Chapter 23: Research in Mathematics Education: What Can it Teach us about Human Learning?

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Among the diverse domains of human knowing, mathematics stands out as a hothouse for insights about teaching and learning in general. This chapter summarizes the ways that research on mathematics learning contributes to our understanding of how people learn. We begin with a brief historical overview, which explains what it is about mathematics that has made it the content area of choice for the study of human learning. We then focus on two approaches in education research, the *acquisitionist* and the *participationist* approaches, which have influenced our understanding of learning and of the practices of teaching in all content areas. Mathematics education research affected learning sciences at large by challenging the first approach (which is still dominant in much education research) and helping to introduce and develop the second one.

Mathematics Education Research – A Historical Overview

Mathematics education research is an applied discipline aiming at improving the practice of learning and teaching mathematics. It established itself as full-fledged academic discipline about 60 years ago. However, various scholars began studying mathematical learning at the end of the 19th century, when the influential idea of evidence-based pedagogy first came to the fore. The first half of the 20th century saw numerous experimental and quasi-experimental studies (see Campbell & Stanley, 1963, for a review) in which researchers endeavored to compare and evaluate the effectiveness of different teaching approaches. That research, conducted mainly by mathematics educators, attracted the attention of professional mathematicians. Whereas successful mathematical learning was commonly regarded as requiring special abilities, it was the frequent failure that puzzled mathematicians: "How does it happen that there are people who do not understand mathematics?" (Poincaré, 1929/1952, p. 47). The early studies proved unable to answer the mathematicians' query, in large part because they focused on an "input-output" model that assumed simple causal relations between teaching methods (inputs) and student achievement (outputs), that is, relied on a *transmission and acquisition* conception of teaching and learning. This approach gave little attention to *processes* of learning in the context of instruction (on processes, see Chinn & Sherin, this volume).

A small number of psychologists studied processes of mathematics learning that had been ignored by mathematics educators during the first part of the 20th century. The psychologists hoped that this particular, discipline-specific case of learning would bring insights into human learning in general. Their efforts aimed at identifying and describing the development of those aspects of cognition that were common to all humans, regardless of their sociocultural backgrounds and personal histories. With this goal in mind, psychologists focused their empirical studies on human skills and understandings that they assumed would be more or less the same in every culture. Mathematics, hailed as "the only true universal language" (Rees, 2009), seemed to be ideally suited.

By the 1960s, a reasonable merger was achieved between three academic communities interested in mathematics teaching and learning: psychologists, mathematicians, and mathematics educators. This was the origin of the modern discipline of mathematics education research. Since the 1960s, most studies tended to focus either on teaching or on learning, with the researchers asking questions such as 'How do people learn mathematics?', 'How can our understanding of human cognition inform the way we teach math?', 'What learning environments and curricula are more effective in producing desired learning outcomes?'. These questions have been studied within two very different conceptual frameworks that reflect contrasting interpretations of what mathematics is, and consequently, what it means to learn mathematics (Sfard, 1998). The first framework comprises approaches that portray mathematics as pre-given structures and procedures, and view learning mathematics as the *acquisition* of these structures and procedures. The acquired entities may be called *knowledge*, *schemes* or *conceptions*, and the process of acquisition can be either passive, happening through mere "transmission", or active, achieved by the learner's own constructive efforts. We call this framework *acquisitionism*. The second framework portrays mathematics as a form of human activity rather than as something to be "acquired", and views learning mathematics as the process of *becoming a full-fledged participant* in this distinct type of activity. We call this framework *participationism*.

Acquisitionism holds that mathematics was discovered or constructed by mathematicians and is acquired or reconstructed by the learner. In contrast, participationism holds that mathematics is one of many human ways of doing things, and that it has evolved historically and continues to undergo change. As will be shown in the review that follows, each of these two frameworks has very different consequences for both research on mathematics learning and for pedagogy.

Acquisitionist Research in Mathematics Education

Learning as constructing conceptions

Beginning in the middle of the 20th century, educational researchers considered the then well-developed Piagetian theory of learning to be the perfect framework for conducting studies that aimed to “look inside the human head” in a theoretically-sound manner. *Constructivism*, as an elaborated version of this approach became known (von Glasersfeld, 1989; Kafai, 2006), was the predominant framework until the end of the 1980s. The central assumption of constructivism is that learners build their understanding of the world primarily through direct interaction with their environment. In constructivist pedagogy, the teacher is portrayed as a provider of opportunities for learning rather than as someone who delivers information to a passive learner. This characterization of learning in instructional situations pushed the individual learner to center stage and seemed to suggest that there was little need to study instruction per se.

Two complementary lines of constructivist research have made systematic contributions to mathematics education. The first has investigated the non-canonical student *conception*s that often emerge as the products of mathematics learning. The second has modeled the *processes* of conceptual change with the help of empirical data from clinical studies, the most insightful of which are known as *teaching experiments*.

Studying Learners’ Conceptions

This strand of research started in science education with studies of student *misconceptions* (see diSessa, this volume), a notion derived from Piaget’s contention that inadequate understandings of reality can be found consistently across situations and people (Piaget, 1962). The term misconception referred to student’s *“own meanings*—meanings that are not appropriate at all” (Davis, 1988, p. 9). The assumption that learners’ meanings were inappropriate, and the term “misconception” itself, began to be questioned by science educators in the 1970s (Driver & Easley, 1978). Resilient student conceptions that seemed to violate the scientific canon were later described more charitably as *alternative* or *naïve* conceptions, and were characterized as possibly inevitable early version of formal concepts. In mathematics education, researchers spoke in terms of *tacit models* (Fischbein, 1989) or of students' *concept-images*, which were contrasted with *concept-definitions* (Tall & Vinner, 1981). There is hardly a mathematical concept learned in school that has not been studied using this framework.

Researchers who engaged in this kind of study found surprising consistencies in alternative conceptions across classrooms, schools, and even nations. This was taken as evidence that idiosyncrasies in how students deal with mathematical concepts were not just a matter of faulty teaching. For instance, research on student conceptions has repeatedly shown that many high school students identify as functions only those mappings that can be represented by simple formulas. This is often the case even for learners who can recite the formal definition of function and know that this definition does not require a formula (Malik, 1980; Vinner & Dreyfus,1989). Similarly, young children were found to resist the idea that multiplication can yield a value lower than either of the multiplied numbers (think, for instance, about $\frac{1}{2}\cdot 6$, which is 3) or that division can do the opposite (as is the case with 3 divided by ½; Fischbein, 1989; Harel, Behr, Post, & Lesh, 1989).

Misconceptions research has been widely disseminated in teacher education programs and has been quite influential, in part due to its methodological simplicity. Initially, research on misconceptions used surveys composed of open-ended questions, or sometimes multi-choice questions that could be administered to large groups of people and processed statistically. Later, surveys were complemented or replaced by structured and semi-structured interviews. The pedagogical implications of these studies were also enticingly simple: to repair a misconception, the teacher was advised to elicit a *cognitive conflict*, a situation in which students' current conceptions clash with empirical evidence or with a mathematical definition. It was believed that when faced with such conflicts, students would come to realize that their current conception was not accurate.

Investigations of students' conceptions have been central in science education for some time (see diSessa, this volume). In mathematics education, misconceptions research began relatively late, peaked in the 1980s, and started to decline in the early 1990s. It was at this point that mathematics education research began parting company with research in science education. Science education researchers were aware of various foundational weaknesses of misconceptions studies, including discrepancies with constructivist principles and the lack of a sound theoretical basis (Smith, diSessa, & Rochelle, 1993). However, they opted for repair rather than replacement, and the study of learners' scientific misconceptions evolved into theoretically grounded research on *conceptual change* (see diSessa, this volume). In contrast, mathematics educators gradually began to reject the misconceptions approach, possibly as a result of insights they gained through investigating processes of concept formation that we review next.

Studying Processes of Constructing Conceptions

In mathematics education, research on students' conceptions has been accompanied almost from the beginning by studies of the cognitive processes that take place as the conceptions develop. The pioneers of this research in mathematics education translated Piaget's theory of human development into discipline-specific research frameworks (see e.g. Skemp, 1971). Their followers proposed a number of homegrown theories of mathematical thinking and its development, among them the theory of conceptual fields (Vergnaud, 1990), theories of process-object duality of mathematical conceptions (Sfard, 1991; Dubinsky, 1991; Gray & Tall, 1993), and a theory of the growth of mathematical understanding (Pirie & Kieran, 1994). The ultimate purpose of the resulting studies was to produce models of students' evolving conceptions (Steffe, Thompson, & von Glassersfeld, 2000). In these studies, students' progress through a sequence of developmental levels was characterized as process of individual construction that can be precipitated by interactions with others.

This line of inquiry has made two lasting contributions. First, it has produced a number of models of mathematical learning that have informed researchers' thinking about how students learn a range of different mathematical topics. Second, it produced methodological innovations that, with time, became stepping stones to current research methods, notably to that of teaching experiments and design based research (Barab, this volume; see the next section).

Models that describe how learners gradually construct different mathematical conceptions differ widely in their scope. Probably the best known model, proposed by the Dutch researchers Dina and Pierre van Hiele, portrayed development of students' geometric thinking (van Hiele, 2004/1959). In the spirit of Piaget's staged theories of development, this model was predicated on the assumption that five qualitatively different levels of understanding are successively attained as students advance in learning geometric concepts. However, the van Hieles departed from Piaget in one important respect. They argued that although progress from one level to the next is as inevitable as the transitions between Piagetian stages, these changes result from teaching rather than from the learners' independent construction. According to the van Hieles, it is in the context of proper instruction that the student proceeds from the level of *visualization*, where she can recognize a shape as a whole, to that of *analysis*, where she can also name its parts; to that of *abstraction*, where she recognizes the role of a formal definition in establishing the name of a shape; and to that of *deduction*, where she can also derive and justify her claims with the help of the rules of logic. Eventually, the learner may reach the level of *rigor*, where the whole edifice of geometry becomes a formal theory, derivable from a small set of axioms. The van Hieles' model spurred a flurry of research that indicated that very few secondary school graduates reach the fourth level, deduction, and the great majority end at the second level, analysis, or the third level, abstraction (Battista, 2007).

The van Hiele model of mathematical concept construction is particularly robust, and today, it remains as influential as ever. This is remarkable, given that the Piagetian approach that inspired this model has declined in influence. The key to this resiliency may be the van Hieles' emphasis on the critical role of teaching in cognitive change, together with their focus on developmental changes in *language*. As a consequence, the van Hiele model can be reconciled with current sociocultural approaches (see below, in the section on participationist framework).

The study of learners' development of arithmetical and algebraic conceptions, inspired by Piaget's pioneering work on numerical thinking, has been a particularly active area of research. Researchers have zoomed in on specific, clearly delineated forms of mathematical activity such as counting (Gelman & Gallistel, 1978), using fractions (Kieren, 1992), and solving problems with unknowns (Filloy & Rojano, 1989). This research typically provides meticulous descriptions of how the relevant conceptions and skills evolve (for comprehensive reviews see Kilpatrick, Swafford, & Findell, 2001 and Kieran, 2007).

Finally, the method of the *teaching experiment* was introduced in the 1980s by researchers who were trying to map trajectories in the development of students' mathematical conceptions. This technique is premised on the assumption that it is only through teaching that researchers can come into contact with those critical moments when students reorganize their thinking (Steffe et al., 2000). And if so, researchers must engage in a long-term process of teaching individual students one-on-one in order to study the process of concept construction. Consistent with the constructivist principles that inspired this work, the pedagogies used in teaching experiments involved designing special mathematical tasks and pressing students to reflect on their problem-solving activity. Video-recorded interactions were analyzed after each teaching session to inform the selection of tasks for subsequent sessions (also see Chinn & Sherin, this volume, on microgenetic methods). Once the experiment was completed, retrospective analyses of video-recordings and transcriptions were conducted to model the participating students' learning "in terms of coordinated schemes of actions and operations" (Cobb & Steffe, 1983/2010, p. 24). In spite of its focus on a small number of individual learners, the models were intended to be "both general and specific" (ibid, p. 27). The impact of the method of teaching experiments has extended beyond mathematics education and marked the beginning of the intensive study of learning processes in instructional situations.

To summarize, research on the learner's mathematical conceptions and on their development has extended our knowledge of mathematical learning in two important ways. First, it resulted in thorough records of most common idiosyncrasies of the learner's mathematical thinking, and second, it yielded important insights about mechanisms of learning mathematics. In addition, this research gave rise to several quandaries, with the surprising resilience of student non-standard ideas, or "misconceptions," being perhaps the most puzzling. Research findings indicated that students' non-standard ideas are often impervious to both preventive and remedial efforts of the teacher. The vision of learning as "acquisition of conceptions" seemed ill-suited for dealing with this phenomenon. Indeed, when conceptions are understood as discrete entities "transmitted" to or "reconstructed" by learners, it is difficult to explain the non-linear nature of learning, the fact that it often cannot proceed by a simple extension of what the student already knows. In the next section, we present additional reasons for the emergence of the alternative participationist approach in mathematics education and beyond.

Re-conceptualizing learning: From Acquisitionism to Participationism

By the early 1990s, some researchers concluded that it would not be possible to answer certain core questions about mathematics learning without a thorough revision of the tacit assumptions underpinning their work. Mathematics-related critiques of the acquisitionist approach came from a variety of cross-cultural and cross-situational studies, all of which indicated the untenability of Piagetian claims about universality of human intellectual development and about the primacy of development over learning. Piagetians reserved the term *development* for changes that happen spontaneously and are common to all people. Numerical thinking, considered to be a universal human capability, was an obvious candidate for cross-cultural and cross-situational examination. However, numerical thinking soon proved to be anything but culturally invariant. In fact, the very idea of cross-cultural comparisons proved problematic. Investigations of numerical thinking often led to one conclusion if the participants were presented with school-like numerical tasks, and to quite a different one if they were asked to solve mathematically-equivalent everyday problems (Cole, 1996, p. 74).

Evidence about the diversity rather than universality of numerical thinking, and thus about its cultural rather than developmental sources, came from numerous investigations, including studies of the counting practices of Oksapmin people in Papua New Guinea (Saxe, 1982), the money transactions and paper-and-pencil arithmetic of unschooled Brazilian street vendors (e.g., Nunes, Schliemann, & Carraher, 1993), and the use of numbers and measurement in a range of work (e.g., Scribner, 1997) and everyday tasks (e.g., Lave, 1988). These studies suggested that mathematics learning is fundamentally *situated*, that is, people tend to recruit what they once learned only in situations highly reminiscent of those in which the learning originally took place (see Engeström, this volume; also Brown, Collins, & Duguid, 1989). Furthermore, the findings of these studies strong indicate that contextual changes that appeared mathematically irrelevant frequently result in previously successful students becoming helpless, and that the diversity of learning and its outcomes is much greater than anticipated by Piaget and his followers. Many researchers saw this as a challenge to the idea of primacy of development over learning.

Mathematics education researchers voiced their own doubts about some of the tenets of constructivism and their findings called for revision of the limited role attributed to instruction. In one of these studies (Erlwanger, 1973), the researcher followed a 6^{th} grader learning on his own as he worked through a "teacher-proof" series of mathematics booklets. The findings shocked mathematics educators by showing that the child produced correct answers by inventing ingenuous, locally effective but mathematically faulty solution procedures. Such findings led many scholars to conclude that mathematics education research had to acknowledge the central role of the teacher in learning.

Growing dissatisfaction could also be felt among researchers who favored the teaching experiment methodology over other investigative techniques. They felt that the restricted role of the teacher in one-on-one teaching experiments was at odds with research on teaching that highlighted the importance of teachers proactively supporting students learning. In addition, it was becoming increasingly apparent that interactions in one-on-one teaching experiments might be qualitatively different from what occurs in the classroom as the teacher and multiple students interact (Cobb, 2012).

Another reason for concern was that acquisitionist researchers, who assumed the primacy of development over learning, seemed unable to account for either the cross-cultural and cross-situational diversity of individual learning, or for learning that occurs at the level of society. This latter type of learning expresses itself in the historical change of human ways of thinking and acting and in the incessant growth of their complexity. A solution to both these problems, to be found mainly in the work of Lev Vygotsky and his associates, was to reconceptualize learning as a process of becoming capable of acting in uniquely human ways (see Nathan & Sawyer, this volume). An important aspect of human activities is that they are *mediated by artifacts*, that is, are performed with the help of material tools--such as hammers or computers on the one hand, and of symbolic systems, such as language, counting systems, and writing, on the other. The artifacts, and thus the activities themselves, are constantly refined and passed from one generation to the next. The activities are historically constituted rather than predetermined, and they can differ from one culture to another. For Vygotsky, therefore, to learn meant to become a competent participant in activities that characterize the times and the culture into which people are born. From this theoretical perspective, learning mathematics is reconceptualized as becoming a competent participant in mathematical activity.

To sum up, the contemporary theories we call *participationist* emphasize joint participation in shared cultural activities. As will be illustrated in the review that follows, participationist researchers focus on social, predominantly linguistic interaction (see Enyedy and Stevens, this volume), and they study learning, as it occurs in classroom interaction or in everyday activities (see Engestrom, this volume).

Participationist Research in Mathematics Education

In this section, an introduction to participationist research is followed by summaries of this strand's contributions to what is known today on processes of learning and teaching mathematics and on factors that shape these processes.

Participationist researchers view personal growth as originating on "the social plane" (Vygotsky, 1987, p. 11) rather than in the direct encounter of a person with the world. These researchers thus emphasize the importance of the learner's interactions with more knowledgeable others, such as parents and teachers, and more generally, refuse to investigate mathematical learning in separation from its social context. The systematic study of learning processes occurring in such diverse settings as classrooms, children's homes and playgrounds, museums and workplaces has been supported by new technologies with which one can document these processes in all their complexity and then analyze them at any level of detail.

Participationist researchers distinguish between several types of learning occurring in the classroom. In addition to individual *student* learning, there is the type of learning that can be called *collective* because it involves overall changes in what is considered in the classroom as acceptable ways of doing things; and there is *teacher* learning, which had previously been given only marginal attention. Collective learning is often conceptualized as a *change in practice,* whereas individual students' and the teacher's learning are seen as a *change in ways of participating in collective practices* (cf. Rogoff, 1990). Practice, the term that refers to the unit of analysis accepted by most participationist researchers, is understood as referring to a patterned, historically established form of human activity. Mathematics can be seen as a particular set of historically established practices.

Participationist researchers study mathematics learning that takes place in different types of settings and involves different forms of practices. While much effort is invested in observing learning that occurs in daily circumstances, sometimes without intentional teaching, the study of school learning is still the leading strand of this research. Inspired by philosophers who suggested that human knowledge should be viewed as “a kind of discourse” (Lyotard, 1993, p. 3), some participationist researchers conceptualize mathematics learning as the process of mastering the communicational practices of the mathematical community. Mathematics education researchers who adopted this conceptualization (Lerman, 2001; Kieran, Forman,& Sfard, 2001; Sfard 2008) treat mathematics learners’ talk as an object of study in its own right, and not just as a “window” to something else – conceptions, mental schemes, and so forth. From this perspective, studying mathematics learning is synonymous with investigating processes of discourse development. In its most “radical” version, this discursive approach rejects the strict ontological divide between what is going on “inside” the human mind and what is happening “outside” (see Enyedy & Stevens, this volume; Nathan & Sawyer, this volume; Abrahamson & Lindgren, this volume). To forestall misunderstanding, let us stress that equating mathematics with discourse does not entail the denial of processes that happen inside human heads. It only means that discourse becomes the superordinate category, with mental phenomena no longer considered as belonging to separate ontological category. Admittedly, not all discursively-oriented researchers embrace this uncompromising, “radical” form of non-dualism).

Two important principles guide participationist studies of classroom learning. First, this learning cannot be investigated without considering what happens on both the collective and the individual level; or, to use Vygotsky’s language, without considering the interaction between the social and individual planes (note the parallels with the distinction between elemental and systemic approaches described by Nathan & Sawyer, this volume). Second, learning and teaching are two sides of one process, and as such, must always be studied in tandem. True, the researcher can choose whether to focus on the student and the process of learning or the teacher and the process of teaching. Yet, she must also keep in mind that none of these aspects of learning-teaching processes stands on its own and none can be ignored, whatever the focus of the study.

To investigate the interdependence of collective and individual learning and to modify instruction in response to evolving classroom mathematical practices, participationist researchers developed a method called *design based research* or sometimes the *design experiment* (see Barab, this volume). This method involves both instructional design and research. Throughout the investigation, the researchers assume the responsibility for a class and its mathematical learning (Cobb, Confrey, diSessa, Lehrer, & Schauble. 2003; Stephen, McClain, & Gravemeijer, 2001). Rather than planning the entire teaching unit in advance, they prepare for instruction by specifying learning goals, anticipating a possible collective learning trajectory, and identifying possible types of learning tasks and tools. Decisions about what will actually be done during each lesson are then grounded in ongoing analyses of what happened in prior classroom sessions.

One of the overall goals of a design experiment is to see how students become capable of performing such mathematical tasks as finding and justifying solutions, evaluating the reasonableness of solutions, generalizing from solutions, and making connections between multiple representations of a mathematical idea (Kilpatrick, Swafford, & Findell, 2001). The design experiment methodology makes it possible for the researchers to both support and observe successive patterns in the development of these mathematical capabilities, and to tie the patterns to the specific means used to support the students' learning. The analyses of the data collected in an experiment therefore emphasize that both the process of students' mathematical learning and the mathematical capabilities they develop are situated with respect to the classroom learning environment, and are highly dependent on the students' interactions with the teacher.

In participationist research, the basic type of data is the carefully transcribed communicational event. Its methods of analysis are mainly adaptations of techniques developed by applied linguists or by discursively-oriented social scientists (see Enyedy & Stevens, this volume). There is also a rapidly expanding assortment of analytic tools that are tailor-made to fit the particular needs of mathematics education research (Moschkovich, 2010). In spite of their disciplinary specificity, most of these techniques can be easily transferred to the study of other subjects.

Learning mathematics

Participationist research has already brought many important insights on learning-teaching processes. In one of the most comprehensive participationist studies, Cobb and his colleagues introduced the notion of *classroom norms*, unwritten rules that define acceptable ways of acting in a classroom. Three types of norms have been identified and studied. The *social* and *sociomathematical* norms differ in the degree of generality: social norms regulate students' and teachers' social conduct and are discipline-independent, whereas sociomathematical norms are unique to the learning of mathematics (Yackel & Cobb, 1996). The third type of norms, those collectively labeled *mathematical practices of the classroom*, pertain to mathematical ways of doing things and encompass the purpose for engaging in mathematical activity, standards of mathematical argumentation, and ways of reasoning with tools and symbols (Cobb et al., 2001). These practices have been shown to evolve in the course of classroom interactions (Cobb et al., 2001). The notion of norm has proved useful both in describing collective learning and in explaining individual participants' actions and interactions. In fact, all types of classroom learning may be thought of as resulting from the mutual shaping of collective norms and the teacher's and students' individual actions.

The discursive strand of participationist research has been generating a high-resolution picture of the development of mathematical discourses, as this development takes place in a classroom and beyond. Discursive methods, while demanding and time-consuming, allow the analyst to see what often escapes the teacher's attention during a real-time classroom conversation. By making these interactional details visible, discursive analyses often reveal differences between things or situations that previously appeared identical. When this difference is recognized, students' discursive actions that so far did not seem to make much sense may appear reasonable, even if non-standard.

An important insight generated by discursive research is that historical and ontogenetic developments of mathematical discourse both involve two types of transformations. *Object-level developments* extend what is known about already constructed mathematical objects. This type of discoruse growth is mainly accumulative: it expresses itself in systematic addition of endorsed narratives, and possibly of routines. *Meta-level developments* are those that involve changes in the rules of the discourse. The transition from the discourse of unsigned numbers to the discourse of signed numbers is a good example of this latter type of change. The need for meta-level learning poses special challenges for the learner, which are likely to be aggravated by their invisibility not only to the learner, but also to the teacher.

The term *identity*, which can be described as referring to a collectively built image of an individual, has been introduced to participationist research in the recognition of the inherently social nature of learning. Although identity does not have an agreed operational definition, there seems to be a consensus about a number of descriptors that, when taken together, delineate this concept and its possible uses (Darragh, 2016). In informal terms, a person's identity is a set of features that one is likely to consider when answering the question of who this person is. While building identities, either their own or those of others, people draw on an assortment of identity templates available in their social milieu. These templates reflect what is valued in the given society and what may be expected from its members. In societies that appreciate mathematical competence, learning mathematics becomes an important part of the process of building students' identities. In research on learning, thinking about interpersonal processes in terms of identity and power relations makes it possible to deal with emotional, social, and political aspects of educational processes under a single, well-integrated conceptual and methodological umbrella, one that can also accommodate epistemological and ontological issues of mathematics learning.

Clearly, different people may attribute different identities to the same person. Another important trait of identities is their being in a constant flux. Whereas omnipresent, identity transformations are a matter of interpersonal dynamics and as such, cannot be attained at will. Further, identities tend to function as self-fulfilling prophecies and may have a long-term effect on learning. Once a student is identified as "weak" in mathematics, she will be more likely to fail in the future; in contrast, the learner who has been labeled as "strong" will be more determined to achieve success. The result will reinforce the previously constructed identities, reducing the chances for a change in a reverse direction (Heyd-Metzuyanim, 2015). Aware of the co-constitutive relation between the learning of mathematics and the processes of building the learner's identity, some researchers propose that direct attempts to mold students' mathematical identities may have positive impact on the students' readiness and ability to participate in mathematics discourse. According to those who promote *critical* mathematics education, templates for "identities of success" that have currency in today's societies may stymie mathematics learning rather than be helpful (Gutierrez 2007). Disrupting the widely held views of what counts as desirable student identities may thus be a precondition for any decisive improvement in mathematics learning.

Teaching mathematics

Participationist research has contributed to the emergence of a broad consensus about how mathematics lessons should be organized to support students' development of key mathematical capabilities and productive identities as doers of mathematics (Hiebert & Grouws, 2007; Langer-Osuna & Esmonde, 2017). The findings of a number of studies indicate the value of the teacher introducing mathematical tasks that are challenging for students, then students working to solve the tasks individually or in small groups, and finally the teacher orchestrating a whole class discussion of the students' solutions. This strand of research has also made a significant contribution by analyzing the major aspects of productive classroom learning environments, including instructional tasks, classroom norms for each phase of lessons, the nature of classroom discourse, and students' use of notations and other types of tools (Lehrer & Schauble, 2011). It is apparent from this work that the various aspects of the classroom learning environment are interdependent. For example, instructional tasks as they are actually implemented in the classroom and experienced by students depend on the tools that are available to students and on whether the teacher simply grades students' solutions or leads a whole class discussion in which students are pressed to explain and justify their reasoning. In this regard, participationist research focuses on both the nature of instructional tasks and on the classroom activities within which the tasks come to have meaning and significance for students. Although this work is content-specific, the conceptualization of classroom learning environment as composed of interdependent aspects can be adapted to inform research on learning in other content areas.

Drawing on research on student learning, participationist researchers have made substantial progress in delineating key classroom instructional routines that are likely to give rise to significant learning opportunities for students. As an illustration, let us consider a whole class discussion, the aim of which is to present, compare, and consolidate students' solutions. Orchestrating such discussion is challenging because it involves drawing on students' contributions while ensuring that classroom discourse supports students in deepening their understanding of central mathematics ideas. Stein, Engle, Smith, and Hughes (2008) found that accomplished teachers could anticipate the most common types of student solutions and that they purposefully sequenced the order in which solutions were discussed so that significant mathematical issues came to the fore. In addition, these teachers treated student contributions differentially based on whether they would advance their instructional agenda, and also pressed students to explain and justify their reasoning, assess others' solutions, and make connections between different solutions. As this illustration indicates, analyses that identify specific instructional routines make a significant contribution by specifying potential goals for teachers' learning and thus for teacher professional development.

Discursive research provides insights on the proactive role of the teacher in supporting students' learning. What has been learned from this research thus far casts doubt upon several widespread beliefs about teaching. For instance, at those times when a further development of mathematical discourse requires meta-level learning, the support of the teacher or of the "more knowledgeable other" cannot be limited to encouraging the learner's own invention. Instead, it is imperative that students are intentionally inducted into the discourse of the accomplished participant, and that they are encouraged to persist in participating in this discourse even though it is incompatible with their own. Such persistent guided participation is necessary if they are ever to fathom the reasons for the historical emergence and for the wide acceptance of the new discourse (see Reiser & Tabak, this volume).

Learning to Teach Mathematics

In recent years, participationist research has focused increasingly on teachers' as well as students' learning. This development is a direct response to the realization that teachers' development of the classroom instructional routines outlined above involves substantial learning, and that this learning requires sustained support. Participationist research has clarified the goals for teachers' learning by identifying the types of knowing inherent in the enactment of instructional routines. An influential finding in this regard concerns what Hill and her colleagues have labeled as *mathematical knowledge for teaching* and involves much more than just solving the types of problems that they use with students (Hill, Sleep, Lewis, & Ball, 2007). Researchers who study mathematical knowledge for teaching also ask whether teachers "understand mathematics in the particular ways needed for teaching, whether they know what their students are likely to make of the content, and whether they can craft instruction that takes into account both students and the mathematics" (p. 125). Hill and colleagues' finding that there is significant connection between teachers' mathematical knowledge for teaching, the quality of instruction, and student achievement indicates the value of viewing teachers' ways of knowing as situated with respect to the types of decisions and judgments that they make in the course of their work.

More recent investigations have identified two additional aspects of teachers' ways of knowing that appear to be critical if they are to develop the above instructional routines. The first concerns teachers' development of an initial *vision or image of high-quality instruction* that encompasses the types of cognitively demanding tasks that students are expected to solve, the nature of classroom discourse, and the role of the teacher in the successive phases of the lesson (Munter, 2014). Intuitively, findings that indicate the importance of teachers developing an initial vision of high-quality instruction are reasonable in that it is difficult if not impossible for teachers to develop new forms of practice if they do not have a vision of their function in supporting students' learning The second aspect of teachers' ways of knowing concerns their *views of their students' current mathematical capabilities*, particularly students who they perceive are currently struggling (Jackson, Gibbons, & Sharp, 2017). This aspect is important because some teachers who have developed a relatively sophisticated vision of high quality instruction nonetheless eschew opportunities to develop such instructional routines because they contend that their students are incapable of participating in and learning from such instruction. These teachers appear to view the limited capabilities they attribute to their students as inherent, context independent characteristics of the students rather than as situated with respect to the instruction in which their students are currently participating. In contrast, teachers who have developed what Jackson et al. term *productive views* of their students' current capabilities anticipate that their students will be able to participate in and learn from high quality instruction that is organized around challenging tasks, provided they receive appropriate support. As a consequence, these teachers typically capitalize on opportunities to develop such instructional routines.

A closely related line of research has investigated designs for supporting teachers' development of *high-leverage instructional routines* that are likely to give rise to significant learning opportunities for students (Kazemi, Franke, & Lampert, 2009). Examples include eliciting and responding to students' contributions, managing small group work on challenging tasks, and leading whole-class discussions of students' solutions to challenging tasks. This line of research has made considerable progress in the last ten years or so and has deepened our understanding of high quality school- and system-level professional development, school-based teacher collaborative meetings, and one-on-one content-focused coaching in teachers' classrooms.

One of the key findings is that high quality supports for teachers' learning connect student learning goals, students' mathematical reasoning, and instruction. As an illustration, meetings that coaches conduct with teachers to debrief on the enactment of lessons that they had planned together typically focus almost exclusively on the instruction during the lesson. However, the findings of two recent studies indicate the value of the coach and teacher first analyzing students' reasoning during the lesson to assess the extent to which they attained their goals for the students' learning (Russell, Stein, & Correnti, et al., 2017; Kochmanski, 2020). In productive debriefs, the coach and teacher do not focus on instruction during the lesson until they have completed this analysis, and they then do so in order to explain why the students learned what they actually learned in the lesson. In the process, the coach and teacher identify weaknesses in the lesson and propose instructional changes that are justified in terms of their potential to support students' learning. Horn, Garner, Kane, and Brazel (2017) found that discussions in productive teacher collaborative meetings that have the potential to support substantial teacher learning have a similar structure, and Borko, Jacobs, and Koellner (2010) report parallel findings for professional development sessions. Taken together, these findings highlight the interdependence of students' learning and teachers' learning. This is a significant development given that research on students' learning and research of teaching were, until quite recently, largely separate areas of investigation.

The findings of participationist research on teacher learning have broad implications for our understanding of learning more generally. They indicate both that learning is fundamentally situated and that people's learning in one situation (e.g., professional development sessions) can influence what they do in another situation (e.g., the classroom). These findings suggest that learning to use particular routines across a wide range of situations is a significant achievement and requires the support of more accomplished others who ensure that there is a bi-directional interplay between learners' activity in the different settings (Kazemi & Hubbard, 2008).

Concluding remarks

Today, we know a great deal about what is likely to happen as students make their way into the world of mathematics and about how particular teachers' instructional routines may support or hinder their learning. Moreover, mathematics education researchers now seem more capable than ever of dealing with the complexities of human learning without compromising the standards of scientific quality that such work is expected to meet. This said, much work still lies ahead, and there is the constant need for innovations that can make a significant difference both in research and in the practice of teaching and learning. New opportunities seem to be opening, thanks to recent advances in brain-imaging technology and in methods of data collecting and analyzing. Research in mathematics education, it seems, will continue to make contributions, the importance and applicability of which go beyond its own boundaries, to the sciences of learning in general.

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