique à ce sujet:

H. P McKean, *Stochastics integrals*, Academic Press, New York, 1969.

Le calcul d'Ito (excepté les équations différentielles) a été étendu aux martingales de carré intégrable dans.

H. Kunita, and S. Watanabe, *On square integrable martingales*, Nagoya Math. J. **30** (1967).

L'extension aux martingales locales et aux semimartingales, initiée par P. A. Meyer, trouve sa forme quasi-définitive dans

C. Doléans-Dade, et P. A. Meyer, *Intégrales stochastiques par rapport aux martingales locales,* Séminaire de Probabilité IV, Lecture Notes in Math., Vol. 124, Springer-Verlag, Berlin and New York, 1970.

Mais l'invariance de la notion de semimartingale par changement de loi équivalente a été découverte plus tard par Jacod et Memin, tandis que l' étude générale des équations différentielles stochastiques sort de ses balbutiements avec

C. Doléans-Dade, *On the existence and unicity of solutions of stochastic integral equations.* Z. Wahrscheinlichkeitstheorie and Verw. Gebiete **36** (1976).

Un point de vue original de l'intégrale stochastique, considérée comme intégrale vectorielle dans L^0, a été développé par Metivier et Pellaumail dans une série d'articles. Citons

M. Metivier, et J. Pellaumail, *Mesures stochastiques à valeurs dans des espaces* L^0. Z. Wahrscheinlichkeitstheorie and Verw. Gebiete **40** (1977).

Metivier et Pellaumail ont aussi étendu le calcul différentiel stochastique à des processus à valeurs banachiques. Le lien entre semimartingale et intégrale dans L^0 fourni par le. Théorème 2 de mon exposé est un résultat nouveau que je n'ai pas encore publié.

Le manque de place ne permet malheureusement pas de citer, comme je l'aurais voulu, bien d'autres travaux, et je m'en excuse auprès de leurs auteurs. On pourra trouver de nombreuses références dans les volumes du Séminaire de Probabilités de Strasbourg (Nos. 39, 51 88, 124, 191, 258, 321, 381, 465, 511, 581, 649 des Lecture Notes in Math. publiés par Springer-Verlag et dans les mémoires suivants.

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Dirichlet Spaces and Additive Functionals of Finite Energy

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1. Introduction. In 1959 A. Beurling and J. Deny [1] introduced the notion of the Dirichlet space and revealed all essential substances of the theory, most of which were amplified and proven later in a fine exposition of J. Deny [3].

The link connecting this theory to the Markov process is in the following remarkable fact ([3], [5]): there is a one-to-one correspondence between the family of all Dirichlet forms $\mathscr{E}$ on an L^2-space and the family of all strongly continuous semigroups $\{T_t, t>0\}$ of Markovian symmetric operators on the same L^2-space, the correspondence being specified by

$$\mathscr{D}[\mathscr{E}] = \left\{u \in L^2 : \lim_{t \downarrow 0} \frac{1}{t}(u, u - T_t u)_{L^2} < \infty\right\},$$

$$\mathscr{E}(u, v) = \lim_{t \downarrow 0} \frac{1}{t}(u, v - T_t v)_{L^2}. \tag{1}$$

If the semigroup $\{T_t, t>0\}$ happens to be transient, then the domain $\mathscr{D}[\mathscr{E}]$ can be extended by completion with respect to the 0-order form $\mathscr{E}$ to a Hilbert space which is continuously embedded into a certain weighted L^1-space. The Dirichlet space in the original sense of [1] can be obtained this way ([3], [15] and Appendices of [5]).

Owing to such connections, the author [4], H. Kunita [11], J. Elliott [10], M. L. Silverstein [15] and Y. Le Jan [13] were able to use the Dirichlet forms on L^2-spaces quite effectively in resolving the so-called "boundary problem of Markov processes", which had been formulated and studied before by W. Feller, A. D. Wentzell et al. mainly in the framework of the semigroup theory on the Banach space $\boldsymbol{C}$ of continuous functions. In the meantime it has been shown that every regular Dirichlet

form $\mathscr{E}$ admits a Hunt process M and moreover potential theoretic notions relevant to $\mathscr{E}$ (quasi-continuity, sets of capacity zero, reduced functions and so on) can be interpreted in the language of the Hunt probabilistic potential theory relevant to M (see [5], [6], [15] for symmetric cases and [2], [13] for nonsymmetric cases).

Two different Hunt processes may correspond to the same regular Dirichlet form but their restrictions outside a certain Borel set of capacity zero have the same transition probability [6]. At present we are content with this sort of loose uniqueness of the associated process since the potential theory of the regular Dirichlet form alone can not control inside a set of capacity zero. However it is still important to know whether one can select a nicest version (for instance a Hunt process with a Hölder continuous resolvent). Probably some methods of E. De Giorgi, G. Stampacchia et al. must be brought in before this point is made clearer. See [7] for some related information.

Now we utilize the above mentioned probabilistic potential theory relating the form $\mathscr{E}$ to the process M and study the structure of some important classes of additive functionals of M, namely the class A_c^+ of positive continuous additive functionals (PCAF's) and new classes of additive functionals of finite energy.

We first characterize the class A_c^+ by means of the family S of smooth measures. Since we relax the definition of AF of M slightly by admitting exceptional sets of capacity zero, the family S becomes wider and simpler than the families specified by H. McKean–H. Tanaka, D. Revuz et al. In fact S contains all positive Radon measures charging no set of capacity zero. As an application, H. Nagai [14] has been able to relate the optimal stopping problem of AF's directly to a variational inequality involving the form $\mathscr{E}$ and measures in S.

The *energy* $e(A)$ of a (not necessarily positive) AF A is introduced by

$$e(A) = \lim_{t \downarrow 0} \frac{1}{2t} E_m(A_t^2). \tag{2}$$

The space $\mathring{\mathscr{M}}$ of martingale additive functionals (MAF's) of finite energy is then seen to be complete with metric e. This makes it possible to define the stochastic integrals based on MAF's more simply than M. Motoo–S. Watanabe. Furthermore this leads us to a unique decomposition

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad M^{[u]} \in \mathring{\mathscr{M}}, \ N^{[u]} \in \mathscr{N}_c, \tag{3}$$

for any function u in the Dirichlet space $\mathscr{F} = \mathscr{D}[\mathscr{E}]$, $\mathscr{N}_c$ being the class of CAF's of zero energy.

The novelty of this decomposition lies in that each AF in $\mathscr{N}_c$ is of quadratic variation zero in a weak sense but not necessarily of bounded variation. Thus we are out of the range of semi-martingales and consequently the generalized Ito formula due to H. Kunita–S. Watanabe does not apply in general. Nevertheless we get the

following variant of the Ito formula with the help of a transformation rule of energy measures due to Y. Le Jan [12]:

$$M^{[\Phi(u)]} = \sum_{i=1}^{n} \Phi_{x_i}(u) \cdot M^{[u_i]} \tag{4}$$

for a composite function $\Phi(u)=\Phi(u_1, u_2, \ldots, u_n)$, $u_i \in \mathscr{F}_{\mathrm{loc}}^b$, $1 \leqslant i \leqslant n$. For simplicity we assume here that the process M is a diffusion or equivalently that the form $\mathscr{E}$ possesses the local property.

Formulae (1), (2) and (3) tell us that the *resurrected Dirichlet space* $(\mathscr{F}, \mathscr{E}^{\mathrm{res}})$ introduced in § 4 is isometrically embedded into the Hilbert space $(\mathring{\mathscr{M}}, e)$. Thus we essentially reduce the study of the Dirichlet space to the study of the space of MAF's of finite energy. In particular a calculation of the energy of the both-hand sides of (4) by setting $u_i(x)=x_i$ (the ith coordinate function) immediately gives us the Beurling–Deny formula [1]

$$\mathscr{E}^{\mathrm{res}}(u, v) = \sum_{i,j=1}^{n} \int_{R^n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx), \quad u, v \in C_0^{\infty}(R^n) \tag{5}$$

h olding when the underlying space is the Euclidean n-space and $\mathscr{F}$ possesses $C_0^{\infty}(R^n$ a s its core.

2. PCAF's and smooth measures. Let X be a locally compact separable Hausdorff) space, m be a positive Radon measure on X with Supp $[m]=X$ ans} $M=(\Omega, \mathscr{M}, X_t, P_x)$ be a Hunt process on X which is *m-symmetric* in the sened that the transition function p_t of M satisfies

$$\int_X p_t f(x) g(x)\, m(dx) = \int_X f(x)\, p_t g(x) m(dx), \quad f, g \in \mathscr{B}^+(X).$$

$\{p_t, t>0\}$ then decides uniquely a strongly continuous semigroup $\{T_t, t>0\}$ of Markovian symmetric operators on $L^2(X, m)$ which in turn defines a Dirichlet form $\mathscr{E}$ on $L^2(X, m)$ by the formula (1). We call $\mathscr{E}$ (resp. $\mathscr{F}=\mathscr{D}[\mathscr{E}]$) the Dirichlet form (resp. Dirichlet space) of the Hunt process M.

Our basic assumption is that $\mathscr{E}$ is *regular* in the following sense: $\mathscr{F} \cap C_0(X)$ is $\mathscr{E}_1$-dense in $\mathscr{F}$ and uniformly dense in $C_0(X)$. Here $C_0(X)$ is the space of all continuous functions on X with compact support and $\mathscr{E}_\alpha(u, v)=\mathscr{E}(u, v)+\alpha(u, v)$, $\alpha>0$, $u, v \in \mathscr{F}$, (u, v) being the L^2-inner product.

We call a set $B \subset X$ *properly exceptional* if B is Borel, $m(B)=0$ and the complementary set $X-B$ is *M-invariant*: $P_x(X_t$ or $X_{t-} \in B, \exists t>0)=0$, $\forall x \in X-B$. It is known that a set is of capacity zero (evaluated by the form $\mathscr{E}_1$) if and only if it is contained in a certain properly exceptional set [6].

By an *additive functional* (AF) of the process M, we mean an ordinary (perfect, right continuous, possessing left limits, finite up to the life time ζ) additive functional A of the Hunt process $M|_{X-B}$, B being some properly exceptional set depending on A in general. Two AF's $A^{(1)}$ and $A^{(2)}$ are identified if $\forall t>0\, P_x(A_t^{(1)}=A_t^{(2)})=1$

for q.e. $x\in X$, that is, for every x except on a set of capacity zero. The set of all nonnegative continuous AF's (PCAF's) is denoted by A_c^+.

Let us call a nonnegative Borel measure μ on X *smooth* if μ satisfies the following conditions: μ charges no set of capacity zero and there exists an increasing sequence $\{F_n\}$ of compact sets such that

$$(\mu.1) \qquad P_x(\lim_{n\to\infty} \sigma_{X-F_n} < \infty) = 0 \quad \text{q.e.} \quad x\in X,$$

$$(\mu.2) \qquad \mu(F_n) < \infty, \quad n = 1, 2, \ldots, \quad \mu\left(X - \bigcup_{n=1}^{\infty} F_n\right) = 0.$$

Denote by S the family of all smooth measures.

THEOREM 1 [8]. *The equivalence class of A_c^+ and S are in one-to-one correspondence by the relation*

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}((f\cdot A)_t) = \langle f\cdot\mu, h\rangle, \quad A\in A_c^+, \quad \mu\in S,$$

for any γ-excessive function h ($\gamma\geqslant 0$) and $f\in\mathscr{B}^+(X)$.

The following inequality holding for $A\in A_c^+$ and the associated measure $\mu_A\in S$ plays an important role in the sequel:

$$(6) \qquad E_\nu(A_t) \leqslant (1+t)\|U_1\nu\|_\infty \cdot \mu_A(X) \quad (\leqslant\infty), \quad \nu\in S_{00},$$

where S_{00} is the set of all probability measures on X of finite energy integrals possessing bounded 1-potentials $U_1\nu$. It is known that a set B is of capacity zero if and only if $\nu(B)=0$, $\forall\nu\in S_{00}$.

3. Completeness of $(\mathring{\mathscr{M}}, e)$ and the stochastic integrals. An AF M is said to be a MAF if $\forall t>0$, $E_x(M_t^2)<\infty$, $E_x(M_t)=0$ q.e. The family of all MAF's is denoted by $\mathscr{M}$. Each $M\in\mathscr{M}$ admits its quadratic variation $\langle M\rangle\in A_c^+ : t>0\, E_x(\langle M\rangle_t)=E_x(M_t^2)$ q.e. Let the energy e of AF be defined by (2), then we easily see

$$(7) \qquad e(M) = \frac{1}{2}\mu_{\langle M\rangle}(X), \quad M\in\mathscr{M}.$$

Furthermore e defines a pre-Hilbertian structure in the space $\mathring{\mathscr{M}} = \{M\in\mathscr{M} : e(M)<\infty\}$. Actually (6) and (7) lead us to

THEOREM 2 [9]. *$(\mathring{\mathscr{M}}, e)$ is a real Hilbert space.*

Consider the family $\mathscr{M}_1 = \{M\in\mathscr{M} : \mu_{\langle M\rangle}\ (\in S)$ is a Radon measure$\}$ $(\supset\mathring{\mathscr{M}})$. We have then for $M, L\in\mathscr{M}_1$, $f\in L^2(X;\mu_{\langle M\rangle})$ and $g\in L^2(X;\mu_{\langle L\rangle})$

$$(8) \qquad \left(\int_X |f\cdot g|\, |d\mu_{\langle M,L\rangle}|\right)^2 \leqslant \int_X f^2 d\mu_{\langle M\rangle} \int_X g^2 d\mu_{\langle L\rangle}.$$

In view of (7), (8) and Theorem 2, there exists for $M\in\mathcal{M}_1$ and $f\in L^2(X;\mu_{\langle M\rangle})$ a unique $f\cdot M\in\mathring{\mathcal{M}}$ such that

$$e(f\cdot M, L)=\frac{1}{2}\int_X f(x)\mu_{\langle M,L\rangle}(dx),\quad \forall L\in\mathring{\mathcal{M}}. \tag{9}$$

$f\cdot M$ is called the *stochastic integral* of $f\in L^2(X;\mu_{\langle M\rangle})$ with respect to $M\in\mathcal{M}_1$. Using the inequality (6) again, we can reduce our stochastic integral to the ordinary one due to Motoo–Watanabe relevant to the Hunt process $\boldsymbol{M}|_{X-B}$, B being a suitable properly exceptional set. This identification justifies the rule $f\cdot(g\cdot M)=(fg)\cdot M$.

We now extend the above stochastic integral to a wider class $\mathcal{M}_{1,\mathrm{loc}}$. We say that an AF M is *locally in* $\mathcal{M}_1$ ($M\in\mathcal{M}_{1,\mathrm{loc}}$) if there exist a sequence of relatively compact open sets G_n such that $\bar{G}_n\subset G_{n+1}$, $G_n\uparrow X$, and a sequence of MAF's $M^{(n)}\in\mathcal{M}_1$ such that $M_t=M_t^{(n)}$, $\forall t<\sigma_{X-G_n}$, P_x-a.s. for q.e. $x\in X$. The quadratic variation $\langle M\rangle\in A_c^+$ of M is then well defined by $\langle M\rangle_t=\langle M^{(n)}\rangle_t$, $\forall t<\sigma_{X-G_n}$, $n=1, 2, \ldots$. By making use of Lemma 10 of [8], we further see

$$\int_X f(x)\mu_{\langle M\rangle}(dx)=\int_X f(x)\mu_{\langle M^{(n)}\rangle}(dx)\quad\text{if}\quad \mathrm{Supp}\,[f]\subset G_{n-1} \tag{10}$$

for a bounded Borel f. In particular $\mu_{\langle M\rangle}$ is Radon and (8) extends to the present M. Therefore the stochastic integral $f\cdot M\in\mathring{\mathcal{M}}$ is still well defined by (9) for $M\in\mathcal{M}_{1,\mathrm{loc}}$ and $f\in L^2(X;\mu_{\langle M\rangle})$.

Finally we can define the stochastic integral $f\cdot M\in\mathcal{M}_{1,\mathrm{loc}}$ for any locally bounded Borel function f and $M\in\mathcal{M}_{1,\mathrm{loc}}$ by the formula

$$g\cdot(f\cdot M)=(gf)\cdot M, \tag{11}$$

g ranging over all bounded Borel functions of compact support.

4. A decomposition of the AF $A_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)$, $u\in\mathcal{F}$. Denote by $\tilde{u}$ a quasi-continuous version of $u\in\mathcal{F}$. The formula (1) means that the AF $A^{[u]}$ for $u\in\mathcal{F}$ is of finite energy and

$$e(A^{[u]})=\mathcal{E}^{\mathrm{res}}(u, u) \tag{12}$$

where

$$\mathcal{E}^{\mathrm{res}}(u,v)=\mathcal{E}(u, v)-\int_X \tilde{u}(x)\tilde{v}(x)k(dx),\quad u, v\in\mathcal{F},$$

k being the vague limit of $t^{-1}(1-p_t1)\cdot m$ as $t\downarrow 0$. k is called the *killing measure* and indicates the killing inside X of the sample paths of $\boldsymbol{M}$.

THEOREM 3 [9]. (i) *For each $u\in\mathcal{F}$, the AF $A^{[u]}$ admits a unique decomposition* (2) *where $\mathcal{N}_c=\{N: N$ is a CAF, $e(N)=0$, $E_x(|N_t|)<\infty$ q.e.$\}$.*

(ii) *$N_t^{[u]}\in\mathcal{N}_c$ is of bounded variation in t if and only if there exist two smooth measures $\nu^{(1)}$ and $\nu^{(2)}$ such that*

$$\mathcal{E}(u, v)=\int_X v(x)\big(\nu^{(1)}(dx)-\nu^{(2)}(dx)\big) \tag{13}$$

for any $v\in\mathcal{F}$ vanishing outside some F_k, $\{F_k\}$ being a common nest for $\nu^{(1)}$ and $\nu^{(2)}$.

When M is the one-dimensional Brownian motion,

$$\mathscr{F}=H^1(R^1),\ \mathscr{E}(u,v)=\frac{1}{2}\int_{R^1}u'(x)v'(x)\,dx$$

and the condition (13) reduces to the condition that u' is of bounded variation.

From (2) and (12) we get the isometry from $(\mathscr{F},\mathscr{E}^{\mathrm{res}})$ into $(\mathring{\mathscr{M}},e)$:

$$e(M^{[u]})=\mathscr{E}^{\mathrm{res}}(u,u),\ u\in\mathscr{F}. \tag{14}$$

Put $\mathscr{F}_b=\{u\in\mathscr{F}: u \text{ is bounded}\}$. Theorem 1 then implies the formula

$$\int_X f(x)\mu_{\langle M^{[u]}\rangle}(dx)=2\mathscr{E}^{\mathrm{res}}(u\cdot f,u)-\mathscr{E}^{\mathrm{res}}(u^2,f),\ f,u\in\mathscr{F}_b. \tag{15}$$

5. A stochastic calculus related to the Dirichlet space. For simplicity we assume that M is a diffusion or equivalently that $\mathscr{E}^{\mathrm{res}}(u,v)=0$ whenever v is constant on a neighbourhood of u [6]. The integral in (15) then vanishes when u is constant on a neighbourhood of Supp $[f]$. Hence we have $M_t^{[u_1]}=M_t^{[u_2]}$, $\forall t<\sigma_{X-G}$, P_x-a.s. for q.e. $x\in X$, if $u_1,u_2\in\mathscr{F}_b$ and u_1-u_2 is a constant on an open set G.

A function u is said to be *locally in* $\mathscr{F}_b$ ($u\in\mathscr{F}_{\mathrm{loc}}^b$) if there exists for any relatively compact open set G a function $w\in\mathscr{F}_b$ such that $u=w$ on G. By the above observation, we can see that each $u\in\mathscr{F}_{\mathrm{loc}}^b$ admits uniquely an AF $M^{[u]}\in\mathscr{M}_{1,\mathrm{loc}}$. If $u_1,u_2\in\mathscr{F}_{\mathrm{loc}}^b$ and $u_1-u_2=$constant, then $M^{[u_1]}=M^{[u_2]}$.

THEOREM 4. *The generalized Ito formula* (4) *holds for any* $u_1,u_2,\ldots,u_n\in\mathscr{F}_{\mathrm{loc}}^b$ *and* $\Phi\in C^1(R^n)$ *with bounded first derivatives.*

Especially when u_i's are in $\mathscr{F}$ and Φ vanishes at the origin, $\Phi(u)\in\mathscr{F}$ and the following equation holds for $f\in C_0(X)$, $v\in\mathscr{F}_b$ [12]:

$$\int_X f\,d\mu_{\langle M^{[\Phi(v)]},M^{[v]}\rangle}=\sum_{i=1}^n\int_X f\cdot\Phi_{x_i}(u)\,d\mu_{\langle M^{[u_i]},M^{[v]}\rangle}. \tag{16}$$

This combined with (9) gives formula (4) since $\{f\cdot M^{[v]};\ f\in C_0(X),\ v\in\mathscr{F}_b\}$ is dense in $(\mathring{\mathscr{M}},e)$. Then Theorem 4 readily follows in view of (10) and (11).

If M is an m-symmetric diffusion on R^n and $\mathscr{F}$ possesses $C_0^\infty(R^n)$ as its core, then $x_i\in\mathscr{F}_{\mathrm{loc}}^b$ and we get from (4)

$$M^{[u]}=\sum_{i=1}^n u_{x_i}\cdot M^{[x_i]},\ u\in C_0^\infty(R^n). \tag{17}$$

Now (9), (14) and (17) give formula (5) with $\nu_{ij}=\frac{1}{2}\mu_{\langle M^{[x_i]},M^{[x_j]}\rangle}$, $1\leqq i,j\leqq n$. When ν_{ij} vanishes for $i\neq j$, we have the expression

$$\mathring{\mathscr{M}}=\left\{\sum_{i=1}^n f_i\cdot M^{[x_i]}:\ f_i\in L^2(R^n,\nu_{ii}),\ 1\leqq i\leqq n\right\}. \tag{18}$$

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Strong Theorems on Coin Tossing

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I. The number of heads in short blocks.

I.1. *The length of the longest head run.* The length of the longest head run in a coin tossing sequence was investigated in the very early days of probability theory (see e.g. [10]). A teaching experiment of T. Varga originated a new attack on this problem. His experiment goes like this:

A class of school children is divided into two sections. In one of the sections each child is given a coin which they throw two hundred times, recording the resulting head and tail sequence on a piece of paper. In the other section the children do not receive coins, but are told instead that they should try to write down a "random" head and tail sequence of length two hundred. Collecting these slips of paper, he then tries to subdivide them into their original groups. Most of the times he succeeds quite well. His secret is that he had observed that in a randomly produced sequence of length two hundred, there are, say, head-runs of length seven. On the other hand, he had also observed that most of those children who were to write down an imaginary random sequence are usually afraid of writing down runs of longer than four. Hence, in order to find the slips coming from the coin tossing group, he simply selects the ones which contain runs longer than five.

This experiment led T. Varga to ask: What is the length of the longest run of pure heads in n Bernoulli trials?

Introduce the following notations:

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $P(X_1=0)=P(X_1=1)=1/2$, let $S_0=0$, $S_n=X_1+X_2+\ldots+X_n$ $(n=1, 2, \ldots)$ and

$$I(N, K) = \max_{0\leqq n\leqq N-K}(S_{n+K}-S_n) \quad (N\geqslant K).$$

Define the r.v.'s Z_N $(N=1, 2, \ldots)$ as follows: let Z_N be the largest integer for which $I(N, Z_N)=Z_N$. This Z_n is the length of the longest head-run.

A characterization of the behaviour of Z_n was given by Erdős and Révész. It is presented in this section.

THEOREM 1.1 [7]. *Let ε be any positive number. Then for almost all $\omega\in\Omega$ (the basic space) there exists a finite $N_0=N_0(\omega, \varepsilon)$ such that*[1]

$$Z_N \geqslant [\log N-\log\log\log N+\log\log e-2-\varepsilon] = \alpha_1(N) = \alpha_1$$

if $N\geqslant N_0$.

This result is quite near to the best possible one in the following sense:

THEOREM 1.2 [7]. *Let ε be any positive number. Then for almost all $\omega\in\Omega$ there exists an infinite sequence $N_i=N_i(\omega, \varepsilon)$ $(i=1, 2, \ldots)$ of integers such that*

$$Z_{n_i} < [\log N_i-\log\log\log N_i+\log\log e-1+\varepsilon] = \alpha_2(N) = \alpha_2.$$

Theorems 1.1 and 1.2 together say that the length of the longest head-run is larger than α_1 but in general not larger than $\alpha_2\leqslant\alpha_1+2$. Clearly enough, for some N the length of the longest head-run can be larger than α_2. In our next theorems the largest possible values of Z_N are investigated.

THEOREM 1.3 [7]. *Let $\{\alpha_3(n)\}$ be a sequence of positive numbers for which*

$$\sum_{n=1}^{\infty} 2^{-\alpha_3(n)} = \infty.$$

Then for almost all $\omega\in\Omega$ there exists an infinite sequence $N_i=N_i(\omega, \{\alpha_3(n)\})$ $(i=1, 2, \ldots)$ of integers such that

$$Z_{N_i} \geqslant \alpha_3(N_i).$$

This result is the best possible in the following sense:

THEOREM 1.4 [7]. *Let $\{\alpha_4(n)\}$ be a sequence of positive numbers for which*

$$\sum_{n=1}^{\infty} 2^{-\alpha_4(n)} < \infty.$$

Then for almost all $\omega\in\Omega$ there exists a positive integer $N_0=N_0(\omega, \{\alpha_4(n)\})$ such that

$$Z_N < \alpha_4(N)$$

if $N\geqslant N_0$.

I.2. *The length of blocks containing at most T tails.* Theorems 1.1–1.4 are characterizing the length of the longest run containing no tails at all. One can also ask similar questions about the length of the longest run containing at most T tails.

[1] Here and in what follows log means logarithm with base 2; [x] is the integral part of x.

In order to formulate our results, we introduce the following notations: Let $Z_N(T)$ be the largest integer for which

$$I(N, Z_N(T)) \geqslant Z_N(T)-T.$$

This $Z_N(T)$ is the length of the longest run containing at most T tails. Theorems 1.1–1.4 can be generalized as follows:

THEOREM 1.1* [7]. *Let ε be any positive number. Then for almost all $\omega\in\Omega$ there exists a finite $N_0=N_0(\omega, T, \varepsilon)$ such that*

$$Z_N(T) \geqslant [\log N+T\log\log N-\log\log\log N-\log(T!)+\log\log e-2-\varepsilon] = \alpha_1(N, T)$$

if $N\geqslant N_0$.

THEOREM 1.2* [7]. *Let ε be any positive number. Then for almost all $\omega\in\Omega$ there exists an infinite sequence $N_i=N_i(\omega, T, \varepsilon)$ of integers such that*

$$Z_{N_i} < [\log N+T\log\log N-\log\log\log N-\log(T!)+\log\log e-1+\varepsilon] = \alpha_2(N, T).$$

THEOREM 1.3* [7]. *Let $\{\alpha_3(N, T)\}_{N=1}^{\infty}$ be a sequence of positive integers for which*

$$\sum_{N=1}^{\infty}(\alpha_3(N, T))^{T}2^{-\alpha_3(N,T)} = \infty.$$

Then for almost all $\omega\in\Omega$ there exists an infinite sequence $N_i=N_i(\omega, T, \{\alpha_3(N, T)\})$ of integers such that

$$Z_{N_i}(T) \geqslant \alpha_3(N_i, T).$$

THEOREM 1.4* [7]. *Let $\{\alpha_4(N, T)\}_{N=1}^{\infty}$ be a sequence of positive integers for which*

$$\sum_{N=1}^{\infty}(\alpha_4(N, T))^{T}2^{-\alpha_4(N,T)} < \infty.$$

Then for almost all $\omega\in\Omega$ there exists a positive integer $N_0=N_0(\omega, T, \{\alpha_4(N, T)\})$ such that

$$Z_N(T) < \alpha_4(N, T)$$

if $N\geqslant N_0$.

I.3. *A result of Erdős and Rényi.* Erdős and Rényi were the first ones who investigated the problem proposed by T. Varga, proving strong theorems. One of their results goes like this:

THEOREM 1.5 [6]. *We have*

$$\lim_{N\to\infty}\frac{I(N, c\log N)}{c\log N} = \frac{\alpha(c)+1}{2} \quad (c > 0)$$

with probability 1 *where* $\alpha(1)=1$ *and, if* $c>1$ *then it is the only solution of the equation*

$$\frac{1}{c} = 1-h\left(\frac{1+\alpha}{2}\right)$$

with

$$h(x) = -x\log x-(1-x)\log(1-x) \quad (0<x<1).$$

The herewith defined $\alpha(\cdot)$ *is a decreasing function with*

$$\lim_{c\searrow 1}\alpha(c) = 1 \quad \textit{and} \quad \lim_{c\nearrow\infty}\alpha(c) = 0.$$

I.4. *The number of head runs.* In Theorems 1.1 and 1.2 we have seen that for all N, big enough, there exists a block of size $\alpha_1(N)$ containing only heads but it is not true with $\alpha_2(N)$. Now we ask what is the number of disjoint blocks of size $\alpha_1(N)$ containing only heads.

Let $\nu_N(k)$ be the number of blocks of size k (in the interval $[0, N]$) containing only heads, that is to say $\nu_N(k)=j$ if there exists a sequence $0\leqslant l_1<l_1+k\leqslant l_2<l_2+k\leqslant\ldots\leqslant l_j<l_j+k\leqslant n$ such that

$$S_{l_i+k}-S_{l_i} = k \quad (i=1,2,\ldots,j)$$

but

$$S_{m+k}-S_m<k \quad \text{if} \quad l_i+k\leqslant m<l_{i+1} \quad (i=1,2,\ldots,j).$$

At first we study the case $k=\alpha_1(N)$ and prove

THEOREM 1.6. *For any* $\varepsilon>0$ *there exist constants* $0<c_1=c_1(\varepsilon)\leqslant c_2=c_2(\varepsilon)<\infty$ *such that*

$$c_1 = \liminf_{N\to\infty}\frac{\nu_N(\alpha_1(N))}{\log\log N} \leqslant \limsup_{N\to\infty}\frac{\nu_N(\alpha_1(N))}{\log\log N} = c_2$$

with probability 1.

This theorem says that in the interval $[0,N]$ there are $O(\log\log N)$ blocks of size $\alpha_1(N)$ containing only heads. This fact is quite surprising, knowing that it does happen for infinitely many N that there is not any block of size $\alpha_1(N)+2\geqslant\alpha_2(N)$ containing only heads.

Theorem 1.2 also implies that

$$\liminf_{N\to\infty}\nu_N(k_N) = 0$$

with probability 1 if $k_N\geqslant\alpha_2(N)$. However, Theorems 1.3 and 1.4 imply that

$$\limsup_{N\to\infty}\nu_N([\log N+(1+\delta)\log\log N])\begin{cases}=0 & \text{if} \quad \delta>0,\\ \geqslant 1 & \text{if} \quad \delta\leqslant 0.\end{cases}$$

Now we are interested in

$$\limsup_{N\to\infty}\nu_N([\log N+\log\log N])$$

and formulate our

THEOREM 1.7. *We have*

$$\limsup_{N\to\infty} \nu_N([\log N+\log\log N]) \leqslant 2$$

with probability 1.

II. The number of heads in longer blocks.

II.1. *The most irregular blocks.* As we have already remarked, Theorem 1.5 stated that for any $c>0$ there is a block of size $[c \log N]$ whose density is more than 1/2. This Theorem also says that in any block of size $a_N \gg \log N$ the density of heads is equal to 1/2. That is to say we have

A CONSEQUENCE OF THEOREM 1.5.

$$\lim_{N\to\infty} \frac{I(N, a_N)}{a_N} = 1/2$$

with probability 1, *provided that*

$$\lim_{N\to\infty} \frac{a_N}{\log N} = \infty.$$

Clearly, in order to get more precise results, we have to investigate the properties of the sequence

$$J(N, a_N) = 2I(N, a_N) - a_N.$$

Introduce the following notations:

$$Y_i = 2X_i - 1,\ T_n = \sum_{i=1}^{n} Y_i = 2S_n - n,$$

$$J(N, K) = \max_{0\leqq n\leqq N-K} (T_{n+K} - T_n) = 2I(N, K) - K,$$

$$J^+(N, K) = \max_{0\leqq n\leqq N-K} |T_{n+k} - T_n|,$$

$$\bar{J}(N, K) = \max_{0\leqq n\leqq N-K} \max_{0\leqq l\leqq K} (T_{n+l} - T_n),$$

$$\bar{J}^+(N, K) = \max_{0\leqq n\leqq N-K} \max_{0\leqq l\leqq K} |T_{n+l} - T_n|.$$

The limiting behaviour of these sequences was studied in [3], where the main result says:

THEOREM 2.1 [3]. *Let* $\{a_N\}_{N=1}^{\infty}$ *be a nondecreasing sequence of positive integers for which*

(i) $0 < a_N \leqslant N$ $(N=1, 2, \ldots)$,
(ii) a_N/N *is nonincreasing,*
(iii) $\lim_{N\to\infty} [a_N/\log N] = \infty$.

Then

$$\limsup_{N\to\infty} \beta_N J(N, a_N) = \limsup_{N\to\infty} \beta_N J^+(N, a_N)$$
$$= \limsup_{N\to\infty} \beta_N \bar{J}(N, a_N)$$
$$= \limsup_{N\to\infty} \beta_N \bar{J}^+(N, a_N) = 1$$

with probability 1, *where*

$$\beta_N = (2a_N)[\ln N/a_N + \ln\ln N])^{-1/2}.$$

If we also have

(iv) $\lim_{N\to\infty}[\ln N/a_N/\ln\ln N] = \infty$

then

$$\lim_{N\to\infty} \beta_N J(N, a_N) = \lim_{N\to\infty} \beta_N J^+(N, a_N)$$
$$= \lim_{N\to\infty} \beta_N \bar{J}(N, a_N) = \lim_{N\to\infty} \beta_N \bar{J}^+(N, a_N) = 1$$

with probability 1.

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Some Problems of Large Deviations

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1. Introduction. The problem of large deviations arises naturally in probability theory in different contexts. We shall first look at some typical examples.

If $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed random variables with mean zero, then for any $a>0$

$$\lim_{n\to\infty} P\left[\frac{X_1+\ldots+X_n}{n} \geqslant a\right] = 0 \tag{1.1}$$

by the law of large numbers. Under the assumption that for every real θ

$$M(\theta) = E[e^{\theta X_1}] < \infty$$

Cramér [3] showed that the probability in (1.1) has the specific behavior:

$$\lim_{n\to\infty} \left\{P\left[\frac{X_1+\ldots+X_n}{n} \geqslant a\right]\right\}^{1/n} = \varrho_a \tag{1.2}$$

exists and is given by

$$\varrho_a = \inf_{\theta\geqq 0} M(\theta) e^{-\theta a}.$$

For the second example we consider the process $x_\varepsilon(t)=\varepsilon\beta(t)$ defined in the time interval $0\leqslant t\leqslant 1$, where $\beta(\cdot)$ is the standard one-dimensional Brownian motion. We denote by P_ε the measure corresponding to $x_\varepsilon(t)$, which lives on the space $C_0[0, 1]$ of continuous functions vanishing at $t=0$. As $\varepsilon\to 0$, $x_\varepsilon(t)$ tends to the zero function uniformly and therefore

$$\lim_{\varepsilon\to 0} P_\varepsilon(A) = 0 \tag{1.3}$$

for sets A that are disjoint from some neighborhood of the zero function. One can again show that for a large class of sets $A \subset C_0[0, 1]$

$$\lim_{\varepsilon \to 0} 2\varepsilon^2 \log P_\varepsilon(A) = -h(A) \tag{1.4}$$

exists where

$$h(A) = \inf_{f \in A} \int_0^1 [f'(t)]^2 \, dt \tag{1.5}$$

and the infimum is taken over absolutely continuous functions in f in A with a square integrable derivative. Such a result in a slightly different form can be found in [14].

For the third example we will consider a Markov chain on a finite state space X and transition probabilities π_{ij}. For simplicity we shall assume that $\pi_{ij} > 0$ for all i and j. Let $\vec{q} = \{q_i\}$ be the unique invariant probability vector. If $X_0, X_1, \ldots, X_n, \ldots$ is a realization of the Markov chain, we denote by $f_i^{(n)}$ the frequency of occurrence of the state i during the first n steps of the Markov chain and by $p_i^{(n)}$ the proportion of visits to the state i during the same time. The vector $\vec{p}^{(n)}$ with components $\{p_i^{(n)}\}$ is a random probability distribution on X and is the empirical distribution based on the first n steps. By the ergodic theorem, for large values of n, the vector $\vec{p}^{(n)}$ is close to the invariant probability vector $\vec{q}$ with a very high probability. The theorem on large deviations states in this context that for suitable sets A in space of probability distributions on X

$$\lim_{n \to \infty} \frac{1}{n} \log P\{\vec{p}^{(n)} \in A\} = -h(A) \tag{1.6}$$

exists, where

$$h(A) = \inf_{p \in A} I(\vec{p}) \tag{1.7}$$

and the function $I(\vec{p})$ is defined for probability distributions $\vec{p}$ on X by

$$I(\vec{p}) = -\inf_{u > 0} \sum_{i \in X} p_i \log \left(\frac{\sum_{j \in X} \pi_{ij} u_j}{u_i} \right) \tag{1.8}$$

where the infimum is taken over vectors $u = \{u_i\}$ which have strictly positive components. The function $I(\cdot)$, while it is not explicit, is a nonnegative convex function of $\vec{p}$ which vanishes only when $\vec{p}$ equals the invariant probability $\vec{q}$.

The first example can be generalized by allowing the random variables to take values in more general linear spaces. The second can be generalized to cover processes $x_\varepsilon(t)$ governed by stochastic differential equations of the form

$$dx_\varepsilon(t) = \varepsilon \sigma(x_\varepsilon(t)) \, d\beta(t) + b(x_\varepsilon(t)) \, dt,$$

$$x_\varepsilon(0) = x.$$

Actually one can even combine the two classes in a single large class of problems of large deviations for Markov processes. See for instance [15]–[20], [1], [2], and [13].

As I am hoping that Professor Ventcel will cover this ground in some detail I shall proceed to a discussion of the ideas connected with the third example.

2. Large deviation for occupation times. Let the state space X of the Markov chain be a complete separable metric space instead of a finite set. Let the transition probabilities be given by $\pi(x, dy)$. We shall denote by π, the corresponding operator on the space $B(X)$ of bounded measurable functions on X, defined by

$$(\pi f)(x) = \int f(y)\pi(x, dy). \tag{2.1}$$

We denote by $\mathcal{M}$ the space of all probability distributions on X and we will view $\mathcal{M}$ as a complete separable metric space with weak convergence as the underlying convergence notion. We make the following assumptions on $\pi(x, dy)$.

(H1) *(Feller Property).* If f is bounded and continuous then so is πf.

(H2) *(Strong transitivity).* There is a reference measure $\alpha(dy)$ on X such that $\alpha(dy)$ and $\pi(x, dy)$ are mutually absolutely continuous for every $x \in X$.

(The above condition can be relaxed somewhat.)

For $\mu \in \mathcal{M}$ we define $I(\mu)$ by

$$I(\mu) = -\inf_{u \in U} \int_X \log\left(\frac{\pi u}{u}\right)(x)\mu(dx) \tag{2.2}$$

where U consists of functions in the space $C(X)$ of bounded continuous functions which have a positive lower bound. One can verify that $I(\cdot)$ is a convex, nonnegative, lower semicontinuous functional on $\mathcal{M}$ which vanishes only at invariant probability distributions. The chain of course may not possess an invariant probability distribution in which case $I(\cdot)$ is never zero.

We shall denote by ω, a realization $X_0, X_1, \ldots, X_n, \ldots$ of the chain and by $L_n(\omega, \cdot)$ the empirical distribution based on the first n steps

$$L_n(\omega, E) = \frac{1}{n}\sum_{j=1}^{n} \chi_E(X_j), \quad E \subset X. \tag{2.3}$$

Fixing an arbitrary starting point $x \in X$ for the chain we have the measure P_x corresponding to the process on the space Ω of all possible realizations. One can view $L_n(\cdot, \cdot)$ as a map of Ω into $\mathcal{M}$ and this induces a measure $Q_{n,x}$ on $\mathcal{M}$ which is the distribution of the empirical distribution. If A is a subset of $\mathcal{M}$ then we are interested in the asymptotic behavior of $Q_{n,x}(A)$ as $n \to \infty$. We have the following results:

If $C \subset \mathcal{M}$ is compact, then

$$\limsup_{n\to\infty} \frac{1}{n} \log Q_{n,x}[C] \leq -\inf_{\mu \in C} I(\mu). \tag{2.4}$$

If $G \subset \mathcal{M}$ is open, then

$$\limsup_{n\to\infty} \frac{1}{n} \log Q_{n,x}[G] \geq -\inf_{\mu \in G} I(\mu). \tag{2.5}$$

The reason that (2.4) can be established only for compact sets is that we have no assumption of positive recurrence. Perhaps the measures $Q_{n,x}$ are dissipative and (2.4) really measures the rate of dissipation. It is even possible that $\inf_\mu I(\mu)>0$ and in that case (2.4) cannot hold when C is taken to be the whole space $\mathcal{M}$. However when X is not compact, we can impose a strong positive recurrence condition that will enable us to prove (2.4) for closed sets.

There are continuous time analogs to these results for Markov processes $x(t)$ with transition probabilities $p(t, x, dy)$ on a state space X. We have the corresponding operators

$$(T_t f)(x) = \int f(y)\, p(t, x, dy) \tag{2.6}$$

we make the following assumptions regarding the semigroup $\{T_t\}$.

(H1*) T_t maps the space of bounded continuous functions into itself.

(H2*) There is a reference measure $\alpha(dy)$ such that $\alpha(dy)$ and $p(t, x, dy)$ are mutually absolutely continuous for every $t>0$ and $x\in X$.

(H3*) The strongly continuous center of the semigroup T_t is sufficiently large.

We denote by L the infinitesimal generator acting on the domain $\mathcal{D}\subset C(X)$. We denote by $\mathcal{D}^+$ those functions in $\mathcal{D}$ with a positive lower bound. We define the I-function as the analog of (2.2) by

$$I(\mu) = -\inf_{u\in\mathcal{D}^+} \int \left(\frac{Lu}{u}\right)(x)\, \mu(dx). \tag{2.7}$$

We look at the occupation distribution

$$L_t(\omega, E) = \frac{1}{t}\int_0^t \chi_E(x(s))\, ds \tag{2.8}$$

as the analog of (2.3) and the distribution $Q_{t,x}$ of $L_t(\cdot,\cdot)$ as the analog of $Q_{n,x}$. We then have the exact analogs of (2.4) and (2.5).

For C compact in $\mathcal{M}$,

$$\limsup_{t\to\infty} \frac{1}{t}\log Q_{t,x}(C) \leqslant -\inf_{\mu\in C} I(\mu). \tag{2.9}$$

For G open in $\mathcal{M}$,

$$\liminf_{t\to\infty} \frac{1}{t}\log Q_{t,x}(G) \geqslant -\inf_{\mu\in G} I(\mu). \tag{2.10}$$

Again if X is not compact one can impose a strong recurrence condition and obtain (2.9) for closed sets instead of just compact sets.

If the transition probabilities $p(t, x, dy)$ are such that they have a symmetric density with respect to the reference measure $\alpha(dy)$ then the semigroup T_t is a family of self adjoint contractions in the Hilbert space of functions that are square integrable with respect to the measure $\alpha(dy)$. Then the infinitesimal generator L can be thought of as a nonpositive self adjoint operator. Therefore the operator $\sqrt{-L}$ is well defined. In such a context one can show that $I(\mu)$ is finite if and

only if μ is absolutely continuous with respect to α and the square integrable function $(d\mu/d\alpha)^{1/2}$ is in the domain of $\sqrt{-L}$. In such a case

$$I(\mu) = \left\| \sqrt{-L} \left(\frac{d\mu}{d\alpha} \right)^{1/2} \right\|^2 \tag{2.11}$$

where $\| \ \|$ is the mean square norm with respect to α. For the Brownian motion in R^d this becomes

$$I(f) = I(\mu) = \frac{1}{8} \int \frac{|\nabla f|^2}{f} dx = \frac{1}{2} \int |\nabla \sqrt{f}|^2 dx \tag{2.12}$$

where $\mu(dx) = f(x)\, dx$.

The details of these results can be found in [4], [5], [21].

3. Connections with the principal eigenvalue. Let us consider the case when X is compact. If $x(t)$ is the Markov process corresponding to the generator L and $I(\cdot)$ is the corresponding I-functional defined by (2.7) then one can show [see [15]] that

$$\lambda(V) = \lim_{t \to \infty} \frac{1}{t} \log E_x \left[\exp \left\{ \int_0^t V(x(s))\, dx \right\} \right] \tag{3.1}$$

$$= \sup_{\mu \in \mathcal{M}} \left[\int V(x)\, \mu(dx) - I(\mu) \right]$$

for all $V \in C(X)$. The quantity $\lambda(V)$ can also be identified as the principal eigenvalue (i.e. the point in the spectrum with the largest real part) of the operator $L+V$. From the maximum principle for L one can conclude that $\lambda(V)$ is a convex functional of V and $I(\cdot)$ is its conjugate convex functional. This is the explanation of why $I(\cdot)$ is readily computable in the self-adjoint case. The variational formula (3.1) in that case reduces to the classical Rayleigh–Ritz variational formula for the principal eigenvalue of self adjoint operators. See in this context [6] and [12].

4. Applications. If $\beta(s)$ is the d-dimensional Brownian motion then the Wiener sausage $C_\varepsilon^t(\omega)$ for the trajectory $\omega = \beta(\cdot)$ up to time t is the set

$$C_\varepsilon^t(\omega) = \left\{ y : \inf_{0 \leq s \leq t} |y - \beta(s)| \leq \varepsilon \right\}.$$

A problem that comes from statistical mechanics [11] is the behavior of

$$E[\exp\{-\nu |C_\varepsilon^t(\omega)|\}]$$

where $|A|$ stands for the Lebesgue measure of the set A. Using the techniques developed in the preceding section (see [7], [8]) one can show that

$$\lim_{t \to \infty} \frac{1}{t^{d/(d+2)}} \log E[\exp\{-\nu |C_\varepsilon^t(\omega)|\}] = -k(\nu, d) \tag{4.1}$$

exists and compute $k(v, d)$ as

$$k(v, d) = \inf_{\substack{f \in L_1(R^d) \\ f \geqq 0}} [v|x: f(x) > 0| + I(f)] \tag{4.2}$$

where $I(f)$ is given by (2.12).

One can consider a random walk on the lattice Z^d in R^d of points with integral coordinates. If the distribution of the single step has mean zero and covariance identity then under mild irreducibility assumptions the number $D(n)$ of distinct sites visited by the random walk in the first n steps can be shown to satisfy

$$\lim_{n \to \infty} \frac{1}{n^{d/(d+2)}} \log E[\exp\{-vD(n)\}] = -k(v, d) \tag{4.3}$$

where $k(v, d)$ is again given by (4.2). There are analogs when the Brownian motion is replaced by a symmetric stable process (see [9] for details).

Another application is to derive laws of the iterated logarithm for local times. Let $\omega = \beta(\cdot)$ be one dimensional Brownian motion. Let

$$l(t, y) = \int_0^t \delta(\beta(s) - y)\, ds$$

be the local time of the Brownian path as a function of t and y. We define

$$L(t, y) = \frac{1}{\sqrt{t \log\log t}}\, l\left(t, y\sqrt{\frac{t}{\log\log t}}\right).$$

We can view $L(t, y)$ as a random probability density on the line. One can show that the set of limit points of $L(t, \cdot)$ as $t \to \infty$ coincides as functions of y (in the topology of uniform convergence on compact sets) almost surely with the set of subprobability densities $f(\cdot)$ satisfying

$$\int f(x)\, dx \leqslant 1 \quad \text{and} \quad \frac{1}{8}\int \frac{[f'(x)]^2}{f(x)}\, dx \leqslant 1.$$

These results and similar ones for certain stable processes can be found in [10].

5. General remarks. In the case of continuous time processes the I-functional plays the role of the Dirichlet integral to which it reduces in the self adjoint case. For instance $G \subset X$ is an open set with compact closure and one is interested in the exponential decay rate of $P_x[x(s) \in G, 0 \leqslant s \leqslant t]$ as $t \to \infty$. One can under mild conditions compute it as

$$\lambda(G) = \inf_{\substack{\mu \in \mathcal{M} \\ \mu(G) = 1}} I(\mu).$$

There is a close connection, at least in spirit, to the theory developed by Lanford in [13] in the context of statistical mechanics.

If we start with a Markov process and reverse it in time using an invariant probability distribution then the new reversed Markov process has the same I-functional as the old one. It is an interesting question to examine how much information about the process can be recovered by a knowledge of its I-functional.

In case the process is transient one can consider the total occupation time i.e.

$$L(\omega, E) = \int_0^\infty \chi_E(x(s))\, ds$$

which will be a σ-finite measure on the state space. The tail behavior of its distribution on the space of all σ-finite measures on the state space should again be related to the I-functional. The details of the connection are being currently worked out.

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Большие Уклонения для Случайных Процессов

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1. Пусть ξ_t^h — семейство случайных процессов, зависящее от параметра h, и пусть, скажем, при $h \to 0$ реализации ξ_t^h сходятся по вероятности к неслучайной функции x_t (речь идет о сходимости в некотором метрическом функциональном пространстве).

Простейший пример такого детерминированного в пределе семейства случайных процессов. Пусть ξ_t, $0 \leqslant t < \infty$, — непрерывный справа случайный процесс, такой, что $\xi_0 = x$ неслучайно. Полагаем для $h > 0$: $\xi_t^h = \xi_{ht}$; при $h \to 0$ семейство ξ_t^h сходится к неслучайной константе x.

Задачи о больших уклонениях для семейства процессов ξ_t^h — это задачи о предельном поведении вероятностей $P\{\xi^h \in A\}$ для множеств A, находящихся на положительном расстоянии от неслучайной предельной функции x. Эти задачи аналогичны задачам о больших уклонениях для сумм независимых случайных величин.

В докладе будет говориться о некоторых результатах в этой области, полученных в Москве.

2. Вопросы о том, как получать предельные теоремы о больших уклонениях для случайных процессов, и о том, какие применения они могут иметь, в значительной степени независимы друг от друга. Поговорим о первом.

Среди результатов о больших уклонениях можно выделить два противоположных типа: тип I, аналогичный случаю сложения независимых случайных величин, обладающих конечными экспоненциальными моментами; и тип II, аналогичный случаю сложения случайных величин, скажем, притягивающихся к ненормальному устойчивому распределению. Во II типе ве-

роятности маловероятных событий $P\{\xi^h \in A\}$ образуются в основном за счет реализаций ξ_t^h, совершающих один или несколько больших скачков, тогда как в I типе они образуются в основном за счет реализаций, близких к непрерывным и даже гладким функциям. В частности, если ξ_t — диффузионный процесс, для $\xi_t^h = \xi_{ht}$ имеют место результаты о больших уклонениях I типа; а если ξ_t — марковский процесс, который вблизи точки x может совершать скачки, то для $\xi_t^h = \xi_{ht}$ можно получить результаты II типа.

Почти все полученные результаты относятся к типу I; приведем формулировку одного результата типа II, полученного в [1].

Пусть $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ — последовательность независимых одинаково распределенных случайных величин с функцией распределения $F(x)$, обладающей асимптотикой вида $F(x) = 1 - \gamma_+ x^{-\alpha} + o(x^{-\alpha})$ при $x \to \infty$, $F(x) = \gamma_- |x|^{-\alpha} + o(|x|^{-\alpha})$ при $x \to -\infty$, где $0 < \alpha < 1$ или $1 < \alpha < 2$; в последнем случае предполагается, что $M\xi_i = 0$. Введем на отрезке [0, 1] случайную функцию

$$S_{n,x}(t) = (\xi_1 + \ldots + \xi_{[nt]})/x.$$

Здесь вместо одного числового параметра h речь идет о двух, n и x. Легко доказать, что при $n \to \infty$, $xn^{-1/\alpha} \to \infty$ случайная функция $S_{n,x}$ сходится к $f(t) \equiv 0$. Пусть A — множество в пространстве функций, находящееся на положительном расстоянии от тождественного нуля. Задача о нахождении асимптотики вероятностей вида $P\{S_{n,x} \in A\}$ — это задача о больших уклонениях.

Оказывается, при больших n и $xn^{-1/\alpha}$ среди реализаций случайной функции $S_{n,x}$, далеких от тождественного нуля, наиболее вероятными будут те, которые близки к ступенчатым функциям, делающим один скачок между 0 и 1; среди остальных — реализации, близкие к ступенчатым функциям с двумя скачками; и т. п.

Чтобы точно сформулировать это, введем следующие обозначения Для. $t \in (0, 1]$ и действительного $u \neq 0$ через f_{tu} обозначим ступенчатую функцию со скачком величины u в точке t: $f_{tu}(s) = 0$ при $0 \leqslant s < t$, $f_{tu}(s) = u$ при $t \leqslant s \leqslant 1$. Введем на полосе $(0, 1] \times (R^1 \setminus \{0\})$ меру ν, определяемую соотношениями:

$$\nu(ds\, du) = ds \cdot \gamma_+ \, d(-u^{-\alpha}) \quad \text{на} \quad (0, 1] \times (0, \infty),$$

$$\nu(ds\, du) = ds \cdot \gamma_- \, d|u|^{-\alpha} \quad \text{на} \quad (0,1] \times (-\infty, 0).$$

Для множества A в пространстве функций без разрывов второго рода на [0, 1] обозначим через $[A]$ его замыкание, (A) — множество его внутренних точек, ∂A — границу в смысле равномерной сходимости (соответствующую метрику будем обозначать ϱ). Рассмотрим следующие подмножества полосы $(0, 1] \times (R^1 \setminus \{0\})$:

$$C_+ = \{(t, u): f_{tu} \in [A]\}, \; C_- = \{(t, u): f_{tu} \in (A)\},$$

$$D_\delta = \{(t, u): \varrho(f_{tu}, \partial A) \leqslant \delta\}.$$

Пусть выполнены следующие условия: расстояние от A до тождественного

нуля положительно; множества C_+, C_- измеримы по Жордану относительно меры ν, и $\nu(C_+)=\nu(C_-)$; ν-мера замыкания множества D_δ стремится к 0 при $\delta\downarrow 0$.

Тогда при $n\to\infty$, $xn^{-1/\alpha}\to\infty$,

$$P\{S_{n,x}\in A\}\sim nx^{-\alpha}\cdot\nu(C_+)\quad\left(=nx^{-\alpha}\cdot\nu(C_-)\right).$$

Если A находится на положительном расстоянии от множества всех функций f_{tu}, можно рассмотреть все ступенчатые функции с двумя скачками $f_{t_1u_1t_2u_2}$, попадающие в $[A]$, (A), и при определенных условиях получить, что $P\{S_{n,x}\in A\}$ эквивалентно какой-то константе, умноженной на $(nx^{-\alpha})^2$; и т. д.

3. Большинство полученных результатов I типа — грубые, т. е. с точностью до догарифмической эквивалентности. Один из возможных путей получения таких результатов — следующий. Для случайных процессов ξ_t определенного класса, независимо от того, входят они в какие-либо семейства или нет, вводится функционал $I(\varphi)$, который призван показывать степень трудности прохождения реализации процесса вблизи функции φ. Выводятся оценки такого типа:

$$P\{\varrho(\xi,\varphi)<\delta\}\geqslant\exp\{-I(\varphi)-R_1(\varphi,\delta)\},$$
$$P\{\varrho(\xi,\Phi(i))>\delta\}\leqslant\exp\{-i+R_2(i,\delta)\},$$

где ϱ — метрика в функциональном пространстве, $\Phi(i)$ — множество функций φ, для которых $I(\varphi)\leqslant i$.

Для семейства процессов ξ_t^h функционал I и остаточные члены R_1, R_2 будут зависеть от параметра h: $I^h(\varphi)$, $R_1^h(\varphi,\delta)$, $R_2^h(i,\delta)$. Может оказаться, что функционал I^h асимптотически разбивается на множители: $I^h(\varphi)\sim f(h)\cdot S(\varphi)$. Если при этом остаточные члены $R_1^h(\varphi,\delta)$ и $R_2^h(f(h)\cdot s,\delta)$ бесконечно малы по сравнению с $f(h)$, то отсюда получаются грубые предельные теоремы о больших уклонениях для семейства ξ_t^h.

Оценки указанного типа для определенного класса марковских случайных процессов получены в первой из статей [2]. При их выводе используется обобщение техники, предложенной в [3]. Во второй статье [2] из этих оценок выведены грубые теоремы о больших уклонениях для некоторого класса семейств детерминированных в пределе марковских процессов; но они могут быть применены к получению грубых предельных теорем и в других схемах.

Из немарковских схем, в которых были получены грубые результаты о больших уклонениях, следует упомянуть схему, связанную с принципом усреднения. Если ξ_t — скажем, стационарный процесс с достаточно хорошими свойствами перемешивания, то решение уравнения

$$\dot{x}_t^\varepsilon=b(x_t^\varepsilon,\xi_{t/\varepsilon})$$

при $\varepsilon\to 0$ сходится к неслучайному решению уравнения $\dot{x}_t=\bar{b}(x_t)$, где $\bar{b}(x)$ определяется как математическое ожидание $b(x,\xi_s)$. Результаты о больших уклонениях x_t^ε от x_t получены в [4].

Некоторые точные результаты для марковского случая получены в [5]. Здесь. также применяется обобщение техники [3].

4. Среди применений теорем о больших уклонениях для случайных процессов можно упомянуть обширную область, связанную с поведением однородных по времени процессов с малой случайностью на больших отрезках времени; для диффузионных процессов с малой диффузией (см. [6]). Эти задачи связаны с асимптотическими задачами для уравнений в частных производных эллиптического типа, зависящих от параметра.

Интересная новая область приложений — приложения к вопросам сходимости с вероятностью 1 процедур стохастической аппроксимации [7]. А. П. Коростелев получает необходимые и достаточные условия сходимости с вероятностью 1 в широком классе схем стохастической аппроксимации в случае, когда у «помехи» есть экспоненциальные моменты, с использованием оценок [2]

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Minimal Surfaces: Tangent Cones, Singularities, and Topological Types

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This article reports on some recent contributions of geometric measure theory [**FH1**], [**FH2**] to the study of minimal surfaces and several questions in nonlinear partial differential equations.

Some new regularity results. (1) The most massive (in terms of pages of exposition) of the new regularity results is that k dimensional mass minimizing integral currents in R^n have supports which are almost everywhere regular [**AF2**]. The methods are the first which have successfully treated higher dimensional generalized branching behavior. In particular, the proof contains strong estimates on the branching of functions taking values in the Q-fold symmetric product of a vectorspace and minimizing a generalized Dirichlet integral as well as a new generation of approximation estimates in terms of the "area excess" parameter of [**AF1**], [**FH1**].

(2) New integral curvature estimates were used in [**ASS**] to show that the singular sets of n dimensional oriented hypersurfaces minimizing the integrals of elliptic integrands have zero $n-2$ dimensional Hausdorff measure; in particular, for 2 dimensional surfaces in R^3 there are no singularities. Additionally, the maximum Hausdorff dimension of singular sets corresponding to a particular integrand is upper semicontinuous as a function of the integrand, a fact exploited in [**SL**] to obtain a Bernstein type theorem (that global nonparametric solutions are affine) for the Euler–Lagrange equations associated with integrands close to area in dimensions up to 7.

(3) Solutions to the constant coefficient problems of (2) are regular near any extreme boundary point according to [**HR**], while by [**HS**] area minimizing hyper-

surfaces are everywhere regular at the boundary—a new result even in dimension 2 and leading to other *a priori* estimates.

(4) A novel decomposition technique in [**WB**] shows that for even integrands any minimizing hypersurface modulo 4 locally can be written as the sum of two intersecting oriented minimizing hypersurfaces of the type to which the results of (2) and (3) apply.

Uniqueness of tangent cones to minimal surfaces. Although minimal surfaces long have been known to have tangent cones at all points, the general uniqueness of such cones remains an open question. However, for the case of an m dimensional stationary surface V in R^{p+1} having an isolated singular point at 0 it is shown in [**AA**] that:

(a) If one of the tangent cones to V at 0 is of the form $0 \divideontimes M$ where M is an m dimensional minimal submanifold of S^p and if also for each Jacobi normal vectorfield Z of M in S^p there is a one parameter family of minimal surfaces in S^p having velocity Z at M, then $0 \divideontimes M$ is the unique tangent cone to V at 0 with V converging to $0 \divideontimes M$ for small radii r with rate r^μ.

(b) In case M is the cartesian product of two standard spheres of appropriate radii then each Jacobi vectorfield on M arises from isometric motions of S^p and the Jacobi vectorfield hypothesis in (a) is satisfied. The example of [**BD**] shows this hypothesis is not satisfied for some M.

Relations between boundary curves and minimal surfaces. For a uniformly extreme smooth simple closed boundary curve C there is always a minimal embedding of a surface S with k handles anytime adding the last handle decreases area [**AS**] (with similar results for collections of boundary curves). For S with no handles (i.e. a disk) embedding S minimizes area among immersions as well [**MY**], although this is false in general for minimizing oriented elliptic integrals [**TJ4**]. Examples show that total curvature restrictions on C cannot dominate the genus of absolutely minimizing S.

For C not necessarily extreme the "convex hull genus" of [**AT**] provides a lower bound for the genus of any minimal S with boundary C. The convex hull genus is computed explicitly in [**HJ**] for "almost convex" C, while for general C [**KF**] suggests a clever experimental procedure for obtaining an upper bound.

In case C bounds two surfaces, each locally of least area, variational methods in the large [**PJ**] show it must also bound a third, generally unstable, embedded minimal surface, possibly of different topological type than either of the first two. Although there are infinite parameter families of curves for which surfaces of least area are not unique, the chance of picking such a curve of nonuniqueness at random is zero according to [**MF**].

Modeling of physical phenomena. (1) *Soap bubbles, etc.* The partitioning hypersurfaces of [**AF1**] realistically model soap bubble configurations. The possible local geometries of this model were analyzed in [**TJ1**] providing a mathematical verifica-

tion of the century old observations of J. Plateau. Higher differentiability of singular soap film curves is shown in [**NJ**] while various boundary regularity estimates appear in [**TJ2**].

(2) *Grain boundary migration.* A mathematical analysis [**BK**] has been made in general dimensions of configurations of surfaces of no inertial mass driven by surface tension and opposed by frictional forces proportional to velocity. The motion of grain boundaries in an annealing pure metal exhibits such behavior, and appendix B of [**BK**] corrects a calculation in the metallurgy literature.

(3) *Crystal growth.* The normal growth velocity functions $S^2 \to R^+$ which are possible for a general time parametrized family of solids are characterized in [**FJ**] resulting in a constructive model for the growth of physical crystals under conditions of near thermodynamic equilibrium and arbitrary initial shape. Variational problems modeled on crystal phenomena are also a central theme of [**TJ3**].

(4) *Turbulent fluid flow.* Corresponding to initial conditions for any finite energy divergence free velocity vectorfield there is a fluid flow weakly satisfying the Navier–Stokes equations of viscous incompressible fluids with velocity which is continuous in 4 dimensional space-time except on a closed set of Hausdorff dimension at most 2 [**SV2**]. Furthermore, the closed set of times at which velocity discontinuities can occur has Hausdorff dimension not exceeding 1/2 [**SV1**].

(5) *Elastic deformation of solids.* In deforming a general nonlinearly elastic body, the amount by which the motion differs from an isometry is dominated by the nonlinear elastic work done (expressed only as a weak volume integral with no pointwise hypotheses) according to [**KR**].

(6) *Explicit computation of solutions.* Significant progress towards methods of explicit geometric computation is being made [**PH**], [**TJ3**].

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Propagation of Singularities of Solutions of Symmetric Hyperbolic Systems

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1. We report certain results about the propagation of singularities (WF's) of solutions of symmetric hyperbolic systems of the first order in a "free space" as well as near the boundary on which adequate (e.g. strictly dissipative) boundary conditions are given. Here two analytic theorems are of primary importance; all the other results are their corollaries though by no means simple ones; the proofs of these corollaries are based on the examination of bicharacteristics, bicharacteristical billiards and on the construction of auxiliary functions; they do not refer to the theory of pdo's. The proofs of the main theorems are based on energy estimates. Our results do not depend on lower terms and boundary conditions; they are determined only by the symplectic geometry of the characteristic variety or the pair comprised by the characteristic variety and a boundary. Moreover, we will report a number of finer results about the propagation of WF's of solutions of a wave equation near the boundary; the latter results are based on special analytic theorems which we are obliged to leave out because of a lack of space.

1.1. Let us first consider the case of the "free space". Let P be a classical $d\times d$-matrix pdo of the first order, P_1 being its principal symbol, $g=\det P_1$. It is known that $\mathrm{WF}(u)\setminus\mathrm{WF}(Pu)\subset\mathrm{Char}\,P=\{g=0\}$ and if g is real then $\mathrm{WF}(u)$ is invariant of flow along pieces of bicharacteristics of g anti-crossed with $\mathrm{WF}(Pu)$.

THEOREM. *Let P_1 be a selfadjoint matrix, Ω be an open conic with respect to ξ bounded with respect to x set, φ be a real smooth function positively homogeneous of degree 0 with respect to ξ whereby*

$$\langle\{P_1,\varphi\}(\varrho)v,v\rangle>0\quad\forall v\in\mathrm{Ker}\,P_1(\varrho)\setminus 0,\quad\forall\varrho\in\mathrm{Char}\,P\cap\Omega. \tag{1}$$

Then if $u\in D'$, $\mathrm{WF}(Pu)\cap\Omega\cap\{\varphi\leqslant 0\}\cap\mathrm{Char}\,P=\emptyset$, $\mathrm{WF}(u)\cap\partial\Omega\cap\{\varphi\leqslant 0\}\cap\mathrm{Char}\,P=\emptyset$, *then* $\mathrm{WF}(u)\cap\Omega\cap\{\varphi\leqslant 0\}\cap\mathrm{Char}\,P=\emptyset$.

The same holds true for WF^s.

We will assume that P_1 is a self-adjoint matrix and that (1) is true with $\varphi=x_0$. Then (1) is equivalent to the condition

$$(H_\varphi^k H_{x_0}^{r-k}g)(\varrho)(H_{x_0}^r g)(\varrho)>0,\ k=1,\dots,r(\varrho),\ \varrho\in\mathrm{Char}\,P\cap\Omega,$$

where H_f is a Hamiltonian field generated by f, $r(\varrho)=\dim\mathrm{Ker}\,P_1(\varrho)$.

It follows from here that if microlocally $g=eh^r$, e is the elliptic symbol, h is a real symbol of the principal type then WF's are flown along the bicharacteristics of h.

1.2. Let g be microlocally factorised, $g=eh_1^{r_1}\dots h_s^{r_s}$, e is the elliptic symbol, $h_1,\dots,h_s$ are real symbols of the principal type. It is natural to expect that WF's propagate along broken lines composed of pieces of bicharacteristics of $h_1,\dots,h_s$ and that the closure of the union of all such broken lines radiating from ϱ makes up the influence set of ϱ whose structure can vary widely depending on the symplectic structure of $h_1,\dots,h_s$. This hypothesis is confirmed in a number of cases. Let $s=2$, $\{h_1,h_s\}\neq 0$ when $h_1=h_2=0$, or let $s=2$, $\{h_1,\{h_1,h_2\}\}\cdot\{h_2,\{h_1,h_2\}\}>0$ when $h_1=h_2=\{h_1,h_2\}=0$, or finally let $s>2$ and $\{h_i,h_j\}>0$ when $i>j$. In all these cases as well as in some others WF's propagate along "trees" composed of pieces of bicharacteristics of $h_1,\dots,h_s$. If, on the other hand, co-multipliers are in involution, i.e. $\{h_i,h_j\}=0$ when $h_i=h_j=0$ and H_{h_i} $(i=1,\dots,s)$ are linearly independent modulo $\xi\partial/\partial\xi$ then the influence set of $\varrho\in\cap\{h_i=0\}$ will be an s-dimensional bicharacteristical leaf—a piece of a variety tangent to vector fields H_{h_i} $(i=1,\dots,s)$ and bounded by $(s-1)$-dimensional varieties Γ_j tangent to H_{h_i} $(i=1,\dots,s,\ i\neq j)$. In case the comultipliers are divided into groups so that the comultipliers of the same group are in involution whereas the comultipliers of different groups are not, and $\{h_i,h_j\}>0$, if h_i belongs to a group with a smaller number than h_j, then WF's propagate along the "trees" composed of bicharacteristical leaves corresponding to different groups.

However, as the example of J. Ralston shows, this hypothesis is not always true, and WF's may propagate not only along bicharacteristic broken lines; in this example $s=2$, h_2 has 0 of the infinite order along the bicharacteristics of h_1. Other examples of this sort are also possible; the comultipliers in these examples are not in involution but an infinite number of switchings from bicharacteristics of one to the bicharacteristics of another comultiplier on a limited piece of a broken line is still possible.

1.3. If the characteristic symbol is not completely factorised then the study is considerably complicated. The analysis of the behaviour of bicharacteristics constitutes here one of the principal difficulties. Nevertheless, it is possible to investigate the propagation of WF's in the situation of a locally general position when the characteristics multiplicity is no higher than 2. The same situations are those of

a locally general position in the cases of crystal optics and the theory of elasticity in heterogeneous anisotropic medias systems.

Let $\Sigma=\{g=H_g=0\}$. Suppose that Σ is C^∞-variety, contraction on Σ of a symplectic form has a defect 0 or 1 and $T_\varrho\Sigma=\operatorname{Ker} F(\varrho)$, $\operatorname{Ker} F^2(\varrho)\cap\operatorname{Im}F^2(\varrho)=0$ in every point $\varrho\in\Sigma$, where $F(\varrho)$ is the fundamental matrix of g in ϱ. Then, either $F(\varrho)$ has two real proper numbers different from 0, and the other proper numbers are purely imaginary, or all the proper numbers are purely imaginary. The latter case can only occur if codim Σ is odd. In the former case, Σ is a stationary variety of the saddle type: there exist the C^∞-varieties Σ_1, $\Sigma_2\subset\operatorname{Char}P$, $\Sigma_1\cap\Sigma_2=\Sigma$, $\dim\Sigma_1=\dim\Sigma_2=\dim\Sigma+1$, Σ_1 and Σ_2 are transversally crossed such that the bicharacteristics of g crossing Σ_j are contained in Σ_j and have the limit points belonging to Σ, bicharacteristics lying in $\operatorname{Char}P\setminus(\Sigma_1\cup\Sigma_2)$ have no limit points in Σ. Into every point $\varrho\in\Sigma$ there enter two bicharacteristics parametrised by x_0 and two bicharacteristics leave it. In this case WF's propagate along the bicharacteristics lying in $\operatorname{Char}P\setminus(\Sigma_1\cup\Sigma_2)$ and along the bicharacteristics lying in $\Sigma_1\cup\Sigma_2$ with a branching to Σ.

In the second case Σ is a stationary variety of the centre type. The simple bicharacteristics of g have no limit points in and are continually at a distance of the same order from Σ, "winding" onto curves lying in Σ along which $d\varrho/dt\in\operatorname{Im}F(\varrho)$. (This condition in our assumption defines the family of curves covering Σ only once; we will call them limit bicharacteristics.) In this case WF's propagate along bicharacteristics and limit bicharacteristics.

Thus the conical refraction can only occur in the situations of ungeneral position. In particular, it takes place when Σ is an involutive C^∞-variety. The question of the propagation of WF's where Σ changes type (centre for saddle or vice versa) remains unsolved.

2. 2.1. Let us come now to the discussion of boundary value problems for symmetric hyperbolic systems. Let $X=\{x_1\geqslant 0\}=\overline{R}^+\times X'$; $Y=\{x_1=0\}$, $x'=(x_0,\ldots,x_{l-1})$, $j\colon T^*X|_Y\to T^*Y$ is a natural map. Let $P=K(x,D')D_1+A(x,D')$, K,A are classical $d\times d$-matriced pdo's of orders 0 and 1, respectively; P,K,A consist of diagonal blocks $P_{(\nu)},K_{(\nu)},A_{(\nu)}$. Suppose that $K_0|_Y$ is invertible, K_0,A_1 are selfadjoint matrices. Let $B(x',D')$ be a classical $d_+\times d$-matrix pdo of order 0 on Y, d_+, the number of positive eigenvalues of $K_0|_Y$. We assume that B is a strictly dissipative boundary operator for P, i.e. $\langle K_0(\varrho)v,v\rangle<0$ $\forall v\in\operatorname{Ker}B_0(\varrho)\setminus 0$.

Assuming that B is dissipative the results obtained will not be true, generally speaking. In fact there exist conditions intermediary between dissipativity and strict dissipativity under which our results hold true. Many boundary value problems relate to the ones described here, in particular, the problems for wave equation meeting the Šapiro–Lopatinskii condition, also the Neumann problem, some problems for the Maxwell system etc.

To describe the singularities let us introduce $\mathrm{WF}_b(u)\subset T^*Y\setminus 0$; $\varrho\notin\mathrm{WF}_b(u)$ if there exists $q(x,D')$, i.e. a classical scalar order 0 pdo such that $q_0(\varrho)\neq 0$ and

$qu \in C^\infty(X)$. Let us introduce a full wave front $\mathrm{WF}_f(u) = \mathrm{WF}(u) \cup j^{-1}\mathrm{WF}_b(u)$. Thus here already we glue up T^*X on Y equalizing the points $\varrho, \varrho' \in T^*X|_Y$ such that $j\varrho = j\varrho'$.

THEOREM. *Let the assumptions described above be fulfilled in the neighbourhood of* ϱ^*. *Let* $\varphi_\nu(x, \xi')$ *be real smooth positively homogeneous functions of degree* 0 *with respect to* ξ' *coinciding on* T^*Y *such that* $\varphi_\nu(\varrho^*) = 0$, $\langle \{P_{(\nu)1}, \varphi_\nu\}(\varrho)v, v\rangle > 0$

$$\forall \varrho \in j^{-1}\varrho^* \ \forall v \in \mathrm{Ker}\, P_{(\nu)1}(\varrho) \setminus 0 \ \forall \nu. \tag{2}$$

Then if $u \in C^\infty(\overline{R}^+, D'(X'))$, $\varrho^* \notin \mathrm{WF}_b(Pu) \cup \mathrm{WF}(Bu|_Y)$,

$$\mathrm{WF}_f(u_{(\nu)}) \cap \{\varphi_\nu < 0\} \cap \Omega \cap \mathrm{Char}\, P = \emptyset \quad \forall \nu$$

where $u_{(\nu)}$ *are blocks of* u, Ω *is a sufficiently narrow neighbourhood of* $j^{-1}\varrho^*$ *in* T^*X *then* $\varrho^* \notin \mathrm{WF}_b(u)$.

The same is true for WF^s.

We will assume that (2) is fulfilled with $\varphi_\nu = x_0$. Then (2) can be reformulated in terms of g as in the case of (1).

As seen from the theorem, if B is a suitable boundary operator then its choice does not influence the propagation of WF's although in some cases this can result in a poorer propagation. Nevertheless the description of the propagation of WF's near the boundary is considerably more complex than in the free space since the symplectic geometry of the pair $\{\mathrm{Char}\, P, Y\}$ is more complex than the geometry of Char P, and the analysis of the behaviour of (branching) bicharacteristics billiards is more complex than that of bicharacteristics.

2.2. Let us first consider systems with the characteristics of constant multiplicity assuming for the sake of simplicity that the multiplicity equals 1. Let in the region that is of interest to us there be no $\mathrm{WF}_f(Pu)$, $\mathrm{WF}(Bu|_Y)$.

From the theorem there follows the well-known result that if there exist only the transversal bicharacteristics then WF's come to Y along the incoming bicharacteristics and go away from Y along the outgoing ones and propagate along bicharacteristic billiards branching on Y. In general transversal bicharacteristics of constant multiplicity can be excluded from the microlocal analysis which we will do for simplicity.

Let over $\bar{\varrho} \in T^*Y$ there lie one tangent bicharacteristic and the polynom $g(\bar{\varrho}, \xi_l)$ have a double root $\bar{\xi}_l$. Then if X is strictly bicharacteristically convex in $\hat{\varrho} = (\bar{\varrho}, \bar{\xi}_l)$ i.e. if $(H_g^2 X_l)(\hat{\varrho}) < 0$ then WF's propagate (near $\bar{\varrho}$) both along bicharacteristic billiards of g and along boundary bicharacteristic, i.e. those of the boundary symbol $g_b(\varrho) = g(\varrho, \xi_1(\varrho))$ given on T^*Y where $\xi_1(\varrho)$ is the root of the equation $g_{\xi_1}(\varrho, \xi_1) = 0$. In the case of X being bicharacteristically concave in $\hat{\varrho}$, $(H_g^2 X_l)(\hat{\varrho}) > 0$ the principal theorem does not give a precise description of the propagation of WF's; here the results are likely to correspond to those of § 1.1.

Now let over $\bar{\varrho}$ there lie several tangent bicharacteristics each of which meets the assumptions given above. Then there may occur various effects depending on the interaction of the boundary symbols of $g_{b,j}$. We have only managed to analyse the case when $g_{b,j}$ are in involution and $H_{g_{b,j}}$ are linearly independent modullo $\xi'\partial/\partial\xi'$; then there appears the propagation along boundary bicharacteristic leaves constructed according to the set of $g_{b,j}$.

Finally, let over $\bar{\varrho}$ there lie one tangent bicharacteristic, and the multiplicity of ξ_l, i.e. the root of $g(\bar{\varrho}, \xi_1)$ is equal to 3, $g_{\xi_0}(\hat{\varrho})>0$. Then if $g_{\xi_l\xi_l\xi_l}(\hat{\varrho})<0$ and $(H_g^2 X_l)(\hat{\varrho})>0$ one tangent bicharacteristic comes into $\hat{\varrho}$ and if $\mathrm{WF}(u)$ does not come along it then $\bar{\varrho}\notin \mathrm{WF}_b(u)$. In the case of $g_{\xi_l\xi_l\xi_l}(\hat{\varrho})>0$ and $(H_g^2 X_l)(\hat{\varrho})<0$ there are no bicharacteristics coming into $\hat{\varrho}$ and $\bar{\varrho}\notin \mathrm{WF}_b(u)$. Unfortunately we have not managed to analyse the two other cases. It should be noted that in the case of $(H_g^2 X_l)(\hat{\varrho})>0$ (X is strictly bicharacteristically concave) we have only obtained the results for the points with the multiplicity of the root of $g(\varrho, \xi_1)$ equal to 3, but not for those lying near the points where there is one simple and one double root.

2.3. The propagation of WF's of the solutions of boundary value problems for systems with the characteristics of variable multiplicity is rather poorly investigated (we do not discuss the case of systems with constant coefficients in a half- or quarter-space).

As before, let there be no WF (Pu), $\mathrm{WF}(Bu|_Y)$ in the region which is of interest for us.

Let the characteristic symbol be factorised, $g=eh_1^{r_1}\dots h_s^{r_s}$, e is elliptic on $j^{-1}\bar{\varrho}$, h_j are real symbols of the principal type such that the equation $h_j(\varrho, \xi_1)=0$ has only one root, $h_{j,\xi_0}>0$ and the bicharacteristics h_j are transversal to Y. Suppose that if $h_{i_1}=\dots=h_{i_t}=0$ then $H_{h_{i_1}}\dots H_{h_{i_t}}$ are linearly independent modulo $\xi\partial/\partial\xi$, H_X and that the comultipliers are in involution. Let us first assume that $h_i(\bar{\varrho}, \xi_1)=0$ for all i. Then WF may come into $\bar{\varrho}$ along a Γ^--bicharacteristic leaf constructed according to $h_1, \dots, h_s$ (more precisely along its intersection with T^*X) if there are symbols with incoming bicharacteristics among the comultipliers. If among them there is one and only one symbol with incoming bicharacteristics (h_s, for example) then $j\Gamma^-|_Y$ itself is a bicharacteristic leaf constructed according to $f_1, \dots, f_{s-1}$, where f_j are symbols on T^*Y, $f_j=\alpha_j h_j+\beta_j h_s$, $\alpha_j>0$, $f_{j,\xi_1}\equiv 0$.

Now let the comultipliers $h_1, \dots, h_s$ be broken into groups so that in every group there is not more than one symbol with an incoming bicharacteristic and only the symbols belonging to the same group may vanish simultaneously. Let all the boundary symbols $f_1, \dots, f_m$ collected from all the groups be in involution and $H_{f_1}, \dots, H_{f_m}$ are linearly independent modulo $\xi'\partial/\partial\xi'$. Then WF may come into $\bar{\varrho}$ either along a bicharacteristic leaf corresponding to any group of comultipliers or along a boundary bicharacteristic leaf constructed according to $f_1, \dots, f_m$.

Now let $s=2$, the bicharacteristics of $h_1(h_2)$ are incoming (outgoing) and $\{\{h_1, h_2\}^2, f\}<0$ where f is a boundary symbol. Then WF may come into $\bar{\varrho}$ along

the bicharacteristic of h_1 or along the boundary bicharacteristic of f. It is not known whether WF may really come along the boundary bicharacteristic.

Finally, let g be a polynom of the 2 degree with respect to ξ_l, $\Sigma=\{g=H_g=0\}$ is involutive C^∞-variety, $T_\varrho\Sigma=\operatorname{Ker} F(\varrho)$, $\operatorname{Im} F(\varrho)\cap\{\alpha\xi\partial/\partial\xi+\beta H_{x_l}\}=0$, X be bicharacteristically strictly convex in the neighbourhood of $\bar\varrho$, $j^{-1}\bar\varrho\cap\operatorname{Char} P=j^{-1}\bar\varrho\cap\Sigma$ consist of one point $\hat\varrho$. Then the influence set for $\bar\varrho$ will be that for $\hat\varrho$ constructed in the free space intersected with T^*X. This result allows us to describe the propagation of WF's for the systems with constant coefficients in strictly convex domains such that the polynom $g(\varrho,\xi_l)$ has not more than one multiple root and its multiplicity is equal to 2.

3. For a wave equation in cylindrical domains with respect to x_0 we have obtained more precise results. If the boundary operator B meets the Šapiro–Lopatinskii condition and Y, the boundary is not degenerated then WF's propagate along generalised billiards containing both pieces of bicharacteristics and boundary bicharacteristics. Here such effects are possible as the entrance of WF's into the boundary and their exit from it.

If the Šapiro–Lopatinskii condition is violated then the mixed problem is, generally speaking, incorrect and one should use not the energetic method but the pseudo-convexity method. Let $P=-D_0^2+\sum_j D_j^2$, $B=D_n+\mu$, $D_n=-i\partial/\partial n$, n be a unit normal to Y, μ be a classical 1 st order pdo on Y. Then out of $\Sigma=\operatorname{Char} P\cap T^*Y$ the study of the propagation of WF's is reduced to that for some *pd*-equations on Y. Let us discuss the propagation of WF's near Σ, more precisely near $\Lambda=\{\varrho\in T^*Y, D=v=\sigma=0\}$ where the Šapiro–Lopatinskii condition is violated. If $\xi_0\{P,\sigma\}>0$, $\{P,v\}<0$ then WF's propagate as before along generalised billiards. Let $\sigma\equiv0$, $\{P,v\}<0$; then WF's propagate along generalised billiards into which however new boundary pieces-bicharacteristics of the symbol $(-\sqrt{-P}\operatorname{sign}\xi_0+v)$ are being involved. Let $v\equiv0$ (then B is dissipative but not strictly dissipative), $\{P,\sigma\}\neq0$. Then WF's propagate along generalised billiards containing also boundary pieces-bicharacteristics of the symbol $(\sqrt{P}+\sigma)$.

For a two-dimensional wave equation we have managed to study the propagation of WF's in regions having angular points with angles other than $0,\pi,2\pi$ whose smooth pieces of boundary are not degenerated. Let $Pu=0$, $u|_Y=0$. Then WF's propagate along generalised billiards containing pieces of bicharacteristics; in corners these billiards may reflect from sides or simply continue, or they may diffract as well. When diffracting the smoothness of the solutions increases by 1/2.

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Algebraic Curves of Infinite Genus Arising in the Theory of Nonlinear Waves

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1. Introduction. The study of nonlinear waves on the circle leads naturally to very special classes of algebraic curves of infinite genus $g=\infty$. § 2 describes the general scheme whereby the original flow is converted into a straight-line motion at constant speed on the Jacobi variety of the curve. §§ 3–5 describe how the scheme is applied to three examples: (1) Korteweg–de Vries $\partial q/\partial t=3q\partial q/\partial x-(1/2)\partial^3 q/\partial x^3$; (2) sine–Gordon $\partial^2 q/\partial t^2-\partial^2 q/\partial x^2+\sin q=0$; and (3) Boussinesq $\partial^2 q/\partial t^2=(\partial^2/\partial x^2)(q^2+\partial^2 q/\partial x^2)$. (1) is completely understood; see Dubrovin–Novikoff [4], Its–Matveev [12], McKean–van Moerbeke [16], and Novikoff [23] for $g<\infty$, and McKean–Trubowitz [17] and [18] for $g=\infty$. (2) is discussed for $g<\infty$ in McKean [19]; $g=\infty$ offers difficulties not yet overcome. (3) is in a very primitive state: see Zakharov [24] and McKean [14]. The purpose of this talk is two-fold: (a) to describe the state of the art, and (b) to try to interest geometers in these particular curves and their Jacobi varieties. It seems too difficult at the present time to form any very general conception of curves of infinite degree or of varieties in an infinite number of dimensions. It is my hope that the present curves and Jacobi varieties may be sufficiently manageable as to provide some hints in that direction; see § 6. I must not neglect to mention the existing literature on curves of infinite genus: Ahlfors–Sario [2] is for general reference. I cite, particularly, Heins [6], Hornich [8]–[11]; Myrberg [20]–[22]; and Accola [1] as being very close to the concerns of § 6.

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However, with the exception of Hornich who studied the curve $\mathfrak{y}^2=\cos\lambda$, it is my belief that the curves envisaged are not sufficiently special as to be *really algebraic*, though, naturally, I must leave vague the actual meaning of this phrase.

2. Multiplier curves. The nonlinear wave equations I have in mind may be expressed in the form[1] $Q^{\cdot}=[Q,K]$, in which Q is, *e.g.*, a differential operator of degree 1 with 2×2 infinitely differentiable matrices as coefficients, and K is a skew-symmetrical operator of the same type. This fact has the consequence that the flow of Q is *isospectral*, meaning that it preserves the spectra of Q determined by fixing a number $m\neq 0,\infty$ and solving $Qf=\lambda f$ in the class of vectorial functions f: $[0,1)\to C^2$ with $f(x+1)=mf(x)$. The matter can be described alternatively as follows: Let $M(1)$ be the (2×2) elementary solution of $Qf=\lambda f$ evaluated at $x=1$, *i.e.*, the *monodromy matrix* of Q. The possible values of the *multiplier* m for fixed λ are the roots of $\det[M(1)-mI]=0$. The vanishing of the determinant suggests looking at m as a several-valued function of λ; in fact, it defines a curve $\mathfrak{M}$, with points $\mathfrak{p}=(\lambda,m)$, usually of infinite genus and with numerous singularities. The desingularization $\mathfrak{K}$ of $\mathfrak{M}$ is the so-called *multiplier curve* of Q, and isospectrality requires that it is preserved by the flow. The fiber $\mathfrak{Q}$ of the map $Q\to\mathfrak{K}$ is the *isospectral class* of Q. The experience is that $\mathfrak{M}$ can be recovered from $\mathfrak{K}$ so that no information is lost by desingularization. The next ingredient of the recipe is a correspondence between Q and a divisor of $\mathfrak{K}$ of degree h closely related to the genus g of $\mathfrak{K}$. The divisor comprises those points $\mathfrak{p}=(\lambda,m)$ of $\mathfrak{K}$ such that, *e.g.*, $M_{12}(1)$ vanishes. The divisor $\mathfrak{p}_i$ $(i=1,\dots,h)$ so produced is now mapped over to the Jacobi variety of $\mathfrak{K}$. This involves a careful choice of differentials ω of the first kind [DFK]. Then the sum

$$\int_{\mathfrak{o}}^{\mathfrak{p}}\omega=\sum_{i=1}^{h}\int_{\mathfrak{o}_i}^{\mathfrak{p}_i}\omega=\varkappa(\omega),$$

with base points $\mathfrak{o}_i$ $(i=1,\dots,h)$ selected so as to control the tail of the sum and paths $\mathfrak{o}_i\mathfrak{p}_i$ $(i=1,\dots,h)$ not winding too much on $\mathfrak{K}$, is viewed as an element of the dual space (DFK)′ of DFK and construed *modulo* the lattice L of images of closed paths. The divisor is mapped thereby into the Jacobi variety $\mathfrak{J}=(\mathrm{DFK})'/L$, and the experience is that if this recipe can actually be carried out, *the image in* $\mathfrak{J}$ *of the flow* $Q^{\cdot}=[Q,K]$ *will be a straight line traversed at constant speed.* The map $\mathfrak{Q}\to\mathfrak{J}$ is not always 1:1. The experience is that (a) the cotangent bundle of Q is spanned by the vector fields $X\colon Q\to J(\partial H/\partial Q)$, in which the Hamiltonian H is a convenient multiple of sp $M(1)$, $\partial H/\partial Q$ is the gradient in function space, and J is a skew-symmetrical operator; (b) the different values of H employed are involutive relative to a symplectic structure associated with J so that the corresponding flows $Q^{\cdot}=J(\partial H/\partial Q)$ commute; and (c) the map to the Jacobi variety converts

[1] $\cdot=d/dt$.

the flows into straight-line motions at constant speed; in particular, the original flow $Q^{\cdot}=[Q, K]$ is seen to be *integrable* in the sense of the Hamilton–Jacobi theory.

3. The Korteweg-de Vries equation. The equation is $\partial q/\partial t=3q\,\partial q/\partial x-(1/2)\,\partial^3 q/\partial x^3$. This may be put in (nonclassical) Hamiltonian form as $q^{\cdot}=D(\partial H_2/\partial q)$ with the Hamiltonian

$$H_2=\int_0^1 [(1/2)q^3+(1/4)(q')^2]\,dx.$$

The symplectic structure is derived from the skew-symmetric operator $J=D$, the associated (nonclassical) symplectic 2-form being $\omega[\partial/\partial q(x), \partial/\partial q(y)]=\pm(1/2)$ according as $x<y$ or not.

The associated differential operator[2] Q is the Hill's operator $-D^2+q(x)$, the expression of the flow by commutators being $Q^{\cdot}=[Q, K_2]$ with $K_2=2D^3-(3/2)(qD+Dq)$. The multiplier curve $\mathfrak{M}$: $m^2-2\Delta m+1=0$ is defined by a single transcendental integral function Δ of order 1/2 and type 1: the so-called[3] *discriminant* of Q, *alias* 1/2 the spur of its monodromy matrix $M(1)$. The roots of $\Delta=+1$ $[-1]$ determine the periodic [anti-periodic] spectrum of Q: the ground state λ_0 is simple and periodic, while the remainder of the spectra comes in alternately anti-periodic and periodic pairs $\lambda_{2j-1}\leqslant\lambda_{2j}$ $(j=1, 2, \ldots)$ tending to $+\infty$, the equality $\lambda_{2j-1}=\lambda_{2j}$ signifying a 2-dimensional eigenspace. The multiplier curve $[m=\Delta\pm\sqrt{\Delta^2-1}]$ exhibits *simple ramifications* over the simple spectra, plus *double points* over the double spectra. The existence of double spectra is exceptional, and I leave this possibility out, so that $\mathfrak{R}=\mathfrak{M}$ is a nonsingular hyperelliptic curve with infinitely many branch points accumulating to ∞, which must also be considered to be ramified. The ramifications are well controlled by virtue of the expansion $\lambda_{2j-1}, \lambda_{2j}=j^2\pi^2+c_0+c_1 j^{-2}+c_2 j^{-4}+\ldots$; in particular, the length of the so-called *lacuna* $[\lambda_{2j-1}, \lambda_{2j}]$ vanishes rapidly as $j\uparrow\infty$, so that, while $\mathfrak{R}$ has an infinite number of handles $[g=\infty]$, the distant ones are placed in a very special way and, besides, are very small, *i.e., $\mathfrak{R}$ is doing its best to be compact.* The divisor is now formed from the points $\mathfrak{p}=(\lambda, m)$ of $\mathfrak{R}$ for which Q has an eigenfunction, with eigenvalue λ and multiplier m, vanishing at $x=0$, and it is a fact due to Borg [3] that the map from Q to the divisor so produced is 1: 1. The location of the divisor is very precise: it contains just one point $\mathfrak{p}_i$ on the circle of $\mathfrak{R}$ covering the lacuna $[\lambda_{2i-1}, \lambda_{2i}]$ $(i=1, 2, \ldots)$, so that *the degree of the divisor is precisely* $h=g=\infty$. The choice of DFK is more technical: what works well is the class of differentials $\omega=\varphi(\lambda)\,d\lambda/\sqrt{\Delta^2(\lambda)-1}$ with integral φ of (a) order 1/2 and (b) type $\leqslant 1$, such that (c) $\varphi(\lambda^*)=[\varphi(\lambda)]^*$, (d) $\int_0^\infty |\varphi(\lambda)|^2\lambda^{3/2}d\lambda<\infty$, and (e) the values of $\varphi(\lambda)$ in the lacuna $[\lambda_{2j-1}, \lambda_{2j}]$ vanish rapidly as $j\uparrow\infty$. Then the recipe

$$Q\to\mathfrak{p}_i\quad(i=1, 2, \ldots)\to\sum_{i=1}^{\infty}\int_{\mathfrak{o}_i}^{\mathfrak{p}_i}\omega=\varkappa(\omega)$$

[2] $D=d/dx$.

[3] Δ is actually 1/2 the conventional discriminant.

with base points $\mathfrak{v}_l$ covering λ_{2l-1} $(i=1, 2, \ldots)$ maps the isospectral class $\mathfrak{Q}$ 1:1 onto the, so to say, *real part* of the Jacobi variety $\mathfrak{J}$ of $\mathfrak{R}$ and straightens out the flows, as advertised in § 2. The details may be found in McKean–Trubowitz [17] and [18].

4. The sine-Gordon equation. This is more complicated: $\partial^2 q/\partial t^2 - \partial^2 q/\partial x^2 + \sin q = 0$ is equivalent to the system $q^{\cdot} = p$, $p^{\cdot} = q'' - \sin q$. This is already in (classical) Hamiltonian form $(q^{\cdot}, p^{\cdot}) = J(\partial H_2/\partial q, \partial H_2/\partial p)$ with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and

$$H_2 = \int_0^1 [(1/2)p^2 + (1/2)(q')^2 - \cos q]\, dx.$$

The associated symplectic form is the classical $\int_0^1 dq \wedge dp$. Faddeev–Takhtadzhyan–Zakharov [5] introduced the associated differential operator

$$Q = -\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} D + \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \frac{\sqrt{-1}}{4}(q' + p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \frac{1}{4}\begin{pmatrix} e^{\sqrt{-1}q/2} & 0 \\ 0 & e^{-\sqrt{-1}q/2} \end{pmatrix},$$

the flow being expressed by commutators as $Q^{\cdot} = [Q, K_2]$ with

$$K_2 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} D - 2\begin{pmatrix} 0 & JB \\ BJ & 0 \end{pmatrix}.$$

The 4-dimensional problem $Qf = \lambda f$ reduces to a 2-dimensional one: $(-JD + A + B^2/\lambda)f = \lambda f$. The multiplier curve $\mathfrak{M}$ is defined by $m^2 - 2\Delta m + 1 = 0$, as before, Δ being 1/2 the spur of the monodromy matrix of the *reduced* problem. Δ is integral in the punctured plane $C - 0$ and of exponentual type 1 [1/16] in the vicinity of ∞ [0]. The ramifications and singularities of $\mathfrak{M}$ are specified by the roots of $\Delta = \pm 1$, as before, only now these come in quartets of simple roots $\pm a \pm \sqrt{-1}b$ $(a, b \neq 0)$ or pairs of double roots $\pm a$ $(a \neq 0)$ over which lie, respectively, the simple ramifications and double points of $\mathfrak{M}$; it is also possible to have a quartet of simple or a pair of double imaginary roots, but I will ignore this possibility. The roots now accumulate both to ∞ $[a \sim n\pi,\ b = o(1)]$ and to 0 $[a \sim 1/16n\pi,\ b = o(1)]$. The desingularized curve $\mathfrak{R}$ is of genus $g = 2n - 1$, n being the number of quartets of simple roots. The chief complication arises from the fact that the projection of the divisor is not so precisely located as before. The divisor contains a point $\mathfrak{p}$ for each pair $a \pm \sqrt{-1}b$ and a point $-\mathfrak{p}$ for the reflected pair $-a \pm \sqrt{-1}b$, these being trapped at $\pm a$ if $b = 0$, *i.e.*, if the root is double, so that the degree of the active part of the divisor is $h = 2n = g + 1$. Now it is unfortunate that *the projections of the active points of the divisor do not move along preassigned 1-dimensional loci, such as the lacunae of* § 3, *under, e.g., translation or the flow* $Q^{\cdot} = [Q, K_2]$; indeed, it can be proved that if a single active point of the divisor projects onto a fixed curve, then $g = 1$. This raises technical difficulties in the choice of DFK which I have not

overcome for $g=\infty$, though I *can* prove that if $g<\infty$, then the isospectral class is a g-dimensional real torus and that the map into the Jacobi variety is 2:1 and straightens out the flow. The case of the sinh-Gordon equation $\partial^2 q/\partial t^2-\partial^2 q/\partial x^2+\operatorname{sh} q=0$ is simpler. Now the ramifications of $\mathfrak{K}$ come in *real* pairs and their reflections, and the projections of the points of the divisor are confined to the intervening lacunae, as in § 3. This simplification enables a complete working out of the infinite genus case closely parallel to the results reported in § 3; see McKean [15].

DFK is now the class of differentials $\omega=\varphi(\lambda)(\Delta^2-1)^{-1/2}d\lambda/\lambda$, much as before, where φ is entire of exponential type (a) 1 at ∞ and (b) 1/16 at 0, such that (c) $\varphi(\lambda^*)=[\varphi(\lambda)]^*$, (d) $\int_{-1}^{+1}\lambda^{-2}|\varphi(\lambda)|^2 d\lambda+\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)|\varphi(\lambda)|^2 d\lambda<\infty$, and (e) the values of $\varphi(\lambda)$ in the lacunae vanish rapidly upon nearing 0 or ∞. The 2 :1 character of the mapping to the Jacobi variety suggests some connection with Prym varieties, but I have not looked into that.

5. Boussinesq's equation. The equation is taken in the form

$$\partial^2 q/\partial t^2=(\partial^2/\partial x^2)[(4/3)q^2+(1/3)\partial^2 q/\partial x^2].$$

The Hamiltonian form is $(q^{\cdot}, p^{\cdot})=J(\partial H_2/\partial q, \partial H_2/\partial p)$ with $J=\left(\begin{smallmatrix}D & 0\\ 0 & D\end{smallmatrix}\right)$ and Hamiltonian

$$H_2=\int_0^1[(1/2)p^2+(4/9)q^3+(1/6)(q')^2]\,dx.$$

The associated differential operator introduced by Zakharov [24] is $Q=-D^3+qD+Dq+p$, the flow being expressed by commutators as $Q^{\cdot}=[Q, K_2]$ with $K_2=-D^2+(4/3)q$. The multiplier curve $\mathfrak{M}$ is now defined by the cubic $m^3-3\Delta m^2+3\Delta^{\dagger}m+1=0$, in which Δ is 1/3 the spur of the 3×3 monodromy matrix of Q, while $\Delta^{\dagger}$ is the same thing for $Q^{\dagger}$=the dual of Q. $\mathfrak{M}$ is ramified or has a singular point over the simple [double or triple] roots of the discriminant of the cubic. The latter is an integral function of order 1/3 and type 1 with roots $\sim 8\pi^3 n^3/3\sqrt{3}$ $(n\to\pm\infty)$. These far roots are always real and are either double or come in simple pairs ; in addition, $\mathfrak{M}$ may have an indefinitely large number of triple points. $\mathfrak{M}$ is no longer a 2-sheeted but rather a 3-sheeted covering of the complex line, and very little is proved about it even if $g<\infty$. The function theory for $g<\infty$ may be found in Hensel–Landsberg [7]. The papers of Zakharov [24] and McKean [14] contain a little preliminary information.

6. Questions of function theory. The present section is devoted to function-theoretical questions raised by the examples of §§ 3–5.

DFK. The proper choice of DFK was explained in the Korteweg–de Vries case [§ 3]. This class is *not* the same as the differentials of the first kind with $-\sqrt{-1}\int\omega\wedge\omega^*<\infty$, though it is closely related. The following desiderata are emphasized: (a) the map from divisors into $\mathfrak{J}=(\mathrm{DFK})'$ *modulo* periods should be locally 1:1 or close to that; and (b) (DFK)′ should be so large as to contain the images of

the interesting cotangent vectors to $\mathfrak{Q}$, *e.g.*, the image of $q\cdot=3qq'-(1/2)q'''$. This all works out nicely in the case of § 3 for divisors p_i $(i\leqslant g=\infty)$ in *real position, i.e.*, with projections falling in the lacunae, and gives rise to an isomorphism between the isospectral class $\mathfrak{Q}$ and the so-called *real part* of the Jacobi variety $\mathfrak{J}$, the latter being a highly compact $(g=\infty)$-dimensional real torus. The same can be said about the sinh-Gordon equation of § 4: everything works out nicely except that the map from $\mathfrak{Q}$ to $\mathfrak{J}$ is now 2:1 which is *not* so nice.

Complex Jacobi variety. The *complex* Jacobi variety awaits investigation: for the case of § 3, it is *always* unbounded if $g=\infty$. The Mathieu case $[q(x)=\cos 2\pi x]$ is probably a nice example to begin with; see McKean–Trubowitz [**18**] for more information.

The function field. The proper class of meromorphic functions on K is not yet settled: for the case of § 3, the cheap half of Abel's theorem, stating that for the roots $\mathfrak{q}_i$ $(i=1, \ldots, n)$ and poles $\mathfrak{p}_i$ $(i=1, \ldots, n)$ of a meromorphic function. $\varkappa(\omega)=\sum_{i=1}^{n}\int_{\mathfrak{q}_i}^{\mathfrak{p}_i}\omega$ is a period, is discussed in McKean–Trubowitz [**18**] but the converse is left open.

Riemann–Roch. The theorem of Riemann–Roch is also open. The only aspect which is nailed down is that, in the case of § 3, every differential of the first kind with a root of finite multiplicity $2m$ at ∞ has $2g-2-2m$ additional finite roots on $\mathfrak{K}$ in a very special and precise sense, *i.e.*, it has $2g-2$ *roots in all.* ω from DFK *can* have a root of infinite multiplicity at ∞ and not vanish identically, but that is not possible if, *e.g.*, the lacunae vanish exponentially fast: $\lambda_{2j}-\lambda_{2j-1}\leqslant a\exp(-bj)$ $(j\uparrow\infty)$; see McKean–Trubowitz [**18**]. The most elementary difficulty about Riemann–Roch is that all the dimensions and/or codimensions involved may be infinite, and it is unclear how to rephrase the statement to give it a sufficiently precise content as to be useful.

The theta function. The Jacobi variety of § 3 has a nice theta function Θ in infinitely many variables; in particular, the Riemann period relations, upon which the quadratic form in the Θ-sum depends for its positivity, hold, and Θ itself satisfies a Riemann-type vanishing theorem. Θ plays an important role in inverting the map $\mathfrak{Q}\to\mathfrak{J}$; see Its–Matveev [**12**] for $g<\infty$ and McKean–Trubowitz [**18**] for $g=\infty$. The existence of such a theta function for the examples of §§ 4–5 awaits investigation.

$\mathfrak{J}$ *as a variety.* The Jacobi variety of § 3 may be expressed as the intersection of a countable number of quartics[4]; it also carries a sufficient number of theta functions to embed it, in the manner of Lefschetz, into an ∞-dimensional projective space, so it seems to be doing as well as it can to be a *variety*; naturally, I leave the meaning vague.

[4] Moser-Trubowitz [**19**].

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Singularities of Solutions to Boundary Value Problems

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In this talk some of the recent results on the behavior of the singularities of boundary value problems are reviewed in a somewhat more general setting than is usual.

We shall introduce a space of pseudodifferential operators which is coordinate free on M, a given manifold with boundary.

First, for $Z=\boldsymbol{R}_+\times\boldsymbol{R}^n$, the elements of $L_P^m(Z)$ are maps

$$(1)\quad Au=(2\pi)^{-n-1}\int\int\int_0^\infty e^{i(x-x')\xi+i(y-y')\cdot\eta}a(x,y,x',y',x\xi,\eta)u(x',y')\,dx'\,dy'\,d\xi\,d\eta$$

where $a(x,y,x',y',\lambda,\eta)\in S^m(Z\times Z\times\boldsymbol{R}^{n+1})$ is a properly supported symbol. The important point to note here is the form of the ξ-dependence of the symbol $\tilde{a}=a(x,y,x',y',x\xi,\eta)$ in (1)—only through the product $x\xi$. The operators in $L_P^m(Z)$ enjoy many of the properties of pseudodifferential operators on closed manifolds. As oscillatory integrals they map the space $\dot{\mathscr{D}}'(Z)$ of distributions supported by Z into $\mathscr{D}'(Z)$, the space of distributions extendible from $\mathring{Z}$ to $\boldsymbol{R}^{n+1}$, and reduce to the usual type of pseudodifferential operator away from the boundary. Most importantly for the applications to boundary value problems, the elements of $L_P^m(Z)$ preserve regularity in the normal variable:

$$(2)\qquad A\colon C^\infty(\bar{\boldsymbol{R}}_+;\mathscr{D}'(\boldsymbol{R}^n))\to C^\infty(\bar{\boldsymbol{R}}_+;\mathscr{D}'(\boldsymbol{R}^n)),$$

in the sense that $Au|_Z$ extends to be in $C^\infty(\bar{\boldsymbol{R}}_+;\mathscr{D}'(\boldsymbol{R}^n))$ and if $a\in S^{-\infty}$ is properly supported, then $\tilde{a}(z,D_z)C^\infty(\bar{\boldsymbol{R}}_+;\mathscr{D}'(\boldsymbol{R}^n))\subset C^\infty(Z)$. We shall denote by $\tilde{a}(z,D_z)=a(x,y,xD_x,D_y)$ the operator with symbol $\tilde{a}$ in (1), where $z=(x,y)$.

(3) DEFINITION. Let $\mathcal{N}_c(Z)\subset C_c^\infty(\bar{\mathbf{R}}_+;\mathscr{E}'(\mathbf{R}^n))\cup\mathscr{E}'(\mathring{Z})$ be the subspace of those distributions u for which there exist $\delta,\varepsilon>0$ such that if $a(x,y,\lambda,\eta)=0$ when $|\lambda|\leqslant(1+|\eta|)/\delta$ then $\tilde{a}(z,D_z)u\in C^\infty([0,\varepsilon)\times\mathbf{R}^n)$; we denote by $\mathcal{N}(Z)$ the subspace of those distributions locally in $\mathcal{N}_c(Z)$.

(4) THEOREM. *$L_P^*(Z)$ acts on $\mathcal{N}(Z)$ as a filtered ring of operators with complete symbol map, $\tilde{\sigma}_F$, satisfying the usual symbol calculus. $\mathcal{N}(Z)$, and $L_P^m(Z)$ acting upon it, are coordinate-free so $L_P^m(M)$, acting on $\mathcal{N}(M)$, is well defined for any paracompact manifold M. If $A\in L_P^m(M)$ and $u\in\mathcal{N}(M)$ then $Au|_{\partial M}=(\partial A)(u|_{\partial M})$ (mod $C^\infty(\partial M)$) where $\partial A\in L^m(\partial M)$ is a properly supported pseudodifferential operator with principal symbol satisfying*

$$(\iota^*)^*\,\sigma_m(\partial A)=\tilde{\sigma}(A)|_{\partial T^*M},\ \iota^*\partial T^*M\to T^*\partial M$$

being the natural projection.

For distributions $u\in\mathcal{N}(M)$ we introduce an invariant notion of wavefront set (used in special cases in [1], [2], [11]). Put $BM=T^*\mathring{M}\setminus 0\cup T^*\partial M\setminus 0$, with topology given by the identification map ι^*: $\partial T^*M\to T^*\partial M$. The symbol $\tilde{\sigma}_m(A)$ of $A\in L_P^m(M)$ is well defined (and continuous) on BM; we define $\Sigma(A)=\{\varrho\in BM;\ \tilde{\sigma}_m(A)(\varrho)=0\}$ and say that A is elliptic outside this characteristic set. Then, set

$$\mathrm{WF}_b(u)=\bigcap\{\Sigma(A);\ A\in L_P^m(M)\text{ and }Au\in C^\infty(M)\}\subset BM. \tag{5}$$

From Definition 3, in local coordinates, $\mathrm{WF}_b(u)$ is always contained in a cone of the form $\{(x,y,\xi,\eta);$ if $0<x<\varepsilon$ then $|\xi|\leqslant|\eta|/x\varepsilon$ for some $\varepsilon>0\}$. The usual results on wavefront sets hold, $\mathrm{WF}_b(u)\setminus\Sigma(A)\subset\mathrm{WF}_b(Au)\subset\mathrm{WF}_b(u)$, $\mathrm{WF}_b(u)=\emptyset$ exactly when $u\in C^\infty(M)$ and moreover, $\mathrm{WF}(Au|_{\partial M})\subset\mathrm{WF}_b(u)\cap T^*\partial M$.

If $P\in\mathrm{Diff}^k(M)$ is a differential operator of order k on M and $x\in C^\infty(M)$ vanishes simply on ∂M then $x^kP\in L^k(M)$. This suggests the following notion of a characteristic boundary for these pseudodifferential operators.

(6) DEFINITION. $A\in L_P^m(M)$ is said to be characteristic of (exactly) order k at $\varrho\in T^*\partial M\setminus 0$ if, in any local coordinates at $\varrho=(y^0,\eta^0)$, $\tilde{\sigma}_F(A)=[\tilde{a}]$ with $a(x,y,\lambda,\eta)\in S^m(Z\times\mathbf{R}^{n+1})$ satisfying $\partial_x^r\partial_\lambda^{l-r}a(0,y,0,\eta)=0$ near ϱ if $l<k$, $\partial_\lambda^k a(0,y^0,0,\eta^0)\neq 0$.

We shall give results below for operators of this type. Of course, if $P\in\mathrm{Diff}^k(M)$ and ∂M is non-characteristic for P, then $x^kP\in L_P^k(M)$ is characteristic of order k at all boundary points. To complete the connection between the differential boundary value problems usually considered and the pseudodifferential problems discussed here we note the following extension of a well-known theorem of Peetre (see Hörmander [5, Corollary 4.3.1]).

(7) THEOREM. *If $P\in\mathrm{Diff}^k(M)$ is noncharacteristic on $N^*(\partial M)$ and $u\in\mathcal{D}'(M)$ is such that $Pu\in\mathcal{N}(M)$ then $u\in\mathcal{N}(M)$.*

From now on, for simplicity, we shall consider only classical operators, the ring $L^{\cdot}_{\mathrm{cl}}(M)\subset L^{\cdot}_{P}(M)$ of those A for which the symbols $a\in S^m(Z\times Z\times \boldsymbol{R}^{n+1})$ defining A in local coordinates have complete asymptotic expansions with integral steps in order. Thus, $A\in L^m_{\mathrm{cl}}(M)$ has a homogeneous principal symbol $\tilde{a}_m\in\tilde{S}^m$. We shall suppose that

(8) $\tilde{a}_m$ is real and $d\tilde{a}_m$, α are linearly independent on $\tilde{a}_m=0$.

If A is characteristic of order k at $\varrho\in T'^*\partial M\setminus 0$, then the polynomial

$$a^{\partial,k}(y,\eta;\xi)=\sum_{j\leqq k}\partial_x^j\,\partial_\lambda^{k-j}a_m(0,y,0,\eta)\xi^j/\partial_\lambda^k a(0,y,0,\eta)$$

is easily seen to be well defined in $\imath^{*-1}(\gamma)\cap T^*M\setminus N^*\partial M$, for γ a conic neighborhood of ϱ in $T^*\partial M\setminus 0$. We shall further suppose

(9) $a^{\partial,k}$ has simple zeros, except possibly for one double zero, in $\imath^{*-1}(\varrho)\otimes\boldsymbol{C}$.

Assuming, as we may, that (9) holds for all $\varrho'\in\gamma$, let $G'\subset\partial T^*M$ be the set of double zeros (all real), in $\imath^{*-1}(\gamma)$. We assume that

(10) $a^{\partial,k}$, α are linearly independent on $T_\mu(\partial T^*M\cap\{a^{\partial,k}=0\})\ \forall\mu\in G'$,

the glancing surface $G=\imath^*G'$ is then a conic hypersurface.

We proceed to various results on the propagation of singularities for operators characteristic of order one or two; Γ' will always be a sufficiently small conic neighborhood of the base point $\varrho\in BM$. Using factorization arguments, these results apply to A satisfying (8), (9), (10).

(11) THEOREM. *If $A\in L^m_{\mathrm{cl}}(M)$ is characteristic of order 1 at $\varrho\in T^*\partial M\setminus 0$ and the boundary form $a^{\partial,1}=\xi+e(y,\eta)$ has $\operatorname{Im}e>0$ then*

$$\big(\mathrm{WF}_b(u)\setminus\mathrm{WF}_b(Au)\big)\cap\Gamma'=\emptyset\quad\forall u\in\mathcal{N}(M);$$

if $\operatorname{Im}e<0$ then

$$\big(\mathrm{WF}_b(u)\setminus\mathrm{WF}_b(Au)\big)\cap\Gamma'=\mathrm{WF}(u|_{\partial M})\cap\Gamma'\cap T^*\partial M\setminus 0,\quad\forall u\in\mathcal{N}(M).$$

When A is characteristic of order 1 at ϱ and (8) holds, the characteristic set $\Sigma(A)\subset BM$ has a natural foliation by the (closure of the) integral curves of the Hamilton field $H_{\tilde{a}_m}$ in T^*M, the bicharacteristics, which are transversal to $T^*\partial M$ near ϱ. Hörmander's theorem [6] then extends as follows.

(12) THEOREM. *If $A\in L^m_{\mathrm{cl}}(M)$ is characteristic of order 1 at $\varrho\in T^*\partial M\setminus 0$ and (8) holds then $\Gamma'\cap\big(\mathrm{WF}_b(u)\setminus\mathrm{WF}_b(Au)\big)\subset\Sigma(A)$ is a union of maximal bicharacteristic segments with*

$$\big(\mathrm{WF}_b(u)\setminus\mathrm{WF}_b(Au)\big)\cap\Gamma'\cap T^*\partial M=\big(\mathrm{WF}(u|_{\partial M})\setminus\mathrm{WF}_b(Au)\big)\cap\Gamma'\cap T^*\partial M$$

for all $u\in\mathcal{N}(M)$.

These two theorems are extensions of the results of Chazarain [2] and Lax–Nirenberg [14], see also Lax [7].

When $A\in L^2_{cl}(M)$ satisfying (8) is characteristic of order 2 at $\varrho\in T^*\partial M\setminus 0$ and essentially so, i.e., $\varrho\in G$, and (10) holds, there is a symplectic curvature invariant at ϱ. To see this directly we note that $A=Cx^2P$ where $C\in L^0_{cl}(M)$ is elliptic at ϱ, P is a differential-pseudodifferential operator in local coordinates,

$$P=D_x^2+a(x,y,D_y)D_x+b(x,y,D_y),\ a\in C^\infty(\overline{R}_+;L^1_{cl}(R^n)),\ b\in C^\infty(\overline{R}_+;L^2_{cl}(R^n))$$

and the equivalence means that $\mathrm{WF}_b(Au-Cx^2Pu)\cap\Gamma'=\emptyset$ for all $u\in\mathcal{N}(M)$. That $\varrho=(y^0,\eta^0)\in G$ means that the discriminant $(a^2-4b)(y^0,\eta^0)=0$. Then $\varkappa(y,\eta)=\partial_x(a^2-4b)(0,y,\eta)$ is invariantly defined on G up to a nonvanishing smooth factor. The surface G is divided into the subsets G_d where $\varkappa>0$ (diffractive points or points of bicharacteristic concavity) G_g where $\varkappa<0$ (gliding points or points of bicharacteristic convexity) and G_h where $\varkappa=0$ (points of higher order bicharacteristic tangency). Near $\varrho\in G$ we define the closed surface $\Sigma_b(A)=(\Sigma(A)\cap T^*\overset{\circ}{M})\cup(H\cup G)\subset BM$, and note that the open subset $\Sigma_b(A)\setminus G$ has a natural 1-foliation by "broken bicharacteristics" obtained by joining the two bicharacteristics, one from each factor, through each point of H, where the zeros of $a^{\vartheta,z}$ are real.

(13) Proposition. *The broken Hamilton foliation of $\Sigma_b(A)\setminus G$ extends by continuity to a 1-foliation of $\Sigma_b(A)\setminus G_\infty$, where G_∞ is the set of points of infinite order bicharacteristic tangency.*

The new curves added through the points of G can be locally free bicharacteristics, gliding rays (i.e., curves in the Hamilton foliation of G) or, at points of bicharacteristic inflexion, half one and half the other. Near G_∞ the obvious extension of Proposition 13 is not always valid (see Taylor [15] for the original example); there may not be a unique broken bicharacteristic through each point. Even so, by examining the differential equations satisfied by the broken bicharacteristics near G one can define, in a natural invariant way, generalized bicharacteristics, such that through each point of $\Sigma_b(A)$ there is at least one such curve, with local uniqueness through points of $\Sigma_b(A)\setminus G_\infty$. Moreover, if $\varrho'_k\to\varrho$ in $\Sigma_b(A)$ then for a subsequence $k(n)\to\infty$, the generalized bicharacteristic through $\varrho'_{k(n)}$ converges to a generalized bicharacteristic through ϱ.

(14) *Open question.* Is every compact segment of generalized bicharacteristic in $\Sigma_b(A)$ the limit of a sequence of bicharacteristic segments completely within $\Sigma_b(A)\setminus G_\infty$?

(15) Theorem. *If $A\in L^m_{cl}(M)$ is characteristic of order 2 at $\varrho\in T^*\partial M\setminus 0$ and* (8)–(10) *hold with $\varrho\in G$ then for all $u\in\mathcal{N}(M)$,*

$$\Gamma'\cap(\mathrm{WF}_b(u)\setminus(\mathrm{WF}_b(Au)\cup\mathrm{WF}(u|_{\partial M})))\subset\Sigma_b(A)$$

is a union of maximally extended generalized bicharacteristics.

This result, proved in nearly this generality in Melrose Sjöstrand [12], contains earlier results near G_d (Melrose [9], Taylor [15]) and G_g (Andersson–Melrose [1],

Eskin [3]), but is not quite as precise as Theorem 12 and the main significance of the question posed above is that its affirmative resolution would allow one to show Theorem 15, the "generalized Huygens principle", to be essentially optimal. Together, Theorems 11, 12 and 15 provide a reasonably complete description of the singularities of the solution to the Dirichlet problem for P a differential operator of second order (see [11]).

Before proceeding to outline the proof of Theorem 15 we note that similar results are also available for a large class of other boundary conditions. (See Melrose–Sjöstrand [15], and for results near G_d, Taylor [16].)

The main results, Theorems 11, 12 and 15 can all be proved using a transformation theory based on the Fourier integral operator analogues of the pseudodifferential operators (1) (an outline of which is given in [11], see also [10]). This is the advantage of introducing the general operators of Theorem 4. The "normal forms" to which the operator A can be reduced at $(0, \eta^0) \in T^*\partial Z, \eta^0 = (0, \ldots, 0, 1)$ are: for Theorem 11, following work of Friedlander [4] and Ludwig–Morawetz [8], $x(D_x \pm iD_{y_1})$; for Theorem 12, xD_x; near G_d, $x^2(D_x^2 - xD_{y_n}^2 + D_{y_1}D_{y_n})$; near G_g, $x^2(D_x^2 + xD_{y_n}^2 + D_{y_1}D_{y_n})$ and near G_h, $x^2\left(D_x^2 + xa(x, y, D_{y'}, xD_x) + D_{y_1}D_{y_n}\right)$ where $y' = (y_2, \ldots, y_n)$.

The last three forms occur in Theorem 15, the first two allow a constructive approach and the last direct energy estimation because of the differential nature of the dependence on y_1—one can also arrange that $\partial a/\partial\lambda$ vanishes to infinite order at $x=0$ so that $\tilde{a}$ is actually a classical pseudodifferential operator vanishing to all orders at $x=0$ (cf. [1]). In reducing P microlocally to this form one has sufficient freedom in choice of the canonical transformation involved to place the "time" surfaces $y_1 = \text{const.}$ so that global regularity from the energy esimates implies the microlocal regularity stated in Theorem 15.

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Critical Points of Indefinite Functionals and Periodic Solutions of Differential Equations

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We shall describe a method for finding periodic solutions for a class of differential equations. The underlying existence mechanism is provided by minimax theory from the calculus of variations. Thus the solutions are obtained as critical points of real valued functionals defined on an appropriate Hilbert space. These functionals are indefinite in a highly degenerate fashion as will be seen later.

We begin with two examples illustrating some of the problems in which we are interested. First consider a boundary value problem for a semilinear wave equation:

$$\begin{aligned} u_{tt}-u_{xx}+f(x,u) &= 0,\ 0<x<l, \\ u(0,t) &= 0 = u(l,t), \end{aligned} \tag{1}$$

where f is smooth and $f(x,0)=0$. Then $u\equiv 0$ is a solution of (1). The problem we pose is are there nontrivial time periodic solutions of (1), i.e. so-called *free vibrations*. Since the period, T, of any such solution is a priori unknown, it is convenient to make the change of time variable $t\to 2\pi T^{-1}t\equiv\lambda^{-1}t$. This transforms (1) to

$$\begin{aligned} u_{tt}-\lambda^2(u_{xx}-f(x,u)) &= 0,\ 0<x<l, \\ u(0,t) &= 0 = u(l,t) \end{aligned} \tag{2}$$

and we now want a real number $\lambda\neq 0$ and a 2π periodic function u such that (λ, u) satisfies (2).

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As a second and related example, consider a Hamiltonian system of ordinary differential equations:

$$\dot{z} = \mathscr{J} H_z(z), \quad \mathscr{J} = \begin{pmatrix} 0 & -\mathscr{I} \\ \mathscr{I} & 0 \end{pmatrix} \tag{3}$$

where $p, q \in \boldsymbol{R}^n$, $z=(p, q)$, $\cdot$ denotes d/dt, $H: \boldsymbol{R}^{2n} \to \boldsymbol{R}$, and $\mathscr{I}$ is the identity matrix in $\boldsymbol{R}^n$. Scaling time as with (1) leads to

$$\dot{z} = \lambda \mathscr{J} H_z(z) \tag{4}$$

and we again seek a pair (λ, z) satisfying (4) where z is 2π periodic. In this setting, there are two further natural constraints which we can try to impose on our problem, namely either the energy H or the period T can be prescribed. Of course the same observation can be made about (2) but unlike (4), thus far the fixed energy problem has proved to be intractable for (2).

The connection between (2) or (4) and the calculus of variations is that solutions of (2) or (4) can be obtained as critical points of corresponding functionals. For example if λ is prescribed, formally solutions of (2) are critical points of

$$\int_0^{2\pi} [(p, \dot{q})_{R^n} - \lambda H(z)]\, dt \tag{5}$$

where $(\cdot, \cdot)_{R^j}$ denotes the usual $\boldsymbol{R}^j$ inner product. Likewise solutions of (2) are critical points of

$$\int_0^{2\pi} \int_0^{l} [\frac{1}{2}(u_t^2 - \lambda^2 u_x^2) - \lambda^2 F(x, u)]\, dx\, dt \tag{6}$$

where $F(x, r) = \int_0^r f(x, s)\, ds$. Similar observations can be made for (4) if the energy is prescribed. E.g. if $H(z(t)) \equiv 1$, then any critical point of the action integra

$$A(z) \equiv \int_0^{2\pi} (p, \dot{q})_{R^n}\, dt \tag{7}$$

subject to the constraint

$$\frac{1}{2\pi} \int_0^{2\pi} H(z)\, dt = 1 \tag{8}$$

and having a nonzero Lagrange multiplier λ provides a nontrivial solution (λ, z) of (4).

All of the above functionals are indefinite in the following sense: if e.g. H and f are smooth, (5), (6), and (7)–(8) are not bounded from above or from below. Indeed there are infinite dimensional submanifolds of any reasonable underlying function space on which each of these functionals is positive and on which it is negative. Moreover if $f(x, 0)=0$ or $H_z(0)=0$, the corresponding functional (5) or (6) possesses the trivial critical point, 0, which must be avoided in any existence proof. Hence finding a critical point for these variational problems is a subtle matter. We do not know a direct way in which to do this; instead we employ

a variant of the time honored recipe of replacing the given problem (in an infinite dimensional function space) by an approximate finite dimensional one, solving the new problem, and obtaining strong enough estimates for its solution so as to be able to pass to a limit to solve the original problem.

Before describing our procedure more fully, we state an existence result for the fixed period problem for (3). We will present our method in this setting since it involves the smallest number of technicalities.

THEOREM 9 [1]. *Suppose* $H \in C^1(\boldsymbol{R}^{2n}, \boldsymbol{R})$ *and satisfies* (H_1) $H(z) \geqslant 0$; (H_2) $H(z) = o(|z|^2)$ *at* $z=0$, *and* (H_3) $0 < H(z) \leqslant \theta(z, H_z(z))_{R^{2n}}$ *for large* $|z|$ *where* $\theta \in (0, \frac{1}{2})$. *Then for any* $T>0$, (3) *possesses a nonconstant* T *periodic solution.*

REMARK 10. Integrating the inequality (H_3) shows

$$H(z) \geqslant a_1 |z|^{1/\theta} - a_2 \tag{11}$$

for some constants $a_1, a_2 > 0$.

SKETCH OF THE PROOF OF THEOREM 9. (A complete proof of the theorem can be found in [1] or [2].) Let $E \equiv (W^{1,2}(S^1))^{2n}$, i.e. E is the Hilbert space of $2n$-tuples of 2π periodic functions which together with their first derivatives are square integrable. We seek a solution of (4) in E. Our procedure essentially consists of five steps. However the fact that (H_3) does not imply an upper bound for the rate at which $H(z) \to \infty$ as $|z| \to \infty$ creates technical problems which we bypass by adding a preliminary step 0° and final step 6° as indicated below.

Step 0°: *Modify the problem.* We replace $H(z)$ by

$$H_K(z) = \chi(|z|) H(z) + (1 - \chi(|z|)) \varrho(K) |z|^4 \tag{12}$$

where $\chi(s)=1$ if $s \leqslant K$; $=0$ if $s \geqslant K+1$; $\chi' < 0$ if $s \in (K, K+1)$, and $\varrho(K)$ is a constant chosen so that (among other things) H_K satisfies (H_3) with a new constant θ independent of K. The choice of $|z|^4$ in (12) is not crucial but the power of $|z|$ must be larger than two and small enough so that the argument of step 3° (c) below can be carried out. We now replace (4) by

$$\dot{z} = \lambda J H_{Kz}(z) \tag{4}_K$$

and (5) by

$$I_K(z) = \int_0^{2\pi} [(p, \dot{q})_{R^n} - \lambda H_K(z)]\, dt. \tag{5}_K$$

Step 1°: *Make a finite dimensional approximation.* Instead of seeking critical points of I_K in E, we do the same for I_K in E_m, a finite dimensional subspace of E. Since E consists of $2n$-tuples of Fourier series, a natural choice for E_m is a truncation thereof at order m. Then $E_{m+1} \supset E_m$ and $\overline{\bigcup_{m \in N} E_m} = E$.

Step 2°: *Find a nontrivial critical point of the approximate problem.* This is achieved via a minimax argument. Lack of space prohibits providing much detail here.

Therefore we will merely give a characterization of a positive critical value, c_m, of $I_K|_{E_m}$. Let E_m^+, E_m^-, E^0 denote respectively the (L^2 orthogonal) subspaces of E_m on which $A(z)$ is positive definite, negative definite, and null. Select any $\varphi \in E_m^+ \setminus \{0\}$ and set $V_m = E_m^- \oplus E^0 \oplus \text{span}\{\varphi\}$. By (H_3), $R = R(m)$ can be chosen so that $I_K \leqslant 0$ outside the (L^2) ball of radius R. If

$$\Gamma_m = \{h \in C(B_R \cap V_m, E_m) | h(z) = z \text{ if } I_K(z) \leqslant 0\},$$

then

$$c_m = \inf_{h \in \Gamma_m} \max_{z \in B_R \cap V_m} I_K(h(z)) \tag{13}$$

is a positive critical value of $I_K|_{E_m}$. Let z_m denote a corresponding critical point. The motivation behind (13) lies in the qualitative properties that I_K possesses, in particular as provided by (H_1)–(H_3).

Step 3°: *Estimates.* Three types of estimates are obtained in a typical P.D.E. bootstrap fashion. One novelty is that the first estimate exploits (13) to get:

(a) An upper bound for c_m independent of m and K. This is obtained by choosing $h(z) = z \in \Gamma_m$ and using the definitions of c_m and V_m, the form of I_K, and (H_3).

(b) A weak bound for z_m independent of m and K. Here we mean an upper bound for $\|(z_m, H_{Kz}(z_m))_{R^{2n}}\|_{L^1}$ independent of m and K. This estimate follows from the fact that $I_K'(z_m)z_m = 0$ (where I_K' denotes the Frechet derivative of I_K), (a), and the choice of $\varrho(K)$.

(c) A bound for $\|z_m\|_E$ independent of m. It is at this point that the replacement of (4) by $(4)_K$ is crucial for otherwise we do not know how to get (c) from (b). The form of H_K and (b) give an m independent but K dependent bound for $\|z_m\|_{L^4}$. Then $(4)_K$ projected on E_m with $z = z_m$ combined with the L^4 bound and an interpolation argument using the Gagliardo–Nirenberg inequality yields the bound for $\|z_m\|_E$.

Step 4°: *Pass to a limit.* The bounds on $\|z_m\|_E$ imply, via standard compactness and simple regularity arguments, that a subsequence of z_m converges to z_K, a solution of $(4)_K$.

Step 5°: *Show the solution is nontrivial.* This is achieved by a comparison argument. By (H_2) and the definition of (H_K), for all $\varepsilon > 0$ and $z \in R^{2n}$,

$$H_K(z) \leqslant \varepsilon |z|^2 + A_\varepsilon |z|^4 \equiv G_K(z)$$

where A_ε is a constant depending on ε and K. Therefore

$$I_K(z) \geqslant \int_0^{2\pi} [(p, \dot{q})_{R^n} - \lambda G_K(z)]\, dt \equiv J_K(z)$$

for all $z \in E$. Arguing as in 2° yields a critical value b_m of $J_K|_{E_m}$ with corresponding critical point w_m and such that $0 \leqslant b_m \leqslant c_m$. If $z_K \equiv$ constant, then $c_m \to c_K = I_K(z_K) = 0$. Hence $b_m \to 0$ and by an easy argument, $\|w_m\|_E \to 0$. However an analysis

of $(4)_K$ with H replaced by G shows $\|w_m\|_E$ is bounded away from 0. Hence z_K is a nontrivial solution of $(4)_K$.

Step 6°: *Solve the original problem.* It suffices to obtain a bound for $\|z_K\|_{L^\infty}$ independent of K for then $H_K(z_K)=H(z_K)$ for large K. By (H_3),

$$H_K(z) \leqslant \theta\big(z, H_{Kz}(z)\big)_{R^{2n}}+a_3. \tag{14}$$

Integrating (14) over $[0, 2\pi]$ with $z=z_K$, using 3° (b), the fact that $H_K(z_K)\equiv$ constant since $(4)_K$ is a Hamiltonian system, and (11), the desired estimate follows and the proof is complete.

Essentially the same procedure can be used to obtain existence theorems for (1). A more complicated modification is needed in step 0° and of course the estimates involved are considerably more intricate especially in 3° and 6°. The direct analogue of Theorem 9 for (1) (with $f=f(u)$) is:

THEOREM 15 [3]. *Let $f\in C(\boldsymbol{R}, \boldsymbol{R})$ and satisfy (f_1) f is strictly monotonically increasing and $f(0)=0$, (f_2) $f(r)=o(|r|)$ at $r=0$, and (f_3) $0<F(r)\leqslant \theta r f(r)$ for large $|r|$ where $\theta\in(0, \frac{1}{2})$. Then for any T which is a rational multiple of l, (1) has a nontrivial classical T periodic solution.*

The reason for the restriction $Tl^{-1}\in \boldsymbol{Q}$ is that in the course of the proof we invert the linear wave operator in the class of functions satisfying the periodicity and boundary conditions. If $Tl^{-1}\in \boldsymbol{Q}$, this operator has a compact inverse on the orthogonal complement of its null space while if $Tl^{-1}\notin \boldsymbol{Q}$, one gets involved in small divisor problems which we do not know how to handle. There are versions of Theorems 9 and 15 which cover cases where H or f depend explicitly on t in a T periodic fashion.

Modifications of our procedures also suffice to treat (3) in the fixed energy setting. A major change is required in step 2° since we now consider a functional subject to a constraint. The new minimax argument relies heavily on the index theory introduced in [4] by Fadell and the author. To illustrate our results for this case we have:

THEOREM 17 [1]. *Suppose $H\in C^1(\boldsymbol{R}^{2n}, \boldsymbol{R})$ with $H^{-1}(1)$ a manifold diffeomorphic to S^{2n-1} under radial projection. Then (3) possesses a periodic solution on $H^{-1}(1)$.*

REMARK 18. A proof of Theorem 17 for $H^{-1}(1)$ convex has been given by A. Weinstein [5] using arguments from differential geometry. The theorem can also be obtained as a simple consequence of Theorem 9 [2]. Other variants of Theorems 9 and 17 can be found in [2] and [6].

Much work remains to be done on these problems especially for (1) and its higher dimensional analogues.

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Eigenvalues for Hypoelliptic Operators and Related Methods

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If one wants to study a pseudodifferential operator P it is often useful to try to construct the exponential $\exp(-tP)$. For instance, if P is of order 1 with real principal symbol on a compact manifold, then for $t\in\boldsymbol{R}$, $\exp(-itP)$ is a Fourier integral operator, and when P is elliptic and selfadjoint, one can get the asymptotic distribution of eigenvalues by studying the trace for small t (Hörmander [3]). When P is of real principal type microlocal parametrices can be obtained formally as $\pm i\int_0^{+\infty}\exp(\pm itP)\,dt$.

When P is of order 1 with complex principal symbol p, it is still possible to construct $\exp(-tP)$ for $t\geqslant 0$, by using Fourier integral operators with complex phase, provided that $\operatorname{Re} p\geqslant 0$ (Kucherenko [5], Maslov [7], Melin–Sjöstrand [9]). If in addition P is of principal type, a microlocal parametrix may be obtained as $\int_0^{+\infty}\exp(-tP)\,dt$ ([6], [9]). In the note [6], Kucherenko also states some results when P is no more of principal type, then the singularities of $\exp(-tP)$ may be stationary and the behaviour when $t\to+\infty$ has to be studied carefully, in order to define the integral. Naturally the subprincipal symbol of P now plays an essential role. See also Helffer [2]. The same difficulty is encountered already in the construction of $\exp(-tP)$ for small t, when P is of order >1 and we shall discuss such a situation for a selfadjoint operator. Most of the following is joint work with A. Menikoff [10], [11], [14]. (See also Metivier [12] and Bolley–Camus–Pham [1] for related results.)

Let X be a compact manifold of dimension n. All inner products and L^2-norms will be defined using some fixed positive smooth density on X. Let P be a classical pseudodifferential operator of order $m>1$, whose symbol in local coordinates can be written $\sim p_m(x,\xi)+p_{m-1}(x,\xi)+\ldots$, where p_{m-j} is homogeneous of degree

$m-j$. We shall always assume that $p=p_m \geqslant 0$ and that P is formally self-adjoint. Next after the elliptic case, the simplest possibility is that P is hypoelliptic with loss of 1 derivative. We also want P to be semibounded:

$$(Pu, u) \geqslant -C\|u\|^2, \ u \in C^\infty(X).$$

It follows from Melin [8] and Hörmander [4], that these two properties hold if and only if

$$S_p + \frac{1}{2}\widetilde{\mathrm{tr}} > 0 \quad \text{on} \quad p^{-1}(0).$$

Here $S_p = p_{m-1} - (2i)^{-1}\sum \partial^2 p/\partial x_j \partial \xi_j$ is the subprincipal symbol and $\widetilde{\mathrm{tr}} = \sum \mu_j$, where $\mu_j > 0$ and $\pm i\mu_j$ are the nonvanishing eigenvalues of the Hamilton matrix of p (which is defined as the matrix of the linearization of the Hamilton field at $p^{-1}(0)$).

Under this assumption P becomes a self-adjoint operator on $L^2(X)$ with domain $\{u \in L^2;\ Pu \in L^2\}$, and the spectrum is discrete and bounded from below. Let $N(\lambda)$ be the number of eigenvalues $\leqslant \lambda$.

In the case when $\Sigma = p^{-1}(0)$ is a submanifold and p vanishes precisely to the second order on Σ (i.e., $C^{-1} d_\Sigma^2 \leqslant p \leqslant C d_\Sigma^2$ locally if d_Σ is the distance to Σ in some Riemannian metric), we have obtained the following result ([10], [11]).

THEOREM 1. *Let* $k = \operatorname{codim} \Sigma$:

(1°) *If* $mk - 2n > 0$, $N(\lambda) = (1 + o(1)) C_1 \lambda^{n/m}$, $\lambda \to +\infty$,

(2°) *If* $mk - 2n = 0$, $N(\lambda) = (1 + o(1)) C_2 \lambda^{n/m} \log \lambda$, $\lambda \to +\infty$,

(3°) *If* $mk - 2n < 0$, $N(\lambda) = (1 + o(1)) C_3 \lambda^{(n-k/2)/(m-1)}$, $\lambda \to +\infty$.

Here C_1 is the same constant as in the elliptic case, C_2 can be expressed as the integral over the image of Σ in S^*X of a certain density which only depends on p. C_3 also depends on the subprincipal symbol:

$$C_3 = \frac{1}{(2\pi)^{n-k/2}\Gamma((n-k/2)/(m-1)+1)} \int_\Sigma e^{-(\tilde{\mathrm{tr}}/2 + S_p)\pi} \frac{\mu_j}{(1-e^{-\mu_j})} \omega(d\theta').$$

Here $\omega(d\theta')$ is the invariant density on Σ, defined as $(\det p''_{\theta''\theta''})^{-1/2} d\theta'$ in local coordinates $\theta = (\theta', \theta'')$ such that Σ is given by $\theta'' = 0$ and $d\theta = dx\, d\xi$. In all three cases $N(\lambda)$ and

$$\int_{p + |\xi|^{m-1} \leqq \lambda} dx\, d\xi$$

are of the same order of magnitude. Hörmander suggested to us to express the theorem in a unified way by using some modification of the measure $dx\, d\xi$.

This way of formulating the result becomes important when we drop the assumption that $p^{-1}(0)$ should be a manifold and we shall now consider this case. Consider for simplicity S^*X as a submanifold of $T^*X \setminus 0$ and fix a Riemannian metric on S^*X. Let (x_0, ξ_0) be a point where p vanishes and let k be the rank of p'' at

(x_0, ξ_0). Let δ be the smallest non-vanishing eigenvalue of $p''(x_0, \xi_0)$ and choose $\varepsilon>0, \varepsilon<\delta^4$. Choose a k-dimensional foliation $\mathfrak{g}$ of S^*X in an ε-neighbourhood of (x_0, ξ_0), such that the leaf through (x_0, ξ_0) is orthogonal to $\ker p''(x_0, \xi_0)$. We extend $\mathfrak{g}$ by homogeneity to a k-dimensional foliation in $T^*X\setminus 0$ and write $p=p_{\mathfrak{g}}+q_{\mathfrak{g}}$, where $q_{\mathfrak{g}}$ is constant on each leaf and $p_{\mathfrak{g}}\geqslant 0$ and vanishes at precisely one point on each leaf. The set where $p_{\mathfrak{g}}$ vanishes is a submanifold $\Sigma_{\mathfrak{g}}$ of codimension k. Choose local coordinates $(r, \theta)=(r, \theta', \theta'')$ such that r is homogeneous of degree 1 and equal to 1 on S^*X, θ is homogeneous of degree 0, (r, θ') is constant on each leaf of $\mathfrak{g}$ and $p_{\mathfrak{g}}=r^m\theta''^2$. Write $k=2d$ or $k=2d+1$ and let $\pm i\mu_j(r, \theta')$, $1\leqslant j\leqslant d$, be the eigenvalues of the Hamilton matrix of $p_{\mathfrak{g}}$ on $\Sigma_{\mathfrak{g}}$ arranged so that $0\leqslant\mu_1\leqslant\ldots\leqslant\mu_d$. (If there are fewer than $2d$ nonvanishing eigenvalues, we allow some of the μ_j's to be 0.) Write $\theta''=(\theta''_1, \ldots, \theta''_k)$ and define the map γ by $\gamma(r, \theta', \theta'')=(r, \theta', \tilde{\theta}'')$, where $(\tilde{\theta}''_{2j-1}, \tilde{\theta}''_{2j})=c_j(r, \theta)(\theta''_{2j-1}, \theta''_{2j})$, $1\leqslant j\leqslant d$, and $\tilde{\theta}''_{2d+1}=\theta''_{2d+1}$ if k is odd. Here $c_j=1$ when $\mu_j(r, \theta')=0$ and otherwise the non-negative number such that $r^m(\tilde{\theta}''^2_{2j-1}+\tilde{\theta}''^2_{2j})=k_j\mu_j(r, \theta')$, where k_j is the integer defined by

$$k_j\mu_j(r, \theta') \leqslant r^m(\theta''^2_{2j-1}+\theta''^2_{2j}) < (k_j+1)\mu_j(r, \theta').$$

Now define the measure $\Omega_{\mathfrak{g}, R}$ to be $dx\,d\xi$ in the region $r^m\theta''^2\geqslant Rr^{m-1}$ and to be the direct image of $dx\,d\xi$ under γ in the region $r^m\theta''^2<Rr^{m-1}$. (The measure also depends somewhat on the choice of the coordinates θ''.) For $\alpha>0$ (small) we can finally define a measure $\Omega_{\alpha, R}$ on $T^*X\setminus 0$ (nonunique) by piecing together $dx\,d\xi$ outside $p^{-1}(0)$ with various local measures as above: $\Omega_{\mathfrak{g}_j, R}$, $1\leqslant j\leqslant M$, by means of a partition of unity. Each measure $\Omega_{\mathfrak{g}_j, R}$ is here defined in a conic ε_j-neighbourhood of some point $(x_j, \xi_j)\in p^{-1}(0)$ and $\varepsilon_j\leqslant\alpha$.

THEOREM 2 [**14**]. *Under the assumption* (1):

$$N(\lambda) = \big(1+\mathcal{O}(\alpha^{3/8n}+1/R)\big) \int\limits_{p+S_P+\widetilde{\mathrm{tr}}/2\leqq\lambda} \Omega_{\alpha, R}(dx\,d\xi),\ \lambda\to+\infty.$$

Here $S_P+\frac{1}{2}\widetilde{\mathrm{tr}}$ is defined outside $p^{-1}(0)$ to be some continuous homogeneous extension of degree $m-1$.

REMARK 3. $N(\lambda)$ and $\int_{p+|\xi|^{m-1}\leqq\lambda} dx\,d\xi$ are still of the same order of magnitude. The term $\alpha^{3/8n}$ is certainly not the smallest possible.

In the proof we first want to construct $\exp(-tP)$ by solving at least approximately

$$\Big(\frac{d}{dt}+P\Big)v(t, x) = 0,\ t\geqq 0,$$

$$v(0, x) = u(x).$$

In local coordinates we try the "Fourier integral operator"

$$v(t, x) = A_t u(x) = \int e^{i\varphi(t, x, \eta)}a(t, x, \eta)\hat{u}(\eta)\,d\eta/(2\pi)^n,$$

where the phase should satisfy $\operatorname{Im}\varphi \geqslant 0$ and with some negligible error:

$$\partial\varphi/\partial t + i^{-1}p(x, \varphi'_x) = 0, \quad \varphi|_{t=0} = \langle x, \eta\rangle. \tag{3}$$

Although we only want to construct $\exp(-tP)$ for small positive times, the quasi homogeneity property

$$\varphi(t, x, \lambda\eta) = \lambda\varphi(t\lambda^{m-1}, x, \eta), \quad \lambda > 0, \tag{4}$$

leads us to a global problem in t with $|\eta|=1$. The results of [9] (together with an easy reduction to the case when p is homogeneous of degree 1) permit us to solve (3) on arbitrarily long finite time intervals with an error which is $\mathcal{O}(\operatorname{Im}\varphi^N)$ for all N (when $|\eta|=1$), but the constants involved in these estimates of the error may depend on the length of the interval. (The factor i^{-1} in (3) is essential, without that factor the phase becomes real but will in general only exist for small times, because of the appearance of caustics.)

A more careful application of the methods of [9] shows however that (3) can be solved for $|\eta|=1$, $0 \leqslant t < T(x, \eta)$, with an error which is $\mathcal{O}((1+t)^{q_N}\operatorname{Im}\varphi^N)$ for every $N \geqslant 0$, and such that φ and its derivatives grow at most polynomially in t. Here T is a certain lower semicontinuous function with values in $\boldsymbol{R}_+ \cup \{+\infty\}$, such that $\operatorname{Im}\varphi(t, x, \eta) \geqslant (C(1+T(x, \eta)))^{-4}$ when t is close to $T(x, \eta)$. It is then rather easy to extend φ to a solution with the same properties, defined for all t.

The transport equations can be solved similarly and one finds a symbol $a \sim \sum_0^\infty a_k(t, x, \eta)$, where $a_k(t, x, \lambda\eta) = \lambda^{-k}a_k(t\lambda^{m-1}, x, \eta)$ and the a_k are exponentially decreasing in t for $|\eta|=1$. (Here is where the assumption (1) is used.) This exponential decrease compensates for the powers of $(1+t)$ in the estimates for the error term in the characteristic equation and we get a smooth family of operators $A_t: \mathscr{D}'(X) \to \mathscr{D}'(X)$, $t \geqslant 0$, such that

$$(d/dt+P)A_t: \mathscr{D}'(X) \to C^\infty(\boldsymbol{R}_+; \mathscr{D}'(X)) \text{ has } C^\infty \text{ kernel and } A_0 = \mathrm{Id}. \tag{5}$$

The proof is then completed by computing (approximate expressions for) the trace and using the identity $\operatorname{tr}(\exp(-tP)) = \int e^{-t\lambda}\, dN(\lambda)$.

In the proof of Theorem 1 one can work more directly with Taylor expansions on Σ and this case is particularly interesting because one can see a little more explicitly what is going on. In particular when Σ is symplectic, $\varphi(t, x, \eta)$ converges exponentially fast with all its derivatives to a limit $\varphi(+\infty, x, \eta)$ as $t \to +\infty$ [10], while in general we only have polynomial bounds on the derivatives [11]. In the symplectic case Treves [13] has used the heat equation approach (after reduction to a model) to establish the analytic hypoellipticity.

REMARK 4. The construction of A_t also gives a new proof of Melin's inequality and it also gives a direct construction of a parametrix.

REMARK 5. It can be shown, using the general results of [4], that (1) is also a necessary condition for hypoellipticity with loss of one derivative, except where

$p^{-1}(0)$ is a symplectic manifold and p vanishes to precisely second order on $p^{-1}(0)$. We hope to be able to treat this remaining symplectic case in the near future and then obtain a general result for selfadjoint operators of order >1, hypoelliptic with loss of one derivative, and with nonnegative principal symbol.

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Eigenvalues of the Laplacian Plus a Potential

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To Rufus Bowen, in memoriam

0. Introduction. We report here on some recent work concerning the spectrum of operators of the form $\Delta + V$, where Δ is the (non-negative) laplacian on a compact riemannian manifold X, and $V: X \to \boldsymbol{R}$ is a C^∞ "potential" function. Such operators arise, for example, in the quantum mechanics of rigid bodies and rotators.

1. Does the spectrum determine the potential? We shall denote by $\mathrm{Spec}\,(X, V)$ the sequence $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ of eigenvalues of $\Delta + V$, repeated according to their multiplicities. In this section, we discuss the inverse problem: if $\mathrm{Spec}\,(X, V_1) = \mathrm{Spec}\,(X, V_2)$, is there an isometry $f: X \to X$ such that $V_2 = V_1 \circ f$?

The answer to our question in full generality is *no*. If X is the circle S^1, then the flow of the Korteweg–de Vries equation gives nontrivial 1-parameter families $V_t: S^1 \to \boldsymbol{R}$ such that $\mathrm{Spec}\,(S^1, V_t)$ is independent of t. On the other hand, it is known that a constant potential *is* determined by its spectrum: if $\mathrm{Spec}\,(S^1, V_1) = \mathrm{Spec}\,(S^1, V_2)$, and V_1 is constant, then $V_2 = V_1$. (See [M–vM] for a discussion of this 1-dimensional case.)

Attempts have been made to find isospectral deformations of Korteweg–de Vries type for potentials on higher dimensional manifolds, with no success as yet (except for direct products with S^1). Some uniqueness results have been obtained, however.

THEOREM 1.1. *Let X be a negatively curved surface such that no two periodic geodesics on X have the same length. If* $\mathrm{Spec}\,(X, V_1) = \mathrm{Spec}\,(X, V_2)$, *then* $V_2 = V_1$.

This is due to V. Guillemin and D. Kazhdan [G–K]. Note that the only isometry of such a surface can be the identity.

THEOREM 1.2. *Let X be a symmetric space of rank* 1. *If* Spec $(X, V_1)=$Spec (X, V_2), *and V_1 is constant, then $V_1=V_2$.*

Most of Theorem 1.2. is the work of V. Guillemin **[G1]**, **[G2]**, with the finishing touch in the case $X=S^2$ provided by H. Widom **[WM2]**.

2. Integrals around periodic goedesics. An important ingredient in Theorem 1.1 is the following theorem of Guillemin **[G1]**.

THEOREM 2.1. *If $\gamma: \boldsymbol{R}\to X$ is a nondegenerate periodic geodesic of least period L such that no other periodic geodesic has period L, then* Spec (X, V) *determines* $\int_0^L V(\gamma(t))\,dt$.

If the periodic geodesics of period L form a smooth non-degenerate family, then the parameter space F of the family has a natural volume element **[D–G]**, and it is the integral $\int_F \left[\int_0^L V(\gamma(t))\,dt\right] d\gamma$ which is determined by Spec (X, V).

When all the geodesics on X are periodic with the same least period L, then we can go further. Denote by $\overline{V}: G\to \boldsymbol{R}$ the function on the space G of closed geodesics defined by

$$\overline{V}(\gamma) = \frac{1}{L}\int_0^L V(\gamma(t))\,dt.$$

Then not only is the integral $\int_G \overline{V}(\gamma)\,d\gamma$ determined by Spec (X, V), but so is $\int_G \varphi(\overline{V}(\gamma))\,d\gamma$ for any C^∞ function $\varphi: \boldsymbol{R}\to\boldsymbol{R}$; i.e. the spectrum determines the measure μ_V on $\boldsymbol{R}$ defined by

$$\mu_V([a, b]) = \mu\{\gamma | a \leqslant \overline{V}(\gamma) \leqslant b\},$$

where μ is the measure on G.

In fact, as was first suggested to the author by the numerical experiments of G. Chachere **[CH]**, the measure μ_V can be "seen" quite clearly in the spectrum if X is a symmetric space of rank one. We recall that the eigenvalues of Δ in this case are of the form $\nu_k=(k+\alpha)^2-\beta^2$, $k=0, 1, 2, \ldots$; the multiplicity m_k of ν_k is a polynomial in k of degree $(\dim X)-1$. Since the potential V is a bounded operator, the addition of V to Δ causes the multiple eigenvalues in Spec $(X, 0)$ to split into clusters of bounded width about the ν_k's. The structure of the clusters is related to V according to the following theorem.

THEOREM 2.2. *Let X be a symmetric space of rank* 1, $V\in C^\infty(X, \boldsymbol{R})$.

(a) *For any $\varepsilon>0$, all but finitely many of the eigenvalues of $\Delta+V$ lie in the intervals $I_k=[\nu_k+\min \overline{V}-\varepsilon, \nu_k+\max \overline{V}+\varepsilon]$. For k sufficiently large, the number of eigenvalues in I_k is precisely m_k.*

(b) *If a and b are regular values of $\overline{V}$, then*

$$\frac{\#\{\lambda\in \text{Spec}\,(X, V) | \nu_k+a \leqslant \lambda \leqslant \nu_k+b\}}{m_k} = \frac{\mu\{\gamma\in G | a \leqslant \overline{V}(\gamma) \leqslant b\}}{\mu(G)} + O(k^{-1}).$$

In other words, the probability, for a randomly selected eigenfunction of $\Delta+V$,

that the contribution of V to the eigenvalue lies in $[a, b]$ is asymptotically equal to the probability, for a randomly selected periodic geodesic on X, that the average value of V lies in $[a, b]$.

In its present form, Theorem 2.2 is due to Y. Colin de Verdière [CL], based on earlier versions by Guillemin [G1] and the author [WE]. (See also [WM 1].)

3. Methods. Although Theorem 2.2 has the flavor of classical perturbation theory (see, for example, pp. 86–89 of [WL] for a discussion of the role of time averages), a proof with a good remainder estimate requires tools which are more precisely adapted to differential operators on manifolds.

All of the results described above rely on the theory of pseudodifferential and Fourier integral operators to give a geometric description of the eigenfunctions of Δ which may be used for analyzing the effect of adding V. Another possible approach is via Lie group theory. In his thesis [L], W. Lichtenstein has used purely algebraic methods to find, for any compact symmetric space X, eigenfunctions of Δ which are concentrated near the maximal totally geodesic flat tori in X. Using such eigenfunctions as a basis, one can analyze the effect of V in terms of integrals of V over the tori.

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Formal and Analytical Integral Sets

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Introduction. Let the analytical system of ordinary differential equations have an invariant irreducible k-dimensional torus T, filled with conditionally-periodic solutions ($k \geqslant 0$). There are many methods for investigation of the solutions of the system in a neighbourhood of the torus T. The most general and perspective one is apparently the investigation of the system by means of its normal form. Here we shall indicate some new results obtained by the method. For simplicity all formulations are given only for the case $k=0$, when T is a fixed point; the formulations for the cases $k=1$ (T is a periodic solution) and $k>1$ are similar.

1. The normal form. So we consider the analytical system

$$\dot{x}_i = \varphi_i(X, M), \quad i = 1, \ldots, n, \tag{1}$$

in a neighbourhood of the fixed point $X=0$, $M=0$. Here $X=(x_1, \ldots, x_n)$ are the local coordinates and $M=(\mu_1, \ldots, \mu_m)$ are the small parameters. Let $(\lambda_1, \ldots, \lambda_n)=\Lambda$ be eigenvalues of the matrix $(\partial\varphi_i/\partial x_j)$ for $X=0$, $M=0$.

THEOREM 1 [1]. *There exists a formal invertible transformation* $X=\Xi(Y, M)$ *which reduces the system* (1) *to the normal form*

$$\dot{y}_i = \psi_i(Y, M) \equiv y_i \sum g_{iQ}(M) Y^Q, \quad i = 1, \ldots, n, \tag{2}$$

$$Q = (q_1, \ldots, q_n), \quad Y^Q = y_1^{q_1} \ldots y_n^{q_n},$$

where coefficients $g_{iQ}(M)$ *are power series in* M, *they do not vanish only for such integral values of* Q *for which the scalar product*

$$\langle Q, \Lambda \rangle \equiv q_1\lambda_1 + \ldots + q_n\lambda_n = 0.$$

The normal form (2) has resonance terms only and is equivalent to a system of lower than n order [2]. It can be further simplified and reduced to an integrable system [3]. If the system (1) is real, then real values of the coordinates X correspond to such complex values of the coordinates Y, which satisfy the specific real conditions [4]. Methods of Delaunay and von Zeipel and many averaging schemes are the versions of (complete or partial) reducing into the normal form [5].

Usually the normalizing transformation is divergent in each neighbourhood of the torus T [2]. Therefore a correspondence between the solutions of the system (1) and (2) requires a further investigation. The results of such investigations can be classified in the following way:

(1) *The investigation of the stability of the torus* T. (a) Instability of the system (1) is deduced from instability of the system (2).

(b) Asymptotic stability of the system (1) is deduced from the asymptotic stability of the system (2).

(c) Neutral stability of the system (2) involves formal stability of the system (1) [6], and in some cases it also involves Ljapunov's stability [7].

Results of this kind are applied, for example, for investigation of stability of Lagrangian solutions of the restricted threebody problem [8].

(2) *Estimation of the instability effects of the system* (1) *by means of such effects of the system* (2). *Examples:* the influence of nutation's oscillations on the drift rate of a gyroscope in the Cardan suspension was calculated [9]; the mechanism of formation of the gaps in the asteroid's belt was considered [10].

(3) *Determination of periodic and quasi-periodic solutions of the system* (1) *from the system* (2). We shall consider this question in detail assuming for simplicity that all eigenvalues are pure imaginary, i.e. $\operatorname{Re}\Lambda=0$.

2. Sets of periodic and quasi-periodic motions. Let functions $f_1(X, M), \ldots, f_r(X, M)$ be analytical in point $X=0$, $M=0$ and equal to zero in it. Then the system of equations

$$f_j(X, M) = 0,\ j = 1, \ldots, r, \tag{3}$$

determines the *analytical set* $\mathcal{M}$, which contains the point $X=0$, $M=0$. If $f_1, \ldots, f_r$ are the formal power series, then we shall say, that the system of equations (3) determines the *formal set* $\mathcal{M}$.

Problem. What formal invariant sets of the system (1) are analytical?

The matter is that formal invariant sets can be calculated comparatively simply for the normal form (2). We have only to select from them those sets which are analytical for the initial system (1).

For the normal form (2) we define the formal set

$$\mathcal{A} = \{Y, M:\ \psi_i = \lambda_i y_i a,\ i = 1, \ldots, n\} \tag{4}$$

where a is a free parameter. It may be excluded from these equations; then set

$\mathscr{A}$ will be written in the form of (3). We shall say that the coordinate subspace $\{Y, M: y_i=0, i=i_1, \ldots, i_s\}$ is *rational,* if the corresponding eigenvalues $\lambda_j, j\neq i_1, \ldots, i_s$ $(1\leqslant j\leqslant n)$ are mutually commensurable. Let $\mathscr{K}$ denote the union of all rational subspaces. We denote $\tilde{\mathscr{A}}=\mathscr{A}\cap\mathscr{K}$.

THEOREM 2 [11]. *In the system* (1) *the formal set* $\tilde{\mathscr{A}}$ *is analytical.*

On set $\mathscr{A}$ we consider the matrix

$$B = (\partial\psi_j/\partial y_i - \delta_{ij}\lambda_i a),\ i, j = 1, \ldots, n,$$

where δ_{ij} is the Kronecker symbol and a is the same parameter as in equations (4). We define the formal set $\mathscr{B}$ as a subset of set $\mathscr{A}$ such that the matrix B is nilpotent. That is $\mathscr{B}=\{Y, M: Y, M\in\mathscr{A}, B^n=0\}$.

THEOREM 3 [11]. *In the system* (1), *the formal set* $\mathscr{B}$ *is analytical if eigenvalues* Λ *satisfy "small divisor" condition:*

$$|\langle Q, \Lambda\rangle| > c\|Q\|^{-\nu}$$

for all integral vectors Q (c *and* ν *are some positive constants*).

Properties of sets $\mathscr{A}$ *and* $\mathscr{B}$.

(1) All solutions from set $\operatorname{Re}\mathscr{A}$ are conditionally periodic (including periodic and fixed solutions). Indeed, the value of parameter a for each solution is constant and

$$y_i = y_i^0 \exp \lambda_i a t,\ i = 1, \ldots, n. \tag{5}$$

(2) Set $\operatorname{Re}\tilde{\mathscr{A}}$ consists of fixed and periodic solutions only.

(3) The value of parameter a for the conditionally periodic solution (5) from set $\mathscr{B}$ determines its basis of frequencies and its eigenvalues.

EXAMPLE. Let a Hamiltonian system with s degrees of freedom and with no parameters have k-dimensional torus T, and all its eigenvalues are purely imaginary. In the case of general situation set $\mathscr{B}$ can have the real part, which differs from torus T. Then torus T lies on the analytical one-parameter family $\operatorname{Re}\mathscr{B}$ of invariant s-dimensional tori. It is a refinement of Arnold's result [12], that there are many invariant s-dimensional tori near torus T.

If $\operatorname{Re}\Lambda\neq 0$, then the normal form (2) has an invariant coordinate subspace corresponding to all pure imaginary eigenvalues. As above we determine sets $\mathscr{A}$, $\tilde{\mathscr{A}}$ and $\mathscr{B}$ in that subspace; in this case Theorems 2 and 3 are valid. For investigations of the analytical perturbations and bifurcations of periodic and conditionally-periodic solutions it is sufficient to consider sets $\tilde{\mathscr{A}}$ and $\mathscr{B}$ in the systems with small parameters M.

[13], [14] show that set $\tilde{\mathscr{A}}$ contains periodic solutions found by H. Poincaré, A. M. Ljapunov, C. L. Siegel, J. Henrard and others; and set $\mathscr{B}$ contains quasi-periodic solutions found by A. N. Kolmogorov, V. I. Arnold, J. Moser, N. N. Bogolubov and others. For some generalisations and applications see [15]–[18].

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Résultats Récents sur la Conjugaison Différentiable

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1. Introduction. On pose $T^n = R^n/Z^n$ et on désigne la mesure de Haar de T^n par m.

1.1 Si M est une variété compacte connexe R-analytique, on désigne par $\mathrm{Diff}^r(M)$ (resp. $\mathrm{Diff}^r_+(M)$) le groupe des difféomorphismes de classe C^r de M (resp. qui sont C^r-isotopes à l'identité).

On munit $\mathrm{Diff}^r(M)$ de la C^r topologie et $\mathrm{Diff}^r_+(M)$ est alors la composante connexe de l'Id. Ici $r \in \{0, +\infty, \omega\} \cup \{x \in R \mid x \geqslant 1\}$; si $r=0$, c'est le groupe des homéomorphismes de T^n; si $r \geqslant 1$, $r \in R^* - N$, c'est le groupe des difféomorphismes de classe C^r vérifiant une condition d'Hölder d'exposant $r-[r]$ sur la $[r]$ème dérivée; si $r=\omega$ c'est le groupe des difféomorphismes R-analytiques de M. Si $r \in N \cup \{+\infty\}$ alors $\mathrm{Diff}^r(M)$ est un groupe topologique Polonais pour la C^r-topologie.

1.1 *Problème* 1. *Etudier la structure de groupe de* $\mathrm{Diff}^r(M)$.

Par exemple. Quand f et g sont-ils C^r-conjugués (i.e. il existe $h \in \mathrm{Diff}^r_+(M)$, tel que $f = h^{-1} \circ g \circ h$)?

Quelle structure peuvent avoir les C^r-centralisateurs

$$\left(\text{i.e. } \mathrm{Cent}^r(f) = \{g \in \mathrm{Diff}^r(M) \mid g \circ f \circ g^{-1} = f\}\right)?$$

1.2 Remarque. Si f et $g \in \mathrm{Diff}^\infty(M)$, il y a une infinité de problèmes: à quelles conditions f et g sont-ils $C^0, C^1, \ldots$ etc conjugués?

1.3. Exemple. Soient $T^2 = R^2/Z^2$ et $f(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)) \mod Z^2$, avec $\alpha \in T^1 - Q/Z$ et $\varphi \in C^\infty(T^1)$ vérifiant $\int_{T^1} \varphi(\theta)\, d\theta = 0$. On montre (voir [H])

que f est C^r-conjugué à $R_{(\alpha,0)}: (x_1, x_2) \to (x_1+\alpha, x_2)$ dans $\mathrm{Diff}^r_+(T^2)$ si et seulement s'il existe $\psi \in C^r(T^1)$ vérifiant $\psi - \psi \circ R_\alpha = \varphi$ (considérer $(x_1, x_2) \to (x_1, x_2+\psi(x_1))$).

Si α est un nombre de Liouville (i.e. α est irrationnel et pour tout entier $i \geqslant 1$, il existe $p_i/q_i \in \boldsymbol{Q}, (p_i, q_i)=1$, $q_i \geqslant 2$ vérifiant $|\alpha-(p_i/q_i)| < q_i^{-i}$) et si $r \in \boldsymbol{N}$ est donné, on peut choisir φ tel que ψ soit de classe C^r mais non $C^{r+\varepsilon}$. Dans ce cas on peut montrer que $\mathrm{Cent}^\infty(f)$ contient un sous-groupe fermé pour la C^∞-topologie, qui est monothétique, non localement compact et totalement discontinu.

1.4 Pour le problème de la simplicité des groupes $\mathrm{Diff}^r_+(M)$, voir [**Ma**$_1$], [**H**$_1$], [**H**$_2$], [**Th**]; pour les applications aux feuilletages voir [**Ma**$_2$], [**Law**].

1.5 Une question importante est la suivante:

Problème 2. Soit $f \in \mathrm{Diff}^\infty(M)$, *à quelles conditions sur* M *et* f *existe-t-il un* C^∞*-voisinage ouvert de* l'Id O, *tel que l'ensemble* $O^\infty_{f,V} = \{g^{-1} \circ f \circ g \mid g \in V \subset \mathrm{Diff}^\infty_+(M)\}$ *soit localement fermé et de codimension finie au voisinage de* f *(le tout pour la* C^∞*-topologie)?*

On posera dans la suite: $O^\infty_f = \{g^{-1} \circ f \circ g \mid g \in \mathrm{Diff}^\infty_+(M)\}$.

2. Cas de T^n. Nous allons rapidement décrire les exemples connus sur $T^n = R^n/Z^n$; on considère les translations $R_\alpha: x \to x+\alpha$, $\alpha \in T^n$.

2.1 Définition. La translation R_α de T^n satisfait à une condition diophantienne, s'il existe $\beta > 0$, $C > 0$, tel que pour tout $(k_1, \ldots, k_n) \in Z^n - \{0\}$ on ait

$$\left\| \sum_{i=1}^n k_i \alpha_i \right\| \geqq C(\sup |k_i|)^{-\beta}$$

avec $\alpha = (\alpha_1, \ldots, \alpha_n)$ et si $x \in T^1$, $\|x\|$ est la distance à l'entier le plus proche d'un relevé de x à $\boldsymbol{R}$.

2.2. Théorème (Kolmogorov [**Ko**$_1$], Arnold [**A**$_1$], Moser [**Mo**$_1$].). *Soit* R_α *une translation de* T^n *satisfaisant à une condition diophantienne. Il existe un voisinage* V_{R_α} *de* R_α *dans* $\mathrm{Diff}^\infty_+(T^n)$ *tel que si* $f \in V_{R_\alpha}$, *il existe* $\lambda \in T^n$ *et* $g \in \mathrm{Diff}^\infty_+(T^n, 0) = \{f \in \mathrm{Diff}^\infty_+(T^n) \mid f(0)=0\}$ *vérifiant*

$$f = R_\lambda \circ g^{-1} \circ R_\alpha \circ g.$$

De plus (voir [**Mo**$_2$]*) cette décomposition est localement unique.*

2.3. Pour démontrer le Théorème 2.2 Kolmogorov [**Ko**$_1$], [**Ko**$_2$] a proposé d'utiliser la démonstration des fonctions implicites en remplaçant la méthode d'itération de Picard par la méthode de Newton et en utilisant des opérateurs de lissage (pour pouvoir continuer l'itération); ou, ce qui revient au même en $\boldsymbol{R}$-analytique, en diminuant les domaines de convergence des complexifiés. Nash [**N**] a utilisé une idée semblable pour résoudre le problème du plongement isométrique en C^∞ (les principales) réferences pour ce problème sont dans [**GR**]). Dans [**Mo**$_2$], Moser à la suite de J. T. Schwartz [**Sch**$_1$], [**Sch**$_2$] a donné une forme

abstraite à la version de Nash. Ceci a suscité de nombreux travaux, par exemple Sergereart [Se], Hamilton [Ha], Hörmander [$\mathbf{Hor}_1$] et [$\mathbf{Hor}_2$]. Plus proche des travaux d'Arnold [$\mathbf{A}_1$], [$\mathbf{A}_2$], [$\mathbf{A}_3$], [AA] et de Moser [$\mathbf{Mo}_1$], [$\mathbf{Mo}_3$] sur les tores invariants (voir aussi [J]), on trouvera une très belle simplification due à Rüssmann [$\mathbf{R}_1$]. L'idée de Rüssmann [$\mathbf{R}_1$] a été reprise dans Zehnder [$\mathbf{Z}_1$], [$\mathbf{Z}_2$] et [$\mathbf{R}_2$]. On trouvera une démonstration de 2.2 en suivant [$\mathbf{R}_1$] dans [H, annexe].

La difficulté de la démonstration est la suivante: Soit l'application

$$\Phi_{R_\alpha}\colon (\lambda, g)\in \boldsymbol{T}^n\times \mathrm{Diff}_+^\infty(\boldsymbol{T}^n) \to R_\lambda \circ g^{-1}\circ R_\alpha \circ g \in \mathrm{Diff}_+^\infty(\boldsymbol{T}^n),$$

la «dérivée» de Φ_{R_α} pour $\lambda=0$ et $g=\mathrm{Id}$ est l'application linéaire

$$(\lambda, \varphi)\in \boldsymbol{R}^n\times C^\infty(\boldsymbol{T}^n, \boldsymbol{R}^n) \to \lambda+\varphi-\varphi\circ R_\alpha \in C^\infty(\boldsymbol{T}^n, \boldsymbol{R}^n).$$

Si on donne

$$\eta(x) = \sum_{k\in Z^n} \hat{\eta}(k) e^{2\pi i\langle k, x\rangle} \in C^\infty(\boldsymbol{T}^n, \boldsymbol{R}^n),$$

on peut résoudre l'équation

$$\varphi-\varphi\circ R_\alpha+\lambda = \eta$$

formellement par

$$\lambda = \hat{\eta}(0) \quad \text{et}$$

$$\varphi(x) \sim \sum_{k\neq 0} \hat{\eta}(k)(1-e^{2\pi i\langle k, \alpha\rangle})^{-1} e^{2\pi i\langle k, x\rangle},$$

(R_α est une translation ergodique de $\boldsymbol{T}^n \Leftrightarrow \forall k\neq 0, 1-e^{2\pi i\langle k, \alpha\rangle}\neq 0$). On montre que si R_α satisfait à une condition diophantienne, alors $\varphi\in C^\infty(\boldsymbol{T}^n, \boldsymbol{R}^n)$ et si $\eta\in C^r(\boldsymbol{T}^n, \boldsymbol{R}^n)$, $r>\beta$, alors φ est «en général» seulement $C^{r-\beta-\varepsilon}$ pour tout $\varepsilon>0$.

Si R_α est une translation ergodique de $\boldsymbol{T}^n$ ne satisfaisant pas à une condition diophantienne «en général», (pour la catégorie de Baire) pour $\eta\in C^\infty(\boldsymbol{T}^n, \boldsymbol{R}^n)$ il n'existe pas de φ dans $L^1(m)$ ni même m-mesurable (voir [$\mathbf{H}_5$]). La perte de dérivabilité en C^r (r fini) est une des sources principales des difficultés de 2.2. Il est aussi indispensable pour un Théorème dès Fonctions implicites dans les Fréchet que les applications satisfassent des conditions restrictives (cf. [LZ]) mais dans le cas considéré, l'action venant du groupe des difféomorphismes, une suite d'inégalités se trouve automatiquement vérifiée pour Φ_{R_α} (voir [Se], [$\mathbf{Z}_2$]).

2.4 *Problème* 2 (suite). *Si $O_{f,V}^\infty$ est localement fermé et de codimension finie pour un C^∞-ouvert V, est-ce-que M est difféomorphe à $\boldsymbol{T}^n$? Une question analogue se pose pour l'équation linéarisée de la conjugaison.*

2.5. En utilisant 2.2 on peut montrer (voir [H], [$\mathbf{H}_2$] et l'unicité locale de [$\mathbf{Mo}_2$]):

PROPOSITION. *Si R_α est une translation de $\boldsymbol{T}^n$ satisfaisant à une condition diophantienne, alors il existe un $r_\alpha\in \boldsymbol{R}_+^*$ tel que si $f\in \mathrm{Diff}_+^\infty(\boldsymbol{T}^n)$ est C^{r_α}-conjugué à R_α, f est C^∞-conjugué à R_α.*

2.6 *Problème* 3. *Déterminer le plus petit réel $r_\alpha\geqslant 0$ tel que* 2.5 *soit vraie.*

Nous nous proposons dans ce qui suit d'aborder ces problèmes dans le cas particulier du cercle $\boldsymbol{T}^1$; c'est-à-dire le seul cas que l'on sache étudier à l'heure actuelle.

3. Nombre de rotation des homeomorphismes du cercle [P] (voir aussi [H]).

3.1 On considère $D^r(T^1)=\{f\in \mathrm{Diff}^r(R) | f-\mathrm{Id}=\varphi\in C^r(T^1)\}$ où $C^r(T^1)$ est identifié aux fonctions de R dans R, Z-périodiques.

On a $\mathrm{Diff}^r_+(T^1)=D^r(T^1)/C$ (ou encore $D^r(T^1)$ (mod 1)) où $C=$Centre de $D^r(T^1)=\{R_p | p\in Z\}$.

3.2 Si $f\in D^0(T^1)$ avec $f=\mathrm{Id}+\varphi$ $(\varphi\in C^0(T^1))$ et si $n\in N$ on a $f^n=\mathrm{Id}+\sum_{i=0}^{n-1}\varphi\circ f^i$ (f^n dénote l'itérée n-ième de f).

On montre que si $n\to+\infty$, $(f^n-\mathrm{Id})/n$ converge uniformément vers une constante $\varrho(f)\in R$. La fonction $\varrho(f)$ *est appelée nombre de rotation de* f.

3.3 On montre les propriétés suivantes (qui sont dues à Poincaré):

(a) L'application ϱ de $D^0(T^1)$ dans R est continue pour la C^0-topologie.

(b) $\varrho(f+1)=\varrho(f)+1$.

(c) $\varrho(f)=\varrho(g^{-1}\circ f\circ g)$ pour tout élément g de $D^0(T^1)$. ϱ est donc un invariant de conjugasion.

(d) $\varrho(R_\alpha)=\alpha$.

D'après la propriété (b), ϱ définit par passage au quotient sur $\mathrm{Diff}^0_+(T^1)=D^0(T^1)/C$ un invariant de conjugaison à valeur dans T^1, que l'on appelle nombre de rotation (et que l'on note encore ϱ).

(e) Soit $f\in\mathrm{Diff}^0_+(T^1)$: les deux propriétés suivantes sont équivalentes:

$\varrho(f)\in T^1-Q/Z$.

f n'admet pas de point périodique sur T^1.

En outre $\varrho(f)=p/q$ (mod 1), $(p,q)=1$, est équivalent à: q est le plus petit entier >0 tel que f^q ait un point fixe.

(f) Soit $f\in\mathrm{Diff}^r_+(T^1)$ et supposons que f soit conjugué à une rotation R_α, avec α irrationnel (mod 1), par deux homéomorphismes h_1 et h_2 (i.e. $f=h_i^{-1}\circ R_\alpha\circ h_i$, $i=1,2$), alors il existe $\lambda\in T^1$ tel que $h_1=R_\lambda\circ h_2$. (Le centralisateur d'une rotation irrationnelle est le groupe des rotations.)

(g) Si $\alpha\in T^1-Q/Z$ et si $\varrho(R_\lambda\circ f)=\varrho(f)=\alpha$ alors $\lambda=0$.

Si $f=h^{-1}\circ R_\alpha\circ h$ alors $\varrho(f)=\alpha$. On est ainsi amené à voir dans quelle mesure ϱ caractérise la conjugaison à une rotation.

4. Cas $\varrho(f)=p/q$ (mod 1).

4.1 PROPOSITION. *Les deux affirmations suivantes sont équivalentes.*

$f\in\mathrm{Diff}^r_+(T^1)$ *est* C^r*-conjugué à la rotation* $R_{p/q}$.

$f^q=\mathrm{Id}_{T^1}$.

Considérons $U^r=\{f\in\mathrm{Diff}^r_+(T^1) | \varrho(f)=p/q\in Q/Z$ et $f^q-\mathrm{Id}$ change de signe$\}$.

4.2 PROPOSITION [H]. *Si* $0\leqslant r\leqslant\omega$ U^r *est un ouvert* C^r*-dense de* $\mathrm{Diff}^r_+(T^1)$.

De plus on montre (voir [A_1]) que si $r\geqslant 1$, U^r contient l'ouvert dense $V^r=\{f\in U^r$ les points périodiques de f sont hyperboliques$\}$ et tout $g\in V^r$ est structurellement stable.

4.3 Par 4.1 et 4.2 on voit que la fonction ϱ nombre de rotation ne caractérise pas «en général» la conjugaison à une rotation rationnelle. On peut montrer (cf. [Ma$_1$]) que si $f\in \mathrm{Diff}^\infty_+(T^1)$ vérifie $\varrho(f)=p/q\in Q/Z$ alors $\bar{O}^\infty_f$ (l'adhérence de O^∞_f dans $\mathrm{Diff}^\infty_+(T^1)$) est de codimension infinie dans $\mathrm{Diff}^\infty_+(T^1)$.

5. Cas $\varrho(f)=\alpha\in T^1-Q/Z$. A. Denjoy a démontré le théorème suivant (voir [H] pour plusieurs remarques):

5.1 Théorème de Denjoy ([D$_1$], [D$_2$]). *Soit $f\in\mathrm{Diff}^2_+(T^1)$. Supposons que $\varrho(f)=\alpha\in T^1-Q/Z$ alors il existe $h\in\mathrm{Diff}^0_+(T^1)$ tel que $f=h^{-1}\circ R_\alpha\circ h$. De plus pour tout $\alpha\in T^1-Q/Z$ il existe $f\in\mathrm{Diff}^1_+(T^1)$ tel que f ne soit pas C^0-conjugué à R_α.*

5.2. Le problème qui se pose est le suivant (voir 3.3 f):

Problème 4. *Soit $f\in\mathrm{Diff}^r_+(T^1)$ vérifiant $\varrho(f)=\alpha\in T^1-Q/Z$, quelle est la classe de différentiabilité de l'homéomorphisme h du théorème de Denjoy?*

5.3 Arnold [A$_1$] a montré qu'il existe un difféomorphisme R-analytique de nombre de rotation irrationnel tel que l'homéomorphisme h du théorème de Denjoy soit singulier par rapport à la mesure de Haar m de T^1 (la dérivée de h existe m-presque partout et on a $Dh=Dh^{-1}=0$ presque partout). On peut monter [H] qu'un tel exemple existe dans la famille de difféomorphismes R-analytique $x\to x+a\sin 2\pi x+b \pmod 1$ $0<a<1/2\pi$, $b\in T^1$.

5.4 On montre [H] que pour tout nombre de Liouville α il existe un difféomorphisme f de classe C^∞, de nombre de rotation α, qui ne soit pas C^1-conjugué à R_α. Pour l'étude du cas C^r avec perte de différentiabilité (voir [H]).

5.5 Il résulte de 2.2 et 3.3 (g) que si α satisfait à une condition diophantienne et si $f\in\mathrm{Diff}^\infty_+(T^1)$, $\varrho(f)=\alpha$, et si f est «suffisamment» C^∞-proche de R_α, alors f est C^∞-conjugué à R_α.

5.6 *Problème* 5. *Si $f\in\mathrm{Diff}^\infty_+(T^1)$ et $\varrho(f)=\alpha$ satisfait à une condition diophantienne alors f est-il C^∞-conjugué à R_α?*

6. Le resultat principal de conjugaison.

6.1 Soit A l'ensemble des nombres de T^1 qui satisfont à la condition A suivante: $\alpha\in A$ si α est irrationnel et si le développement en fractions continues de $\alpha=a_0+1/(a_1+1/(a_2+\dots$ vérifie

$$\lim_{B\to+\infty}\limsup_{N\to+\infty}\Big(\sum_{\substack{1\leqq i\leqq N\\ a_i\geqq B}}\mathrm{Log}\,(1+a_i)\Big/\sum_{1\leqq i\leqq N}\mathrm{Log}\,(1+a_i)\Big)=0.$$

On montre que l'ensemble A est de mesure de Haar égale à 1 (voir [H]).

6.2 Théorème [H]. *Soient $3\leqslant r\leqslant\omega$ et α satisfaisant à la condition* A. *Si $f\in\mathrm{Diff}^r_+(T^1)$ vérifie $\varrho(f)=\alpha$ alors $f=h^{-1}\circ R_\alpha\circ h$ avec $h\in\mathrm{Diff}^{r-1-\beta}_+(T^1)$ (pour tout $\beta>0$) si f est un difféomorphisme de classe C^∞ (resp. C^ω) alors h est de classe C^∞ (resp. C^ω).*

6.3 COROLLAIRE [H]. *Soit $\alpha \in A$ (par exemple un nombre algébrique de degré 2). Alors tout $f \in \mathrm{Diff}_+^\infty(T^1)$ s'écrit de façon unique*

$$\oplus \qquad f = R_\lambda \circ g^{-1} \circ R_\alpha \circ g$$

où $\lambda \in T^1$ et $g \in \mathrm{Diff}_+^\infty(T^1, 0)$. De plus la décomposition $\oplus$ dépend de façon C^∞ de paramètres variant dans une variété C^∞ de dimension finie.

6.4 *Conjecture. Le Théorème* 6.2 *reste valable si α satisfait à la condition du théorème de Roth: Pour tout $\varepsilon > 0$, il existe $C_\varepsilon > 0$ tel que pour tout $p/q \in \boldsymbol{Q}$ on ait $|\alpha - (p/q)| \geqslant C_\varepsilon q^{-2-\varepsilon}$.*

6.5 La démonstration de 6.2 se fait en deux étapes. On commence par démontrer que f est C^1-conjugué à R_α; pour cela on cherche à *appliquer* la proposition suivante (ceci est la partie la plus difficile de la démonstration):

PROPOSITION. *$f \in \mathrm{Diff}_+^1(T^1)$ est conjugué à une rotation si et seulement si la suite $\{f^i\}_{i \in N}$ est bornée dans la C^1-topologie* (i.e. si $\mathrm{Sup}_{i \in N} \|Df^i\|_{C^0} < +\infty$ *où* $\|Df^i\|_{C^0}$ *est la norme C^0 de la dérivée de l'itérée i-ième de f*).

Noter que f est C^1-conjugué à une rotation s'il existe $\psi \in C^0(T^1)$, $\psi > 0$, tel que l'on ait $\psi \circ f \cdot Df = \psi$.

La démonstration de 6.2 suit d'une version en C^r de 2.2 et 2.5. Pour tout ceci voir [H], on peut aussi consulter [Del].

7. Mesure de Lebesgue et nombre de rotation.

7.1 Kolmogorov [$\mathbf{Ko}_2$] puis Arnold [$\mathbf{A}_1$], [$\mathbf{A}_2$], [$\mathbf{A}_3$], pour surmonter les difficultés des mesures sur les groupes de difféomorphismes, ont proposé d'étudier la mesure de Lebesgue sur des «sous variétés différentiables» de dimension finie de ces groupes. C'est, me semble-t-il, la partie la moins étudiée et la plus prometteuse de la théorie des «petits dénominateurs». Voir à ce propos l'article de Lazutkin [Laz].

7.2 Soit pour $f \in \mathrm{Diff}_+^\infty(T^n)$ la fonction $M(f)$ égale à la mesure de Haar de l'ensemble $\{\lambda \in T^n \mid R_\lambda \circ f$ est C^∞-conjugué à une translation ergodique$\}$. Le théorème suivant généralise un théorème d'Arnold [$\mathbf{A}_1$]:

THÉORÊME [$\mathbf{H}_3$]. *Si $f \to \mathrm{Id}$ dans la C^∞-topologie, alors $M(f) \to 1$.*

7.3. Soit $t \in [0, 1] \to f_t \in \mathrm{Diff}_+^\infty(T^1)$ un chemin de classe C^1. On pose $M(f_t) =$ la mesure de Lebesgue de l'ensemble $\{t \in [0, 1] \mid \varrho(f) \in A\}$.

Le théorème suivant montre que la conclusion de 4.3 «oublie» que la mesure d'un ouvert dense peut être petite.

THÉORÈME [$\mathbf{H}_3$]. *Pour tout chemin f_t, tel que $\varrho(f_0) \neq \varrho(f_1)$ alors $M(f_t) > 0$.*

En s'inspirant de [C] on peut en un certain sens «calculer» une minoration de $M(f_t)$.

8. Nombre de rotation et categorie de Baire.

8.1 Considérons $O^\infty(T^1) = \{g^{-1} \circ R_\alpha \circ g \mid \alpha \in T^1, g \in \mathrm{Diff}_+^\infty(T^1)\}$ et $\bar{O}^\infty$ l'adhérence de $O^\infty(T^1)$ dans $\mathrm{Diff}_+^\infty(T^1)$ (pour la C^∞-topologie). Il suit de 6.2 (voir [H]) que

$\bar{O}^\infty$ est l'adhérence de l'ensemble $\{f\in \mathrm{Diff}_+^\infty(T^1)|\varrho(f)\in T^1-Q/Z\}$; par 4.1 et 4.2 $\bar{O}^\infty$ est un fermé sans point intérieur dans $\mathrm{Diff}_+^\infty(T^1)$ et par 7.3 $\bar{O}^\infty$ est néanmoins «assez» gros.

8.2 En contrepartie on a la proposition (très simple) suivante:

PROPOSITION [H]. *$O^\infty(T^1)$ est maigre dans $\bar{O}^\infty$ pour la C^∞-topologie induite ($\bar{O}^\infty$ est un espace de Baire).*

8.3 REMARQUES. (a) Noter que l'ensemble $\{\alpha\in T^n|R_\alpha$ satisfait à une condition diophantienne$\}$ est de mesure de Haar égale à 1, mais c'est un ensemble maigre de T^n!

(b) Pour l'équation linéarisée de la conjugaison C^∞ à une rotation on a une situation tout à fait semblable (cf. 2.3).

8.4 La situation que l'on vient juste de décrire est tout à fait générale. Le point important est de noter que $O^\infty(Q/Z)=\{g\circ R_{p/q}\circ g^{-1}|p/q\in Q/Z,\ g\in\mathrm{Diff}_+^\infty(T^1)\}$ est dense dans $\bar{O}^\infty$ pour la C^∞-topologie.

8.5. *Situation générale.* On se donne un espace topologique X non vide, localement compact, à base dénombrable et sur X une mesure $\mu>0$ de Radon (μ pouvant être de masse infinie). On suppose qu'un groupe topologique G, homéomorphe à un espace métrique complet, agit continûment et effectivement à gauche sur X ($G\times X\to X$ est continue) et de plus, tout $g\in G$ préserve μ. On suppose aussi que G possède un tore de dimension 1 et que $t\in T^1\hookrightarrow R_t\in G$ est continue. On suppose que G satisfait à la condition suivante:

Pour tout $p/q\in Q/Z$, pour toute fonction φ continue à support compact vérifiant $\int_X\varphi\,d\mu=0$ et pour tout $\varepsilon>0$, alors il existe $h\in G$ vérifiant:

(C)
$$\text{(a)}\quad \ldots h\circ R_{p/q}=R_{p/q}\circ h.$$
$$\text{(b)}\quad \Big|\int_{T^1}\varphi\circ h\circ R_t\,dt\Big|_{L^1(\mu)}<\varepsilon.$$

On pose $O_G(T^1)=\{g\circ R_t\circ g^{-1}|t\in T^1,\ g\in G\}$ et $\bar{O}_G$ est l'adhérence de $O_G(T^1)$ dans G avec la topologie induite de G.

THÉORÈME. *Soit G satisfaisant aux conditions précédentes. Alors l'ensemble $\{f\in\bar{O}_G|f$ est μ-ergodique$\}$ est un G_δ dense de $\bar{O}_G$ (cet ensemble est donc non vide).*

Nous démontrerons ultérieurement ce théorème [H_4] qui est d'ailleurs très simple.

8.6 *Applications* [H_4]. (a) On peut ainsi démontrer à nouveau un Théorème d'Anosov et Katok [An], [AK], [Ka]:

THÉORÈME. *Toute variété M, C^∞, compacte, connexe, admettant une action effective C^∞ de T^1 préservant une mesure $\mu>0$ de densité C^∞ (i.e. dans chaque carte μ a une densité C^∞ et >0 par rapport à la mesure de Lebesgue) admet un difféomorphisme C^∞ f préservant μ et μ-ergodique.*

G est ici le groupe des difféomorphismes C^∞ de M préservant la mesure μ agissant sur (M, μ).

(b) On retrouve aussi le résultat de Krygin [**KR**]: On prendra pour le groupe $G=\{f\in \mathrm{Diff}^\infty(T^1\times R) | f(\theta, y)=(\theta+\alpha, y+\varphi(\theta))$ avec $\alpha\in T^1$, $\varphi\in C^\infty(T^1)\}$ agissant sur $(T^1\times R, d\theta\otimes dy)$.

On peut généraliser (voir [$\mathbf{H}_4$]) ce résultat de la façon suivante:

Si H est un groupe de Lie, connexe, continûement moyennable (cf. [**G**]) et m une mesure de Haar invariante par les translations a gauche de $T^1\times H$, alors on peut prendre $G=T^1\times C^\infty(T^1, H)$ agissant sur $(T^1\times H, m)$ par $(\theta, g)\to(\theta+\alpha, g_\theta\cdot g)$ avec $(\alpha, g_0)\in G$ et $R_t(\theta, g)=(\theta+t, g)$.

(c) Soit M une variété C^∞ paracompacte connexe et $\nu>0$ une mesure (de Radon) de densité C^∞.

DÉFINITION. *Un difféomorphisme f de classe C^∞ de M est de type* III_1 *si le difféomorphisme* $S(f): M\times R\to M\times R$, *défini par* $S(f)(x, y)=(f(x), y+\mathrm{Log}\, d(f_*^{-1}\nu)(x)/d\nu)$ ($S(f)$ *preserve la mesure* $e^{-y}d\nu\otimes dy=\mu$) *est μ-ergodique.*

On montre qu'il existe $f\in G=\mathrm{Diff}_+^\infty(T^1)$ (agissant sur $T^1\times R$ par $S(f)$) tel que f soit un difféomorphisme se type III_1 de T^1. f n'admet pas de mesure σ-finie $\neq 0$ invariante absolument continue par rapport à la mesure de Haar de T^1. On retrouve ainsi un résultat de Katznelson [**Kat**] (les théorèmes sont incorrectement énoncés dans [**Kat**] par oubli de 4.2 et 4.3).

On peut montrer que toute variété C^∞ paracompacte connexe de dimension $\geqslant 3$ admet un difféomorphisme de classe C^∞ de type III_1.

8.7 *Morale.* Si M est une variété C^∞ compacte connexe de dimension $\geqslant 1$ admettant une action effective C^∞ de T^1 et $G=\mathrm{Diff}^\infty(M)$ alors, bien que $\bar{O}_G$ soit un fermé sans point intérieur dans G, par une propriété analogue à 8.4, on peut souvent construire des difféomorphismes dans $\bar{O}_G$ ayant des propriétés «chaotiques» (voir [**FH**], [**AK**], [**Kat**]). Ceci est l'idée fondamentale des théorèmes de catégorie d'Oxtoby et Ulam [**OU**], Halmos et Rohlin, voir [**Hal**], [$\mathbf{KAS}_1$] et [$\mathbf{KAS}_2$].

Néanmoins, en classe C^∞, la théorie de Kolmogorov—Arnold—Moser nous amène à poser la question de nature globale:

Question. Est-ce que tout nombre de rotation (à définir) peut être réalisé?

Donnons un *problème précis.*

Problème 6. *Existe-t-il un difféomorphisme C^∞ de D^2 ou T^2 préservant la mesure de Lebesgue m, qui soit m-ergodique et métriquement conjugué à la rotation R_α de T^1 où $\alpha=2^{1/2}$? (Pour le cas où α est super-Liouville voir* [**AK**]*).*

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Global and Local Aspects of the Theory of Complex Differential Equations*

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In this report we deal with three related subjects: absolute rigidity of algebraic differential equations over C, the topology of their foliations near a singular point, and reasons for the divergence of formal powerseries linearization.

1. Absolute rigidity. We study the set $\mathscr{A}_N^n$ of differential equations on C^n defined by

$$\dot{Z} = P(Z), \tag{1}$$

where $P(Z)$ is polynomial in $Z=(z_1, \ldots, z_n)$ of degree $\leqslant N$. The vector field $P(Z)$ determines a complex line element field, and therefore a cross-section (with singularities) in the projective tangent bundle $\mathrm{PT}(CP^n)$ of CP^n, which is the projective n-space obtained by completing C^n with a CP^{n-1} "at infinity". Such a birational cross-section for CP^n, or in general for an algebraic n-manifold X, is called a *differential equation* on X.

DEFINITIONS. The solutions of differential equations on X form in the complement of the singular set the *leaves* of a foliation.[1]

Two analytic differential equations on a manifold X are called *topologically (respectively analytically) equivalent,* if the corresponding foliations are conjugate by a homeomorphism (resp. holomorphism) $X \to X$.

The topological classification of analytic differential equations in the real domain is rather rough in comparison with the analytic one. For the equations in the neigh-

* Delivered by N. Kuiper.

[1] So the leaf does not contain the singular points, even if adding such points to the solution would give again a holomorphic curve.

bourhood of the singular point $0 \in R^n$, this follows from the classical theorems of Andronov–Pontrjagin and Grobman–Hartman.

For the complex differential equations these classifications are often near to each other and they may even coincide.

DEFINITION. Let A be some class of differential equations on a complex manifold X. An equation $\alpha \in A$ is called *absolutely rigid* in A, if there exists a neighbourhood $U \subset A$ of α and a neighbourhood $\mathscr{H}$ of the identity in the space of all homeomorphisms $X \to X$ with the topology of uniform convergence on compact sets, such that if the equations α and $\alpha' \in A$ are conjugate by a homeomorphism $H \in \mathscr{H}$, then they are analytically equivalent.

THEOREM [9]. *Almost all equations* $\alpha \in \mathscr{A}_N^2$, $\alpha \in \mathscr{A}_1^2$, *are absolutely rigid.*

2. Common properties. In spite of the absolute rigidity, almost all equations of the class $\mathscr{A}_N^n$ may possess some similar, "common" properties.

A. *Density.*

DEFINITION. An analytic differential equation on the manifold X has the "density property", if each of its leaves except a finite set is dense in X.

THEOREM [17]. *Almost all the equations* $\alpha \in \mathscr{A}_N^2$ *have the density property.*

THEOREM [13]. *There is a domain* $\Omega \subset \mathscr{A}_N^3$ *such that almost all equations* $\alpha \in \Omega$ *have the density property.*

B. *Cycles.*

DEFINITIONS. A cycle on a leaf φ of an analytic differential equation is any non-trivial element of the fundamental group $\pi_1(\varphi)$.

The cycles $\gamma_1, \ldots, \gamma_n, \ldots$ on the solutions of an analytic differential equation are called homologically independent, if any finite set of these cycles, belonging to the same leaf, is homologically independent on this leaf.

Let γ be a cycle on the solution φ, let (Γ, P) $(P \in \varphi)$ be a germ of a manifold at P, transversal to φ. By the general theory of foliations, there is a germ of holomorphic map $(\Gamma, P) \to (\Gamma, P)$ corresponding to the cycle γ, called the holonomy of γ and denoted by Δ_γ, and defined like the classical Poincaré mapping in the case of real differential equations [14].

The germs $\{\Delta_\gamma | \gamma \in \pi_1(\varphi)\}$ with superposition as operation form a group called the "monodromy group" of the leaf φ, and it is a natural homomorphic image of $\pi_1(\varphi)$.

The cycle γ is said to have identity holonomy in case $\Delta_\gamma = \mathrm{id}$; otherwise γ is called a limit cycle.

THEOREM [9]. *Almost all equations* $\alpha \in \mathscr{A}_N^2$ *have a countable set of homologically independent limit cycles.*

3. Commentary. For almost all $\alpha \in \mathscr{A}_N^2$ the straight line CP^1 "at infinity" in CP^2 contains $N+1$ singular points of the equation α. After deleting these we obtain the "leaf at infinity" which is denoted by F_α. The fundamental group $\pi_1(F_\alpha)$ is free with N generators. The existence of this leaf with such a rich fundamental group makes the study of the class $\mathscr{A}_N^2$ possible. The monodromy group of the leaf F_α is called the monodromy group of the equation α at infinity. The complexity of this group (action) is the cause of the intertwining of the solutions. It involves the density of the solutions and the existence of countably many homologically independent limit cycles for almost all equations from $\mathscr{A}_N^2$. These cycles may be found in arbitrarily small neighbourhoods of the leaf at infinity. Furthermore, the monodromy group at infinity is the topological invariant which distinguishes those equations that have no solutions with compact closure a real 2-manifold, except the solution at infinity. This is the reason for absolute rigidity.

4. Conjectures.

A. *Almost all equations from $\mathscr{A}_N^n$ possess the density property, are absolutely rigid and have countably many homologically independent limit cycles.*

B. *Almost all equations from $\mathscr{A}_N^n$ have limit cycles only.*

Together with the theorem about a countable set of limit cycles this conjecture means, that for almost all equations from $\mathscr{A}_N^2$ all the solutions, besides, perhaps, a countable set, are simply connected, and the sum of the one dimensional Bett numbers of the solutions, which are not simply connected, is equal to infinity.

Now we shall discuss some topics of local theory.

5. The topology near a singular point in C^n.

A. The linear case. Contrary to the real case, the topological classification of autonomous linear systems with complex time has complex moduli, even if we restrict the study to equations of "general type". This fact was discovered independently by C. Camacho, N. H. Kuiper, J. Palis [10], [18], and by N. N. Ladis [12]. Here we recall some definitions.

The point $\lambda=(\lambda_1, \ldots, \lambda_n) \in C^n$ belongs to the Siegel domain, if the convex hulli $\mathscr{H}(\lambda_1, \ldots, \lambda_n)$ contains $0 \in C$; otherwise it is said to belong to the Poincaré domain.

A linear equation $\dot{Z}=AZ$, $Z \in C^n$, is of strict Siegel type, if the convex hull of the spectrum of A has $0 \in C$ in its interior.

A singular point of a differential equation is of strict Siegel type, if the linear part at that point is of that type.

Even the class of linear differential equations contains absolutely rigid ones as we see in the

THEOREM (N. N. LADIS). *If two nondegenerate linear differential equations in the strict Siegel domain whose matrices have at least one nontrivial Jordan cell are topologically equivalent, then they are linearly conjugate.*

THEOREM ([10], [12], [8]). *If two linear (diagonalisable), differential equations of strict Siegel type: $\dot{z}_j=\lambda_j z_j$ and $\dot{w}_j=\mu_j w_j$, $j=1, \ldots, n$, are topologically equivalent*

then the sets of inverse eigenvalues $\{\lambda_j^{-1}\}$ *and* $\{\mu_j^{-1}\}$ *are linearly equivalent.*[2] *The converse also holds.*

The topological classification of complex linear differential equations is at present almost complete [6], [8], [10]–[12]; only the degenerate equations and those lying on the boundary of the Siegel domain are not yet classified.

B. *The nonlinear case.* The first statement of the last theorem (the topological equivalence of two equations of strict Siegel type in the neighbourhood of a singular point involves the linear equivalence of the inverse spectrums of their linear parts) is valid also for nonlinear systems (V. A. Naišul'). Consequently, the topological classification problem near singular points of nonlinear equations of strict Siegel type yields at least $n-2$ complex moduli.

We were informed that for $n=3$ the topological classifications in the nonlinear and the linear (strict Siegel domain) case are the same [18].

C. *Siegel singular points of algebraic differential equations.* I have no proof of the statement that almost all equations from $\mathscr{A}_N^n$ have at least one singular point of strict Siegel type for $n>2$. Nevertheless we propose the following conjecture.

CONJECTURE. *There is a constant* θ (perhaps, $\theta=\frac{1}{2}$), *such that each equation* $\alpha\in\mathscr{A}_N^n$, *that has in* $\mathbf{C}^n$ *the largest possible (equal to* N^n*) number of singular points, has at least* θN^n *singular points of strict Siegel type.*

This conjecture is a transcendental question about an algebraic variety. A point of this variety is the set of spectra of the linear parts of all singular points of the equation α.

D. *Some reasons for the conjecture about the absolute rigidity of almost all equations* $\alpha\in\mathscr{A}_N^n$.

We shall fix the equation $\alpha\in\mathscr{A}_N^n$. Let the equation $\alpha'\in\mathscr{A}_N^n$ be topologically equivalent to the equation α.

According to Naišul's theorem, each singular point of strict Siegel type of α imposes $n-2$ (complex) conditions on α', for being topologically conjugate. If the conjecture is true, then α' satisfies $W=(n-2)\theta N^n$ conditions. For large n then $W>\dim\mathscr{A}_N^n$. Independence of the conditions will yield the absolute rigidity of almost all $\alpha\in\mathscr{A}_N^n$.

6. Convergence and divergence of normalizing series. The general theory of normal forms and invariant manifolds of analytic differential equations in the neighbourhood of a singular point was developed by A. D. Bruno. Here we discuss some geometric questions.

The formal change of variables, for reducing a given equation to the linear normal form near a singular point (briefly—the normalizing series) may be divergent only

[2] This means the existence of $g\in\mathrm{GL}(2, R)$, which maps the set $\{\lambda_j^{-1}\}$ onto $\{\mu_j^{-1}\}$.

if the eigenvalues of the linear part of the equation is closely approximated by the resonances [7]. V. I. Arnold [1] gave a geometric picture, which explains the divergence of the normalising series for small n ($n \leqslant 3$). Some conjectures from [1] are proved by A. S. Pjartly and A. D. Bruno [15], [16], [4]. Recently V. I. Arnold discovered a parallelism in the "geometric" linearization theories of the singular points of analytic differential equations, of the fixed points of holomorphic mappings, and of the zero type embeddings of elliptic curves. The Arnold's program was carried out by A. S. Pjartly and the author. It appears that the theorems, proved in one theory, may be immediately transferred to the two others. In the next section we shall sketch the third theory mentioned above.

7. Imbeddings of elliptic curves. An elliptic curve is a Riemann surface of genus 1. In what follows Γ is an elliptic curve, M, M_1 and M_2 are analytic surfaces. For any imbedding $f: \Gamma \to M$ there is a normal bundle V_f over Γ.

DEFINITIONS. 1. Two imbeddings $f_i: \Gamma_i \to M$, $i=1, 2$, are said to be (analytically) *equivalent,* if two holomorphically equivalent neighbourhoods $U_i \subset M_i$ of $f_i\Gamma$ exist, such that the holomorphism $U_1 \to U_2$ maps $f_1\Gamma$ on $f_2\Gamma$.

2. The imbedding $f: \Gamma \to M$ is *of zero type,* if $f\Gamma$ has zero index of self intersection in M.

3. The linear bundle N over Γ is said to be

(a) *Resonant,* if it may be trivialised over some finite covering of the base;

(b) A *bundle of zero type,* if the imbedding $\Gamma \to N$ as zero section is of zero type;

(c) *Rigid,* if every imbedding $f: \Gamma \to M$ with normal bundle $V_f = N$ is equivalent to the zero section imbedding $\Gamma \to V_f$.

The space of all linear bundles over elliptic curves is an analytic manifold; we shall fix some smooth measure on it (all such measures are equivalent).

THEOREM. *Almost all linear bundles over Γ are rigid.*

Nonrigidity of a linear bundle over an elliptic curve has the following geometric explanation:

THEOREM. *Let $f: \Gamma \to M$ be an imbedding of zero type, and let the corresponding normal bundle be pathologically near to countably many resonant bundles. Then each tubular neighbourhood $U \subset M$ of $f\Gamma$ with the smooth projection $\pi: U \to f\Gamma$ contains a countable set of elliptic curves, which form finite branch coverings $f\Gamma$ by the projection π, and which are the obstruction to the equivalence of the imbedding f and the imbedding $\Gamma \to V_f$ as zero section.*

The proofs use the traditional techniques for differential equations: small denominators, invariant manifolds and so on.

8. Where is seen the algebraicity of algebraic differential equations? There are trivial algebraic answers e.g. the number of isolated singular points of the equation $\alpha \in \mathcal{A}_N^n$ does not exceed some constant which depends only on n and N.

A "transcendental" answer is the conjecture, dealt with in Hilbert's sixteenth problem:

CONJECTURE. *For the real equations of class* $\mathscr{A}_N^2$ *there exists a constant, depending only on* N, *such that the number of limit cycles of any equation* $\alpha \in \mathscr{A}_N^n$ *does not exceed this constant.*

The analogue of this conjecture in the complex case is not yet formulated. The following conjecture seems to be related to the preceding.

CONJECTURE. *There exists a constant* C, *depending only on* N *such that every cycle of any complex equation* $\alpha \in \mathscr{A}_N^2$ *has multiplicity less than* C *(the stable points of the corresponding monodromy transformation have multiplicity less than* C*), unless it has holonomy identity.*

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Singularities in Classical Celestial Mechanics

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Consider the motion of n point masses in a ν-dimensional Euclidean space under the laws of Newtonian physics. Denote the mass of particle i by $m_i>0$ and its position by $q_i \in R^\nu$. The gravitational potential energy can be written

$$U(q) = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|},$$

where $q=(q_1, \ldots, q_n)\in(R^\nu)^n$, and where $|\cdot|$ denotes the Euclidean norm. The motion of the particles is determined by the system of equations

$$m_i \ddot{q}_i = -\operatorname{grad}_i U(q_1, \ldots, q_n), \quad i = 1, \ldots, n, \tag{1}$$

where grad_i denotes the gradient with respect to q_i. Thoughout this paper we use a single dot over a variable to represent its derivative with respect to time t and a double dot to represent its second derivative with respect to t.

The potential energy U has a singularity whenever $q_i=q_j$. We write this singular set

$$\Delta_{ij} = \{q \in (R^\nu)^n : q_i = q_j\},$$
$$\Delta = \bigcup_{i<j} \Delta_{ij}.$$

The function U is real-analytic on $(R^\nu)^n - \Delta$. Applying the standard existence and uniqueness theorems for systems of ordinary differential equations to (1), we obtain

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the following result: Given $q(0) \in (R^\nu)^n - \Delta$ and $\dot{q}(0) \in (R^\nu)^n$, there exists a unique solution $q(t)$ defined for $t \in [0, \sigma)$, where σ is maximal. A similar statement can be made for negative t.

If $\sigma = \infty$, then the solution exists for all positive time. However, if $\sigma < \infty$, then the solution is said to experience a *singularity* at time σ. Henceforth we assume that $\sigma < \infty$.

The singularities of the solutions must correspond to the singularities of the potential function. In fact, a classical result shows that $q(t)$ approaches Δ as $t \to \sigma$ [13]. An important special case occurs if $q(t)$ actually approaches a particular point in Δ, *i.e.*

$$q(t) \to q_\sigma \in \Delta \quad \text{as} \quad t \to \sigma.$$

Such a singularity is called a *collision* and has the property that each of the particles approaches a limiting position. It is called a "collision" because at least two of the particles must coincide in the limit.

We can now state a question which has remained unanswered to this date. Does there exist a solution with a non-collision singularity? The work of Painlevé [8], von Zeipel [20], and Sperling [15] shows that, if $q(t)$ is such a solution, then $q(t) \to \infty$ as $t \to \sigma$. Thus an equivalent question may be stated. Does there exist a solution which becomes unbounded in finite time? For $n = 3$ Painlevé [8] showed that the answer is no. Saari [9] has shown that, for $n = 4$, the set of singular solutions must have zero measure. The proof depends strongly on the result that, if a noncollision singularity exists, all four particles must lie asymptotically along a fixed straight line [9], [12]. We shall return to the above question at the end of this paper.

Even the singularities due to collision are not fully understood when more than two of the particles are involved. Double collisions, however, are completely understood. From a variety of viewpoints it makes sense to extend a solution through a double collision by what is most easily described as an "elastic bounce" [2], [3], [6], [13], [17], [19]. It should be noted, however, that this extension depends strongly on the interaction between the particles. For potential functions other than gravitational, such an extension may not be possible [1], [7].

We shall now describe a technique which can be used to understand collisions. We shall discuss the behavior of the system when all n particles approach a simultaneous collision at their common center of mass. Such a collision is often called "total collapse". A similar technique can be applied when a subset of the particles approaches a common collision. For other examples where this technique has been applied, see [1], [7], [10], [11], [14].

We digress momentarily to discuss some notation. Let Q be a finite dimensional real linear space with inner product $\langle \cdot, \cdot \rangle$. Let Q^* be the dual space of Q. We shall use a star to denote both the natural map from Q to Q^* and the natural map from Q^* to Q. In other words, for any $p \in Q^*$, let p^* be the unique element of Q which satisfies

$$\langle p^*, q \rangle = pq \quad \forall q \in Q.$$

The star also denotes the inverse map, *i.e.*

$$(p^*)^* = p, \quad \forall p \in Q^*,$$
$$(q^*)^* = q, \quad \forall q \in Q.$$

We then define the induced inner product on Q^*,

$$\langle p, p' \rangle = \langle p^*, (p')^* \rangle, \quad \forall p, p' \in Q^*.$$

We use $\|\cdot\|$ to denote the norm induced by the inner product $\langle \cdot, \cdot \rangle$, both in Q and in Q^*.

We now return to the system of particles. Let

$$Q = \left\{ q \in (R^\nu)^n : \sum_{i=1}^n m_i q_i = 0 \right\},$$

the subspace of $(R^\nu)^n$ determined by fixing the center of mass at the origin. We define the inner product on Q whose norm corresponds to the moment of inertia:

$$\langle q, q' \rangle = \sum_{i=1}^n m_i (q_i, q_i'),$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product on R^ν. Introducing the momentum $p = \dot{q}^*$, we can write (1) as a first order system on $Q \times Q^*$

$$\dot{q} = p^*, \quad \dot{p} = -DU(q). \tag{2}$$

The total energy is a constant of motion and can be written

$$H(q, p) = \frac{1}{2} \|p\|^2 + U(q).$$

A solution (2) is confined to the set

$$E(h) = \{(q, p) \in Q \times Q^* : H(q, p) = h\}$$

for some fixed real h. For most values of h, $E(h)$ is a smooth submanifold of $Q \times Q^*$ of codimension 1, and (2) is a vector field on $E(h)$.

Although we do not use here the Hamiltonian structure, we note in passing that (2) is a Hamiltonian system with Hamiltonian function H.

Since the center of mass is fixed at the origin, total collapse can occur only at $q = 0$. System (2) is of course undefined at $q = 0$. We now introduce new variables so that the transformed equations can be extended to points corresponding to total collapse. Let

$$r = \|q\|, \qquad s = \|q\|^{-1} q,$$
$$v = \|q\|^{-1/2} pq, \quad w = \|q\|^{1/2} p - \|q\|^{-3/2} (pq) q^*. \tag{3}$$

Note that $r > 0$, $v \in R^1$, $\|s\| = 1$, and $ws = 0$. Let

$$S = \{s \in Q : \|s\| = 1\},$$
$$T_s^* S = \{w \in Q^* : ws = 0\},$$
$$T^* S = \{(s, w) \in Q \times Q^* : s \in S, w \in T_s^* S\}.$$

Then equations (3) define a real analytic diffeomorphism from $(Q-\{0\})\times Q^*$ to $(0, \infty)\times(-\infty, \infty)\times T^*S$. The inverse of this diffeomorphism can be written

$$q = rs, \quad p = r^{-1/2}(w+vs^*).$$

We further introduce a time transformation

$$dt = r^{3/2}\,d\tau$$

and use a prime to denote differentiation with respect to τ. Let V be the restriction of U to S, let

$$\pi(s)\colon Q \to \{q \in Q\colon \langle q, s\rangle = 0\}$$

be orthogonal projection, and let

$$\pi(s)^*\colon T_s^*S \to Q^*$$

be the dual of $\pi(s)$. Now (2) can be written

$$\begin{aligned} &r' = rv, && s' = w^*, \\ &v' = \frac{1}{2}v^2+\|w\|^2+V(s), && w' = -\frac{1}{2}vw-\|w\|^2 s^*-\pi(s)^*DV(s). \end{aligned} \tag{4}$$

The computation is facilitated if one notes that U is homogeneous of degree -1 and if one recalls the Euler formula $DU(q)q = -U(q)$.

The energy relation $H(q, p)=h$ transforms to

$$\frac{1}{2}(v^2+\|w\|^2)+V(s) = rh \tag{5}$$

which determines an invariant manifold for (4),

$$M(h) = \{(r, v, (s, w)) \in [0, \infty)\times(-\infty, \infty)\times T^*S\colon (5) \text{ holds}\}.$$

Note that this manifold contains points with $r=0$, but that the manifold $E(h)$ contains no points with $q=0$. Transformation (3) defines a diffeomorphism between $E(h)$ and

$$M^0(h) = \{(r, v, (s, w)) \in M(h)\colon r > 0\}.$$

This diffeomorphism carries the vector field (2) on $E(h)$ to the vector field (4) on $M^0(h)$.

Points of total collapse now correspond to the manifold

$$N(h) = \{(r, v, (s, w)) \in M(h)\colon r = 0\}.$$

Note that $N(h)$ forms the boundary of $M(h)$. Letting $r=0$ in (5) we have

$$\frac{1}{2}(v^2+\|w\|^2)+V(s) = 0. \tag{6}$$

Thus $N(h)$ is independent of h, and we may write

$$N = \{(v, (s, w)) \in (-\infty, \infty) \times T^*S\text{: (6) holds}\}.$$

The vector field (4) extends to N, with N as an invariant set.

We have now accomplished the following. We have pasted a boundary corresponding to total collapse onto the constant energy manifold. We have scaled the vector field so that it can be extended to the boundary. The scaling acts to slow down orbits ending in total collapse so that they now take an infinite amount of time to reach collision. By studying the flow on the boundary we can draw conclusions about the flow near the boundary and hence about orbits near total collapse.

Using (6), we rewrite the vector field on N.

$$\begin{aligned} v' &= \frac{1}{2}\|w\|^2, \\ s' &= w^*, \\ w' &= -\frac{1}{2}vw - \|w\|^2 s^* - \pi(s)^* DV(s). \end{aligned} \tag{7}$$

First note that the critical points occur when $w=0$, $DV(s)=0$. Values of s for which $DV(s)=0$ are called "central configurations" [13], [19]. Next note that the value of v strictly increases except at these critical points. Thus the flow on N is "gradient-like". These observations correspond to Sundman's result that solutions beginning or ending in total collapse asymptotically approach central configurations [13], [16], [19].

A thorough analysis of the flow on N has been performed only for $v=1$ and $n=3$, *i.e.* for the collinear three-body problem [5], [6]. In this case, Q is two-dimensional, $M(h)$ is three-dimensional, and N is two-dimensional.

Although S is now a circle, it breaks naturally into six segments, corresponding to the permutations of the three particles. As mentioned above, orbits can be extended through double collisions. Since the extension does not change the order of the particles, we may restrict our attention to one of the six segments. After this extension and this restriction N becomes homeomorphic to a two-sphere with four points removed, as illustrated in Figure 1. (For details, see [5].)

The coordinates labeled in Figure 1 are analogous to those of system (7). In fact, v is exactly the same, corresponding to radial velocity in the plane Q. Thus v measures the rate at which the particles are moving away from each other. The coordinate s is a parametrization of the segment of S described above and measures the configuration of the system. Suppose that the particles are numbered so that $q_1 \leqslant q_2 \leqslant q_3$. Then $s=-1$ corresponds to a configuration with $q_1=q_2$, while $s=+1$ corresponds to one with $q_2=q_3$. The saddle points in Figure 1 occur at a value of s corresponding to the collinear central configuration with $q_1<q_2<q_3$. The coordinate w corresponds to tangential velocity in the plane Q and measures the rate at which the configuration is changing.

The only two rest points for the flow on N are the saddles. Orbits in $M(h)$ ending in triple collision are asymptotic to the lower saddle, while orbits beginning in triple collision are asymptotic to the upper saddle. An application of the stable manifold theorem to these rest points yields the result that the set of orbits beginning or ending in triple collision forms a smooth immersed submanifold of $M(h)$. This result is a special case of a theorem of Siegel [13].

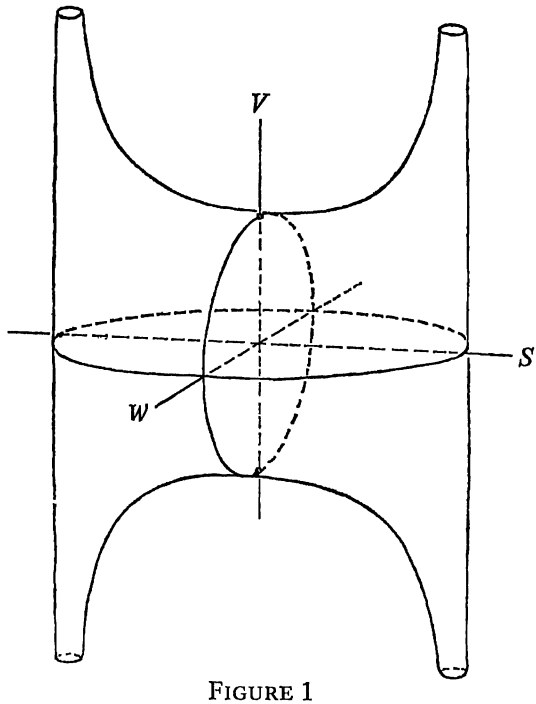

FIGURE 1

Now recall that v increases along orbits in N. Therefore most orbits on N move up one of the two arms. Consider an orbit passing close to triple collision. The continuity of the flow implies that this orbit must remain close to N for a long time. It therefore must move up the arm a large distance and emerge from a near-collision with large v. The coordinate s is near ± 1, which means that two of the particles form a very tight binary pair while the third is some distance away. The fact that v is large corresponds to a high velocity of the third particle away from the binary pair.

This last observation yields a somewhat unexpected result. By passing close to triple collision the system may transfer an arbitrary amount of energy from potential to kinetic. This new kinetic energy resides in the motion of the third particle with respect to the binary pair. Note that the system of particles with $\nu=1$ forms an invariant set for the system with $\nu \geqslant 2$. Thus the same result holds for arbitrary ν, as was proved independently by Waldvogel [18]. Since for $\nu \geqslant 2$ the set of orbits

encountering a double collision has positive codimension, orbits as described above exist without double collisions.

As a final remark, let us return to the question of whether there exist orbits which become unbounded in finite time. For $v=1$ and $n=4$ an uncountable number of such orbits was constructed by Mather and McGehee [4]. What follows is a description of one of these orbits.

Consider the four particles on the line with $q_1 \leqslant q_2 \leqslant q_3 \leqslant q_4$. Think of particles 1 and 2 as forming a binary pair. Particle 3 first moves toward the pair, coming close to triple collision. After interacting with the binary, it emerges with such a large velocity that it travels over to particle 4 and bounces back to the binary in time less than t_1. The process is repeated infinitely often, the kth occurrence taking time t_k. Choose the times t_k so that $t_1+t_2+t_3+\ldots=\sigma<\infty$. This orbit becomes unbounded as $t\to\sigma$ (Figure 2). Note, however, that the orbit experiences an infinite number of double collisions through which the solution has been extended and therefore does not provide a final answer to the question of the existence of a noncollision singularity.

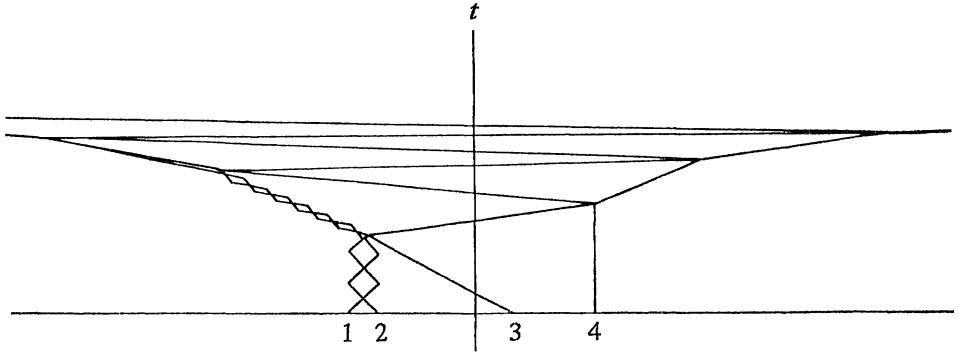

FIGURE 2

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Moduli of Stability and Bifurcation Theory

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To Rufus Bowen, in memoriam

Bifurcation theory of dynamical systems goes as far back as Poincaré. One aims at describing the changes in the phase portrait (space of orbits) of a system depending on parameters when such parameters vary. This lecture concerns a line of recent developments in this direction, especially on bifurcations of one parameter families of vector fields and diffeomorphisms. A relevant role, in this context, is played by certain differentiable invariants of topological equivalence for systems that exhibit non-transversal saddle connections. Such invariants bring up, in a natural way, the notion of moduli of stability in dynamical systems. They also imply, as explained in one of the topics below, the existence of moduli of stability for holomorphic vector fields near a singularity.

We begin with some simple examples of such invariants. Let X, X' be vector fields of class C^2 of R^2, each with two hyperbolic singularities of saddle type and trajectories γ and γ' connecting them. Let μ, μ' and λ, λ' be the eigenvalues of dX, dX' at the saddles along transversal directions to γ, γ'. Then [13], there is a flow conjugacy between X and X' near γ and γ' if and only if $\mu/\lambda=\mu'/\lambda'$.

We recall that, for a flow conjugacy, we require a homeomorphism h such that $hX_t=X'_t h$, where X_t and X'_t for $t\in R$ are the flows generated by X and X'. For a topological equivalence, we just require a homeomorphism sending orbits of X into orbits of X'.

We say that, up to conjugacy, the modulus of stability of X near γ as above is one, since there is one parameter family of different conjugacy classes for the vector fields C^2 near X. It follows that, up to topological equivalence, the modulus of stability of X near γ is zero.

This fact can be generalized to higher dimensions as follows. We now require

the connecting orbit γ to be a quasi-transversal (second order contact) intersection of the unstable manifold of a saddle p_1 and the stable of a saddle p_2, contained in the "weak" parts of both these submanifolds. We assume that the eigenvalues of dX at p_1 and p_2 are real. If $\mu_1, \mu_2, \ldots, \mu_s$ are the negative eigenvalues of dX at p_1, we also assume that $\mu_i < \mu_1$, $2 \leqslant i \leqslant s$. Similarly, $\lambda_1 < \lambda_j$ for $2 \leqslant j \leqslant u$, where $\lambda_1, \lambda_2, \ldots, \lambda_u$ are the positive eigenvalues of dX at p_2. Then, up to conjugacy, the modulus of stability of X near γ is one, given by the ratio μ_1/λ_1. Up to topological equivalence, it is zero.

Let us see the role of these invariants in the study of one parameter families of gradient flows. Let M be a smooth compact manifold and Grad (M) its C^∞ gradient vector fields, with the C^∞ topology. Consider the set of C^1 arcs $\varphi: I \to \text{Grad}(M)$, $I=[0, 1]$, with the C^1 topology. Two such arcs φ and φ' are conjugate, resp. equivalent, if there are a homeomorphism $\varrho: I \to I$ and a continuous map $h: I \to \text{Homeo}(M)$ such that, for each $s \in I$, $h(s)$ is a conjugacy, resp. a topological equivalence, between $\varphi(s)$ and $\varphi'(\varrho(s))$. An arc φ has finite modulus of stability if we can parametrize all equivalence classes of arcs near φ with finitely many real parameters; the minimum number of parameters is called the modulus of φ. Very recently, in a work still in production, Newhouse, Takens and the author proved the following result.

THEOREM [11]. *There is an open and dense subset B of one parameter families in* Grad (M) *such that if $\varphi \in B$ then the modulus of stability of φ, up to conjugacy, is finite. The modulus is given by the number of values $s \in I$ for which $\varphi(s)$ has a saddle connection.*

As a corollary we have

THEOREM. *There is an open and dense subset of stable one parameter families of gradient vector fields, up to topological equivalence.*

One of the main motivations for pursuing the ideas above was to obtain this result. It implies the stability of the simple arcs constructed by Newhouse–Peixoto connecting Morse–Smale vector fields, (see [14]). Note that, for dim $M=2$, it is a consequence of the work of Sotomayor [16]. The following are, in my opinion, very pretty questions in this direction. Are the stable k-parameter families of gradient flows dense for $k>1$? If not, are the ones with finite modulus of stability dense?

There are analogous invariants for saddle connections of diffeomorphisms. Let f and f' be C^2 diffeomorphisms of R^2, each with two fixed saddles and orbits γ and γ' of parabolic contact between their unstable and stable manifolds. As before, we indicate by μ, μ' and λ, λ' the eigenvalues of df, df' at the saddles taken along transversal directions to these invariant submanifolds. Then, there exists a conjugacy between f and f' near γ and γ' if and only if $\log|\mu|/\log|\lambda| = \log|\mu'|/\log|\lambda'|$.

That such a relation among the eigenvalues is necessary for a conjugacy is in [13] and the converse in [7]. Earlier, Newhouse had pointed out that a conjugacy as above does not in general exist. We remark that a similar invariant can be obtained in higher dimensions [11]. Some global results for dim $M=2$ are in [7]; for higher dimensions, Melo and the author believe it is possible to describe all Axiom A diffeomorphisms (see definition in [12]) with modulus of stability one. An interesting question is to obtain invariants of conjugacy for saddle connections with higher order contact. We also remark that all these facts and questions can be considered for saddle connections of closed orbits of vector fields. A particularly interesting case is discussed in the next topic.

We now describe a parallel development for holomorphic flows near a singularity. Let F be a holomorphic vector field near the origin 0 of C^n, with 0 as a singularity. Consider the real flow F induces by intersecting its orbits with a small sphere S^{2n-1} centered at the origin. We assume that any two eigenvalues of $dF(0)$ are independent over the reals. If the convex hull of these eigenvalues does not contain the origin of C, Guckenheimer showed that the real flow on S^{2n-1} is Morse–Smale, hence stable. It follows that F is locally stable. In the other case, the real flow on S^{2n-1} has pairs of closed orbits with nontransversal saddle connections, which persist under small perturbations of F. This fact implies the existence of moduli of stability: the modulus of stability of such F in C^n near the singularity is at least $2n-4$. For linear flows, it was shown in [2] and independently by Ladis (see [5]) that the modulus of stability is precisely $2n-4$. We conjecture that the same is true for non linear flows; for $n=3$ this is proved in [2]. Using the results for linear flows on C^{n+1}, a complete classification of holomorphic vector fields on the complex projective space CP^n is also carried out in [2].

Another interesting example of moduli is in [3], where the Lorenz attractor is proved to have modulus two of stability.

Let us mention some results of a global nature on bifurcations of diffeomorphisms. See [1], [12], [14], [17] for background, related accounts and references. In [9], [10] we studied how a family of diffeomorphisms depending on a parameter bifurcates from the Morse–Smale ones, these forming the best known class of stable diffeomorphisms. Consider C^1 families $\{\varphi_\mu\}$ of diffeomorphisms, $\mu\in[0, 1]$, such that φ_0 is Morse–Smale. Call $b=\inf\{\mu; \varphi_\mu$ is not Morse–Smale$\}$ the first bifurcation point. The problem is then to describe the phase portraits of φ_b and φ_μ, $b<\mu<b+\varepsilon$ for some $\varepsilon>0$. We assume the limit set $L(\varphi_b)$ to be a finite number of orbits. This is true in many examples and we conjecture that it is so for a Baire (residual) subspace A of arcs. We prove that A contains an open subset of all arcs, the remaining part of the conjecture being its density. The other results can be briefly summarized as follows. For a residual (generic) set of arcs, we have

(i) if $L(\varphi_b)$ has no cycles, then for some $\varepsilon>0$ φ_μ is Morse–Smale for μ in an open and dense subset of $(b, b+\varepsilon)$;

(ii) if $L(\varphi_b)$ is hyperbolic and has an equidimensional cycle, then for some

$\varepsilon>0$ there are infinitely many distinct classes of stable φ_μ's with $L(\varphi_\mu)$ infinite, $\mu\in(b, b+\varepsilon)$.

In case (ii), the subset of $(b, b+\varepsilon)$ for which φ_μ is stable has a large measure relative to ε. However, on its complement there are residual subsets of small intervals for which φ_μ exhibits infinitely many sinks [8]. We remark that some of the above questions were also treated by Afraimovic and Silnikov (see [12] for references). In his forthcoming thesis [6], M. Levi shows that several of the above facts occur in a model, previously introduced by Levinson, of the forced van der Pol equation.

As for the stability of such arcs, a characterization is given in [11] of the stable ones up to and beyond the first bifurcation point. In doing so, we provide universal models for the unfoldings of the elementary bifurcations, the novelty being the surprising rigidity of the saddle-node case. From our previous discussion on moduli of stability, we conclude that arcs of diffeomorphisms going through nontransversal saddle connections are not stable. One of our results, proving the stability of a certain class of arcs, has been generalized by C. Robinson [15].

In our analysis, a specially interesting question is the stability of arcs of diffeomorphisms going through a saddle-node with cycle. In some cases, there appears an invariant circle and using Herman's results [4], it is shown that such arcs are not stable. In many other cases (in fact, we conjecture: in all other generic cases), nontransversal homoclinic orbits and horse-shoes appear. We succeeded in showing this fact for normally attracting (repelling) saddle-node. In this case, the saddle-node is contained in a normally attracting annulus. Through projection, the question is reduced to the study of a class of endomorphisms of degree one of the circle, with varying rotation "interval". Such a concept is similar to that of rotation number for a homeomorphism of the circle.

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On the Structure of Optimal Feedback Systems

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The basic optimal control problem is given by a system

$$\dot{x} = f(x, u),\ x \in R^n,\ u \in R^m, \tag{1}$$

a control domain

$$U \subset R^m \tag{2}$$

a performance index

$$J(u) = \int_0^T f^0(x, u)\, dt\ \ (f^0 \colon R^n \times R^m \to R), \tag{3}$$

initial and target states x_0, x_1 respectively. By an admissible control we understand a piecewise continuous function, defined on some interval of the real line with values in U. Under suitable regularity conditions on f^0, f every admissible control $u \colon [0, T] \to U$ when substituted into (1) defines a unique solution $x(t, u)$ starting at x_0 for $t=0$ (called the response of u). Substituting the control and its response into (3) for u, x respectively, gives a real value to J. One is interested in finding and studying the properties of the optimal control which steers the system from x_0 to x_1 (i.e. its response $x(t)$ called the optimal trajectory satisfies $x(T)=x_1$) for some $T>0$ and minimizes the performance index J.

From the very beginning of the optimal control theory one of the approaches to study this problem has been to imbed it in a family of problems with a varying initial state x_0. This approach is based on the simple observation (frequently called Bellman's optimality principle) that if u is an optimal control on $[0, T]$, then its restriction to any interval $[t_0, T]$, $t_0 \geqslant 0$, is an optimal control for the initial state $x(t_0, u)$. If for each initial state x in some region G the optimal control u_x (and, consequently, its response ξ_x starting at x) is unique, from the optimality principle

we obtain immediately that the optimal control can be expressed, independently of the initial state, as a function of the present state of the system, i.e. there exists a function $v: G \to U$ such that $u_x(t) = v(\xi_x(t))$ for $x \in G$. Therefore the optimal trajectories satisfy in G the differential equation

$$\dot{x} = f(x, v(x)). \tag{4}$$

Let us note that in many applications the ultimate goal of solving the optimal control problem is to find the function v, which is called the closed-loop optimal control, the optimal feedback law or the synthesis of optimal control.

Formally, one can consider (4) as an equation for optimal trajectories. In order to utilize it, it is important to know something about the properties of the function v. For example, for the classical existence and uniqueness theory of ordinary differential equations it would be useful if v were continuous. However, simple examples in which v can be constructed explicitly (cf. [1, Chapter III] or [11, Chapter 2]) show that due to unilateral constraints, which are typical for the optimal control theory, v is frequently discontinuous.

A deeper reason for studying the structure of v is the problem of sufficiency of the variational necessary conditions of optimality, in particular of the Pontrjagin maximum principle (PMP). Assume that for every initial state $x \in G$ there exists a unique control steering the system from x to x_1 and satisfying PMP, thus being the unique candidate for the optimal control. If we define $v(x) = u_x(0)$, we may ask whether u is the closed-loop optimal control, i.e. whether (4) yields optimal trajectories (and only optimal trajectories) as its solutions. As it is shown in [1], [2] this problem is closely connected with the problem of the sufficiency on the dynamic programming equation (which corresponds to the Hamilton–Jacobi equation of the classical calculus of variations).

When trying to resolve this question one is again confronted with the problem of the regularity of the behaviour of v. Bolt'anski observed that one can work also with a discontinuous synthesis, provided its set of discontinuities is sufficiently regular. This led him to introduce the concept of regular synthesis for the time-optimal control problem ($f^0 = 1$) (cf. [1], [2]). By a regular synthesis for the time-optimal control problem in a region G we understand a pair $(\mathscr{S}, v)$, where $\mathscr{S}$ is a locally finite partition of G into C^1 connected submanifolds of G (called cells), v is a function $G \to U$ satisfying the following conditions:

A. The set $\bar{G}'$ (where G' is the union of the cells of dimension $< n$) admits a stratification in G. (By a stratification $\mathscr{P}$ of a subset H of G we understand a locally finite partition of H into C^1 connected submanifolds of G (called strata) such that $P \cap \bar{Q} \neq \emptyset$ implies $P \subset \bar{Q}$ and $\dim P < \dim Q$ for any $P, Q \in \mathscr{P}, P \neq Q$.)

B. The function v is C^1 on each $S \in \mathscr{S}$ and can be extended to a C^1 function in some neighbourhood of S. The cells of $\mathscr{S}$ are of type I and type II. If S is

of type I, then $f(x, v(x)) \in T_x S$ (the tangent space of S at x) for every $x \in S$ and there is a uniquely defined cell $\pi(S)$ such that every solution of (4) starting at any point $x \in S$ enters $\pi(S)$ transversally for some $\tau > 0$ (after staying in S on $(0, \tau)$) which is a continuous function of x. If S is of type II then $f(x, v(x)) \notin T_x S$ for all $x \in S$ and there is a unique cell $\Sigma(S)$ of type I such that v is C^1 on $S \cup \Sigma(S)$ and every solution of (4) starting in S lies in $\Sigma(S)$ for sufficiently small positive times.

C. Every trajectory $x(t)$ of (4) starting at some point $x \in G$ (which is by B uniquely defined until it stays in G) eventually reaches x_1 in finite time $T(x) \geqslant 0$ passing through a finite number of cells only and together with the control $u(t) = v(x(t))$ satisfies PMP.

D. $T(x)$ is continuous in G. Let us note that this definition differs somewhat from Bolt'anski's one as well as from that of [3]. (For details, cf. [3] and the forthcoming Erratum to [3].)

In [2] (cf. also [1]) Bolt'anski proved that if $(\mathcal{S}, v)$ is a regular synthesis, then v is the closed-loop optimal control in the following sense:

The trajectory ξ_x (in the Carathéodory sense) on $[0, T(x)]$ of equation (4) starting at $x \in G$ is the optimal trajectory and $u_x(t) = v(\xi_x(t))$ is the optimal control.

Virtually in all the simple examples in which it has been possible to construct the synthesis explicitly, the latter has satisfied the conditions of regularity. However, except for some studies of the local structure of v near x_1 (cf. e.g. [14]) no attempt has been made to prove that a more general class of problems would globally admit a regular synthesis. Such a result has been made possible by Hironaka's theory of subanalytic sets [7], [9], [10]. It concerns linear control systems

$$\dot{x} = Ax + Bu \tag{5}$$

with

$$U = \operatorname{co} \{w_1, \ldots, w_p\} \tag{6}$$

being a convex polytope. Such a problem is called normal if for every $i \neq j$, k,

$$\det \left(b_k(w_i - w_j), Ab_k(w_i - w_j), \ldots, A^{n-1} b_k(w_i - w_j)\right) \neq 0,$$

where $B = (b_1, \ldots, b_m)$. Let us note that normality is a generic property (cf. [11, Chapter 2, Theorem 11]).

THEOREM 1 [3]. *Assume that the control system defined by* (5), (6) *is normal and that* U *contains* 0 *in its interior. Then the time-optimal control problem with the target point* $x_1 = 0$ *admits a regular synthesis in the domain* G *of points that can be steered to* 0.

As mentioned above, the proof of this theorem makes use of the theory of subanalytic sets. A subset M of an analytic manifold is called subanalytic if it can be locally (in A) expressed as a finite union of sets of type $f(Y) \setminus g(Z)$, where Y, Z are analytic manifolds and f, g are analytic proper. By the central theorem of the

theory of subanalytic sets, every subanalytic subset of A admits an analytic stratification, the strata of which are subanalytic (cf. also [13]).

The cells of the synthesis are obtained by an inductive construction. The sets of continuity of v are shown to be subanalytic and the synthesis cells are obtained by a sequence of partitions of these sets into connected analytic submanifolds. In addition to the standard theory of subanalytic sets one needs the following

LEMMA. *Let M be a subanalytic subset of an analytic manifold A and let $X_1, \ldots, X_r$ be analytic vector fields on A. Then M admits a locally finite partition $\mathscr{P}$ into connected analytic submanifolds of A, which are subanalytic in A, such that for every $P \in \mathscr{P}$ and $i=1, \ldots, r$, X_i is either everywhere or nowhere tangent to P.*

This lemma, an improved version of which has been proved by Sussmann, appears to be crucial also for other application of the theory of subanalytic sets in control theory (cf. [12]). From this theorem it immediately follows that the minimum steering time to x_1, $T(x)$, is analytic in G everywhere except for a stratified set (G') of dimension $n-1$ (=maximal dimension of the strata).

If one tries to extend the concept of regular synthesis to problems where PMP yields controls with corners which are not jumps (like time optimal control problems with control domains having piecewise analytic curvilinear boundaries, or linear-quadratic problems with linear constraints), one immediately sees that the transversality assumptions as well as the C^1 extendability of v to the neighbourhood of S in B cannot be required. Instead, one has to assume their consequences, namely that the time $\tau(x)$ at which $\xi_x(t)$ enters $\pi(S)$ for S of type I and $\pi(\Sigma(S))$ for S of type II, the trajectory $\xi_x(t)$ and the control $u_x(t)$ are C^1 functions of x, t for $x \in S$, $t \in [0, \tau(x))$ and can be extended to C^1 functions of x, t for $t \geqslant \tau(x)$ close to $\tau(x)$. With this difference, the definition of regular synthesis can be literally extended to control problems with other performance indices (T in D replaced by J, the performance index). Bolt'anski's proof can be extended easily to yield an extension of his sufficiency theorem to general performance indices.

Employing essentially the same induction techniques as in the linear time-optimal problem case, one can prove an abstract existence theorem. However, due to the lack of transversality mentioned above, in order to obtain the C^1 dependence of the required quantities one has to construct auxiliary partitions in the product space of the state and adjoint space. By suitable partitions one can achieve that the product flow of the system and its adjoint enters the cells in the product space transversally, thus yielding analyticity of the required quantities.

Because of lack of space we desist from introducing this theorem, which has a rather cumbersome formulation. This is due to technical assumptions, which are needed for the extendability of the solutions of certain vector fields to sufficiently long intervals. Rather we note that the most serious requirements (in addition to analyticity, of course) for a system to admit a regular synthesis in some region G are the following ones:

1. For every initial state $x \in G$, there has to be a unique control u_x satisfying PMP which steers the system from x to x_1.

2. The number of switchings (which are roughly speaking the points of non-analyticity) of the controls u_x has to be locally uniformly bounded.

The first requirement makes the range of applications of such a result rather limited. Indeed, although singular controls (not minimizing the Hamiltonian strictly), which are quite typical for nonlinear control problems, are not excluded in principle, when they appear the first requirement is usually not satisfied. On the other hand the second requirement, the validity of which is difficult to prove for more general classes of systems, is virtually always satisfied in particular problems.

The following theorem concerns a model class of problems in which these difficulties can be overcome—linear-quadratic optimal control problems with linear constraints.

THEOREM 2. *Consider the optimal control problem*

$$\dot{x} = Ax + Bu,$$

$$J = \int_0^T [x^* Q x + u^* R u]\, dt \quad (R > 0,\ Q \geqslant 0),$$

$$U = \{u \in R^m \mid \langle l_j, u \rangle \leqslant m_j,\ j = 1, \ldots, p\},$$

$x(T) = 0$, *T fixed, and assume that this system is normal. Then the problem admits a regular synthesis.*

The normality assumption here consists in the non-vanishing of certain polynomials involving the entries of A, B, Q, l_j, m_j, as in the case of the linear time-optimal control problem it is a generic property.

Of course, this theorem has a similar impact on the regularity of the minimal value of the performance index as Theorem 1 had on the regularity of the minimal steering time.

Let us note that neither Theorem 1 nor Theorem 2 contribute anything to sufficient conditions of optimality (the sufficiency of PMP in both cases can be proved by other, simpler means). Their value lies rather in the insight they give into the structure of the closed-loop optimal control.

Finally let us note that in Bolt'anski's sufficiency results one understands the solutions in the classical Carathéodory sense. However, it has been demonstrated by several authors in the fifties that this concept is inadequate in the case of equations with discontinuities in the dependent variable. Because of the discontinuity of v this is the case for equation (4) in many control problems. Several concepts of solutions for such equations have been proposed, the most elaborate being that of Filippov [6]. Therefore it is natural to ask whether the optimal trajectories (which are the usual solutions of (4)) coincide with the Filippov trajectories or not. This problem is related to the problem of stability of the behaviour of the solutions of (4) with respect to perturbations (cf. [8], [4]). Using a slight improvement of Theorem 1 this question can be answered positively for the linear time-optimal control problem

with $\dim u=1$ (cf. [3], [5]). However, the results of [4], where the problem is completely solved for the two-dimensional linear time-optimal control problem, show that there is a non-exceptional class of problems for which the optimal trajectories do not coincide with the Filippov trajectories of (4).

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Nonsmooth Analysis and Optimization*

Frank H. Clarke

1. Introduction. The title of this talk refers not to the mere existence of nonsmoothness in analysis and optimization, which is of course not new, but to the attempts to consider differential properties of functions that are not differentiable in the usual sense(s) of the term, and tangential properties of sets that are not smooth. Precedents for such a study do exist: the classical Dini derivates provide an early example, and the study of convex sets and functions a more recent one. Convex analysis (as the latter is called) is an excellent illustration of how such a study can lead to new insights and inspire new methods. The present discussion concerns the systematic development of a calculus for functions that need be neither differentiable nor convex, and the applications of the ideas involved to analysis and optimization.

Sometimes we are led quite directly to consider differentiation of nonsmooth functions. The norm in a Banach space, for example, is a function which may or may not be differentiable (away from the origin), and the question of whether or not it is proves to be highly interesting. As another example, consider the following design problem: a system in state x is subject to distortion $f(x)$. In manufacture, the state x can only be specified up to a possible error (tolerance) $e \in E$. We wish to choose the specified state x so as to minimize the worst distortion that might ensue; i.e., to minimize

$$F(x) = \max \{f(x+e): e \in E\}.$$

The function F so defined will generally not be differentiable, so if one would like

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to apply numerical minimization techniques analogous to the familiar ones that employ derivatives of F, then a substitute for the derivative must be found.

To illustrate a more indirect way in which such considerations arise, suppose we are given a family of optimization problems depending on a parameter u:

$$\text{minimize } f(x) \text{ subject to } g(x) = u.$$

Let $\alpha(u)$ be the minimum in the above problem for parameter value u. In applying Lagrange multiplier techniques to this mathematical programming problem, an informal argument suggests that the multiplier is related to the derivative of α when it exists (such interpretations are important in mathematical economics). However, $\alpha(\cdot)$ cannot be guaranteed differentiable by any reasonable hypotheses, so that any attempt to give a precise meaning to this relationship must be couched in a generalization of derivative (see for example [3] and [37]).

Extended real-valued functions (i.e., ones which may attain infinite values) may seem far removed from differentiability. Yet (as first shown in convex analysis) there is much to be said for being able to calculate their "derivatives". Consider for example, for a fixed u, the optimization problem of the preceding paragraph. It amounts to minimizing $L(x)$, where L is defined via

$$\begin{aligned} L(x) &= f(x) \quad \text{if} \quad g(x) = u, \\ &= +\infty \quad \text{otherwise.} \end{aligned}$$

If a point minimizing L must be a "stationary point" (in a generalized sense), then a necessary condition for optimality ensues. As another example, consider the situation in which one wishes to minimize a functional which admits a restricted domain of definition (such as certain integrals). Calculating a "derivative" relative to this domain can be viewed as differentiating an extended real-valued function. It has been pointed out to me by J. F. G. Auchmuty that this problem arises for example in deriving variational principles in astrophysics [4], as well as in several other physical contexts, where ad hoc methods were found for its solution.

2. Generalized gradients. We begin with a real-valued function f, defined on a Banach space X, that we suppose to be locally Lipschitz, which is to say that every point x in X admits a neighbourhood N_x and a scalar K_x such that, for every y and z in N_x, $|f(y)-f(z)| \leqslant K_x|y-z|$. Given v in X, the generalized directional derivative of f at x in the v direction, denoted $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) = \limsup_{y \to x;\, \lambda \downarrow 0} [f(y+\lambda v)-f(y)]/\lambda.$$

Note that this is a finite quantity. It is easy to see that $f^\circ(x; \cdot)$ is positively homogeneous and subadditive, which suggests the following definition. The *generalized gradient* of f at x, denoted $\partial f(x)$, is given by

$$\partial f(x) = \{\zeta \in X^* : f^\circ(x; v) \geqslant \langle \zeta, v \rangle \text{ for all } v \text{ in } X\}$$

where X^* is the continuous dual of X. Of course, f may not admit a derivative

in a customary sense, but it turns out that $\partial f(x)$, in spite of being a set, behaves very much like a derivative of f. There are other ways of associating to various classes of nondifferentiable functions a concept of derivative, and many others have done so. Some prominent examples are the Dini derivatives mentioned above, and also approximate derivatives (see [40]), the subdifferential of convex analysis [37] the semidifferential of Neustadt and Halkin (see for example [36]). Other, more recent work in this direction has been done by J. Warga [41], [42] and H. Halkin [26], [27].

We believe that in general the generalized gradient is the most useful concept for the analysis of nonsmooth, nonconvex functions. This is due to several factors: its generality, the fact that it is intrinsic, and, especially, the fact that a satisfactory calculus can be developed (formulas for sums, compositions, mean values, etc.). The wide application that it has found tends to confirm the impression that very often the properties of the generalized gradient are "right".

Since space limitations do not permit a discussion of the calculus of generalized gradients, we refer the reader to [9], [14], [16], and also to [3], and to the references therein. Before going on to the uses of generalized gradients, we mention some recent significant work dealing with the case in which f is not necessarily locally Lipschitz, but extended real-valued, as in the example of § 1. It is possible to define ∂f for such f indirectly from the Lipschitz case, as was done in [9] in finite dimensions. Now, R. T. Rockafellar [38] has shown how an extended definition may be given which unifies the two (see also related work of Hiriart-Urruty [30]). This is based upon an alternate (and necessarily more complicated) definition of $f^\circ(x; v)$, which reduces to the one given here when f is locally Lipschitz.

3. Some uses of generalized gradients.

(A) *Tangents and normals to arbitrary sets.* Let C be a nonempty subset of X and let $d_C(\cdot)$ be its distance function: $d_C(x)=\inf\{\|x-c\|: c\in C\}$. It is not hard to prove that d_C is locally Lipschitz, so that its generalized gradient as defined in § 2 is available to us. One use we may make of it is to define the *normal cone* [9] $N_C(x)$ to C at any point $x\in C$; we define $N_C(x)$ as the closed cone in X^* generated by $\partial d_C(x)$. We then define the *tangent cone* $T_C(x)$ as the dual of $N_C(x)$; i.e., the set of all v in X such that $\langle v, \zeta\rangle \leqslant 0$ for all $\zeta\in N_C(x)$. An equivalent characterization of $T_C(x)$ would describe it as consisting of all v such that $d_C^\circ(x; v)\leqslant 0$. Here is a direct definition bypassing generalized gradients [9, Proposition 3.7]: v belongs to $T_C(x)$ iff for every sequence x_n of points in C converging to x and λ_n converging to 0, there is a sequence v_n converging to v such that $x_n+\lambda_n v_n$ belongs to C for each n. This approach has recently been clarified by R.T. Rockafellar [39].

(B) *Mathematical programming.* Consider the problem of minimizing $f(x)$ subject to $x\in C, h_j(x)=0$ $(j\in J), g_i(x)\leqslant 0$ $(i\in I)$, where the functions involved are locally Lipschitz and I, J are finite index sets. The following is proven in [14]:

THEOREM. *If x solves the above problem, then there exist scalars $\lambda \geq 0, s_j$ $(j \in J)$, $r_i \geq 0$ $(i \in I)$ not all zero such that $r_i g_i(x) = 0$ and such that*

$$\lambda \partial f(x) + \sum_J s_j \partial h_j(x) + \sum_I r_i \partial g_i(x) \in -N_C(x).$$

The proof of this very general Lagrange multiplier rule uses nondifferentiability in a fundamental way, as well as the theorem of Ekeland cited below. Alternate approaches to multiplier rules with generalized gradients have been developed by J. Hiriart–Urruty [30], [31], who has also applied them to stochastic programming [29] and "marginal functions" [28]. Numerical methods using generalized gradients have been developed by A. Feuer [23], A. A. Goldstein [25], R. Mifflin [35] and A. Auslender [6]. They have been employed in sensitivity analysis by J. Gauvin [24], by A. Auslender [5], and by J. P. Aubin and F. H. Clarke [3].

(C) *Analysis.* A question that has attracted much interest is the following: under what conditions and to what extent is a convex function defined on a Banach space differentiable? A more general question concerns the differentiability properties of locally Lipschitz functions. G. Lebourg [34], using generalized gradients as the basic tool, has made significant progress in this area in the case of separable Banach spaces.

Given a locally Lipschitz function $f: R^n \to R^n$, we define the *generalized Jacobian* [15] $\partial f(x)$ to be the set of $n \times n$ matrices given by

$$\operatorname{co} \{\lim Df(x_i): x_i \to x\}$$

(i.e. we consider all sequences x_i converging to x such that f is differentiable at x_i and such that the sequence $Df(x_i)$ of Jacobians converges, and we take the convex hull of all such limits). For various reasons, this is one of the natural ways of extending the definition of generalized gradient to vector-valued functions. No completely satisfactory definition is known; see however A. D. Ioffe [33] for a new approach. We say $\partial f(x)$ is *nonsingular* when every matrix in $\partial f(x)$ is nonsingular.

THEOREM [15]. *If $\partial f(x)$ is nonsingular, then in a neighbourhood of x, f is one-to-one and onto, and its inverse is Lipschitz.*

Just as in the classical case, this result immediately yields an implicit function theorem, as has been noted by Hiriart–Urruty [30, Theorem 11]. Further generalizations of this and other classical results appear in J. Warga [43] and H. Halkin [26].

An interesting approximation result on solutions of equations has been given by A. D. Ioffe [32] in terms of generalized gradients and normals. This result, as well as a fixed-point theorem couched in derivate terms [19], make use of an elegant theorem of I. Ekeland [21] (see also these Proceedings). Ekeland's theorem, which has been found to have wide application in analysis and optimization, is useful in conjunction with generalized gradients, since nondifferentiability intervenes in

it in a fundamental way. Our disappointment in not being able to include a look at this topic is assuaged by the opportunity to refer the reader to the discussion in [22].

4. Optimal control and the calculus of variations. Consider the "basic problem" of minimizing $\int_0^1 L(x, \dot{x})\,dt$ over the absolutely continuous functions $x(\cdot)$ with bounded derivative satisfying $x(0) \in C_0$, $x(1) \in C_1$, and where L is locally Lipschitz.

THEOREM [10], [20]. *If x solves the above problem, then there exists $p(\cdot)$ such that*
(a) *$p(\cdot)$ is absolutely continuous; $p(t) \cdot \dot{x}(t) - L(x, \dot{x}) = $ constant;*
(b) *$L(x, \dot{x}+v) - L(x, \dot{x}) \geq v \cdot p(t)$ for all v, for almost all t;*
(c) *$p(t) \in \partial_{\dot{x}} L(x, \dot{x})$; $\dot{p}(t) \in \partial_x L(x, \dot{x})$;*
(d) *$p(0) \in N_{C_0}(x(0))$; $-p(1) \in N_{C_1}(x(1))$.*

We discern in this theorem the analogues of all the first-order necessary conditions of the classical calculus of variations: the two Erdmann conditions, the conditions of Legendre and Weierstrass, the Euler–Lagrange equation, and the natural boundary (or transversality) conditions. In [18] generalized gradients were used to prove a strengthened form of the variational multiplier rule for inequality constraints, and in [17] the case of multiple integrals was broached.

A variety of control and constrained variational problems can be cast in the form of the above basic problem by the device of allowing the integrand L to assume extended real values (i.e., $+\infty$) (see [10]). We define the *Hamiltonian* in this general setting via $H(x, p) = \sup \{p \cdot v - L(x, v) : v \in R^n\}$. It is possible to obtain an analogue of the classical Hamiltonian equation in the form of a Hamiltonian inclusion [12], [13]. Space permits only a sample result; we choose one that pertains to the "differential inclusion" $\dot{x}(t) \in E(x(t))$, where E is a given "multifunction". We define $\mathcal{A}(C)$, the attainable set, as the set of all values $x(1)$, where x is a solution of the differential inclusion satisfying $x(0) \in C$. We assume that E is compact-valued and locally Lipschitz.

THEOREM [20]. *If x is a solution of the differential inclusion with $x(0) \in C$ such that $x(1)$ lies on the boundary of $\mathcal{A}(C)$, then there exists a nonzero $p(\cdot)$ such that $p(0) \in N_C(x(0))$, $(-\dot{p}, \dot{x}) \in \partial H(x, p)$ a.e., $H(x(t), p(t)) = $ constant.*

The above controllability result yields necessary conditions for optimality [13], [20]; in this connection see also V. G. Boltjanskii [8] and V. I. Blagodatskih [7]. An existence theorem for the differential inclusion in which the trajectory is invariant with respect to a given set and monotone with respect to a given preference ordering is given by J. P. Aubin and F. H. Clarke [1]. Necessary conditions for nondifferentiable problems in standard form have been proved by F. H. Clarke [11], J. Warga [42] and H. Halkin [27].

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Problèmes Variationnels Non Convexes

Ivar Ekeland

dedié à mon ami Alexandre Ioffe
„Solch ein Gewimmel möcht ich sehn,
Auf freiem Grund mit freiem Volke stehn".

Goethe, Faust II, 11579-80

Ceci est un bref compte rendu des progrès realisés depuis mes travaux de 1972 sur le principe variationnel. De nombreux mathématiciens y ont prêté la main, et les applications se sont étendues pratiquement à toutes les branches de l'analyse non linéaire. Je devrai me contenter d'en esquisser quelques-unes, renvoyant pour plus de détails aux articles originaux ou à une publication ultérieure [15].

I. Les résultats de base. Voici le point de départ ([11], [12]):

THÉORÈME A. *Soit V un espace métrique complet, et $F: V \to \mathbf{R} \cup \{+\infty\}$ une fonction semi-continue inférieurement, $\not\equiv +\infty$, bornée inférieurement. On se donne $\varepsilon > 0$ et un point $u \in V$ tel que:*

$$F(u) \leqslant \inf_V F + \varepsilon.$$

Alors pour tout $\lambda > 0$ il existe un point $v \in V$ tel que:

$$F(v) \leqslant F(u),$$

$$d(u, v) \leqslant \lambda,$$

$$\forall w \neq v, \quad F(w) > F(v) - \varepsilon d(v, w)/\lambda. \ \square$$

On a immédiatement le corollaire ([11], [12]).

THÉORÈME B. *Soit $F: V \to \mathbf{R} \cup \{+\infty\}$ comme ci-dessus. Pour tout $\varepsilon > 0$, il existe un point $v \in V$ tel que:*

$$F(v) \leqslant \inf_V F + \varepsilon,$$

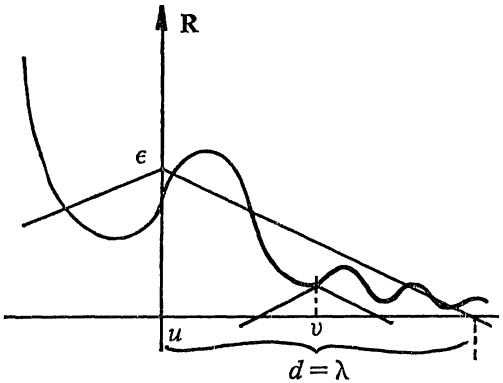
$$\forall w \in V, \quad F(w) \geqslant F(v) - \varepsilon d(v, w). \ \square$$

Il existe enfin de ces résultats une version locale [16]. Je dirai qu'une fonction $F: V \to R \cup \{+\infty\}$ est ε-soutenue au point v si $F(v) < +\infty$ et s'il existe $v^* \in V^*$ (dual topologique) et $\eta > 0$ tels que:

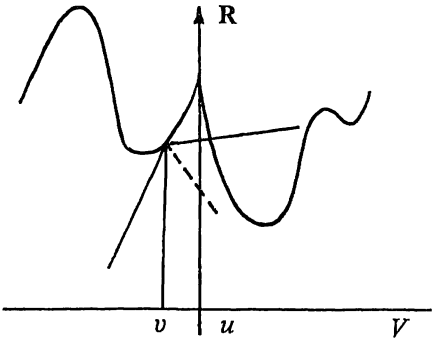
$$\|w-w\| \leqslant \eta \Rightarrow F(w) \geqslant F(v) + \langle v^*, w-v\rangle - \varepsilon\|w-v\|.$$

THÉORÈME C. *Soit V un espace de Banach. On suppose qu'il existe sur V une fonction Fréchet-differentiable à support borné, non identiquement nulle. Alors, pour toute fonction s.c.i. $F: V \to \boldsymbol{R}$ et tout $\varepsilon > 0$, l'ensemble des points où F est ε-soutenue est dense dans V.* □

Ces diverses situations sont illustrées par les diagrammes suivants.



Théorèmes A et B



Théorème C

La démonstration du Théorème A introduit une relation d'ordre sur $V \times \boldsymbol{R}$, et trouve dans l'épigraphe de F (qui est une partie fermée) un élément maximal. Ce genre d'argument est dû à Bishop et Phelps ([1], [2]), et a été utilisé dans d'autres circonstances par Brøndsted et Rockafellar [3], et par F. Browder [5].

Je vais classer les applications, non pas dans l'ordre chronologique, mais suivant le théorème auquel elles se rattachent.

II. Les applications. 1. Version faible (Théorème B). Caristi [6] a démontré un théorème de point fixe qui ne requiert pas de continuité. Soit V un espace métrique complet, et f une application de V dans lui-même vérifiant

$$\forall u \in V, \quad d(u, f(u)) \leqslant \varphi(u) - \varphi(f(u))$$

ou $\varphi: V \to R$ est s.c.i. Alors f admet un point fixe.

F. Clarke [9] a également démontré un théorème de point fixe, sur lequel je ne m'étendrai pas.

Brézis et Browder [4] se sont servis de ce genre d'arguments pour étudier un semi-groupe non linéaire $S(t)$, $t \geqslant 0$, agissant sur un espace métrique complet V où l'on a isolé un fermé F. En particulier, s'il s'agit d'un semi-groupe de contractions,

ils montrent que la vitesse maximum à laquelle les trajectoires quittent F est plus grande que la vitese moyenne à laquelle elles s'en éloignent.

J'ai moi-même [12] utilisé le théorème B pour montrer, dans les problèmes d'optimisation non convexes, l'existence de solutions presque minimisantes qui vérifient presque les conditions nécéssaires du premier ordre. Par exemple, si F est une fonction Gâteaux-différentiable et bornée inférieurement sur un Banach V, il existe une suite v_n telle que $F(v_n)\to\inf F$ et $F'(v_n)\to 0$.

Ceci jette un jour nouveau sur la fameuse condition (C) de Palais-Smale [18], qui implique que la suite v_n converge vers une limite, ou que l'un des $F'(v_n)$ est nul. Elle suffit à assurer l'existence d'un minimum $v\in V$, sans qu'il soit besoin pour cela que F soit continûment différentiable [12].

2. Version forte (Théorème A). L'intérêt de la version forte est qu'elle permet de localiser le point v par rapport au point u (prendre $\lambda=\sqrt{\varepsilon}$ par exemple). F. Clarke l'a utilisée, en théorie de la commande, pour démontrer le principe du maximum sous des hypothèses extrêmement faibles ([7], [10]) et en programmation mathematique, pour démontrer des conditions nécéssaires à la Fritz-John en présence de liaisons [8]. Ses démonstrations, fort élégantes, reposent sur l'idée suivante. La conclusion du Théorème A (avec $\lambda=\sqrt{\varepsilon}$) signifie que la fonction $G(w)=F(w)+\sqrt{\varepsilon}d(w,v)$ atteint son minimum en v, avec $d(v,u)\leqslant\sqrt{\varepsilon}$. Il ne reste plus qu'à écrire les conditions nécéssaires d'optimalité pour G en v, et à faire tendre ε vers 0.

A. Ioffe [17] a démontré un théorème de la moyenne en plusieurs dimensions, et l'a utilisé en programmation mathématique. Soit F une function de classe C^1 sur R^n, et E l'ensemble de ses zéros; pour tout u, il existe v tel que $\|v-u\|\leqslant d(u,E)$ et $\|F'(v)\|\leqslant|F(u)|/d(u,E)$.

3. Version fine (Théorème C). En collaboration avec Lebourg [16], j'ai étudié les fonctions du type $F(u)=\inf_{x\in X}f(u,x)$, où, pour chaque $x\in X$, la fonction $u\to f(u,x)$ est C^1 sur un Banach V. Si V satisfait au théorème C, et si la famille $\{f(\cdot,x)|x\in X\}$ est équi-différentiable, la fonction F sera Fréchet-différentiable en chaque point d'un G_δ dense. Par exemple, si V possède une norme équivalente Fréchet-différentiable (sauf en 0), et si F est une fonction convexe continue sur V (enveloppe supérieure de fonctions affines continues), alors F est génériquement Fréchet-différentiable. Ceci résout la conjecture d'Asplund dans un sens.

On peut également adopter un autre point de vue, et s'intéresser, pour chaque $u\in V$, à la fonction $x\to f(u,x)$ sur X. Moyennant certaines conditions supplémentaires, dire que F est Fréchet-différentiable au point $u\in V$ impliquera que la fonction $f(u,\cdot)$ atteint son minimum sur X en un point unique. C'est en particulier le cas pour les fonctions

$$f(x,u)=g(x)+\langle x,u\rangle,\quad \text{ou}$$
$$f(x,u)=g(x)+a\|x-u\|^p,$$

ou V est un Hilbert, $X\subset V$ est fermé borné, $g: X\to R$ est s.c.i., $a\in\boldsymbol{R}$ et $1<p<\infty$. En dépit de l'absence de compacité, même faible, l'ensemble des $u\in V$ pour lesquels

le problème d'optimisation $\inf_{x \in X} f(x, u)$ admet une solution unique contient un G_δ dense.

En analyse globale, j'ai donné une version du théorème de Hopf—Rinow en dimension infinie ([13], [14]): si V est une variété riemannienne forte, complète et connexe, l'ensemble des points de V qui peuvent être joints à un point donné par une géodésique minimale et une seule contient un G_δ dense. Des contre-exemples montrent que ce résultat est le meilleur possible.

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Control of the Diffusion Type Processes

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1. Introduction. The application of mathematics to the study of a real system subject to imperfectly known disturbances begins frequently by considering the differential equation

$$\dot{x}_t = b(s+t, x_t)+\sigma(s+t, x_t)\dot{\xi}_t, \quad t \geqslant 0, \quad x_0 = x \tag{1}$$

where the vector x_t describes the system state at time t, the random vector ξ_t corresponds to the disturbances. It is convenient to write (1) in the integral form

$$x_t = x+\int_0^t b(s+r, x_r)\,dr+\int_0^t \sigma(s+r, x_r)\,d\xi_r. \tag{2}$$

One usually tries to choose the parameters x_t so that for every $t_0 \geqslant 0$ the evolution of x_t for $t>t_0$ should be independent of x_t for $t<t_0$ if the value x_{t_0} is known. If this condition is satisfied then x_t contains all sufficient current information about our system and x_t is a Markov process. In that very case, as a rule, one can take the Wiener process w_t instead of ξ_t in (1), (2) and consider the second integral in (2) as Ito stochastic integral. Note that the derivative $\dot{w}_t$ does not exist and equation (1) has no direct sense if $\xi=w$.

We say that x_t is a diffusion process if it satisfies an equation of type (2) for $\xi=w$. We have a controlled diffusion process if b and σ in (2) are dependent on parameters whose values we can change during the evolution of the system. The theory of controlled diffusion processes is discussed in many books and articles (cf. [1], [2] and references cited there). In this report we shall be considering extensions of the results of [2].

2. Controlled process definition. Classes of policies. Let (Ω, F, P) be a complete probability space, $\{F_t, t\geqslant 0\}$ an increasing family of complete σ-algebras $F_t\subset F$. Let (w_t, F_t) be a d_1-dimensional Wiener process, A a separable metric space, E_d a d-dimensional Euclidean space with a fixed orthonormal basis. Suppose that for $\alpha\in A$, $t\geqslant 0$, $x\in E_d$ there are defined a $d\times d_1$ matrix $\sigma(\alpha, t, x)$ and $b(\alpha, t, x)\in E_d$. Suppose that σ, b are Borel functions of α, t, x are Lipschitz continuous in x with the Lipschitz constant independent of α, t and $\sigma(\alpha, t, 0)$, $b(\alpha, t, 0)$ are bounded. An A-valued process $\alpha=\alpha_t=\alpha_t(\omega)$, $t\geqslant 0$, $\omega\in\Omega$, is called a policy if it is progressively measurable with respect to $\{F_t\}$. Let $\mathfrak{A}$ be the set of all policies. For every $s\geqslant 0$, $x\in E_d$, $\alpha\in\mathfrak{A}$ there exists a solution of

$$x_t = x+\int_0^t b(\alpha_r, s+r, x_r)\,dr+\int_0^t \sigma(\alpha_r, s+r, x_r)\,dw_r. \tag{3}$$

Denote this solution by $x_t^{\alpha, s, x}$. We shall also consider classes of policies other than $\mathfrak{A}$.

Let $C([0, \infty), E_d)$ be the space of all E_d-valued continuous functions $\xi=\{x_t, t\geqslant 0\}$ defined on $[0, \infty)$ with the topology of uniform convergence on bounded subsets of $[0, \infty)$. A function $\beta=\beta_t(\xi)\colon [0, \infty)\times C([0, \infty), E_d)\to A$ is called a natural policy admissible for the point (s, x) if it is Borel measurable, for every $t\geqslant 0$, β_t is independent of x_u for $u>t$ and there exists at least one F_t-measurable solution of (3) where $\beta_t(\xi)$ for $\xi=\{x_t, t\geqslant 0\}$ is taken in place of α_t. The set of all natural policies admissible for s, x is denoted by $\mathfrak{A}_N(s, x)$; $\mathfrak{A}_M(s, x)$ is its subset consisting of all functions β for which $\beta_t(\xi)=\beta_t(x_t)$. Fix some solution $x_t^{\beta, s, x}$ of (3) (where $\alpha_t=\beta_t(\xi)$) for every s, x, $\beta\in\mathfrak{A}_N(s, x)$. Then the formula $\alpha_t(\omega)=\beta_t(\xi(\omega))$ for $\xi(\omega)=\{x_t^{\beta, s, x}(\omega), t\geqslant 0\}$ defines an embedding $\mathfrak{A}_N(s, x)\subset\mathfrak{A}$ such that $x_t^{\beta, s, x}=x_t^{\alpha, s, x}$ (a.s.). Let $\mathfrak{M}(t)$ be the set of all stopping times $\tau\leqslant t$.

3. Performance indices. Define real Borel functions $c^\alpha(t, x)\geqslant 0, f^\alpha(t, x)$, $g(t, x)$ for $\alpha\in A$, $t\in[0, \infty)$, $x\in E_d$. Fix $T\in(0, \infty)$ and for $s\in[0, T]$, $\alpha\in\mathfrak{A}$, $\tau\in\mathfrak{M}(T-s)$ put

$$\varphi_t^{\alpha, s, x} = \int_0^t c^{\alpha_r}(s+r, x_r^{\alpha, s, x})\,dr,$$

$$v^{\alpha, \tau}(s, x) = E_{s, x}^{\alpha}\Big[\int_0^\tau e^{-\varphi_t} f^{\alpha_t}(s+t, x_t)\,dt+e^{-\varphi_\tau} g(s+\tau, x_\tau)\Big]$$

where α, s, x near E mean that α, s, x must be written in the interior of the brackets in all the places where α, s, x have been omitted. If the suppositions of §4 (below) are satisfied then $v^{\alpha, \tau}(s, x)$ is well defined. If we measure the quality of the policy α on the time interval $[0, \tau]$ by means of $v^{\alpha, \tau}(s, x)$ then four problems arise: to find α maximizing $v^{\alpha, T-s}(s, x)$, to find α, τ maximizing $v^{\alpha, \tau}(s, x)$ and to find the functions

$$\begin{aligned} v(s, x) &= \sup_{\alpha\in\mathfrak{A}} v^{\alpha, T-s}(s, x),\\ w(s, x) &= \sup_{\alpha\in\mathfrak{A}}\sup_{\tau\in\mathfrak{M}(T-s)} v^{\alpha, \tau}(s, x). \end{aligned} \tag{4}$$

If $\mathfrak{A}$ in (4) is replaced by $\mathfrak{A}_N(s,x)$ $(\mathfrak{A}_M(s,x))$ then the left sides are denoted by $v_{(N)}(s,x)$, $w_{(N)}(s,x)$ $(v_{(M)}(s,x)$, $w_{(M)}(s,x))$. The embeddings $\mathfrak{A}_M(s,x)\subset\mathfrak{A}_N(s,x)\subset\mathfrak{A}$ imply $v_{(M)}\leqslant v_{(N)}\leqslant v$, $w_{(M)}\leqslant w_{(N)}\leqslant w$.

4. Sufficiency of natural and Markov policies. From now on we suppose that σ, b, c, f are continuous in (α, x) and continuous in x uniformly with respect to α for every t, g is continuous in (t,x) and $c+|f|+|g|\leqslant K(1+|x|)^m$ for all α, t, x where K, m are constants. Define $H_T=(0,T)\times E_d$.

THEOREM 1. (a) *v, w are continuous in $\overline{H}_T$, $|v|+|w|\leqslant N(1+|x|)^m$, $v(T,x)=w(T,x)=g(T,x)$,* (b) *$v_{(N)}=v$, $w_{(N)}=w$.* (c) *For $\varepsilon\geqslant 0$ let τ_ε be the exit time of $(s+t, x_t)$ from $Q_\varepsilon=\{(t,x)\in\overline{H}_T: w(t,x)>g(t,x)-\varepsilon\}$. Then*

$$\sup_{\alpha\in\mathfrak{A}} v^{\alpha,\tau_\varepsilon}(t,x)=\sup_{\alpha\in\mathfrak{A}_N(t,x)} v^{\alpha,\tau_\varepsilon}(t,x)\geqslant w(t,x)-\varepsilon$$

in $\overline{H}_T$ for $\varepsilon>0$.

It is not known whether $v_{(M)}=v$, $w_{(M)}=w$ or not in the general situation. The next example shows the typical difficulties.

EXAMPLE 1. Let $d=d_1=T=1, b=c=f=0, g=x^2, \sigma(x)=\operatorname{sgn} x$ for $|x|\geqslant 1$, $\sigma(x)=x$ for $|x|\leqslant 1$, $\sigma(\alpha,t,x)=\sigma(\alpha+x)$, $A=\{-1\}\cup\{+1\}$. It is easy to see that $v=x^2+1-s$ and an optimal policy, if it exists, must be the Markov policy $\alpha_t(x)=\operatorname{sgn} x$. This policy is not admissible if $F_t=F_t^w$ and it is not clear whether the equality $v=v_{(M)}$ is valid.

THEOREM 2. *Suppose* (a) *the first and second order derivatives in x of the functions $\sigma^{ij}, b^i, c, f, g$ exist, are continuous in x and can be estimated by $K(1+|x|)^m$ in $\overline{H}_T$.* (b) *The same is true for the first order derivatives in t.* (c) *A is a convex set in a Euclidean space, σ and b satisfy Lipschitz condition in α uniformly in (t,x) and either* (d) *$\sigma\sigma^*$ does not depend on x or* (e) *$\sup_{\alpha\in A}|\sigma^*(\alpha,t,x)\lambda|>0$ if $(t,x)\in\overline{H}_T$, $\lambda\neq 0$. Then $v_{(M)}=v$, $w_{(M)}=w$ in $\overline{H}_T$.*

5. Smoothness of the value functions. From now on we also suppose that the condition (a) of Theorem 2 is satisfied.

THEOREM 3. (a) *The first order derivatives of v, w in t and the second order derivatives in x in the sense of distribution theory are measures which are finite on compacts $\Gamma\subset H_T$,* (b) *Sobolev derivatives v_{x^i}, w_{x^i} exist and can be estimated by $N(1+|x|)^{2m}$* (a.e. H_T), (c) *Sobolev derivatives $\partial v/\partial t\in L_{2,\mathrm{loc}}(H_T)$, $\partial w/\partial t\in L_{2,\mathrm{loc}}(Q_0)$.*

EXAMPLE 2. Let $d_1=d$, $T=2$, $A=[-1,1]$, σ be a unit matrix for $t\in[0,1)$, $\sigma=0$ for $t\in[1,2]$, $b=0$ for $t\in[0,1)$, $b=\alpha e_1$ for $t\in[1,2]$ where e_1 is the first basis vector. Let $c=f=0$, $g=(x^1)^2$. Here $v=(|x^1|+2-t)^2$ for $t\in[1,2]$, $v=E(|w^1_{1-t}+x^1|+1)^2$ for $t\in[0,1]$. It is easy to check that $\partial v/\partial t\in L_{2+\varepsilon,\mathrm{loc}}(H_2)$ for $\varepsilon\in[0,1)$ but not for $\varepsilon=1$.

The functions v, w have classical derivatives.

THEOREM 4. *The classical derivatives* $\partial v/\partial t$, $\partial w/\partial t, v_{x^i}, w_{x^i}$ *exist a.e. in* H_T. *The classical derivatives* $v_{x^ix^j}, w_{x^ix^j}$ *exist also a.e. in* H_T *if in the process of calculating* $v_{x^ix^j}, w_{x^ix^j}$ *one does not consider the points where* v_{x^i}, w_{x^i} *do not exist.*

To formulate stronger results about the derivatives of v, w let

$$\mu(l) = \inf_{\lambda: l\lambda=1} \sup_{\alpha\in A} |\sigma^*(\alpha, t, x)\lambda|^2,$$

$$\mu = \inf_{|\lambda|=1} \sup_{\alpha\in A} |\sigma^*(\alpha, t, x)\lambda|^2.$$

THEOREM 5. *Let a domain* $Q\subset H_T$ *and* $\mu(l)\geqslant\varepsilon$ *on* Q *for some* $l\in E_d$, $\varepsilon>0$. *Then the second order Sobolev derivatives of* v, w *in* x *with respect to the direction of* l *are of class* $L_{2,\mathrm{loc}}(Q)$ *respectively (for* w*) of class* $L_{2,\mathrm{loc}}(Q\cap Q_0)$. *If* $\mu\geqslant\varepsilon$ *on* Q *then* $v\in W^{1,2}_{2,\mathrm{loc}}(Q)$, $w\in W^{1,2}_{2,\mathrm{loc}}(Q\cap Q_0)$. *If condition* (b) *of Theorem* 2 *is satisfied then one can take* L_∞, W_∞ *instead of* L_2, W_2 *in the present theorem.*

6. Bellman equation. Denote $a=\frac{1}{2}\sigma\sigma^*$,

$$F[u] = \sup_{\alpha\in A}\left(a^{ij}u_{x^ix^j}+b^iu_{x^i}-cu+f+\frac{\partial u}{\partial t}\right).$$

THEOREM 6. $F[v]=0$, $F[w]\leqslant 0$ *(a.e.* H_T*),* $F[w]=0$ *(a.e.* Q_0*) where the classica derivatives are used in* F.

By using classical derivatives one cannot characterize v uniquely as a solution of the Bellman equation with the boundary data $v(T, x)=g(T, x)$.

EXAMPLE 3 (cf. [3]). Let $d=T=1$, $\sigma=c=g=0$, $b=2\alpha$, $f=-\alpha^2$, $A=[-2, 2]$. In this case $v=0$. But the function $(1-t-|x|)_+$ also is equal to 0 for $t=1$ and satisfies the Bellman equation a.e.

In Example 3 the Bellman equation is a Hamilton–Jacoby equation. There already is a theory of such equations (cf. [4] and references cited there) and in this theory a unique solution is characterized by the requirement of being representable as a sum of a convex function and a smooth function.

THEOREM 7. *There exists a constant* N *such that for* $u(x)=N(1+|x|^2)^{3m/2}$ *the functions* $v+u$, $w+u$ *are convex in* x.

In the general case the described requirement does not provide the uniqueness of the solution.

EXAMPLE 4. Let $h(x)$ be the Cantor function on $[0, 1]$, $h(x)=0$ if $x\leqslant 0$, $h(x)=1$ if $x\geqslant 1$, $u(t, x)=h(1-t)\varphi(x)$ where

$$\varphi(x) = \int_{-\infty}^{x} h(r)\,dr.$$

Then $\partial u/\partial t+u_{xx}=0$ (a.e. $(0, 1)\times E_1$)), $u(1, x)=0$. At the same time $u\not\equiv 0$ and u is convex in x.

THEOREM 8. *For every* $\alpha \in A$ *a measure with the differential*

$$a^{ij}(\alpha, t, x) v_{x^i x^j}(dt\, dx) \quad (a^{ij}(\alpha, t, x) w_{x^i x^j}(dt\, dx))$$

is absolutely continuous in H_T *(in* Q_0*).*

This property of v, w resembles a property of the solutions of degenerated quasilinear equations, introduced in [5].

7. Bellman equation solution uniqueness.

THEOREM 9. *Let* z *be a function for which the statements* (a), (c) *of Theorems* 1, 3 *and the statements of Theorems* 7, 4, 6, 8 *are valid if there* v *is replaced by* z. *Then* $v = z$.

An analogous result is valid for w.

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Analytic Stratifications and Control Theory

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Optimal control problems often have closed-loop (i.e. feedback) solutions which fail to be smooth, but whose non-smoothness exhibits a substantial degree of regularity. The solution is smooth on an open, dense set, whose complement is, locally, a finite union of submanifolds. Such submanifolds—known as "switching loci" if the solution is actually discontinuous there—occur in the simplest problems, but the really surprising fact is that, for fairly general classes of problems, no wilder behaviour occurs. At least, this is the natural conclusion to be drawn from the vast amount of existing knowledge of particular problems. However, no general "existence and regularity theorem" is known which enables us to prove that this type of behavior is indeed necessarily found whenever reasonable conditions are satisfied.

Recently, progress in the direction of proving such theorems has been made thanks to the discovery, by P. Brunovsky, of the possibility of applying the theory of subanalytic sets to the study of this question. In this paper, we report on recent results along similar lines.

Roughly speaking, a semianalytic subset of R^n—or of a finite dimensional analytic manifold—is a set which is locally defined by finitely many equalities or inequalities involving real analytic functions. (See [2] or [4] for the precise definition.) The family of subanalytic sets is the smallest one that contains the semianalytic sets, and is closed under the operations of locally finite unions and intersections, complements, inverse images under analytic mappings, and under the operation of taking the image of a set S by an analytic mapping which is proper on the closure of S. The useful facts about such sets are (a) that they have a nice structure, and (b) that almost anything reasonable that one may define using analytic functions ends up being subanalytic.

Let us first make (b) more precise. Consider an arbitrary formal expression F that involves certain variables $x_1, \ldots, x_n$, and which is obtained from "atomic formulae" of the type $(x_{i(1)}, \ldots, x_{i(m(j))}) \in S_j$ by the logical operations of conjunction, disjunction, negation, universal or existential quantification. Suppose that $x_1, \ldots, x_k$ are the free variables, i.e. those that are not under the scope of a quantifier. Then if $M_1, \ldots, M_n$ are analytic manifolds, and the symbols S_j are interpreted as subanalytic sets, the set of those $(x_1, \ldots, x_k)$ for which F holds is necessarily subanalytic, *provided only that the quantifications are locally bounded* (i.e., that every time a quantifier Qx_i occurs, with $Q = \exists$ or $Q = \forall$, then, if $S_Q(x_i, y)$ is the scope of Qx_i and y are the other variables that are free in S_Q, it follows that for every compact set K of the y domain there is a compact J of the x_i domain such that, for each $\bar{y} \in K$, "$(Qx_i)\, S_Q(x_i, \bar{y})$" happens if and only if "$(Qx_i \in J) S_Q(x_i, \bar{y})$"). This is easily seen to be a restatement of the properties of subanalytic sets that were described before, since disjunction, conjunction and negation correspond to union, intersection and complementation, existential quantification to the operation of taking the image of a set A by the projection $(x_i, y) \to y$, universal quantification can be expressed in terms of negation and existential quantification and, finally, the boundedness of the quantifiers implies that, before projecting, one can replace A by a set A' such that the projection is proper on $\operatorname{Clos} A'$. Therefore, the proof that a set A of interest to us is subanalytic will, in general, reduce to the tedious task of exhibiting a definition of A which is of the desired form, and showing that the quantifiers are bounded.

On the other hand, subanalytic sets have nice properties, of which the most important one is that they make it possible to stratify analytic maps. Precisely, a *stratification* of a smooth manifold M is a locally finite partition $\mathscr{P}$ of M such that (S1) each $P \in \mathscr{P}$ is a regular smooth submanifold of M ("regular" means that it is a topological subspace of M) (S2) the frontier $\operatorname{Fron} P$ of each $P \in \mathscr{P}$ is a union of members of $\mathscr{P}$ (here $\operatorname{Fron} P = (\text{Closure } P) - P$), and (S3) if $P \in \mathscr{P}$, $Q \in \mathscr{P}$, $Q \subseteq \operatorname{Fron} P$, then $\dim Q < \dim P$. A family $\mathscr{A}$ of subsets is *compatible* with a set B if every $A \in \mathscr{A}$ is either contained in or disjoint from B. We call $\mathscr{A}$ compatible with another family $\mathscr{B}$ if $\mathscr{A}$ is compatible with each $B \in \mathscr{B}$. Let us use the abbreviation "CASA" to refer to a connected analytic submanifold which is a subanalytic set. Let us write H_S to denote the restriction of H to S, whenever H is an object for which this makes sense (e.g. a function, or vector field, or stratification compatible with S). If M, N are analytic manifolds, and $g: M \to N$ an analytic map, a *stratification of* g is a pair $(\mathscr{S}, \mathscr{T})$ such that: (i) $\mathscr{S}$ is a stratification of M, (ii) $\mathscr{T}$ is a stratification of N, and (iii) for every $S \in \mathscr{S}$, the image $g(S)$ belongs to $\mathscr{T}$, and $g_S: S \to g(S)$ is a submersion.

The main result of interest to us is:

THEOREM 1. *Let $g: M \to N$ be an analytic map between analytic manifolds. Let $\mathscr{A}, \mathscr{B}$ be locally finite families of subanalytic subsets of M, N, respectively. Let L be open, subanalytic, and such that $g_{\operatorname{Clos} L}$ is proper. Then there are stratifications*

$\mathscr{S}, \mathscr{T}$ of M, N by CASA *sets, such that* (i) *$\mathscr{S}$ is compatible with $\mathscr{A}$ and with L,* (ii) *$\mathscr{T}$ is compatible with $\mathscr{B}$, and* (iii) *$(\mathscr{S}_L, \mathscr{T})$ is a stratification of g_L.*

This result is essentially Corollary 4.4 of Hardt [2]. (Hardt proves the existence of a stratification $(\mathscr{S}', \mathscr{T})$ of g_L, compatible with $\mathscr{A}$ and $\mathscr{B}$. Apply this to some open subanalytic L' such that g is proper on Clos L' and Clos $L \subseteq L'$, and with $\mathscr{A}$ replaced by $\mathscr{A}' = \mathscr{A} \cup \{\text{Fron } L\} \cup \{L\}$. Let $(\mathscr{S}', \mathscr{T})$ be the resulting stratification of $g_{L'}$. Let $\mathscr{S}''$ be a stratification of M compatible with $\mathscr{A} \cup \mathscr{S}'_{\text{Fron } L}$. Let $\mathscr{S} = \mathscr{S}'_L \cup \mathscr{S}''_{M-L}$. Then $(\mathscr{S}, \mathscr{T})$ satisfies the desired conclusion.)

A simple but useful consequence of Theorem 1 is as follows. Let f be a partially defined map into N, with domain $D(f) \subseteq M$. Call f *subanalytic* if its graph $G(f)$ is a subanalytic subset of $M \times N$. Call f *locally bounded* if f maps relatively compact sets into relatively compact sets (we are *not* assuming that f is continuous). Let f be subanalytic and locally bounded. Let $g: M \times N \to M$ be the projection. Then g is proper on Clos $G(f)$, so we can find an open subanalytic L such that $g_{\text{Clos } L}$ is proper, and that $G(f) \subseteq L$. Apply Theorem 1 to this case, with $\mathscr{A} = \{G(f)\}$ and $\mathscr{B}$ arbitrary. We get:

COROLLARY 2. *Let $f: M \to N$ be subanalytic and locally bounded. Let $\mathscr{B}$ be a locally finite family of subanalytic subsets of M. Then there exists a* CASA *stratification of M which is compatible with $\mathscr{B}$ and is such that f is analytic on each stratum.*

A particular example of a subanalytic function is obtained by minimizing with respect to a parameter. Suppose that N, M are analytic manifolds, $F: N \to M$ and $\varphi: N \to \mathbf{R}$ are analytic mappings. Let C be a subanalytic subset of N such that F is proper on Clos C. Define a function $\tilde{\varphi}$ with domain $D(\tilde{\varphi}) = F(C)$ by $\tilde{\varphi}(m) = \inf \{\tilde{\varphi}(n) : F(n) = m\}$.

COROLLARY 3. *$\tilde{\varphi}$ is subanalytic and locally bounded.*

PROOF. The point (m, t) belongs to the graph of $\tilde{\varphi}$ if and only if

$$(\forall n)\big(n \in C \wedge F(n) = m \Rightarrow \varphi(n) \geqslant t\big) \wedge (\forall \varepsilon)\big[\varepsilon > 0 \Rightarrow (\exists n)\big(n \in C \wedge F(n) = m \wedge \varphi(n) < t + \varepsilon\big)\big].$$

The conclusion follows if we prove that the quantifiers are locally bounded, and this is trivial for the ε-quantifier (we can take $\varepsilon \leqslant 1$) and is a direct consequence of the properness of F for the n-quantifier. Q.E.D.

Now, if a control problem is given with dynamical law $\dot{x} = f(x, u)$, $x \in M$, and Lagrangian $L(x, u)$, we get a situation that closely resembles the one that gave rise to the definition of $\tilde{\varphi}$. Fix a terminal point x_0 and consider, for each initial x, the problem $\Gamma(x, x_0)$ of minimizing $\int L(x(t), u(t))\,dt$ among all trajectories that go from x to x_0. Let $\Phi(x)$ be the optimal cost for $\Gamma(x, x_0)$ (assuming it exists). Let N be the class of open loop controllers $u = \{u(t) : a \leqslant t \leqslant b\}$, a, b arbitrary.

Let F be the map which to each u, with domain $[a, b]$, associates $\xi_u(a)$, where $t \to \xi_u(t)$ is the trajectory for which $\xi_u(b) = x_0$. Let $\varphi(u) = \int_a^b L(\xi_u(t), u(t))\, dt$. Then $\Phi(x)$ is the infimum of $\varphi(u)$ as u ranges over $F^{-1}(x)$. So Φ is very much like a function $\tilde{\varphi}$ to which Corollary 3 applies.

There are, however, substantial differences, of which the most important one is that N is not a finite dimensional analytic manifold. However, this difficulty can be overcome whenever the class N of all controllers can be replaced by a smaller class N_0 (in the sense that it is known a priori that $\inf\{\varphi(u): u \in N, F(u) = x\}$ equals $\inf\{\varphi(u): u \in N_0, F(u) = x\}$), and that N_0 can be parametrized by a finite dimensional variable α. In this case (and if, in addition, the necessary analyticity is established, and a $C \subseteq N_0$ is found such that F is proper on Clos C and that the infimum of $\varphi(u)$ for $u \in N_0$, $F(u) = x$ remains unchanged when the constraint "$u \in C$" is added) Corollaries 2 and 3 can be applied to get the conclusion that Φ is "piecewise analytic", i.e., that there is a locally finite partition $\mathscr{P}$ of $D(\Phi)$ into CASA's such that Φ is analytic on each member of $\mathscr{P}$.

The simplest situation where this can be done is the case of a calculus of variations problem defined on $\boldsymbol{R}^n$, whose Lagrangian $L(x, \dot{x})$ satisfies:

(A1) L is analytic on $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

(A2) L is bounded below by a positive constant.

(A3) The matrix $\left(\left(\partial^2 L / \partial \dot{x}_i\, \partial \dot{x}_j\right)(x, \dot{x})\right)$ is positive definite for each $(x, \dot{x})$, and

(A4) $\operatorname{Lim}_{|\dot{x}| \to +\infty} [L(x, \dot{x})/|\dot{x}|] = +\infty$ uniformly in x.

Let $M = \boldsymbol{R}^n \times (0, \infty)$. Let $x_0 \in \boldsymbol{R}^n$ be fixed, and, for $\bar{t} > 0$, let $\Phi(x, \bar{t})$ be the infimum of $\int_{-\bar{t}}^0 L\left(\xi(\tau), \dot{\xi}(\tau)\right) d\tau$, taken over all C^1 curves $\tau \to \xi(\tau)$ such that $\xi(-\bar{t}) = x$, $\xi(0) = x_0$. Then one can limit the search for the infimum to the class of *extremals,* i.e. of those trajectories $t \to \xi(t)$ that satisfy the Euler–Lagrange equations. But this class is "finite dimensional", the parameter being $\dot{\xi}(0)$. On the other hand, it is not hard to show that, for each compact $K \subseteq M$, there is a $c > 0$ such that, for $(x, \bar{t}) \in K$, the minimizing extremal satisfies $|\dot{\xi}(0)| \leqslant c$. This gives the necessary properness, and one gets:

THEOREM 4. *For a calculus of variations problem whose Lagrangian satisfies* (A1), ..., (A4), *the value function Φ is piecewise analytic.*

It is also possible to prove that Φ is piecewise analytic whenever there exist a priori bounds on the number of switchings. Precisely, consider a problem $\dot{x} = f(x, u)$, with Lagrangian $L(x, u)$, and suppose that

(B1) L is bounded below by a positive constant,

(B2) f and L are analytic functions of x and u, and

(B3) there exists a family $\mathscr{F}$ of analytic functions $u(x)$ such that, for every $C > 0$, there are (1) a finite family $\mathscr{F}(C)$, (2) a positive integer $N(C)$, and (3) a compact set $K(C)$ with the property that, whenever x can be steered to x_0 with cost $\leqslant C$, then $x \in K(C)$ and x can be optimally steered to x_0 by a concatenation of at most $N(C)$ trajectories of members of $\mathscr{F}(C)$.

Now let $A_C(x_0)$ denote the set of points that can be steered to x_0 with cost$\leqslant C$, and let $A(x_0)$ be the union of all the $A_C(x_0)$. Then we have:

THEOREM 5. *If* (B1), (B2), (B3) *hold, then there is a countable partition* $\mathscr{P}$ *of* $A(x_0)$, *such that each* $A_C(x_0)$ *meets finitely many members of* $\mathscr{P}$, *and that each* $P\in\mathscr{P}$ *is a* CASA *on which* Φ *is analytic.*

To prove Theorem 5, one reparametrizes trajectories using the cost as the new parameter, thereby reducing the general case to the time-optimal one. If $u=u(x)$ is analytic, and K compact subanalytic, let $F(K, u)$ denote the set of all triples (x, y, t) such that $t\geqslant 0$, that $y=\gamma(t)$, where γ is the integral curve of $f(\cdot, u(\cdot))$ which passes through x when $t=0$, and that $\gamma(\tau)\in K$ for τ between 0 and t. Then $F(K, u)$ is subanalytic. If U is a finite sequence $(u_1, \ldots, u_m)$, define $F(K, U)$ by declaring (x, y, t) to be an element of $F(K, U)$ if and only if

$$(\exists t_1, \ldots, t_m, y_0, \ldots, y_m)\big((y_{i-1}, y_i, t_i)\in F(K, u_i)$$

$$\text{for } i = 1, \ldots, m \wedge x = y_0 \wedge y = y_m \wedge t = t_1+\ldots+t_m\big).$$

Then $F(K, U)$ is subanalytic. Now let $\mathscr{U}(C)$ be the set of all finite sequences of u's in $\mathscr{F}(C)$ of length at most $N(C)$. Define $G_1(C)$ by choosing a compact subanalytic $J(C)$ that contains $K(C)$, and declaring (x, t) to belong to $G_1(C)$ if $(x, x_0, t)\in F(J(C), U)$ for some $U\in\mathscr{U}(C)$ and if $t\leqslant C$. Then $G_1(C)$ is subanalytic, since $\mathscr{F}(C)$ is finite. Finally, let

$$(x, t)\in G(C) \Leftrightarrow (x, t)\in G_1(C) \wedge (\forall\tau)[\tau < t \Rightarrow (x, \tau)\notin G_1(C)].$$

Then $G(C)$ is subanalytic, and it is clear that $G(C)$ is the graph of the restriction of Φ to $A_C(x_0)$. The conclusion follows from Corollary 2.

The method of analytic stratifications can also be used to prove piecewise regularity of closed-loop solutions of the optimal control problem. Let M be an analytic manifold. Suppose that P, Q are regular analytic submanifolds of M such that $P\cap Q=\emptyset$, and that X is an analytic vector field on P. We say that Q is a *nice exit manifold for* (P, X) if: (I) for every $p\in P$, there is a $T>0$ such that $\gamma(T-)$ exists and belongs to Q, where γ is the integral curve of X such that $\gamma(0)=p$ and (II) for every $q\in Q$ there exists a neighborhood V of q in Q, an $\varepsilon>0$, and an analytic diffeomorphism H from $V\times(-\varepsilon, \varepsilon)$ onto a submanifold W of M, such that, if $q'\in V$, then

(a) $t\to H(t, q')$, $-\varepsilon<t<0$, is an integral curve of X (hence, in particular, contained in P),

(b) $H(0, q')=q'$.

A *piecewise analytic vector field* on a subset S of the analytic manifold M is a 6-tuple $(\mathscr{P}, \mathscr{P}_1, \mathscr{P}_2, \Sigma, \Pi, X)$ such that

(PAVF1) $\mathscr{P}$ is a finite or countably infinite partition of S into connected regular analytic submanifolds of M.

(PAVF2) $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$, $\mathscr{P}_1\cap\mathscr{P}_2=\emptyset$.

(PAVF3) Σ: $\mathscr{P}_2\to\mathscr{P}_1$ and Π: $\mathscr{P}_1\to\mathscr{P}$ are mappings.

(PAVF4) $X=\{X_P: P\in\mathscr{P}_1\}$ where, for each $P\in\mathscr{P}_1$, X_P is an analytic vector field on P.

(PAVF5) If $P\in\mathscr{P}_2$ then P is a nice exit manifold for $(\Sigma(P), -X_{\Sigma(P)})$.

(PAVF6) If $P\in\mathscr{P}_1$, then $\Pi(P)$ contains a regular analytic submanifold Q which is a nice exit manifold for (P, X_P).

If $V=(\mathscr{P}, \mathscr{P}_1, \mathscr{P}_2, \Sigma, \Pi, X)$ is a piecewise analytic vector field, then there is an obvious definition of *trajectories* of V. We say that $x_0\in S$ is a *terminal point* for V if $\{x_0\}\in\mathscr{P}$ and if, for every $p\in S$, the trajectory of V that starts at p reaches x_0 in a finite time, after going through a finite number of manifolds $P\in\mathscr{P}$.

THEOREM 6. *If conditions* (B1), (B2), (B3) *hold, then there exists a piecewise analytic vector field* V, *on* $A(x_0)$ *with terminal point* x_0, *such that, for each* $x\in A(x_0)$, *the trajectory of* V *that starts at* x *solves the optimal control problem* $\Gamma(x, x_0)$.

For a proof, see Sussmann [6]. Brunovsky proved this result in [1] for normal linear time-optimal problems with a polyhedral control set. Since conditions (B1), (B2), (B3) hold for such problems even without the normality assumption, we see that this hypothesis is not needed. More generally, any problem for which a bang-bang theorem holds, with a priori bounds in the number of switchings, satisfies the conditions of Theorem 6. A particular example is as follows: consider a system $\dot{x}=f(x)+ug(x)$, where f and g are analytic vector fields, and the control u is restricted by $|u|\leqslant 1$. Suppose that, for each j, the vector field $[g, (\mathrm{ad}\, f)^j(g)]$ can be written as a linear combination of the fields $(\mathrm{ad}\, f)^i(g)$, $0\leqslant i\leqslant j+1$ with analytic coefficients, in such a way that the coefficient α of $(\mathrm{ad}\, f)^{j+1}(g)$ satisfies $|\alpha(x)|<1$ for all x. Then it is proved in Sussmann [7] that conditions (B1), (B2), (B3) hold for the associated time-optimal problem, so that a piecewise analytic feedback solution exists.

It is often the case that conditions (B1), (B2), (B3) hold if certain "singular controls" are added to the bang-bang ones. Whenever this happens, we can still conclude that a piecewise analytic feedback solution exists.

It is known from the study of many particular problems that the open loop solutions can often be expressed as finite concatenations of bang-bang trajectories and singular arcs. It should be possible to prove a general theorem giving reasonable conditions under which this is true, and the proof will probably yield bounds on the number of switchings. However, no such theorem seems to be known at the moment.

There is an even more challenging question for the future. Situations are known where bounds in the number of switchings do not exist. A typical example is Fuller's problem, whose solution involves infinitely many switchings. Despite the fact that the hypothesis of Theorem 6 fails to be satisfied, Fuller's problem has a very nice

feedback solution. This suggests that it should be possible to extend Theorem 6 further. However, such an extension is not likely to be possible using the technique of subanalytic sets, since this technique depends on the reduction to finite-dimensional spaces of controls. A totally different idea is probably needed.

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Some Topics in Quantum Statistical Mechanics

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Some topics in quantum statistical mechanics of spin lattice systems are reviewed from a mathematical point of view

1. C^*-dynamical systems. The mathematical object studied in quantum statistical mechanics of a spin lattice system is a C^*-dynamical system consisting of a C^*-algebra $\mathfrak{A}$ and a strongly continuous one-parameter group τ_t, $t\in R$, of *-automorphisms of $\mathfrak{A}$. The "spin lattice" refers to a system of a finite dimensional full matrix algebra $\mathfrak{A}_x$ (the spin algebra) associated with each point x in a lattice L such as the n-dimensional cubic lattice $L=Z^n$. The algebra $\mathfrak{A}$ is the inductive limit of tensor products of $\mathfrak{A}_x$ and has the following system of local subalgebras: For each finite subset Λ of the lattice L, there corresponds a finite dimensional subalgebra $\mathfrak{A}(\Lambda)$ with the trivial center (corresponding to the tensor product of $\mathfrak{A}_x$ for $x\in\Lambda$ and the identity operators 1_y of $\mathfrak{A}_y$ for $y\notin\Lambda$), satisfying (1) $\Lambda_1\supset\Lambda_2$ implies $\mathfrak{A}(\Lambda_1)\supset\mathfrak{A}(\Lambda_2)$, (2) the union of all $\mathfrak{A}(\Lambda)$ is dense in $\mathfrak{A}$, (3) $a\in\mathfrak{A}(\Lambda_1)$ and $b\in\mathfrak{A}(\Lambda_2)$ commute if Λ_1 and Λ_2 are disjoint. The algebra $\mathfrak{A}$ is simple and has a unique tracial state, which we denote by ω_0. We denote the conditional expectation relative to ω_0 from $\mathfrak{A}$ to $\mathfrak{A}(\Lambda)$ by E_Λ (the partial trace on the complement of Λ).

The τ_t (describing the development of the system with the time t) can be uniquely described by its infinitesimal generator δ (the derivative of τ_t at $t=0$), which is a *-derivation. A model of a quantum spin lattice system is specified by giving a *-derivation δ_Φ with its domain $D(\delta_\Phi)=$ the union of all $\mathfrak{A}(\Lambda)$ (Λ finite subsets of L) in terms of a "potential" Φ as follows: Φ is a map from non-empty subsets of L into selfadjoint elements of $\mathfrak{A}$ such that $\Phi(\Lambda)\in\mathfrak{A}(\Lambda)$ and

$$H_\Lambda=\lim_{\Lambda'}\sum_{\Lambda_1}\{\Phi(\Lambda_1)\colon \Lambda'\supset\Lambda_1, \Lambda\cap\Lambda_1 \text{ nonempty}\}$$

exists in $\mathfrak{A}$ for all finite subsets Λ of L as the finite subset Λ' of L tends to L. For $a\in\mathfrak{A}(\Lambda)$, the *-derivation δ_Φ in question is defined by the following commutator:

$$\delta_\Phi a = i[H_\Lambda, a].$$

The complicated definition of H_Λ in terms of Φ is just to guarentee the consistency of this definition when the same a is considered as an element of a different local algebra $\mathfrak{A}(\Lambda_1)$ with $\Lambda_1\supset\Lambda$.

Any *-derivation whose domain is the union of $\mathfrak{A}(\Lambda)$ can be described in this manner. The correspondence of such *-derivations with potentials becomes bijective if we impose the following condition on Φ: $E_{\Lambda_1}(\Phi(\Lambda))=0$ for all Λ with non-vanishing intersection with the complement of Λ_1. The set of all such Φ form a Fréchet space relative to seminorms $r_\Lambda(\Phi)\equiv\|H_\Lambda\|$. ($L$ is assumed to be countable.) For any UHF algebra, of which our $\mathfrak{A}$ is a special case, it is also known that the domain of the generator of a strongly continuous one-parameter group of *-automorphisms contains $\bigcup_\Lambda\mathfrak{A}(\Lambda)$ for a suitable choice of the local algebras $\mathfrak{A}(\Lambda)$.

A real mathematical problem in describing τ_t in this manner lies in the construction of τ_t from δ_Φ, i.e. the extension of δ_Φ to a generator of a one-parameter group of *-automorphisms of $\mathfrak{A}$. Each of the following conditions is known to be sufficient for the closure of δ_Φ to be a generator.

(A) The following expression is finite for some $s>0$:

$$\sum_{n=1}^{\infty}\sup_x\left\{\sum_\Lambda\{\|\Phi(\Lambda)\|e^{sn}\colon \Lambda\ni x, |\Lambda|=n\}\colon x\in L\right\}$$

where $|\Lambda|$ denotes the number of points in Λ. The proof is by explicitly constructing $\tau_t a$ in terms of power series of t and proving the absolute convergence of the series for sufficiently small t for any $a\in\cup\mathfrak{A}(\Lambda)$ [1], [2], [3].

(B) There exists a monotone sequence of finite subsets Λ_n tending to L such that the distance of H_{Λ_n} from $\mathfrak{A}(\Lambda_n)$ is bounded uniformly in n. The proof is by using an abstract characterization of a generator [4]. This applies to some 1-dimensional models.

(C) There exists a monotone sequence of finite subsets Λ_n tending to L such that the distance of H_{Λ_n} to $\mathfrak{A}(\Lambda_{n+m})$ is bounded by $Cne^{-\alpha m}$ for all positive integers n and m for some constants $C>0$ and $\alpha>0$. This condition applies to a class of models on two dimensional lattices. The proof is by first constructing a candidate for $\tau_t a$ by a kind of perturbation expansion, next proving an integral equation which it satisfies and finally appealing to an abstruct characterization of a generator [5].

(D) There exists a monotone sequence of finite subsets Λ_n tending to L such that $\delta(H_{\Lambda_n})=\lim_m i[H_{\Lambda_m}, H_{\Lambda_n}]$ exists and is bounded uniformly in n. The H_{Λ_k}, $k=n$ and m, in this condition can be replaced by $E_{\Lambda_{p(k)}}H_{\Lambda_n}$ for any $p(k)$ tending to ∞ as $n\to\infty$. The proof depends on a simple abstract argument [5]. This condition

applies to the classical interaction where $\Phi(\Lambda)$ belongs to an abelian subalgebra of $\mathfrak{A}$ and hence H_Λ's mutually commute [6].

2. General properties of τ_t. The specification of the model in terms of the potential Φ refers to a local behavior of the time translation τ_t. For a given Φ, it is difficult to prove some global property of the time translation τ_t. We mention some such properties, which do not necessarily hold but are of some interest.

(i) The fixed point subalgebra $\mathfrak{A}^\tau$ of $\mathfrak{A}$ under τ_t. If all $\Phi(\Lambda)$ belongs to an abelian subalgebra of $\mathfrak{A}$ (as in classical interactions), then that abelian subalgebra certainly belongs to $\mathfrak{A}^\tau$. On the other hand, for typical quantum interactions $\mathfrak{A}^\tau$ is expected to be trivial.

(ii) The GNS representation associated with the unique trace state ω_0 carries a unitary representation $U_0(t)$ of $t\in R$ implementing τ_t and leaving the canonical cyclic vector invariant. The spectral property of $U_0(t)$ is of some interest. For example, if the only invariant vectors are multiples of the canonical cyclic vector, then $\mathfrak{A}^\tau$ must be trivial.

(iii) If $\|[\tau_t(a), b]\|$ tends to 0 as $t\to\infty$ for any a and b in $\mathfrak{A}$, τ_t is said to be (strongly) asymptotically abelian. If $\varphi([\tau_t(a), b])$ tends to 0 as $t\to\infty$ for all state φ instead, τ_t is said to be weakly asymptotically abelian. Either property implies that the set of the $U_0(t)$ invariant vectors is one-dimensional and the spectrum of the selfadjoint generator of $U_0(t)$ is the whole real line.

Since the asymptotic abelian property of the time translation τ_t is a handy assumption in a general discussion and is often assumed, it is desirable to have a general method of proving (or disproving) it for a reasonable class of potentials for quantum interactions.

An important class of potentials are those having a symmetry property. Let G be a group of *-automorphisms of $\mathfrak{A}$ and π be a homomorphism of G into bijections of L such that $g\mathfrak{A}(\Lambda)=\mathfrak{A}(\pi_g\Lambda)$. A potential Φ is said to be G-invariant if $g\Phi(\Lambda)=\Phi(\pi_g\Lambda)$ for all $g\in G$ and all Λ. Under the condition $E_{\Lambda_1}\Phi(\Lambda)=0$ stated earlier, this is the case if and only if δ_Φ commutes with all $g\in G$ (and hence if and only if τ_t commutes with $g\in G$ under any of the conditions (A)–(D)).

Typically G contains a sequence g_n such that $\pi_{g_n}\Lambda$ becomes disjoint with any given finite set as $n\to\infty$. In that case the asymptotic abelian property

$$\lim_n \|[g_n a, b]\| = 0$$

holds for all a and b in $\mathfrak{A}$. Usually G is a locally compact group and the asymptotically abelian property holds for $g\to\infty$ in G.

For sufficiently complicated interactions, it is probably a reasonable guess that the commutant of τ_t in the set of all *-automorphisms of $\mathfrak{A}$ is generated by τ_t and the obvious symmetry group G of the above kind. However, no such cases have been established, nor any method of using such information is known. In the case of a classical interaction, the commutant of τ_t clearly contains a large number

of *-automorphisms, i.e. the group of unitaries in an abelian subalgebra of $\mathfrak{A}$ modulo the circle group. The commutant of a certain group of quasifree automorphisms of the CAR algebra has been determined [7], but this does not provide an example of the commutant of a one-parameter group of automorphisms.

3. Equilibrium states. For our purpose, an equilibrium state at a nonzero temperature is a state ω_β of $\mathfrak{A}$ satisfying the following KMS condition for some real number β (inversely proportional to the absolute temperature): For any $a\in\mathfrak{A}$ such that $\tau_t a$ is entire in t, $\omega_\beta(ab)=\omega_\beta(b\tau_{i\beta}a)$ for all $b\in\mathfrak{A}$. Such a state is called a (τ_t,β)-KMS state. The $(\tau_t,0)$-KMS state is unique and is the trace state ω_0.

The equilibrium state at the zero temperature is a τ_t-invariant state ω (i.e. $\omega(\tau_t a)=\omega(a)$ for all $a\in\mathfrak{A}$) such that $\omega(a\tau_t b)$ is the Fourier transform $\int e^{itp}d\mu_{a,b}(p)$ of a complex measure $\mu_{a,b}$ whose support is in the positive half line for all $a, b\in\mathfrak{A}$ (a ground state) or in the negative half line for all $a, b\in\mathfrak{A}$ (a ceiling state). This condition, apart from the invariance, is the same as requiring one-sided spectrum for the unitary one-parameter group $U_\omega(t)$ implementing τ_t and leaving the cyclic vector invariant in the GNS representation of ω.

If τ_t is an inner automorphism, i.e. $\tau_t a=e^{itH}ae^{-itH}$ for some $H\in\mathfrak{A}$, then there exists a unique (τ_t,β)-KMS state $\omega_{\beta H}(a)=\omega_0(e^{-\beta H}a)/\omega_0(e^{-\beta H})$ for all finite β (the canonical ensemble). If τ_t can be approximated by inner automorphisms, i.e.

$$\tau_t a=\lim_n e^{itH_n}ae^{-itH_n}$$

uniformly over compact sets of t for a dense set of a in $\mathfrak{A}$ for a sequence $H_n\in\mathfrak{A}$, then any accumulation point of states $\omega_{\beta H_n}$ as $n\to\infty$ is a (τ_t,β)-KMS state, which exists for all β. Since any accumulation point of (τ_t,β)-KMS states as $\beta\to+\infty$ (or $-\infty$) is a ground (or ceiling) state, ground and ceiling states also exist in this case. If the closure of δ_Φ is a generator (hence if one of (A)–(D) holds), then the above situation holds with H_Λ serving for H_n as $\Lambda\to L$ [8].

Under the condition (A), the (τ_t,β)-KMS state is known to be unique for small $|\beta|$. Under the condition (B), it is unique for all β [9], [10], [11]. There are examples of Φ (satisfying conditions (A, C, D)) for which (τ_t,β)-KMS states are not unique for large $|\beta|$. The β-dependence (and in particular the critical point) of (τ_t,β)-KMS states is of great importance in physics.

It is of both mathematical and physical interest to find out the set S_τ of all τ_t-invariant states in relation with the sets S_β of all (τ_t,β)-KMS states and the sets $S_{\pm\infty}$ of all ground and ceiling states. All of them are compact convex. S_β for any finite β is a Choquet simplex and its extremal points are exactly primary KMS states. If τ_t is asymptotically abelian, S_τ is also a Choquet simplex. Extremal points of S_τ are called ergodic states. A primary (τ_t,β)-KMS state is ergodic if and only if τ_t has a weak form of (η-) asymptotic abelian property in the associated GNS-representation. In particular, if τ_t is asymptotically abelian, primary KMS states are ergodic.

It is difficult to pin down all ergodic states for any given τ_t. Clearly, any unique

KMS state of any one-parameter subgroup of the commutant of τ_t is a primary τ_t-invariant state. (See § 4.) For a certain group of quasifree automorphisms of the CAR algebra, the set of all ergodic states is known [7].

The (τ_t, β)-KMS states have other equivalent characterizations. We briefly list some of them. We assume that the closure of δ_Φ is the generator of τ_t whenever δ_Φ or H_Λ appears explicitly.

(I) A characterization in terms of δ_Φ [12]: $\omega_\beta(\delta_\Phi a)=0$ and $-i\beta\omega_\beta(a^*\delta_\Phi a)\geqslant S(\omega_\beta(aa^*):\omega_\beta(a^*a))$ for all $a\in\cup\mathfrak{A}(\Lambda)$ where $S(v:u)$ is $u\log(u/v)$ if u and v are positive, 0 if $u=0$ and $+\infty$ if $u>0$ and $v=0$.

(II) The local variational principle of minimal free energy [12], [13]: For any finite set Λ and for any state φ of $\mathfrak{A}$, which has the same restriction to $\mathfrak{A}(\Lambda_1)$ as the given ω_β as long as Λ_1 is disjoint with Λ, the minimality $\tilde{F}_{\Lambda,\beta}(\omega_\beta)\leqslant\tilde{F}_{\Lambda,\beta}(\varphi)$ holds for the free energy $\tilde{F}_{\Lambda,\beta}$ defined by $\tilde{F}_{\Lambda,\beta}(\varphi)=\beta\varphi(H_\Lambda)-\tilde{S}_\Lambda(\varphi)$, where the open system entropy $\tilde{S}_\Lambda(\varphi)$ is the limit of the difference $S_{\Lambda'}(\varphi)-S_{\Lambda'\setminus\Lambda}(\varphi)$ as Λ' tends to L, which exists, the closed system entropy $S_{\Lambda'}(\varphi)$ is defined as $-\omega_0(\varrho_{\Lambda'},\log\varrho_{\Lambda'})$ in terms of the density matrix $\varrho_{\Lambda'}\in\mathfrak{A}(\Lambda')$ uniquely determined by $\varphi(a)=\omega_0(\varrho_{\Lambda'},a)$ for all $a\in\mathfrak{A}(\Lambda')$, and $\Lambda'\setminus\Lambda$ denotes the relative complement of Λ in Λ'.

For ground states, (I) reduces essentially to our definition: $\omega(\delta_\Phi a)=0$ and $-i\omega(a^*\delta_\Phi a)\geqslant 0$ for all $a\in\cup\mathfrak{A}(\Lambda)$. The other becomes

(II)′ For any finite set Λ and for any state φ of $\mathfrak{A}$, which has the same restriction to $\mathfrak{A}(\Lambda_1)$ as the given state ω as long as Λ_1 is disjoint with Λ, the minimality $\omega(H_\Lambda)\leqslant\varphi(H_\Lambda)$ holds [14].

The case of ceiling states is obtained by reversing $\geqslant$ and $\leqslant$.

If we consider a fixed symmetry group G such that π_G is transitive on L, the set of all G-invariant potentials form a Banach space under the norm $\|\Phi\|=\|H_{\{x\}}\|$ (independent of $x\in L$) due to the estimate $\|H_\Lambda\|\leqslant|\Lambda|\|\Phi\|$. If we restrict our attention to G-invariant states, then we have the following characterization of (τ_t,β)-KMS states for all finite β under a certain condition on π_G (usually satisfied):

(III) The global variational principle: $\beta e(\omega_\beta)-s(\omega_\beta)\leqslant\beta e(\varphi)-s(\varphi)$ for all G-invariant states φ of $\mathfrak{A}$ where the mean energy $e(\varphi)$ and the mean entropy $s(\varphi)$ are the limits of $|\Lambda|^{-1}\varphi(H_\Lambda)$ and $|\Lambda|^{-1}S_\Lambda(\varphi)$ as $\Lambda\to L$ in a manner appropriate with respect to the action of π_G (á la Van Hove).

For ground states,

(III)′ $e(\omega)\leqslant e(\varphi)$ for all G-invariant states φ of $\mathfrak{A}$.

The value $P(\beta\Phi)=s(\omega_\beta)-\beta e(\omega_\beta)$ is called the pressure function, which coincides with the limit of $|\Lambda|^{-1}\log\omega_0(e^{-\beta H})$ as $\Lambda\to L$ and is convex and continuous in Φ.

We now discuss characterizations which do not involve the parameter β. A state is called passive [15] if for any self-adjoint $\mathfrak{A}$-valued differentiable function h_t of $t\in R$ with a compact support,

$$L_h(\omega)\equiv\int_{-\infty}^{\infty}\omega(\tau_t^h(dh_t/dt))\,dt\geqslant 0$$

where τ_t^h is the *-automorphism of $\mathfrak{A}$ defined by the following perturbation formula:

$$\tau_t^h a = \sum_{n=0}^{\infty} i^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n$$

$$\tau_t[\tau_{-t_1}(h_{t_1}), \dots, [\tau_{-t_n}(h_{t_n}), a] \dots].$$

The (τ_t, β)-KMS state with $\beta \geqslant 0$ as well as ground states are passive. The converse is not true because convex combinations of (τ_t, β)-KMS states with mixed values of β are also passive. The converse holds if we impose some conditions on ω such as the weak clustering property:

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} dt(\omega(a\tau_t b) - \omega(a)\omega(b)) = 0.$$

Alternatively, the complete passivitiy condition that $\omega^{\otimes n}$ on $\mathfrak{A}^{\otimes n}$ is passive relative to $\tau_t^{\otimes n}$ for all positive integers n implies that ω is either a (τ_t, β)-KMS state for some $\beta \geqslant 0$ or a ground state. The passivity condition expresses the property of equilibrium state that it can not produce an energy in a cyclic process, i.e. a form of the second law of thermodynamics.

A characterization of KMS states for $\beta \leqslant 0$ and ceiling states together is obtained by reversing the inequality in the passivity condition.

Another characterization of equilibrium states is the stability under a (localized) perturbation of dynamics [16], [17], [18], [19]. For simplicity of exposition, assume the L_1-asymptotic abelian property for τ_t: $\int_{-\infty}^{\infty} \|[b, \tau_t a]\| \, dt < \infty$ for all a, b in a dense subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$. If ω is an equilibrium state, then the stability condition $\int_{-\infty}^{\infty} \omega([P, \tau_t a]) \, dt = 0$ for all P and a in $\mathfrak{A}_1$ holds. For primary states ω, for example, the converse also holds.

For some other characterization of KMS states, see for example [20], [21].

It is an interesting question to find a condition under which the third law of thermodynamics holds, i.e. whether $s(\omega_\beta) \to 0$ as $\beta \to \infty$, and whether $s(\omega) = 0$ for ground states and ceiling states ω.

4. Chemical potential. The original meaning of equilibrium is that the state is stable. The canonical ensemble prescription and variational principle are its characterization heuristically obtained many years ago in the framework of statistical mechanics and thermodynamics, respectively. The KMS condition is an abstract characterization of the canonical ensemble by Kubo and by Martin and Schwinger about 20 years ago. The equivalence of these different characterizations is now more or less established in the mathematical framework explained above.

There is another thermodynamic variable used by physicists to describe an equilibrium state, namely the chemical potential. It appears in the following physical circumstances. If a number of particles is preserved under isolated circumstances but can change when brought into a contact with some other substances such as

a catalyzer, the equilibrium states are parametrized by chemical potentials (in addition to β), which serves as a "potential" in chemical reactions. Such a state will be unstable under catalytic perturbation but stable under perturbations preserving the number of particles. The following is a summary of theory of chemical potentials in the present mathematical framework [22], [23].

We consider a compact group G_0 of *-automorphisms of $\mathfrak{A}$ (called the gauge group) such that $g\mathfrak{A}(\Lambda)=\mathfrak{A}(\Lambda)$ for all Λ and $g\tau_t=\tau_t g$ for all $g\in G_0$. For example G_0 may be a circle group $T=\{\alpha_\theta: \theta \bmod 2\pi\}$ and any operator $a\in\mathfrak{A}$ satisfying $\alpha_\theta a=e^{in\theta}a$ is interpreted as changing the number of particles by n. We are interested in those states which satisfies the stability condition for G_0-invariant perturbation P. The restriction of the state to the G_0-fixed point subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ must then be an equilibrium state for the restriction of τ_t to $\mathfrak{A}_0$ (which is τ_t-invariant). Hence we shall treat the extension of extremal (τ_t, β)-KMS states ω of $\mathfrak{A}_0$ to a τ_t invariant state of $\mathfrak{A}$. Since such a state can be decomposed into ergodic states, we actually discuss extensions to ergodic states. The traditional prescription of physicists gives $(\tau_t\alpha_{\mu t}, \beta)$-KMS states in the above mentioned example with the real parameter μ, which is the chemical potential. (The states known as grand canonical ensembles.) Our aim is to justify this mathematically.

We state the result under the assumption of asymptotically abelian τ_t. Then an ergodic extension of a primary (τ_t, β)-KMS state of $\mathfrak{A}_0$ to $\mathfrak{A}$ exists and is unique up to gauge i.e. two extensions φ_1 and φ_2 are related by $\varphi_1(ga)=\varphi_2(a)$ for all $a\in\mathfrak{A}$ for some $g\in G_0$. (This result holds for any ergodic states of $\mathfrak{A}_0$.) If the state is faithful on $\mathfrak{A}_0$, then the extension is a $(\tau_t g_t, \beta)$-KMS state where g_t is a one-parameter subgroup of the center of the stabilizer $G_\varphi=\{g\in G_0: \varphi(ga)=\varphi(a)$ for all $a\in\mathfrak{A}\}$ of the extension φ. (In the above example $g_t=\alpha_{\mu t}$.) If the state is not faithful, the only possible complication is that the extension may have a "one-sided" spectrum for a normal closed subgroup N_φ of G_φ and is a $(\tau_t g_t, \beta)$-KMS state for g_t in the center of G_φ/N_φ when the extended state is restricted to the N_φ-fixed point subalgebra of $\mathfrak{A}$. The analysis can also be carried out without the assumption of asymptotic abelian for τ_t under some other assumptions which are more complicated but can be verified for concrete models more easily.

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Geometrical Aspects of Gauge Theories

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1. Introduction. Gauge theories are, broadly speaking, physical theories of a geometrical character such as Einstein's theory of general relativity. In a narrower sense however they correspond to the differential geometry of fibre bundles and were first introduced by H. Weyl in 1916. In recent years they have become increasingly strong candidates to provide a fundamental description of elementary particles. The hope is that, when quantized, gauge theories will explain elementary particles in the same way as quantization of Maxwell theory leads to photons.

The basic difficulty in this programme lies in the nonlinearity of the classical field equations. The successes so far achieved in quantum field theory have depended on sophisticated perturbation techniques which involve expanding about the linearized equations. For many purposes however perturbation theory is inadequate and physicists are exploring alternative nonperturbative approaches to the quantization problem.

One hopeful development has been the discovery that gauge theories possess some remarkable classical solutions which, in simple cases, can be explicitly described. One example is the 'tHooft–Polyakov "magnetic monopole" (see [9]) which is a static solution for an $SU(2)$-gauge theory behaving asymptotically like a usual Dirac monopole (for the group $U(1)$) but having no singularities. Another example is the instanton solution of Belavin et al. [8] for an $SU(2)$-gauge theory in Euclidean 4-space (as opposed to Minkowski space).

The physical significance of instantons is different from that of monopoles and is best understood in terms of the Feynman functional integral approach to quantization. This involves expressing physical quantities as functional integrals involving e^{iS} where S is the (Minkowski space) action or Lagrangian. A standard way to

attempt to ascribe meaning to such an integral is to continue analytically into Euclidean space in which case the exponential becomes e^{-S} where S is now the (positive) Euclidean action. Instantons correspond to minima of the Euclidean action. Of course this does not solve the problem of giving a mathematical meaning to the functional integral but it is a step in the right direction and is becoming increasingly popular.

The general hope is in any case that a thorough understanding of the classical nonlinear equations of gauge theories may cast light on the difficult problem of quantization.

2. Instantons. Mathematically instantons, for a given compact Lie group G, correspond to fibre bundles with group G over S^4 (the compactified 4-space, having a connection A which minimizes the L^2-norm $\|F\|^2$ of the curvature F. For G simple and non-abelian (e.g. $SU(2)$) the fibre bundle is classified topologically by an integer k and the minimum of $\|F\|^2$ is $8\pi^2|k|$.

In physical terminology A is the potential, F is the field and $\|F\|^2$ the action. Different integers k correspond to different asymptotic conditions for A in R^4, and k is referred to as the instanton number. Since $\|F\|^2$ depends only on the conformal structure of R^4 it is natural to pass to the conformal compactification S^4.

For $k=1$ and $G=SU(2)$ a solution spherically symmetric about the origin in R^4 was discovered by Belavin et al. [8]. This was extended by 'tHooft and others (see [10]) to give solutions for $k>1$. Their solutions can be regarded as a non-linear supposition of k single instantons located at different points of R^4, the superposition being achieved by an ingenious but rather mysterious Ansatz.

A parameter count, using infinitesimal deformation theory, showed that, for $k>2$, this Ansatz did not yield the most general k-instanton [3], [13], for which more sophisticated methods have had to be used. These methods arise naturally from Penrose's twistor theory [2], [12] and have led to a complete solution of the instanton problem, not just for $SU(2)$, but for all compact classical groups [4].

The geometry underlying Penrose's theory goes back to Plücker and Klein and hinges on the fact that 4-dimensional space can be viewed as the parameter space of lines in 3-space. More precisely, for our purposes, there is a fibration $P_3(C)\to S^4$, where $P_3(C)$ is complex projective 3-space and the fibres are complex projective lines (i.e. 2-spheres). Instanton bundles on S^4 correspond by this map to holomorphic bundles over $P_3(C)$, satisfying suitable constraints [6]. Such bundles are necessarily algebraic by a basic theorem of Serre and the constraints are also expressible algebraically (over the real numbers). Thus the instanton problem gets reduced to a problem of real algebraic geometry in 3 dimensions.

Using the powerful techniques of modern algebraic geometry, and following in particular the work of G. Horrocks and W. Barth [7], one obtains the complete solution of the instanton problem referred to above. The main technique involved is the use of sheaf cohomology groups, some of which have a direct interpretation

in 4-space as solutions of standard linear equations (Laplace, Maxwell, Dirac, etc.). The 'tHooft Ansatz, for example, which is based on Laplace's equation, has a natural cohomological counterpart, and can be generalized to the other equations [6].

The final outcome [4] of this algebraic geometry is a very explicit description of all k-instantons. The solutions, i.e. the connections A and curvature F are given by rational functions of the 4-space coordinates, and depend on a suitable matrix of parameters. The 'tHooft solutions for $SU(2)$ correspond to the special case of a diagonal matrix. The general k-instanton does not have the same local or pointwise appearance due to the presence of the off-diagonal terms.

This solution of the general instanton problem provides a tangible mathematical justification of the Penrose approach. The technical reasons for its success lie in the fact that sheaf cohomology is a more flexible tool than its counterparts in 4-space. In the long run this may well mean that the process of quantization should, as Penrose argues, be carried out in the twistor framework.

There are two caveats that should perhaps be made. In the solution of the instanton problem one key step has to be carried out in the S^4-picture. This involves proving that a certain operator is positive and is not so easy to see in the $P_3(C)$-picture. Secondly the Penrose transformation works well for instantons because these are given by the self-dual Yang–Mills equations (Euler equations for the Lagrangian $\|F\|^2$). The full Yang–Mills equations involve also the anti-self-dual case and because the equations are non-linear we cannot, as in Maxwell theory, combine the two together. However Witten, in a very interesting paper [15] (see also [16]), has recently shown how to interpret the full Yang–Mills equations in a twistor framework. This involves studying the product of $P_3(C)$ with its dual and looking at the 5-dimensional "incidence" hypersurface together with some normal derivatives. Interestingly enough this use of normal derivatives corresponds to a supersymmetric approach involving formal anti-commuting variables.

Finally I should mention a very intriguing paper by Manton [11] which shows that, in a certain precise sense, an infinite superposition of instantons produces a magnetic monopole. To see how this might happen recall that a monopole is time-independent and so can also be viewed as an infinite action solution in Euclidean 4-space.

3. Topological aspects. In the Feynman approach to quantization we have to integrate over the function space of all classical fields. In the Euclidean version of gauge theory, extended to S^4, this space would be the space $\mathscr{A}$ of all connections. There is however a large group of symmetries of this space, namely the group $\mathscr{G}$ of gauge transformations[1] (bundle automorphisms), which preserves the Lag-

[1] For technical reasons it is convenient here to use only gauge transformations equal to the identity at ∞ (i.e. at the base point of S^4).

rangian $\|F\|^2$. Integration should therefore be carried out on the quotient space $\mathscr{C}=\mathscr{A}/\mathscr{G}$. Now $\mathscr{A}$ is a linear space but $\mathscr{C}$ is only a manifold and has to be treated with more respect. Thus for integration purposes a Jacobian term arises which, in perturbation theory, gives rise to the well-known Faddeev–Popov "ghost" particles. Nonperturbatively it seems reasonable that global topological features of $\mathscr{C}$ will be relevant.

Homotopically $\mathscr{C}\sim\Omega^3(G)$ the function space of based maps $S^3\to G$, the components $\mathscr{C}_k$ of $\mathscr{C}$ corresponding to maps of degree k. The k-instantons define a finite-dimensional manifold $M_k\subset\mathscr{C}_k$ given by the minima of the action. For $G=SU(2)$ and $k\to\infty$ it is a remarkable fact [5] that all the homology of $\mathscr{C}_k$ lies in M_k. It is not unreasonable to conjecture that a similar result should hold for all G and that one should even have a homotopy equivalence $M_k\sim\mathscr{C}_k$ as $k\to\infty$. This would tell us that all the global complication in $\mathscr{C}_k$ was already present in M_k (for $k\to\infty$), and might indicate that instanton contributions, suitably interpreted, would converge to the required functional integral.

The results of [5], for $G=SU(2)$, use the 'tHooft solutions depending on configurations of k distinct points, and the striking theorem of G. B. Segal [14] that, for $k\to\infty$, $C_k(R^3)\approx\Omega^3_k(S^3)$ where $C_k(R^3)$ is the configuration space of k points of R^3 and $\approx$ denotes homology equivalence. It is important to note that the fundamental group of $C_k(R^3)$, namely the permutation group Σ_k, gets abelianized on passage to $\Omega^3_k(S^3)$. It seems likely that this already happens in M_k as a result of the nondiagonal matrices in the description of the general k-instanton (for $k\geqslant 3$). This indicates that a "particle" interpretation of the space $\mathscr{C}_k$, while valid for homology purposes, is inadequate in other respects.

Homotopically the space $\Omega^3(G)$ simplifies, for $G=SU(n)$, if we take $n\to\infty$. This suggests that the quantum theory might be soluble in some sense for this limit case and one might then take the $1/n$-expansion to derive information about the finite levels. One very significant feature of the limit theory is that all the homotopy of the function space $\mathscr{C}$ is then contained in the family of (massless) Dirac operators coupled to the (iso-spin representation of the) gauge field. More precisely if we assign to $A\in\mathscr{C}$ the corresponding Dirac operator D_A we get a homotopy equivalence between $\mathscr{C}$ and the space of Fredholm operators in Hilbert space. This is related to the index theorem for elliptic operators and the Bott periodicity theorems concerning the homotopy of the unitary groups [1]. Even for finite n one can derive interesting consequences [5].

In conclusion, and to put matters in proper perspective, I would like to emphasize that my discussion of geometrical aspects of gauge theories does not imply that geometry or topology alone will unlock the secrets of physics. Many other insights coming from analysis, statistical mechanics and of course experimental physics are needed. The geometrical point of view is however a comparatively new one in the context of quantum field theory and I hope it can provide some useful ideas. It is at least encouraging that the mathematical study of classical gauge theories, geo-

metric, topological and analytic, has over the past decades developed powerful new approaches and techniques. If gauge theories turn out to provide the right explanation of the basic forces of nature, physicists may find that the work of their mathematical colleagues has not been entirely irrelevant.

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Model Equations for Waves in Nonlinear Dispersive Systems

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The present discussion focuses on models for unidirectional wave propagation in which nonlinear, dispersive and dissipative effects are simulated realistically and in such a way that shock formation and other singular behaviour is avoided. While comparatively narrow, the range of discussion nevertheless covers a number of interesting and challenging scientific issues, several of which still remain open. The model equations take one of the following forms:

$$u_t+f(u)_x+Hu_t = 0, \tag{1a}$$

or

$$u_t+f(u)_x-Hu_x = 0. \tag{1b}$$

Here $u=u(x, t): \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ and subscripts denote partial differentiation. The function $f: \boldsymbol{R}\to\boldsymbol{R}$ represents nonlinear effects in the physical system being modeled, while H is a linear operator representing dispersive effects, and dissipative effects when they are considered. The best known model in the form (1b) is the KdV equation

$$u_t+u_x+uu_x+u_{xxx} = 0 \tag{2}$$

which was introduced by Korteweg and de Vries [17] and has in recent years been the subject of prolific study.

The derivations of approximate equations such as (1a, b) differ from one modeling situation to another. Nevertheless, one may appreciate in general why such models arise, at a certain level of approximation, without relying on the details of a particular application (cf. Benjamin [3]; Benjamin, Bona and Mahony [6]).

Upon linearization of the full equations of motion around a rest state, a dispersion relation is determined for plane simple-harmonic waves which relates frequency

$\omega = kc(k)$ to wave-number k. For example, two-dimensional surface waves in water of uniform depth have the dispersion relation

$$c(k) = \omega(k)/k = \{\tanh(k)/k\}^{1/2} \tag{3}$$

in suitably scaled coordinates. Here the phase velocity $c(k)$ has a maximum $c(0)$, corresponding to the limit of large wavelengths. In general the propagation of infinitesimal waves in such a system will be governed by an equation of the form

$$u_t + Mu = 0, \tag{4}$$

where $\widehat{Mu}(k) = ikc(k)\hat{u}(k)$ and the circumflexes denote Fourier transforms. Note that if $c(k)$ has a nonzero imaginary part, then (4) will contain a dissipative term. In many applications attention is restricted to a long-wave régime $k \ll 1$, and it is then justified to approximate $c(k)$ near $k=0$ in order to obtain a more tractable model equation. In the case of water waves as mentioned above, two relevant models are

$$u_t + u_x + \tfrac{1}{6}u_{xxx} = 0 \quad \text{and} \quad u_t + u_x - \tfrac{1}{6}u_{xxt} = 0, \tag{5}$$

which correspond respectively to the dispersion relations $c(k) = 1 - \frac{1}{6}k^2$ and $c(k) = 1/(1+\frac{1}{6}k^2)$ approximating (3).

If the effects of dispersion due to finite wavelength are ignored and attention is concentrated solely on the effects of nonlinearity, then it is a general attribute of the systems in question that waves propagate along characteristics which depend on the value of the dependent variable: thus

$$\left.\frac{dx}{dt}\right|_{u=\text{constant}} = g(u).$$

This property is equivalent to

$$u_t + f(u)_x = 0, \tag{6}$$

where $f' = g$. If the régime of interest includes in its characterization an assumption that the waves be of small amplitude, then it is justified to use a simpler model obtained by approximating g for small values of its argument. If a linear approximation to g is presumed to be adequate over the range of amplitudes in question, then we take $g(u) = 1 + u$, say, and so obtain

$$u_t + u_x + uu_x = 0. \tag{7}$$

If the nonlinear, dispersive and dissipative effects are of a similar order of smallness, then normally the interaction between these effects is of a yet higher order of smallness. Accordingly, it is warranted simply to add the extra terms appearing respectively in (5) and (7), so to obtain the model equation (1b).

It is noteworthy that nonlinear, dispersive and dissipative effects are generally small corrections to the basic one-way propagator $u_t + u_x = 0$, which is just a factor, governing propagation in the $+x$-direction, of the one-dimensional wave equation.

Thus, if $g(u)=1+g_1(u)$, where $g_1(u)=O(u)$ as $u\to 0$, and if $c(k)=1+c_1(k)$, where typically $c_1(k)=O(k^2)$ as $k\to 0$, then (1b) may be written as

$$u_t+u_x+f_1(u)_x-Lu_x=0, \tag{8}$$

where $f_1'=g_1$ and $\widehat{Lv}(k)=c_1(k)\hat{v}(k)$. Provided u and k are required to be small, then f_1 and L are of higher order of smallness than the leading terms u_t and u_x. In such a situation, the basic level of approximation will be unaltered if the approximate relation $u_x=-u_t$ is utilized to alter the higher order terms. Hence the equation

$$u_t+u_x+f_1(u)_x+Lu_t=0 \tag{9}$$

of the form (1a) may be inferred as a model for the unidirectional propagation of small-amplitude long waves. References to specific examples where (1a, b) have been derived as models may be found in the review article of Jeffrey and Kakutani [15] and in the two collections of articles on nonlinear waves edited respectively by S. Leibovich and R. Seebass [19] and A. Newell [21].

Some care is necessary in the use of the approximations outlined in the preceding discussion. The stated hypotheses are invariably pivotal to the derivation of these equations as rational models, and they should therefore be respected in using the models to gain insight into a physical situation. In the particular case of irrotational surface waves on shallow water, if the independent variables x and t and the dependent variable u, which represents the height of the wave above the undisturbed depth, are scaled so that u and its derivatives are order one, there appear the two model equations

$$u_t+u_x+\varepsilon uu_x+\delta^2u_{xxx}=0 \tag{10a}$$

and

$$u_t+u_x+\varepsilon uu_x-\delta^2u_{xxt}=0, \tag{10b}$$

corresponding to the different approximations to the dispersion relation given in (5). The parameter ε is a measure of the amplitude of the waves and δ^{-1} is a measure of their wavelength. It is appropriate to assume both ε and δ are small, in this scaling, and that ε and δ^2 are of the same magnitude. In the literature on water waves this is sometimes expressed by demanding that the Stokes number $S=\varepsilon\delta^{-2}$ is order one. The assumption concerning the Stokes number being valid, an order-one change of the dependent variable gives the special case $\varepsilon=\delta^2$. In this scaling it is apparent that the nonlinear and dispersive terms represent small corrections to the basic propagator $u_t+u_x=0$, the smallness of the corrections being measured explicitly by ε. Needless to say, the zero on the right-hand side of (10a, b) represents an approximation to terms that are of order ε^2.

On time scales of order ε^{-1}, the nonlinear and dispersive corrections can accumulate and have an order-one influence on the wave profile. Equally, on time scales of order ε^{-2}, the higher order terms not included in (10a, b) can have an

order-one effect on the wave profile. Hence, on such time scales, the models may have become unreliable and predictions made on longer time scales should be viewed with caution.

At first sight it might appear somewhat contradictory that two different models purport to describe the same physical phenomena, as in (10a, b). However, it has been shown by Bona, Pritchard and Scott [11] that if u and v denote respectively the solution of (10a) and (10b) corresponding to the same order-one initial wave profile, then $u-v$ is of order ε over the time scale ε^{-1}. Numerical studies indicate that the difference $u-v$ grows linearly to order one on the time scale ε^{-2}. Hence it appears that the two models may indeed simultaneously provide accurate predictions at least over time scales where either model may be expected to apply. A further conclusion is that expediency should govern the choice of (10a) or (10b) in a particular situation where a model for small-amplitude long waves is needed. For instance, the inverse scattering methodology and the infinite collection of polynomial conserved densities for (10a) may both be very useful for various theoretical considerations (cf. Miura [20] for an account of the inverse-scattering method, and Whitham [27] and Segur [26] for some applications). Olver [22] has shown that (10b) has only the three polynomial conversed densities corresponding to mass, momentum and energy in the original physical problem that is modeled. This and certain other facts indicate that there is also no inverse scattering formalism for (10b), at least as we presently understand such a formalism. On the other hand, (10b) is far easier to handle numerically than (10a).

In the task of comparing the predictions of the models (10a) or (10b) with experimental data, the most natural mathematical formulation is an initial- and boundary-value problem to be explained presently. In this setting, (10b) appears definitely easier to use and quantitative comparisons using (10b) have been made by Bona, Pritchard and Scott [12]. Their work supplements earlier comparisons, made using the pure initial-value problem for (10a), by Zabusky and Galvin [28] and by Hammack and Segur [14], which showed good qualitative agreement between measured data and theoretical predictions.

The experimental configuration used in all the above-mentioned comparisons was a rectangular channel containing water with a wavemaker at one end. For the experiments reported by Bona et al. [12], the water was initially at rest when the wavemaker was set in motion. At several stations along the channel, temporal records of the passage of the waves generated by the wavemaker were taken. An appropriate mathematical problem was suggested and analyzed by Bona and Bryant [9]. In dimensionless but unscaled coordinates, it is

$$\begin{aligned} &u_t+u_x+uu_x-u_{xxt}=0, \\ &u(x,0)\equiv 0,\ u(0,t)=g(t), \end{aligned} \tag{11}$$

for $x, t\geqslant 0$. In a numerical scheme for (11), the function g is a discretization of

the measurement of the wave taken closest to the wavemaker. The numerical integration of the model (11) will then predict $u(x_0, t)$ for any station x_0 further from the wavemaker than the station at which g is measured. Such a prediction may then be directly compared with the measurement taken at the station x_0 and the model judged on the basis of the discrepancy between the two.

The Stokes number S for these experiments ranged from 1/4 to 30. Dissipative effects proved to be generally of the same importance as nonlinear and dispersive effects, and accordingly had to be incorporated into the model. The proper form of dissipation term in equations of the types (10a) or (10b) has been derived by Kakutani and Matsuuchi [16] and it is non-local in character. For the experiments in question, where most of the energy was manifested at one frequency, an *ad hoc,* local form of dissipation represented by a term $-\nu u_{xx}$ in (11) can be justified. This form was used by Bona et al. [12], although comparisons are desirable between measured data and predictions from the model incorporating the proper form of dissipation. The latter need poses an interesting mathematical and numerical challenge which is presently under study.

The agreement between the experimental and numerically predicted values was quite good. For S in the range [1/4, 10], the difference between the measured and computed wave traces was about 8% of the size of the physical wave. The agreement was less striking as S became large. The difference between the measured and the computed wave was 22% of the size of the measured wave when $S=30$. These comparisons are all respective to the L_∞ norm. Even for such large values of S, some of the qualitative properties of the wave profile were still modeled well, although quantitatively the situation had deteriorated.

Turning now to a different aspect, we recall one of the most fascinating properties of many of the equations in (1). It is that, when dissipative effects are ignored, the balance between nonlinearity and dispersion admits the possibility of a special class of waves moving at constant velocity and without change of shape. These waves were called solitary waves by Scott Russell [25] who first observed them on the surface of a canal in the early 1830s. Scott Russell subsequently conducted experiments which showed the solitary wave to be a very stable waveform, which could sustain repeated complicated interactions without losing its identity. The existence of such permanent waves was at variance with the surface-wave theory known in the middle of last century. Indeed, one of the main accomplishments of Korteweg and de Vries in the 1890s was to resolve the paradox of the solitary wave, at least at the level of their model equation. Even so, the importance of this class of waves was not recognized until the 1960s when computer studies by Kruskal and Zabusky [18] of the KdV equation (2) showed that an initial profile of elevation broke up into a sequence of solitary waves and very little else. An analogous result for the sine-Gordon equation had been obtained earlier by Perring and Skyrme [23]. The celebrated inverse-scattering theory for (2), first discovered by Gardner, Greene, Kruskal and Miura [13], subsequently established this result, and a host

of others including the fact that solitary-wave solutions of (2) emerge from interaction with each other with only a phase shift. This type of exact result is true for a number of other wave equations admitting an inverse-scattering theory, including the sine-Gordon equation.

Recently a class of equations of the form given in (1) has been shown to possess solitary-wave solutions (cf. Benjamin, Bona and Bose [5]; Bona and Bose [8]), even when the operator H is not a differential operator. Moreover, numerical studies show that while the tidy situation regarding interaction of solitary waves for the KdV equation (2) does not in general obtain, the solitary wave nevertheless plays a distinguished role in the long-term evolution of an initial profile of elevation.

To take a concrete example, consider the model equation (10b). It has been proven by Benjamin [2] and Bona [7] that the solitary-wave solution of both KdV and (10b) is stable in the following sense. Let φ denote a solitary-wave profile and let ψ be a perturbation of φ, say in the norm defined by

$$\|w\|^2 = \int_R [w^2(x) + w_x^2(x)]\, dx.$$

Let $\eta > 0$ be given. Then there exists a $\delta > 0$ such that if $\|\varphi - \psi\| < \delta$, then $d(\varphi, u) < \eta$ for all $t > 0$, where u is the solution, of (10b) say, with initial profile ψ and

$$d(w, v) = \inf_{y \in R} \|w(\cdot) - v(\cdot + y)\|$$

is a pseudo-metric that compares the shape of two functions. One may think of d as being defined on the product of the quotient space H^1/τ with itself, where H^1 is the space of measurable functions $f: \boldsymbol{R} \to \boldsymbol{R}$ such that $\|f\| < \infty$ and τ is the translation group in $\boldsymbol{R}$. Moreover, Abdulloev, Bogolubsky and Makhanov [1] and Bona, Pritchard and Scott [10] have produced numerical results indicating that when a pair of solitary-wave solutions of (10b) interact, the bulk of the mass emerges as a slightly different pair of solitary waves, shifted in phase, with a very small dispersive tail lagging behind. Finally, numerical experiments indicate than an initial wave of elevation evolving under (10b) breaks up into a finite number of solitary waves followed by a dispersive tail. Similar numerical results hold good for other models of the type given in (1).

Exactly what we should make of all this is still unclear. What is it in common to the models (1a) and (1b) that causes waves to evolve into solitary waves? Whatever this may be, it is probably more fundamental and at the same time less powerful than the inverse-scattering theory. A satisfactory answer to this question might have implications for the more complex models such as the various versions of the Boussinesq equations and ultimately the full equations of motion for various physical systems.

A final point deserves mention. Equations (1) have natural multi-dimensional versions which are of interest. The case of a system of two equations, for example,

can serve as a model for the two-way propagation of one-dimensional waves. An existence, uniqueness and regularity theory for such systems has been given by Saut [24] in the case (1b) and by Benjamin and Bona [4] in the case (1a). However, the qualitative properties of solutions of such systems are still largely unknown.

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On the Mathematics of Phase Transitions and Critical Phenomena

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I. Introduction. In this review I try to summarize (mostly in the form of references) a tiny part of the mathematics of equilibrium statistical mechanics (ESM), phase transitions and critical phenomena. Many important contributions are not mentioned, because of ignorance, bias, or, most importantly, page limitations. The only purpose of these notes is to provoke the reader to learn more about ESM, to provide him with a few basic, recent references and to outline some recent developments in which I have personally been involved. In this connection I shall emphasize the concept of reflection positivity which has proven equally important in relativistic quantum field theory and ESM.

I start with a short list of some of the main circles of problems in ESM each requiring further research efforts.

1. *The general framework of ESM.* Some basic references are [1]–[7]. Interesting open problems concern: A good theory of the third law of thermodynamics (Nernst: Entropy at 0 Kelvin=0) and its violations; a concrete theory of the Gibbs phase rule of satisfactory generality.

2. *Time evolution of infinite systems, approach to equilibrium, irreversibility, open systems.* Many (or most?) basic questions asked by the masters (Boltzmann, Planck, ...) may be regarded as unanswered. Huge heuristic and rigorous efforts have been invested in understanding these problems. I do not know a short list of basic references, but should like to recommend [8]. Related to 2 are

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3. *Foundational problems in ESM.* (Derivation of the ergodic hypothesis in classical ESM; see [9] for a short review, of the KMS condition in quantum ESM; see [4], [10] and references given there. The second problem has deep links to Tomita–Takesaki theory and its applications [11]. The central questions have not been answered, yet.)

4. *Structural problems of ESM.* See [1]–[3], [6], [7]. (Structure of manifold of equilibrium states, general theory of phase transitions, symmetry breaking, critical points, ...)

5. *General techniques to determine the phase diagrams, the critical behaviour, ... of a sufficiently general class of systems in thermal equilibrium of interest to physics.* This review is centered around that circle of problems, and references can be found in the main part of the text.

6. *Mathematical study of special, physical systems and physical phenomena at thermal equilibrium.* (Stability of nonrelativistic, quantum mechanical matter [12], [13]; thermodynamic functions of Coulomb systems [14]–[17]; existence of correlation functions [18]–[20]; screening [14], [21], etc. Two marvelous references are [22], [23]. Mathematical foundation of solid state physics: Rigorous theory of magnetism, super conductivity, Bose–Einstein condensation, super fluidity, melting, liquids, ... For inspiration one may consult any honest, modern text book on solid state physics.)

Henceforth I concentrate on 5, emphasizing those techniques that are linked to the concept of reflection positivity, in field theory known as Osterwalder–Schrader positivity [24]; see [25]–[29]. It is obviously impossible to give any complete arguments.

II. Classical lattice systems and reflection positivity. I illustrate the main points of this review in terms of classical lattice systems, a class of dynamical systems defined below. The techniques described in the following extend however to quantum mechanical lattice systems and to relativistic quantum field theory in the Euclidean description [24] which has inspired them in the first place; see [30], [25]. Those techniques do *not* seem to be applicable to nonrelativistic continuous systems in a satisfactory way. Here, new ideas are very much needed.

Let $\mathscr{L}$ be an infinite, ν-dimensional lattice; in the following $\mathscr{L}=\mathbf{Z}^\nu$ (simple, cubic lattice), but other lattices can usually be accomodated.

At each site $x\in\mathscr{L}$ we are given a copy K_x of some configuration space K and a fixed probability measure $d\varrho(\chi_x)$, $\chi_x\in K_x$. Sometimes K will be a finite dimensional Hilbert space carrying a representation, U, of some compact, topological group G. Then $d\varrho$ is assumed to be invariant under the action of G on K. Given a subset $X\subset\mathscr{L}$, we define

$$K_X = \bigtimes_{x\in X} K_x,\quad K_\infty = K_{X=\mathscr{L}}.$$

(It is technically useful to choose a topology on K such that it is a compact Hausdorff space. Then K_X and K_∞ have the same property.)

The class of continuous functions on K_X is denoted $C(K_X)$, the "observables localized in X". We denote by "tr" the state on $C(K_\infty)$ determined by the product measure $\prod_{x\in\mathscr{L}} d\varrho(\chi_x)$.

A classical dynamical system is defined in terms of the "algebra of observables" $C(K_\infty)$ and the set of states on $C(K_\infty)$=probability measures on K_∞. The dynamics of such a system is specified by an *interaction* Φ. This is a map from bounded subsets $X\subset\mathscr{L}$ to $C(K_\infty)$ satisfying

(a) $\Phi(X)\in C(K_X)$, real-valued;

(b) $\Phi(X+a)=\tau_a(\Phi(X))$, where τ_a denotes translation by a vector $a\in\mathscr{L}$, i.e. τ_a is the natural identification map from $C(K_X)$ to $C(K_{X+a})$.

(c) $\sum_{X\ni 0}(1/\text{vol. } X)\sup|\Phi(X)|<\infty$, and $\text{tr}_Y(\Phi(X))\equiv\int\prod_{x\in Y} d\varrho(\chi_x)\Phi(X)(\chi)=0$, when $Y\cap X\neq\emptyset$.

Interactions form a real Banach space, B. The *Hamilton function* of such a system confined to a bounded region $\Lambda\subset\mathscr{L}$ is given by

$$H_\Lambda^\Phi=\sum_{X\subset\Lambda}\Phi(X),$$

the *free energy* at inverse temperature β by

$$f_\Lambda(\beta,\Phi)=-(1/\beta\cdot\text{vol. }\Lambda)\log\text{tr}\,(e^{-\beta H_\Lambda^\Phi}).$$

For a clean discussion of interesting phenomena such as *phase transitions* and critical phenomena one must first take the *thermodynamic limit* $\Lambda\uparrow\mathscr{L}$. The limit $f(\beta,\Phi)=\lim_{\Lambda\uparrow\mathscr{L}} f_\Lambda(\beta,\Phi)$ (in the sense of van Hove [1], [6]) exists and defines a *concave, bounded* functional on B. ESM of classical lattice systems is by and large the theory of the functional f.

A *tangent functional* $\alpha_{\beta\Phi}$ *to* f *at* $\beta\Phi$ *is a linear functional on* B *satisfying*

$$f(\beta,\Phi)+\alpha_{\beta\Phi}(\Psi)\geqslant f(\beta,\Phi+\Psi),$$

for all $\Psi\in B$. Associating with Ψ the observable

$$A_\Psi=\sum_{X\ni 0}(1/\text{vol. }X)\Psi(X)\in C(K_\infty)$$

one can show that

$$\omega_{\beta\Phi}(A_\Psi)\equiv\alpha_{\beta\Phi}(\Psi) \tag{1}$$

defines a *translation invariant* state of the infinite ($\Lambda=\mathscr{L}$) system.

The *(translation invariant) equilibrium states* of the infinite system with dynamics given by Φ, at inverse temperature β, are precisely the *tangent functionals* to f at $\beta\Phi$, i.e. the states given by (1). (Equivalently, equilibrium states can be characterized by the "Gibbs variational principle" or the "Dobrushin–Lanford–Ruelle equations" [1]–[3], [6].)

One says that a system with dynamics Φ has a *phase transition with order parameter* if $\omega_{\beta\Phi}$ is *unique* for small β and *non-unique* for large β. (Uniqueness for

small β is known if $\sum_{X \ni 0} \sup |\Phi(X)| < \infty$, [36], [3].) It has a *phase transition without order parameter* if $\omega_{\beta\Phi}$ is unique for all β, but the curvature of f at $\beta\Phi$ is finite (on finite range interactions) at small β and infinite, in some direction, at large β. A *critical point* β_c for Φ is a point such that the curvature of f at $\beta_c\Phi$ is infinite, in the direction of some finite range interaction (but $\omega_{\beta_c\Phi}$ is unique, whereas $\omega_{\beta\Phi}$ is generally nonunique, for $\beta > \beta_c$).

Deep structure theorems about equilibrium states and phase transitions with order parameter follow from the fact that f is a concave functional on a real Banach space the tangent functionals of which are the equilibrium states: See [6]. In particular, it is shown in [6] that equilibrium states are norm-dense in the space of all translation-invariant states on $C(K_\infty)$, that phase transitions with order parameter are in some sense generic and that interactions with phase transitions are not isolated in B; see also [31]. Next, I introduce the concept of RP *(reflection positivity):* Consider a decomposition of $\mathscr{L}$ into two disjoint sublattices, Γ_+, Γ_- (in the following half-lattices separated by a hyperplane); r is the reflection taking $\Gamma_\pm$ to $\Gamma_\mp$, and θ_* the obvious reflection map from K_{Γ_-} to K_{Γ_+}. For $F \in C(K_{\Gamma_+})$ set

$$(\theta F)(\chi_-) = \overline{F(\theta_* \chi_-)}, \quad \chi_- = \{\chi_x\}_{x \in \Gamma_-}.$$

An interaction Φ is called RN (reflection negative) iff $\theta\Phi(X) = \Phi(rX)$, and

$$\sum_{X \cap \Gamma_\pm \neq \emptyset} \operatorname{tr}\left(F(\theta F)\Phi(X)\right) \leqslant 0, \tag{2}$$

for all $F \in C(K_{\Gamma_+})$.

Note that RN interactions form a *convex cone.* A state ω on $C(K_\infty)$ is called RP *(reflection positive)* iff

$$\omega\left(F(\theta F)\right) \geqslant 0 \quad \text{for all } F \in C(K_{\Gamma_+}). \tag{3}$$

Loosely speaking one has the following general

THEOREM 1. (1) [28], [29]. *For all inverse temperatures $\beta \geqslant 0$ there exists an RP equilibrium state $\omega_{\beta\Phi}$ if and only if Φ is RN. (Remark: That Φ is RN does not imply that all equilibrium states for $\beta\Phi$ are RP!)*

(2) [32]. *Any RP equilibrium state for $\beta\Phi$ can be decomposed into extremal-invariant equilibrium states for $\beta\Phi$ which are RP.* □

It turns out that it is considerably easier to prove theorems on the structure of $f(\beta, \Phi)$ and of the equilibrium states when Φ belongs to the cone of RN interactions. This will be illustrated with a few examples. It requires rather much knowledge of the Euclidean description of relativistic quantum field theory [24] (r.q.f.t.) to understand how the concept of RP arises and why it is natural, at least in r.q.f.t. (There it guarantees the existence of a quantum mechanical Hilbert space and a self-adjoint, positive Hamilton operator [24].)

All applications of RP described below are more or less tricky applications of a *Schwarz inequality:* Consider the vector space $C(K_{\Gamma_+})$ and an RP state

ω on $C(K_\infty)$. Then, by (3), $(F, G)\to\omega(F(\theta G))$ clearly defines a positive semi-definite inner product on $C(K_{\Gamma_+})$. Hence

$$|\omega(F(\theta G))| \leqslant \omega(F(\theta F))^{1/2}\omega(G(\theta G))^{1/2}. \tag{4}$$

III. The chess board estimate. Let $F(\chi)\in C(K)$ be real-valued, and let $\tau_x(F)$ denote the corresponding function in $C(K_x)$. For *translation-invariant* states ω, define

$$z(\omega, F) = \lim_{\Lambda\uparrow\mathscr{L}} |\omega(\prod_{x\in\Lambda}\tau_x(F))|^{1/|\Lambda|}.$$

Existence of this limit is standard. A rather immediate consequence of infinitely many applications of (4) and translation invariance is

THEOREM 2 [27]–[29]. *Let $F_x\in C(K)$ be real-valued, for all $x\in\mathscr{L}$. If ω is RP then*

$$|\omega(\prod_x \tau_x(F_x))| \leqslant \prod_x z(\omega, F_x). \ \square$$

This inequality is remarkable, because it provides upper bounds for expectations of local observables in terms of thermodynamic functions ($z(\omega, F)$ is a constrained thermodynamic function) which are relatively easy to estimate.

Theorem 2 is a special case of the so called "chess-board estimates" which follow from a generalization of the Hölder inequality proven in [27] and RP. The generalized Hölder inequality provides, moreover, simple (real analysis) proofs of the Hölder inequality for traces and many other inequalities for matrices important in ESM [28]. (It might be a starting point for an L_p theory for general von Neumann algebras.)

IV. Gaussian domination and infrared bounds. Let $J(x)$ be a real-valued function on $\mathscr{L}$ with $\sum_x J(x)=0$. Let S be some real-valued function in $C(K)$ and S_x the corresponding function in $C(K_x)$. The interaction Φ_2, Φ_2: $\{x, y\}\to\Phi_2(\{x, y\})=-S_x J(x-y)S_y$, $\Phi_2(X)=0$, for $X\neq\{x, y\}$, is RN if and only if

$$\sum_{x,y\in\Gamma_+} Z_x J(x-ry)\bar{Z}_y \geqslant 0, \tag{5}$$

for all complex-valued functions $Z: x\to Z_x$ on Γ_+.

Examples of J's satisfying (5) are ($|x|$=Euclidean length of x):

(a) the nearest neighbor interaction

$$J(x) = \begin{cases} 1, & |x| = 1, \\ -2\nu, & x = 0, \end{cases} \quad J(x) = 0, \ |x| > 1. \tag{6}$$

(b) $$J(x) = \text{const.}\,|x|^{-\nu+2-\eta}, \ \eta \geqslant 0, \text{ for } x \neq 0.$$

(c) $$\begin{aligned} &J(x) = (-1)^{\Sigma x^j - 1}J_0(x), \\ &J(x) = J_1(x)J_0(x), \end{aligned} \quad J(x) = -(J_0)^{-1}(x),$$

where J_0 and J_1 satisfy (5); see [29].

Let $\omega_{\beta\Phi_2}$ be an arbitrary translation-invariant, RP equilibrium state for Φ_2 at inverse temperature β. (By Theorem 1, (1) such a state exists.) To $\omega_{\beta\Phi_2}$ we may thus apply inequality (4) and Theorem 2. This yields (via some not quite trivial arguments.)

THEOREM 3 [25], [26], [29] (GAUSSIAN DOMINATION). *Let $h=\{h_x\}$ be a real-valued function on $\mathscr{L}$ of compact support. Then*

$$\omega_{\beta\Phi_2}(\exp[\beta\sum S_x J(x-y)h_y]) \leqslant \exp[-\frac{\beta}{2}\sum h_x J(x-y)h_y]. \quad \square$$

Taking second derivatives at $h=0$ one obtains

COROLLARY 4.

$$\sum_{u,x,y,z} h_u J(u-x)\omega_{\beta\Phi_2}(S_x S_y)J(y-z)h_z \leqslant -\beta^{-1}\sum_{x,y} h_x J(x-y)h_y. \quad \square$$

After Fourier transformation one arrives at the *infrared* bound [25], [29]

$$d\lambda(k) \leqslant [\alpha\delta(k)-(\beta\hat{J}(k))^{-1}]d^\nu k, \tag{7}$$

where $d\lambda(k)$ is the Fourier transform of $\omega_{\beta\Phi_2}(S_0 S_x)$, and α some nonnegative number.

Theorem 3 and Corollary 4 remain true, verbally, if $\omega_{\beta\Phi_2}$ is replaced by $\omega_{\beta(\Phi_2+\Phi')}$, where Φ' is an arbitrary RN interaction, and $\omega_{\beta(\Phi_2+\Phi')}$ any translation-invariant RP equilibrium state for $\Phi_2+\Phi'$ at inverse temperature β. A particularly direct proof, based on Theorem 2, of a yet somewhat more general result is also given in [33].

V. Applications of RP, chess board estimates and infrared bounds.

1. *Universal diamagnetism in relativistic q.f.t.* Let K be a Hilbert space carrying a representation, U, of a compact group G. Let $g=\{g_{xy}\}$ be a *connection,* i.e. to each pair, xy, of nearest neighbors there corresponds $g_{xy}\in G$, with $g_{xy}=g_{yx}^{-1}$, and $U(g_{xy})$ defines an isomorphism: $K_y\to K_x$. The trivial connection $\mathbf{1}=1$ is obtained by choosing for g_{xy} the identity in G, for all xy.

Let Φ_g be a connection-dependent interaction defined, for example, by $\Phi_g(\{xy\})=-(\chi_x, U(g_{xy})\chi_y)$, where $(\cdot,\cdot)$ is the scalar product on K; $\Phi_g(X)=0$, otherwise.

THEOREM 5 [34] (UNIVERSALITY OF DIAMAGNETISM). *Let Φ' be any g-independent, RN interaction. Then*

$$f(\beta, \Phi_g+\Phi') \geqslant f(\beta, \Phi_{g=1}+\Phi'). \quad \square$$

This is a special case of a general theorem proven in [34] as a consequence of RP.

In q.f.t., χ is interpreted as a *lattice matter field,* and the connection g is a *lattice gauge field.* In this interpretation, universality of diamagnetism is the statement that when matter fields are coupled to a gauge field by an RN interaction, Φ_g,

the "free energy" f (=vacuum energy density) rises. This has remarkable consequences for the construction and the physical contents of gauge theory models. See [34], [35] and references given there.

2. *Phase transitions with order parameter and the critical point in the N-vector models.* Let $K=S^{N-1}$, the unit sphere in $\boldsymbol{R}^N$, $\vec{S}$ the vectors in K, and $\vec{S}_x$ the vectors in K_x. Let $d\varrho$ be the uniform distribution on S^{N-1}. Furthermore J, satisfying (5), is as in § IV, and Φ_2 is given by $\Phi_2(\{x,y\})=J(x-y)\vec{S}_x\cdot\vec{S}_y$, $\Phi_2(X)=0$, otherwise. Clearly $\vec{S}\cdot\vec{S}=1$, so that $\omega_{\beta\Phi_2}(\vec{S}_0\cdot\vec{S}_0)=1$. On the other hand, inequality (7) says

$$\omega_{\beta\Phi_2}(S_0^jS_0^j)\leqslant\alpha^{(j)}+\beta^{-1}I(\nu,J),$$

with $I(\nu,J)=-\int\hat{J}(k)^{-1}d^\nu k$. Setting $\alpha=\sum_{j=1}^N\alpha^{(j)}$, one obtains

$$1=\omega_{\beta\Phi_2}(\vec{S}_0\cdot\vec{S}_0)\leqslant\alpha+\beta^{-1}NI(\nu,J).$$

Thus, when $I(\nu,J)$ is finite and $\beta>(NI(\nu,J))^{-1}$, α is positive. Now, there exists, for all β, at least one RP $\omega_{\beta\Phi_2}$ for which $\omega_{\beta\Phi_2}(\vec{S}_0)=0$. If $\omega_{\beta\Phi_2}$ were the *unique* equilibrium state (i.e. extremal-invariant) then $\alpha=|\omega_{\beta\Phi_2}(\vec{S}_0)|^2$; see e.g. [26], [28]. Hence, for $\beta>(NI(\nu,J))^{-1}$, $\omega_{\beta\Phi_2}$ cannot be extremal-invariant. On the other hand, for β small enough, $\omega_{\beta\Phi_2}$ is definitely unique [36]. Hence there exists a *phase transition with order parameter.* This transition is accompanied by the breaking of a continuous symmetry group of the dynamics, $O(N)$, (and there exist $(N-1)$ so called Goldstone excitations), [25], [29]. These results can be extended to interactions $\Phi=\Phi_2+\Phi'$, with Φ' RN and $O(N)$-invariant. Analogous results have been established for $(\vec{\Phi}\cdot\vec{\Phi})^2$ quantum field models, the study of which has inspired the whole method [25]. It remains to decide when $I(\nu,J)$ is *finite.* Sufficient conditions for $I(\nu,J)<\infty$ are: For $\nu\geqslant 3$, $I(\nu,J)$ is finite even in the case that J is a nearest neighbor interaction; see (6). For $\nu=1,2$, $I(\nu,J)$ is finite if e.g. $J(x)\geqslant\text{const.}(|x|+1)^{-\gamma}$ and $\nu<\gamma<2\nu$. This is a relatively easy problem in Fourier analysis; see [29].

The following asserts the existence of a critical point for some special N-vector models (a result also based in part on RP).

THEOREM 6. *Let* $\nu\geqslant 3$, *and let* J *be nearest neighbor; see* (6). *Then, for* $N=1,2$, *the corresponding N-vector models have a critical point,* β_c, *with the property that* $\sum_{x\in\mathscr{L}}\omega_{\beta\Phi_2}(\vec{S}_0\cdot\vec{S}_x)$ *is finite for* $\beta<\beta_c$, *but diverges at* $\beta=\beta_c$ [37]. (*See* [38] *for* $N=2,3,4$.) *For* $\beta<\beta_c$, *the equilibrium states are unique* [39], [40]; *the state* $\omega_{\beta_c\Phi_2}=\lim_{\beta\uparrow\beta_c}\omega_{\beta\Phi_2}$ *is extremal-invariant.* □

A natural question is then: What is the situation, for these *(nearest neighbor)* models when $\nu=1,2$. For $\nu=1$, these models are explicitly soluble, the equilibrium state is unique, for all $\beta<\infty$, and there are no critical points. The model $\nu=2$, $N=1$ *(Ising model)* is explicitly soluble, too (a very nontrivial task!!). There is a phase transition with order parameter and a unique critical point.

The model $\nu=2$, $N=2$ (two-dimensional *rotator model*) has a *unique* equilibrium state, for all $\beta<\infty$ [40]. Existence of an interval $[\beta_c, \infty]$ of critical points is conjectured. It has been proven that $\beta_c>(8\pi+\varepsilon)^{-1}$ (ε small) [41]. Partial results towards proving the conjecture are given in [41], [42]. For a review of rather sophisticated, heuristic results see [43] and references.

For $\nu=2$, $N\geqslant 3$ it is conjectured that there is a unique RP equilibrium state and *no* critical point, i.e. $\beta_c=\infty$. Partial results—at least rather sharp lower bounds on β_c—have been obtained in [44], [45]; see also [46]. The following result, partly based on RP and Theorem 2, has grown out of attempts to prove the $\nu=2$, $N\geqslant 3$ conjecture.

3. *Applications of RP to exclude the occurrence of phase transitions and critical points.* Let $K=R^N$. Vectors in K are denoted by $\vec{S}$; $d\varrho(\vec{S})=g(\vec{S}^2)d^NS$, where g is a nonconstant, monotone decreasing function on $(0, \infty)$. Let Φ_2 and J be as in § V. 2.

THEOREM 7 [44], [45]. *For* $\nu\geqslant 2$ *and all* $\beta<\infty$,

$$|\omega_{\beta\Phi_2}(\vec{S}_0\cdot\vec{S}_x)|\leqslant \text{const.}\exp[-m(\beta)|x|],$$

for some $m(\beta)>0$. □

For the N-vector models ($K=S^{N-1}$) of § V. 2, Theorem 7 is only proven for $\beta<\max(\beta_0, (N-2)/2\nu)$ [45], for some positive β_0 estimated in [45]. Moreover, it has been shown, in this case, that for $\nu=2$

$$m(\beta)\leqslant \text{const.}\exp[-\text{const.}(\beta/N)] \quad [47],$$

mainly a consequence of infrared bounds! These results rather favour the conjectures in § V. 2. The general problem of which the above are special, but possibly *typical* examples is: *Develop a general, mathematical theory of phase transitions without order parameter and of critical points!* We have a few reasons to believe that, again, RP is going to play a very useful role. The reader should consult Sinai's contribution for relevant information and references concerning this general problem. It certainly is one of the main problems in present day mathematical physics, being at the core of some of the central questions in ESM and relativistic q.f.t. For this reason it is most unfortunate that we still do *not* have a general mathematical theory of this sort.

4. *Applications of RP to the Peierls argument.* The Peierls argument [48] (which cannot be described here) is one of the most powerful machines for proving the existence of phase transitions with *discrete* order parameter. It was invented by Peierls to prove the existence of a phase transition with order parameter in the $\nu=2$ Ising model. Its marriage with RP, pioneered in [30] and developed into a systematic tool in [27], [49], [29] has simplified and strengthened it in a most satisfactory way, as references [30], [27], [49], [29] clearly show. For a complementary,

very powerful extension of the Peierls arguments see [50]. A short review of this and other matters in the theory of phase transitions is [51].

Much of this review describes work done in collaboration with, learnt from or due to D. Brydges, J. Glimm, R. Israel, A. Jaffe, E. H. Lieb, E. Seiler, B. Simon and T. Spencer.

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Introduction to Gauge Theories

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This report is divided into three parts. In the first part, we describe some nonlinear equations of classical physics. Certain of these equations, for example the Yang–Mills equations, have a geometrical significance. Their solutions are harmonic connection forms on a principal bundle; the structure group of the bundle is called the "gauge group" in physics.

The purely mathematical questions concerning the solutions to these equations have an interest of their own. In the second part of this report, however, we shall sketch the relation between these solutions and physics. Here it becomes necessary to introduce the notion of quantization. The classical equations we consider do not have a direct interpretation in physical terms; rather they yield insight into quantum field theory. In order to explain this connection, we shall describe the constructive field program. Many steps in this program have been established over the past ten years; now the construction of quantized gauge theories (i.e. quantization of geometry) poses interesting, new problems.

Finally, we shall return in part three to the interpretation of the solutions described in part one. We sketch how these solutions may provide an explanation of quark confinement, i.e. the lack of observation of the quark particles, suggested by Gell-Mann and others to be the basic building blocks of nuclear particles.

1. Classical nonlinear equations from physics.

I. 1 *Some examples.* Let us begin with typical equations from physics. We let M denote a subset of R^4 on which the equations are defined, and denote $x \in M$

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by space-time coordinates $x=(t, \vec{x})=(t, x_1, x_2, x_3)$. If we wish to ensure that solutions have limits at infinity, we replace R^4 by the four sphere S^4. We occasionally denote coordinates x_μ, $\mu=0, 1, 2, 3$, where $x_0=t$.

(i) *Nonlinear wave equations.* Let $P(s)$ denote a real polynomial of $s\in R^1$. The equation

$$\varphi_{tt}-\sum_{j=1}^{3}\varphi_{x_j x_j}+P'(\varphi)=0 \tag{1}$$

for the function $\varphi=\varphi(x)$ is a nonlinear wave equation with potential energy density $P(\varphi(x))$. Equations of the type (1) play a role in the description of meson interactions, as well as in the classical nonlinear oscillation of a membrane. Such equations have been studied extensively.

(ii) *Maxwell's equations.* The equations of classical electrodynamics, i.e. Maxwell's equations, describe a two-form F, the electromagnetic field,

$$F=\frac{1}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu. \tag{2}$$

(Here and in the following we use a summation convention.) The Maxwell equations are

$$dF=0,\quad d*F=*J, \tag{3}$$

where $*$ denotes Hodge duality, and where J is the "current" one form. Given J, these equations are linear. In case that the form F is exact, i.e. there is a one form A such that

$$F=dA, \tag{4}$$

then the equation $dF=0$ is automatically satisfied. In physics, A is called the "potential." Thus the solution of Maxwell's equations is reduced to the solution of the equation

$$d*dA=*J. \tag{5}$$

Any exact differential may be added to A yielding the same F, so the solution to (5) is not unique,

$$A'=A+d\Lambda,\quad F'=F. \tag{5'}$$

In physics, the arbitrariness (5') in A is called a "gauge transformation", and Maxwell's electromagnetic field is the simplest example of a gauge theory.

To concretely identify these equations, write Maxwell's equations in standard form, where the electric field $\vec{E}$ and magnetic field $\vec{B}$ have components defined as various components $F_{\mu\nu}$ of F. Writing these components as an antisymmetric matrix,

$$F_{\mu\nu}=\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{6}$$

Then the equation $dF=0$ can be written

$$\text{(7)} \qquad \operatorname{div}\vec{B}=0,\ \frac{\partial\vec{B}}{\partial t}+\operatorname{curl}\vec{E}=0,$$

while $d*F=*J$ is the pair of equations

$$\text{(8)} \qquad \operatorname{div}\vec{E}=\varrho,\ \operatorname{curl}\vec{B}-\frac{\partial\vec{E}}{\partial t}=\vec{J},$$

where $J=\varrho\,dt+\sum_{i=1}^{3}J_i\,dx^i$.

(iii) *Pure Yang–Mills equations.* These equations are a nonlinear generalization of the Maxwell equations. The generalization is most easily understood in a geometrical context. We can regard the potential A of Maxwell's equations as a connection form on a circle bundle over M. Locally the bundle is $M\times U(1)$, and the connection form $A=A_\mu(x)\,dx^\mu$ has sections with values in the Lie algebra $i\mathbf{R}^1$ of the circle group $U(1)$. The electromagnetic field $F=dA$ is the curvature defined by A. To obtain a Yang–Mills theory, we replace the circle bundle by a principal bundle $P(M, G)$. Locally P is the product $M\times G$, and the case $G=U(1)$ reduces to Maxwell's equations above. In general, we take G to be a compact Lie group.

The potential A is now defined to be a connection on P, i.e. to be a Lie algebra valued one form, $A=A_\mu(x)\,dx^\mu$, where the sections $A_\mu(x)$ belong to the Lie algebra $\mathcal{G}$ of G. A connection defines parallel transport of the fibre over a point $x\in M$, and a covariant exterior derivative D_A on Lie algebra valued forms, cf. [8], [25].

Such a differential form θ is tensorial if it transforms under the adjoint action of G, namely $\theta\to\theta^g=g^{-1}\theta g$, where $g: M\mapsto G$. Then D_A is covariant in the sense that for tensorial θ,

$$(D_A\theta)^g=D_{A^g}\theta^g,$$

where

$$\text{(9)} \qquad A^g=g^{-1}Ag+g^{-1}\,dg.$$

The transformation (9) is the generalization of the gauge transformation (5′); here the exact differential dA is generalized to the cocycle $g^{-1}\,dg$. In case $G=U(1)$, the formulas reduce to $D=d$, $A^g=A'$, and $\theta^g=\theta$. Groups $G\neq U(1)$ enter physics as "gauge groups" of particles. An extremely important example is the group of internal "color" symmetry, $G=\mathrm{SU}(3)$.

The generalization of Maxwell's equation (3) is obtained by replacing d by the covariant derivative D_A, namely the requirement that

$$\text{(10)} \qquad D_AF=0,\ D_A*F=*J.$$

Often these equations are considered in the "free space" case, meaning $J\equiv 0$. We restrict our attention to this problem. Thus we study $DF=0$, $D*F=0$, which can be regarded as the requirement that F be harmonic, $\Delta_AF=0$, where Δ_A is the covariant operator $\Delta_A=D_AD_A^*+D_A^*D_A$.

The assumption that F is the curvature form defined by the connection A on P, automatically assures that $D_A F=0$. In fact, this requirement means that

$$F = D_A A = dA + 1/2[A, A]. \tag{11}$$

Then, although $D_A^2 \neq 0$, in general, it is an identity that $D_A^2 A = D_A F = 0$. In fact this identity is the Jacobi identity of differential geometry, the generalization of $d^2=0$ to covariant differentiation.

With the assumption (11), the Yang–Mills equations are reduced to finding a curvature F satisfying

$$D_A * F = 0. \tag{12}$$

This equation, because of (11), can be regarded as a nonlinear system for A, namely

$$D_A * D_A A = 0.$$

In terms of components,

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0, \tag{13}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Here ∂_μ denotes $\partial/\partial x^\mu$.

(iv) *Particle physics.* The actual interaction of particles is described in physics by a Lagrangian combining the nonlinear wave equation, the Yang–Mills theory and a current J arising from a Dirac field. The mathematical abstraction of isolating parts of the complete system above is made to simplify the analysis. Ultimately the complete coupled system is of interest to physics. We now restrict attention to pure Yang–Mills theories with $J=0$.

I.2. *Action functionals.* Equations as (1) or (12) above can be derived from a variational principle ("action principle" in physics). The standard action functional $\mathscr{A}$ is defined as the integral of the Lagrangian density

$$\mathscr{A} = \int_M \mathscr{L}\, d\vec{x}\, dt. \tag{14}$$

In the case of equation (1) above,

$$\mathscr{L} = -1/2\, \partial_\mu \varphi\, \partial^\mu \varphi - P(\varphi). \tag{15}$$

In the case of the Yang–Mills equations (15), we let A be a connection and define

$$\mathscr{L}(A) = \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}, \tag{16}$$

$$\mathscr{A} = -\|F\|^2 = \operatorname{Tr} \int_M F \wedge * F.$$

In each case, the metric to raise or lower indices has the Minkowski signature $(-+++)$. The flat metric $ds^2 = -dt^2 + (d\vec{x})^2$ (special relativity) has been studied extensively; a general pseudo-Riemannian metric is the case of general relativity. We restrict attention here to the flat case. The requirement that $\mathscr{A}$ has vanishing first variation, is equivalent to the equations of motion (1) or (15). In classical physics the action plays no more fundamental role; however in section II we discover that the action has another, central significance in quantization.

Associated with each action functional above is a second functional, the *Euclidean* action

$$\mathscr{A}_E = \int_M \mathscr{L}_E \, d\vec{x}\, dt. \tag{17}$$

Here $\mathscr{L}_E$ differs from $\mathscr{L}$ only in that it is calculated with the Euclidean (flat) metric with signature $(++++)$. Here again, the requirement that the first variation of $\mathscr{A}_E$ vanishes is equivalent to variational equations. Unlike the case above, these equations are elliptic (rather than hyperbolic) on account of the metric. For the two cases above, they are

$$\Delta\varphi + P'(\varphi) = 0, \tag{1'}$$

$$D*F = 0, \tag{12'}$$

where $*$ depends upon the metric, and Δ is the Laplace operator.

In physics we are used to the role of classical hyperbolic equations of propagation. For the remainder of this talk, we consider the elliptic Yang–Mills case (12′). One might ask, "Why?" Aside from the mathematical appeal of the problem, there is an important role for these equations in physics: The elliptic problem provides a basis for understanding the quantization of the hyperbolic, classical equation. Here we study the mathematics of the elliptic, classical problem. In §§ II, III we discuss the application.

I.3. *Classical elliptic problems.* We return to the Yang–Mills equations (12′) with action $\mathscr{A}_E$ of the form (16). Two cases:

Case A. $F \in L_2$, $*F = \pm F$. This is the case of instantons (antiinstantons). The first explicit example of such a connection form was discovered in 1975 by Polyakov and coworkers [4]. Recently, Atiyah–Hitchin and independently Manin–Drinfeld gave explicit formulas for all solutions [2], [10], see also [6]. The condition $F = \pm *F$ means that any curvature (which must satisfy $DF=0$) automatically satisfies $D*F=0$. The condition $*F = \pm F$ becomes the integrability condition for a complex structure, and the classifications of these structures is dealt with by methods of algebraic geometry. See Atiyah in these PROCEEDINGS.

It has been conjectured that *all* solutions with $F \in L_2$ must satisfy $*F = \pm F$. In that case, all square-integrable solutions would already be known. A partial proof of this conjecture (i.e. that local maxima of $\mathscr{A}_E$ satisfy $*F = \pm F$) has been announced recently [5].

Case B. $F \notin L_2$. In case $F \notin L_2$, then $*F \neq \pm F$, in general. This problem is no longer purely algebraic, but involves hard analysis. These solutions appear to have interesting interpretations and applications. We shall concentrate here entirely on $F \notin L_2$. Only partial results are known for $F \notin L_2$, and many general questions remain open.

The non-L_2 character of F arises from a singular set S for F. We can regard (12′) as equation on $M = \mathbf{R}^4 \setminus S$, with specified growth near S. Alternatively we

can consider (12′) as an equation for generalized functions defined on R^4 with singular support S. The simplest cases are:

(i) S=*finite set of points* (dim $S=0$). In this case we interpret the solutions as having point charges at S. In known examples these charges are called "merons" and F has poles at S.

(ii) S=*curve* (dim $S=1$). This case of "line charges" can also be interpreted as dipole charges. This case has only been studied in detail for $G=U(1)$ and on a lattice [15].

(iii) S=2-*surface*. In this case the topology of $R^4\setminus S$ may be nontrivial, for example the fundamental group of $R^4\setminus S$ may not be the identity. In this case the Yang–Mills connections are said to have "vortices" on S. For lack of space, however, we shall not discuss them here. See [22], [11], [18].

I.4. *Parallel transport.* An important notion in the physical interpretation of gauge theories is the *holonomy,* defined by the connection A on the bundle P. Let C be a closed curve in $M=R^4\setminus S$, starting and ending at x. A point g in the fibre over x is moved by parallel transport along C into a point g'. Thus by varying g, parallel transport along C yields a mapping of the fibre over x into itself. This mapping $U(C)$ is by definition an element of the holonomy group of A with respect to P, and with base point x. (If C_1, C_2 are two such curves, and $C=C_1\circ C_2$ is their composition, then $U(C)=U(C_1)U(C_2)$.) The character, Tr $U(C)$, is in fact independent of the base point.

As parametric representation for C, take $s\in[0,1]$, $s\mapsto x(s)\in C$. The usual expression for $U(C)$ is the solutions to the differential equation

$$\frac{dV(s)}{ds}=A\big(X(s)\big)V(s),\quad V(0)=I,$$

where $X(s)$ is the tangent vector to C at $x(s)$. There is a special formula in the physics literature for $U(C)$, namely

$$U(C)=P\exp\Big(\int_C A\Big). \tag{18}$$

Here P denotes a "path ordered" or "multiplicative" integral. This notation is motivated by the fact that for smooth C,

$$U(C)=\lim_{n\to\infty}\exp\left(n^{-1}A_n\right)\exp\left(n^{-1}A_{n-1}\right)\ldots\exp\left(n^{-1}A_1\right),$$

where $A_k=A\big(X(k/n)\big)$.

I.5. S=*points in a plane.* We consider the special case where the singular set S consists of points lying in a place. Meron connections are characterized vanishing Chern class on M,

$$\mathrm{Tr}\,F\wedge F=0. \tag{19}$$

REMARK. All known examples have the property that for a curve C in the place of S, the holonomy is

$$U(C) = (-I)^n, \tag{20}$$

where n is the number of points enclosed by C.

We first consider the subcase that all the points of S lie on a line l. The construction of the connection A then simplifies by the use of cylindrical symmetry to eliminate redundant coordinates. In the case the group $G=\mathrm{SU}(2)$, the elliptic system $D*F=0$ of 12 equations simplifies to a single equation. We describe this case in a series of steps.

Step 1. *Reduction* [16]. Choose the t coordinate axis along l. Map the upper half plane $z=t+i|\vec{x}|$ onto the unit disc D in the standard conformal way and consider the equation

$$\Delta\Psi = \Psi^3 - \Psi \tag{21}$$

in D with $\Psi=\pm 1$ on ∂D. The discontinuities of Ψ occur exactly on the image on ∂D of the singular set S. We now denote this image by S. Here Δ is the Laplace–Beltrami operator on D with constant negative curvature,

$$\Delta = 1/4(1-|z|^2)^2\,\partial\bar{\partial}. \tag{22}$$

THEOREM 1. *Every solution to* (21) *yields an* SU (2) *Yang–Mills connection on M, given by*

$$A = 1/2(\Psi+1)|\vec{x}|^{-2}(\vec{x}\times d\vec{x})\cdot i\vec{\sigma}. \tag{23}$$

Here $i\vec{\sigma}$ are Lie algebra generators for SU (2).

Step 2. *Existence*; Jonsson, Zirilli, McBryan, Hubbard [24]. Let C denote the C^∞ functions on the interior of D which are continuous on $D\setminus S$ and which satisfy the boundary condition for Ψ as described above.

THEOREM 2. *There exists a solution* $\Psi\in C$ *to* (21).

The proof of this theorem involves the adaptation of variational methods to singular problems. The naive action functional is not defined on the solution Ψ to (21). Let S_ε denote an ε-neighborhood of S in D, and define

$$\mathscr{A}_\varepsilon(\Psi) = \int_{D\setminus S_\varepsilon} [|\partial\Psi|^2 + 2(1-|z|^2)^{-2}(\Psi^2-1)^2]\,dz\,d\bar{z}. \tag{24}$$

Although $\mathscr{A}_\varepsilon(\Psi)$ has no limit as $\varepsilon\to 0$, there exists a constant a_ε independent of Ψ such that

$$\mathscr{A}_{\mathrm{Ren}}(\Psi) = \lim_{\varepsilon\to 0}(\mathscr{A}_\varepsilon(\Psi) - na_\varepsilon),\quad n = \text{cardinality } S, \tag{25}$$

exists for $\Psi\in C$. The resulting "renormalized" functional $\mathscr{A}_{\mathrm{Ren}}$ is bounded from below and can be minimized to obtain a solution to (21). Uniqueness is still an interesting open question.

I mention two steps to generalize these results: In the case $G=\mathrm{SU}(3)$, there are solutions which are not SU (2) imbeddings. These solutions satisfy two coupled equations, as has been shown by Imbrie [23]. For general points S in a plane, Taubes [30] has reduced the 12-equation SU (2) system to a 4-equation system in a space of constant negative curvature. While only partial existence results have been established in this case, presumably existence can be proved. The boundary conditions will automatically yield the holonomy (20).

There are many open questions of interest both for geometry and for physics.

II. The constructive field program. Main goals of the constructive field program are to quantize classical equations, to prove existence of solutions to these equations (quantum fields) and to establish properties of these solutions. Ultimately, of course, one hopes to shed new light on physics, by dealing with issues such as those discussed in § III.

Quantum fields are not completely understood either as laws of nature or from a mathematical point of view. For this reason, mathematicians sometimes fail to appreciate the fundamental role that quantum fields play in physics. Physicists, however, are convinced that quantum fields provide an accurate picture of nature. The basic reasons are persuasive and close to the heart of physics. First, the rules for calculation given in field theory texts explain a variety of phenomena, from elementary particles, to atoms to macroscopic matter. Also, these rules yield numbers far more precise than any other physical theory—numbers which can be compared with observations made in the most accurate experiments on nature. In the case of the magnetic moment of the electron, for example, we have eleven decimal place agreement between calculational rules and observations. Both experiment and calculation in this case have been developed over a thirty year period.

This very success, however, places a great constraint on possible mathematical formulations of quantum field theory. In particular, when specialized to the case of these perturbation calculations, the mathematical theory must predict and reproduce the existing rules. This is the case with constructive field models, in that existing models have asymptotic expansions which agree with the rules (perturbation theory) of the textbooks. Presently, we have examples in two and three space-time dimensions, and a framework for a theory in four dimensions. Much of this has been joint work with J. Glimm and by our collaborators.

With this background, let us begin our discussion of quantization. The problem is to construct a one parameter unitary group $U(t)=\exp(-itH)$ on a Hilbert space $\mathscr{H}$, such that the fields are linear operators on $\mathscr{H}$. Also

$$\varphi(t+t_0) = U(t)^{-1}\varphi(t_0)U(t) \tag{26}$$

is the solution to the quantized field problem with initial data $\varphi(t_0)$.

A number of methods have been developed in the last ten years to solve such problems. I shall describe only one quantization procedure, starting from the Euclidean action functional $\mathscr{A}_E$. Basically it generalizes the Feynman–Kac repre-

sentation for the kernel of the heat operator $\exp(t\Delta - tV)$, written as a Wiener integral. Ultimately it yields a formula for $\exp(-tH)$.

For quantum fields one must replace Wiener integration by a measure on some space of generalized functions $\mathscr{F}$. We can take these functions as sections of a bundle associated with P, and take $M = \boldsymbol{R}^4$. In the case of the nonlinear wave equations, for example, we take $\mathscr{F} = \mathscr{S}'(M)$, the space of tempered distributions.

Without giving details, one defines a translation invariant probability measure $d\mu$ on $\mathscr{F}$, which formally has density $\exp(\mathscr{A}_E)$. Generally, the measure has the form

$$d\mu = \lim_{\varepsilon \to 0} Z(\varepsilon)^{-1} \exp(\mathscr{A}_E) \prod_{x \in M(\varepsilon)} dg(x).$$

Here $Z(\varepsilon)$ is a normalizing factor, and $M(\varepsilon)$ denotes a discretized M (e.g. a lattice approximation with lattice spacing ε). Also $dg(x)$ denotes Haar measure on the fibre over x. The hard work of constructive field theory goes into the estimates proving existence and establishing properties of such measures.

The construction of Osterwalder and Schrader [28], [27] gives simple sufficient conditions for the measure $d\mu$ to yield a quantum field theory. Under these conditions there exists a Hilbert space $\mathscr{H}$, a canonical projection π from $L_2(d\mu)$ to $\mathscr{H}$, and a commutative diagram which defines the Hamiltonian $\mathscr{H}$, cf. also [19].

$$\begin{array}{ccc} L_2(d\mu) & \xrightarrow{t\text{-translation}} & L_2(d\mu) \\ \pi \downarrow & & \downarrow \pi \\ \mathscr{H} & \xrightarrow{e^{-tH}} & \mathscr{H} \end{array}$$

This is the quantization of $\mathscr{A}$, since in the perturbation limit it agrees with the standard physics textbook quantization of $\mathscr{A}$.

The most powerful method to analyze $d\mu$ is based on expansions. In particular, these expansions have the character

(28) Quantum Theory = Classical + Fluctuation,

$$d\mu = d\mu_{\text{classical}} \times d\mu_{\text{fluctuation}}.$$

Here $d\mu_{\text{classical}}$ is the Gaussian measure about a stationary point of $\mathscr{A}_E$. Thus the classical Euclidean solutions of § I provide the zero order term in the expansion, and they yield approximate quantum fields.

Using these expansions, going by the generic title of cluster expansions, existence theorems and many properties of quantum field examples have been established. I mention in particular two results established since 1974, namely existence of dimension $d = 3$ quantum fields [11], [26] and nonuniqueness (existence of phase transitions) for $d = 2$ quantum fields [21], cf. [9]. Both these subjects have developed into whole areas, cf. [19], [20], [13], [12] for references.

Osterwalder and Seiler have established the existence of lattice gauge models [29]. However, little has been proved in the continuum limit. The analysis of classical Euclidean solution is the first of many steps in this constructive program.

III. Quark confinement. If quarks are the basic building blocks of protons, neutrons, mesons, etc., why are they not observed in the products of high energy particle collisions? This is one major unanswered question-of-principle in particle theory. Physicists propose that for nonabelian groups G, a gauge theory can provide the answer to this puzzle. The method is to show that pairs (and certain triples) of quarks are attracted to each other by a potential $V(L)$ which increases with distance L between the quarks. Thus if one attempts to separate the quarks, the attractive force between them increases, keeping them in a bound state (quark confinement). One would then only observe these bound states (protons, neutrons, mesons, etc.).

We now outline an argument that

$$V(L) \sim \alpha L, \; L \to \infty, \quad \alpha > 0. \tag{29}$$

An asymptotically linear potential (29) would ensure confinement. We reduce our discussion of (29) to three basic hypotheses. Establishing these three properties of a quantized gauge theory would yield a mathematical proof of (29). While these particular hypotheses are tentative and may not be correct, it is likely that a variation of the theme is correct. We present this argument because it illustrates a potentially important physical application of classical solutions to the Euclidean Yang–Mills problem, such as those described in § I. The proposal given here appears in [18], [17]. Other proposals can be found in [7], [22], [18].

The first hypothesis is that $V(L)$ can be computed from the expectation of the holonomy group. Let C be an $L \times T$ rectangle and define

$$V(L) = -\lim_{T \to \infty} \frac{1}{T} \ln \int \operatorname{Tr} U(C) \, d\mu. \tag{30}$$

K. Wilson [31] suggested that $V(L)$, defined by (30) and computed in a pure gauge theory, is the physically correct potential between heavy quarks in a gauge theory including both quarks and gauge particles (gluons). One can plausibly justify this assumption, so we accept (30) as our potential.

Our second hypothesis is that the solutions with holonomy (20), i.e. $U(C) = (-I)^n$, dominate the contribution to the integral in (30). In this case,

$$\int U(C) \, d\mu \sim \Pr(C)_+ - \Pr(C)_-, \; T \to \infty, \tag{31}$$

where $\Pr_\pm$ are the probabilities in measure $d\mu$ that C encloses an even or an odd number of points.

The third hypothesis is that the distribution of the number of points inside C obeys a Bernoulli-type distribution law with a given density. (Such a statistical

property is commonly true in statistical physics, e.g. in Ising-type models.) By this assumption

$$\Pr(C)_+ - \Pr(C)_- \sim \exp[-O(\text{Number of Points in} C)] \tag{32}$$

$$= \exp[-O(\text{Area } C)]$$

$$\sim \exp(-\alpha TL),$$

which ensures (29).

We thus confront the most challenging open mathematical problem: To understand qualitatively and quantitatively quantum fields in dimension four.

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Polynomial Interpolation*

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We discuss two recent developments in this ancient and basic subfield of Numerical Analysis.

1. Proof of the Bernstein and Erdős conjectures. Let $C[a, b]$ be the Banach space of continuous functions on the finite interval $[a, b]$ with the usual norm, $\|f\| := \max\{|f(x)|: a \leq x \leq b\}$, and let $n \geq 2$. Corresponding to each point t in the open simplex

$$T := \{t \in R^{n-1}: a < t_1 < \ldots < t_{n-1} < b\},$$

we construct the linear map P_t of polynomial interpolation in $C[a, b]$ at the $n+1$ points or nodes $a =: t_0, t_1, \ldots, t_n := b$. In its Lagrange form,

$$P_t f = f(t_0)l_0 + \ldots + f(t_n)l_n,$$

with

$$l_i(x) := \prod_{j \neq i} \frac{x - t_j}{t_i - t_j}, \quad i = 0, \ldots, n.$$

We wish to determine *optimal nodes,* i.e., a point or points $t^* \in T$ for which

$$\|P_{t^*}\| = \inf_{t \in T} \|P_t\|.$$

Here $\|P_t\| := \sup_{f \in C} \|P_t f\|/\|f\|$, as usual. Consideration of this problem is motivated by the fact that the interpolation error admits the sharp bound

$$\|f - P_t f\| \leq (1 + \|P_t\|) \operatorname{dist}(f, \pi_n)$$

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since P_t is a linear projector with range π_n, the subspace of polynomials of degree $\leq n$. Further, $\|P_t\|$ bounds the effect which errors in the data $f(t_i)$, $i=0, \ldots, n$, have on the interpolant constructed. In fact,

$$\|g\| \leq \|P_t\| \max_i |g(t_i)| \quad \text{for any } g \in \pi_n.$$

Thus (and this was Bernstein's motivation) a choice of t for which $\|P_t\|$ is small allows a "cheap" and effective bound on the size of a polynomial $g \in \pi_n$ in terms of just $n+1$ of its values.

The last inequality is connected with the well known observation that

$$\|P_t\| = \|\Lambda_t\|,$$

with

$$\Lambda_t := |l_0| + \ldots + |l_n|$$

the *Lebesgue function* of the process. (L. Fejér [8] lists Runge, Borel, Lebesgue, de la Vallée Poussin, Faber, S. Bernstein, D. Jackson, Tietze and Hahn, in that order, as having dealt with this function. Tietze, in turn, acknowledges J. Radon for having pointed out to him this formula for $\|P_t\|$.) It is easy to see that, for each $i=1, \ldots, n$, Λ_t has exactly one local maximum on the interval $[t_{i-1}, t_i]$, at τ_i, say,

$$\lambda_i(t) := \max_{t_{i-1} \leq x \leq t_i} \Lambda_t(x) = \Lambda_t(\tau_i), \quad i = 1, \ldots, n.$$

This allows the conclusion (apparently due to Morris and Cheney [13]) that

$$\partial \lambda_i / \partial t_j = -F_i'(t_j) l_j(\tau_i),$$

with $F_i := \sum_j (\text{signum } l_j(\tau_i)) l_j$, which shows the map

$$\Gamma: T \to \boldsymbol{R}^n: t \mapsto (\lambda_i(t))_1^n$$

to be smooth.

Bernstein [1] conjectured that $\|P_t\|$ *is minimal when* Λ_t *equioscillates,* i.e., when $\lambda_1(t) = \lambda_2(t) = \ldots = \lambda_n(t)$.

Erdős [7] added to this the conjecture that *there is exactly one choice of* t *for which* Λ_t *equioscillates and* that, *for all* t,

$$\min_i \lambda_i(t) \leq \inf_{s \in T} \|P_s\|.$$

Luttmann and Rivlin [12] and Cheney and Price [6] summarize subsequent work on these conjectures. At the last congress, Cheney [5] ended his talk with a description of some progress achieved by him and Kilgore.

In the meantime, Kilgore [10] has established what turned out to be the crucial result: An *optimal* Lebesgue function, i.e., a Λ_t for which $\|\Lambda_t\| = \inf_s \|\Lambda_s\|$, must necessarily equioscillate. On first sight, this is rather less than what is wanted since it only establishes the converse of the Bernstein conjecture. But, in the proof, Kilgore shows by an ingenious argument that all the $(n-1)$-minors in the Jacobian of Γ,

$$\partial \Gamma = (\partial \lambda_i / \partial t_j),$$

are nonzero on T. It is this fact which can be made to yield a proof of both the Bernstein and Erdős conjectures, as is shown in de Boor and Pinkus [2]. Kilgore [11] uses the fact (together with a result from de Boor and Pinkus [2]) in a different way for a proof of the Bernstein conjecture.

The problem of determining an optimal t is equivalent to finding the best approximation in $l_\infty(n)$ to the point $\mathbf{0}$ from the nonlinear (connected and smooth) manifold

$$M := \{\Gamma(t): t \in T\}.$$

Viewed this way, the Bernstein and Erdős conjectures generalize well known facts about best approximation in $l_\infty(n)$ from an $(n-1)$-dimensional linear Haar manifold, i.e., from a set

$$L := \{At+b: t \in R^{n-1}\}$$

with b some n-vector and A an $n\times(n-1)$ matrix all of whose $(n-1)$-minors are nonzero.

The specific results proved in de Boor and Pinkus [2] are as follows.

THEOREM 1. *The map* $\Delta: T \to R^{n-1}: t \mapsto (\lambda_{i+1}(t)-\lambda_i(t))_1^{n-1}$ *is a homeomorphism from* T *onto* R^{n-1}.

In particular, there is exactly one choice of t for which Λ_t equioscillates, and this, together with the converse of Bernstein's conjecture established by Kilgore, proves the Bernstein conjecture and the uniqueness assertion in Erdős' conjecture. The other part of Erdős' conjecture follows from

THEOREM 2. *If* $\lambda_i(t) \leqslant \lambda_i(s)$ *for* $i=1, \ldots, n$, *then* $t=s$.

The proof of Theorem 1 consists in verifying that the map Δ is a local homeomorphism (a consequence of Kilgore's result) which maps the boundary of T to the boundary of R^{n-1}, i.e., for which $\lim_{t \to \partial T} \|\Delta(t)\|_\infty = \infty$. The conclusion that therefore Δ must be a (global) homeomorphism onto R^{n-1} is called Palais' Theorem by some (because of Corollary 4.3 in Palais [15]), but can be found already in Browder [3].

Finally, similar results are proved for trigonometric interpolation, for which the uniform distribution of interpolation points is shown to be optimal.

Given the truth of the Bernstein conjecture, it is not all that hard to compute the optimal t^* for the first few n. But, Theorem 2 implies that such t^* is not of any practical importance. For, Brutman [4] has shown that, with

$$t_i^c := \left(a+b+(a-b)\left[\cos\frac{2i+1}{2n+2}\pi\right]\Big/\left(\cos\frac{\pi}{2n+2}\right)\right)\Big/2, \quad i=0, \ldots, n,$$

the zeros of the Chebyshev polynomial of degree $n+1$, adjusted to the interval $[a, b]$ in such a way that the first and last zero fall on the endpoints of the interval,

$$\max_i \lambda_i(t^c) - \min_i \lambda_i(t^c) < 0.201.$$

Numerical evidence strongly suggests that even

$$\max_i \lambda_i(t^c) - \min_i \lambda_i(t^c) < 0.0196$$

which would mean that this easily constructed node vector t^c gives rise to an interpolation map P_{t^c} whose norm is within 0.02 (and, in any case, within 0.201) of the best possible value for all n.

Since, for these expanded Chebyshev points, $\|P_{t^c}\| \sim (2/\pi) \ln n + 0.6$, this reaffirms interpolation at these points as a very good way to construct near best polynomial approximants (for values of n considered in practice). The construction of best polynomial approximants in several variables is much more expensive; hence such a way of constructing near best approximations in several variables would be very desirable. But, unfortunately, almost nothing is known about good, let alone optimal, points for polynomial interpolation in two or more variables. The space π_n of polynomials of total degree $\leq n$ in k variables has dimension $\binom{n+k}{n}$, but not much is known about how one might choose $\binom{n+k}{n}$ interpolation points so that, at least, an interpolant is defined. For this reason, it is intriguing to consider the following different and striking generalization of polynomial interpolation.

2. Kergin interpolation. The following theorem is proved in Kergin [9] as part of the author's thesis.

THEOREM (KERGIN). *Let $t_0, \ldots, t_n \in \mathbf{R}^k$, not necessarily distinct. There is one and only one map P on $C^{(n)}(\mathbf{R}^k)$ to π_n, the subspace of polynomials on $\mathbf{R}^k$ of total degree $\leq n$, with the following properties:*

(i) *P is linear.*

(ii) *For every $f \in C^{(n)}$, every m with $0 \leq m \leq n$, every mth degree homogeneous polynomial q_m in k variables, and every $J \subseteq \{0, \ldots, n\}$ with $|J| = m+1$, there exists a point in the convex hull of $\{t_j\}_{j \in J}$ at which $q_m(D)Pf$ matches $q_m(D)f$.*

Here,

$$q_m(D) = \sum_{|i|=m} \alpha_i (\partial/\partial x_1)^{i_1} \ldots (\partial/\partial x_k)^{i_k}$$

for certain numbers α_i.

In particular, we get for $m=0$ in (ii) that Pf must interpolate f at the points t_i. Also, if all the points t_i coincide, then Pf is necessarily the Taylor polynomial for f around that common point.

Kergin also proves that such a P is necessarily continuous on $C^{(n)}$. This allows him to restrict attention to f in

$$S := \{g \circ \lambda \colon g \in C^{(n)}(\mathbf{R}),\ \lambda \colon \mathbf{R}^k \to \mathbf{R} \text{ linear}\},$$

since span S is dense in $C^{(n)}$.

The matching conditions (ii) yield fairly easily that there can be at most one such map. The existence of such a map is, offhand, much harder to establish. Here is one way to do it which developed in discussions about Kergin's theorem with

M. Golomb, C. A. Micchelli and J. Robbin and which brings out the close relationship to univariate polynomial interpolation.

We use the abbreviations $\Delta x_i := x_{i+1} - x_i$ and

$$\int_{[x_0,\dots,x_m]} f := \int_0^1 \int_0^{t_1} \dots \int_0^{t_{m-1}} f(x_0 + t_1 \Delta x_0 + t_2 \Delta x_1 + \dots + t_m \Delta x_{m-1})\, dt_m \cdot dt_1.$$

In terms of this notation, the Hermite–Genocchi formula for the *mth* divided difference at the points $x_0, \dots, x_m \in \boldsymbol{R}$ of a function $g \in C^{(m)}(\boldsymbol{R})$ can be written

$$[x_0, \dots, x_m]g = \int_{[x_0,\dots,x_m]} g^{(m)}$$

(see, e.g., Nörlund [**14**; p. 16]). In particular, for $f = g \circ \lambda \in S$,

$$\int_{[x_0,\dots,x_m]} f = [\lambda x_0, \dots, \lambda x_m] g^{(-m)}.$$

Recall now that divided differences occur in the *Newton form* for the interpolating polynomial:

$$P_t g := \sum_{m=0}^{n} (\cdot - t_0)(\cdot - t_1) \dots (\cdot - t_{m-1})[t_0, \dots, t_m]g$$

is the polynomial of degree $\leqslant n$ which agrees with g at the points $t_0, \dots, t_n$ (repeats in the t_i's giving rise to matching of derivatives in the usual way). Recall further that therefore

$$(*) \qquad [t_{i_0}, \dots, t_{i_m}] P_t g = [t_{i_0}, \dots, t_{i_m}] g, \quad \text{all} \quad \{i_0, \dots, i_m\} \subseteq \{0, \dots, n\}.$$

Now consider the map P on $C^{(n)}(\boldsymbol{R}^k)$ given by the following formula due to C. A. Micchelli:

$$(Pf)(x) := \sum_{m=0}^{n} \int_{[t_0,\dots,t_m]} D_{x-t_0} D_{x-t_1} \dots D_{x-t_{m-1}} f,$$

with $D_y f := \sum_{r=1}^{k} y_r (\partial / \partial x_r) f$. The map P is clearly linear and continuous on $C^{(n)}$. We can therefore understand it by its action on S. For $f = g \circ \lambda \in S$,

$$D_{x-t_0} \dots D_{x-t_{m-1}} f = \lambda(x - t_0) \dots \lambda(x - t_{m-1}) g^{(m)} \circ \lambda.$$

Consequently,

$$P(g \circ \lambda)(x) = \sum_{m=0}^{n} \lambda(x - t_0) \dots \lambda(x - t_{m-1})[\lambda t_0, \dots, \lambda t_m] g,$$

i.e.,

$$(**) \qquad P(g \circ \lambda) = (P_{(\lambda t_i)} g) \circ \lambda.$$

This shows that P maps S into $\pi_n \cap S$; hence P maps $C^{(n)}$ into π_n. If now $q_m(x) = \sum_{|j|=m} \alpha_j x^j$ for some $0 \leqslant m \leqslant n$, and $\{i_0, \dots, i_m\} \subseteq \{0, \dots, n\}$, then for $f = g \circ \lambda \in S$ (with $\lambda x =: \sum_r \lambda_r x_r$)

$$\int_{[t_{i_0},\dots,t_{i_m}]} q_m(D)(f) = \int_{[\dots]} \sum_{|j|=m} \alpha_j \lambda^j g^{(m)} \circ \lambda = \sum_{|j|=m} \alpha_j \lambda^j [\lambda t_{i_0}, \dots, \lambda t_{i_m}] g,$$

hence (*) and (**) imply that

$$\int_{[t_{i_0},\dots,t_{i_m}]} q_m(D)f = \int_{[t_{i_0},\dots,t_{i_m}]} q_m(D)Pf$$

for all $f \in S$, hence for all $f \in C^{(n)}$.

This shows that P has all the properties demanded of the map in Kergin's theorem.

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Approximation des eindimensionalen Stefan-Problems durch finite Elemente

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0. Einführung. Die mathematische Formulierung zahlreicher in der Praxis auftretender Probleme führt auf Randwertprobleme — speziell für parabolische Differentialgleichungen — mit dem Charakteristikum, daß der Rand zum Teil nicht von vornherein gegeben ist, sondern von Eigenschaften der Lösung selbst abhängt. Das älteste solche „freie" Randwertproblem geht auf Stefan [1889] zurück. Bei einer Raumdimension lässt es sich so beschreiben: In dem Gebiet — o.B.d.A. sei $s(0)=1$ angenommen —

$$\Omega = \{(y,\tau) | \tau > 0 \wedge 0 < y < s(\tau)\} \tag{1}$$

ist nach einer Funktion U gefragt, welche die Gleichung

$$U_\tau - U_{yy} = 0 \quad \text{in } \Omega \tag{2}$$

erfüllt. Neben der Anfangsbedingung

$$U(y,0) = f(y) \quad \text{für} \quad y \in I = (0,1) \tag{3}$$

ist im einfachsten Fall am linken Rand

$$U_y(0,\tau) = 0 \quad \text{für} \quad \tau > 0 \tag{4}$$

vorgeschrieben. Längs des freien Randes $y=s(\tau)$ muss einerseits U verschwinden

$$U(s(\tau),\tau) = 0 \quad \text{für} \quad \tau > 0, \tag{5}$$

andererseits ist die Funktion $s(\tau)$ durch die zusätzliche Bedingung

$$s_\tau + U_y(s(\tau),\tau) = 0 \quad \text{für} \quad \tau > 0 \tag{6}$$

an die Lösung U gekoppelt. Das Schmelzen bzw. Gefrieren eines Eisblocks ist eine der physikalischen Interpretationen.

Freie Randwertprobleme haben in den vergangenen Jahren zunehmendes Interesse gefunden. Der Zusammenhang mit Variationsungleichungen wirkte als zusätzlicher Stimulus. Wir verweisen auf den Übersichsartikel von Magenes [1976], in dem auch der Zusammenhang mit weiteren freien Randwertproblemen dargestellt ist. Hinsichtlich numerischer Methoden zur Approximation der Lösung des Stefan-Problems sei auf die Literatur-Hinweise in Nitsche [1978] verwiesen.

In der letztgenannten Arbeit wurde eine Finite-Element-Methode vorgeschlagen, welche für „reguläre" Lösungen optimale Konvergenz sicherstellt, d.h. ist die Lösung hinreichend regulär, so ist die sich ergebende Konvergenz-Ordnung optimal. Bei dem hier betrachteten Problem ist die Regularität der Lösung nur abhängig von der Regularität der Anfangswerte einerseits und dem Erfülltsein von Kompatibilitätsbedingungen bei $y=0$ und $y=1$ andererseits. In aufsteigender Folge handelt es sich um die Bedingungen (i) $f(1)=0$, (ii) $f'(0)=0$, (iii) $f''(1)=f'^2(1)$ usw.

Selbst bei linearen parabolischen Problemen mit festen Rändern erfordert die Analyse der Konvergenz von Galerkin–Verfahren bei reduzierter Regularität der Anfangswerte und/oder nichterfüllten Kompatibilitätsbedingungen besondere Methoden. Wir verweisen auf die Arbeiten Babuska—Fix [1972], Bramble et al. [1977], Helfrich [1974], Thomee [1974]. Es ergibt sich, daß für positive τ-Werte auch dann optimale Konvergenz vorliegt; die auftretenden Koeffizienten divergieren jedoch gegen unendlich bei $\tau \to 0$.

Gegenstand dieser Note ist ein erster Schritt in der Herleitung derartiger Aussagen für das Stefan-Problem. Da sich dieses auch als nicht-lineares Problem mit festem Rand auffassen lässt — siehe # 1 — ist damit auch eine Möglichkeit zur „lokalen" Konvergenz — Analyse von Galerkin-Verfahren nicht-linearer parabolischer Aufgaben aufgezeigt.

1. Schwache Formulierung des Stefan-Problems, Finite-Element-Methode. Durch die Transformation

$$x = s^{-1}(\tau)y \tag{7}$$

wird das Stefan-Problem in ein solches für das Gebiet

$$Q = \{(x,\tau)|0 < x < 1 \wedge \tau > 0\}, \tag{8}$$

d.h. mit festem Rand, übergeführt. Wird zusätzlich anstelle von τ die neue Variable t gemäss

$$d\tau/dt = s^2(\tau),\ \tau(0) = 0 \tag{9}$$

eingeführt — vgl. Friedman [1976] — so genügt die Funktion $u(x,t)=U(y,\tau)$ der Differentialgleichung

$$u_{xx} - u_t = xu_x(1,t)u_x \quad \text{in } Q \tag{10}$$

und den weiteren Bedingungen

$$\begin{aligned} u_x(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned} \quad \text{für} \quad t>0, \tag{11}$$

$$u(x,0)=f(x) \quad \text{für} \quad x\in I.$$

Der freie Rand ergibt sich aus

$$ds/dt=-u_x(1,t)s \quad \text{mit} \quad s(0)=1. \tag{12}$$

Die Funktion $v=u_x$ liegt im Raum

$$\dot{H}_1=\{w|w\in H_1(I)\wedge w(0)=0\}. \tag{13}$$

Eine „schwache" Charakterisierung von v ist gegeben durch das

Problem P_v. Gesucht ist v mit $v(\cdot,t)\in\dot{H}_1$ derart, dass

$$(\dot{v},w)+(v',w')=v(1)(xv,w') \quad \text{für} \quad w\in\dot{H}_1 \quad \text{und} \quad t>0 \tag{14}$$

erfüllt ist zusammen mit der Anfangsbedingung

$$v(\cdot,0)=g:=f'. \tag{15}$$

Durch einen Punkt bzw. Strich ist dabei die Differentiation nach t bzw. x angezeigt; $(\cdot,\cdot)$ ist das $L_2(I)$-Skalarprodukt mit der Norm $\|\cdot\|$. Weiterhin ist zur Abkürzung $v(1)=v(1,t)$ benützt.

Die Finite-Element-Methode zu P_v bietet sich an: Es sei S_h ein Teilraum von $\dot{H}_1$, wir denken etwa an Splines mit verschwindenden Werten bei $x=0$. Die Näherung $v_h=v_h(\cdot,t)\in S_h$ wird durch (14) festgelegt, wobei jetzt w nur den Teilraum S_h durchläuft. Hinzu kommt die Anfangsbedingung

$$v_h(\cdot,0)=g_h \tag{16}$$

mit einer geeigneten Approximation $g_h\in S_h$ an g.

2. A priori Abschätzungen. Gemäss unserer Zielsetzung wollen wir den Fall einer reduzierten Regularität diskutieren. Demgemäss machen wir über $g=f'$ (15) nur die Voraussetzung

$$g\in L_2(I). \tag{17}$$

Als Approximation g_h (16) wählen wir die L_2-Projektion $g_h=P_hg\in S_h$ definiert durch

$$(g_h,\chi)=(g,\chi) \quad \text{für} \quad \chi\in S_h. \tag{18}$$

Es lässt sich zeigen

SATZ 1. *Es existiert ein nur von* $\|g\|$ *abhängiges* $T>0$, *so dass* v *und* v_h *samt allen Ableitungen nach* t *für* $t\in(0, T]$ *in* $L_2(I)$ *liegen. Dabei gilt:*

$$t^{2k}\{\|\partial_t^k v\|^2+\|\partial_t^k v_h\|^2\} \leqslant c, \tag{19}$$

$$\int_0^T t^{2k-1}\{\|\partial_t^k v\|^2+\|\partial_t^k v_h\|^2\}\,dt \leqslant c. \tag{20}$$

Hier wie später gibt c *Konstanten an, die nur von* $\|g\|$ *abhängen.*

Wir benötigen später Konvergenzeigenschaften der L_2-Projektion. Aus (19), (20) und der Tatsache, daß bis auf Terme niederer Ordnung die Ableitungen $\partial_t^k v$ und $\partial_t^{2k} v$ einander entsprechen, ergibt sich unmittelbar für den Fehler $\varepsilon=\varepsilon_h=v-P_h v$

$$\|\varepsilon\|^2 \leqslant c\,\mathrm{Min}\{h^{2k}t^{-k}|0 \leqslant k \leqslant r\}, \tag{21}$$

$$\int_0^T t^k\|\varepsilon\|^2\,dt \leqslant ch^{2k+2} \quad \text{für} \quad 0 \leqslant k \leqslant r-1. \tag{22}$$

Hierbei ist unterstellt, daß S_h ein Spline-Raum der Ordnung r ist (stückweise Polynome vom Grade $<r$).

In der Supremums-Norm $|\cdot|$ ergibt sich wegen der a priori Abschätzung $|w|^2 \leqslant 2\|w\|\,\|w'\|$ für $w\in\dot{H}_1$ speziell

$$|v|^2+|v_h|^2 \leqslant ct^{-1/2}. \tag{23}$$

3. Fehleranalyse. Der Fehler $e=e_h=v-v_h$ genügt der Beziehung

$$(\dot{e}, \chi)+(e', \chi') = v(1)(xe, \chi')+e(1)(xv_h, \chi) \quad \text{für} \quad \chi\in S_h. \tag{24}$$

Die Aufspaltung

$$e = (v-P_h v)-(v_h-P_h v) = \varepsilon-\Phi \tag{25}$$

liefert für den Korrektur-Term $\Phi\in S_h$ die definierende Relation

$$(\dot{\Phi}, \chi)+a(\Phi, \chi) = a(\varepsilon, \chi) \quad \text{für} \quad \chi\in S_h, \tag{26}$$

wobei die Bilinearform $a(\cdot, \cdot)$ durch

$$a(w, z) := (w', z')-v(1)(xw, z')-w(1)(xv_h, z') \tag{27}$$

erklärt ist. Die Wahl $\chi=\Phi$ führt auf die Abschätzung

$$\frac{d}{dt}(t\|\Phi\|^2)+t\|\Phi'\|^2 \leqslant c\|\Phi\|^2+ct\{\|\varepsilon'\|^2+t^{-1/2}\|\varepsilon\|^2\}. \tag{28}$$

Der Term $\|\Phi\|^2$ rechts läßt sich eliminieren. Dazu führen wir die durch

$$\begin{aligned} -w'' &= \Phi \quad \text{in } I, \\ w(0) &= w'(1) = 0 \end{aligned} \tag{29}$$

festgelegte Funktion $w=w(\cdot, t)\in\dot{H}_1$ und die Ritz-Approximation $\varphi=R_h w\in S_h$

ein. Es ergibt sich

$$\|\Phi\|^2 = (\Phi, -w'') = (\Phi', w') = (\Phi', \varphi') \tag{30}$$

$$= a(\Phi, \varphi) + v(1)(x\Phi, \varphi') + \Phi(1)(xv_h, \varphi')$$

und damit weiter

$$\|\Phi\|^2 = -(\dot{\Phi}, \varphi) + a(\varepsilon, \varphi) + v(1)(x\Phi, \varphi') + \Phi(1)(xv_h, \varphi'). \tag{31}$$

Wesentlich ist die Umformung des ersten Gliedes rechts:

$$-(\dot{\Phi}, \varphi) = (\dot{w}'', \varphi') = -(\dot{w}', \varphi') \tag{32}$$

$$= -(\dot{\varphi}', \varphi') = -\frac{1}{2}\frac{d}{dt}\|\varphi'\|^2.$$

Unter Heranziehung des Gronwallschen Lemma folgt aus (28) in Verbindung mit (30) nach einigen Umformungen der

SATZ 2. *Zu α mit $\alpha<1$ existieren Konstante $\varkappa, \gamma$ und c nur abhängig von α und $\|g\|$ derart, dass gilt*

$$\frac{d}{dt}\{t\|\Phi\|^2 + \varkappa e^{-\gamma t^{1-\alpha}}\|\varphi'\|^2\} + \{t\|\Phi'\|^2 + \|\Phi\|^2\} \tag{33}$$

$$\leqslant ct^{\alpha}\{\|\varepsilon'\|^2 + t^{-1/2}\|\varepsilon\|^2\}.$$

Unter Ausnützen der Abschätzungen für ε aus $\#2$ lässt sich folgern.

SATZ 3. *Der Fehler $e=v-v_h$ der Finite-Element-Methode für das Problem P_v genügt für $\alpha<1$ der Abschätzung*

$$\|e\| \leqslant c_\alpha h^\alpha t^{-1/2} \quad \text{für} \quad t\in(0, T]. \tag{34}$$

Von natürlichem Interesse ist die Approximation an den freien Rand. Indem — vgl. (12) — die Näherung s_h durch

$$\dot{s}_h = -v_h(1, \cdot)s_h, \; s_h(0) = 1 \tag{35}$$

definiert wird, ergibt sich der

SATZ 4. *Der Fehler $s-s_h$ für den freien Rand ist bei $\alpha<1$ durch*

$$|s-s_h| \leqslant c_\alpha h^\alpha \quad \text{für} \quad 0 \leqslant t \leqslant T \tag{36}$$

beschränkt.

Die Approximation $U_h(y, \tau)$ an die Lösung U des Stefan-Problems ergibt sich durch Integration von v_h bezüglich x und Rücktransformation $(x, t)\to(y, \tau)$ vermöge $y=xs_h(t_h(\tau))$, wobei $t_h(\tau)$ die Umkehrfunktion von $\tau_h(t)$ definiert durch — siehe (9) —

$$\dot{\tau}_h = s_h^2, \; \tau_h(0) = 0 \tag{37}$$

ist. Die Abweichung $U-U_h$ in der L_2-Norm ist wieder von der Ordnung $h^\alpha t^{-1/2}$, wobei in einer h^α-Umgebung des freien Randes U bzw. U_h geeignet zu extrapolieren ist.

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Eléments Finis et Dualité

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1. Introduction. De manière classique la méthode des éléments finis (M.E.F.) est basée sur un principe variationnel connu en elasticité sous le nom de principe du minimum de l'énergie potentielle. L'analyse numérique de cette M.E.F. est désormais bien connue: cf. par exemple les livres de Ciarlet [9] et de Strang et Fix [23]. Parallèlement, les ingénieurs ont popularisé d'autres M.E.F. basées sur des principes variationnels différents tels que le principe de l'énergie complémentaire, le principe d'Hellinger—Reissner etc., ... qui jouent un rôle important en élasticité: pour cela, cf. par exemple Oden et Reddy [15]. Ces M.E.F. ont une importance croissante dans les applications et leur analyse numérique a considérablement progressé ces toutes dernières années. Le but de cet article est de fournir une brève introduction aux développements mathématiques récents sur le sujet.

2. Un exemple simple. Considérons le problème suivant:

$$\begin{aligned} -\Delta p &= g \quad \text{dans } \Omega, \\ p &= 0 \quad \text{sur } \Gamma, \end{aligned} \tag{2.1}$$

où Ω est un ouvert borné de R^N de frontière Γ et où la fonction g est donnée dans $L^2(\Omega)$. Il est bien connu que la solution p de (2.1) peut être caractérisée à l'aide du principe variationnel

$$I(p) = \min_{q \in H_0^1(\Omega)} I(q), \quad I(q) = \int_\Omega \left\{ \frac{1}{2} |\mathbf{grad}\, q|^2 - gq \right\} dx. \tag{2.2}$$

En fait, dans les applications, on est souvent plus intéressé par la détermination de la fonction $\boldsymbol{u} = \mathbf{grad}\, p$ que par la connaissance de la fonction p elle-même. On cherche

donc à utiliser le principe suivant qui donne une caractérisation directe de u:

$$J(u) = \min_{v \in (L^2(\Omega))^N,\ \operatorname{div} v + g = 0} J(v), \quad J(v) = \frac{1}{2} \int_\Omega |v|^2 \, dx. \tag{2.3}$$

c'est la formulation *duale* du problème (2.1). En pratique, il n'est pas toujours aisé de satisfaire exactement la contrainte $\operatorname{div} u + g = 0$. Par dualisation de cette contrainte on vérifie facilement que le couple (u, p) est l'unique point-selle de la fonctionnelle quadratique

$$\mathcal{L}(v, q) = J(v) + \int_\Omega q(\operatorname{div} v + g) \, dx \tag{2.4}$$

sur l'espace produit $H(\operatorname{div}; \Omega) \times L^2(\Omega)$ où $H(\operatorname{div}; \Omega) = \{v \in (L^2(\Omega))^N;\ \operatorname{div} v \in L^2(\Omega)\}$. Ainsi p n'est autre que le *multiplicateur de Lagrange* associé à la contrainte $\operatorname{div} u + g = 0$. De manière équivalente, le couple (u, p) est caractérisé par les équations

$$\begin{aligned} &\int_\Omega \{u \cdot v + p \operatorname{div} v\} \, dx = 0 \quad \forall v \in H(\operatorname{div}; \Omega), \\ &\int_\Omega q(\operatorname{div} u + g) \, dx = 0 \qquad \forall q \in L^2(\Omega). \end{aligned} \tag{2.5}$$

Le principle variationnel (2.4) (ou (2.5)) est le fondement de la construction des M.E.F. *mixtes* de résolution du problème (2.1).

3. Une formulation générale des méthodes d'éléments finis. La situation précédente étant en fait très générale, nous allons introduire un principe variationnel abstrait et une méthode d'approximation associée qui englobent la plupart des exemples rencontrés dans les applications.

On se donne deux espaces de Hilbert (réels) X et W, et on note $(\cdot, \cdot)$ la dualité entre X et son dual X' ou entre W et son dual W'. On se donne en outre deux formes bilinéaires continues $a(\cdot, \cdot): X \times X \to \mathbf{R}$ et $b(\cdot, \cdot): X \times W \to \mathbf{R}$. On considère alors le problème suivant: étant donné $f \in X'$ et $\chi \in W'$, trouver un couple $(u, p) \in X \times W$ tel que

$$\begin{aligned} a(u, v) + b(v, p) &= (f, v) \quad \forall v \in X, \\ b(u, q) &= (\chi, q) \quad \forall q \in W. \end{aligned} \tag{3.1}$$

On introduit ensuite le sous-espace V de X défini par

$$V = \{v \in X;\ b(v, q) = 0 \ \forall q \in W\}. \tag{3.2}$$

Alors on a le résultat fondamental suivant qui est en fait un cas particulier d'un résultat plus général de Brezzi [4] (cf. aussi Babuška [2]).

THEOREME 1. *On suppose qu'il existe deux constantes* $\alpha, \beta > 0$ *telles que*

$$a(v, v) \geqslant \alpha \|v\|_X^2 \quad \forall v \in V, \tag{3.3}$$

$$\sup_{v \in X} \frac{|b(v, q)|}{\|v\|_X} \geqslant \beta \|q\|_W \quad \forall q \in W. \tag{3.4}$$

Alors le problème (3.1) *admet une solution unique* (u, p) *et l'application* $(f, \chi) \to (u, p)$ *est continue de* $X' \times W' \to X \times W$.

Donnons ensuite un cadre général de l'approximation du problème (3.1) inspiré de [8]. Introduisons deux autres espaces de Hilbert $\tilde{X}$ et $\tilde{W}$ tels que $X \subset \tilde{X}$, $\tilde{W} \subset W$ avec injections continues et denses puis deux formes bilinéaires continues $\tilde{a}(\cdot, \cdot): \tilde{X} \times \tilde{X} \to \boldsymbol{R}$ et $\tilde{b}(\cdot, \cdot): \tilde{X} \times \tilde{W} \to \boldsymbol{R}$ telles que

$$\tilde{a}(u, v) = a(u, v) \quad \forall u, v \in X, \ \tilde{b}(v, q) = b(v, q) \quad \forall v \in X, \quad \forall q \in \tilde{W}. \tag{3.5}$$

Une méthode d'approximation de (3.1) consiste alors à se donner deux espaces de dimensions finies X_h et W_h tels que

$$X_h \subset \tilde{X}, \ W_h \subset \tilde{W} \tag{3.6}$$

et, en supposant $f \in \tilde{X}'$, à chercher un couple $(u_h, p_h) \in X_h \times W_h$ solution de

$$\begin{aligned} \tilde{a}(u_h, v_h) + \tilde{b}(v_h, p_h) &= (f, v_h) \quad \forall v_h \in X_h, \\ \tilde{b}(u_h, q_h) &= (\chi, q_h) \quad \forall q_h \in W_h. \end{aligned} \tag{3.7}$$

Posons

$$V_h = \{v_h \in X_h; \ \tilde{b}(v_h, q_h) = 0 \ \forall q_h \in W_h\}. \tag{3.8}$$

On a le théorème d'approximation qui étend des résultats de [4], [8].

THEOREME 2. *On suppose que la solution* (u, p) *de* (3.1) *vérifie* $p \in \tilde{W}$ *et qu'il existe deux constantes* $\alpha_*, \beta_* > 0$ *indépendantes de* h *telles que*

$$\tilde{a}(v_h, v_h) \geqslant \alpha_* \|v_h\|_{\tilde{X}}^2 \quad \forall v_h \in V_h, \tag{3.9}$$

$$\sup_{v_h \in X_h} \frac{\tilde{b}(v_h, q_h)}{\|q_h\|_{\tilde{X}}} \geqslant \beta_* \|q_h\|_W \quad \forall q_h \in W_h. \tag{3.10}$$

Alors le problème (3.7) *admet une solution* (u_h, p_h) *unique et il existe une constante* $C > 0$ *indépendante de* h *telle que*

$$\|u - u_h\|_{\tilde{X}} + \|p - p_h\|_W \leqslant C \left\{ \inf_{v_h \in X_h} \left\{ \|u - v_h\|_{\tilde{X}} + \sup_{q_h \in W_h} \frac{|\tilde{b}(u - v_h, q_h)|}{\|q_h\|_W} \right\} + \inf_{q_h \in W_h} \|p - q_h\|_W \right\}. \tag{3.11}$$

Notons que si $\tilde{X} = X$, $\tilde{W} = W$, on a sous les hypothèses (3.9), (3.10) le résultat naturel d'approximation

$$\|u - u_h\|_X + \|p - p_h\|_W \leqslant C \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in W_h} \|p - q_h\|_W \right\}. \tag{3.12}$$

D'après le Théorème 2, il reste dans chaque cas — mais c'est là le point délicat! — à construire des formes $\tilde{a}(\cdot, \cdot)$, $\tilde{b}(\cdot, \cdot)$ et des espaces X_h, W_h ayant de bonnes propriétés d'approximation de façon à satisfaire les conditions (3.9) et (3.10). Notons d'ailleurs que si, dans les applications, l'hypothèse d'ellipticité est souvent

assez simple à vérifier, il n'en est pas de même du moins en général, de l'hypothèse de compatibilité (3.10) entre les espaces X_h et W_h (cf. [11] à ce sujet).

4. Applications. Nous allons illustrer les résultats généraux du paragraphe précédent à l'aide de deux examples représentatifs.

4.1. *Problèmes elliptiques d'ordre* 2. Reprenons l'exemple introduit au paragraphe 2. Il correspond à $X=H(\mathrm{div};\Omega)$, $W=L^2(\Omega)$, $a(u,v)=\int_\Omega u\cdot v\,dx$, $b(v,q)=$ $=\int_\Omega \mathrm{div}\, v\,dx$, $f=0$ et $(\chi,q)=-\int_\Omega qg\,dx$.

On suppose alors que $\mathscr{T}_h$ est une triangulation *régulière* de $\bar{\Omega}$ à l'aide de triangles K de diamètres $\leqslant h$. La M.E.F. *mixte* consiste à choisir $\tilde{X}=X$, $\tilde{W}=W$. Il s'agit alors de construire des sous-espaces de dimensions finies de $H(\mathrm{div};\Omega)$ et de $L^2(\Omega)$ formés de fonctions ayant un comportement polynomial dans chaque triangle K de la triangulation $\mathscr{T}_h$. Pour la construction explicite de tels espaces X_h et W_h vérifiant les conditions (3.8), (3.10) nous renvoyons à [18], [25]. L'inégalité (3.12) fournit alors une majoration de $p-p_h$ dans $L^2(\Omega)$. On peut en déduire une majoration dans $L^\infty(\Omega)$ à l'aide des techniques de Nitsche, cf. [20].

Par ailleurs, il faut noter que l'appartenance d'une fonction v à l'espace $X_h\subset H(\mathrm{div};\Omega)$ impose la continuité des traces normales $v\cdot n$ aux interfaces des éléments finis K. On peut également dualiser cette contrainte de continuité ce qui donne naissance aux M.E.F. mixtes hybrides, hybrides duales, équilibres etc. Pour tout cela, voir [3], [18], [19], [24], [25].

Toutes ces M.E.F. se généralisent facilement aux équations et systèmes d'équations elliptiques du 2ème ordre. Dans le cas du système de l'élasticité, on peut en outre dualiser la symétrie du tenseur des contraintes, cf. [1].

4.2. *Problèmes elliptiques d'ordre* 4. Donnons maintenant un exemple où $X_h\not\subset X$. Considérons alors le problème biharmonique

$$\begin{aligned} \Delta^2\psi &= f && \text{dans } \Omega,\\ \psi &= \partial\psi/\partial n = 0 && \text{sur } \Gamma, \end{aligned} \tag{4.1}$$

où la fonction f est donnée dans $H^{-1}(\Omega)$. *Une* formulation variationnelle du problème (4.1) consiste à poser

$$X = H_0^2(\Omega)\times L^2(\Omega),\ W = L^2(\Omega)$$

et pour $u=(\psi,\omega), v=(\varphi,\theta)\in X$

$$a(u,v)=\int_\Omega \omega\theta\,dx,\quad b(v,q)=-\int_\Omega q(\Delta\varphi+\theta)\,dx,\quad (f,v)=\langle f,\varphi\rangle,\quad \chi=0$$

où $\langle\cdot,\cdot\rangle$ désigne la dualité entre $H^{-1}(\Omega)$ et $H_0^1(\Omega)$. Le problème (3.1) correspondant admet alors une solution unique $(u=(\psi,\omega),p)$ où ψ est la solution de (4.1) et $\omega=p=-\Delta\psi$. On obtient ainsi une formulation *mixte* du problème (4.1).

Pour éviter d'avoir à construire des sous-espaces de dimensions finies de $H_0^2(\Omega)$, ce qui nécessite l'utilisation d'éléments finis C^1, on introduit les espaces et les formes bilinéaires (cf. [10])

$$\tilde{X} = H_0^1(\Omega)\times L^2(\Omega),\ \tilde{W} = H^1(\Omega),$$

$$\tilde{a}(u, v) = a(u, v),\ \tilde{b}(v, q) = \int_\Omega (\mathbf{rot}\,\varphi \cdot \mathbf{rot}\, q - \theta q)\, dx,$$

de sorte qu'il suffit de construire des sous-espaces de $H^1(\Omega)$ pour obtenir une méthode d'approximation.

Soit alors l un entier $\geqslant 1$; on pose par exemple:

$$W_h = \{q\in C^0(\bar{\Omega});\ q_{|K}\in P_l(K)\ \forall K\in\mathscr{C}_h\},\ X_h = (W_h\cap H_0^1(\Omega))\times W_h,$$

où $P_l(K)$ désigne l'espace des restrictions à K des polynômes de degré $\leqslant l$. Les conditions (3.9) et (3.10) sont alors vérifiées de sorte que le problème (3.7) admet une solution unique $(u_h=(\psi_h, \omega_h), p_h=\omega_h)$. En utilisant les techniques développées dans [21] pour évaluer le 2ème membre de (3.11), on obtient si la fonction ψ solution de (4.1) est assez régulière

$$\|\psi-\psi_h\|_{H^1(\Omega)}+h^{1/2}\|\omega-\omega_h\|_{L^2(\Omega)} = \begin{cases} O(h^l) & \text{si } l\geqslant 2,\\ O(h^{1-\varepsilon}) & \text{si } l=1,\end{cases} \tag{4.2}$$

ce qui améliore les résultats de [21].

Pour d'autres formulations variationnelles du problème (4.1) et d'autres M.E.F. mixtes, hybrides, ou équilibres, nous renvoyons à [7], [8], [16].

5. Problèmes non linéaires. Les méthodes d'approximation introduites plus haut peuvent s'étendre de manière naturelle aux problèmes non linéaires. Donnons deux exemples d'application.

5.1. *Inéquations variationnelles.* Considérons le problème unilatéral (problème de l'obstacle associé à (2.1)): trouver $p\in C$ solution de l'inéquation variationnelle

$$\int_\Omega \mathbf{grad}\, p\cdot\mathbf{grad}\,(q-p)\, dx \geqslant \int_\Omega g(q-p)\, dx \quad \forall q\in C, \tag{5.1}$$

où C est le convexe fermé supposé non vide $C=\{q\in H_0^1(\Omega);\ q\geqslant\psi$ p.p. dans $\Omega\}$.

Ici aussi on peut introduire une formulation variationnelle analogue à (2.5): le couple $(\boldsymbol{u}=\mathbf{grad}\, p, p-\psi)$ est l'unique élément de $H(\mathrm{div};\Omega)\times L^2_+(\Omega)$ tel que

$$\begin{aligned} &\int_\Omega (\boldsymbol{u}\cdot\boldsymbol{v}+p\,\mathrm{div}\,\boldsymbol{v})\, dx = 0 \quad \forall \boldsymbol{v}\in H(\mathrm{div};\Omega),\\ &\int_\Omega (q-(p-\psi))(\mathrm{div}\,\boldsymbol{u}+g)\leqslant 0 \quad \forall q\in L^2_+(\Omega), \end{aligned} \tag{5.2}$$

où $L^2_+(\Omega)$ est le cône $\{q\in L^2(\Omega); q\geqslant 0$ p.p. dans $\Omega\}$. La forme abstraite de ce

problème (5.2) consiste, avec les notations du § 3, à se donner un *cône* convexe K fermé non vide de W et à chercher un couple $(u, p) \in X \times K$ tel que

$$\begin{aligned} a(u, v) + b(v, p) &= (f, v) \qquad \forall v \in X, \\ b(u, q-p) &\leqslant (\chi, q-p) \quad \forall q \in K. \end{aligned} \tag{5.3}$$

Pour des résultats généraux sur l'approximation du problème (5.3) et sur la convergence de la M.E.F. pour les problèmes unilatéraux nous renvoyons à [5] (cf. aussi [22]). Pour l'approximation par des M.E.F. équilibres des problèmes unilatéraux en élasticité, voir [13]. La situation est beaucoup plus délicate lorsque le convexe K n'est pas un cône; c'est le cas en particulier des problèmes d'élastoplasticité (cf. [6], [14]).

5.2. *Equations de Navier-Stokes.* La M.E.F. mixte de résolution de (4.1) peut être étendue à l'approximation des équations de Navier-Stokes en dimension N=2 et en formulation fonction courant

$$\begin{aligned} &\nu \Delta^2 \psi - \frac{\partial}{\partial x_1}\left(\Delta\psi \frac{\partial \psi}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\Delta\psi \frac{\partial \psi}{\partial x_1}\right) = f \quad \text{dans } \Omega, \\ &\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{sur } \Gamma. \end{aligned} \tag{5.4}$$

Il suffit avec les notations du § 4.2 de remplacer $a(u, v)$ par

$$a(u, v) = \nu \int_\Omega \omega\theta \, dx + \int_\Omega \omega \left(\frac{\partial \psi}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \varphi}{\partial x_1}\right) dx.$$

On peut établir la convergence de la M.E.F. mixte dès que ψ est une solution *isolée* de (5.4) avec les mêmes ordres de convergence (4.2) que dans le cas linéaire (cf. [12]). Pour un panorama des M.E.F. de résolution des équations de Navier—Stokes, voir [17] et la bibliographie de cet article.

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Труды Международного Конгресса Математиков
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Численное Решение Задач Математической Физики

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Доклад посвящен, в основном, вопросам общей теории разностных методов и построения разностных схем для линейных и нелинейных задач. Излагаются результаты, полученные в последние годы автором и его сотрудниками.

1. Устойчивость операторно-разностных схем. Основная теорема. Рассмотрим, следуя [1], двухслойную операторно-разностную схему

$$B\frac{y_{n+1}-y_n}{\tau}+Ay_n=0,\ n=0,1,\ldots,\ \text{задан}\ y_0\in H, \tag{1}$$

с линейными ограниченными операторами A, B, зависящими от параметров $h, \tau>0$ и $t_n=n\tau$ и заданными в гильбертовом пространстве $H=H_h$. Теория устойчивости схем (1) разработана в [1]—[3]. Цель теории — найти необходимые и достаточные условия устойчивости и априорные оценки без предположений о структуре A и B. Схема (1) называется устойчивой в пространстве H_D со скалярным произведением $(y, v)_D=(Dy, v)$, где $D:H\to H$, $D^*=D>0$, если для решения задачи (1) и любых $y_0\in H$

$$(Dy_{n+1}, y_{n+1}) \leqslant (Dy_n, y_n)\| \text{ или } \|y_{n+1}\|_D \leqslant \|y_n\|_D. \tag{2}$$

Теорема 1. *Пусть A и B не зависят от n, $A^*=A>0$ и B^{-1} существует. Тогда для устойчивости схемы* (1) *в H_A, необходимо и достаточно, чтобы выполнялось операторное неравенство* $B_0=\mathrm{Re}\,B\geqslant 0.5\tau A$.

2. Устойчивость в случае несамосопряженных операторов A и B. Требуется указать свойства операторов A и B схемы (1), устойчивой в каком-либо прост-

ранстве H_D. Показано, например, что при $A^*=-A$, $B^*=B>0$ или $A^*=A>0$, $B^*=-B$ схема (1) не является устойчивой ни в одной из норм H_D (абсолютно неустойчива).

Теорема 2 ([4]). *Пусть* $A^*=A$, $B^*=B$ *и хотя бы один из операторов* A *или* B *положителен. Если схема* (1) *устойчива в каком-либо пространстве* H_D, *то выполняется операторное неравенство* $B\geqslant(\tau/2)A$.

Требование самосопряженности A и B существенно (имеется пример устойчивой схемы с $A^*=A>0$, не удовлетворяющей условию $B\geqslant(\tau/2)A$).

Теорема 3 (А. В. Гулин). *Пусть* $B=D+\tau AC$, $D^*=D>0$, *существуют операторы* C^{-1}, B^{-1}. *Тогда устойчивость* (1) *в* H_D *эквивалентна выполнению операторного неравенства* $A_0+\tau A^*((C-0.5E)D^{-1})_0A\geqslant 0$, *где* $A_0=\operatorname{Re}A=0.5(A^*+A)$, E — *единичный оператор. Если* $A^*=-A$, A^{-1} *существует, то условие* $DC+C^*D\geqslant D$ *необходимо и достаточно для устойчивости в* H_D.

Результаты теории могут быть использованы, например, для исследования устойчивости двухпараметрического семейства схем

$$u_t+\sigma_1\hat{P}_{\bar{x}}+(1-\sigma_1)P_{\bar{x}}=0,\quad \frac{1}{c_0^2}P_t+\sigma_2\hat{u}_x+(1-\sigma_2)u_x=0$$

(обозначения см. [1]), аппроксимирующих систему уравнений акустики $\partial u/\partial t+\partial P/\partial x=0$, $(1/c_0^2)\partial P/\partial t+\partial u/\partial x=0$. Найдены операторы D и C, при которых $B=D+\tau AC$; условие устойчивости $DC+C^*D\geqslant D$ приводит к неулучшаемым условиям для σ_1, σ_2:

$$\sigma_1+\sigma_2\geqslant 1,\quad 1+4\gamma^2(\sigma_1-0.5)(\sigma_2-0.5)\geqslant 0,\quad \gamma=\tau c_0/h.$$

Устойчивость трехслойных схем с несамосопряженными операторами исследована в [4].

4. Итерационные методы. Общая теория итерационных методов для решения линейного операторного уравнения $Au=f$, $A\colon H\to H$ в гильбертовом пространстве H изложена в [1], [3]. Двухслойный (одношаговый) итерационный метод записывается в канонической форме

$$B\frac{y_{k+1}-y_k}{\tau_{k+1}}+Ay_k=f,\quad k=0,1,2,\ldots,\quad \forall y_0\in H;\quad B\colon H\to H. \tag{3}$$

Если $A^*=A>0$, $B^*=B>0$ и заданы γ_1, γ_2 из условий $\gamma_1B\leqslant A\leqslant\gamma_2B$, то существует набор оптимальных параметров τ_1^*, τ_2^*, ..., τ_n^* при котором метод вычислительно устойчив и число итераций, гарантирующих точность $\varepsilon>0$, есть $n\geqslant n_0(\varepsilon)$, $n_0(\varepsilon)=\ln(2/\varepsilon)/2\xi$, $\xi=\gamma_1/\gamma_2$. Оператор B выбирается из условия максимума ξ и минимума числа операций при определении y_{n+1} из (3). Выбирая матрицу B в виде произведения нижней и верхней треугольных матриц (см. [1])

$$B=(E+\omega R_1)(E+\omega R_2),\quad R_1+R_2=A_1,\quad R_2^*=R_1,$$

получаем универсальный попеременно-треугольный метод (ПТМ) (А. А. Самарский, 1964; см. [1]). Так как $B^*=B>0$, то для ПТМ можно использовать оптимальный чебышевский набор $\{\tau_k^*\}$. Если предположить, что $A\geqslant\delta E$, $R_1R_2\leqslant(\Delta/4)A$, $\delta,\Delta>0$, то выбирая $\omega=2/\sqrt{\delta\Delta}$ и параметры $\{\tau_k^*\}$, получим

$$n_0(\varepsilon)\approx\ln\frac{2}{\varepsilon}/(2\sqrt{2}\sqrt[4]{\eta}),$$

где $\eta=\delta/\Delta$. В частности,

$$n_0(\varepsilon)\approx 0.28\ln\frac{2}{\varepsilon}/\sqrt{h}$$

в случае разностной задачи Дирихле для уравнения Пуассона в единичном p-мерном ($p\geqslant 2$) кубе на кубической сетке с шагом h.

А. Б. Кучеровым и Е. С. Николаевым в [5], [6] предложена модификация ПТМ:

$$B=(D+\omega R_1)D^{-1}(D+\omega R_2),\quad D^*=D>0;\quad R_1^*=R_2,\quad R_1+R_2=A.$$

Если $A\geqslant\delta D$, $\delta>0$, $R_1D^{-1}R_2\leqslant(\Delta/4)A$, то при $\omega=2/\sqrt{\delta\Delta}$ и $\{\tau_k=\tau_k^*\}$ оценка для $n_0(\varepsilon)$ остается в силе. В качестве $D=(d_{ij})$ можно взять диагональную матрицу, выбирая d_{ij} из условия максимума $\eta=\delta/\Delta$. Модифицированный ПТМ оказался весьма эффективным при решении разностной задачи Дирихле в произвольной области для уравнения $\operatorname{div}(k\operatorname{grad}u)=f$ с сильно меняющимся коэффициентом $k=k(x)$; так, в случае уравнения Пуассона число итераций практически совпадает с числом итераций для той же задачи в квадрате, сторона которого равна диаметру области. Для третьей краевой задачи и для уравнения со смешанными производными в прямоугольнике число итераций $n_0(\varepsilon)\approx((1/\sqrt{h})\ln 2)/\varepsilon$.

5. Разностные методы для задач с негладкими решениями. Производные решений эллиптических задач могут быть неограничены в окрестностях точек смены типа краевых условий, разрыва краевых значений и правой части, в угловых точках линии разрыва коэффициентов, в многоугольнике в окрестностях вершин углов и т.д. Это приводит к понижению погрешности аппроксимации и точности схемы. Возникают две проблемы: 1) исследование известных разностных схем для указанных задач; 2) разработка новых более эффективных методов решения.

Лаасонен в [7] показал, что разностная задача Дирихле для уравнения Лапласа в плоской области с углом $\nu\pi$ при $\nu>1$ в случае пятиточечной схемы и непрерывной граничной функции имеет погрешность $O(h^{2/\nu-2\varepsilon}/r^{1/\nu-\varepsilon})$, где h — шаг сетки, r — расстояние до вершины угла, $\varepsilon=\text{const}>0$ — любое число, и $O(h^{1/\nu-2\varepsilon}/r^{1/\nu-\varepsilon})$ в случае разрывной граничной функции.

Цикл работ по исследованию классических схем для отыскания негладких решений выполнен В. Б. Андреевым и Е. Ю. Архиповой в [8], [9]. В [8] изучена задача Дирихле для уравнения Лапласа в полуплоскости $y>0$ с граничным

условием $u=\operatorname{sign} x$, имеющим разрыв в начале координат, и показано, что в случае пятиточечной схемы ее погрешность есть $O(h/r)$, если разрыв не в узле сетки, и $O(h^2/r^2)$, если разрыв попадает в узел сетки. Указан способ повышения точности путем сглаживания (немонотонного) граничной функции. Для смешанной задачи Неймана—Дирихле получена оценка погрешности $O(h/\sqrt{r})$ и указан способ исправления краевых условий, при котором погрешность есть $O(h^2/r^{3/2})$.

Разработка новых более эффективных методов отыскания негладких решений идет по трем направлениям: 1) использование обычных схем с выбором в окрестности особых точек специальных полярных сеток с измельчением (по степенному закону) шага по радиусу (Е. А. Волков [10]); 2) выделение особенности и использование метода конечных элементов, что приводит к нелокальным сеточным аппроксимациям (Фикс и Стренг, [11]); 3) построение на обычных шаблонах разностных схем, несущих особенность (И. В. Фрязинов [12]). В [12] уравнение Лапласа $\Delta u=0$, $u=u(x_1, x_2)$ аппроксимируется разностной схемой $(a_1 y_{\bar{x}_1})_{x_1}+(a_2 y_{\bar{x}_2})_{x_2}=\varphi$ с переменными коэффициентами a_1, a_2, причем a_1 и a_2 отличны от 1, а φ — от нуля лишь в конечной окрестности особой точки. Главная часть разложения функции u в окрестности особой точки удовлетворяет разностному уравнению. Эта схема дает хорошую точность как в малой окрестности особой точки, так и вне нее. Аналогично строятся схемы для уравнения Пуассона в многоугольнике.

6. Векторные схемы ([13], [14]). Изучение И. В. Фрязиновым краевых задач для эллиптических уравнений общего вида в областях произвольной формы, а также экономичных схем для многомерных параболических уравнений привело к понятию векторной разностной схемы и существенно расширило класс разностных схем. Это расширение связано с построением схем в пространстве сеточных функций более сложной структуры, являющемся прямой суммой нескольких сеточных пространств. Пусть заданы сетки $w_h^{(s)}$, зависящие от h, и соответствующие пространства $H_h^{(s)}$ сеточных функций $y^{(s)}$, $s=1, 2, \ldots, m$. Пусть $H_h=H_h^{(1)}\oplus H_h^{(2)}\oplus\cdots\oplus H_h^{(m)}$ — прямая сумма пространств $H_h^{(s)}$ и вектор $\vec{y}=(y^{(1)}, y^{(2)}, \ldots, y^{(m)})\in H_h$ с компонентами $y^{(s)}\in H_h^{(s)}$; $s=1, 2, \ldots, m$.

Под векторной разностной схемой будем понимать операторное или операторно-разностное (разностное по аргументу $t_n=n\tau$, $n=0, 1, \ldots$) уравнение с операторами в H_h (см. [14]). Формальный прием построения векторных схем состоит в следующем. Задаче для исходного уравнения ставится в соответствие эквивалентная корректная задача для системы дифференциальных уравнений: единственным ее решением является вектор-функция, все компоненты которой совпадают с решением исходной задачи. Аппроксимируя каждое из уравнений этой системы на соответствующей сетке $w_h^{(s)}$, приходим к векторной схеме. Если векторная схема корректна, то каждая ее компонента приближает соответствующую компоненту эквивалентной системы, а значит

и исходной задачи. Семейство векторных схем существенно более широкое, чем семейство различных аппроксимаций исходной задачи для одного уравнения на одной сетке. Большое число векторных схем для эллиптических и параболических уравнений построено И. В. Фрязиновым. Сюда относятся, в частности, экономичные аддитивные схемы, обладающие аппроксимацией в суммарном смысле [1], для решения многомерных уравнений параболического типа.

В случае нерегулярных сеток разностные схемы для эллиптического уравнения

$$Lu = \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x)\frac{\partial u}{\partial x_\beta}\right) = -f(x), \quad x \in (x_1, x_2),$$

обычно строятся методом конечных элементов. Однако эти схемы (особенно простейшие из них) можно, как показано И. В. Фрязиновым, получить методом баланса (интегроинтерполяционным методом [1]). Они являются векторными схемами.

7. Вариационно-разностные схемы для уравнений газодинамики и магнитной гидродинамики (МГД). Для этих уравнений хорошей точностью обладают полностью консервативные схемы [15], для которых на сетке выполняется не только закон сохранения полной энергии, но и уравнения баланса внутренней и кинетической энергии, а также энергии электромагнитного поля. Для вывода полностью консервативных разностных схем в переменных Лагранжа эффективным оказался вариационный подход, основанный на полудискретном аналоге принципа наименьшего действия Гамильтона—Остроградского [16], [17]. Получающиеся при этом вариационно-разностные схемы (ВРС) имеют второй порядок аппроксимации на гладких решениях, обладают полной консервативностью, легко алгоритмизируются. ВРС имеют ковариантную форму записи, так что при переходе от одной системы координат к другой изменяется только вид выражения объема лагранжевой ячейки как функции координат ее вершин. Это обеспечивает возможность расчета течений в системах координат, автоматически подстаривающихся к рассчитываемому потоку. ВРС позволяют ввести искусственную вязкость в ковариантной форме, не зависящей ни от числа измерений, ни от системы координат [18]. Для других часто встречающихся многомерных уравнений математической физики (теплопроводности, диффузии магнитного поля и т. д.) также могут быть получены разностные схемы, имеющие ковариантную форму записи, обладающие вторым порядком аппроксимации и сохраняющие свойство самосопряженности и знакоопределенности.

ВРС для многомерных МГД уравнений с теплопроводностью позволяет решить ряд модельных и прикладных задач в различных системах координат: о развитии неустойчивости Рэлея—Тейлора в несжимаемой жидкости [19], о магнитной кумуляции [20], о симметрии сферических мишеней при сжатии их лазерным излучением [21] и др.

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Galerkin-Finite Element Methods for Parabolic Equations

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The purpose of this paper is to present a survey of error estimates for Galerkin-finite element methods applied to parabolic initial-boundary value problems. In doing so we shall depend on known results pertaining to the corresponding elliptic problem. We shall concentrate on the error originating from the discretization in the space variables and only quote at the end some work related to the discretization in time.

Before we state our parabolic problem we consider briefly the elliptic problem

$$Au \equiv -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right)+a_0 u = f \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in R^N and where the coefficients are smooth with (a_{jk}) positive definite and a_0 nonnegative in $\bar{\Omega}$. This problem may also be stated in weak form: Find $u \in H_0^1(\Omega)$ such that

$$A(u,\varphi) = (f,\varphi) \quad \text{for} \quad \varphi \in H_0^1(\Omega),$$

where

$$A(u,v) = \int_\Omega \left(\sum_{j,k=1}^{N} a_{jk}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_j}+a_0 uv\right)dx, \quad (u,v) = \int_\Omega uv\,dx.$$

Let $\{S_h\}$ denote a family of finite dimensional subspaces of $H_0^1(\Omega)$, depending on the "small" parameter h, with the property that for some integer $r \geqslant 2$,

$$\inf_{\chi \in S_h} \{\|w-\chi\|+h\|w-\chi\|_1\} \leqslant Ch^r\|w\|_r \quad \text{for} \quad w \in H_0^1(\Omega)\cap H^r(\Omega),$$

where $\|\cdot\|_s$ denotes the norm in $H^s(\Omega)$ and $\|\cdot\| = \|\cdot\|_0$. A simple example (with $r=2$) of such a family is obtained by approximating the domain Ω from the interior

by a union Ω_h of triangles with diameter at most h, and considering continuous functions which are linear on each triangle and vanish outside Ω_h. More generally, one may consider continuous functions which reduce to polynomials of degree $r-1$ on triangles. Nontrivial modifications near the boundary are then often necessary.

The "standard" Galerkin-finite element method for our boundary value problem is then to find $u_h \in S_h$ such that

$$A(u_h, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h.$$

Setting $e = u_h - u$ we have at once $A(e, \chi) = 0$ for $\chi \in S_h$ and hence

$$C^{-1}\|e\|_1^2 \leqslant A(e, e) = A(e, \chi - u) \leqslant C\|e\|_1 \inf_{\chi \in S_h} \|u - \chi\|_1,$$

so that by our assumptions on $\{S_h\}$,

$$\|e\|_1 \leqslant Ch^{r-1}\|u\|_r \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^r(\Omega).$$

A famous duality argument by Aubin [1], Nitsche [18] and Oganesjan and Ruchovetz [23] shows the L_2 estimate needed to conclude the optimal order error estimate

$$\|e\| + h\|e\|_1 \leqslant Ch^r\|u\|_r \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^r(\Omega).$$

We now turn to our main target, the initial boundary value problem ($u_t = \partial u/\partial t$, $R_+ = \{t \geqslant 0\}$).

(1) $u_t + Au = f$ in $\Omega \times R_+$, $u = 0$ on $\partial\Omega \times R_+$, $u(x, 0) = v(x)$ in Ω,

which we write in weak form, with $u(\cdot, t) \in H_0^1(\Omega)$,

$$(u_t, \varphi) + A(u, \varphi) = (f, \varphi) \quad \text{for} \quad \varphi \in H_0^1(\Omega).$$

The corresponding "standard" Galerkin-finite element semidiscrete problem is then to find $u_h(t) \in S_h$ such that

(2) $$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h, \quad t \geqslant 0, \quad u_h(0) = v_h,$$

where v_h is a suitable approximation of v in S_h. This may be considered as an initialvalue problem for a system of ordinary differential equations in the coefficients of u_h with respect to some basis in S_h.

Error estimates for (2) were given in e.g. Price and Varga [24], Douglas and Dupont [11], Fix and Nassif [15], Wheeler [32] and Dupont [14]. We show the following for $e = u_h - u$.

THEOREM 0. *For u sufficiently smooth in $\Omega \times [0, t_0]$ and with a suitable choice of v_h we have*

$$\|e(t)\| + h\|e(t)\|_1 \leqslant C(u)h^r \quad \textit{for} \quad 0 \leqslant t \leqslant t_0.$$

PROOF. Following Wheeler [30] we define the elliptic projection $P_1 : H_0^1(\Omega) \to S_h$ by $A(P_1 u - u, \chi) = 0$ for $\chi \in S_h$. By above we then have

(3) $$\|P_1 u - u\| + h\|P_1 u - u\|_1 \leqslant Ch^r\|u\|_r \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^r(\Omega).$$

In the parabolic case, set $\theta=u_h-P_1u$ and $\varrho=P_1u-u$. By our definitions we have

$$(\theta_t,\chi)+A(\theta,\chi)=-(\varrho_t,\chi) \quad \text{for} \quad \chi\in S_h.$$

Choosing in particular $\chi=\theta_t$ we find

$$\|\theta_t\|^2+\frac{1}{2}\frac{d}{dt}A(\theta,\theta)=-(\varrho_t,\theta_t)\leqslant\frac{1}{2}\|\varrho_t\|^2+\frac{1}{2}\|\theta_t\|^2.$$

Hence after integration, with $v_h=P_1v$ so that $\theta(0)=0$, in view of (3),

$$C^{-1}\|\theta\|_1^2\leqslant A(\theta,\theta)(t)\leqslant A(\theta,\theta)(0)+\int_0^t\|\varrho_t\|^2\,d\tau\leqslant C(u)h^{2r}.$$

This completes the proof since $e=\theta+\varrho$ so that, using (3) once more,

$$\|e\|+h\|e\|_1\leqslant\|\varrho\|+h\|\varrho\|_1+\|\theta\|_1\leqslant C(u)h^r.$$

In the rest of this paper we shall, following Bramble, Schatz, Thomée and Wahlbin [8], write the semidiscrete equation in a somewhat different form. Thus let $T_h\colon L_2(\Omega)\to S_h$ denote the solution operator of the discrete elliptic problem, defined by

$$A(T_hf,\chi)=(f,\chi) \quad \text{for} \quad \chi\in S_h. \tag{4}$$

The semidiscrete problem (2) may then be written

$$T_hu_{h,t}+u_h=T_hf \quad \text{for} \quad t\geqslant 0, \quad u_h(0)=v_h. \tag{5}$$

The continuous problem (1) may analogously be put into the form

$$Tu_t+u=Tf \quad \text{for} \quad t\geqslant 0, \quad u(0)=v, \tag{6}$$

where $T=A^{-1}$. The operator T_h has the properties:

(i) T_h is selfadjoint, positive semidefinite on $L_2(\Omega)$ and positive definite on S_h,

(ii) There is an integer $r\geqslant 2$ such that

$$\|T_hf-Tf\|\leqslant Ch^s\|f\|_{s-2} \quad \text{for} \quad 2\leqslant s\leqslant r.$$

We may now consider the discrete problem (5) assuming only that T_h is an approximate solution operator of the elliptic problem satisfying (i) and (ii). In this fashion we also include into our considerations methods other than the standard Galerkin method described above. For instance, one may cover situations when the functions in S_h cannot easily be made to satisfy the homogeneous boundary conditions. One way of dealing with such a situation, which is contained in the above framework, is to use in the discrete problem rather than the bilinear form $A(\cdot,\cdot)$ a form with boundary terms, such as the following form proposed by Nitsche [19],

$$B_h(v,w)=A(v,w)-\left\langle v,\frac{\partial w}{\partial\nu}\right\rangle-\left\langle\frac{\partial v}{\partial\nu},w\right\rangle+\beta h^{-1}\langle v,w\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L_2(\partial\Omega)$, $\partial/\partial\nu$ the conormal derivative on $\partial\Omega$ and β a positive constant. Another method included is the Lagrange

multiplier method of Babuška [2] which employs a separate family of approximating functions on $\partial\Omega$.

Subtracting (6) from (5) we find for the error

$$T_h e_t + e = \varrho \equiv (T_h - T)Au = (P_1 - I)u, \tag{7}$$

where the elliptic projection is now defined by $P_1 = T_h A$. Using the properties (i) and (ii) one proves easily by the energy method (cf. [8]):

THEOREM 1. *We have for* $t \geqslant 0$,

$$\|e(t)\| \leqslant C\Big\{\|e(0)\| + h^r\Big[\|v\|_r + \int_0^t \|u_t\|_r \, d\tau\Big]\Big\}.$$

In particular, for the homogeneous equation ($f=0$) with $v_h = P_1 v$ or $P_0 v$ (P_0 denotes the L_2-projection onto S_h), we find under the appropriate compatibility conditions on v at $\partial\Omega$,

$$\|e(t)\| \leqslant C_\varepsilon h^r \|v\|_{r+\varepsilon};$$

a somewhat more precise argument yields this inequality with $\varepsilon = 0$. In this case, it is in fact possible to show convergence of order r, even for time derivatives, under much weaker regularity assumptions than above, when t is bounded away from zero (cf. [8]).

THEOREM 2. *Let* $j \geqslant 0$ *and* $v_h = P_0 v$. *Then for the homogeneous equation,*

$$\|D_t^j e(t)\| \leqslant C h^r t^{-r/2-j} \|v\| \quad (D_t = \partial/\partial t).$$

Results of this nature were also discussed by spectral representations in Blair [4], Thomée [28], Helfrich [17], Fujita and Mizutani [16] and recently by the energy method in Sammon [25].

The estimate of Theorem 2 for the homogeneous equation may be combined with Theorem 1 to derive error estimates for the nonhomogeneous equation for t bounded away from zero, which require smoothness of the solution only near t (cf. [30]).

THEOREM 3. *Let* $j \geqslant 0$ *and* $v_h = P_0 v$. *Then for the general nonhomogeneous equation, for* $t \geqslant \delta > 0$,

$$\|D_t^j e(t)\| \leqslant C h^r \Big\{\sum_{l=0}^{j} \|D_t^l u(t)\|_r + \int_{t-\delta}^{t} \|D_t^{j+1} u\|_r \, d\tau + \|v\| + \int_0^t \|f\| \, d\tau\Big\}.$$

In one application below, we shall need an error estimate in H^1 (cf. [31]).

THEOREM 4. *Consider the standard Galerkin method* (2) *for the nonhomogeneous equation and let* $v_h = P_0 v$. *Then for* $t \geqslant \delta > 0$,

$$\|D_t^j e(t)\|_1 \leqslant C h^{r-1} \Big\{\sum_{l=0}^{j} \sup_{t-\delta \leqq \tau \leqq t} \|D_t^l u(\tau)\|_r + \Big(\int_{t-\delta}^{t} \|D_t^{j+1} u\|_{r-1}^2 \, d\tau\Big)^{1/2}\Big\}$$

$$+ C h^r \Big\{\|v\| + \int_0^t \|f\| \, d\tau\Big\}.$$

The above estimate in H^1 shows an order of approximation in the gradient of the solution which is one order less than that for u itself. We shall now present a result from [30] which shows that if the finite element spaces are based on uniform partitions in a specific sense (which we shall here only refer to as uniform) in the interior domain Ω_0, then difference quotients of u_h may be used to approximate any derivative of u in the interior of Ω_0 to order $O(h^r)$. This generalizes a result in the elliptic case by Bramble, Nitsche and Schatz [6]. In addition to the global norms used above we use for $\tilde{\Omega}\subset\Omega$ the norms $|\cdot|_{\tilde{\Omega}}$ and $\|\cdot\|_{s,\tilde{\Omega}}$ in $L^\infty(\tilde{\Omega})$ and $H^s(\tilde{\Omega})$, respectively, and set $N_0=[N/2]+1$.

THEOREM 5. *Let S_h be uniform on $\Omega_0\subset\Omega$ and assume that T_h is such that*

$$A(T_h f,\chi)=(f,\chi) \quad \text{for} \quad \chi\in S_h \quad \text{with} \quad \operatorname{supp}\chi\subset\Omega_0.$$

Let $v_h=P_0 v$ and let Q_h be a finite difference operator approximating D^α with order of accuracy r. Then for $t\geqslant\delta>0$, $\Omega_2\subset\subset\Omega_1\subset\subset\Omega_0$,

$$|Q_h u_h(t)-D^\alpha u(t)|_{\Omega_2}\leqslant Ch^r\Big\{\sup_{t-\delta\leqq\tau\leqq t}\|u(\tau)\|_{r+|\alpha|+N_0,\Omega_1}$$

$$+\Big(\int_{t-\delta}^{t}(\|u_t\|^2_{r+|\alpha|+N_0-1,\Omega_1}+\|u\|^2_r+\|f\|^2)\,d\tau\Big)^{1/2}+\|v\|+\int_0^t\|f\|\,d\tau\Big\}.$$

Notice the local character of the stringent regularity assumptions.

We shall now turn to global estimates in the maximum-norm and denote by $|\cdot|$ and $|\cdot|_r$ the norms in $L_\infty(\Omega)$ and $W^r_\infty(\Omega)$, respectively. The following result was proved in [8] (for $N=1$, cf. [33]).

THEOREM 6. *Assume that T_h satisfies*

$$|T_h w|\leqslant C|Tw|_1, \quad \|T_h w\|\leqslant C\|Tw\|_1.$$

Then for $t\geqslant 0$ we have

$$|e(t)|\leqslant C\Big\{\sum_{j=0}^{N_0-1}|(I-P_1)D_t^j u(t)|+\|D_t^{N_0}e(t)\|\Big\}.$$

The proof consists of a simple iteration argument using the error equation (7) and noticing that T_h is a bounded operator from $L_q(\Omega)$ to $L_p(\Omega)$ if $0<q^{-1}-p^{-1}<N^{-1}$.

Combining this with a property such as

$$|(I-P_1)v|\leqslant Ch^r(\log h^{-1})^{\delta_{2,r}}|v|_r$$

(for a survey of such estimates, see Nitsche [20]) and the above estimates for time derivatives we have under the appropriate assumptions, for $t\geqslant\delta>0$,

$$|e(t)|\leqslant C(u)h^r(\log h^{-1})^{\delta_{2,r}}.$$

Using weighted norms, Nitsche [21] (cf. also Dobrowolski [10]) proved the following result which is uniform for small t and in which the number of derivatives entering is independent of N. Here we are concerned with the standard Galerkin method

with $A=-\Delta$ and the subspaces are assumed to consist of C^0 piecewise polynomials of degree $r-1$ on a quasiuniform partition into simplices, or isoparametric modifications.

THEOREM 7. *For T_h and S_h as stated and with $v_h=P_1v$, $r\geqslant 3$, we have for any N,*

$$|e(t)|\leqslant Ch^r\Big\{|u(t)|_r+|u_t(t)|_r+\Big(\int_0^t|u_{tt}|_r^2\,d\tau\Big)^{1/2}\Big\}.$$

Recent work [26] shows that under the present types of assumptions the following discrete a priori estimate holds for solutions of the homogeneous semidiscrete equation (for $N\geqslant 5$ under an additional assumption about the discrete elliptic problem), namely

$$|u_h(t)|\leqslant C(\log h^{-1})^{p_N}|v_h|.$$

This has as a consequence for the error in the nonhomogeneous problem:

THEOREM 8. *Under the above assumptions, and with $v_h=P_1v$, we have*

$$|e(t)|\leqslant Ch^r\Big(\log\frac{1}{h}\Big)^{p_N+\delta_{2,r}}\Big\{|u(t)|_r+\int_0^t|u_t|_r\,d\tau\Big\}.$$

In the analysis of different finite element methods for elliptic problems the duality argument quoted above for showing L_2 error estimates from the basic H^1 estimates, also yields error estimates in negative norms. To state such an estimate, set for $s\geqslant 0$, $\|v\|_{-s}=(T^sv,v)^{1/2}$. This norm can be shown equivalent to

$$\sup\Big\{\frac{(v,\varphi)}{\|\varphi\|_s};\ \varphi\in C^\infty(\bar\Omega),\ A^j\varphi=0\ \text{ on }\partial\Omega\ \text{ for }\ j<s/2\Big\}.$$

The negative norm estimate for the elliptic problem can then be expressed as

$$\|(I-P_1)u\|_{-(r-2)}\leqslant Ch^{2r-2}\|u\|_r\quad\text{for}\quad u\in H_0^1(\Omega)\cap H^r(\Omega),$$

and thus shows convergence in $\|\cdot\|_{-(r-2)}$ which is of higher order than that in the L_2-norm if $r>2$. In the rest of the paper we now assume this estimate to hold in addition to (ii) or that now

(ii′) $$\|T_hf-Tf\|_{-p}\leqslant Ch^{p+q+2}\|f\|_q,\ 0\leqslant p,\ q\leqslant r-2.$$

One may show similar estimates for the parabolic problem (cf. [31]).

THEOREM 9. *With $v_h=P_0v$ or P_1v we have for $t\geqslant 0$,*

$$\|e(t)\|_{-(r-2)}\leqslant Ch^{2r-2}\Big\{\|v\|_r+\int_0^t\|u_t\|_r\,d\tau\Big\}.$$

For the purpose of proof, one introduces the semi-inner product $(v,w)_{-s,h}=(T_h^sv,w)$ and the corresponding seminorm $\|\cdot\|_{-s,h}$. It can easily be seen by (ii) that

$$\begin{aligned}\|v\|_{-s,h}&\leqslant C\{\|v\|_{-s}+h^s\|v\|\},\\ \|v\|_{-s}&\leqslant C\{\|v\|_{-s,h}+h^s\|v\|\},\end{aligned}\qquad 0\leqslant s\leqslant r-2.$$

It is therefore sufficient to show the desired estimates in $\|\cdot\|_{-(r-2),h}$. This is done by the energy method similarly to the proof for $s=0$, using the fact that T_h is selfadjoint and positive semidefinite with respect to $(\cdot,\cdot)_{-(r-2),h}$.

One may also derive negative norm estimates for time derivatives. These require additional smoothness only near t.

THEOREM 10. *Let $j\geqslant 0$ and $v_h=P_0v$ or P_1v. Then for $t\geqslant\delta>0$,*

$$\|D_t^j e(t)\|_{-(r-2)} \leqslant Ch^{2r-2}\Big\{\sum_{l=0}^{j}\|D_t^l u(t)\|_r+\int_{t-\delta}^{t}\|D_t^{j+1}u\|_r\,d\tau+\int_0^t\|u_t\|_r\,d\tau\Big\}.$$

We shall now give two examples from [31] utilizing the above negative norm error estimates to show pointwise convergence of order $O(h^{2r-2})$ for certain approximation procedures. Following Douglas and Dupont [12] such procedures are referred to as superconvergent, in as much as they are of higher order than the optimal order basic error estimates in L_2 or L_∞. The first estimate in the literature of this nature for Galerkin methods for parabolic equations (cf. [27]) concerned the pure initial value problem in one space dimension, with S_h consisting of smooth splines on a uniform mesh, and shows that if v_h is taken as the interpolant of v, then an associated finite difference operator has accuracy of order $2r-2$ and used known results from finite difference theory. For collocation methods for ordinary differential equations a similar phenomenon was observed by de Boor and Swartz [5].

Our first example here concerns superconvergence at knots for C^0 elements in one space dimension. This was proved first by Douglas, Dupont and Wheeler [13] using as a comparison function a so called quasi-projection of the exact solution into the subspace. Their approach required a more special choice of discrete initial data and somewhat higher regularity of the exact solution than the one described here.

Recall first the following simple fact for the solution of the two-point boundary value problem

$$Au=f \quad \text{on} \quad (0,1), \quad u(0)=u(1)=0,$$

and the corresponding semidiscrete solution $u_h=T_hf\in S_h$ where T_h is defined by (4) and where S_h consists of piecewise polynomials of degree $r-1$, with $\chi(0)=\chi(1)=0$ and with only continuity required at the knot $x=x_0$. With $g=g_{x_0}$ the Green's function of A with boundary values zero, we have for $e=u_h-u$,

$$e(x_0)=A(e,g)=A(e,g-\chi) \quad \text{for} \quad \chi\in S_h.$$

Since $g\in H^r(0,x_0)\cap H^r(x_0,1)\cap C^0(0,1)$ one finds easily

$$|e(x_0)|\leqslant Ch^{r-1}\|e\|_1\leqslant C(u)h^{2r-2}.$$

We now state a corresponding result for the parabolic equation.

THEOREM 11. *With the above assumptions on* T_h *and* S_h, *we have in the parabolic case for any* $n \geqslant 0$,

$$|e(x_0, t)| \leqslant C\Big\{h^{r-1} \sum_{j=0}^{n} \|D_t^j e(t)\|_1 + h^r \|D_t^{n+1} e(t)\| + \|D_t^{n+1} e(t)\|_{-2n}\Big\}.$$

It follows by our previous estimates that under suitable regularity assumptions,

$$|e(x_0, t)| \leqslant C(u) h^{2r-2} \quad \text{for} \quad t > 0.$$

The proof uses the representation

$$e(x_0, t) = \sum_{j=0}^{n} (-1)^j L(D_t^j e, T^j g) + (-1)^{n+1} (D_t^{n+1} e, T^n g),$$

where $g = g_{x_0}$ is as above, and

$$L(e, v) = (e_t, v) + A(e, v) = L(e, v - \chi) \quad \text{for} \quad \chi \in S_h,$$

and depends on the fact that $T^j g$ may be well approximated by an element of S_h

In the case that the finite element spaces are based on uniform partitions in the way quoted in connection with Theorem 8 in the interior domain $\Omega_0 \subset\subset \Omega \subset R^N$, it is possible to show that for any derivative D^α one may find a local approximation of $D^\alpha u$ from u_h in Ω_0. To see this we first quote the following lemma (Theorem 3 in [29]) which generalizes to the case of derivatives a construction due to Bramble and Schatz [7].

LEMMA. *Let* ∂_h^α *denote the forward difference quotient corresponding to* D^α *and* ψ *the B-spline in* R^N *of order* $r-2$. *Then there exists a function* K_h *of the form*

$$K_h(x) = h^{-N} \sum_{\gamma} k_\gamma \psi(h^{-1} x - \gamma),$$

with $k_\gamma = 0$ *when* $|\gamma_j| \geqslant r-1$ *such that for* $\Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$, $e = u_h - u$,

$$|K_h * \partial_h^\alpha u_h - D^\alpha u|_{\Omega_1} \leqslant C\Big\{h^{2r-2} |u|_{2r-2+|\alpha|, \Omega_0} + \sum_{|\beta| \leqq r-2+N_0} \|\partial_h^{\alpha+\beta} e\|_{-(r-2), \Omega_0} + h^{r-2} \sum_{|\beta| \leqq r-2} |\partial_h^{\alpha+\beta} e|_{\Omega_0}\Big\},$$

with $\|\cdot\|_{-(r-2), \Omega_0}$ *the dual norm to that in* $H_0^{r-2}(\Omega_0)$.

In order to use this estimate we need to have at our disposal the appropriate estimates for $\partial_h^\beta e$. Such estimates may be derived by the techniques developed for elliptic problems in Nitsche and Schatz [22], Bramble, Nitsche and Schatz [6] and yield:

THEOREM 12. *Under the above assumptions, we have for the parabolic problem*

$$|K_h * \partial_h^\alpha u_h(t) - D^\alpha u(t)|_{\Omega_1} \leqslant C(u) h^{2r-2} + C \sum_{l=0}^{m} \{\|D_t^l e(t)\|_{-(r-2), \Omega_0} + h^{2r-2} \|D_t^l e(t)\|_{\Omega_0}\},$$

where $C(u)$ depends on u and certain of its derivatives on Ω_0 at time t and m is a positive integer.

Combining this with our above global estimates we have e.g. with $v_h = P_0 v$, for $t \geq \delta > 0$,

$$|K_h * \partial_t^\alpha u_h(t) - D^\alpha u(t)|_{\Omega_1} = O(h^{2r-2}) \quad \text{as} \quad h \to 0.$$

Several of the papers referred to above complete the error analysis by also discussing the error introduced by discretizing in the time variable, particularly by means of the backward Euler or Crank–Nicolson methods. For more general time discretizations than these we quote in particular Crouzeix [9] for Runge–Kutta type methods in the nonhomogeneous case and Baker, Bramble and Thomée [3] for estimates for homogeneous equations with smooth and nonsmooth data.

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Combinatorial Aspects of Some Mathematical Problems

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It is my guess that my being invited to give this talk has something to do with the Four Color Theorem. This encourages me to present a discussion of our proof although it has appeared already in print [**A.H.**] and [**A.H.K.**]. First I should remark that Appel, Koch, and myself are members of the fifth generation working on the Four Color Problem and that we have made extensive use of many results due to earlier as well as contemporary investigators. If I would try to give proper credit to everybody who has in this way contributed to the proof then this would require about 45 minutes and I could not discuss anything else. Consequently I have to ask everybody who is interested to read the introduction to our paper [**A.H.**] in which this is described and to permit me to concentrate in this talk mainly on questions which are not explicitly discussed in our paper.

We know now that every planar map is four-colorable, or, using the equivalent dual formulation of the theorem, that every planar triangulation is vertex-four-colorable (so that any two vertices which are joined by an edge receive different colors). I must admit that I could perfectly live without knowing this. The importance of the Four Color Problem seems to lie entirely in the challenge it has provided to mathematicians to eventually develop proper methods to deal with such a simple question, rather than in the answer to the question itself. Did we recently find the proper methods and if so, what are they? This question is so controversial that it may be interesting.

Our proof consists of two parts, I: Discharging (our method of proving unavoidability of sets of configurations in planar triangulations), and II: Reducibility. When Appel and I began our joint work the methods of Part II had been developed already to a high degree of perfection. I begin therefore with a discussion of that part.

In a planar triangulation Δ, we consider a simple closed circuit R_n of n vertices and n edges ($n>1$), a so-called n-ring, so that the "interior" of R_n (in the plane) contains precisely m vertices of Δ ($m>0$), that no "diagonal" edges of Δ go through the interior, and that the exterior contains at least m vertices of Δ. The m interior vertices, together with those edges of Δ which join pairs of them and those triangles of Δ which join triplets, form a so-called n-ring configuration C in Δ. For some special configurations C one can prove by standard methods that they cannot occur in any minimally five-chromatic triangulation Δ, i.e., in any hypothetical counterexample Δ to the Four Color Conjecture that has a minimal number, say N, of vertices. Such configurations are called reducible (the precise technical term is "C-reducible", see [H] or [T.W.]). The standard proof of reducibility is, in principle, simple but in most cases requires a large number of case-distinctions: The configuration C is removed from Δ and the "hole" is closed (by "collapsing", filling in "diagonals", and/or filling in a configuration smaller than C) so as to obtain a triangulation Δ' with fewer than N vertices. Then by hypothesis, Δ' permits a (vertex-) four-coloration, say δ'. Now the old triangulation Δ is reconstructed (by reversing the operations which changed Δ into Δ') and the coloration δ' provides us with a coloration, say ε, of the exterior of C in Δ, which includes in particular a coloration, say ζ, of the ring R_n. If the coloration ζ can be extended over C (i.e., over the interior of R_n) then we have already the desired contradiction (by having colored the hypothetical counterexample Δ). If ζ does not extend over C then we have several possibilities of changing ε (and correspondingly ζ) by so-called Kempe-interchanges (for details see [T.W.] or [H]), interchanging the two colors in some of the so-called Kempe blocks (which are the maximal connected regions of the exterior of C being colored by one pair of colors). If the changed coloration, say ζ^*, extends over C then we have again the desired contradiction. Thus in order to prove the reducibility of C, we must consider all possible colorations ζ of the ring R_n (the number of which increases exponentially with n); those extending over C are called "initially good" or "good$_0$" and do not require any further treatment. For each coloration ζ which is not good$_0$, we must consider all possible "Kempe chain dispositions" (since the hypothetical triangulation Δ is not explicitly given); then for each of these Kempe chain dispositions we must try to find some suitable Kempe-interchange that changes ζ into a good$_0$ coloration; if this succeeds then we call ζ "good$_1$". The remaining colorations ζ (which are neither good$_0$ nor good$_1$) are treated again and those of them which can be changed (under all possible Kempe chain dispositions) into good$_0$ or good$_1$ colorations are called good$_2$. If continuing in this way we can change all colorations ζ into good ones then the reducibility proof for C is finished.

All configurations with ring size $n=2, 3$, or 4 are reducible. This implies in particular that a minimally five-chromatic triangulation Δ cannot contain any vertices of degree smaller than 5 (i.e., with less than 5 incident edges of Δ). Thus we may restrict our attention to triangulations Δ in which every vertex has degree at least 5.

On the other hand, Euler's formula (which is the basis for all work done in Part I) yields the "unavoidability" result that every planar triangulation contains some vertices of degree smaller than 6. Thus if the standard reducibility proof would work for the case $(n=5, m=1)$, then the single degree-5 vertex would be reducible, implying that every planar triangulation contains some reducible configuration and hence cannot be minimally five-chromatic; then the proof of the Four Color Theorem, as attempted by Kempe in 1879, would be complete. But for $n=5$, the standard proof of reducibility works if and only if $m>1$.

All attempts of finding essentially stronger methods of proving reducibility have failed. But the standard method yields the reducibility of many individual configurations with $n>5$ and $m>3$; moreover, the method is quite suitable to be carried out by a digital computer and several investigators have written effective programs for n up to 15.

At this point Appel and I applied a method which is not really new but certainly not very popular either, the method of probabilistic estimates (regarding the truth or falsity of particular statements in pure mathematics). Since this method has essentially provided the key to solving the Four Color Problem I think it deserves some attention, as controversial as it may be, and I would like to consider it somewhat more closely.

Can one make a prediction in which cases the method of proving reducibility (as considered above) will be successful? The ideal achievement would be a precise prediction, i.e., a proof of reducibility for certain configurations C without the necessity of carrying out the algorithm for every one of them. But this goal has been reached (in spite of considerable efforts) only in very few exceptionally suitable cases. In general, the question whether a coloration ζ of R_n which is not good_i can be changed under every Kempe chain disposition into some good_j $(j \leqslant i)$ coloration seems combinatorially too complicated as to permit a precise prediction. Thus we have the alternative of either making no prediction at all or helping ourselves in some way. It is close at hand to convert the absence of an obvious pattern into an assumption of randomness. If we select a coloration ζ^* of R_n at random then the probability that ζ^* is good_0 is g_0/h where g_0 is the number of good_0 colorations of R_n and h is the number of all possible colorations of R_n. Now we assume that those colorations which we obtain by Kempe interchanges are 'randomly distributed" and thus each of them is good_0 with a probability g_0/h.

REMARK. The crucial point is that we interpret our disability of making a precise prediction as randomness of the distribution. What is wrong with that? You cannot blame a mathematician for being naive—this is one of our privileges. But we must recognize the following serious objection: "While these considerations may be use ful to a certain extent, they are so vague that they should not be accepted in mathematics or else the rigor and the reputation of all of mathematics would suffer." (A similar objection was made against infinitesimal calculus in the 18th century.) But I am confident that with some care and some good will one can make these considerations

as rigorous as anything else in mathematics. In particular I suggest that we should replace the word "probability" by "pseudo-probability" or, somewhat shorter and friendlier, "paraprobability" whenever some questionable assumption of randomness is involved. Moreover, when we formulate a result concerning the paraprobability for the truth of some proposition we should always explicitly state all assumptions of randomness which have been made.

Now it is not too difficult to estimate (see § 4 of [**A.H.**]) that the paraprobability for a configuration C of being reducible depends quite critically on g_0/h where h can be exactly computed from the ring size n (and is approximately proportional to 3^n) while another estimate can be made for g_0 in terms of $m+n$ ("the more vertices in $C \cup R_n$ the more colorations"). This yields the following simple rule.

R1. If $m > n-5$ then C is likely to be reducible, in particular, for every unit by which m exceeds $n-4$, the likelihood of reducibility increases considerably.

It is, of course, important that an estimate of this type can be tested by the computation of samples as well as by theoretical work. The computational tests of our rule R1 are disappointing for small values of n: not only the single degree-5 vertex ($m=1$, $n=5$) is irreducible (i.e., the C-reduction algorithm fails) but also the only configurations with $n=6$ and $m=2$ or $m=3$ (the "5-5 edge" and the "5-5-5 triangle") are irreducible. The detailed work of several investigators has resulted in a theory of "local reduction obstacles"; for instance, if a configuration C contains a vertex which has more than three neighbors in the ring R_n, then C is extremely likely to be irreducible. This means a serious deficiency of our above assumption of randomness which, however, is easily corrected by adding to R1 the restriction "provided that C is obstacle-free". Amended in this way, the rule is excellently supported by computational tests. (Only one failure among the about 200 configurations with $n \leqslant 11$.) In order to remove the somewhat annoying restriction to obstacle-free configurations one can prove that if $m > 3n/2-6$ then C always contains an obstacle-free subconfiguration and one obtains the rule

R2. If $m > 3n/2-6$ then C is extremely likely to be reducible.

Here the words "extremely likely" mean that we have the

CONJECTURE. Every configuration C with $m > 3n/2-6$ is reducible.

This conjecture is an excellent candidate for a mathematical proposition which is true but not provable (or in this case, at least not admitting a proof which is shorter than a proof of the Four Color Theorem itself). Why is the conjecture likely to be true? Because the configurations which fulfill the hypothesis of the conjecture can be recursively enumerated in a sequence $C_1, C_2, \ldots, C_i, \ldots$ so that the paraprobability for C_i to be a counterexample to the conjecture is small for $i=1$ and decreases exponentially with i; moreover, for the small values of i (at least up to 200), the conjecture has been confirmed by computation. Thus the paraprobability for the existence of any counter-example to the conjecture is an infinite sum which

has a very small limit. Why is the conjecture likely to be not provable? Because it is not enough that the conjecture can be proved for every single value of i, even if each one of these proofs can be obtained by one and the same algorithm, since the juxtaposition of all these proofs would be infinitely long. (It seems to be the Achilles heel of mathematical theorem proving that proofs of finite length are required.) For provability of the conjecture, the configurations C_i ought to have so extremely strong similarities that one and the same (finite) argument can yield the reducibility of all (the infinitely many) of them. So it appears plausible that truth is a much weaker property than provability and that, regarding true propositions, unprovability should be regarded the general case while provability is the special case. These considerations may be regarded as an attempt to intuitively digest the unsolvability results of mathematical logic and to estimate the extent to which they may affect the working mathematician. Quite in opposition to this, there seems to be a wide-spread belief that for "practically" all "naturally arising" conjectures truth implies provability. Certainly I cannot prove that this is wrong. I cannot even prove that there is no Santa Claus; but this does not justify the assumption that there is one, and we better make preparations for the event that there is none. All I can say is that exactly this skeptical attitude regarding our possibilities of proving theorems has opened the way to the solution of the Four Color Problem; it leads immediately to a more positive appreciation of the probabilistic estimates, for the simple reason that they seem to be better than nothing and that there might be no essentially better insight available in that area. This does not mean at all to think that there is no progress possible in reduction theory; in the opposite! A great variety of interesting progresses have been made decades ago as well as very recently, and certainly can be made in the future; but yet it may be that none of them can lead to a proof of our conjecture or of any theorem of similar strength. Furthermore, it should not be suggested that a probabilistic estimate can be "as good as a proof". The only "first class knowledge" is obtained through mathematical proof. However, it may be good to remind ourselves that even our "first class knowledge" is not "absolute certainty", its two weak points being

(1) the question of consistency of the underlying axioms,

(2) the question of correctness of the proof.

The second question seems to be of a much lower intellectual level than the first one but nevertheless at least as serious. We shall have to come back to this point later. Compared with this, the "second class knowledge" (of the truth of a mathematical proposition) as obtained through probabilistic estimates has two additional weak points,

(3) the question of justification of the randomness assumptions,

(4) the remaining non-zero paraprobability for the proposition to be false.

REMARK. Regarding the seriousness of points (3) and (4) as compared with (1) and (2), an interesting variety of opinions is available; some mathematicians with a highly developed sense of dignity prefer the view that our second class insight is

no insight at all. Of course, it is perfectly all right to say "if I cannot go first class then I refuse to go at all"; but it amounts to a greater risk of missing the train.

After these remarks we are now ready for the crucial question regarding the Four Color Problem: can we reasonably expect that every planar triangulation must contain some reducible configuration (which implies the non-existence of a counter-example to the Four Color Conjecture)? The answer is obviously affirmative: We look at all 0th, 1st, 2nd, and 3rd neighborhoods of vertices in all possible planar triangulations. (The 0th neighborhood is the vertex itself; the 1st neighborhood consists of the vertex and its neighbors together with the joining edges and triangles; the 2nd neighborhood contains the vertex, its neighbors and all their neighbors; etc.) By Euler's formula, the average degree of the vertices is six, which allows us to estimate the average m- and n-values, $\bar{m}$ and $\bar{n}$, of the neighborhoods (see § 4 of **[A.H.]**, the averages being taken over all vertices and all triangulations). For the 0th neighborhoods we have $\bar{m}=1$, $\bar{n}=6$; for the 1st neighborhoods is $\bar{m}=7$, $\bar{n}=12$; i.e., no indication of likelihood of reducibility. But for the 2nd neighborhoods we have $\bar{m}=19$, $\bar{n}=18$; this means that the hypothesis $m>n-5$ of our rule R1 is already fulfilled with a margin of 5 by the average second neighborhood; however, the hypothesis of our extremely cautious conjecture, $m>3n/2-6$, is not yet fulfilled. For 3rd neighborhoods we have $\bar{m}=37$, $\bar{n}=24$, and the hypothesis of the conjecture is fulfilled by a margin of 6. We need not go on; clearly, the "area" m increases much faster than the "circumference" n with the "radius", or size class, of the neighborhoods.

These considerations indicate that not only should every planar triangulation contain some reducible configuration but that from a certain size class on, the majority of all neighborhood-configurations should be reducible. Again, one can criticize the assumptions under which we have obtained this estimate, for instance our averaging over all triangulations; one can certainly construct individual triangulations in which the averages of m and n for 2nd neighborhoods are significantly different from our $\bar{m}$ and $\bar{n}$. Such critical remarks are extremely important for testing and improving probabilistic estimates; in fortunate cases, such improvements may eventually lead to mathematical proofs superseding the probabilistic estimates. For instance, W. Stromquist found a quite amazing method for treating 2nd neighborhoods and arrived at a proof that every planar triangulation contains some 2nd neighborhood of a vertex of degree $\leqslant 30$ which in turn contains a configuration with $m>n-4$ and with no vertex degree over 30. This implies the existence of a finite "unavoidable set" of configurations which fulfill the hypothesis of our rule $R1$ by a margin of 1. Appel and I developed the method of "discharging" to such a degree that it allowed us to construct explicitly unavoidable sets of likely reducible configurations. This development had certainly not been possible had we not relied essentially on the estimated likelihood of reducibility but only on computed results. A refinement of the above estimates had shown that finite unavoidable sets of reducible configurations should exist so that all configurations had

ring sizes $n \leqslant 14$. This guided us to refine the discharging method to the point that it actually yielded such a set (of 1482 configurations). The reducibility of these configurations was established by Appel and Koch (during the last 6 months of the 4 years long development of the discharging method) using a computer-algorithm which devoted only a limited amount of time to each configuration. In this way we treated many configurations as "reduction failures" which could have been proved reducible with more time. But since—as predicted by the above estimates—unavoidable sets of reducible configurations exist in vast numbers, it meant relatively little extra work to find one such set whose members could be treated relatively easily.

REMARK. The role of the computer in the proof of the Four Color Theorem seems to have a disturbing effect on many mathematicians. Indeed, a computer-run (with a particular program and a particular input) may be regarded as a physical experiment. The experiment may tell us that a particular configuration C is reducible. How can a physical experiment be part of a mathematical proof? Of course, it cannot. We do not suggest to change the concept of mathematical proof; by no means! I suggest however, for a further discussion of the issue, to distinguish between the concept of mathematical proof (the finite sequence of elementary logical steps leading from the hypothesis to the conclusion) and the actually printed material which (assuming that no confusion can arise) is also called "proof". Let me call the latter a demonstration. If the demonstration is valid then it is the demonstration of a proof. This again means that there is an obvious routine procedure for filling in all the trivial details and thus expanding the given demonstration into a mathematical proof (which is formally perfect and likely unbearably long). But what means "obvious routine"? This has led to disagreements between authors and referees as well as to undetected errors, and the "validity problem of proofs" (here I should say of demonstrations) might deserve some serious consideration in general. In our special case (reducibility of a configuration), there is no principal difference between it and any "conventional" case of a proof: The demonstration is very short; only the configuration itself (and perhaps a so-called C-reducer) are given; the rest is trivial routine. In this case, it is explicitly spelled out what "trivial routine" means: the reduction algorithm. The computer is used for verifying the validity of the demonstration; but its use is not absolutely necessary; the work could be done by hand, however with less reliability and requiring about 3000 man-hours for a configuration of ring size $n=14$. (The computer could also print the mathematical proof, perhaps on 30,000 pages; but who wanted to see it?) Some mathematical proofs are longer than others, for simple reasons of combinatorics. But since many true propositions do not have any proofs (of finite length) it is plausible that many will have long proofs but no short ones. In such cases, computer-verification may be superior to conventional methods. This leads to the following controversial conclusion: There are problems in pure mathematics which can be solved by methods of physical experiment rather than conventional reasoning. Of

course, most physicists like this aspect whereas many mathematicians do not. That is just too bad! All I can do in order to compensate for this is to ask the physicists whether the day will come on which they know more elementary particles than the botanists know plants, and what will then be the celebrated difference between the exact and the descriptive sciences. Then most physicists do not like this, and we are even again. I personally like the idea that certain serious difficulties may bring different sciences closer together rather than farther apart and that a complex of superiority is merely a handicap rather than a blessing.

It seems obvious that the method of probabilistic estimates can be applied to many other mathematical problems. It appears that in some cases, this is rather easy and trivial, while in other cases it is quite difficult and challenging.

As an example of an easy case, I like to consider Fermat's Last Theorem. I admit that I feel strange when I address myself to this problem not being a number theorist at all. But if I can give this talk in the section on aspects of computer science barely six months after I learned programming, then I can talk as well on Fermat's Last Theorem without knowing any number theory. It looks as follows: Given three positive integers, a, b, n, $a \leqslant b$, $n > 2$, we define the paraprobability for the existence of an integer c so that $a^n + b^n = c^n$ as the density of the nth powers near the value $a^n + b^n$. It follows by very simple computation that the paraprobability for the falsehood of Fermat's Last Theorem is $p < 3^{-N}$ where the number 3 comes from my knowledge that for a counter-example, b must be at least 3 (which can certainly be greatly improved), and N means the smallest value of n for which the theorem is not yet proved, i.e., $N = 125{,}001$. Thus we have $p < 10^{-50{,}000}$. Although this is only second-class knowledge, it seems to me that we know the truth of Fermat's Last Theorem with a rather high degree of certainty, (how does this compare with our knowledge, say about the truth of the principle of conservation of energy or of the second law of thermodynamics?)

A more difficult case for an estimate seems to be the Riemann Hypothesis—at least for the outsider. The fact that the first few millions of zeros of the ζ-function lie on the critical line seems to support the conjecture, but the fact that extremely close pairs of zeros occur again and again seems to point in the opposite direction, and it requires an expert to produce a reasonable estimate.

Finally regarding the Poincaré Conjecture, I may regard myself an expert. But all I could produce so far was first a crude estimate which supported the falsity of the conjecture, then a computational test which reversed it and yielded a refined estimate supporting the truth of the conjecture, and then another computational test which revealed the refined estimate as totally unsatisfactory. This indicates that the method can be quite stimulating long before it yields reasonable estimates.

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О Некоторых Результатах в Теории Функциональных Систем

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Одним из важных разделов дискретной математики является теория функциональных систем с операциями.

Функциональные системы с операциями могут быть определены следующим образом.

Пусть $E=\{e\}$ — множество элементов e. Рассмотрим функции $f(x_1, \ldots, x_n)$, которые определены на подмножестве $\mathscr{E}_f \subset E \times \ldots \times E$, значения которых принадлежат так же E. Обозначим через $P=\{f\}$ множество всех таких функций. Обычно функциональная система P берется вместе с некоторой совокупностью операций R, которая позволяет исходя из любого подмножества $\mathfrak{M}$ функций из P порождать подмножество $[\mathfrak{M}]_R$ функций из P называемого замыканием $\mathfrak{M}$. Пара (P, R) называется функциональной системой с операциями.

Функциональные системы с операциями возникли в математической логике как аппарат для изучения некоторых логических проблем. Укажем некоторые из них.

(P_2, C) — система всех булевских функций с операцией суперпозиции.

(P_k, C) — система всех функций k-значной логики с операцией суперпозиции.

$(P_{\text{ч.р.}}, C, \text{Пр}, \mu)$ — система всех частично-рекурсивных функций с операциями суперпозиция, примитивная рекурсия и минимизация.

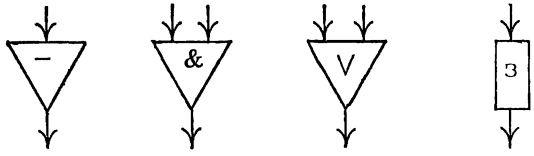
В средине нашего века возник новый мощный источник формирования функциональных систем с операциями. Речь идет об изучении управляющих систем в кибернетике. Последние можно грубо говоря, трактовать, как устройство, построенное из элементов и ячеек памяти, которые имеют входы и выходы (см. Чертеж 1).

Чертеж 1

Кроме того, данное устройство осуществляет преобразование состояний входов и ячеек памяти, которое можно характеризовать некоторой функцией f. Пусть $P=\{f\}$. Данные управляющие системы можно соединять друг с другом определенным образом и получать новые управляющие системы. Эти правила соединений (композиций) приводят к системам операций R над подмножествами из P. Мы получаем функциональную систему с операциями (P, R). Дадим некоторые примеры.

Класс параллельно последовательных двухполюсных контактных схем очевидно связан с системой $(P_2, (ПП, \&, \vee))$. Каждая такая схема реализует булевскую функцию. Параллельному соединению соответствует операция $\vee$, последовательному $\&$, $ПП$ — операция переименования переменных.

Класс автоматов, построенных из элементов (см. Чертеж 2)



Чертеж 2

Работа каждого из автоматов характеризуется набором функций $f_1, \ldots, f_m$ каждая из которых описывает преобразование входных последовательностей в выходную (эти функции будет называть автоматными). С классом всех автоматов в данном базисе связана функциональная система $(P_{авт.}, C, 0)$ с операциями суперпозиции и обратной связи.

В настоящее время появились обобщения функциональных систем. Эти обобщения идут в двух направлениях.

1. Для каждого переменного x_i берут свою область изменения E_i и тогда $f(x_1, \ldots, x_n)$ определяется на $\mathscr{E}_f \subset E_1 \times \ldots \times E_n$.

2. Наряду с функцией f рассматривают некоторую скалярную ее характеристику. Например

(f, τ) — преобразование f происходит за время τ ([1])

(f, p) — преобразование f имеет вероятность ошибки равную p ([2]).

Обычным образом введем ряд понятий.

Определение. Класс $\mathfrak{M}$ функций из P называется замкнутым, если $[\mathfrak{M}]_R=\mathfrak{M}$.

Определение. Система $\mathfrak{N}$ функций из замкнутого класса $\mathfrak{M}$ называется полной в $\mathfrak{M}$, если $[\mathfrak{N}]_R=\mathfrak{M}$.

Определение. Полная в $\mathfrak{M}$ система $\mathfrak{N}$ называется базисом в $\mathfrak{M}$, если всякая собственная ее подсистема не будет полной в $\mathfrak{M}$.

Определение. Разбиение замкнутого класса $\mathfrak{M}$ в прямую сумму подмножеств называется конгруенцией, если оно сохраняется операциями из R.

Теперь мы можем сформулировать некоторые общие задачи для функциональных систем:

1. проблема полноты,
2. исследование базисов,
3. изучение структуры всех замкнутых классов в P,
4. изучение изоморфизмов, гомоморфизмов и конгруенций,
5. выявление особенностей индивидуальных функциональных систем,
6. сравнение возможностей операций из R,
7. исследование метрических свойств замкнутых классов, и т. п.

Получение общих результатов для функциональных систем, хотя и возможно (см. например Саломаа [3]), но в виду чрезмерной широты этого понятия мало, что дает для конкретных функциональных систем. Здесь можно провести аналогию с универсальными алгебрами (функциональные системы с операциями — некоторый подкласс алгебр). В виду этого представляется целесообразным сузить понятие функциональной системы но так, чтобы оно охватывало все важнейшие конкретные случаи. Например, можно потребовать, чтобы операции R включали в себя операцию отождествления переменных. Последние очень естественно с прикладной точки зрения, поскольку сводится к объединению входов. В этом случае R таково, что все замкнутые классы являются инвариантными в смысле [4] классами. Интересные исследования в таком направлении ведутся В. Б. Кудрявцевым.

В настоящее время приходится идти по пути детального изучения небольшого числа специально отобранных (модельных) функциональных систем.

Здесь следует назвать четыре основных объекта

$$(P_2, C), (P_k, c) \quad \text{при} \quad k > 2, \ (P_{\text{авт}}, C, 0), (P_{\text{ч.р.}}, C, \text{Пр}, \mu)$$

Значительная часть работ по данной тематике приходится на изучение именно этих функциональных систем. Перечисленные функциональные системы интересны тем, что каждая следующая является усложнением предыдущей: усложняется либо понятие функции, либо понятие функции и множество операций. Это обстоятельство позволяет результаты и методы применяемые для одной функциональной системы, использовать для изучения более сложной функциональной системы. Указанное соотношение функциональных систем дает отпечаток и на степень их изученности (P_2, C) — изучена весьма полно (Пост [5]); (P_k, C) — изучена значительно, но есть вопросы на

которые мы сегодня еще не имеем ответов; $(P_{\text{авт}}, C, 0)$ и $(P_{\text{ч.р.}}, C, \text{Пр}, \mu)$ — изучены существенно хуже, чем (P_k, C).

Ввиду этого центральным объектом в настоящий момент является (P_k, C) и в докладе в основном мы будем говорить о ней.

Как уже говорилось, простейшей функциональной системой в этом смысле является (P_2, C), которая была весьма полно изучена Постом. В его работе [5] построена структура всех замкнутых классов. Пожалуй главным результатом является следующая теорема.

Теорема. *Каждый замкнутый класс $\mathfrak{M}$ из P_2 имеет конечный базис.*

К сожалению само доказательство Поста (см. [6]) весьма громоздко и не может быть использовано в общем курсе дискретной математики. Однако докладчику удалось найти существенно более простое доказательство (не нужно брать все замкнутые классы и строить структуру).

Определение. Функция $f(x_1, \ldots, x_n)$ удовлетворяет свойству A_μ (a_μ), если для любых μ наборов $(\alpha_1^i, \ldots, \alpha_n^i)$ $(i=1, \ldots, \mu)$ таких, что $f(\alpha_1^1, \ldots, \alpha_n^1)= \ldots =f(\alpha_1^\mu, \ldots, \alpha_n^\mu)=1$ $(=0)$ существует общая единичная (нулевая) компонента, т.е. существует j такое, что $\alpha_j^1=\ldots=\alpha_j^\mu=1$ $(=0)$.

Определение. Функция φ называется подфункцией функции f, если φ может быть получена из f путем отождествления переменных.

Указанное доказательство опирается на следующий факт.

Теорема. *Если булевская функция f не удовлетворяет свойствам A_2 и a_2, то она порождается своими подфункциями зависящими не более, чем от* 3 *переменных.*

Перейдем теперь к обзору некоторых результатов для (P_k, C).

Проблема полноты. а) Для классов с конечным базисом существует алгоритм позволяющий для любой конечной системы функций из $\mathfrak{M}$ узнать будет она полной или нет.

б) Т. к. алгоритм распознавания полноты является весьма трудоемким, то возник подход построения критерия полноты в терминах предполных классов.

Определение. Подкласс $\mathfrak{M}'$ замкнутого класса $\mathfrak{M}$ называется предполным, если $\mathfrak{M}'$ не является полной системой в $\mathfrak{M}$, но присоединение любой функции f из $\mathfrak{M} \setminus \mathfrak{M}'$ делает систему $\mathfrak{M}' \cup \{f\}$ полной в $\mathfrak{M}$.

Теорема ([7]). *Если замкнутый класс $\mathfrak{M}$ имеет конечный базис, то он имеет только конечное число предполных классов и система $\mathfrak{N}$ будет полной тогда и только тогда, если она не содержится целиком ни в одном из предполных в $\mathfrak{M}$ классов.*

Данная теорема является обобщением теоремы А. В. Кузнецова. Она связана с трудоемкими построениями. Поэтому она не давала возможности явно найти все предполные классы для (P_3, C). Они были построены в [8] и их оказалось 18.

Следующий шаг был связан с попыткой явного описания предполных классов для (P_k, C). Здесь имеется серия работ (А. В. Кузнецов и С. В. Яблонский [7], В. В. Мартынюк [9], Ло Чжу-кай, Пан Юп-цзе, Ван Сян-хао и Лю Сюй-хуа [10]—[14], Е. Ю. Захарова [15]). Окончательный шаг был сделан И. Розенбергом [16] и состоит в следующем.

Теорема. *Все предполные в P_k классы разбиваются на 6 явно описуемых семейств.*

Детальные исследования Е. Ю. Захарова, В. Б. Кудрявцев и С. В. Яблонский [17] показали, что число $\pi(k)$ предполных классов имеет асимптотику вида

$$\pi(k) \sim \delta(k) k 2^{C_{k-1}^{[(k-1)/2]}}, \text{ где } \delta(k) = \begin{cases} 2 \text{ при } k \text{ четном} \\ 1 \text{ при } k \text{ нечетном} \end{cases}$$

и следовательно быстро возрастает, кроме того установили поведение $\pi(k)$ для малых k:

k	2	3	4	5	6	7
$\pi(k)$	5	18	82	643	15182	$> 7 \cdot 10^6$

Если рассматривать классы с точностью до типов, т. е. перестановок чисел $0, 1, \ldots, k-1$ множества E, то критерии полноты в терминах предполных классов практически приемлем до $k \leqslant 6$.

в) Возникает вопрос о построении более эффективных критериев полноты. Последнее возможно, если учитывать более полную информацию о рассматриваемой системы $\mathfrak{N}$ функций из P_k. Здесь хорошо известны ставшие классическими результаты Слупецкого [18] и Саломаа [19] и их обобщения [20] в этих критериях делается предположение о том, что исходная система $\mathfrak{N}$ содержит определенные подмножества функций от одного переменного. Например: $\mathfrak{N}$ содержит все функции от одного переменного и выпускающие хотя бы одно значение или $\mathfrak{N}$ содержит все взаимно-однозначные функции одного переменного (подстановки). Из других результатов следует упомянуть работу В. Б. Кудрявцева [21], в которой дается критерий полноты для систем содержащих одну функцию (критерий шефферовости).

Проблема базиса. Число базисов в P_k как легко видеть бесконечно. Поэтому можно стремиться получить классификацию базисов по определенным свойствам (например, по вхождению в те или иные предполные классы). Для P_2 такие построения делались. В случае P_k это так же вполне реально.

Некоторые продвижения имеются для задачи о максимальной длине базиса в случае $k=3$.

Весьма интересным оказалось результаты, касающиеся простых базисов.

Определение. Базис $\mathfrak{N}$ класса $\mathfrak{M}$ называется простым, если каждая система $\mathfrak{N}'$ полученная из $\mathfrak{N}$ путем замены произвольной её функции на любое подмножество собственных подфункций будет неполной.

Все простые базисы для $k=2$ были найдены в [22]. Конечность числа простых базисов для $k\geqslant 3$ установлена Саломаа [23] и для произвольных замкнутых классов С. В. Яблонским [24]. Для числа $s(k)$ простых базисов в P_k В. Б. Алексеевым [25] показано, что $\log s(k)\sim k^{k^{k-1}}$.

Изучение структуры замкнутых классов. Весьма затруднено тем, что P_k при $k\geqslant 3$ в отличие от P_2 содержит не $\aleph_0$, а $\mathfrak{c}$ замкнутых классов [26]. Поэтому исследования направлены на получение информации об этой структуре. В частности, выясняют как может быть устроена окрестность произвольного замкнутого класса $\mathfrak{M}$, т. е. описать его максимальные подклассы (предполные классы) или его минимальные надклассы, или какого вида встречаются в ней цепочки вложенных друг в друга классов.

Если конечность числа предполных классов связана с существованием конечного базиса, то конечность числа минимальных надклассов для класса $\mathfrak{M}$ содержащего функцию вида $g(x)=x$ эквивалентна предикатной описуемости класса $\mathfrak{M}$ (см. С. В. Яблонский [27]). Интересно отметить, что данный результат тесно переплетается с некоторыми понятиями для инвариантных классов [4].

Исследование изоморфизмов, гомоморфизмов и конгруенций. Начато для (P_k, C) в известной работе А. И. Мальцева [28]. Им найдены все конгруенции для некоторого семейства замкнутых классов, содержащего так же и P_k.

Последнее позволило описать все изоморфизмы и гомоморфизмы P_k в P_e. Интересные результаты в дальнейшем изучении конгруенций получены недавно В. В. Горловым [29]. Им найдены все конгруенции в P_2. Для замкнутых классов содержащих только тривиальные конгруенции получены

1) теорема дающая критерий, позволяющий узнавать имеет ли данный класс только тривиальные конгруенции или нет в случае, когда $\mathfrak{M}$ содержит функции вида $g(x)=x$, и

2) теорема показывающая, что для каждого k существует конечное число минимальных классов все собственные надклассы которых имеют только тривиальные конгруенции.

Выявление особенностей индивидуальных функциональных систем. Эта проблема возникает уже в рамках систем (P_k, C). Сейчас накопилось много фактов, свидетельствующих о существенном отличии (P_2, C) и (P_k, C) при $k>2$. Приведем сводку некоторых результатов такого рода.

Таблица

	(P_2, C)	(P_k, C), $k > 2$
Мощность множества замкнутых классов	$\aleph_0$	$\mathfrak{c}$
Базис	всегда конечен	может быть конечным, счетным и отсутствовать
Конгруенции	всегда конечное число	может быть конечным, счетным и может быть континуальным

Любопытный результат, связанный с анализом явления скачка мощности множества замкнутых классов, получен В. Б. Кудрявцевым [30]. Рассмотрим функциональную систему (P_Σ, C), где P_Σ множество всех функций $f(x_1, \ldots, x_n)$ аргументы которых двух типов один — пробегают множество $E_1=(a, b)$ другие — $E_2=(c, d)$, значения функции принадлежат целиком одному из множеств. Операция суперпозиции уточняется естественным образом: функция может быть подставлена вместо тех переменных, если значения которые она принимает входят в область определения переменного.

Теорема. *Система* (P_Σ, C) *имеет континиуум замкнутых классов тогда и только тогда, когда* $(a, b) \neq (c, d)$.

К данному кругу вопросов относится результат, полученный в работе Е. Ю. Захаровой и С. В. Яблонского [31]. Как известно в (P_2, C) из всякой функции существенно зависящей более чем от одного переменного можно получить путем суперпозиций функции существенно зависящую от $r \geqslant N$ переменных, где N наперед заданное число. При $k \geqslant 3$ наряду с такой возможностью появляются две возможности вырождения суперпозиции

1) хотя и возможно построить функцию, зависящую от $r \geqslant N$ переменных, где N напередзаданное число, она не будет существенно зависеть от части перемепных;

2) из функции нельзя получить функции существенно зависящие более чем N_0 переменных.

Оказывается, что для каждой конечной системы функций можно выяснить за конечное число шагов возможно ли из нее получить функцию, существенно зависящую более, чем N переменных, где N данное число. Далее эти понятия применяются для получения более хорошей нижней оценки сложности реализации функций из P_k формулами в произвольных базисах.

В данном обзоре невозможно перечислить многие из результатов для функциональных систем.

Мы упомянем ряд направлений исследований.

Г. Бурош и его ученики (ГДР) занимались исследованием (P_k, C) и их обобщениями.

Деметрович (ВНР) изучал т. н. предельные логики М. И. Кратко, В. Б. Кудрявцев, С. В. Алешин и В. А. Буевич получили ряд интересных результатов для $(P_{\text{авт}}, C, 0)$ и их модификаций. Много сделано в изучении $(P_{\text{ч.р.}}, C, \text{Пр}, \mu)$ и их модификаций американскими математиками и группой математиков из Новосибирска.

В заключение следует заметить, что теория функциональных систем имеет многочисленные приложения как внутри математики так и за её пределами.

Отметим результат С. В. Алешина [32], который дал новое решение проблемы Бернсайда о периодических группах — дал явное определение группы из $P_{\text{авт}}$ бесконечной периодической и имеющей две образующих.

Перечислим некоторые другие приложения.

Ю. А. Виноградов использовал аппарат многозначной логики для описания электрических схем.

В. В. Тарасов получил ряд теорем о функциональных системах связанных с проблемами надежности.

В. Б. Кудрявцев и Л. А. Бирюкова разработали теорию для функциональной системы $(\{(f, \tau)\}, C)$ в которой учитывается задержка τ при вычислении функции f.

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Some Recent Developments in Formal Language Theory

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Given a finite alphabet Σ, any subset of Σ^* (the set of all words over Σ) is called a *language*. This is how language is defined in formal language theory. Hence in mathematical terms, formal language theory investigates subsets of free monoids. In traditional formal language theory (initiated by the linguist Chomsky) languages are defined by various finitary processes such as grammars and automata, which in general are rather involved combinatorial objects. However in recent years one witnesses a vigorous development of a new area—theory of L systems (initiated by the biologist A. Lindenmayer)—where language definitions have very natural algebraic character.

The aim of this note is to survey a fragment of the theory of L systems. Although we survey only a small portion of available results, it is chosen (and arranged) in such a way as to give the reader some picture of how the theory is built up. We start with the most rudimental systems (consisting of iterations of single homomorphisms) and then consider their three most natural generalizations (namely iterations of a finite number of homomorphisms, iterations of finite substitutions and iterations of a finite number of finite substitutions). In the first section we present several results and research areas typical for this part of the theory which considers sequences of words rather than their sets (languages). This is a subject matter very novel in formal language theory. In the remaining three sections we consider various types of L languages. We concentrate on results describing their combinatorial structure.

We do not quote here the origins of listed results as they all appear in [1] or [2] where they are properly referenced. We use mostly standard terminology and notation. For a word x we use $|x|$ to denote its length and $\#_B x$ to denote the number

of occurrences of letters from B in x, Λ denotes the empty word. For a language K, $\pi_n(K)$ denotes the number of different subwords of length n occurring in words from K and $\text{Length}(K)=\{n: n=|x| \text{ for some } x\in K\}$. Given an alphabet Σ and $\Delta\subseteq\Sigma$, Pres_Δ is a homomorphism from Σ^* into Σ^* defined by $\text{Pres}_\Delta(a)=a$ for $a\in\Delta$ and $\text{Pres}_\Delta(a)=\Lambda$ for $a\in\Sigma\setminus\Delta$.

I. Single homomorphisms iterated. The simplest way to define a sequence of words or a language in L systems theory is to iterate a homomorphism on a free monoid. It is formalized as follows. A DOL *system* is a triple $G=(\Sigma, h, \omega)$ where Σ is a finite alphabet, $h: \Sigma^*\to\Sigma^*$ is a homomorphism and $\omega\in\Sigma^+$. The *sequence of* G, denoted $E(G)$, is defined by $E(G)=\omega_0, \omega_1, \ldots$ where $\omega_0=\omega$ and $\omega_{i+1}=h(\omega_i)$ for $i\geqslant 0$. The *language of* G, denoted $L(G)$, is defined by $L(G)=\{h^n(\omega): n\geqslant 0\}$. The *length sequence of* G, denoted $LS(G)$, is defined by $LS(G)=|\omega_0|, |\omega_1|, |\omega_2|, \ldots$. The *growth function of* G, denoted f_G, is the function from nonnegative integers into nonnegative integers defined by $f_G(n)=|h^n(\omega_0)|$, $n\geqslant 0$. $E(G)$ is referred to as a DOL *sequence*, $L(G)$ as a DOL *language*, $LS(G)$ as a DOL *length sequence* and f_G as a DOL *growth function*.

We shall now briefly review several research areas concerning DOL systems.

I.1. *DOL growth functions.* Growth functions of DOL systems form a very natural object to investigate (also motivated by biological considerations which started the development of the L systems theory). The relationship between DOL length sequences and Z-rational sequences of numbers is by now quite well understood. Typical results here are:

THEOREM. *Assume that an N-rational sequence of numbers has a matrix representation* $u(n)=\pi M^n\eta$, $n=0, 1, 2, \ldots$, *with either only positive entries in* π *or only positive entries in* η. *Then* $u(n)$ *is a DOL length sequence.* □

THEOREM. *Every Z-rational sequence can be expressed as the difference of two DOL length sequences.* □

Generating functions form a very useful tool in investigating DOL growth functions. The following result is typical in characterizing generating functions of DOL growth functions.

THEOREM. *A rational function* $F(x)$ *with integral coefficients and written in lowest terms is the generating function of a DOL growth function not identical to the zero function if and only if either* $F(x)=a_0+a_1x+\ldots+a_nx^n$ *where* $a_0, a_1, \ldots, a_n$ *are positive integers, or else* $F(x)$ *satisfies each of the following conditions:*

(i) *The constant term of its denominator equals* 1.

(ii) *The coefficients of the Taylor expansion* $F(x)=\sum_{n=0}^{\infty}a_nx^n$ *are positive integers and, moreover, the ratio* a_{n+1}/a_n *is bounded by a constant.*

(iii) *Every pole* x_0 *of* $F(x)$ *of the minimal absolute value is of the form* $x_0=r\varepsilon$ *where* $r=|x_0|$ *and* ε *is a root of unity.* □

I.2. *Locally catenative DOL systems.* A very natural way to generalize linear homogeneous recurrence relations to words is as follows.

A *locally catenative formula* (LCF for short) is an ordered k-tuple $v=(i_1, \ldots, i_k)$ of positive integers where $k \geqslant 2$ (we refer to k as the *width of* v and to $\max\{i_1, \ldots, i_k\}$ as the *depth of* v). An infinite sequence of words $\omega_0, \omega_1, \omega_2, \ldots$ satisfies v with a *cut* $p \geqslant \max\{i_1, \ldots, i_k\}$ if, for all $n \geqslant p$, $\omega_n = \omega_{n-i_1} \ldots \omega_{n-i_k}$. A sequence of words satisfying some LCF v with some cut is called (v-) *locally catenative.* A DOL system G is called (v-) *locally catenative* if $E(G)$ is (v-) locally catenative. We say that G is *locally catenative of depth* d if G is v-locally catenative for some LCF v with depth of v equal to d.

First of all we get the following correspondence between locally catenative DOL sequences and languages.

THEOREM. *A DOL system G is locally catenative if and only if $L(G)^*$ is a finitely generated monoid.* □

The following result illustrates the relationship between a global property of a DOL sequence (namely its locally catenative property) and a local property of the underlying DOL system (namely the way its homomorphism is defined). Let $G=(\Sigma, h, \omega)$ be a DOL system with $\omega \in \Sigma$ where for no a in Σ, $h(a)=\Lambda$. The *graph of* G, denoted $\mathcal{G}(G)$, is an ordered graph the nodes of which are elements of Σ and, for $a, b \in \Sigma$, (a, b) is an edge in $\mathcal{G}(G)$ if and only if $h(a)=\alpha b \beta$ for some $\alpha, \beta \in \Sigma^*$.

THEOREM. *If there exists $a \in \Sigma$ such that $h^n(\omega)=a$ for some $n \geqslant 0$ and every cycle in $\mathcal{G}(G)$ goes through a then G is locally catenative.* □

The most important open problem concerning locally catenative DOL systems is the decidability status of the question: "Is an arbitrary DOL system locally catenative?" The best known result in this direction is:

THEOREM. (1) *It is decidable whether or not an arbitrary DOL system is locally catenative of depth d, where d is an arbitrary positive integer.* (2) *It is decidable whether or not an arbitrary DOL system is locally catenative of width d, where d is an arbitrary positive integer larger than* 1. □

I.3. *DOL equivalence problem.* One of the more challenging problems in the theory of DOL systems is the DOL sequence (respectively language) equivalence problem: "Given two arbitrary DOL systems G_1, G_2 is it decidable whether or not $E(G_1)=E(G_2)$ (respectively $L(G_1)=L(G_2)$)?"

The problem was solved quite recently (by K. Culik and I. Fris).

THEOREM. *The DOL sequence and language equivalence problems are decidable.* □

Various efforts to solve the above mentioned problems created quite a number of notions and results which are of interest also in traditional formal language theory.

For example one gets the following representation theorem for recursively enumerable languages.

THEOREM. *Let K be a recursively enumerable language. There exist an alphabet Σ, homomorphisms g, h_1, h_2 and a regular language M such that $K=g(\mathrm{Eq}(h_1,h_2)\cap M)$, where $\mathrm{Eq}(h_1,h_2)=\{x\in\Sigma^+: h_1(x)=h_2(x)\}$.* □

II. Single finite substitutions iterated. A natural way to generalize DOL systems is to consider the iteration of a finite substitution rather than a homomorphism. (The difference between a finite substitution and a homomorphism is that the former maps each letter of the alphabet into a finite set of words whereas the latter maps each letter into a single word.) Since in such a case the generated language (rather than the sequence) becomes the primary concept, one therefore considers (as usual in formal language theory) an additional (terminal) alphabet.

An EOL *system* is a construct $G=(\Sigma, h, \omega, \Delta)$ where Σ, Δ are finite alphabets, $\Delta\subseteq\Sigma$, $\omega\in\Sigma^+$ and h is a finite substitution from Σ^* into 2^{Σ^*}. The *language of* G is defined by $L(G)=\{x\in\Delta^*: x\in h^n(\omega)$ for some $n\geqslant 0\}$. $L(G)$ is referred to as an EOL *language.*

The following results illustrate the combinatorial structure of EOL languages. They are especially useful for proving in general that various "concrete" languages are not EOL languages (which is often a difficult task). Let K be a language over Σ and let B be a nonempty subset of Σ. Let $N(K,B)=\{n: (\exists x)_K(\#_B x=n)\}$. We say that B *is numerically dispersed* in K if $N(K,B)$ is infinite and, for every natural number k, there exists a natural number n_k such that whenever u_1 and u_2 are in $N(K,B)$ and $u_1>u_2>n_k$ then $u_1-u_2>k$. B *is clustered* in K if $N(K,B)$ is infinite and there exist natural numbers k_1, k_2 both larger than 1 such that whenever a word x in K satisfies $\#_B x\geqslant k_1$ then x contains at least two occurrences of letters from B, which lie at a distance smaller than k_2 from each other.

THEOREM. *Let K be an EOL language over Σ and B a nonempty subset of Σ. If B is numerically dispersed in K then B is clustered in K.* □

Let K be a language over an alphabet Σ and let B be a nonempty subset of Σ. We say that K is *B-determined* if for every positive integer k there exists a positive integer n_k such that for every x, y in K if $|x|, |y|>n_k$, $x=x_1ux_2$, $y=x_1vx_2$ and $|u|, |v|<k$ then $\mathrm{Pres}_B(u)=\mathrm{Pres}_B(v)$.

THEOREM. *Let K be a B-determined EOL language. There exist positive integer constants c and d such that, for every $x\in K$, if $\#_B x>c$ then $|x|<d^{\#_B x}$.* □

THEOREM. *Let K be an EOL language over an alphabet Σ. If K is Σ-determined then there exists a constant c such that, for every nonnegative integer n, $\pi_n(K)\leqslant c.n^3$.* □

III. Several homomorphisms iterated. The language of a DOL system is obtained by applying to a fixed word an arbitrary homomorphism from the semigroup gene-

rated by a single homomorphism. Semigroups generated by a finite number of homomorphisms form a natural next step.

A DTOL *system* is a construct $G=(\Sigma, H, \omega)$ where Σ is a finite alphabet, $\omega\in\Sigma^+$ and H is a finite set of homomorphisms from Σ^* into Σ^*. The *language of* G is defined by $L(G)=\{x\in\Sigma^*: x=h_n\dots h_1(w)$ for $n\geqslant 0, h_i\in H\}$.

The following result describes a rather basic property of the set of all subwords of a DTOL language.

THEOREM. *Let Σ be a finite alphabet such that $\#\Sigma=n\geqslant 2$. If K is a DTOL language over Σ then* $\lim_{l\to\infty}(\pi_l(K)/n^l)=0$. □

Adding an extra (terminal) alphabet one can define a richer class of languages. An EDTOL *system* is a construct $G=(\Sigma, H, \omega, \Delta)$ where (Σ, H, ω) is a DTOL system and $\Delta\subseteq\Sigma$. The language of G is defined by $L(G)=\{x\in\Delta^*: x=h_n\dots h_1(\omega)$ for $n\geqslant 0$, $h_i\in H\}$; it is referred to as an EDTOL *language*.

The following two results are very useful results on the combinatorial structure of EDTOL languages.

A function f from $\boldsymbol{R}_+$ into $\boldsymbol{R}_+$ is called *slow* if for every $\alpha\in\boldsymbol{R}_+$ there exists $n_\alpha\in\boldsymbol{R}_+$ such that for every $x\in\boldsymbol{R}_+$ if $x>n_\alpha$ then $f(x)<x^\alpha$.

Let Σ be a finite alphabet and let $f:\boldsymbol{R}_+\to\boldsymbol{R}_+$. A word w over Σ is called *f-random* if every two disjoint subwords of w which are longer than $f(|w|)$ are different.

THEOREM. *For every EDTOL language K and for every slow function f there exists a constant s such that for every f-random word x in K longer than s there exist a positive integer t and words $x_0, \dots, x_t, \sigma_1, \dots, \sigma_t$ with $\sigma_1\dots\sigma_t\neq\Lambda$ such that $x=x_0\dots x_t$ and, for every nonnegative integer n, $x_0\sigma_1^n x_1\sigma_2^n\dots\sigma_t^n x_t$ is in L.* □

THEOREM. *Let K be an EDTOL language over an alphabet Σ, where $\#\Sigma=n\geqslant 2$. If* Length (K) *does not contain an infinite arithmetic progression then*

$$\lim_{l\to\infty}\frac{\#\{w\in K: |w|=l\}}{n^l}=0.\ \square$$

IV. Several finite substitutions iterated. In the same way as one generalized DOL systems to EOL systems one extends DTOL systems to obtain ETOL systems.

An ETOL *system* is a construct $G=(\Sigma, H, \omega, \Delta)$ where Σ is a finite alphabet, $\omega\in\Sigma^+$, $\Delta\subseteq\Sigma$ and H is a finite set of finite substitutions from Σ^* into 2^{Σ^*}. The *language* of G is defined by $L(G)=\{x\in\Delta^*: x\in h_n\dots h_1(\omega)$ for $n\geqslant 0$, $h_i\in M\}$.

Here is a typical result concerning combinatorial structure of ETOL languages.

THEOREM. *Let K be an ETOL language over an alphabet Σ. Then for every nonempty subset Δ of Σ there exists a positive integer k such that for every x in K either* (i) $|\mathrm{Pres}_\Delta x|\leqslant 1$, *or* (ii) *there exists a, b in Δ and w in Σ^* such that $x=x_1 awbx_2$ for some x_1, x_2 in Σ^* with $|awb|\leqslant k$, or* (iii) *there exists an infinite subset M of K such that, for every y in M, $|\mathrm{Pres}_\Delta y|=|\mathrm{Pres}_\Delta x|$.* □

The following result is a typical "bridging" result. It allows one to construct examples of non ETOL languages providing that one has examples of languages that are not EDTOL.

THEOREM. *Let* Σ_1, Σ_2 *be two disjoint alphabets and let* $K_1 \subseteq \Sigma_1^*, K_2 \subseteq \Sigma_2^*$. *Let* f *be a surjective function from* K_1 *into* K_2 *and let* $K = \{wf(w) : w \in K_1\}$. *Then*

(1) *If* K *is an ETOL language then* K_2 *is an EDTOL language.*

(2) *If* f *is a bijection, then also* K_1 *is an EDTOL language (if* K *is an ETOL language).* □

We hope that through this short note the reader acquires a taste of the theory of L systems—a new and rapidly developing area of discrete mathematics with interesting connections to computer science and biology.

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Group-Theoretic Algorithms, A Survey

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The development of the modern digital computer has had more influence on group theory than on any other branch of algebra except number theory. There is a growing list of results in group theory which, at least at the present time, could not have been obtained without the use of a computer. Three examples are the determination of the four-dimensional space groups [3], the calculation of the orders of certain restricted Burnside groups such as announced in [1] and the construction of some of the newly discovered sporadic finite simple groups such as described in [12]. The use of computers in group theory has stimulated research in the area of group-theoretic algorithms. The purpose of this paper is to survey some of the achievements in this field.

Many computational problems in group theory do not have algorithmic solutions. Probably the best known problem of this type is the word problem for finitely presented groups. Thus often a proof of the existence of an algorithm for solving a particular problem is a nontrivial result. For example, recently Grunewald [10] showed among other things that it is possible to decide whether or not two given elements of the unimodular group $GL(n, Z)$ are conjugate. However, this paper will be limited primarily to algorithms which have been implemented on a computer and applied to a significant mathematical problem. I shall discuss seven algorithms and one data structure which seem to me to be applicable to a particularly wide class of problems. Space does not permit giving either formal definitions of the algorithms or a complete bibliography. Fortunately an extensive bibliography on computers in group theory has been compiled by V. Felsch [8]. Updated copies are available from Felsch at the Technische Hochschule, Aachen. The references given here are to the papers which provide the best introduction to the algorithms discussed.

In order to describe the algorithms it is useful to introduce the concept of a monoid with involution. Let M be a monoid, or semigroup with identity. An involution on M is an antiautomorphism of period two, that is, a map $x \to x'$ of M into itself such that $(xy)' = y'x'$ and $(x')' = x$ for all x and y in M.

In this paper we shall encounter three types of monoids with involution. The first type consists of groups with the inverse map as the involution. The second type is obtained by choosing a set Δ and a point δ in Δ. Given α in Δ and $f: \Delta \to \Delta$ we shall write the image of α under f as α^f. Let $M(\Delta, \delta)$ denote the set of maps $f: \Delta \to \Delta$ such that $\delta^f = \delta$ and such that whenever $\alpha^f = \beta^f \neq \delta$, then $\alpha = \beta$. The set $M(\Delta, \delta)$ is a submonoid of the monoid of all maps of Δ into itself. The group of invertible elements in $M(\Delta, \delta)$ is isomorphic to the symmetric group on $\Delta - \{\delta\}$. Given f in $m(\Delta, \delta)$, define $f': \Delta \to \Delta$ as follows:

$$\alpha^{f'} = \beta, \quad \text{if} \quad \alpha \neq \delta \quad \text{and} \quad \beta^f = \alpha,$$

$$= \delta, \quad \text{otherwise.}$$

Then f' is in $M(\Delta, \delta)$ and $'$ is an involution on $M(\Delta, \delta)$.

We can also define the free monoid with involution generated by a set X. Let $X^{\pm}$ be the cartesian product $X \times \{1, -1\}$ and let M be the set of all words, or finite sequences, $a_1, \ldots, a_r$, $r \geqslant 0$, where each a_i is in $X^{\pm}$. Let $U = a_1, \ldots, a_r$ and $V = b_1, \ldots, b_s$ be in M. The product of U and V, written here as U, V, is the word $a_1, \ldots, a_r, b_1, \ldots, b_s$. If $a = (x, \varepsilon)$ is in $X^{\pm}$, then define $a' = (x, -\varepsilon)$ and set $U' = a_r', \ldots, a_1'$. Then $'$ is an involution on M and the obvious universal property holds for the pair $(M, ')$. There is a natural homomorphism of M onto the free group F generated by X. The image of the word U will be denoted by $[U]$. We normally identify X with $X \times \{1\}$ and identify a word of length 1 with its single term. Thus we have $X \subseteq X^{\pm} \subseteq M$. If X is finite and the elements of $X^{\pm}$ are linearly ordered in some manner, then we can well-order M by defining $U < V$ to mean that either $r < s$ or $r = s$ and for some i, $1 \leqslant i \leqslant r$, we have $a_j = b_j$, $1 \leqslant j < i$, and $a_i < b_i$.

The first five algorithms described below answer questions about a group defined by a finite presentation. We shall maintain the following notation in discussing these algorithms. Assume we are given a finite set X and a finite subset R of the free monoid with involution M generated by X. We shall also assume that $X^{\pm}$ has a fixed linear order and order M as described above. Let F be the free group generated by X and let N be the normal subgroup of F generated by all conjugates of the elements $[U]$ with U in R. Set $G = F/N$. If U is in M, then the element $N[U]$ of G will be denoted $\overline{U}$.

Given X and R, it is not hard to construct an integer matrix A such that the commutator quotient group G/G' is isomorphic to $\boldsymbol{Z}^m/K$, when $m = |X|$ is the number of columns of A and K is the subgroup of $\boldsymbol{Z}^m$ generated by the rows

of A. The orders of the cyclic direct factors of G/G' are given by the elementary divisors of A. When the presentation X, R comes from an application of the Reidemeister–Schreier algorithm, which is described below, it is quite possible for A to have hundreds of rows and columns. In this case, even when the elementary divisors of A are small, the traditional *elementary divisor algorithm* using integer row and column operations leads to very large integers in the intermediate steps. Work of G. Havas suggests an alternate approach. The i-th elementary divisor of A is the gcd of the determinants of the i-by-i submatrices of A. Using congruence methods similar to those described in [4], one can compute the rank of A and obtain useful information about the elementary divisors of A.

A more recent algorithm for studying the quotient groups of G is the *nilpotent quotient algorithm* [14]. Let p be a prime. Define $\gamma_1^p(G)=G$ and for $i\geqslant 1$ set $\gamma_{i+1}^p(G)=(G, H)H^p$, where $H=\gamma_i^p(G)$, (G, H) is the subgroup of G generated by all commutators $g^{-1}h^{-1}gh$ with g in G and h in H and H^p is the subgroup generated by all h^p with h in H. The series $G=\gamma_1^p(G)\supseteq\gamma_2^p(G)\supseteq\ldots$ is called the lower exponent-p-central series of G. Given X, R, p and a positive integer c, the nilpotent quotient algorithm computes the order p^n of $\bar{G}=G/\gamma_{c+1}^p(G)$, the largest quotient group of G having exponent-p-central class at most c. In addition, a particularly nice type of presentation for $\bar{G}$ is constructed. The difficulty of the computation depends more on the size of c than it does on p or the order of $\bar{G}$. Typical values of c are 10 or 20 while the value of n obtained is often in the hundreds or even larger.

The next three algorithms are concerned with subgroups of finite index in G. To each such subgroup H of G there corresponds the permutation representation of G on the set of right cosets of H. If we number these cosets from 1 to $n=|G:H|$, then we get a transitive representation of G into the symmetric group Σ_n. It is useful to generalize the notion of transitive permutation representation for monoids with involution. We shall need to do this here only for the free monoid with involution M. Let n be a positive integer, let $\Delta_n=\{0, 1, \ldots, n\}$ and let $M_n=M(\Delta_n, 0)$. The group of invertible elements of M_n is isomorphic to Σ_n. A coset table representation of M into M_n is a homomorphism $f: M\to M_n$ such that for each $i\neq 0$ in Δ_n there is a word U_i in M such that $f(U_i)$ maps 1 to i. The associated coset table is the n-by-$2m$ matrix whose i, jth entry is the image of i under $f(a)$, where a is the jth element of $X^{\pm}$. We say f is complete if f maps M into the group of invertible elements of M_n.

Let $f: M\to M_n$ be a coset table representation and let U be a word in M. We shall say f is compatible with U at a point $i\neq 0$ in Δ_n if whenever U can be written as a product V, a, W with V and W in M and a in $X^{\pm}$ such that $j=i^{f(V)}$ and $k=i^{f(W')}$ are both nonzero, then $j^{f(a)}=k$. If f is complete, then f is compatible with U at i if and only if $i^{f(U)}=i$. If f is compatible with U at each i in $\Delta_n-\{0\}$, then we shall say f is compatible with U.

Coset table representations of M which are compatible with each element of

R can be constructed as follows: Let H be any subgroup of G and let $U_1, \ldots, U_n$ be words in M such that $\overline{U}_1, \ldots, \overline{U}_n$ are in distinct right cosets of H and such that $\{U_i | 1 \leqslant i \leqslant n\}$ is a Schreier system in the sense that whenever a product U, V is in the set then U is in the set. For any a in X define $f(a)$ to be the element of M_n mapping $i \neq 0$ to j, where either $j \neq 0$ and $H\overline{U}_i a = H\overline{U}_j$ or $j=0$ and $H\overline{U}_i a$ is not in $\{H\overline{U}_k | 1 \leqslant k \leqslant n\}$. The map $f: X \to M_n$ extends to a coset table representation $f: M \to M_n$, which is compatible with each element of R. We call f a truncation of the action of G on the right cosets of H.

Let $f: M \to M_n$ be a complete coset table representation. For $i \neq 0$ in Δ_n let U_i be the first word such that $f(U_i)$ maps 1 to i. We say f is standard if $U_1 < U_2 < \ldots < U_n$. The following theorem forms the basis of an algorithm for describing the set of subgroups H of G such that $|G:H|$ does not exceed a given bound.

THEOREM. *There is a* 1-1 *correspondence between the set of subgroups of* G *of index* n *and the set of complete standard coset table representations* $f: M \to M_n$ *which are compatible with each element of* R. *The subgroup corresponding to a particular representation* f *is* $\{\overline{U} | f(U)$ *fixes* $1\}$.

The *low index subgroup algorithm* takes as input X, R and an integer N and lists the coset tables corresponding to the complete standard coset table representations $f: M \to M_n$ with $n \leqslant N$ which are compatible with the elements of R. One version of this algorithm is described in [6]. The values of N which can be handled vary from less than 10 to 50 or 100, depending on the presentation.

It is a fundamental result of Reidemeister and Schreier that a subgroup of finite index in a finitely presented group has a finite presentation. Given the presentation X, R and the coset table corresponding to a subgroup H of G, of finite index, the *Reidemeister–Schreier algorithm* [9] constructs a presentation Y, S for H. In most implementations $|Y| = 1 + n(m-1)$ and $|S| = nr$, where $m = |X|$, $n = |G:H|$ and $r = |R|$. Typical values of m, r and n might be 3, 10 and 50, respectively. Due to the large number of generators and relators produced, considerable further processing is required to make the presentations useful. At present only a few *ad hoc* techniques for doing this are available. As mentioned above, the elementary divisor algorithm can be used to determine the structure of H/H'.

Let S be a finite set of words and let H be the subgroup of G generated by the elements $\overline{U}$ with U in S. There is no algorithm for deciding whether $|G:H|$ is finite given only X, R and S. However, if $|G:H|$ is finite, then it is possible to determine $|G:H|$, although there is no way of giving an *a priori* bound for the time needed to complete the computation in terms of some reasonable measure of the size of the inputs X, R and S. If $f: M \to M_n$ is a truncation of the action of G on the cosets of H defined by a Schreier system of words $U_1, \ldots, U_n$, where U_1 is the empty word, then not only is f compatible with the elements of R it is also compatible with the elements of S at 1. The *Todd–Coxeter algorithm,* also referred

to as *coset enumeration,* takes as input X, R, S and a positive integer N, which may be thought of as a bound for the space which can be used during the computation. The output is a coset table corresponding to a representation $f: M \to M_n$ satisfying the following conditions:

(1) $N \geqslant n$.
(2) f is compatible with the elements of R.
(3) f is compatible with the elements of S at 1.
(4) If U and V are words in M such that $f(U)$ and $f(V)$ map 1 to the same nonzero point, then $H\overline{U}=H\overline{V}$.
(5) If $N>n$, then f is complete.

If f is complete, then it follows that $n=|G:H|$. If $|G:H|$ is finite, then for sufficiently large N the Todd–Coxeter algorithm will terminate with f complete. This algorithm has probably been used more extensively in group-theoretic computation than any other algorithm. Descriptions of various implementations may be found in [5].

The algorithms described so far deal with groups which may be infinite. We turn now to algorithms and data structures for handling groups which are obviously finite. An algorithm due to Neubüser [13] constructs the complete lattice of subgroups of a group of moderate size, say of order a few thousand. As formally defined, this *subgroup lattice algorithm* determines only the solvable subgroups. However, the number of nonabelian simple groups which can occur as composition factors of subgroups of a group of order less than 5000 is very limited and using *ad hoc* methods the full lattice of subgroups can be found.

A data structure described in [15] makes it possible to perform many computations in a permutation group G on a finite set Ω even when the degree $n=|\Omega|$ is in the thousands. A base for G is a sequence $\alpha_1, \ldots, \alpha_r$ of points in Ω such that the only element of G fixing each α_i is the identity. Many "interesting" groups have bases with r much smaller than n. An element g in G is determined by the images $\alpha_1^g, \ldots, \alpha_r^g$. Let $G^{(i)}$ be the stabilizer of $\alpha_1, \ldots, \alpha_{i-1}$ in G, $1 \leqslant i \leqslant r+1$, and let Δ_i be the orbit of $G^{(i)}$ containing α_i, $1 \leqslant i \leqslant r$. For each δ in Δ_i let $u_i(\delta)$ be an element of $G^{(i)}$ taking α_i to δ. The sets $U^{(i)}=\{u_i(\delta) | \delta \in \Delta_i\}$ form a concise and useful description of G. If G is defined by a set of generators X, then Δ_1 and $U^{(1)}$ can be found easily, but it is more difficult to find the other $U^{(i)}$. We say X is a strong generating set for G relative to $\alpha_1, \ldots, \alpha_r$, if $G^{(i)}$ is generated by $X^{(i)}=X \cap G^{(i)}$. In this case all the $U^{(i)}$ can be constructed easily. The elements of $U^{(i)}$ can be described by a Schreier system of words which can be efficiently stored using an integer vector v indexed by the points in Ω. If α is in Ω, then $v_\alpha=0$ if α is not in Δ_i. If α is in $\Delta_i-\{\alpha_i\}$, then v_α is the number of the last factor of $U_i(\alpha)$ in some list of the elements of $X^{(i)}$ and their inverses. Finally $v_{\alpha_i}=-1$. The vector v is referred to as a Schreier vector for Δ_i. Bases and Schreier vectors can be quite useful in the construction of algorithms for studying

the elements and subgroups of a given permutation group as well as for determining the automorphism groups of moderate size groups and various types of combinatorial objects. See for example [11].

The *Schreier–Todd–Coxeter algorithm* takes as input a set X_0 of permutations of a finite set Ω and determines a base $\alpha_1, \ldots, \alpha_r$ for the group G generated by X_0, a strong generating set $X \supseteq X_0$ for G relative to $\alpha_1, \ldots, \alpha_r$ and a presentation for G in terms of the generators in X. The algorithm involves repeated coset enumeration. A formal description of the algorithm can be found in [16]. An implementation by J. Leon handled a set of generators for the Suzuki sporadic group of degree 1782 and order about 4.5×10^{11} in 10 seconds on a fairly fast machine.

In conclusion I would like to mention the following additional work on group-theoretic algorithms. Atkinson [2] has suggested an algorithm for deciding whether a given transitive permutation group is primitive. Dixon [7] has described a method for constructing the character table of a moderate size finite group. Several investigators have written programs to assist in the construction of character tables for large finite groups. However, little has been published about these programs, which require close man-machine interaction. Workers in Aachen and Sydney are collaborating on the development of a unified system of programs implementing many group-theoretic algorithms, including the ones discussed here. In addition, a language to facilitate computations with the system is being developed.

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On the Synthesis of Self-Correcting Networks

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1. Introduction. In this paper we consider the following networks: combinatorial networks, contact networks, valve contact networks and valve networks. Precise definitions are given, for example, in [1], [3], [5], [6], [10], [11]. These networks compute Boolean functions. For example, the contact network in Figure 1 computes the function (between the source nodes P_1 and P_2) $x_1x_2x_3 \vee \bar{x}_1\bar{x}_2x_3 \vee \bar{x}_1x_2\bar{x}_3 \vee x_1\bar{x}_2\bar{x}_3 = x_1 \oplus x_2 \oplus x_3$. (We use $\oplus$ or $\bigoplus$ to denote sum modulo 2.) The combinatorial network in Figure 2 computes the function $x_1\&\bar{x}_2 \vee \bar{x}_1\&x_2 = x_1 \oplus x_2$. A valve is a directed edge which is not labelled with a variable. The valve contact network in Figure 3 computes the function $x_1x_2x_3 \vee \bar{x}_1x_3 \vee \bar{x}_1x_4$.

A set Ω consisting of gates is said to be complete if every Boolean function can be realised by some combinatorial network with these gates. For example, the set consisting of the 2-input AND, 2-input OR and the NOT function is complete.

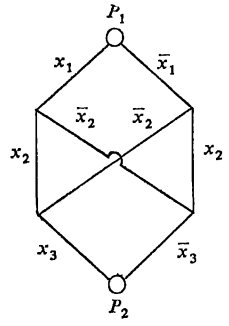

FIGURE 1

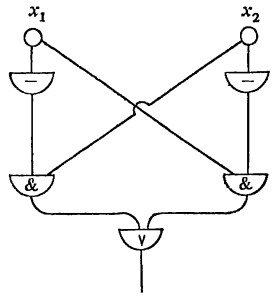

FIGURE 2

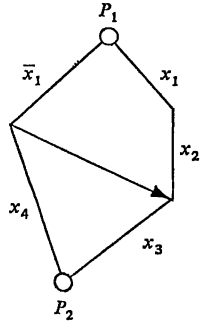

FIGURE 3

We assume that Ω is complete. A cost, P_i, is associated with each of the gates $g_i \in \Omega$. The cost $L_{COM}(\mathfrak{A})$ of a combinatorial network $\mathfrak{A}$ is the sum of costs of its gates. The cost $L_{CON}(\mathfrak{A})$ of a contact network (or the cost $L_{VC}(\mathfrak{A})$ of a valve contact network) $\mathfrak{A}$ is the number of its contacts (or the sum of costs of each of its contacts and of each of its valves). Then, the combinatorial complexity, $L_{COM}(f)$ ($L_{CON}(f)$ or $L_{VC}(f)$, respectively), is the minimum of $L_{COM}(\mathfrak{A})$ ($L_{CON}(\mathfrak{A})$ or $L_{VC}(\mathfrak{A})$, respectively), where $\mathfrak{A}$ ranges over all combinatorial networks (contact networks or valve contact networks, respectively) computing f. Further, the Shannon function $L_{COM}(\mathscr{F})$ ($L_{CON}(\mathscr{F})$ or $L_{VC}(\mathscr{F})$, respectively) of a family $\mathscr{F}$ of Boolean functions is the maximum of $L_{COM}(f)$ ($L_{CON}(f)$ or $L_{VC}(f)$, respectively), where f ranges over all functions in $\mathscr{F}$.

Let Q_n be the set consisting of all Boolean functions of n variables. C. Shannon has shown [11] that $2^n/n \lesssim L_{CON}(Q_n) \lesssim 4(2^n/n)$.[1] O. B. Lupanov has shown [3]–[6] that

$$L_{COM}(Q_n) \sim \varrho \frac{2^n}{n},^{2} \qquad L_{CON}(Q_n) \sim \frac{2^n}{n}, \quad L_{VC}(Q_n) \sim \frac{2^n}{n}^{3} \tag{1}$$

($\varrho = \varrho(\Omega) = \text{const.}$) and that for every positive number ε the number of Boolean functions f having a combinatorial complexity

$$L_{COM}(f) \leqslant (1-\varepsilon)\varrho \frac{2^n}{n}, \quad L_{CON}(f) \leqslant (1-\varepsilon)\frac{2^n}{n}, \quad L_{VC}(f) \leqslant (1-\varepsilon)\frac{2^n}{n},$$

respectively, is $o(2^{2^n})$, i.e. for large n "almost" all Boolean functions of Q_n have a combinatorial complexity which is "almost" equal to $L_{COM}(Q_n)$, $L_{CON}(Q_n)$ or $L_{VC}(Q_n)$, respectively.

We consider networks realising Boolean functions of Q_n and which are self-correcting for errors of any gates, contacts or valves. Other errors, for example, errors of lines, are ignored.

The idea to consider self-correcting networks is due to S. V. Jablonskij [9].

2. Self-correcting contact networks. A contact network $\mathfrak{A}$ is called self-correcting for b interruptions and d shorts or is also called (b, d)-correcting, if $\mathfrak{A}$ realizes a function f in case no contacts of $\mathfrak{A}$ are shorted or interrupted and if $\mathfrak{A}$ realizes the same function f in case at most b contacts are interrupted and at most d contacts are shorted.

A contact network is denoted by $[r, \mathfrak{A}]$ (or $[r, g]$), if it consists of $r+1$ parallel sequences each of which consists of $r+1$ networks $\mathfrak{A}$ (or any $r+1$ realisations of

[1] $a(n) \lesssim b(n)$ (or $b(n) \gtrsim a(n)$) denotes $\overline{\lim}\,[a(n)/b(n)] < 1$.

[2] $a(n) \sim b(n)$ denotes $\lim [a(n)/b(n)] = 1$.

[3] Here is assumed that the cost of a contact is equal to 1.

the Boolean function g) (see Figure 4). Obviously, if $\mathfrak{A}$ realizes $f=f(x_1, \ldots, x_n)$

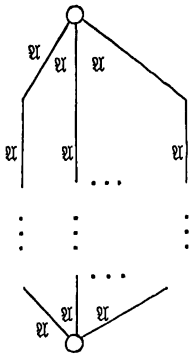

FIGURE 4

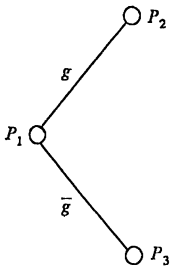

FIGURE 5

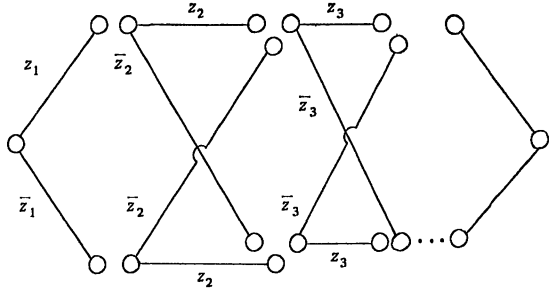

FIGURE 6

(by the assumption that there are no shorts or interruptions), then $[r, \mathfrak{A}]$ also realizes f and is (r, r)-correcting. But the combinatorial complexity of $[r, \mathfrak{A}]$ is $(r+1)^2$ times greater than the combinatorial complexity of $\mathfrak{A}$. However results of some authors show that there are better methods for sufficiently large n.

Let $L_{\text{CON}}^{(b,d)}(f)$ be the number of contacts that are sufficient to construct any (b, d)-correcting contact network realising f. Obviously, $L_{\text{CON}}^{(0,0)}(f)=L_{\text{CON}}(f)$. Let $L_{\text{CON}}^{(b,d)}(Q_n)=\max_{f\in Q_n} L_{\text{CON}}^{(b,d)}(f)$.

Ju. G. Potapow and S. W. Jablonskij have shown [9] that

$$L_{\text{CON}}^{(0,1)}(Q_n) \sim 2^n/n.$$

If $b=0$ and $d=1$, then by (1) we have

$$L_{\text{CON}}^{(b,d)}(Q_n) \sim L_{\text{CON}}^{(0,0)}(Q_n) = L_{\text{CON}}(Q_n). \tag{2}$$

Therefore, if n is a sufficiently large number, then for "almost" all Boolean functions the combinatorial complexity in the case of realisation by contact networks correcting one short is only a little greater than the combinatorial complexity in the case of realisation by contact networks which do not correct errors.

Ch. A. Madatjan has shown [7] that

$$L_{\text{CON}}^{(1,0)}(Q_n) \sim 2^n/n$$

and later E. I. Neciporuk has shown [8] that (2) is true if the number of errors is not constant, namely if $b=o(\log n/\log\log n)$ and $d=3$. (We use $\log a$ to denote $\log_2 a$.) He has also shown that if $b=o(\log n/\log\log n)$ and $d=o(n^{1/2-\delta})$ $(\delta>0)$, then

$$L_{\text{CON}}^{(b,d)}(Q_n) \lesssim 2L_{\text{CON}}^{(0,0)}(Q_n) \sim 2(2^n/n). \tag{3}$$

The author of this paper shows [12], [14] by a special coding that if

$$\log b = o\left(\frac{n}{\log n}\right) \quad \text{and} \quad \log d = o\left(\frac{n}{\log n}\right), \tag{4}$$

then (3) is also true. The idea of proof with the number of errors by (4) is given in §5.

3. Self-correcting combinatorial networks. Assume that a complete set of gates Ω is given and that all elements of Ω are unreliable and may yield their associated Boolean function or may not do so. A combinatorial network is called r-self-correcting if the failure of any r' ($r' \leqslant r$) of its elements does not affect its correct operation. If $r \geqslant 1$ then there does not exist an r-self-correcting combinatorial network $\mathfrak{A}$ constructed only by unreliable gates, because the gate of $\mathfrak{A}$, whose output is also the output of the network $\mathfrak{A}$, is also unreliable, but its errors cannot be corrected. Therefore, for the construction of self-correcting networks we need (abstract) gates which are quite reliable. Assume that in addition to Ω a set Ω^* of erliable gates is given such that symmetrical Boolean functions can be realised using only gates of Ω^*. A cost, P_j^* (it may be very big), is associated with each of the gates $g_j^* \in \Omega^*$. Let $L_{\mathrm{COM}}^{(r)}(Q_n)$ be the Shannon function for r-self-correcting combinatorial networks (i.e. the minimum of costs sufficient for the realisation of every function of Q_n by an r-self-correcting network). G. I. Kirienko has shown [2] that if $r = 2^{o(n)}$, then

$$L_{\mathrm{COM}}^{(r)}(Q_n) \sim L_{\mathrm{COM}}(Q_n) \sim \varrho(2^n/n) \quad (\varrho = \varrho(\Omega) = \text{const.}). \tag{5}$$

But the networks constructed with his method contain very many reliable gates ($\gtrsim 2^n/n^C$ for a constant C).

The author of this paper has shown [13], [14] that if also $r = 2^{o(n)}$ and if Ω^* is a complete set, then combinatorial networks containing only on the order of r reliable gates of Ω^* and satisfying (5) can be constructed. The author of this paper has also shown [14], [16] that it is necessary to use r^* gates of Ω^*, where r^* is on the order of r.

4. Self-correcting valve contact networks and valve networks

E. I. Neciporuk developed methods for the construction of selfcorrecting valve contact networks. He has shown [8] that if $n \to \infty$ then for a certain $a = a_n \to \infty$ and $b = b_n \to \infty$

$$L_{\mathrm{VC}}^{(a,b)}(Q_n) \sim 2^n/n.$$

He has also found the Shannon function for valve networks which are correcting r interruptions for a certain $r = r_n \to \infty$ [8].

5. A code and its application for the construction of selfcorrecting contact networks. Let us consider a sequence $\{(n, q_n, m_n)\}$ where q_n and m_n are integers which are dependent on n.

THEOREM 1. *If* $q_n \to \infty$ *and*

$$\log s_n = o\left(\frac{\log q_n}{\log \log q_n}\right)$$

and if each of (n, q_n, m_n) *is associated with* q_n *Boolean functions* $g_1^{(n)}, \ldots, g_{q_n}^{(n)}$, *which are dependent on the variables* $x_1, \ldots, x_{m_n}$, *then there exist Boolean functions* $h_1^{(n)}, \ldots, h_{r_n}^{(n)}$, *which are also dependent on the variables* $x_1, \ldots, x_{m_n}$ *such that*

(a) $r_n \sim q_n$,

(b) *for every* i $(i=1, \ldots, q_n)$ *there are* s_n *disjoint subsets* $\Pi_1^i, \ldots, \Pi_{s_n}^i$ *of the set* $\{1, \ldots, r_n\}$ *such that*

$$\bigoplus_{j \in \Pi_1^i} h_j^{(n)} = g_l^{(n)} \quad (l = 1, \ldots, s_n).$$

The proof of the Theorem 1 follows from the proof of Theorem 3 of [15] (see also [12], [14]). Let us omit the index n.

Now we give the

IDEA OF PROOF OF THEOREM 1. Let us consider in this paper only the special case $s=2^p$ and $q=u^p$, where u and p are integers. We argue by induction. If $p=1$, then we set

$$h_j = \begin{cases} g_j, & \text{if } j = 1, \\ g_{j-1} \oplus g_j, & \text{if } 2 \leqslant j \leqslant u, \\ g_{j-1}, & \text{if } j = u+1. \end{cases} \tag{6}$$

Obviously, from $\Pi_1^i = \{1, \ldots, i\}$ and $\Pi_2^i = \{i+1, \ldots, u+1\}$ follow

$$\Pi_1^i \cap \Pi_2^i = \Lambda, \quad \bigoplus_{j \in \Pi_1^i} h_j = g_i, \quad \bigoplus_{j \in \Pi_2^i} h_j = g_l \quad \text{and} \quad (\text{if } q \to \infty)\ r = q+1 \sim q.$$

For the reader's convenience let us illustrate the general case by $p=2$. (For the general case see [14], [15].) We take for each integer u a one to one correspondence H_u between $A = \{1, 2, \ldots, u^2\}$ and the set B consisting of all pairs (k, l), where k and l are integers at least 1 and at most u. The integer from A corresponding (k, l) by H_u is denoted by $H(u, k, l)$. We take for each k all of the functions $g_l^* = g_{H(u,k,l)}$ (instead of g_l with l instead of j) and take by (6) (i.e. by induction) all of the functions $h_l^* = g_{l,k}^{**}$ $(l=1, \ldots, u+1)$. Now take for each l by (6) all of the functions $h_{l,k}^{**} = h_{H(u+1,k,l)}$ $(k=1, \ldots, u+1)$ (taking for each l $g_{l,k}^{**}$ instead of g_k with k instead of j). Let $Z_1^i = \{1, \ldots, i\}$ and $Z_2^i = \{i+1, \ldots, u+1\}$. We obtain $\Pi_1^i, \ldots, \Pi_4^i$ by $\Pi_{H(2,\alpha,\beta)}^i = \{H(u+1, v, w) | v \in Z_\alpha^i, w \in Z_\beta^i\}$.

Note that it is not very difficult to show the following two statements.

THEOREM 2. *There is a function* α_n *with* $\alpha_n \to 0$ *for* $n \to \infty$ *such that for every Boolean function* g *dependent on* n *variables there is a network* D_g *having three source nodes* P_1, P_2, P_3 *and realising the functions* $g, \bar{g}$ *and zero between* P_1, P_2, *between* P_1, P_3 *and between* P_2, P_3, *respectively (see Figure 5) and which contains only contacts and no more than* $L_{\text{CON}}(Q_n)(1+\alpha_n)$.

THEOREM 3. *Let $z_1, \ldots, z_t$ be Boolean functions. The function $z_1 \oplus \ldots \oplus z_t$ can be realised with no more than two of each of the networks $D_{z_1}, \ldots, D_{z_t}$ respectively (see Figure 6).*

Now we present the method of construction of contact networks which are self-correcting for R shorts and R interruptions with $\log R = o(n/\log n)$ and realising $f(x_1, \ldots, x_n)$. We set $s=(R+1)^2$, $k=o(n)$ (more precisely see [14]), $q=2^k$ and consider all (not necessarily different from each other) 2^k functions $g_1, \ldots, g_{2^k}$ which are obtainable from $f(x_1, \ldots, x_n)$ by setting $x_{n-k+1}, \ldots, x_n$ for constant (and which are dependent on $n-k$ variables). Then we take the functions $h_1, \ldots, h_r$ by Theorem 1 and construct with no more than

$$2rL_{\mathrm{CON}}(Q_{n-k})(1+\alpha_{n-k}) \sim 2 \cdot 2^k \frac{2^{n-k}}{n-k} \sim 2\frac{2^n}{n}$$

contacts two copies of all of the networks $D_{h_1}, \ldots, D_{h_r}$ respectively (as defined above). This is possible by the Theorems 1 and 2 and by (1). It is not difficult to connect the networks $D_{h_1}, \ldots, D_{h_r}$ with networks of the type $[R, x_i^{\alpha}]$, $i=n-k+1, \ldots, n$ (as defined above), containing together no more than $2^{o(n)}$ contacts in such a way that if $x_{n-k+1}=\sigma_{n-k+1}, \ldots, x_n=\sigma_n$ and if $g_l=f(x_1, \ldots, x_{n-k}, \sigma_{n-k+1}, \ldots, \sigma_n)$, then we obtain according to Theorem 3 $(R+1)^2$ realisations of g_l which are connected to a network $[R, g_l]$. The network obtained in this way is self-correcting for R shorts and R interruptions.

6. Realisation of vector functions. With the code considered in § 5 the author of this paper has shown [14], [15] that the Shannon function for vector functions $(f(x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, f(x_1^{(R)}, \ldots, x_n^{(R)}))$, where f is a Boolean function and all of the variables are disjoint, is also $\sim \varrho(2^n/n)$.

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Recent Developments in the Theory of the Shapley Value*

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1. Introduction. The Shapley value is an a priori measure of a game's utility to its players; it measures what each player can expect to obtain, "on the average", by playing the game. Other concepts of cooperative game theory, such as the Core, Bargaining Set [6], and N–M Solution [26] predict outcomes (or sets of outcomes) that are in themselves stable, that cannot be successfully challenged or upset in some appropriate sense. Almost invariably, they fail to define a unique result; and in a significant proportion of the cases, they do not define any result at all.[1] The Shapley value, although it is not in any formal sense defined as an average of such "stable" outcomes, nevertheless can be considered a mean, which takes into account the various power relationships and possible outcomes.

It follows from this that the Shapley value may also be thought of as a reasonable compromise, the outcome of an arbitration procedure. A player should be willing to settle for a compromise that yields with certainty what he otherwise would only have expected in the mean. For example, the symmetric N–M solution of the 3-person majority game predicts one of the three payoff vectors (1/2, 1/2, 0), (1/2, 0, 1/2), and (0, 1/2, 1/2), corresponding to the three possible 2-person majorities. Before the beginning of bargaining, each player may figure that his chances of getting into a ruling coalition are 2/3, and conditional on this, his payoff is 1/2; the "expected outcome" would then be (1/3, 1/3, 1/3), and this is also the Shapley value. It would,

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[1] The Bargaining Set is the only one of these three covered by a general existence theorem.

therefore, also be a reasonable compromise; but it is not in itself stable, since it can be easily improved upon by any two-person coalition.

Mathematically, the Shapley value is perhaps the most tractable of all the concepts of cooperative game theory. This has led to the growth of a considerable theory, which in turn has enabled a wide range of applications to Economics and Political Science. Here we survey some of the more recent of these developments.

2. General definition in the transferable utility case. We begin by recalling that a *coalitional game,* or simply *game* for short, is a real-valued function v on the σ-field $\mathscr{C}$ of a measurable space $(I, \mathscr{C})$, with $v(\emptyset)=0$. Here I is the *player space,* the members of $\mathscr{C}$ are *coalitions,* and $v(S)$ is the *worth* of a coalition S. A game is called *monotonic* if $S \supset T$ implies $v(S) \geqslant v(T)$.

Fix $(I, \mathscr{C})$. An *outcome* (or *payoff vector*) is a finitely additive game.[2] For each game v and automorphism (one-one bimeasurable function) Θ of $(I, \mathscr{C})$, define the game $\Theta_* v$ by $(\Theta_* v)(S)=v(\Theta S)$ for all S.

Let us be given a linear space Q of games, which is *symmetric* in the sense that $\Theta_* Q=Q$ for all Θ. An operator φ from Q to outcomes is called *symmetric* if $\varphi(\Theta_* v)=\Theta_*(\varphi v)$ for all v in Q and all automorphisms Θ; *monotonic* if φv is monotonic whenever v is; and *efficient* if $(\varphi v)(I)=v(I)$ for all v in Q. A *value* on Q is an operator from Q to outcomes that is linear, monotonic, symmetric, and efficient.

3. Finite games. A game v is called *finite* if there is a finite subset N of I (a *support* of v) such that $v(S)=v(S \cap N)$ for all S. The finite games form a linear space on which there is a unique value; it is given by

$$(\psi v)(\{i\}) = E\big(v(S_i \cup \{i\}) - v(S_i)\big), \tag{3.1}$$

where S_i is the set of players (members of N) preceding i in a random order on N, and E is the expectation operator when each order on N has probability $1/|N|!$ [36]. It is easy to check that (3.1) does indeed define a value; as for uniqueness, perhaps the simplest proof is that of Dubey [7], who uses an induction on $|N|$ to show that every finite game is a linear combination of unanimity games (games for which $v(S)=1$ or 0 according as $S \supset N$ or $S \not\supset N$).

4. Nonatomic games, partition values, and the diagonal property. Diametrically opposed to the finite games are the *nonatomic games,* which model situations in which no individual player has any significance [2]. Examples are games of the form $f \circ \mu$, where μ is a nonatomic vector measure, and f is a real-valued function on the range of μ vanishing at 0. One approach to defining a value for a nonatomic game v is via approximations by finite games. Specifically, if Π is a measurable partition of I—i.e. a finite subfield of $\mathscr{C}$—we may define a finite game v_Π, whose

[2] Intuitively, the sharing of proceeds in an additive game involves no difficulties, so that by associating an additive game to a non-additive game, we have essentially specified an outcome.

support consists of the atoms of Π, by $v_\Pi = v|\Pi$; then v_Π is a kind of finite approximant to v. Given a coalition S in $\mathscr{C}$, an increasing sequence $\{\Pi_1, \Pi_2, \ldots\}$ of such partitions is called *S-admissible* if $S \in \Pi_1$ and $\bigcup_i \Pi_i$ generates $\mathscr{C}$. A value φ on a space Q is called a *partition value* [30] if for each game v in Q and each coalition S, there is an S-admissible sequence $\{\Pi_1, \Pi_2, \ldots\}$ such that

$$\lim_{n\to\infty} (\psi v_{\Pi_n})(S) \to (\varphi v)(S), \tag{4.1}$$

where ψ is the value for finite games. If for a specific game v and outcome φv, (4.1) holds for all S and all S-admissible sequences, then we write $v \in \mathrm{ASYMP}$ and call φv the *asymptotic value* [16] of v. Whereas the partition value is defined in terms of the imbedding space Q, the definition of asymptotic value is independent of any imbedding space; its existence depends on the game v only.

A partition value of a non-atomic game is a limit of values of large finite approximants. The asymptotic value is the strongest possible partition value; if it exists, then no matter how the player space is cut up,[3] in the limit result is the same.

Are there values that are not partition values? This leads us to the *diagonal property* of values. Let ν be a nonatomic nonnegative measure on $\mathscr{C}$ with $\nu(I)=1$ ($\nu \in \mathrm{NA}^1$ for short); Π a partition of I into many—say n—"small" sets; and Q_h the union of the first h atoms of Π in a random order on the atoms. For a fixed h, we will have $\nu(Q_h) \approx h/n$ with high probability; moreover, for fixed ε, if Π is sufficiently far out in some S-admissible sequence, then the probability is $>1-\varepsilon$ that $|\nu(Q_h)-(h/n)|<\varepsilon$ *simultaneously* for all h. Thus if $\mu \in (\mathrm{NA}^1)^m$ (i.e. μ is an m-tuple of NA^1 measures), almost all the coalitions occurring in Formula (3.1) as applied to v_Π will have μ-measures very near the "diagonal" $D^m = \{(t, \ldots, t): t \in [0, 1]\}$ of the m-cube. In particular, let φ be a partition value; then

(4.2) *if φ is defined for two games v_1 and v_2 that agree on all coalitions S with $\mu(S)$ in some ε-neighborhood of D^m, then $\varphi v_1 = \varphi v_2$.*

Any value φ satisfying (4.2) for all vectors μ of NA^1 measures is called a *diagonal value.*

All the values treated in [2] were diagonal, and for a long time it was not known whether *all* values are diagonal. Finally, Neyman and Tauman [29] and Tauman [40][4] found examples of nondiagonal values. In particular, not all values are partition values.

What, then, accounts for the diagonality of all previously considered values? In [27], Neyman answered this question by showing that all *continuous* values are diagonal; here continuity is w.r.t. (with respect to) the *variation* norm, defined by

$$\|v\| = \sup\left\{\sum_{i=1}^{k} |v(S_i) - v(S_{i-1})| : \emptyset = S_0 \subset S_1 \subset \ldots \subset S_k = I\right\}.$$

[3] E.g. into n intervals of "length" $1/n$, or into n of length $1/2n$ and n^2 of length $1/2n^2$

[4] [40] avoids a certain undesirable pathology in [29].

This norm plays a crucial role in the theory, and all previously considered values had been continuous w.r.t. it.

Closely related to the diagonal property is the *diagonal formula* for values. Let pNA denote the smallest variation-closed linear space containing all games $f \circ v$, where $v \in NA^1$ and f is absolutely continuous. There is a unique value on pNA, and pNA $\subset$ ASYMP [16]. Suppose now that $\mu \in (NA^1)^m$ and $f \in C^1(R^m)$. Then $f \circ \mu \in$ pNA, and

$$\varphi(f \circ \mu) = \left\langle \mu, \int_0^1 \nabla f(t, \ldots, t)\, dt \right\rangle \tag{4.3}$$

[2, Theorem B]. To understand (4.3), note that it follows from Lyapunov's theorem that for each t in $[0, 1]$ there is a coalition tI with $\mu(tI) = (t, \ldots, t)$: the tI are called *diagonal coalitions,* and may be considered "perfect samples" of I as far as $f \circ \mu$ is concerned. Let us now think of a "player" in a non-atomic game as an infinitesimal coalition ds; the marginal contribution of ds when added to tI is

$$(f \circ \mu)(tI \cup ds) - (f \circ \mu)(tI) = \langle \mu(ds), \quad \nabla f(t, \ldots, t) \rangle.$$

Thus (4.3) says that *the value of a player is his average contribution to a diagonal coalition.*

This principle, which is of fundamental importance in the theory of nonatomic games and its applications, has been extended far beyond the space pNA for which it was originally established. The deepest and furthest-reaching work on this subject is due to J. F. Mertens [20], who has established the existence of a value obeying a suitable analogue of (4.3) on a very large space of games, which even contains games not in ASYMP.

5. Political applications. A *weighted majority* (WM) game is one of the form $f_q \circ v$, where v is a non-negative measure with $v(I) = 1$ (the *vote* measure), $0 < q < 1$ and $f_q(x) = 0$ or 1 according as $x \leqslant q$ or $x > q$. Finite WM games appear already in [26]. Values of finite WM games were first studied by Shapley and Shubik [38], who interpreted them as measures of political power. They have since been applied to many voting situations, such as the UN security council, the US electoral college, state legislatures, multi-party parliaments, etc.; [18] is a good survey. Shapiro and Shapley [35], Milnor and Shapley [21], and Hart [11] studied values of *oceanic* games, i.e. WM games in which v contains a nonatomic part (the "ocean" of small voters) as well as some atoms (large voters); [21] contains an application to corporations with several large stockholders. An interesting qualitative conclusion is that when $q = 1/2$, a *single* atom has value larger than his vote, as might be expected; but this is often reversed when there are several atoms. For example, when v has 2 atoms and an ocean of measure 1/3 each, then the atoms get only 1/4 of the value each.

The above are asymptotic results on the values of the atoms when the largest "small" vote tends to 0. Calculating the values of the small voters themselves,

even approximately, is much more difficult, and even when there are no atoms, the problem was open for many years. Only recently did A. Neyman [28] prove, in a remarkable tour-de-force of combinatorial reasoning, that $f_q \circ v \in \text{ASYMP}$ when $v \in \text{NA}^1$. Intuitively, his result says that the value of a coalition depends only on its total vote, not on the relative sizes of the voters. It can be used to prove that oceanic games are in ASYMP, and also that $f \circ v \in \text{ASYMP}$ when f is monotonic and continuous, and $v \in \text{NA}^1$. Also, there are close connections to renewal theory.

More complex political structures can also often be described by using WM games. A bicameral legislature is the product of 2 WM games, and the electoral college when the players are the individual citizens is a polynomial in WM games. Such games need not be in ASYMP; thus if $\mu, \nu \in \text{NA}^1$ and $\mu \neq \nu$, then $(f_{2/3} \circ \mu)(f_{2/3} \circ \nu) \notin$ ASYMP; however, it is a member of a space with a partition value [30]. Whether there is a partition value on the algebra generated by all nonatomic WM games is an open question.

See [31] for an application using a non-symmetric variant of the value.

A variant of the Shapley value called the *Banzhaf value* has achieved some prominence in connection with political models. For finite games it is defined by (3.1), with the sole difference that now S_i varies over the set of all subsets of $N \setminus \{i\}$, each such coalition receiving probability $1/2^{|N|-1}$. In general, it is not efficient. An account of the theory and a very extensive bibliography may be found in [8].

6. Economic applications. Games arising in economics often have a property called "homogeneity of degree 1;" roughly, this means that two coalitions differing from each other in their size only, but not in their composition, have worths proportional to their sizes. Examples are games $f \circ \mu$, where $\mu \in (\text{NA}^1)^m$ and f is a function of m variables that is homogeneous of degree 1. Suppose now that φ is a partition value. A principle that is basic to many of the economic applications asserts that

(6.1) *if φ is defined for a superadditive[5] game v that is homogeneous of degree* 1, *then φv is in the core of v.*

(Recall that the *core* of a game v is the set of outcomes x such that $x(I) = v(I)$ and $x(S) \geq v(S)$ for all S.)

Let's demonstrate this in the particular case in which $v = f \circ \mu$, where $\mu \in (\text{NA}^1)^m$ and f is a superadditive[6] function defined and homogeneous of degree 1 on the nonnegative orthant of R^m, and C^1 in its interior. Although $f \notin C^1(R^m)$, it can be shown that nevertheless $v \in \text{pNA}$ and the diagonal formula (4.3) holds. Moreover the homogeneity of degree 1 and the superadditivity together yield the concavity of f. Since f is homogeneous of degree 1, $\nabla f(t, \ldots, t)$ is a constant, so (4.3) yields

[5] $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$.

[6] $f(x+y) \geq f(x) + f(y)$.

$\varphi v = \langle \mu, \nabla f(1, \ldots, 1) \rangle$. This means that φv is a function h of $\mu(S)$, i.e. $(\varphi v)(S) = h(\mu(S))$; and in fact h is the linear function with coefficients $\nabla f(1, \ldots, 1)$. By the efficiency of the value, $h(1, \ldots, 1) = (\varphi v)(I) = v(I) = f(1, \ldots, 1)$, and hence it follows that the graph of h is tangent to that of f at $(1, \ldots, 1)$. Since f is concave and h is linear, it follows that the graph of h always lies above that of f; but this implies that $(\varphi v)(S) \geq v(S)$ for all S, which together with the efficiency $(\varphi v)(I) = v(I)$ means that φv is in the core.

In this case a small additional argument, which depends on the actual tangency (i.e. the differentiability of f), yields that v is the *only* member of the core. This is true whenever $v \in \text{pNA}$; pNA expresses a kind of differentiability property of a game. In general, though, the core will contain more than just the value. For example, when v is the minimum of two NA^1 measures, then the core consists of a non-degenerate interval (i.e. the set of all convex combinations of two different outcomes); in this case the asymptotic value exists and is the midpoint of the core. More generally, Hart [12] has proved that if a superadditive game v that is homogeneous of degree 1 has an asymptotic value φv, then φv is the center of symmetry of the core of v.

If the core has no center of symmetry,[7] there will be no asymptotic value; but not all is lost. If ν is an NA^1 measure, an outcome φv is called a *ν-value* if for all S, (4.1) holds for all S-admissible sequences of partitions whose atoms have equal (or in an appropriate sense almost equal) ν-measures. Suppose now that μ in $(\text{NA}^1)^m$ is absolutely continuous w.r.t. ν, with Radon–Nikodym derivative $d\mu/d\nu$ in $(L^2(\nu))^m$. Let f be superadditive and homogeneous of degree 1; then Hart [14] has shown that $v = f \circ \mu$ has a ν-value, which has an interesting expression in terms of the core of v and the m-dimensional normal distribution whose covariance matrix is the same as that of $d\mu/d\nu$.

We come now to the applications. An important model in economic theory is that of the *exchange economy*. Like many economic models, it cannot be expressed as a transferable utility (TU) game as in § 2; a more general concept—the *nontransferable utility* (NTU) game—is required. The most commonly used adaptation of the value to NTU games is that introduced[8] in [37], which culminated a long development to which many contributed; see in particular [24], [9]. We will not define the NTU value here; a brief treatment is in [1, § 4]. It is enough for our purposes to note that the analysis involves the values of certain TU games auxiliary to the given NTU game.

In an exchange economy, the law of supply and demand defines *competitive prices* and, correspondingly, *competitive allocations* of goods and services. The TU games to which we are led from exchange economies are precisely the superadditive homogeneous games, and their cores are closely related to the cores of the "parent" NTU

[7] For example, the core of the minimum of 3 linearly independent measures is a triangle.

[8] For an alternative approach, see Owen [32].

economies. The relationship between the value and the core expressed by (6.1), and the subsequent discussion, thus imply a close relationship between values and competitive allocations. More precisely, it can be proved that all allocations associated with an NTU value of a non-atomic exchange economy—i.e. all *value allocations*—are competitive. When the utility functions of the agents in the economy are sufficiently differentiable, we can assert the converse as well; in that case, therefore, the value allocations are the same as the competitive allocations.

Again, many people contributed to this development; see in particular [39], [5], [2], [4], [12], [13], [19], [14]. An excellent survey up to 1976 is in [13].

Models containing both political and economic elements, including in particular problems of taxation and redistribution, have been considered recently [1]. The TU games to which these models lead are products of pNA games with nonatomic WM games; the methods of Neyman [28] show that they have asymptotic values, and they are also amenable to the diagonal methods of Mertens [20].

Conceptually, these models differ from exchange economies in that threats play an important role. Games of this kind were treated by Nash [25], and much more generally by Harsanyi [9]. The worth $v(S)$ of a coalition S in an auxiliary TU game is now based as much on the harm that S *could* do to the players outside it as the good that it could do for itself. The value is of course efficient, so that it assumes that destructive threats are not actually carried out; this fits well our interpretation of the value as a reasonable compromise.[9] None of the pie gets thrown out, but how it gets cut up may depend on threats.

7. Cost sharing. An interesting practical application of the Shapley value is to problems of cost sharing. For example, Littlechild and Owen [17] have considered the problem of airport landing fees. Runways (and other airport components) must be built large enough to accomodate the largest aircraft that will use them; but obviously it makes no sense to share the cost equally among all users, i.e. to charge the same landing fees to a jumbo jet and a private 4-seater. Here one defines a game v by considering the players to be individual aircraft landings, with $v(S)$ the hypothetical cost of building a facility that will accommodate the set S of landings. Each landing is then charged a fee precisely equal to its Shapley value. The efficiency condition assures that the fees will exactly cover the cost, the symmetry condition assures that similar users are charged the same fee, and the linearity condition assures that the cost of using two different and independent facilities is the sum of the costs of using each one separately. Monotonicity, of course, only says that you don't get paid for landing at an airport.

A spectacular recent application of this type is to telephone billing at large institutions. See Billera, Heath, and Raanan [3]; the system proposed by them has been adopted for internal telephone billing at Cornell University.

[9] Value models in which threats do sometimes get carried out involve incomplete information; see [10], [23].

8. Other contributions. A complete review of recent developments in the theory of the Shapley value is impossible in the space allotted to this paper. The quantifier "some", not "all", should be understood in the title; there have been many important contributions not covered here. We close by mentioning two conceptually innovative recent works: In [34], A. Roth formalized the idea that the value measures a game's utility to its players; and in [22], R. Myerson characterized the value in terms of communication networks connecting the players.

9. Conclusion. Much of the analysis in political and economic science has traditionally proceeded on an ad hoc basis, often using different methods and principles for each model under consideration. A unified approach to these disciplines is provided by game theory. Among the tools it provides, the Shapley value is particularly broadly and systematically applicable, and appears able to account for theoretical principles in widely diverse areas.

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Computer Animation and the Geometry of Surfaces in 3- and 4-Space

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Geometers have always used any available media to help them illustrate their work with diagrams, pictures, and models. Modern computer graphics provides a new medium with great potential both for teaching and research. Older methods of representing curves and surfaces by drawings on a blackboard or models in wire or plaster are frequently found to be inadequate in many important geometric problems, specifically those which involve objects undergoing transformations or objects which exist properly in the fourth dimension or higher. A high-speed graphics computer makes it possible to approach and solve such problems by methods which were unavailable only a few years ago. Producing 30 different pictures per second, such a computer can display on a television tube a sequence of images which the viewer readily interprets as the projections of an object rotating in 3-dimensional space. By turning dials, a mathematician can investigate a curve or surface by having it rotate about different axes and stopping it at especially interesting positions. He or she can "fly inside" the object to focus on some local behavior or proceed to examine some specific singularity by deforming the object through a one-parameter family of curves or surfaces.

Most of the classical objects of the calculus or differential geometry of curves and surfaces take on new meaning when they are reinvestigated using such methods. But these approaches also give insights into some areas where previous descriptive methods were very limited and entirely new aspects of geometry and topology become subjects for exploration.

In this report, we describe five sets of films which give examples of the use of computer graphics techniques. Some are in finished form and have been used in

a variety of teaching situations. Others are in preliminary form, intended for mathematical research discussions.

For the most part the films rely on direct and uncomplicated techniques. Most of the objects in the films are parametrically defined surfaces, given by three or four coordinate functions, each a function of two variables defined over a cross-hatched domain which is usually either a rectangular or polar coordinate patch in the plane or a portion of the Riemann sphere. The images are then projected into a 3-dimensional subspace, orthogonally or centrally, and then projected again into the 2-dimensional plane of the television tube which is the output device. In some cases, for an extremely complicated object, it is possible to project two images which can be viewed with stereoscopic apparatus to give the effect of a single 3-dimensional image. Usually, however, a sense of 3-dimensionality is developed by having the image rotate slowly about an axis in the 3-dimensional space. For almost all viewers, this movement creates a spatial sensation which is interpreted readily as the shadows of a rotating transparent framework in ordinary 3-space. Subsequent deformations take place within the background context provided by this rotation. In particular as slices are made by planes parallel to a fixed direction, the curves of intersection on the rotating figure are perceived to be planar slices.

Although slicing by a linear function, or more generally by some other function, does require some computational ability, the machine operates quickly enough that it is possible to view a sequence of slices in "real time", as if observing an object through a window as it rotated in the next room. The effect, however, is greater if in addition to the slice, the film displays as well the part of the surface lying below or above the slice—the technique of "water-level slicing". Since this technique requires more time per picture and since it is especially well suited to representations using color, this technique is used primarily in the animation mode, where one picture is made at a time, and then the finished film is projected at 24 frames per second. For color, filters are used and each frame is exposed several times for the different portions of the picture.

In addition to projection, rotation, and slicing it is possible to use linear interpolation between figures with the same parametrization. Again this is fast enough that the technique can be employed for real time manipulation of the figures for videotapes or for on-line research.

More complicated programs require a recomputation of the data for every frame and are suited more for a filming mode.

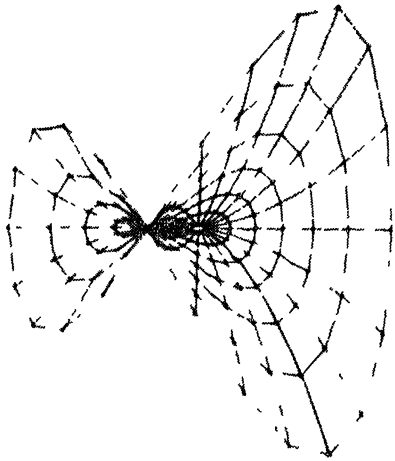
All of these techniques can be handled by a relatively small machine, in this case a META 4 A, B configuration with a Vector General scope, augmented by a parallel processor built at the Brown University Computing Laboratory.

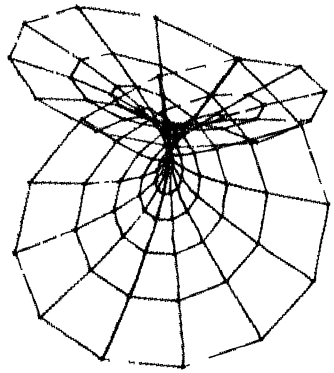
All films described here were produced in collaboration with Charles M. Strauss at Brown University.

THE HYPERCUBE: PROJECTIONS AND SLICING treats the convex hull of the sixteen points $(\pm 1, \pm 1, \pm 1, \pm 1)$ in 4-space, first by orthogonal projection then by

central projection from 4-space to 3-space. In each case we rotate in the coordinate planes xy, yu, xw, yw, and zw ending at the original position. We then slice each figure by hyperplanes perpendicular to the vectors (1, 0, 0, 0) then (1, 1, 0, 0) then (1, 1, 1, 0) and finally (1, 1, 1, 1). For a more thorough description of this film, see [4].

COMPLEX FUNCTION GRAPHS treats graphs of complex functions $w=f(z)$ considered as parametric surfaces (x, y, u, v) in 4-space, where $z=x+iy$ and $w=u+iv$. In each case orthographic projection into (x, y, u) is used to gett he graph of the real part of w (Figure 1), then rotation in the uv plane gives (x, y, v), the graph of the imaginary part of w. Rotating the original graph in the xv plane leads to (y, u, v) the graph of the imaginary part of the inverse function of f, and finally projection to (x, u, v) gives the graph of the real part of the inverse function.

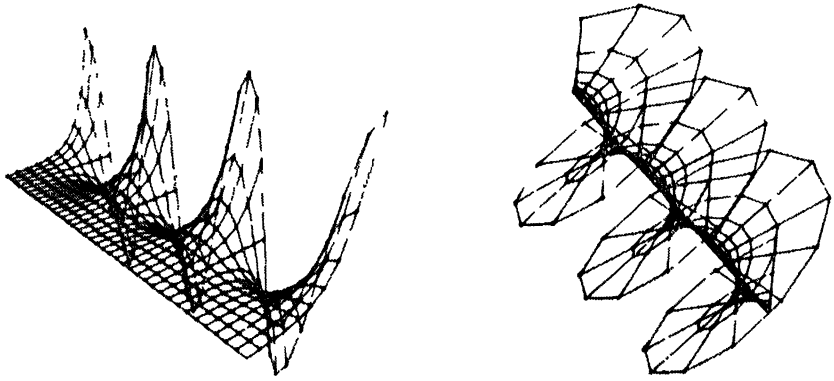




FIGURES 1 AND 2

The first example is the squaring function $w=z^2$ with domain given by the lower half of the Riemann sphere and graph given by $(x, y, x^2-y^2, 2xy)$ in 4-space. Projection of this locus into the (x, y, u) space gives a hyperbolic paraboloid (figure 1). Rotating in the uv plane gives the imaginary part of the squaring function, also a hyperbolic paraboloid. Rotating the original figure in the xv plane gives the imaginary part of the square root as the projection to (y, u, v) space. The graph of the inverse relation has a self-intersection curve along the positive u-axis and a singular point at the origin where the rank is 1. It represents a geometric realization of the Riemann surface of $z=\pm\sqrt{w}$ (Figure 2). In each projection the form of the parabolas $x=$constant is indicated. The special projection to the uv plane has a ramification point of order 2 which resolves into a hypocycloid with three cusps as the graph is rotated.

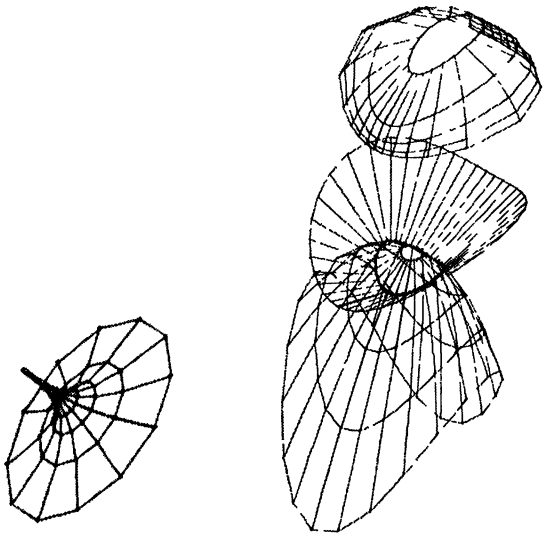
The second example is the exponential function $w=e^z$ with the inverse relation $z=\log(w)$. The domain is $-2\pi \leqslant x \leqslant 4\pi$, $-1 \leqslant y \leqslant 1$ and the graph is $(x, y, e^x \cos(y), e^x \sin(y))$ in 4-space. Projection to (x, y, u) gives the real part of the exponential (Figure 3). The projection (y, u, v) gives a right helicoid which represents the imaginary part of the Riemann surface for the logarithm (Figure 4).



FIGURES 3 AND 4

The projection (x, u, v) gives a surface of revolution of a real exponential function as the real part of the logarithm (Figure 5). This example is also described in [5].

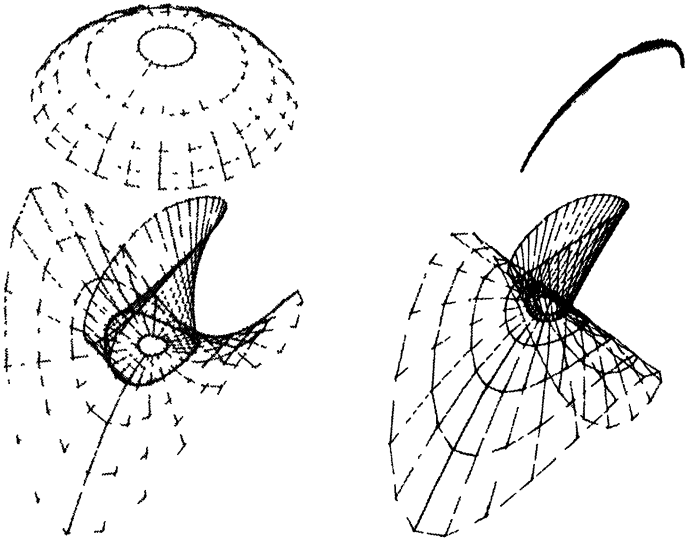
THE GAUSS MAP, A DYNAMIC APPROACH follows geometric ideas originated by Gauss in his paper defining total curvature of embedded surfaces. For the elliptic paraboloid, we show how the unit normals over a curve can be collected at a single point to form the boundary of the spherical image of the region bounded by the curve (Figure 6). The same procedure for the hyperbolic paraboloid produces



FIGURES 5 AND 6

a spherical image with orientation reversed (Figure 7). For a parabolic cylinder, the spherical image degenerates to a single curve (Figure 8).

We consider two examples which include elliptic as well as hyperbolic points and we examine in particular the singularities of the spherical image map. For almost all immersed surfaces, this mapping will have only folds and cusps as singularities and we indicate how two degenerate cases may be deformed to exhibit generic behavior at the cusps of this map.



FIGURES 7 AND 8

First we consider the monkey saddle, with an isolated point of zero Gaussian curvature, and perturb to get the graph of $(x, y, x^3-3xy^2+\varepsilon(x^2+y^2))$ (Figure 9). For $\varepsilon=0$, this surface has a Gauss mapping with a ramification point of order 2, and for $\varepsilon\neq 0$, the image of the parabolic curve will have three cusps (Figure 10).

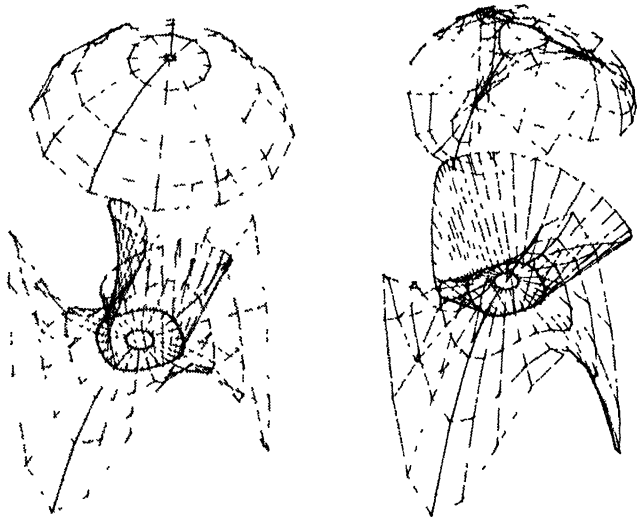
Secondly we consider the biparabolic surface which is the graph of $(x, y, (y-x^2)(y-\varepsilon x^2))$. For $\varepsilon\neq\pm 1$, the Gauss mapping of this surface will have exactly one cusp and the Gauss mapping will be stable (Figure 11). The case $\varepsilon=0$ was first investigated by M. Menn [7] (Figure 12).

In each case we show the spherical image of a circle $x^2+y^2=r^2$ as r changes. We show the linear interpolation between the surface and its Gauss spherical image so that the singularities of the Gauss map are expressed as limits of singularities of homothetic images of parallel surfaces of the original surface. We then show the spherical image of a test curve centered on the curve $r=$constant and indicate the behavior of the asymptotic vectors in a neighborhood of a cusp of the Gauss mapping.

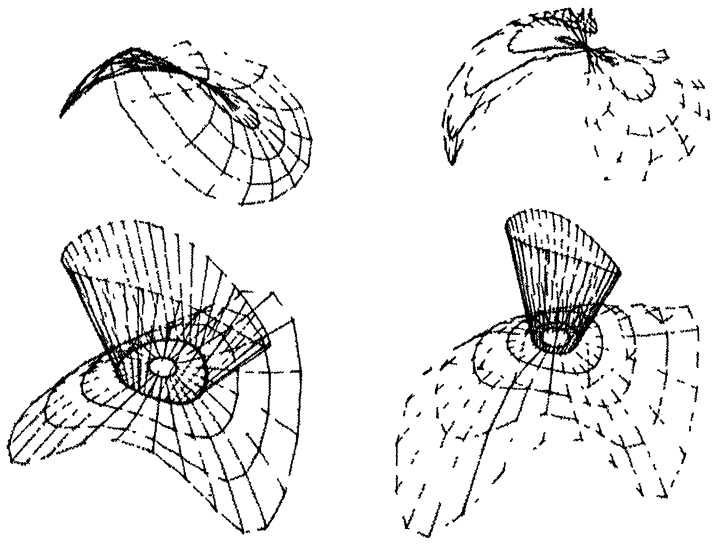
Various characterizations of the singularities of the Gauss map in terms of lines of curvature, ridges, and double tangencies are included in the joint work of the author with T. Gaffney and C. McCrory [3].

THE VERONESE SURFACE is an embedding of the real projective plane which starts with the hemisphere $x^2+y^2+z^2=1$, $z \leqslant 0$ and maps each point (x, y, z) to $(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx)$ in 6-space. The projection of this surface into 4-dimensional space given by

$$(\sqrt{2}xz, \sqrt{2}yz, (1/\sqrt{2})(z^2-x^2), \sqrt{2}xy)$$



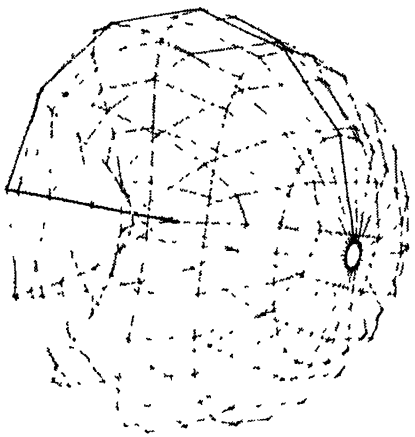
FIGURES 9 AND 10

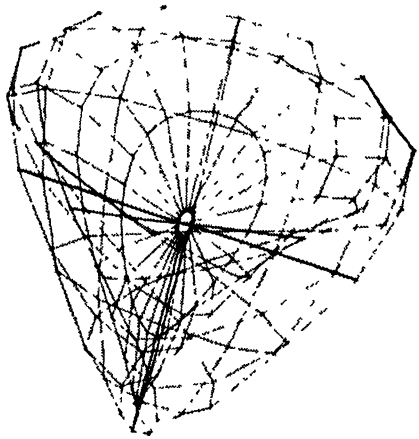


FIGURES 11 AND 12

is again an embedding and we examine a family of projections of this surface into 3-dimensional subspaces (all of which must have local singularities) [1].

The projection into the first three coordinates gives a cross-cap with two pinch points (Whitney umbrella points). The linear interpolation of the lower hemisphere into the cross-cap is a regular homotopy right up to the last instant when opposite points on the equator are identified, forming a segment of double points (Figure 13).





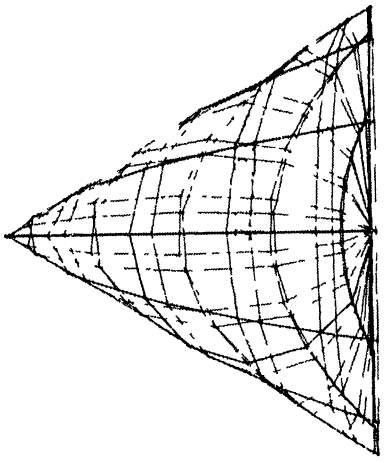
FIGURES 13 AND 14

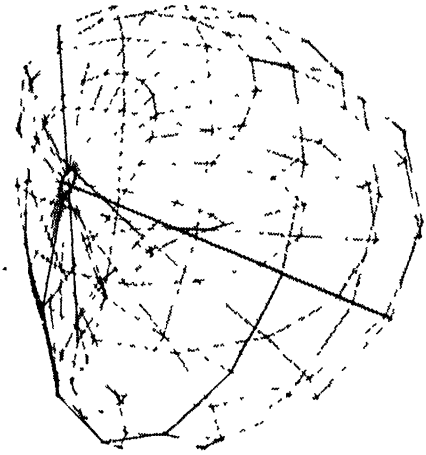
Rotating in the plane of the third and fourth coordinates gives a deformation from the cross-cap to Steiner's Roman surface $(\sqrt{2}xz, \sqrt{2}yz, \sqrt{2}xy)$ (Figure 14) with tetrahedral symmetry. This projection has six pinch points which are the endpoints of three double point segments intersecting in a triple point. These examples are described further in [6].

The embedding in 4-space is tight (i.e. almost every height function when restricted to the surface has exactly one maximum and one minimum) and this property is shared by the images in 3-dimensional subspaces. These examples lead to the conjecture that any stable tight mapping of the real projective plane into 3-space must have either two pinch points or six pinch points. At the position in the rotation where the figure moves from cross-cap form to Roman surface form, the double point locus consists of two straight lines, and one of the orthogonal projections to a 2-plane is an equilateral triangle (Figures 15 and 16).

In the film the cross-cap is sliced perpendicular to (1, 0, 0), then to (1, 1, 0), then to (1, 1, 1). As the first two slices pass through the origin, they contain the line which is the image of the tangent plane at a pinch point and they intersect the surface in a pair of ellipses which are tangent to the line.

After rotation in 4-space, Steiner's Roman surface is sliced in the same three directions, obtaining a cusp in the slice curve whenever the slicing plane passes a pinch point without containing the tangent line. The final slices have threefold symmetry,





FIGURES 15 AND 16

with a maximum, a curve with three cusps, then with three nodes, then with a triple point, then three nodes, ending at a doubly covered projective line with a Möbius band neighborhood containing three pinch points.

THE TORUS is given as the surface of revolution

$$((2+\sqrt{2}\cos\psi)\cos\theta, (2+\sqrt{2}\cos\psi)\sin\theta, \sqrt{2}\sin\psi)$$

and the slices in three different directions describe three different types of critical point behavior. Slicing perpendicular to (1, 0, 0) gives four non-degenerate critical points at different levels and the slice through the origin is a pair of congruent circles. Slicing perpendicular to (0, 0, 1) gives two critical levels, each consisting of a circle of degenerate critical points and the slice through the origin is a pair of circles with the same center. A classical problem in differential geometry asks for a direction for which there are exactly three critical levels, and the film illustrates such a slice, perpendicular to (1, 0, 1). In this case, the slice through the origin is again a pair of circles, this time intersecting in a pair of points [8].

THE FLAT TORUS is an embedding as a product of two circles in 4-space considered as the product of two planes, i.e.

$$(\cos\theta,\ \sin\theta,\ \cos\varphi,\ \sin\varphi).$$

This torus is a surface on the 3-sphere of radius $\sqrt{2}$, as we may project stereographically from $(0, 0, 0, \sqrt{2})$ and we obtain the torus in the previous paragraph (where $\sin\psi=\sqrt{2}\cos\varphi/(\sqrt{2}-\sin\varphi)$). Rotating the flat torus in the plane of the first and fourth coordinates produces a one-parameter family

$$(\cos\alpha\ \cos\theta+\sin\alpha\ \sin\varphi,\ \sin\theta,\ \cos\varphi,\ -\sin\alpha\ \cos\theta+\cos\alpha\ \sin\varphi)$$

which projects to a family of cyclides of Dupin, all conformally equivalent to the original torus. In particular when $\alpha=\pi/2$ and the point $(0, 0, 0, \sqrt{2})$ lies on the torus, the result is a noncompact cyclide which separates all of 3-space into two congruent parts.

The cyclides of Dupin and spheres are the only closed surfaces in 3-space which have the *spherical two-piece property*, so that any sphere separates them into at most two pieces. Their inverse stereographic projections are the only surfaces on the 3-sphere which are tight, so that every hyperplane separates them into at most two pieces [2].

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Хельсинки, 1978

Некоторые Вопросы Истории Теории Аналитических Функций в XIX в.

А. И. Маркушевич

1. В эпохальной работе Римана "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse" (1851) почти отсутствуют ссылки на предшественников. При этом в ней обходятся не только результаты и методы Коши и его последователей, но и «проблема века», которой сам Коши не занимался: изучение абелевых интегралов и их обращение. Но, конечно, Риман жил и воспитывался в идейной атмосфере, созданной всеми этими трудами. Задача настоящего доклада в том, чтобы полнее выявить те идеи и конкретные результаты теории функций середины прошлого века, которые влияли прямо или косвенно на диссертацию Римана.

2. Почти все связанное с именем Коши было уже сделано к этому времени. Это был набор мощных средств исследования различных проблем анализа: интегральная теорема, интеграл Коши, теорема о разложении в степенной ряд, включая и разложение неявных функций, понятие вычета и теорема о вычетах, мажоранты степенных рядов и связанное с ними «исчисление пределов» (calcul des limites), позволявшее обходиться в теории аналитических функций без понятия равномерной сходимости. Многое из указанного было опубликовано почти за четверть века дор рассматриваемого времени. Но и в 40-ые годы Коши усиленно распространял свои прежние и новые результаты, посредством "Exercices d'analyse et de physique mathématique", выходивших отдельными выпусками. На титульных листах четырех томов, каждый из которых составлен из 12 выпусков, значатся 1840, 1841, 1844 и 1847 гг. Необходимо, однако, учитывать, что фактические сроки издания выпусков последних двух томов запаздывали до 3—4 лет по отношению к 1844 и 1847 гг.

Выпуски эти благодаря расторопности их издателя Bachelier своевременно достигали больших городов Европы, в частности Берлина, где Башелье имел своего представителя.

Известно, что Риман, будучи в Берлине с 1847 г., обсуждал с Эйзенштейном основные принципы введения комплексных величин в теории функций, причем Эйзенштейн оставался на точке зрения формальных вычислений, а Риман выдвигал уравнения с частными производными в качестве существенной основы определения функции комплексного переменного. Конечно, формально-калькулятивная точка зрения была свойственна Эйзенштейну в его построении теории эллиптических функций; его центральная работа была опубликована в том же году. Но ясно, что речь у Римана шла не об эллиптических функциях. Мы склонны предполагать, что основным оппонентом Римана был фактически не Эйзенштейн, а Коши. В самом деле, в "Mémoire sur les fonctions de variables imaginaires" Коши, помещенном в 36 выпуске *Exercices* (он вышел по всем признакам к концу 1846 г.), молодые математики могли прочесть следующее определение функции комплексного переменного (по форме более узкое, чем в «Алгебраическом анализе» Коши): "Ces fonctions se trouvent complètement définies quand les opérations ont été définies elles-mêmes, et quand on a complètement fixé le sens de notations employées dans le calcul (*Exercices*, III, 1844, p. 366). Этот тезис прославленного ученого и мог послужить, по нашему предположению, исходным пунктом дискуссии.

3. Гораздо важнее для нас, однако, что Риман вовсе не отбрасывал понятия функций комплексного переменного, как результата операций над величинами. Вообще интерес Римана к старинной проблеме аналитического представления функций и к средствам такого представления удивительно глубок и устойчив. Ведь последней проблеме (правда, в действительной области) посвящена вторая его диссертация "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe" (1854). Но этот интерес был достаточно ясно проявлен и в первой диссертации. Уже в § 1 Риман делает оговорку, что определение функции комплексного переменного посредством требования дифференцируемости не исключает совпадения этого определения с понятием зависимости, выражаемой посредством операций над величинами "für den Begriff einer durch Grössenoperationen ausdrückbaren Abhängigkeit" (*Werke*, 1876, S. 4). Еще дальше он идет в § 20, не содержащем конкретных результатов, но по широте и общности играющем роль теоретического введения к работе. Риман утверждает, что причина и ближайшая цель введения комплексных величин в математику заключается "in der Theorie einfacher durch Grössenoperationen ausgedrückter Abhangigkeitsgesetze zwischen veränderlichen Grössen" (*Werke*, S. 38). Далее, резюмируя, что же нового дает его исследование для теории функций комплексного переменного, Риман указывает, что рассмотрение аналитических выражений он исключает *временно*: "da wir die Betrachtung des Ausdruckes einer Function gegenwärtig ausschliessen" (*ibid.*, S. 39). Наконец, он заключает

этот § задачей: убедиться в полном совпадении понятия функции комплексного переменного, установленного в диссертации "mit dem einer durch Grössenoperationen ausdrückbaren Abhängigkeit" (*ibid.*, S. 39). В сноске поясняется, что речь идет о зависимостях, выражаемых простейшими арифметическими действиями в конечном или бесконечном числе. Выражаясь современным языком, он допускал возможность охарактеризовать класс аналитических функций, как функций, представимых в виде предела последовательности рациональных функций (теорема Рунге). Конечно, в формулировке Римана не хватало требования равномерной сходимости. Однако, Вейерштрасс, от внимания которого это место не ускользнуло, оспаривал Римана в совсем другом, в чем Риман, явно подразумевавший связность области определения функции, был не повинен. В статье 1880 г. "Zur Functionenlehre" Вейерштрасс, приведя пример ряда рациональных функций, представляющего в раздельно лежащих областях различные аналитические функции, прямо пишет, что его цель доказать, что понятие моногенной функции комплексного переменного не полностью совпадает с понятием зависимости, выражаемой посредством арифметических операций над величинами и указывает в сноске что противоположное утверждение было высказано Риманом в конце § 20 его диссертации.

4. Если попытаться отвлечься от сегодняшних сведений в теории функций, то, пожалуй, самое неожиданное впечатление на читателя производят простота и непринужденность, с которой Риман вводит многолистную поверхность, разостланную на плоскости. До Римана математики с ними не встречались! Риман здесь, как и в других случаях, проявил способность гениального конструктора. Мы постараемся показать, однако, что материалы для этой конструкции были в известной степени заготовлены современниками Римана. Листы были выкроены, линии соединения и сам способ перехода от одного листа к другому были указаны. Оставалось объединить их в одно целое, склеить между собой.

Нужно начать все-таки с Коши. В цитированном выше мемуаре, говоря о способах выделения однозначных ветвей многозначных функций таких, как $\ln x$ и x^{α}, он замечает, что в предыдущих работах предполагал всегда, что действительная часть x положительна. Теперь же, вводя значения $p=\arg x$ в интервале $(\varphi-\pi, \varphi+\pi)$, Коши выделяет однозначные ветви функций не в полуплоскости, а в плоскости с разрезом вдоль луча, выходящего из начала координат. Он отмечает, что функция изменяется скачком при переходе x через этот луч. При всей элементарности этого шага вперед, сделанного Коши, в нем можно видеть одну из предпосылок к идеям Римана.

Существенное продвижение в этом направлении было связано с классическим мемуаром V. Puiseux, "Recherches sur les fonctions algébriques", 1850, автор которого, впрочем, настойчиво подчеркивал свою зависимость от идей и методов Коши. Публикация мемуара в распространеннейшем тогда журнале Лиувилля по времени совпадала с первой стадией работы Римана над дис-

сертацией — с октября по декабрь 1850 г. Пройти мимо работы, содержавшей последнее слово в теории алгебраических функций (и их интегралов) и возвещавшей о своем содержании в заголовке, Риман, по нашему убеждению, не мог. Ведь по его же характеристике область исследований, в которых переменным придавались комплексные значения, почти полностью сводилась к таким зависимостям, в которых одна из переменных есть алгебраическая функция другой, или интеграл от алгебраической функции (*Werke*, S. 38).

Мы придаем здесь такое значение высказанной гипотезе о знакомстве Римана в процессе работы над диссертацией с результатами Пюизе потому, что видим в этих результатах как бы заготовки, своего рода выкройки, римановой поверхности (замкнутой). Само появление понятия римановой поверхности в начале диссертации Римана (§ 5) с определением точек разветвления и описанием связи листов между собой, распадающихся на циклы в окрестности каждой такой точки, проницательный читатель того времени мог воспринять как превосходный геометрический комментарий и вместе с тем своего рода резюме основного содержания работы Пюизе.

Заметим, что в этой работе появилось впервые в печати точное описание процесса аналитического продолжения, обычно связываемого с именем Вейерштрасса. Пюизе указывает, что его можно осуществлять также, пользуясь вместо цепи кругов, например, цепью лемнискат. С принципиальной точки зрения важно, что он устанавливает для алгебраических функций эквивалентность двух типов продолжения: с сохранением аналитичности и с сохранением непрерывности. Благодаря этому топологические (гомотопические) идеи играют у Пюизе существенную роль. И в этом отношении он также выступал предшественником Римана. Задача, которую ставит и решает Пюизе заключается в том, чтобы не выполняя фактически процесса продолжения по какому-либо пути, не проходящему через особые точки, и опираясь только на сведения о характере поведения функции в окрестности особых точек, сравнивать конечные результаты продолжения вдоль путей с общими началом и концом. Здесь-то и выявляется необходимость использования топологических взаимоотношений между кривыми, что и делает Пюизе. Но, конечно, Риману целиком принадлежит заслуга вычиленения фундаментальных понятий зарождавшейся теории: порядка связности и рода поверхности и установление соотношений между ними.

Остается добавить, что и после Римана далеко не все математики, изучавшие алгебраические функции и их интегралы, охотно оставляли плоскость и переходили на риманову поверхность, представлявшую, конечно, более высокую ступень абстракции. Так, четверть века спустя, Briot и Bouquet во введении к "Théorie des fonctions elliptiques", 1875, писали, что метод Римана не кажется им имеющим какие-либо преимущества перед идеей Коши, предлагавшего для представления многозначных функций присоединять к значению переменной соответствующее значение функции и различать индексами значения

функции, если они изменяются, когда переменная описывает замкнутую кривую. В этом пассаже верных последователей не только духа, но и буквы Коши, мы видим убеждение современников Римана, что средства, которые предлагал Коши (и развивал Пюизе) равносильны средствам самого Римана. Конечно, это несправедливо по отношению к Риману. Подчеркнем еще раз, что наш экскурс в сторону мемуара Пюизе имел лишь целью показать, что в нем заключался строительный материал, который Риман мог использовать для своей гениальной конструкции, оказавшей столь большое влияние на последующее развитие математики.

5. Рядом с центральными идеями диссертации: концепцией аналитической функции и ее носителя — римановой поверхности, результат, который мы хотим выделить в заключение доклада, имеет частный характер. Однако, в нем зародыш будущей теории граничных свойств аналитических функций, в которой объединяются методы классического анализа и последующей теории функций действительного переменного.

Речь идет о следующей общей теореме § 10, относимой ныне к так называемым теоремам о "стирании особенностей аналитических функций". Вот формулировка Римана, которую он рассматривает как следствие из приведенного им анализа свойств гармонических функций:

«*Если функция* (речь идет об аналитической функции на некоторой римановой поверхности — *А. М.*) *не имеет разрывов непрерывности вдоль линии, и далее для любой точки O′ поверхности, где* $z=z'$, *произведение* $w(z-z')$ *является бесконечно малым вместе с* $z-z'$, *то она необходимо конечна и непрерывна вместе со всеми производными во всех точках поверхности.*»

Доказательство Римана нельзя признать достаточным. Если рассматривать эту теорему с современных позиций, то следует напомнить, что аналитичность в данной области непрерывной функции, для которой конечные производные $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ существуют всюду (кроме, быть может, конечного или счетного множества точек) и удовлетворяют уравнениям

$$\partial u/\partial x = \partial v/\partial y,\ \partial u/\partial y = -\partial v/\partial x,$$

впервые удалось установить вполне строго только в 1932 г. (теорема Looman—Menchoff). При этом доказательство существенно использовало средства теории функций действительного переменного (в частности, теорию меры и интеграла Лебега).

Что касается утверждения Римана о стирании возможных особенностей, расположенных на дуге кривой, на которой функция непрерывна и в окрестности которой она аналитична, то оно неверно без дополнительных ограничений на эту дугу. Однако, оно справедливо, как впервые доказал через полвека P. Painlevé при условии, которое Риман, вероятно, считал само собой разумеющимся — спрямляемости дуги. Во всяком случае, введя необходимые

уточнения в эту теорему, можно полностью сохранить ее значение для последующего изложения.

Факты, предположения и отдельные замечания, составившие содержание этого доклада, конечно, не должны и не могут в какой-то степени умалить значение труда Римана, перед гением которого мы преклоняемся. Но если нам удалось выявить отдельные, пусть небольшие, подробности той творческой атмосферы, в которой эта работа возникла, и указать на то, что связи ее с идеями таких современников Римана, как Коши и Пюизе, воможно были более существенными, чем это обычно принято считать, то наша цель полностью достигнута.

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