Chapter 9

Teaching mathematics as a an exploratory activity – a letter to a teacher

Dear teacher,

I am no longer a professional mathematics teacher myself, but in this brief note I wish to share with you some pedagogical ideas. More precisely, I will now take a few minutes of your time to tell you how I think I would be teaching a particular mathematical topic, quadratic inequalities. True, you do teach the subject yourself, so you probably know a lot about it. I admit to not having a comparable classroom experience, whereas experience, they say, is the best teacher. Still, I consider my pedagogical ideas worth telling, because they have been inspired by many years of research, and in particular, by a careful study of what happened in one classroom in which this topic has been taught. As a researcher, I had the luxury not many teachers can enjoy: I could revisit each classroom event as many time as I wished, I had all the time in the world to ask questions and test possible answers and finally, I could perfect my ways of constructing interpretations and then, after a trial, improve my tools even further. All this made me aware of things that usually escape attention of active participants, who are too busy with moment-to-moment decision-making to notice.

Let me start by imaging that I am a teacher in grade 11. My students are already acquainted with quadratic equations and with the notion of function, and today, I am supposed to initiate them to quadratic inequalities. What can I do? Well, I'd probably begin with a quadratic equation, say $x^2 = 4$. I'd write it on the board, and then, I expect something like the following:

1.	Anna	We want to solve for x. What is our x equal to?	Writes: (x-2)(x+2)=0
2.	learners		All speak together
3.	Anna	We are saying any of these brackets is equal to 0. So we are saying x-2 is equal to 0… OR… x+2 is equal to 0	While saying this, I would be writing on the board: "x-2=0 or x+2=0"
4.	Anna	And then we transpose them. X is equal to?	
5.	learners	2… or x is equal to -2	As the learners are saying this, I would be writing "x=2 or x=-2" on the board
6.	Anna	Are you happy?	
7.	learners	Yes	In unison

From here, I'd proceed to the inequality $x^2 > 4$. But I'll stop here, for now, and ask: Would you agree with this strategy? Would you proceed in the same way in which I did in this hypothetical scene? You may say, "Well, yes, why not?" Indeed, everything seems fine: the mathematics is correct and the learners are clearly with me on the same page. But in this letter, I wish to dig a bit deeper than that, considering every little move I have done and examining its possible alternatives. Yes, when it comes to the effectiveness of teaching, I believe the devil is not in the general principles or even in detailed lesson plans, but in the finest details of the implementation. Before I present my analysis, however, I have to explain some basic things about my approach – I need to make sure that you understand my words the way I understand them.

My basic assumptions about learning and teaching mathematics

My point of departure is that *mathematics can be seen as an activity of telling stories about mathematical objects*. A similar statement can be made about any other school subject. Indeed, in biology one tells stories about "biological objects" called plants and animals, and in physics – about material things. But there is also an important difference between mathematics and all the other sciences: While the objects of biology or physics are concrete and accessible to our senses, the objects of mathematics, such as numbers, functions and sets, are not. We use to say that these latter objects are abstract, and this means that they cannot be seen, smelled, heard or touched, whereas the symbols we use while talking about them – such as tables, algebraic expressions and graphs in the case of function – are only their material proxies. We stress this role of symbol as but avatars of the "real things" by calling them *representations*. Thus, *algebraic expression* x^2 and the *canonic parabola* are representations of the *basic quadratic function* which, rather than being identifiable with any of these symbols, can be defined as *a set of all the ordered pairs of numbers in which the second element is the square of the first*.

This difference between mathematics and the other sciences makes mathematical storytelling particularly difficult. In my opinion, it is this lack of the direct accessibility of mathematical objects that constitutes the main challenge in the teaching and learning of mathematics. If you wish to teach, say, how to move around a new school building, the first thing to do would probably be to take the learners for a tour or, at least, to show them a schematic pictures of the place. Otherwise, how could they find their way to the sport hall or to the mathematics classroom, except if somebody gave them explicit, step-by-step instructions? But what are you going to do when you have a mathematical object instead of a building? How do you "take them for a tour around" a number, a set or a function?

The challenge of teaching about the invisible mathematical objects is so formidable that many teachers and students compromise on the *memorize-symbolic-manipulations* type of teaching, a method comparable to guiding people around a new building by listing the steps necessary to get to specific places rather than letting them explore the building and learn its geography. Indeed, many of my university students recall their mathematical experience as that of “parroting” the teacher and “pairing formulas with exercises”.

This is not the kind of learning I would be aiming at as a teacher. In spite of the difficulty, I would opt for *explorative mathematizing*. Rather than presenting mathematics as a bunch of rituals to be followed with the sole aim of adhering to some inexplicable social norms, *I would like my students to be masters of their mathematical activity*, that is, be able to decide when and how to use mathematics for their own needs. For this to happen, they must emerge from my classroom with a pretty good sense of mathematical objects and with the ability to use these objects as a source of their mathematical stories and of these stories’ ultimate confirmation. Only when they have no longer the need for an “expert opinion” in order to be sure of what they do, will I feel that my job as a teacher has been properly done.

This teaching goal, as ambitious as it already appears, seems even more challenging when one starts thinking about specific steps to be taken. Unlike in physics or biology lessons, to which the student comes with some initial sense of the objects that will be talked about, in mathematics, the objects we are expected to explore, such as functions and derivatives, or even numbers, must be constructed throughout the conversation. To use my former metaphor, whereas a new building exists independently of whether it is explored by the learners or not, mathematical objects can only be brought into being through, and within, out talking about them. So, the learner is supposed to explore mathematical objects while trying to construct them in the first place! This sounds a bit circular, doesn’t it? On the one hand, to get acquainted with a new object the learners have to explore this as-yet-unknown entity; on the other hand, how can one do it, if this object does not yet exist for them?

Here, I have a surprise for you: I am going to vindicate the rituals, at least to some extent. My research has taught me that breaking out of the circularity happens in two stages, and ritualized mathematizing is one of them. Let’s look again at the notion of function. In the first stage, the leaner has no choice but try to participate in a conversation on function simply by imitating what the teacher is doing, thus in a ritualized way. The rituals, therefore, are not to be altogether rejected: they are the necessary basis for future explorations. But the word *basis* must be stressed: ritual is only the foundation, the inevitable point of departure, but it is by no means what the student is supposed to end up with. As the learning goes on, the student’s rituals must gradually morph into explorations. For this transformation to take place, the learners have to engage into incessant attempt to *rationalize their moves*, that is, to try to relate one thing to another – an algebraic expression to a table, a table to graph, etc – so that all these pieces fall in place together as different stories about the same mathematical object.

Easier said than done, of course. The school graduates who spoke to me about mathematics as the activity of parroting the teacher had obviously never made it to explorative mathematizing and left school with a bunch of mathematical rituals. These rituals might have assisted them effectively in passing examinations, but they also gave the students a life-long distaste for mathematics and for themselves as mathematics learners. The question has now to be asked: What can the teacher do in order to promote the transition from ritualized to explorative mathematics? My short personal answer is this:

- ***The teacher has, first, to demonstrate what explorative mathematizing is all about.***

Indeed, this is the teacher's mathematics that the student will start imitating, and if the teacher's mathematizing is ritualized, the learners mathematizing will have little chance to become anything else.

- ***Second, the teacher has to explicitly encourage mathematical explorations and discourage purely ritualized activity.***

But what does all this mean in concrete terms, in the terms of things that should be done and those that should be avoided? To answer, let me go back to the brief hypothetical classroom scene in which I acted as a teacher solving the quadratic equation $x^2 = 4$. In discussing my purported moves, I will have to consider the words I uttered and the momentary decisions I made in finest detail.

Teacher's taks 1: Demonstrating what explorative mathematizing is all about

Before I start my analysis, I have to ask: *What does it mean for a teacher to model explorative mathematizing*? Or even before that, *how do we tell ritualized mathematizng from the explorative?* If mathematizing is the activity of storytelling, one needs to look for an answer in the stories told and in the ways in which these stories have been crafted and judged as true or valid. So, what features should my talk had to have to be up to the standards of explorative mathematics?

Principle 1: Make it clear that your stories are about mathematical objects

> *Principle 1.1: Speak about objects and their properties rather than about your own actions with symbols*

In the light of the principles I have just listed, the first part of my answer will not surprise you: First and foremost, *these stories have to be explicitly about*

mathematical objects and their properties and not about one's own actions with symbols. Let me illustrate.

Here are some of my utterances in the hypothetical classroom scene:

> "We want to solve it for x." ([1])
>
> "We are saying any of these brackets is equal to zero". ([3])
>
> "And then we transpose them." ([5])

Can these utterances be read as statements about mathematical objects? Well, not exactly. They are all about the solver's actions with symbols: solving for x, transposing, equating brackets to zero. This is, therefore, about constructing new symbolic expressions to replace those we actually see on the board. It is like teaching a person how to reach a certain room in an unfamiliar building by listing the necessary leg movements rather than by inviting her to explore the space so as to find her way.

How could I, then, present the same equation, and the same solution, as an exploration of a mathematical object? To begin with, rather than saying "solve for x" ([1]) – something that would give the uninitiated learner no inkling about the nature of the task, I might have said:

> "What is the number the square of which is 4?"

or, using a bit of algebraic formalism,

> "For what number x will x^2 be equal to 4?" ([1a])

This time, I formulated the task as a question about mathematical object called number. Better still, knowing that in the case of inequalities it would be particularly helpful to see things in terms of functions (and remembering that the students have already been introduced to this notion!), I would translate the equation into the query

> "What are the values of x for which the value of function x^2 is 4?" ([1b])

In this new proposition, the focus is on a mathematical object, *function* x^2, and not just on symbols, x and x^2. Thus, in both new versions, an invitation to a specific action of the problem solver has been transformed into a direct question about a property of mathematical objects. Such question, as opposed to the former, invites an explorative action.

And how about my utterance [4], "And then we transpose them"? This was, of course, an abbreviation, and what I really meant to say was "transpose 4 from one side of the equation to another with its sign changed". But there is yet another possibility: I could have said

> "Let us subtract 2 from each of the functions on the two sides of equality sign (y= x+2 and y=0)"([4a])

I definitely think this last option is better. True, it is about human actions, just like the original utterance [4]; and yet, unlike this former instruction, it carries its own justification: subtracting the same number from two equal objects – from the number or functions appearing on the two sides of equality sign – preserves the equality, that is, produces a new pair of equal objects. All this does not transpire from the action on symbols that I, as a hypothetical teacher, tried to describe to the learners in my utterance [4].

At this point you may have much to say on what I have told you. First, you may object, claiming that I pay too much attention to language and that there is no reason to dismiss such neat, concise expressions as "solve for x". Mathematics is full of conventions and this is one of the agreed abbreviations. True. And indeed, if the class has already learned how to unpack these linguistic shortcuts into questions about numbers or function, there is no reason to give them up. A serious problem arises, however, if the phrases "solve for x" and "transpose to the other side" are the only ones the learners have ever heard in the context of equations or inequalities, that is, if the class has never been exposed to the full tale-telling versions of these shortcuts. In this case, how would the students ever know that the task they are facing is that of exploring certain mathematical objects? How would they ever realize that answers to mathematical questions can be derived one from another, and not only memorized?

Second, you may be wondering: If I can see so clearly that the teacher's moves in my hypothetical classroom scene are at odds with my assumptions and principles, why have I presented them in that vignette in the first place? Well, for the sake of this exchange, I did what I thought many teachers are actually doing. I was talking to the learners the way I would be talking to myself while solving an equation. How come? Well, automation and shortcuts are the other side of competence. They come when one has already become adept in explorative mathematizing and, through constant practice, developed proficiency in routine tasks. Many teachers, especially beginners, do not realize that talking to others, and in particular to uninitiated, is a different kind of skill than talking to oneself. My example may thus serve as a means of alerting prospective teachers to this fact. In addition, we sometimes have a tendency to do what mothers are often doing when talking to their children: communicate with our audience on this audience's own terms. As teachers, we feel instinctively that at this early stage in the learning of a new mathematical topic, the students can only act in ritualized way. Thus, ritualized performance is our "teacherese", just as a slightly distorted English is the parents' "motherese".

To recapitulate, the first way to ensure that one's mathematical activity deserves being described as explorative rather than ritualized is to attend to one's language and make clear it is not about actions with symbols but about properties of mathematical objects. Another is to be explicit about the fact that mathematics is about reasoning, that is, deriving facts from facts, rather than about memorizing.[1]

> *Principle 1.2: Connect representations one to another via represented objects, rather than drawing analogies that have to be memorized*

Speak about function as an object. Say you will instantiate why it is important – how these connections turn mathematical objects into a map and the compass - later.

> *Principle 2: Derive your mathematical stories from the properties of mathematical objects rather than trying to get them from somebody else or from your own retrieving from memory*

Decompose into two sub-principles, 2.1 and 2.2

> *Principle 2.1: When unsure of "the answer", try to derive it from other known facts rather than asking others or trying to recall; this, even if you have to say, in the end, "I have to think more about it"*

Here, ask how you arrive at the solution of $x^2 - 4 > 0$;

- algebraic derivation show how one teacher I knew began, but did not make any use of it (and show it to the end)
- graphic – say: even the use of a graph may be ritualized – as a mnemonic, an aid to memory, based on arbitrary analogy.

> *Principle 2.2: Saturate your utterances with logical connectives, such as* therefore*,* from here it follows*,* hence*, etc. Try not to leave any two adjacent utterances without a logical connective.*

For other characteristic features of explorative mathematizing we need to look closely at the sources of one's mathematical stories and the way the person decides whether these stories can be seen as valid.

[1] The parts below marked in blue are not yet ready – these are notes for myself based on which I will write the remaining part of the chapter.

Explorations mean crafting one's own stories about mathematical objects, and thus, one would expect a teacher who wishes to demonstrate what exploratory mathematics is all about to be explicit about his own construction processes. What would this mean in the case of explaining the solution of the inequality $x^2>4$? Probably the simplest way might be to approach the task as one about functions. The teacher could then draw the graphs of the functions $y=x^2$ and $y=4$ and translate the task into one of finding those parts of x-axis for which the corresponding parts of the first graph are above the corresponding parts of the second. Another, legitimate although more cumbersome, option would be to look at the inequality as a relation between numbers and translate it into the question: "For what values of x will the number x^2-4 be positive?"[2]. Mr. Smith, however, took none of these routes. Instead, he simply said "We do likewise. In other words, we are doing exactly what we have done here [in the case of $x^2=4$]". This does not sound as a rational choice of a procedure, informed by the visions of this task as one of exploring numbers or functions. Rather, it is a declaration that, for reasons which have never been stated, some old procedure is now to be recycled. Oh well, there are hints about the source of the decision: in the earlier conversation (see [11] – [15]), Mr. Smith alerts the class to the similarities and differences of the expressions $x^2=4$ and $x^2>4$. His latter request for a general confirmation ([28]), and its immediate granting by his audience ([29]) might have created the impression explanation would be unnecessary. None of these, however, provided the learners with an opportunity to experience a truly explorative mathematizing.

What happened later in this classroom, however, has shown that this decision doomed the classroom, and the teacher himself, to go on acting in a purely ritualized way. His solution routines are symbolic manipulations retrieved from memory or taken from other people rather than being genuine explorations, whereas his "answers" are halting signals that end symbolic manipulations rather than genuine responses to questions about mathematical objects.

#	**who**	**What is said**
170	Mr. Smith	So what you're saying is x minus two, is greater than zero. Is that what you're saying?
171	learners	Yes
172	Mr. Smith	Or?
173	learners	And, and
174	Mr. Smith	Sorry?
175	learners	And
176	Mr. Smith	And see, that's very much interesting. I'm going to ask Mr Pillay to tell, to help us here, this "and" and "or" thing is also confusing me right. I will take "and". Ok let's go on

[2] Translation of x^2-4 into $(x-2)(x+2)$, combined with the rule "multiplication of two numbers is positive iff the two numbers are both positive or both negative" would lead to the conclusion that for the inequality to hold, one of the following needs to be true: either [$x-2>0$ and $x+2>0$] or [$x-2<0$ and $x+2<0$]. The requirements for x could now be simplified by first adding the 2 or -2 to both sides of each of the inequalities and then using number line to picture the situation. All this would lead to the conclusion that the inequality holds when $x>2$ or $x<-2$.

Once you speak about function, and your talk is thus free from reference to any particular symbol, I can now choose my own symbols while trying to solve. Important to make an informed choice and also to make it clear how the different signifiers for the same mathematical object, are interrelated. Thus, in the case of function, it is not enough to draw analogies – the   You need to show what it is that the graph and the expression have in common: the pairs of numbers, (x,y) that are produced by and can be plotted as…Nothing in the statement "solve for x" or "use two-squares formula" would and thus does not associate them with specific steps.

10.	Mr. Smith	You agree with this. Thank you very much. This is quadratic equation.	On the board, points to $x^2=4$ and its solution
11.	Mr. Smith	Now we want to proceed from something we know to something which I think we don't know, but it looks like that.	Points to $x^2>4$ written on the board under the heading "quadratic inequality"
12.	Mr. Smith	Do you see that these two things are not the same?	Points alternatively to $x^2=4$ and $x^2>4$
13.	learners	Yes	chorus
14.	Mr. Smith	Inequality sign, right?	
15.	learners	Yes	chorus
16.	Mr. Smith	OK, so we are saying that that [what] we are going to put here, if we square it, it must be bigger than 4. You get that?	
17.	learners	Yes	chorus
18.	Mr. Smith	OK, do you think there would be a 1 here	
19.	learners	No	chorus
20.	Mr. Smith	It can't be. If you put 2 here, what answer are you going to get?	
21.	learners	4	chorus
22.	Mr. Smith	Is 4 greater than 4?	
23.	learners	No	chorus
24.	……	…………….	………
25.	Mr. Smith	Now, but we need to solve for x here. What do we do? Let's move on. Alright, then what?	
26.	learners	[inaudible]	
27.		Did you hear that? We do likewise. In other words, we are doing exactly what we have done here.	Points to the solution of the equation on the board
28.		How many observed that? You agree this is true?	
29.	learners	Yes	chorus

Teacher's task 2: Encouraging explorations, discouraging rituals

Conclusions

Let me finish this letter with the words of one of the greatest French mathematicians, Henri Poincaré:

> [M]y memory is not bad, but it would be insufficient to make me a good chess player. Why, then, does it not fail me in a difficult mathematical argument in which the majority of chess players would be lost ? Clearly because it is guided by the general trend of the argument. … If I have the feeling, so to speak the intuition, of this order, so that I can perceive the whole of the argument at a glance, I need no longer be afraid of forgetting one of the elements; each of them will place itself naturally in the position prepared for it, without my having to make any effort of memory.