Introduction:
Developing mathematical discourse -
Some insights from communicational research

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Although all the papers in this special issue speak about mathematics and learning, they may appear too diverse in their foci to be bound together in a single volume. At first glance, such issues as the development of algebraic or geometrical thinking, dependence of mathematics on language, the fidelity of curricular implementation, interactions between children trying to learn mathematics in collaborative groups, and the impact of emotions on mathematics learning do not seem to have much in common; all the more so that these topics, when taken together, cover at least three separate research areas, those of cognition, affect, and social interactions. Traditionally, these three types of study differ from each other not just in their themes, but also in their foundational assumptions and methods. Coming from such diverse frameworks, the papers in this volume may appear as targeting several different audiences, with none of these audiences interested in, or even able to access, all seven of them.

It is not by accident, however, that the penultimate sentence of the last paragraph has been qualified with the word "traditionally". One of the main aims of this special issue is to break out from the grip of the separatist tradition, the tradition of using different, often unbridgeable discourses for dealing with different aspects of learning. The importance of the project of bridging and unifying can hardly be overestimated. If the collective effort of those who study learning-teaching processes is to result in a picture of the proverbial elephant rather than in a collection of possibly misleading partial images, researchers need to build on each other's work; to be able to do so, they have to communicate with one another; and in order to communicate, they need a common discourse, one in which cognitive and affective, as well as intra-personal and inter-personal (or individual and social) aspects of teaching-learning processes would all be seen as members of the same ontological category, to be studied with an integrated system of tools, grounded in a single set of foundational assumptions.

A unified discourse of research is what brings the different parts of this special issue together. Between the covers of this volume, the foundational and methodological diversity disappears. Thanks to this discursive unity, the relation between stories told in the different papers is that of continuity and complementarity. When put one next to another, these seven texts combine into a single narrative about mathematics and its development through teaching and learning and, to some extent, also in the course of history. The picture that emerges from this collective tale is more than a sum of the parts. This synergetic effect is made possible by the fact that the seven teams of authors, dissimilar as they are in their interests, are nevertheless united by their ways of talking and their sense of understanding one another. Their common language and the shared set of basic tenets and of research methods make all their papers equally accessible to anybody who is prepared to become a participant in their discourse.

This common approach, inspired by Wittgenstein's (1953, 1988) criticism of mentalism, by Vygotsky's (1978, 1987) vision of learning as originating "on a social plane", and by discursive psychology, as described by Harré and Gillett (1995), was originally introduced as *discursive* (Kieran, Forman, & Sfard, 2001) or *communicational* (Sfard & Kieran, 2001), and more recently, as *commognitive*[1] (Sfard, 2008). Much of this introductory paper is meant for those readers who are not yet acquainted with the commognitive framework but are ready to give it a try. I begin with the basic commognitive ideas of mathematics as discourse and of discourse development. Later, while outlining the collective story of the growth of mathematical discourse that unfolds on the pages of this special issue, I explain additional commognitive notions that appear in several articles but are not fully expounded in any. In a postscript to my sketchy summary of the seven studies I make a few remarks on commognitive methodology, its principles, advantages, and possible pitfalls. I end this review with a description of a number of features that, so we hope, makes the adjective "special" particularly suitable as a descriptor of this issue.

1. Mathematics as discourse

According to the primary commognitive assumption, *thinking can be usefully defined as the activity of communicating with oneself.* This ontological tenet generates its own story of mathematics and of its learning. Thus, one of its immediate entailments is the claim that *mathematical* thinking, or simply *mathematics,* can be seen as a *discourse*, with this latter word referring to a specific *type of communication*. Mathematical discourse, as any other, is identifiable through a number of interrelated characteristic features: its special *keywords*, such as *three*, *triangle, set* or *function*, used in distinctly mathematical ways; its unique *visual mediators*, such as numerals, algebraic symbols, and graphs; its distinctive *routines*, that is, patterned ways in which mathematical tasks are being performed; and its generally *endorsed*

[1] This term is a combination of the words *communication* and *cognition*. The reason for amalgamating these two words will become clear in a moment.

narratives, such as theorems, definitions and computational rules. The descriptor "generally endorsed", used in this last sentence, is to be understood as referring to endorsement by the *mathematical community*, with this latter term signifying all the competent participants of mathematical discourses.[2] Given this definition, the term *mathematics learning* becomes tantamount to becoming a member of a mathematical community.

As already mentioned, the heroes of the story collectively told on these pages are mathematical discourses and the development of these discourses is the main motif of the tale. The seven papers can be divided into two thematic groups: The first three articles highlight selected aspects of the mathematical development while telling stories of children making their first steps as participants of specific mathematical discourses: the discourse of algebra, of geometry, and of function. The other four articles deal with the question of what shapes, puts in motion, spurs or hinders ontogenetic development (learning) of mathematics. The influential factors considered by the authors include the language in which mathematics is being learned; the "expert" mathematical discourse, as offered to the learners by the curriculum and the teacher; and processes of identity building, often quite emotional, that take place in the classroom in parallel to "mathematizing".

2. Development of mathematical discourse

The term *development*, as understood in this volume, refers to a *change in discourses*. Readers are urged to keep this definition in mind and never let themselves be misled by the traditional psychological use of the word *development* as referring to a change in the child. To make the distinction clearer and more memorable, let me say it again: development is understood here as modification of activity, not as an inner change in the actor (the activity may be public - communication with others - or private - thinking)[3]. If I insist on this distinction, it is because it is highly consequential. First, the term *development*, when used in the context of discourse, has a much wider applicability than the psychological idea of development and, unlike the latter, it

[2] The plural form, discourse<u>s</u>, implies that one should speak about a variety of mathematical discourses rather than just one. If I previously used the term *mathematical discourse* in the singular, it is because I meant a generic case.

[3] This disclaimer does not mean the rejection of the possibility that a change may be occurring at the same time *in the child* (think, for example, about physical "imprints" of scripts of activity in the learner's brain) . It only implies that this latter change, if any, is not of interest in our studies.

may refer to both historic and ontogenetic change. Second, in contrast to the traditional idea of development, which is usually contrasted with or complemented by that of *learning,* and is thus central to the time-honored “nature or nurture?” debate, development of discourse is, by definition, a product of collective human actions. As such, it makes the quandary of nature versus culture, or of natural forces versus human intervention, practically irrelevant.[4] Although the authors of this special issue still talk about *learning*, this latter term is not to be understood as complementary to development, but as its subordinate. More specifically, the term *learning* refers to a particular type of development, the one that involves mainly reproductive (as opposed to productive or creative) change in discourse. It is a change aimed at bringing the discourse of the learner closer to a historically established form of discourse. In sum, the term *learning* is almost synonymous with *ontogenetic development* ; it extends the community of discourse rather than changing its discourse. Finally, while speaking about the development of the child’s mathematical discourse rather than about the development of the child herself, one does not make any claims on the child as such and does not, automatically, pass any judgment on her “general abilities”. If such judgment is ever to be made, it has to take into account the history of the *collective* efforts that led to the emergence of the child’s discourse.

2.1 How discourses develop

The use of the single term *development* for both historical and ontogenetic change does not imply that historical growth of discourses is identical with the one that takes place throughout individual lives. These two forms of development are very distinct, if only because of the fact that the former is mainly productive (creative), whereas the other is mainly reproductive. This said, there are reasons to believe that these two forms of development, although happening along different time scales, share certain basic discursive mechanisms and are subject to a number of comparable constraints. Because of this, it is often justified to speak about discourse development without specifying which kind of change is meant, historical or ontogenetic. In this special issue, as long as there is no explicit statement to the contrary, the word *development* should be understood as referring to both of these types of growth.

[4] True, there are some natural constraints on human ability to develop mathematical discourse, but to paraphrase Jerome Bruner’s metaphor, these constraints do not determine the course of discourse development anymore than the mechanical constraints on the movement of a car determine the trajectory of the driver.

With respect to the nature of the resulting change, processes of development can be divided into two main categories: those that happen at the level of discursive objects, "from inside" the discourse, and those that involve meta-discourse (discourse about discourse), that is, result from looking beyond the discourse itself.[5] *Object-level development* is one that expresses itself in the expansion of what is known about the already existing universe of mathematical objects (commognitive definition of mathematical object will be presented in a moment). In commognitive terms, this means exploring the objects so as to be able to formulate and endorse new narratives about them. Object-level growth, therefore, is mainly accumulative. A good example of this kind of development is one that results from an investigation of ever new families of functions, carried out once the object called function has been defined and some basic narratives about it have been endorsed. In contrast, *meta-level* developments are those that change the rules of the game: an expansion of the discourse and the increase in its complexity are accompanied, if not outright conditioned, by a change of meta-discursive rules. These changes are called meta-level because they originate in reflection about the existing discourse in its entirety (as opposed to reflection on its objects) and in particular, in the attempt to find patterns in the existing discourse. The discourse change inspired by such reflection and counting as meta-level is not a matter of a simple accretion. Being governed by different meta-rules, the new discourse is *incommensurable* with the preceding one. In fact, even the old discourse will be, from now, subject to these new rules. In most cases, such meta-level change is necessitated by an introduction of new mathematical objects. Thus, for example, almost any type of number, the rational, the negative, the irrational or the complex, when first introduced, does not seem acceptable unless some of the previously endorsed narratives can be dispensed with. For instance, when negative numbers are introduced, one cannot claim any longer that multiplication preserves order of magnitude between two numbers. Renouncement of this kind of narrative will not appear possible unless there is some change is the rules of endorsement. In the case of negative numbers, one has to give up the idea that "mathematical truth" is imposed by the world itself and agree that consistency of a new narrative with a chosen sub-set of those previously deemed as true is the sufficient condition for its endorsement.

[5] These two types of development, that are called here object-level and meta-level, have been named in Sfard (2008) *endogenous* and *exogenous*, respectively.

It is already quite clear, I assume, that of the two types of discourse, the one that happens at the meta-level involves a more radical and complex kind of change. For this reason, this type of development was more challenging to the mathematicians of the past, as it is also to present day students and teachers. For the researcher, it is an inexhaustible source of wonderings and potential insights about learning. Meta-level developments of mathematical discourse are the main theme of this special issue, and thus some elaboration on this special type of discursive change is in order.

Whereas meta-level development is always a result of reflection on an existing discourse, there are, basically, two ways in which such reflection may lead to an expansion and alteration of that discourse. The first type of meta-level development may be called *vertical*, as it involves combining the existing discourse with its own meta-discourse (which is metaphorically considered as a transition to a *higher* level)[6]. Thus, as argued in the article by Shai Caspi (of which I am co-author), elementary algebra can be seen as arithmetic combined with a formalized discourse about numerical patterns, that is, about arithmetic itself[7]. Indeed, at the closer look, the familiar algebraic expressions, such as $a(b+c)=ab+ac$ or $3x+7=18$ are symbolically encoded texts about numbers: the equality $a(b+c)=ab+ac$ is a generalization of certain numerical relations and the equation $3x+7=18$ is a question about arithmetic calculation. In the same article, the authors posit that the discourse on functions, the one that occupies centre scene in the paper by Talli Nachlieli and Michal Tabach, emerges from a reflection about patterns of elementary algebra. As argued by Nathalie Sinclair and Joan Moss, geometry can also be described as developing in a sequence of shifts from discourse to meta-discourse. Finally, the same assertion is made in the text by Kim, Ferrini-Mundy & Sfard with regard to the development of cardinality discourse, one that starts with the talk about natural numbers and ends with Cantorian stories on sets and infinity.

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[6] Note that the use of the words *vertical* and *horizontal* in this article is different from the one made by Piaget (who spoke about vertical and horizontal *décalage* in development), and also from their use by Treffers and Goffre (1985), who said: "The activity of transforming a problem field into a mathematical problem question is called horizontal mathematisation - the problem field is approached with mathematical methods. The activities of processing within the mathematical system are vertical mathematisations." (p. 109).

[7] The adjective *formalized* means that the rules of the discourse have been explicitly formulated and designed in such a way as to prevent the emergence of ambiguities.

The other type of meta-level development, one that here will be called *horizontal*, involves combining a number of hitherto separate discourses into a single one by subsuming them to a new discourse, populated with new types of mathematical objects. Thus, the discourse on functions can be seen as one that subsumes discourses on computational processes and on lines in a plane. Or, to use an example relevant to the article by Newton, two computational discourses, a discourse about numbers that are known today as *whole*, and a discourse about ratios that are basically pairs of numbers resulting from comparative measurements, may well exist side by side before they turn into sub-discourses of the discourse on *rational* numbers. In mathematics, the majority of discourse-modifying advances involve a combination of vertical and horizontal developments.

2.2 Why discourses develop

The reasons for the object-level development, the one that results from exploring existing mathematical objects, are self evident, and perhaps the most obvious of them is not unlike the one given by George Mallory in response to the question of why he climbed Mont Everest: one explores objects because they are there. The practical usefulness of narratives about mathematical objects is another rationale.

Reasons for meta-level development are somewhat more complex. The first thing to note is that discourses would often change as if of themselves, without any conscious human effort. An ongoing change in forms of communication is an inevitable result of their being constantly produced and reproduced. Much of this unceasing transformation is a product of unintended, often serendipitous variations by individual reproducers. This said, most of the durable developments that took place in mathematical discourse along history were outcomes of mathematicians' desperate quest after the holy grail of perfect communication, one that never fails and that allows to say as much as possible in as little words as possible. This improbable dream was what drove the constant effort to disambiguate mathematical discourse and to make it ever more compact. Whereas the project of disambiguation led to the formalization of mathematics, mathematicians' wish to "compress" the discourse so as to increase its expressive power was the main incentive for their incessant effort to climb to ever higher discursive levels. The best discursive means for saying more with less is the discursive construct known as *mathematical object*. As previously mentioned, introduction of new mathematical objects is often the very gist of meta-level development, and this is the main type of event that leads to discourse compression. Let me elaborate.

To understand what mathematical objects are all about, one needs to understand how they are constructed. Let us look at an example. Creating a mathematical object called *basic quadratic function* may be seen as originating in two types of attempts at discursive compression. First, because the operation of squaring numbers has many general properties, it is reasonable to look for means to talk about the operation of squaring in general, without a constant reference to specific numbers. In the canonical mathematical discourse, this need is satisfied with the help of an algebraic expression x^2 that can be interpreted as both the operation of squaring and as its result (it thus reifies the operation). Second, it turns out that every endorsed narrative about x^2 has an "isomorphic" counterpart in the form of an endorsed narrative about the line called a [elementary] parabola. Stories about the new signifier *basic quadratic function* or *sqr(x)*[8] may, from now on, serve as replacements for both narratives about the operation of squaring and about the parabola. For instance, while saying that "sqr(x) has a minimum for x=0" one says simultaneously that the parabola is u-shaped and that the operation of squaring produces only non-negative values. For this reason, it is justified to claim that the discourse on the quadratic function *subsumes* the discourses on the operation of squaring and on the parabola. All this shows that, indeed, creation of the object called *sqr(x)* is central to the project of compressing mathematical discourse.

Let me add a few words about the way signifiers of mathematical objects are interpreted, or as we prefer to say, *realized*, in the course of discursive activity. Realization of a signifier A is any word or symbol, B, that signifies a material object or of a process on material objects that is pointed to when the signifier A is used. To put it in a more formal way, one signifier counts as a realization of another if there is a closed set of true statements in which the two signifiers are exchangeable. To act as a competent participant in mathematical discourse, one has to realize words that signify numbers, triangles, functions, sets and other mathematical objects with the help of other mathematical words and mediators. More often than not, one signifier may have

[8] The word signifier refers here to the words *basic quadratic function* themselves, not to what is signified by them. To make this subtle distinction we presented these words in italics.

many, perhaps even infinitely many realizations. For example, the signifier *basic quadratic function* or *sqr(x)* can be realized as *a set ordered pairs of numbers where the second member of the pair is a square of the first*; as a line called parabola; as the symbolic formula x^2, as the *operation of squaring numbers*; or even as just the phrase *the square*. These and many additional realizations of basic quadratic function, together with their own realizations and so forth can be considered as what constitutes the mathematical object signified by these words. More generally, *a mathematical object can be defined as a mathematical signifier together with its realization tree*, where the realization tree is a hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations, and so forth.[9] The reader is asked to keep this definition in mind while reading the different papers.

2.3 Models of discourse development

The natural question to ask following the above reflections on the nature of and reasons for meta-level developments of mathematical discourses regards possible trajectories of this development. For instance, what could be said about the sequence of changes that turn the inchoate numerical talk of a child into the canonic discourse on rational numbers? The parallel question about the historical development of numerical discourse can, in principle, be answered through an analysis of historical data, but the question of whether there are any grounds for making general claims about ontogenetic development of discourses requires some deeper thought. Indeed, is it justifiable to assume similarities in the evolution of mathematical communication of different persons? My answer to this last question is yes. Moreover, I wish to argue that some of the claims about regularities in learning may apply also to historical developments.[10]

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[9] In mathematical terms, tree is a kind of graph. Because of the transitive nature of many signifier-realization relations, it may be more useful, for some purposes, to think about realizations as constituting a connected graph that does not necessarily have hierarchical structure and can contain closed cycles.

[10] At this point, it is extremely important to recall the fundamental difference between these present claims and those made by Piaget and his followers about developmental regularities and the parallels between historical and ontogenetic developments: unlike in Piaget's theory, the development discussed in this volume is that of mathematical discourse of the child, not of the child herself; and the regularities to be identified in the processes of ontogentic and historical developments are seen as imposed by the structure of the discourse, not by biological makeup of discourse participant. Thus, our present assertions about development should not be interpreted as in any way related to the evolutionist claim about "ontogeny that recapitulates phylogeny". Rather, it is a statement about structural properties of discourses that make some developmental scenarios more likely than some others.

My above assertion is grounded in the fact that on the sheer force of its definition, the meta-level development is subject to a number of constraints, and these constraints greatly reduce the number of possible developmental trajectories. Perhaps the most obvious of these restrictions regards the order of development. For example, if algebra is a meta-discourse of arithmetic or in simpler terms, the science of patterns discernable in the realm of numbers and numerical operations, it is as unreasonable to think about algebra emerging prior to arithmetic as it is to imagine a house being built from its roof down. The implied order is certainly the one that could be watched along history, and the odds are that it is also the one that should be recommended in schools. After all, trying to introduce students to algebra prior to their being reasonably versed in arithmetic would mean violating the single most important principle acknowledged by all teachers and researchers, regardless of their educational worldview: the principle of constructing new knowledge from the old knowledge or, in commognitive terms, of growing new discourse from old discourse. This violation is likely to result in ritualized algebraic discourse, disconnected from the main source of its meaningfulness. Indeed, unless a special effort is invested in a "retroactive link-building", the objects of thus constructed algebraic discourse may be lacking the vitally important ties to objects from other discourses, and thus may have seriously impoverished realization trees.

Although the different developmental constraints do not determine the course or the pace of discursive growth, they do make some options more likely than some others. Cognizant of constraints on meta-level changes and helped with what is already known from history and from empirical research about children’s learning, one can thus construct models of plausible trajectories of discourse development. Construction of such a model begins with parsing the existing canonic discourse into layers, each of which is a meta-discourse of the one that precedes it; it continues with trying to “undo” those changes that occurred in this preceding layer as a result of subsuming it, once upon a time, under its formalized meta-discourse. This last task constitutes a major challenge to researchers, whose immersion in the present day canonical discourse of mathematics makes them practically impervious to alternative forms of this discourse. To do the necessary unpacking they need to break out from their own deeply rooted discursive habits. In this undertaking, they would typically help themselves with empirical data and specially designed methods of discourse analysis. These techniques are exemplified in the articles by Shai Caspi and Anna Sfard and by Nathalie Sinclair and Joan Moss, both of which deal with the question of how children's mathematical talk differs from that of experienced participants of mathematical discourse. The first of these papers highlights initial stages in the development of algebraic discourse and the second focuses on early geometry.

Three developmental models have been constructed by contributors to this special issue. Shai Caspi and Anna Sfard propose a multilayered scheme that maps the progression from arithmetic to elementary algebra, including its continuation in the form of the discourse on function. The three initial levels of geometric discourse, proposed by Nathalie Sinclair and Joan Moss, can be viewed as a commognitive rendition and a refinement of the well known model of van Hiele. Finally, Kim, Ferrini-Mundy and Sfard, before focusing on university students' talk on infinity, propose a multilevel model of the development of the discourse of cardinality. None of these three models is very detailed. All of them include only those properties of the developmental paths that seem to be necessitated by known constraints and have been corroborated by the existing data. As such, they should be treated as general *templates* for plausible developmental trajectories rather than concrete, fixed trajectories. To put it differently, the models inform us about highly likely common properties of possible historical developments and constitute a tool for evaluating different hypothetical trajectories for teaching and learning as either promising or implausible.

2.4 Development of mathematical discourse in school

The fact that the process of discourse development involves two types of events implies that there are also two types of learning, object-level and meta-level[11]. As stated before, the studies reported in this special issue pay particular attention to the latter type of learning. A few words

[11] Commognitive notion of meta-level learning would probably be interpreted by Piagetians as a major accommodation of mental schemes. Socioculturally minded scholars would likely notice similarity to Vygotsky's description of what he called developmental changes, that is, of these transformations that "lead to a reconstruction of the whole structure of the child's consciousness and in this way change the whole system of relations to external reality and to himself" (Vygotsky, 1998, p. 199; in Daniels, 2007, p. 309). A change that reorganizes what has previously been built is also implicated in the ideas of *scientific revolution* (Kuhn, 1962), of *epistemological obstacle* (as coined by Bachelard; see e.g. Chimisso, 2001), and of *conceptual change* (see e.g. Vosniadou et al., 2007,), except that none of these ideas is grounded in the ontology of thinking as communicating.

about what can support and what may obstruct meta-level learning must be added to help the reader understand the origins of research questions asked by the contributors to this volume.

Unlike object-level learning, which consists in production of narratives that are logical derivatives of those previously endorsed, meta-level learning is supposed to lead to a change that cannot be attained by pure deduction. There is an element of contingency and of choice in every meta-level transformation. For instance, while proceeding from unsigned to signed numbers, the mathematicians had to decide which properties of numbers that had been in force so far should be preserved and which of them could be compromised. Historically, these decisions were hard going and time consuming, and when eventually made, they were grounded in the mathematician's strong intuitions with regard to their prospective advantages. These intuitions were byproducts of the decision-maker's discursive experience. Only rarely can a novice be guided by such helpful intimations. Moreover, since the new rules or new objects of this discourse cannot be deduced from anything the students already know, it seems that the only way for the learners to become participants in this discourse is to actually try to participate, even if just peripherally. On the other hand, how can they participate, if they are unable to figure out the reason for its seemingly counter-intuitive meta-rules and cannot yet sense the prospective advantages of all these seemingly unlikely innovations?

How can the teacher act while ushering the learners into the new discourse, incommensurable with what has already solidified into a "habit of mind"? One can say that in their effort to individualize the new discourse, the students should be encouraged to engage in a thoughtful imitation of expert participants' discursive moves. Here, *individualization* means a gradual transition from a mere observation of discursive practices of others to fully active, autonomous participation in the discourse. *Thoughtful imitation* is what Vygotskian scholars would likely call "instruction of scaffolding": the learner and the teacher work together, whereas the student progresses from mere observation to implementing ever more substantial parts of the task. The learner does the latter by an attempt to recapitulate what was previously done by the expert. The adverb *thoughtfully* has been used in order to stress that the imitation should be accompanied by a constant effort to understand the expert implementer's reasons for acting the way he did. Talli Nachlieli and Michal Tabach, while watching a class of 7^{th} graders entering the discourse on functions hoped to observe this kind of students' engagement. What they found was only partially consistent with the idea of thoughtful imitation. I leave it to the authors to tell the details of the learners' techniques of coping with the paradoxical requirement to participate in conversations about objects that they were yet to construct.

The terms *individualization*, *imitation* and *scaffolding* imply an implausibility of independent meta-learning, that is, of the learning that would take place without active leadership of an expert participant.[12] This said, there is much the learners can do in order to prepare themselves for the transition to a new discursive level. At least, this is the message of the study by Shai Caspi and Anna Sfard, which showed that elementary school students spontaneously develop reasonable command of informal meta-arithmetic (just to remind, this spontaneous meta-arithmetic constitutes, according to basic commognitive tenets, the precursor of elementary algebra). The word *spontaneous* is used here the way it was applied by both Piaget and Vygotsky and their followers. More specifically, spontaneous growth is one that happens without its being explicitly intended and deliberately fostered in a school-like fashion. For Piaget, such growth happens on the force of our biological makeup, for Vygotsky, it occurs through everyday interactions with the human and non-human environment (but the human influence is primary; to dissociate himself from associations with biological determinism, Vygotsky eventually replaced the "spontaneous" with "everyday"). For example, Vygotsky spoke about spontaneous development of concepts, and thus spontaneous concepts "as those that were acquired by the child outside the context in which explicit instruction was in place" (Daniels, 2007*,* pp. 310-11). In our use talk on spontaneity we follow Vygotsky rather than Piaget.

This latter result, that is the fact that children can develop some command of algebraic discourse without being deliberately taught in a school-like fashion, is not surprising. If algebra is a discourse about arithmetic, a person with a good command of numbers and calculations doesn't need any formal introduction to start doing algebra. On the contrary, thinking about regularities in numerical relations and about numbers producing a given result is a natural thing

[12] I interpret Vygotsky as saying exactly this in the context of change we call meta-level learning can only take place in school. This is how I read the following statement, while taking Vygotsky's "development" as referring to meta-level development (see the previous footnote): "Instruction would be completely unnecessary if it merely utilized what had already matured in the developmental processes, if it were not itself a source of development" (Vygotsky, 1987, p. 212)

to do for a competent arithmetician. In some cases, such “preliminary” developments can be spurred by creating situations in which children are likely to start questioning their own discursive habits. In this volume, Nathalie Sinclair and Joan Moss report on how they used dynamic geometric software to stimulate a discussion, in the course of which 5-year old children modified their use of the word *triangle*.

2.5 Factors that shape ontogenetic development of mathematical discourse

Having discussed the nature, types, and typical obstacles to discourse development, as well as the way historical and ontogenetic growth may occur, the obvious question to ask is: *What is it that shapes the actual course of the latter type of development?* Because, as argued above, meta-level developments occur mainly in school, it is natural to try to answer this question in the context of school teaching and learning. Four teams of contributing authors try to do exactly this when they explore a number of factors most likely to impact classroom processes and their outcomes. The shared assumption implicitly present in these four studies is that development of mathematical discourse may be influenced from inside or from outside the discourse itself; that is, it can be dependent on properties of a given mathematical discourse, as it takes place in the school, or by interactions between this discourse and some others.

The first type of intra-discursive influence, unsuspected by those who view mathematics as "universal" and independent of language in which it is practiced, has been found by Dong-Joong Kim, Joan Ferrini-Mundy and Anna Sfard, who compared English and Korean speaking university students' discourses on *infinity*. In the Korean language, unlike in English, there is a disconnection between colloquial and mathematical discourses on infinity, in that the mathematical word for infinity is not a formalized version of a colloquial word but a novel sound, inspired by a Chinese word for infinity. This difference has shown itself in the way the two groups responded to a series of colloquial questions and of mathematical tasks involving the word *infinity* and its cognates. Another type of intra-discursive influences was found by Jill Newton, who in her quest for the shapers of learning investigated the obvious suspect: mathematics curriculum. She decided to focus on the mathematics program of one middle school classroom. Alas, the term *curriculum* proved not unequivocal. As she was able to show, mathematically significant differences, sometimes considerable, may be found between *written* and *enacted curricula*, that is, between the mathematical discourse of the textbook and the mathematical discourse of the classroom. Armed with a detailed list of differences, the author explains how and why discourse development spurred by implemented curriculum may be quite different from what was intended by the authors of the written curriculum.

Interplay of discourses, while mostly invisible, may be a very powerful factor that moulds both historical and ontogenetic growth of mathematics. Two teams of authors ground their studies in the assumption that the learning-teaching process taking place in mathematics classrooms comprise two concomitant activities: that of *mathematizing* – communicating about mathematical objects; and that of *subjectifying* – communicating about participants of mathematical discourse. They also assume that of all subjectifying activities, the most consequential for learning is that of *identifying* – the activity of talking about properties of persons rather than about what the persons do. Marcy Wood and Crystal Kalinec take us to an elementary school classroom and show how the identities that students build for themselves and for others serve as a channel through which other discourses of the given society make their way into mathematics classroom and how these other discourses interfere with the activity of mathematizing. Theirs is a story of a student whose participation in mathematical discourse is hindered by stories others tell about him. Einat Heyd-Metzuyanim and Anna Sfard complement these insights with a narrative about intensive, emotionally charged identifying that evolved around the issue of leadership in learning and prevented students from taking advantage of what appeared as a particularly promising opportunity for learning. In the latter article, emotional expressions are treated as an aspect of communicational activity and are thus metaphorically described as *emotional hue* of utterances. Accordingly, the study of emotions becomes grounded in special types of discourse analysis.

3. Some principles of commognitive research

Let me conclude this introduction with a few remarks on the way commognitive research, as instantiated in this special issue, is done.

The commognitive ontology comes hand in hand with its own epistemological assumptions and research methods. To describe the epistemological status accorded by commognitive researchers to their own stories about the world, let me help myself with a metaphor. In this volume, researchers' stories are seen as connected to the world the way clothes are related to our bodies: they are human-made and should thus not be confused with what they are only supposed to "cover"; they have many versions, and although not everything goes, more than one of these versions would usually pass as a good match; and finally, be the match as satisfactory as it may, none of the versions should be seen as the "ultimate one".[13] This postmodern vision replaces millennia long tradition of treating research as the activity of "discovering" the world's own testimony. As illustratively stated by Richard Rorty, one of the engineers of the epistemological turnaround, "[T]he fact that Newton's vocabulary lets us predict the world more easily than Aristotle's does not mean that the world speaks Newtonian." (Rorty, 1989, p. 6). Inspired by Mikhail Bakhtin (1986), yet another protester against treating researchers' narratives as if they were the world's own "monologues", we may speak about commognitive research as grounded in the *principle of multivocality*. This principle has a clear implication for commognitive research practices: remembering that we are creative storytellers rather than the world's ventriloquists, the researcher is talking in the first person and is always in the quest after new, more convincing versions.

The multivocal approach to research, when combined with commognitive ontology, gives rise to several methodological principles. To begin with, there is the *principle of operationality*: if research is the activity of sharing useful narratives about the world, the researcher's talk must be guarded, as much as possible, against misunderstandings. The first condition for communicational effectiveness is disambiguation and operationalization of the researcher's vocabulary. Another principle, the *principle of completeness*, breaks with the tradition of the study of "concept formation" and requires that when one wishes to explore learning of a given topic, say, function, one should choose the entire discourse related to this topic as the unit of analysis. This is in tune with the holistic vision of discourses, one that precludes the possibility of a change affecting the use of a single word and leaving intact the use of the others. Further, while collecting her data the researcher keeps in mind that any kind of interaction, even that described as "non-interventional interviews", are events of learning. This implies the need to follow the *principle of contextuality,* according to which one has to try to document human interactions as fully as possible, never considering participants' utterances out of their conversational context (this practice stands in stark contrast with the tradition of ignoring

[13] One can take the metaphor of wardrobe even farther and say that our choices of research discourse are not any less a matter of fashion that are our choices of the ways to dress.

interviewer's parts of an interview). The activity of analyzing data, in turn, is guided by the *principle of alternating perspectives:* the analyst, considering the possibility of incommensurability between her mathematical discourse and the discourse of the participants of her study, makes a sustained effort to alternate between being an insider and an outsider to her own ways of using words. Finally, there is *the principle of directness* that regulates the way researchers describe their studies: in their reports, they begin with presenting things said (and done) by the participant, not with their own renderings of these data. They also make it clear that their stories are not directly about the world, but about the participants' narratives about the world (of course, the researcher may also be interested in the relation between participants' stories and her own narratives arising from direct observations). I believe that all these principles are well instantiated by the research work presented in this volume.

4. Some (special) features of the special issue

To strengthen the message of this collective work and facilitate the reading and understanding of individual papers, this volume has been endowed with three special features.

First, the reader is given access to the full, or nearly full, corpora of the data collected in the different studies. These materials may be found at the dedicated website, <the url will be provided by Springer>. One of the advantages of this addition is that it supports the use of this volume for teaching and learning. The readers may try to apply the proposed methods of data analysis before looking at how it was actually done in the original study. They may also approach the data from alternative, non-commognitive, perspectives, asking themselves about relative strengths and weaknesses of the different approaches. In cases in which studies were carried out in language different than English, such as Korean or Hebrew, the access to the original transcripts will allow the speakers of these languages to assess the aptness of translations done by researchers.

Since all the papers in the volume use terminology that is likely to be new for the reader, a commognitive glossary has been compiled and posted on the same website. This will be of help especially to those of the readers who wish to read a particular paper as a stand-alone piece, without the need to read also this introduction (in which the commognitive vocabulary is introduced once for all). With the help of the glossary, it will be possible to read each paper separately.

The last special feature of this special issue is the use of drawings to give the readers an access to visual aspects of events narrated in research reports. To understand interactions analyzed by the authors, it is important to actually see the participants and have a general sense of the way they were positioned in space relatively to each other and to the cameras. Most of the contributors to this volume did not have permission to publish photographs, and the drawings are the replacement. In these pictures, the images of the participants do not bear any significant likeness to the actual faces.

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Research, whether mathematical or in education, is society's deliberate effort to develop its own discourses. Undertaken in the attempt to make these discourses more useful as tools for sense making and as means for improving lives, this incessant pursuit of "a better version" is like our quest after new garments: One simply cannot be pleased forever with one particular wardrobe. The discourse change sought by those who contributed to this volume was not a simple one; the authors' intention was to modify some of the rules of the traditional game – they pursued meta-level development. Not satisfied with the research discourse as it is now, this courageous team of investigators decided to embark at the difficult endeavor of modifying their own talk as observers and analysts. How difficult such endeavor may be they then learned in two ways simultaneously: by inquiring meta-level learning in mathematics classrooms, and by struggling with their own entrenched discursive habits that had been obstructing novel insights. The fact that the group needed almost two years of concerted efforts to put this volume together is perhaps the most direct proof of the difficulty of the task. The team's long history of ups and downs - of cycles of doubt and despair followed by sudden leaps of fight - constitutes another piece of evidence. I wish to thank all the authors for not giving up. I hope that no reader is going to doubt the value - or the very existence - of the new clothes they have sewn together.

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References

Bakhtin, M. (1981). *The dialogic imagination: Four essays* (M. Holquist & C. Emerson, Trans.). Austin: University of Texas Press.

Cristina Chimisso. (2001). *Gaston Bachelard: Critic of Science and the Imagination*, Routledge.

Daniels, H., Cole, M., & Wertsch, J. V. (2007). *The Cambridge companion to Vygotsky*. Cambridge, UK: Cambridge University Press.

Harré, R., & Gillett, G. (1995). *The discursive mind*. Thousand Oaks, CA: Sage Publications.

Kieran, C., Forman, E. A., & Sfard, A. (2001). Learning discourse: Sociocultural approaches to research in mathematics education. *Educational Studies in Mathematics, 46*(1/3).

Kuhn, T. (1962). *The structure of scientific revolutions* (2nd ed.). Chicago: University of Chicago Press.

Rorty, R. (1989). *Contingency, irony, solidarity*. New York: Cambridge University Press.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, UK: Cambridge University Press.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42-76.

Treffers, A., & Goffree, F. (1985). *Rational analysis of realistic mathematics education - The WISKOBAS program* Paper presented at the Ninth International Conference for the Psychology of Mathematics Education, The Netherlands

Vosniadou, S., Baltas, A., & Vamvakoussi, X. (Eds.). (2007). *Reframing the conceptual change approach in learning and instruction*. Amsterdam: Elsevier.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1987). Thinking and speech. In R. W. Rieber & A. C. Carton (Eds.), *The collected works of L. S. Vygotsky* (pp. 39-285). New York: Plenum Press.

Vygotsky, L. S. (1998). *The collected works of L. S. Vygotsky. Volume 5: Child psychology* (M. J. Hall, Trans.). London: Plenum Press.

Wittgenstein, L. (1953). *Taken from Wittgenstein sans the private language argument*, Retrieved from http://www.ul.ie/~philos/vol1/wittsan2.html

Wittgenstein, L. (1988). *Remarks on the Philosophy of Psychology* (G. E. M. Anscomb, Trans.; Vol. 1). Chicago: University Of Chicago Press.