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Growing mathematical objects in the classroom – The case of function☆

Talli Nachlieli [a,*], Michal Tabach [b]

[a] Levinsky College of Education, Israel
[b] Tel-Aviv University, Israel

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ABSTRACT

This article is devoted to some of the educational quandaries stemming from the fact that mathematics is a discourse that creates its own objects. More specifically, we ask how the participants of classroom learning-teaching processes cope with the seemingly paradoxical situation in which they are supposed to talk about objects, of the existence or nature of which they are not yet sufficiently aware (it is through participation in a conversation about them that these objects are being brought into being). To answer this question, we watched videos of 7th grade students as they were making their first steps in the discourse on functions. The learning–teaching processes were followed for nearly two months. Curricular materials and the teachers' discourse were documented and analyzed as well. We found out that the students were able to participate in the discourse on function without ever dealing directly with this as-yet nonexistent object. They managed to cope with problems by associating them with solution routines through all kinds of discursive clues, to which they were sensitized through their former experience. This said, we were also able to conclude that although the learners were still in the early phase of the project of objectification, their participation in the project laid a solid foundation for their future discourse on functions.

1. Introduction: the quandary of mathematical objects' creation

One of the defining characteristics of mathematical discourse that put it apart from almost any other form of communication is that it constitutes an autopoietic system, one that spurs its own development and incessantly grows "from inside". Unlike many other discourses, the objects of mathematics, those things that are being talked about, do not pre-exist the talk; rather, they arise as byproducts of the ongoing mathematical conversation.[1] In this sense, negative numbers, functions and sets are quite unlike plants, animals and stars, or even as force or velocity – as all those real-world phenomena of which the child may be aware before she has linguistic means to deal with them. Clearly, the autopoietic nature of mathematics invites many questions about its learning. This article is devoted to some of these quandaries.

One of the most resilient pedagogical dilemmas related to the introduction of whole new mathematical objects regards the order of this special meta-level learning. To remind (see Section 1), meta-level learning has been defined in the introduction to this volume as one that involves changes in the meta-rules of the discourse; this, as opposed to object-level learning that just expands the existing assortment of routines and endorsed narratives. Meta-level development of a discourse could involve combining the existing discourse with its own meta-discourse (*vertical* development) or involve

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* Corresponding author at: 41 HaGefen St., Sitriya 76834, Israel. Tel.: +972 52 8258214.
E-mail address: talli@nachlieli.net (T. Nachlieli).

[1] We do not mean to say that intra-discursively produced objects can be found only in mathematics. In fact, they are everywhere. However, mathematics is unique in that from a certain point on in the development of this dicourse, the intra-discursive processes are by far the leading, if not the only ways of producing new objects.

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combining discourses which were separate into a single one by subsuming them to a new discourse populated with new types of mathematical objects (*horizontal* development). In this paper we widely discuss the development of the discourse on functions as one that subsumes discourses on algebraic expressions, on lines in a plain, and operations on numbers. The situation and its requirements seem to be somewhat paradoxical. If, on the one hand, mathematical object is a discursive construct, one needs to engage in the discourse about it to bring it into being. More specifically, one has to *begin* with actually using the signifier, say the word *function*, because only through applying it in talk one can consolidate such hereto unrelated things as symbolic expression, table and graph into one whole. Indeed, these diverse entities acquire unity by turning into a *realization tree* of the word *function* (just to remind, it is the realization tree that converts a mere sound or a mark on paper into a signifier of an object). On the other hand, how can one talk about an object that has not yet been constructed? What can a person say about this non-existent thing?[2]

The pedagogical question immediately follows: what can the teacher do when faced with this kind of situation? A pretty straightforward answer has been offered by Vygotsky:

> The development of a scientific concept begins with the verbal definition. As part of an organized system, this verbal definition descends to the concrete; it descends to the phenomena which the word represents (Vygotsky, 1987, p. 168).

Provided we agree that in the case of mathematics, Vygotsky's *scientific concept* translates into a formally defined *mathematical object*,[3] this quote says that new objects make their way into mathematical discourse via their explicit definitions. In the light, however, of what is known today about the seemingly negligible influence of definitions on students' use of words (e.g. Fischbein & Nachlieli, 1998; Hershkowitz & Vinner, 1983, 1984; Tall & Vinner, 1981; Vinner, 1990; Wilson, 1990), this advice may raise some brows. As Lemke (1990) reminds us, ‘‘classroom language is not just a list of technical terms, or even just a recital of definitions. It is the use of those terms in relation to one another, across a wide variety of contexts.'' (p. 12). And, indeed, literature abounds in counter-proposals. For example, some writers (e.g. Skemp, 1971) speak about beginning with specific examples of the object in question rather than with its definition. Considering the fact, however, that examples can only be presented with the help of specific realizations, this proposal, if not coupled with defining, is unlikely to work. It is through defining that the learner may begin to understand that no mathematical object should be identified with a specific concrete thing, including dedicated ideograms, and that mathematical objects exist ‘‘in between'' symbols rather than in any one of them.

Although combining definitions and examples seems as an optimal approach, it is by no means an ultimate solution to the problem of introducing mathematical objects. In fact, there can be no such thing as ‘‘ultimate solution'', in this case. Whereas conjuring something out of nothing seems like a bit of miracle, there is no miraculous way around the inherent difficulty of the task. In the case of the object called *function*, this difficulty may be more acute than for any concept in natural sciences and also for the majority of mathematical objects that populate school curricula.

In the discourse of physics, for example, almost any scientific concept learned in school has its informal precursor. Thus, the student is familiar with words such as *force* or *velocity* (or *speed*) prior to their introduction in physics lessons. Before being initiated to the formal discourse of science, the learner is able to use the terms because she associates them with a whole complex of well-known situations and phenomena. Such palpable, directly experienced precursors of scientific concepts provide an easy way around the formal definitions: the learner can participate in the discourse by relying on her previous informal use of words. Often, the student would indulge in this practice of recycling old uses in the new discourse without qualms, unimpressed by occasional inconsistency with the formal definition. Also in mathematics, innovations may come as extensions of existing, familiar objects. Thus, for example, one can offer *negative number* as a new species within the familiar category of numbers, just as ostrich can be introduced as a (somehow unusual) type of bird. In the case of function, however, the situation is quite different. Here, there is no informal basis and no analogical example on which the new mathematical discourse could be built. Unlike numbers or geometric figures, function has no obvious spontaneously developed precursor – no direct mathematical ‘‘predecessor'' from which it could emerge as this object's ‘‘next generation''.

It seems, therefore, that when the word *function* is uttered in the class for the first time, the learner must start to develop a new discursive object practically from scratch. The aim of this paper is to try to see what happens in the classroom when this situation occurs. To this end, we visit an algebra class of 12–13-year-old 7th graders. While looking at the teaching materials, the teacher and the children making their first steps in the discourse on functions, we ask a number of questions:

(1) What did the curriculum developers and the teacher do to support their participation in the new discourse?
(2) What means did the students use to cope with the inherent circularity of the task while trying to develop the discursive object called function? Here, particular attention will be given to the query of what role, if any, is played by the definition of function in the process of object construction.

[2] This dilemma is a commognitive counterpart of the famous *learning paradox*, known also as the *paradox of Meno* (e.g. Bereiter, 1985).

[3] Vygotsky defined *scientific concepts* as products of an effort to regulate the use of words in such a way as to turn them into a part of an unambiguous, well-organized conceptual system. In result, the use of these words is governed by explicit rules relating the given word to the rest of the system. The counterpart of a scientific concept is described with the adjective *everyday* because it is learned spontaneously through everyday experiences. Based on these descriptions, everyday and scientific concepts can be seen as coming from *informal* and *formal* discourses, respectively, as these discourses have been defined in the introduction to this volume.

(3) *How successful were the participants in their attempts to develop a discourse on functions?* More specifically, what kind of mathematical objects, if any, emerged from the two months attempt to develop students' discourse on function?

Needless to say, since our findings are coming from one particular classroom, we will have to be careful while trying to draw any general conclusions. This said, we expect that with the help of an analytical support derived from our theoretical model of discourse development, some tentative generalizations will be possible. Our generalizable insights, we hope, will contribute to cracking the puzzle of the inherently problematic meta-level learning. They are also likely to engender some initial ideas about what works for learners and what stymies them in their first attempts to come to grips with new mathematical objects.

2. Discourse on function: what do we know about its development?

2.1. What kind of mathematical object is function?

As explained in the introduction to this volume, a mathematical object is a signifier together with its *realization tree*, whereas a realization tree is a hierarchically organized set of realizations of the given signifier together with the realizations of these realizations, these latter realizations' realizations, and so forth. In the case of function, the realizations of the signifier (e.g., *basic quadratic function*) can come in various forms, e.g. as an algebraic expression (x^2), as a curve (called *parabola*), as a table (that matches the numbers in the left-side column with their squares in the right-hand column) or as an encapsulated collection of ordered pairs of numbers, ($<a,b>$) each of which is a reification of the process of squaring ($b = a^2$). The first three of these realizations may also play a role of a signifier: in the present case, the relation signifier/signified[4] is symmetric.

Fig. 1 shows a schematic, and only partial, realization tree of the signifier *basic quadratic function*. Of course, the example given here outlines our own vision of what the basic quadratic function is (just to remind, realization tree of a given signifier is an individual construct, and it may be different from one person to another). This said, we also wish to believe that it would be endorsed by the professional mathematical community, or in other words that it can count as picturing the canonic, expert version of the discursive object called *function*.

2.2. Circularity of the historical process of building the object called function

History of mathematics, if told as a story of a discourse, becomes a source of insights about how the circularity of object creation can be broken. In what follows, we take a look at the historical development of the discourse on function. Let us make clear what is meant, in the present context, by the term *discourse on function*. Although history of mathematics is teeming, from its earliest times, with events that can count as related to the eventual emergence of the discourse on functions, ours is the story of the discourse that foregrounds the topic of function and actually uses the word *function*. We take the introduction of the term as a point of departure simply because this is the moment in history when the deliberate, concentrated effort to build a new object called *function* began.

The birth of this new object was announced not only by the actual appearance of the word *function*, but also by the shift in mathematicians' attention to what may today be considered as the materials from which a realization tree of function was to arise: geometric forms and algebraic symbols. This process began at the turn of the eighteenth century, when Leibniz initiated the use of the word *function* in the geometric context and Johann Bernoulli, possibly as a result of his correspondence with Leibniz (1694–1698), proposed a definition that portrayed function as an algebraic construct: *one calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants* (Johann Bernoulli, 1718; after Kleiner, 1989). While Bernoulli's focus on algebraic symbolism was only implicated (variables and constants are to be understood as letters used in particular mathematical ways), it became quite explicit in Euler's reformulation of 1748, which identified function with *analytic expression* (*A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities*; Euler 1748; after Kleiner, 1989).

One of the things that can be learned from the scrutiny of developments that followed these first attempts at introducing the new object called *function* is that objectification may be a cyclic process, with each cycle initiated by a tentative proposal for the use of a signifier in question, continuing with the actual use of the signifier in the proposed way, and ending with a proposal for a revision of the original proposal. The reason for these persistent recurrent attempts at creating a new discourse built around a new noun was the mathematicians' growing awareness of an isomorphism between discourses on lines in the plane and on algebraic expressions (this awareness found its initial expression prior to the introduction of the notion of function, in Descartes' creation known today as *analytic geometry*). The introduction of the new noun *function* marked the beginning of the effort to unify these two discourses into one. Whereas such unified discourse was supposed to be about functions, each of its narratives was to be interpretable both as a narrative on lines and as a story of algebraic expressions. And there was yet another kind of talk waiting there to be subsumed in the discourse on function: the talk of scientists studying multifarious natural phenomena, such as motion, heat and light.

[4] Our pair *signifier–signified* can be seen as a counterpart of Peirce's *representamen-interpretant* in his triad *representamen-interpretant-object*. Peirce's *object* refers to the extra-discursive entities supposed to be the primary source of the whole triad. Since mathematical objects are produced intra-discursively (originate in the reflection on the discourse itself), this part of the triad is irrelevant in the current discussion.

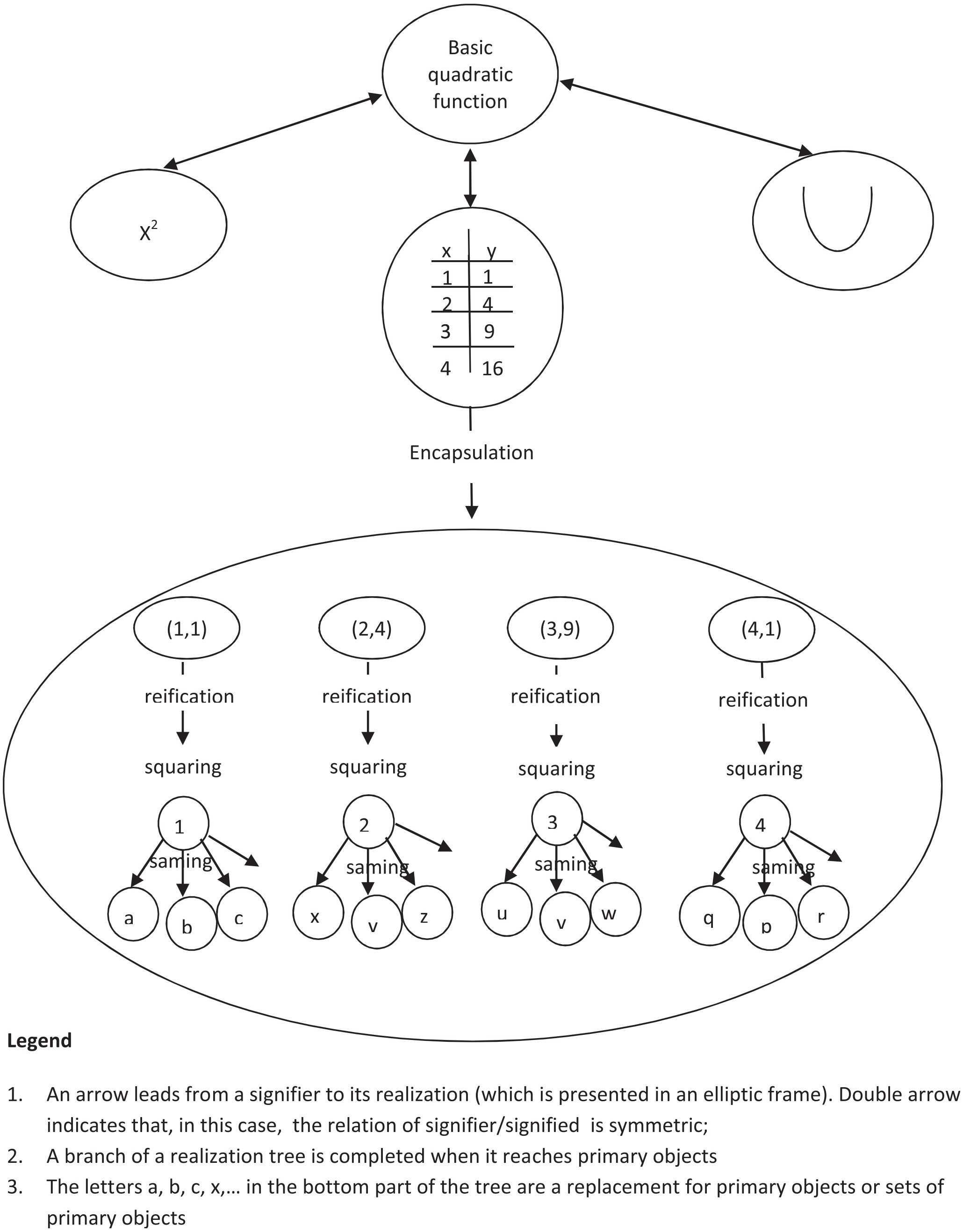


Fig. 1. Realization tree of basic quadratic function. (1) An arrow leads from a signifier to its realization (which is presented in an elliptic frame). Double arrow indicates that, in this case, the relation of signifier/signified is symmetric. (2) A branch of a realization tree is completed when it reaches primary objects. (3) The letters a, b, c, *x*, . . . in the bottom part of the tree are a replacement for primary objector sets of primary objects.

The sustained efforts to create this unifying discourse, and thus to objectify the noun *function*, lasted for more than two centuries and have been fueled by what can be described with the help of metaphor borrowed from Kleiner (1989) as a tug-of-war between the contributing discourse: whenever a version was found that satisfied mathematicians' requirements with respect to the allowed geometric realizations of function, this discourse would fall short of expectations when it came to algebraic or concrete real-world realizations. And vice versa: whenever a definition was found that would satisfy algebraic expectations, the geometrical or real-world side of the story would leave something to wish for. And thus, for example, when Euler realized that some useful lines could not be described with a single "analytic expression" (these were lines that correspond to what is called today *split domain* function), he modified his own definition of function by removing any mention of visual mediators. His new definition spoke of function as "a *quantity* that depends on another quantity" (Kleiner, 1989; emphasis added). Other definitions of function followed, each one of them extending the set of what could count as an instance of function in order to include either geometric or algebraic constructs left out by the former version, but now considered as an important part of the discourse. The process of objectification of function has been completed (at least for now) in the middle of the 20th century, when Bourbaki group formulated its famous definition, made possible by renouncement of any requirement whatsoever for either symbolic–algebraic or geometric realization:

Let E and F be two sets, disjoint or not. A relationship between a variable x of E and a variable y of F will be called a functional relationship in y if for every $x \in E$ there exists a single $y \in F$ that is in the given relation with x (from Kleiner, 1989).

Another insight to be gleaned from history concerns the question of what it is that motivates this cyclic process of construction and gives it its direction. Historical data make us aware of the importance of mathematicians' intuitions regarding the nature of the object yet-to-be-built and of their intimations with regard to its potential uses. As can be learned from the few examples mentioned above, these intuitions were themselves subject to modifications, and every so often, either a desired property or an expected use of the object had to be compromised for the sake of further development, be these concessions as difficult to make as they might.

To sum up, creation of the realization tree for the signifier *function* was a painstaking, prolonged process and it did not happen without struggles. One thing that a teacher may wish to remember while introducing students to this new discourse and trying to help them in overcoming the circularity of the construction process is that historically, this uneasy task has been accomplished through the persistent use of the signifier *function* and its cognates that began well before their realizations have been agreed upon by all the members of the mathematical community.

2.3. Circularity of the ontogenetic process of building the object called function and its consequences for learning

There is no reason to assume that those who learn mathematics can be spared struggles similar to those fought by mathematicians of the past. Educational research provides much evidence for a considerable difficulty experienced by students trying to become full-fledged participants in the discourse on functions. Numerous findings, although usually reported in the language of *conceptions* rather than of discursive objects, can nevertheless be interpreted as showing that the learners' realization trees would often be quite different from what curriculum developers and teachers usually hope for, and that in spite of the teachers' efforts, this would frequently be the case till the students' last days in school.

Indeed, results of many research studies seem to be implying that learners tend to identify function with what for an expert would count as but one of its realizations (or "representations"): with either graph of algebraic formula (Even, 1990; Leinhardt et al., 1990; Sfard, 1992; Vinner & Dreyfus, 1989). Moreover, the word *function* may be seen as tantamount with algebraic formula in one context, as a graph in another context, and only rarely as related to ("represented by") both of them at the same time. One of the recurring motives of research on function is that students have difficulty making transitions from one mediational mode to another (or, as the authors of these studies usually put it, from one *representation* to another; e.g. Keller & Hirsch, 1998; Schwarts & Hershkowitz, 1999). All this means that rather than creating a unified discourse, these students have developed a bunch of discourses that, although evolving around the same keyword, use this keyword in different ways and on different occasions.

A testimony for the unyielding nature of the problem is the fact that all the above phenomena, while ubiquitous, tend to persist, whatever the teaching method. In particular, intensive work with explicit definitions of functions does not seem to have the expected corrective impact. Studies show that even those learners who are able to reproduce definitions may act in ways and endorse narratives that contradict this definition (Tall & Vinner, 1981; Vinner & Dreyfus, 1989).

These, as well as many other function-related phenomena require more research. In spite of the proliferation of educational studies on functions, not much is known about the actual *process* of learning. Only a few researchers tried to follow this process as it actually happens in the classroom (Walter & Gerson, 2000; Yerushalmy, 2006), and a small number of others conducted similar investigations in laboratory settings (Chiu et al., 2001; Moschkovich, 2004; Schoenfeld et al., 1993). Our awareness of the enormity of the challenge awaiting the learner who is being introduced to the discourse on function and the scarcity of the relevant research were the main incentives for the study to be presented on the following pages.

3. When functions enter the discourse – a story of one classroom struggling for objectification

The data we are going to use in the analyses that follow come from the Montreal Algebra Project (MAP).[5] Some readers may already be familiar with this project, which has been carried out as long ago as the years 1992–1996. Its results have already been extensively analyzed and insights gained through their analyses have been reported in a series of papers (see e.g., Kieran & Sfard, 1999; Kieran, 1994; Sfard & Kieran, 2001). If we decided to have a new take on its data, it is because they seemed rich in opportunities for relevant insights that have not been exhausted in the former analyses. If we waited for more than a decade to return to the MAP data with this new attempt to understand the process of objectification, it is because all this time was necessary to develop conceptual tools that can push our thinking farther than the original authors were able to go in the earlier round of analysis. This said, all along the following pages the reader may wish to keep in mind that since MAP

[5] We wish to thank the principal investigators in that project, Carolyn Kieran and Anna Sfard, for letting us use their data.

was not designed for the present study, one may not find in its data all that is necessary if these new analyses are to be implemented in the optimal way.

The data consist of video recordings and transcripts of two months long introductory course in algebra that took place in one 7th grade class. The aim of that project was to investigate students' initial algebraic learning and, at the same occasion, to test certain hypotheses about possible ways of promoting this process. A teaching sequence was designed that introduced algebraic thinking via stories, tables, and graphs, and only much later with the help of algebraic symbols.

The main part of the study took place in one 7th grade class of a private Montreal middle school. All students were pretested and many of them were also interviewed before the teaching began. Each of the subsequent 26 lessons, some of which took place in a regular classroom and some in computer laboratory, began with a brief introductory exposition by the teacher (see Fig. 2), continued with students working in pairs (Fig. 3), and ended with a whole class discussion, during which the results of the small team work were analyzed, assessed, and consolidated. Research assistant served as a teacher, whereas the regular teacher of this class, along with two principal researchers, conducted observations and interviews, and assisted the students in their work, whenever necessary.

All class and computer laboratory sessions were videotaped and all students' paperwork was collected. Two focus pairs were followed during the teamwork by two separate cameras. Much of our data in this article comes from one of these pairs. Ari and Gur, two 12-year-old boys, were chosen at the beginning of the project as quite representative for their class: their pretest scores were similar and close to the class average (Sfard & Kieran, 2001).

In what follows, we organize our report on findings around the three questions asked in the introduction to this paper.

3.1. What did the curriculum developers and the teacher do to support students' participation in the new discourse?

The following description of MAP instructional design is based on its written documentation (teacher's materials, see e.g. course syllabus in Table 1, and students' worksheets), on our analysis of teacher's talk during the lessons, and on personal communications with one of MAP course developers (Sfard). The curricular decisions were grounded in the designers' theoretical assumptions about development of the discourse on functions, on the one hand, and in their vision of the needs of the learner vis-à-vis this development, on the other. The central fact considered in this process was the inherent circularity of object development. The designers assumed that because of this circularity, the learners would have to engage in the conversation about functions as a precondition for objectification of the focal signifier *function*, that is, before they had built its realization tree or even had just figured out how the new discourse is related to the subsuming discourses.

Fig. 2. The Montreal class.

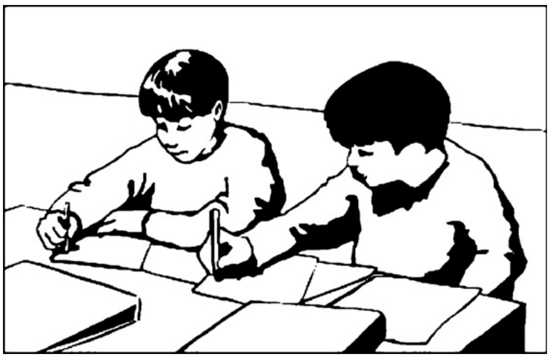


Fig. 3. Ari and Gur.

Table 1
The outline of the introductory algebra course.

Unit	#	Subject	Activities
1 Graphs	1	Discrete graphs, no rule	1) **Plot** points given in table onto prepared plane 2) **Read** qualitative and quantitative information from scaled graph 3) **Construct** graph from table, prepared axes 4) **Construct** graph from table, no prepared plane (challenge)
	2	Continuous graphs, no rule	1) **Read** information from scaled graph 2) **Construct** graph from table, make plane 3) **Match** unscaled graph to story when no specific numbers are given
	3	Story-based graphs with a rule	1) Make a table from a linear story, make the word rule, plot the points, join 2) Do the same for a story in two parts: linear and quadratic
	4	Abstract graphs with a rule	1) Make a table and graph from a linear rule: issues of domain/range 2) Make a table and graph from a quadratic rule
	5	Graphs with rules: on computer	1) Enter table from 3.1, computer plot, compare 2) Enter table from 4.1, computer plot, compare 3) If time: construct a new table from story, computer plot, investigate for new data
	6	Discussion and summary	1) Integer operations 2) Rules from stories 3) Info from graph 4) Plot points from table 5) Table from story 6) Table from rule
2 Algebraic expressions	7	Algebraic expression as a symbolic representation of a rule	1) Make expression from rule
	8	Interplay between story, table, expression, graph	1) Substitution into expression to make table 2) Judge whether expression matches graph 3) Match expressions to stories, tables, graphs
3 Functions	9	Introduction to the concept of function (also D + S from unit 2)	1) Given expression for $f(x)$, find $f(2)$, etc. 2) Plot two expressions which yield the same table and same graph
	10	Categorizing functions: linear, quadratic, other	1) Compare a linear and quadratic function by: expression, graph, table 2) At the computer, enter given functions, and from the shape of the resulting graph, sort the expressions into linear, quadratic, or other. 3) Homework: given several expressions, predict shape of graph
	11	Investigate linear functions	1) At computer, investigate changes in parameters in story context 2) Homework: make expression from graph
	12	Investigate quadratic functions	1) At computer, investigate changes in parameters in $ax2 + b$ 2) Match quadratic graphs and expressions 3) Compare expressions
	13	Consolidating linear graphs and expressions	1,2) Make expression from graph using slope and y intercept 3) Match expressions to graphs (linear) from slope, y intercept
	14	Review for test	
	15	Test	
4 Equivalence of expressions	15	Operations on functions: addition	1) Add two graphs connected to a story, write all 3 expressions 2–5) Add two graphs, write expression for sum 3) Draw graph of 2 functions, add, write expression
	16	Distributivity # 1	1) Write sums given 2 expressions 2) Simplify more than 4 terms 3) Given 2 expressions, make graphs, add graphs, write new expression
	17	Grouping like terms (subtractions)	1) Quiz: simplify expression; write expression for graph 2) Decompose $ax + b$, some negative terms 3) Simplify with negative terms included
	18	Distributivity #2 (multiplication)	1,2) Multiply graph point-wise 3) Simplify expressions with brackets 4) Simplify expressions with combinations of operations
	19	Review: equivalence of expressions (simplification)	Computer-generated randomly: 1) Simplify additions (expressions) 2) Simplify multiplications (expressions)
	20	Consolidation of equivalence of expressions	7 Activities centered on making equivalent expressions
	21	Linear functions in tabular representation	1) Concept of function as an infinite set of points 2) Linear expressions from tables 3) Classifying tables into linear and other 4) Write expressions for functions in tables 5) Judge expression fits table 6) Increasing, decreasing and constant functions in tables

Table 1 (*Continued*)

Unit	#	Subject	Activities
5 Comparing functions: equations and inequalities	22	Concept of equation and inequality	1) Making pointwise comparisons between two functions 2) Writing equations and inequalities to express comparisons 3) Solving equations and inequalities from graphs
	23	Equivalence of equations and inequalities	1) Solving perimeter problem 2) Writing and solving equations from story 3) Solving equations using graphs
	24	Exploration of types of solutions to equations and inequalities; verifying solutions of equations and inequalities	1) Computer graphing to solve equations and inequalities 2) Determining from a graph whether an equation or inequality has only one solution, an infinite number of solutions or no solutions 3) Match graphs with equations to solve
	25	Classifying equations according to their types of solutions	1) Determining from an algebraic equation how many solutions it will have, by simplification
Evaluation	26	Review 1	
	27	Review 2	Computer activities: 1) From expression, make table, write expression 2) From table, make expression
	28,29	Test one, test two	

The principles and the resulting didactic moves of the teacher, all of them meant to support the learners in their struggle for objectification, are listed below. It must be stressed that at the time the curriculum was being constructed, many of these principles remained tacit, exerting their influence from behind the scene. It is thanks to theoretical developments in years that followed the completion of the project that the articulation of the pedagogical intuitions underlying the curriculum became possible. As will be explained in the closing section of this article, these theoretical advances prompted us, and the designers themselves, to take a critical look at some of the principles and at the way they acted upon them.

Principle 1: To objectify function, the student must be competent in discourses on its future realizations, that is, algebraic symbols (symbolic expressions), graphs, tables, etc. According to this principle, the learner's ability to participate in discourses that are to be subsumed under the discourse on function is the precondition for the introduction of the latter. In the present case, this meant that the introduction of functions had to be preceded by a thorough work on the discourses on symbolic expressions, on graphs, and on tables. Discourses on real-world phenomena involving covariant quantities should be promoted as well, in conjunction with the others.

The first task of the designers and the teacher, therefore, was to ensure that students would practice these component discourses. Our assumption, confirmed by the results of a test given to the participants at the beginning of the project, was that none of the children was as a blank slate with respect to any of these discourses. Nevertheless, all of them needed much more practice. During the first six lessons (1–6, Fig. 4) three visual mediators – algebraic expressions, graph, and tables – were introduced, and they were often used in the context of real-world phenomena involving pairs of covariant quantities. The development of respective discourses was the main objective of those lessons.

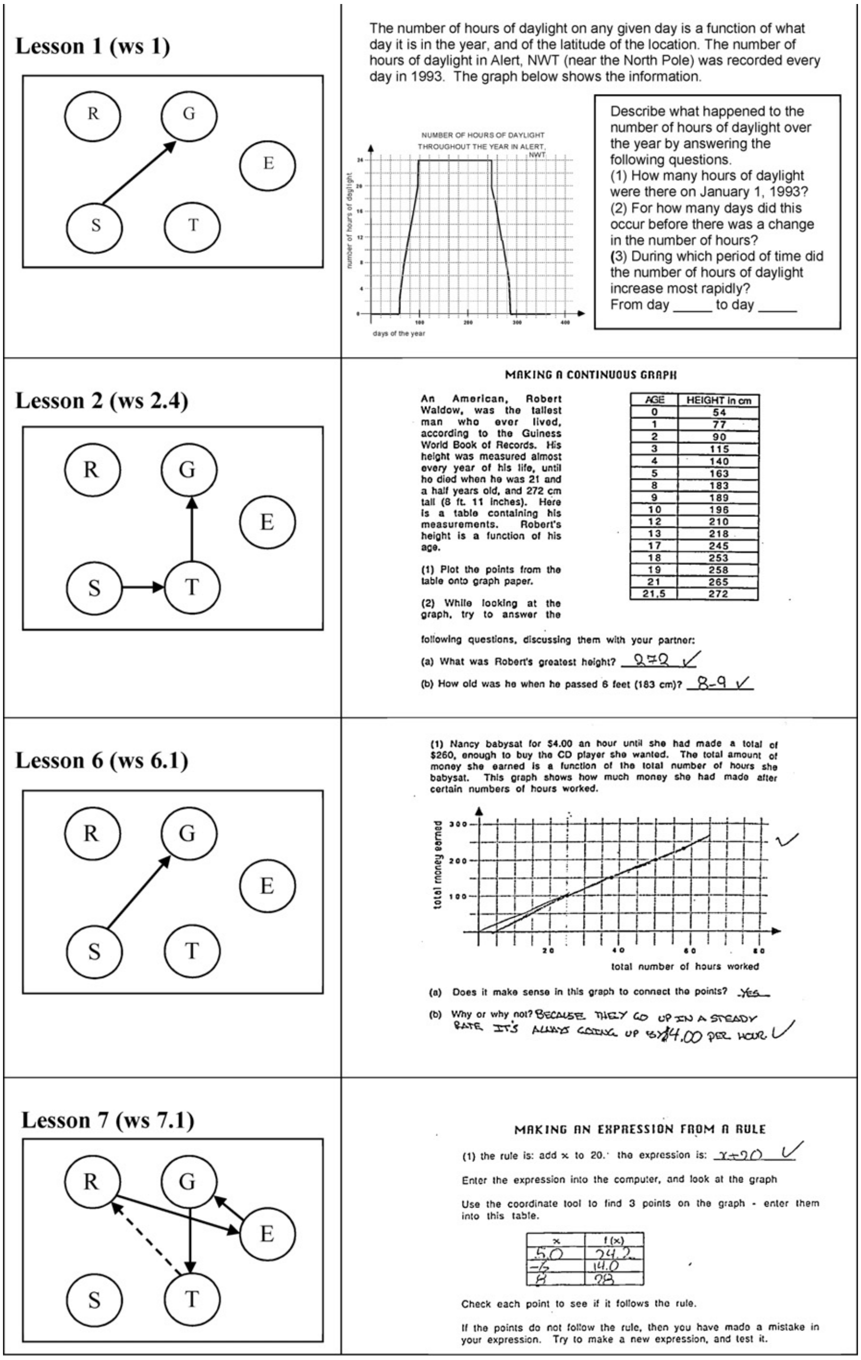
The way symbolic expressions were introduced deserves special mention. During lesson 3, the students got acquainted with a special form of narrative called *rule*. A rule was a sentence in imperative describing an operation on numbers, such as "multiply a number by three and add four". The rules were to be routinely constructed by the children as prescriptions for calculating different missing quantities in multifarious real-world stories. During the following lessons the rules were gradually shortened by a partial replacement of word phrases with symbolic operators and letters, until during the seventh lesson they turned into full-fledged algebraic expressions. From lesson 7 on, these new visual mediators were present and used in multiple ways.

Principle 2: Participation in the discourse on function is precondition for the objectification of function. As explained above, breaking the circularity of the process of objectification required that the learner be given an opportunity to actually participate in the talk on function early on, so that the process of individualization of this discourse starts informally, grounded in examples of the new word uses rather than in any formal definitions.[6]

The word *function* began appearing in the worksheets and teacher's talk as early as lesson 2 and as can be seen from the syllabus of the course in Table 1, this means that the discourse on functions gained residence in the classroom prior to its "official" introduction and before any explicit remark about the nature of its focal object had been made.

Principle 3: The discourse on function needs to be introduced gently and gradually, so that participants arrive at the desired use of the new noun in a series of steps rather than in one leap. Such gradual process could be observed throughout history of

[6] This decision is coherent with Davydov's approach to content-based generalizing and the development of theoretical concepts in mathematics classrooms: "instructional subjects must include, not ready-made definitions of concepts and illustrations of them, but problems requiring the ascertainment of the conditions by which these concepts originated" (Davydov, 1990, p. 162).



Legend:

(1) Capital letters on the left side are abbreviations of mediators of functions (R – rule, G – graph, E – expression, T – table, and S – story).

(2) Arrows indicate connections that were called for by the task and that were performed by the students as they were working on the worksheets.

(3) Dashed line indicates those connections that were required in the task but were not made by the students at the time of implementation.

Fig. 4. Transitions among mediators: examples from worksheets. (1) Capital letters on the left side are abbreviations of mediators of functions (R – rule, G – graph, E – expression, T – table, and S – story). (2) Arrows indicate connections that were called for by the task and that were performed by the students as they were working on the worksheets. (3) Dashed line indicates those connections that were required in the task but were not made by the students at the time of implementation.

Table 2
Expert's uses of the word function throughout the course.

	Worksheet	Teacher	Total
Function as relations ("y is a function of x")	13	20	33
Function as the signifier of an object	33	47	80
Total	46	67	113

function, and it was now to be replicated, at least to some extent, in the classroom. In MAP, the use of the word function has been introduced and molded in three steps.

The first step, corresponding to lessons 2–8, was to use the word *function* not as a stand-alone noun but rather as a part of the phrase "x is a function of y", where x and y were covariant quantities. The term was introduced to help in describing objects (more specifically, pairs of objects) rather than to signify an object in its own right. For example, the worksheet in lesson 2 included the words "Robert's height is a function of age", during the third lesson the teacher said "We say that his height is a *function* of how old he was. As he got older, he got taller," and in the worksheet given to the students during lesson 6, the learners found the sentence "The total amount of money she earned is a function of the total number of hours she babysat". The intensity with which the phrase was used in the curricular materials and by the teacher is reflected in the frequencies summarized in Table 2.

Underlying the decision to restrict the initial use of the word function to its appearances in the phrase "x is a function of y" was the assumption that since no new object was implicated in such use, the introduction of the term *function* would cause no major upheavals and the term would be easily absorbed into discourse, with the principle of its use readily decipherable from illustrative examples. Another support for this didactic move came from history: the initial classroom discourse on function was modeled on Euler's talk about "one quantity as a function of another quantity." Besides, be this use as restricted as it was, it was sufficient for the kinds of problems the students were dealing with at this point in time. During the first eight lessons, each task began with a story of either a real-world phenomenon (e.g. earning money), an algebraic expression, a line in plane or a table. The request was to translate such story into another discourse (see examples in Fig. 4). To make the transition from, say, the discourse on algebraic expressions to the one on lines in the plane, there was no real need for mediating entities, such as function.

The second step began when the worksheets and the teacher started using *function* as stand-alone noun, hinting on the existence of an object "out there" that this noun is signifying. The introduction of this new use became necessary during lesson 9, when problems appeared in which the students began dealing with algebraic equivalences. The designers decided that the best way to define equivalence of pairs such as $3(2x+5)$ and $6x+15$ was to say that these expressions "represent (realize) the same function." They assessed that by that time the students would have gained enough experience with graphs, tables of numbers, stories, rules and the expressions themselves, and would have already developed sufficient awareness of the relations among the various mediators to be prepared for an explicit talk about the new object called *function*. At the same time, however, they did not believe that the learners would be ready for the highly abstract definition, such as the one that equates function with "a set of ordered pairs" and is thus free from any mention of visual mediators. Consequently, the transition to *function* as a stand-alone noun was to be made mainly by means of examples of the new use, and only marginally, if at all, by an explicit meta-talk on this use. The advice for the teacher included specific suggestion for a number of spoken or written statements to be made during lesson 9:

- "A 'situation' may have different representations – story, table, graph, rule, expression."
- "We say that story, table, graph and expression which refer to the same story represent the same *function*."
- "The expression $3x-7$ represents the function $f(x)=3x-7$."
- Write the function on the board: "$f(x)=(x-5)^2$"
- Ask: "What is $f(10)$?" Go through the calculations with them.
- Write "$f(10)=25$". Ask them to find $f(-1)$, $f(6)$, $f(15)$, etc.

Following this advice, the teacher began lesson 9 with a conversation on word stories, rules, expressions, graphs, and tables and then said: "We found that we could have one situation, and that one situation could be represented in all of these five ways. When that happens we say that we have a function." She then wrote the word "FUNCTION" on the board, all in caps to signal its central importance.

The new uses of *function* included phrases that signaled the object-like nature of the entity signified by this word: from now on, functions could be *represented*, *given*, *counted as discrete entities* ("For the following *two* functions. . ."), and *combined one with another* ("Add the following functions"). See Table 2 for the frequency of the different uses of *function* in the course.

The last (third) step, taken during lesson 21, was not meant to make any real change in discourse. For the sake of closure and to let the children have an initial sense of yet another centrally important meta-rule of mathematics, the designers decided that after the learner had an opportunity to get familiar with the use of *function* as a signifier of object, she should also be exposed to its abstract, signifier-independent definition. There was no intention to actually require the children to learn this definition or to make any real use of it. The move was made in recognition of the needs of those of the learners who had a good sense of mathematical game and for whom the formal definition might become a food for thought and an additional trigger for the objectification of *function*. On day 21, after a long debate on what it was that was common to table, graph, and expression which could be converted one into another and were thus said to "represent the same function"; and after repeated attempts to focus children's attention on the fact that for all of these signifiers, a given number x corresponded to the same number y, the teacher said:

> So, we can say that a function is represented by, or is, a collection of pairs of numbers. We haven't used this term (pairs) very much. If I look at the table, I see that for this *x* [*points to a number in the left-hand column*], I have this f of *x* [*points to the corresponding number in the right-hand column*]. So these two numbers are considered to be a pair. If I look at the graph and I want to know what this point is here, I label that point by an *x* value, how far it is over on the *x* axis, and how far up it was up on the *y* axis. So I get a pair of numbers. And even the functional expression gives me a pair of numbers. I can use something for *x*, I will get something for f of *x*. So we say that a function [like this one] is an infinite number of pairs of numbers.

Principle 4: If the objectification is to happen, practicing the new discourse has to be accompanied by reflection on this practice. Only through such reflection the learner can gradually become aware of a (partial) isomorphism between the component discourses on algebraic expressions, graphs, and tables and real-world phenomena involving covariant quantities. In children's edition, the isomorphism of discourses simply means that stories about almost any such phenomenon could be told in either of these discourses. Cognizant of the isomorphism, the student may now be able to appreciate the role of the object called function as a "kingpin" that keeps the partially equivalent discourses together.

To develop their awareness of connections among the component discourses, the students were asked to move back and forth among the mediators, translating narratives about one of the mediators into narratives about another. With the help of tasks such as those presented in Fig. 4, they were repeatedly exposed to the fact that different visual mediators can all correspond to the same story.

3.2. What means did the students use to cope with the inherent circularity of objectification?

In the previous section we have described some of the salient characteristics of the expert discourse on function to which the students were exposed along the 30 lessons. The question now is what the learners did in order to be able to take part in the conversation. The task, just to remind, was far from trivial, since the noun *function*, however used in the worksheets or by the teacher, could not yet signify for the students what it did for those who initiated the use of this word. In what follows, we make a number of observations regarding students' coping strategies.

Observation 1: The learners avoided speaking about functions and kept the discourse at the level of what was supposed to serve as realizations of functions.

The almost total absence of the word *function* in student's talk is the first thing to note while going through the transcripts: although often addressed with utterances featuring the word *function*, the learners almost never actually used the word themselves, not even when responding to questions explicitly asking about functions.[7] This is hardly surprising. Because for the children *function* was just a sound, not yet uniquely associated with anything familiar, there was not much they could say about it. They were even unable to do as much as choose a linguistic form into which the word could be incorporated in a ritualized response to a standard type of question. The situation was initially confusing to all, but to some it was more so than to others. The case of Gur, described in detail by Sfard and Kieran (2001), shows how mixed up and disoriented in his discursive moves a child can become when invited to a conversation in which he cannot figure out what the talk is all about.

The case of Ari, Gur's partner for learning, has shown, however, that the initial confusion could be overcome and that learners did have their ways to participate in the discourse about objects, the nature of which was not yet clear to them. Indeed, the children were able to keep the conversation going by remaining at the level of component discourses, mainly the discourses on graphs and expressions, which they had already objectified (in Table 3, see the high frequency of students' references to graphs, lines and expressions). Although the teacher, as well as worksheet queries, explicitly stressed relations between function and its realizations (recall the teacher's statements on representations made during lesson 9), the students never mentioned functions while implementing the tasks. They didn't seem to need the mediation of the higher-level objects to relate the lower-level objects one to another (once again, see Fig. 4 for examples of the connections the students were able to draw).

Observation 2: The learners were looking for ways to do things rather than asking about properties of mathematical objects.

Because to participate in a discourse means to do something according to the rules of this discourse, the children were single-mindedly focused on the question of "What am I to do now?" This is evidenced, among others, by the fact that whatever they said was about their own actions performed on graphs, tables and expressions rather than about the properties of the abstract object called function. Indeed, the transcripts of the 30 sessions are replete with evidence for the

[7] One exception was observed in all those cases when the students read aloud texts that contained the word. They have also used the term three times during the 21st lesson, while responding to the teacher's question: "What is a function?" (see summary of these uses Table 3).

Table 3
The number of times that students and the teacher used words related to functions.

Words used	Students		Teacher
	Read	Initiated by students	
Number	6	43	65
Line	3	53	61
Rule	0	1	20
Graph	15	30	65
Table	8	5	40
Expression	10	23	71
Function	9	5	67

claim that the students were action-oriented and that, as long as they knew what to do, they did not bother themselves with the question of what the task poser had in mind when speaking of "functions." Below, we bring examples that we find particularly compelling.

Example 1: Looking for what to do

Most questions asked by students as they were solving various problems inquired about how to do things. Questions [1] and [8] in Excerpt 1 are representative in this respect. In the vignette, taken from lesson 21, Ari and Gur are trying to build a formula for a linear function given by a table.

Excerpt 1: Asking about a formula for a function

Turn	Speaker	What is said	What is done
[1]	Ari	Wait, how do we find out the slope again? No, no, no, no. Slope, no, wait, intercept is negative 5. Slope	
...			
[6]	Gur	What‘s that?	
[7]	Ari	It‘s the formula, so you can figure it out.	
[8]	Gur	Oh. How'd you get that formula?	

Consequently, a great proportion of students' assertions was in the first person and spoke about the learners' own actions; some others were framed as second-person prescriptions for solving a problem at hand. Here is a representative example, taken from the same conversation as Excerpt 1.

Excerpt 2: Telling what needs to be done to find a formula for a function

Turn	Speaker	What is said	What is done
[9]	Ari	and you replace the x by 6.	
...			
[13]	Ari	And then you see how many is in between each, like from zero to what	

The next excerpt, taken from Ari and Gur's worksheet for lesson 10, shows that the language of doing was present even when the students were required to provide a description of a visually accessible object, such as algebraic expression.

Excerpt 3: Comparing algebraic realizations of functions

Here are two functions, $f(x) = x^2 - 5$, $g(x) = x - 5$. How are the algebraic expressions of these functions different?

Turn	Speaker	What is said	What is done
	Ari	one you square the x the other you don‘t.	
	Gur	one you square one you don‘t.	

All these examples would probably not be worth mentioning – after all, these are legitimate and almost self-evident questions and assertions to make – if not for their exclusivity: it was difficult to find in the transcripts students' utterance that would feature a mathematical object as its grammatical subject or object and simply describe its properties.

Example 2: When asked to define an object, the students still spoke about doing

This phenomenon could be seen already in the first and the last of the cases presented above. When Ari's said "It's the formula, so you can figure it" (see [7] in Excerpt 1) in response to Gur's question [6], it was the role of the formula in discourse rather than its structural properties or its relation to other mathematical objects that he was attending to.

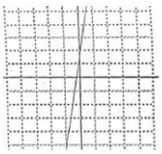
The next example is taken from lesson 21 during which the teacher initiated a debate on "What is function?" The assumption was that after many weeks of actually participating in the discourse on functions, the time has come for the students to start reflecting systematically on the new object. Excerpt 4 exemplifies the conversation that took place following the teacher's question. As can be seen from the parts of utterances highlighted in italics, although this question invited conversation on defining properties of an object (function), the learners spoke about people's actions, and from the form of their utterances it was not even clear that the actions were to be performed on the object in question. Sentences such as Sara's, "When *you do, you do it* and *you find the answer*, it‘s a function" ([7]) and "*A function is where you show how to do something*" ([15]) simply spoke about problem-solver's actions the girl *associated* in an unspecified way with the word *function*.

Excerpt 4: What is a function?

Turn	Speaker	What is said	What is done
[5]	Steven:	*It shows how you can do it.*	
[6]	Teacher:	It shows how you do it. That's part of what a function is.	
[7]	Sara:	When *you do, you do it* and *you find the answer*, it's a function.	
[8]	Teacher:	Ok, we talked about how function can be represented. Do you remember, we had five different ways of representing a function?	
[9]	Sara:	Yeah.	
[10]	Teacher:	How can we represent a function?	
[11]	Sara:	*By multiplying.*	
[12]	Teacher:	I'll give you a hint. One way is by a graph.	
[13]	Sara:	Oh, yeah.	
[14]	Students:	Tables.	
[15]	Sara:	*A function is where you show how to do something.*	

Yet another example, although not concerned with function, is brought here in some detail because it reinforces with particular clarity the above claim about the students' tendency to seek prescriptions for discursive action rather than descriptions of discursive objects. The episode took place during 25th lesson, when the students were working in pairs to solve the problem, "Simplify the following equations, and tell how many solutions the equation will have". The request was accompanied by eight linear equations. After Ari read the task description aloud, Gur asked "What do you mean by solution, though"? This sounded as a meta-level question about the use of a noun (solution).[8] The conversation that followed, however, made it clear that definition of the word *solution* was not what Gur was looking for. Indeed, Gur was familiar with the noun and what he did not know was how to use it in the present context – the word simply did not prime him on what to do. In Excerpt 5, Ari, evidently aware of what Gur was looking for, stated a procedure by which the problem could be solved.

Excerpt 5: What do you mean by solution?

Turn	Speaker	What is said	What is done
[23a]	Ari:	Look. When the slopes are the same, and the intercepts are the same, then the solution will always be the same.	Ari shows Gur the first equation on worksheet 25.1: $7x + 2 = 7x + 2$, and circles the numbers in the expressions.
[23b]	Ari:	Like 	Ari traces line on the given graph.
...		...	
[26]	Gur:	What is the solution, though?	
[27]	Ari:	Well if you look at like from, let's say this is 1. You go up, it will be exactly the same.	Ari picks a spot on the x axis for the graph of $7x + 2$.

In spite of Gur's ostensible insistence on a definition, Ari chose to tell him how he can actually solve such problem. One may argue that there was an inherent ambiguity in Gur's question. This kind of query could be interpreted not only as a meta-level question about the meaning of the word *solution*, but also as an object-level question about the specific solutions of a specific equation. This latter interpretation could explain Ari's decision to show solutions instead of defining the word *solution*. And yet, the ambiguity disappeared when Gur reiterated the question ([26]), evidently not satisfied with Ari's explanations. Alas, although now it should be clear that definition of the word *solution* was expected, Ari went on explaining the way to solve. Moreover, Gur's decision to comply made it clear that, indeed, his question "What do you mean by solution" was, in fact, to be interpreted as "How do you do it?"

Observation 3: Building on their experience, the students forged linguistic ties between tasks and solution routines.

Students' efforts to find something to do evidently did pay: the learners had their ways to decide what to do without necessarily being able to directly cope with the higher-level objects they were required to explore. The last question to ask is how they forged these helpful associations between tasks and implementation routines. In the absence of an access to the higher level object called *function*, and thus lacking a higher point of view from which the optional trajectories would be clearly visible and open to their inspection, they went about the business of choosing a discursive action in ways that would not necessarily be approved by competent participants. The children's technique was to rely on their experience in pairing phrases used in descriptions of tasks with certain ways of acting that could produce legitimate response.

[8] Based on our scrutiny of the transcripts, this was a rare type a query for the student to pose. In fact, this is one of only two cases we found in which the student seemed to be asking for a definition of an object.

Our basis for this claim is, above all, the students' obvious tendency to rely on memory rather than on reasoning while deciding about the required sequence of steps. This tendency is clearly visible in the following fragment of conversation that took place prior to Excerpt 5 above.

Excerpt 6: Turning to memory

Turn	Speaker	What is said	What is done
[2]	Ari:	(reads) "Simplify the following equations, and tell how many solutions the equation will have." "How many solutions will the equation have." Ohhh. Like the thing we just did. Like the solution, will on the graph, will the solution always be the same?	the equation: $5x - 15 = 7x + 2$
[3]	Gur:	What do you mean by solution, though?	
[4]	Ari:	I don't know. You just got me confused. Thanks.	
[5]	Gur:	Well, you have to know what that solution is.	
[6]	Ari	Well, I did until you just said that. So now I've forgotten. So let's just see	

Notice Ari's explicit reference to remembering (or rather forgetting) in [6] and, more importantly, his exclamation, "Ohhh. Like the thing we just did", followed by the words "Like the solution, will on the graph, will the solution always be the same?" ([2]) that referred to one of the tasks the children had recently performed. Clearly, at the sight of a new problem, Ari's first impulse was to search his arsenal of known routines, and this required identifying a similar task encountered in the past. It appears that he built on the assumption inspired and constantly reinforced by his daily classroom experience: the required 'similar task' is one of those recently completed and it can be identified by phrases or words appearing also in the present one. Indeed, Ari was able to spot such task among those the class has dealt with earlier in that same lesson. This former task, just as the one at hand, featured pairs of functions and the words *solution* and *equation*. Ari's reliance of this particular task was evidenced by the fact that the phrases he used to describe solutions, "always", "never", and "only once" (see [35] in Excerpt 7), were identical with those students were required to write down in this former task, even though these expressions did not fit the present question, one that did not inquire about time (the present question was "How many solutions will the equation have?", whereas in the former task, the question was "How many times do the graphs meet?").

Excerpt 7: "How many solutions" or "how many times do the graphs meet"?

Turn	Speaker	What is said	What is done
[28]	Gur:	So the only ones that can be the same	
[29]	Ari:	Cause they'll be on the same	
[30]	Gur:	are if they are the same number.	
[31]	Ari:	Yes, exactly.	
[32]	Gur:	If they're always different that means they never (mumble)	
[33]	Ari:	And if the slopes	
[34]	Gur:	And that'll be "only. . ."	
[35]	Ari:	This will be never. No, no, this would be "only once."	Teacher arrives. The equation: $5x - 15 = 7x + 2$
[36]	Gur:	Only once, yeah, ok.	

It is also significant that Ari became confused when asked about the meaning of the word *solution* (see [4] in Excerpt 6). Clearly, he did not need to know how to define the word to be able to solve a task like this. To sum up, it seems that for the students, *the entire routine for implementing the given type of task was the atomic unit of activity*. Such routine could not be decomposed into meaningful, independently executable parts that would lead to self-contained activities, such as defining or explaining.

3.3. How successful were the participants in objectifying function?

Having recourse to well-established routines proved itself time and again as an effective survival technique for those immersed in other people's discourse in which, at this point, they could only try to follow in the competent participants' footsteps. As anticipated on the basis of analytic considerations (Sfard, 2008; Vygotsky, 1987), the students, when faced with the need for meta-level learning, relied on imitation. That this imitation was usually thoughtful and non-accidental is evidenced by the fact that their concerted attempts to find a proper routine resulted in reasonable choices, in full agreement with the rules of the canonic discourse on function. This said, one has good reasons to suspect that for at least some of the students, these routines were but rituals, that is, well-defined, rigid sequences of actions, which the child did not execute in response to her own need for knowing but rather because they made sense to others (and Gur might well be one of these learners). On the other hand, there is also much evidence in the transcripts that for many students, their discursive actions were not altogether meaningless. While performing on objects with which they were already familiar, such as graphs and algebraic symbols, these learners preserved a good sense of direction and, most importantly, could reflect on what was being done. As argued by Vygotsky (1987) and elaborated by others (Sfard, 2008), a thoughtful imitation, similar to the one applied by MAP participants, is the first necessary step in the process of individualizing a new discourse.

Let us be more specific about what was achieved by the students during the 30-sessions of the two months long course. First, they became competent in the lower-level discourses that the discourse on functions was supposed to subsume. In particular, much progress was made in discourses on graphs and on algebraic symbols. On the post-tests given at the end of the course, over 90% of the students answered correctly to the 10 questions that focused on each of the discourses separately. More than half of the students answered correctly to the remaining 40 questions. It must be noted that the algebraic discourse was almost completely foreign to the learners prior to their arrival to MAP, and whatever algebraic competence they were showing was achieved in the course of the project.

Second, although they did not yet become active, autonomous participants of the discourse on functions, the students acquired considerable familiarity with explicit talk on function and could now perform many of the tasks characteristic of this discourse. This situation will be likely to change when the interdiscursive connections, already quite obvious to the students, become an object of systematic, sustained reflection.

4. Discussion and conclusions

It is now time to summarize our findings, to take critical look on what was done in MAP project, and to ask whether any of the above insights can be generalized beyond the case at hand.

4.1. Summary of findings

The summary of our analyses is presented below in the form of responses to the three questions asked in the beginning of this article.

The first of the questions inquired about the way curriculum developers and the teacher decided to help the students in overcoming the circularity of the process of building new mathematical object such as function. On the basis of written documentation and of analyses of the teacher's talk in the classroom, we claimed that the instructional design was in concert with the vision of mathematical learning as individualization of historically established discourse and, in particular, with the assumption that meta-level learning can only begin with reflective practice and thoughtful imitation. That is, the beginning of meta-level learning is both provoked by, and framed by, the discursive input of the expert teacher. Indeed, while developing students' familiarity with the lower-level discourses on algebraic expressions, graphs and tables, the teacher also began engaging the learners in an explicit conversation on functions. This was a gradual process, with the explicit abstract definition of function appearing only toward its end, and only as a means of encouraging reflection on the relation between the new discourse and the discourses it was supposed to subsume.

The second question inquired about techniques learners employed in their attempt to cope with the inherent circularity of the process of objectification. We found out that the students were able to participate in the discourse on function without ever dealing directly with this as-yet nonexistent object. They did this by associating problems with solution routines through all kinds of discursive clues, to which they were sensitized through their former experience. For the children, each such routine came as an indissoluble "unit of activity" – a package deal, with very limited possibility of modifying the procedure according to the changes in the formulation of the task.

Our final question regarded the state of students' discourse on function by the end of the study. We found out that the learners became reasonably competent in the discourses on algebraic expressions, graphs, tables, and real-world stories, and that they were reasonably skillful in making transition from one of these discourses to another. One can conclude, therefore, that they have developed a good sense of inter-discursive relations. In most cases, the three lower-level discourses, organized around manipulable signifiers, could now be said to be practiced by the student as discourses "for herself".[9] We thus concluded that although the learners were still in the early phase of the project of objectification, their participation in MAP laid a solid foundation for their future discourse on functions.

4.2. Looking back and forward: how to teach for objectification?

To complete the picture, two teaching-related issues will now be addressed. First, we shall revisit the MAP designers' instructional decisions and subject them to critical reassessment; second, we shall give some thought to the question of what should have come next in this classroom if the learners were to make it safely to the objectified discourse on function.

With regard to the first of these issues, let us recall that the principles which guided instructional decisions were in concert with the vision of learning as individualization of a historically developed discourse and of meta-learning as requiring the learner to practice the new discourse before it becomes objectified and turns into a discourse "for oneself." Nothing in the present study could undermine these basic assumptions. Still, the way the principles were translated into an actual instructional sequence deserves reevaluation.

As shown by our analyses, in the span of the two months spent by the children in making their first steps in the discourse on functions, the process of objectification of function had barely begun. This statement is not meant as a critique of what has been done and achieved. As previously argued, objectification is a difficult circular process, and thus whatever the teaching method, it cannot happen overnight, not even in the course of 30 lessons. Indeed, if it is the use of word that is supposed to "teach you

[9] This expression is borrowed from Vygotsky, who distinguished between gestures a young child can be performing for others, solely as tools for *interpersonal* communication, and those she can already perform as gestures for herself – as tools for *thinking*.

meaning'' (Wittgenstein, 1953/2003, p. 181), you need to give this use a proper chance. And yet, even if not much could be done in the MAP classroom to speed up the process of objectification, some modifications in instructional sequence could have reduced students' confusion and their resulting frustration. The first thing that comes to mind in this context is that the students should have probably spent more time getting acquainted with the three lower-level discourses before the subsuming discourse on function was introduced. This is particularly true of algebraic discourse, which at the beginning of the project was almost entirely new to the students. Some support for this call to foster algebraic discourse first comes from a model of algebra development proposed by Caspi and Sfard (see their article in this issue), according to which several additional acts of meta-level learning need to be accomplished before algebraic discourses attains the form in which it can become a stepping stone for the discourse on function. This claim goes against the radical ''functional approach'' to teaching algebra, originally promoted by MAP researchers (Sfard & Kieran, 2001) and favored until this day by many others (see e.g., Yerushalmy & Schwartz, 1993).

With regard to the students' future learning, we may ask what course the process of objectification should now take and what kind of support the teacher might give to the learners. At the time Montreal Algebra Project ended, the students were practicing the talk on function as discourse-for-others. Anything could happen to them now: they could either proceed toward a fully objectified discourse on function or become stuck in the ritualized form of participation. What will actually happen depends, of course, on students' cognitive investment – on how insightful she or he will be with regard to the workings of the discourse, its usefulness and its inner logic. This, however, is but a part of the story. Indeed, at this point it is time to note what has not yet been said in this article: how successful the student is in the task of becoming a participant of a given discourse depends, to great extent, on how determined she is in becoming one. Indeed, the necessary cognitive effort will not be made if there is no wish on the part of the student to turn the discourse into her own – to make the fluency in this discourse a part of her identity. Whether she does or does not display such agentive engagement depends, in turn, on great many factors: on the role she ascribes to this discourse in her life, to the kind of identity how she views and values those who can and those who cannot. Agentive attitude is what will help cope with the necessity to participate in rituals – the kind of doing that some people may find unworthy or even repellent and humiliating (It is thoughtless subsuming oneself to the will of other people as a result of coercion.) To help the learner to make it safely through this difficult threshold the teacher has to be fully aware of the situation and of its inherent risks. If she is to succeed in what is probably most difficult of her tasks, she has to be capable of delicate fine-tuning to the needs and sensitivities of every individual learner in her class. It is at this point in the process of new discourse development, therefore, that the art of teaching mathematics may sometimes be observed at its best and most powerful.

4.3. Looking beyond this study

It is reasonable to assume that the phenomena we observed in this one classroom were not accidental and, more specifically, that the ways MAP students acted while paving their way into the new discourse can be observed in almost any other classroom when new objects are introduced and meta-level learning is about to occur. Indeed, we feel that the results of the analyses reported in this paper contribute to our thinking on mechanisms of meta-level learning and objectification in general, and on the role of definition in this process in particular.

To begin with, these analyses seem to support the four-stage model of word use development proposed by Sfard (2008; see Fig. 5). According to this model, the child exposed to a whole new discourse becomes a passive participant, that is, one who may listen to other people's talk and can try to make sense of what is said, but can hardly contribute to the conversation. From now on, she will gradually develop the ability to recognize, and perhaps even perform, a restricted collection of routines which she associates with tasks featuring the new word. One can speak about her entering the second phase when she arrives at a considerable repertoire of routines and the talk about the new object[10] turns into an integral part of these routines' execution. In the Montreal Algebra Project we had an opportunity to observe the learners going through the first phase and entering the second. Of necessity, this is where the story told in this paper ends. Still we can hypothesize on the future development. According to the model in Fig. 5, the final phase of objectified use of the word *function* would not arrive before the intermediary *phrase-driven* form of word use developed. During this phase, the children would be able to meaningfully use the word *function* as a part of certain constant phrases. Such phrases will thus replace whole routines in the role of atomic units of the discourse on function.

What was observed in this study during the first two phases of concept development seems to provide an explanation to the question of why words' definitions, even if well known to the student, seem to play almost no role in the learner's decisions about the use of the words (e.g. Hershkowitz & Vinner, 1983, 1984; Wilson, 1990). Definitions describe objects, whereas the present study has shown that prescriptions for actions are what students are looking for. In MAP classroom this tendency proved so strong that even when appearing to be asking for a definition of noun, the learners were in fact seeking answers to the question of what should be done; and if one wants to be told what *to do* rather than what something *is*, definitions are not a proper answer. This said, let us immediately add a disclaimer: we should not conclude that defining is useless altogether. Rather, we should probably reconsider its place in the process of learning. As we argued before, once the student is fluent and competent enough to start reflecting on the discourse, definition becomes one of the best foci for such reflection. Thus, rather than being treated as means for introducing new objects into discourses, defining should probably be encouraged at a more advanced stage of learning, as an additional catalysts for objectification.

[10] It is not clear whether this talk is actually about the ''new object'' or it is still dealing with the ''new word''.

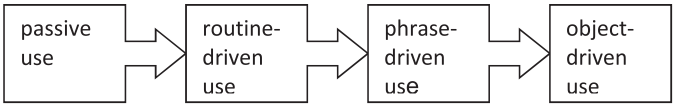


Fig. 5. Four-stage model of the development of word use (Sfard, 2008, p. 182).

4.4. Meta-conclusion

In this article, we revisited the data collected and analyzed more than a decade ago. Although the empirical material remained the same, the story we were able to tell with its help changed, at least to some extent. If we think about research as societal learning (learning by a society), then what happened here can be seen as an evidence of researchers' meta-level learning: the narrative is now different not because of the arrival of new data, but because of a modification in the way we talk about them.

A number of lessons may be learned from this exercise. First, our new take at the old data has shown the importance of meta-discursive revisions in the project of advancing our understanding of mathematical thinking and learning. Second, it has proved the usefulness of looking on old data from new perspectives. Most importantly, however, it gave us an additional glimpse into the mechanisms of meta-level learning, the very same issue we were exploring in this article while looking at the development of school children's discourse on functions.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.ijer.2011.12.007.

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