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Metaphors in mathematical thinking and in research on mathematical thinking: a prop or a trap?

When Rainer Maria Rilke spoke about the “geflügelte Entzücken” (“winged energy of delight”) or the “frühen Abgrund” (“first abysses” of childhood) (Rilke, 1956, 157), he was helping himself with the poets’ favorite device known as *metaphor*. The use of this special linguistic form, although usually associated with literary context, far exceeds the boundaries of literature and poetry. Scientists are probably as frequent users of metaphors as poets, except that they try to conceal this fact by pushing the tropes into the straitjacket of formal definitions. In this way, they deprive the metaphor of this one feature that poets find most endearing: ambiguity. For instance, geneticists use metaphors when they speak about *messenger RNA*, as do physicists when they mention *chain reaction*.

In this paper, I reflect on the role of metaphors in two inter-related domains: that of creative *mathematical thinking* and that of *research on mathematical thinking*. In particular, I aim to show that metaphors shape our thinking, either mathematical or meta-mathematical, and through thinking, they mold our decisions and actions. The focus of these reflections is on one ubiquitous and particularly consequential type of metaphor that can be called *metaphor-of-object* or *MofO*, for short. I am guided by the question of MofO's relative strengths and weaknesses. I begin with stressing the all-important constructive role of MofO in the development of mathematics. Later, while reflecting on the way the same metaphor functions in mathematics education research, I give special attention to those of its uses that are potentially harmful. I conclude with an attempt to answer the question of how to utilize metaphors in both mathematics education research and in teaching, so as to make sure they serve as props rather than traps.

1. The metaphor of object in mathematical thinking

Metaphor

The word *metaphor*, a combination of the Greek terms *meta*, which is equivalent to *trans* or *beyond*, with *phrein* – to *carry*, was first defined by Aristotle as calling something by a name that belongs to something else. While I agree that metaphor is a linguistic device, I prefer a definition that relates it to language *in use*, that is, to discourse: we encounter a metaphor wherever parts of a familiar discourse are used in conjunction with another, seemingly unrelated one. Thus, when we begin speaking about *messenger RNA*, we relate the "source" discourse about carrying telegrams or letters,

In F. Caluori, H. Linneweber-Lammerskitten & C. Streit (Hrsg.), *Beiträge zum Mathematikunterricht 2015* (S. x–y). Münster: WTM-Verlag.

which is what messengers traditionally do, to the "target" discourse of genetics, which is not about people or their actions, but rather about organisms and the ways they develop. When the metaphor comes into being, the resulting change in the target discourse involves much more than addition of the word *messenger*. Together with this single old term, utterances modeled on those about human messengers begin appearing in the relatively new discourse of genetics. This is the first evidence for the utmost importance of metaphor as a discursive device. As I will be arguing all along this talk, metaphors create new discourses rather than just embellishing existing ones, and as such, they shape our thinking, and through our thinking they impact almost anything we do (i.e. Lakoff & Johnson, 1980).

Metaphor of object

One type of metaphor, which we encounter literally everywhere, hides in expressions with which we are so familiar that we are unlikely to notice their metaphorical nature. Consider, for instance, such terms as *falling in love*, *acquiring knowledge*, *transferring learning*, *crystallizing ideas*, *building meaning* or *decomposing numbers*. Although coming from different domains, all these abstract notions have one common feature: because of their being preceded by words such as *falling into*, *acquiring*, *crystallizing*, *building* and *decomposing*, they all sound as is if they were signifying physical objects. Once we become aware of our tendency to think in terms of objects, we start noticing *the metaphor of object* literally everywhere. Wherever we go, we hear people and ourselves speaking about *entities*, either concrete or abstract; and this is true, in particular, in research discourses. To give just a few examples, let me mention such nous as *energy*, *force* or *number*, which we use in the so called "exact" sciences, and the terms *knowledge*, *concept*, *meaning*, *belief*, *attitude*, *value*, *personality*, *ability*, *gift*, *ego*, *superego* that populate our talk about humans. I am aware that calling all these notions metaphors may sound, at first, somewhat implausible, if only because none of these words seems to have an alternative that could be regarded as its "literal" version. In the next two sections devoted, respectively, to mathematical thinking and to thinking about mathematical thinking, I will support this claim with some additional arguments. Special effort will be invested in showing the metaphorical origins of the concept of number.

Before doing this, let me remark on the mechanism of *objectification* through which metaphor of object comes into being. Objectifying may be described as a discursive process of introducing new nouns and using them as if they signified self-sustained, discourse-independent object. This process involves two discursive moves: *reification*, that is replacing talk about process (using verbs) into talk on things (using nouns); and *alienation*, that is, removal of the human performer of the process. See, for instance, the two cases presented in Table 1. In the first example, which comes from mathematical discourse, the transition is from the talk about an operation to the talk about its product; in the second case, taken from the discourse on people doing mathematics, the move is from talk about properties of human action to talk about properties of the actor.

Mathematical discourse	**operation**		**product**
	If I extract a square root from x and raise the result to the third power, I get the same result as when I raise x to the 3^{rd} power and extract square root from it	⇨	The 3^{rd} power of square root equals square root of the 3^{rd}
Discourse about doing mathematics	**property of action**		**property of actor**
	John often solves difficult mathematical problems and always passes test with flying colors	⇨	John has a gift for mathematics

Table 1: Examples of objectifying in mathematics and in research on mathematical thinking

2. Metaphor of object in mathematical thinking

Example: Number as metaphor

The claim that number is but a metaphor may sound unlilely. The reified and alienated way we speak about numbers, and the fact that talk on numbers is governed by strict rules that no person can change makes us convinced that while mentioning, say "number five" we refer to an object that exists in the world independently of our thoughts or will, just like stars and animals. The fact that contrary to the latter, this thing called *five* is not accessible through our senses does not undermine our belief in its independent existence. Still, why is it that when we take a look at pictures as different as the fingers of one hand, the famous building called Pentagon and the American coin called "nickel", we are tempted to claim that something is "the same" in all of them? What is it in these pictures that makes us see "sameness"? One can say that in each of these pictures, she can "see" number five. But what is this number five? In fact, we cannot see, smell or touch it! A similar query with regard to another set of pictures, say, a set of photographs of yourself from different times of your life and possibly quite dissimilar to one another, is much easier to answer: The same object (person) is shown in all them. Back to the first set, we do not find any *concrete object* behind the pictures and, after some additional reflection we are compelled to admit that the common property is different: It is the fact that *if we count the elements of the set*, we end with the same number word (*five*) in all three cases. Thus, if the common property of the photographs is the *object* that features in all of them, the common property of the other set is a certain *process* – process of counting. If so, why do we speak about both sets of pictures as if they were presenting *objects*? This way of speaking makes us able to build on what we are familiar with: concrete objects and processes involving such objects. This, in turn, makes us feel we understand things better. Indeed, thinking in terms of "the same object" about the processes that underlay the equivalence of the fingers of one hand, the Pentagon and the nickel helps us to account for the fact that we view these three things as in some sense "the same" in spite of the fact that they do not look as having anything in common. Below, I argue that using number words as referring to objects has another extremely important advantage: It makes us able to say more with less, thus making our discourses more concise and more likely to develop even further.

MofO in mathematical thinking as a prop

To objectify number means to start using number words within language structures originally devised for the talk about objects. It is thanks to this kind of use that we can create the brief expression "three multiplied by five is fifteen", which in its symbolic form, $5 \cdot 3 = 15$, becomes even briefer. Here, indeed, the words *five*, *three* and *fifteen* are used as if they signified things, two of which combine together to give the third. As long as number words remain unobjectified, the general fact expressed as $5 \cdot 3 = 15$ would have to be stated in so many words: *If I have five sets of objects such that when I count the elements of any of these five sets I stop at the word* three*, then when I put these five sets together and count the elements of the resulting big set, I finish with the word* fifteen.

This shows how the act of objectifying compresses the discourse, and thus increases its manageability. In metaphorical terms, mathematical objects function as compact black boxes which we may conveniently use without bothering about their complex, densely packed interiors. It is now possible to manipulate the newly constructed mathematical objects and combine them together – something that would have been a very awkward thing to do as long as the processes that gave rise to these objects have not undergone objectification. One can say that MofOs are the "whipping boys of mathematics": they take the "punishment" of complexity on themselves, leaving us free to enjoy the simple and elegant sides of mathematics.

It is also because of this newly attained manageability that mathematicians may now look at the existing numerical discourse "from above", searching for its characteristic patterns. They may be intrigued by, say, the operation of squaring numbers. Soon, the explorers will be objectifying this proceedure, and perhaps presenting it with the help of the brief algebraic formula x^2. The objectified operation of squaring will eventually be baptized with the new name *quadratic function* and will become subject to new operations and new complex processes.

What was presented here will repeat itself time and again: processes on already existing mathematical objects will be objectified and then operated upon. The new process, involving the newly created objects will be objectified again, giving raise to yet another extension of mathematical discourse. All this will repeat itself over and over again. It is therefore the metaphor of object that fuels the development of mathematical discourse (Sfard, 1994; Lakoff & Núñez, 2000). One can say that mathematics owes this metaphor its very existence, no less!

MofO in mathematical thinking as a trap

One generally acknowledged downside of metaphors is that they often bring with them undesirable entailments. This is what happens when we automatically ascribe a property of the source object to the one that constitutes the target. Thus, for instance, the students tend to interpret the mathematical term *limit* as referring to what mathematicians call the *upper bond*: they wrongly believe that a sequence cannot, at any point, transcend its limit. Clearly, this belief is fed by such everyday uses of the word as the one we make while speaking about speed limit or when stating emphatically, "My patience has its limit!".

Another MofOs-related problem, probably more difficult to deal with, stems from the counterintuitive nature of the idea of viewing a process as its own product. Indeed, defining the basic complex number *i* as "the square root of –1" may be seen as improbably as the claim that a cake is the same as the recipe according to which it is made. Moreover, to construct a new object, one must fulfill an inherently circular pair of requirements: On the one hand, to bring this object into being and start getting acquainted with its properties, one has to engage in a conversation about it; on the other hand, how can one talk about an object before she knows anything about it and is not even aware of its existence? Many of those who lag behind others in their mathematical development, possibly also suffering from acute mathematical anxiety, may be the victims of this paradox.

3. Metaphor of object in thinking about mathematical thinking

In this part, I turn to the role of metaphor of object in discourses in which we engage while doing research on mathematical thinking. After showing some examples, I will claim that the risks of MofOs in the talk about people may exceed their gains.

Examples: MofOs in discourses on thinking

In the previous part of this paper, I made an effort to show the omnipresence of MofOs in mathematical thinking. The examples in Table 2 make us realize how common this metaphor is also in our thinking about thinking. At a closer look we realize that literally every statement on processes – on what people do (*think*, *see*, *deal*, *fail*) – can be replaced with an allegedly equivalent statement on objects – what people *have* (*ideas*, *concepts*, *discaclulia*).

Talk about processes		Talk about objects
The student *has always been dealing* successfully with tasks involving functions	⇨	The student acquired *the concept of function*
She *has been always been failing* in dealing with numerical tasks	⇨	She has dyscalculia

Table 2: Examples of objectification in our discourses about mathematical thinking

MofO in thinking about mathematical thinking as a prop

Although MofOs are as beneficial in the discourse on mathematical thinking as they are in mathematical discourse with regard to the economy of speech (see Table 2), I now wish to point out the price we pay for replacing the talk about what people *do* with statements about what they we *have*.

MofO in thinking about mathematical thinking as a trap

Those who use the metaphor of object in the talk about people face at least three types of dangers. First, the MofOs may lead to ambiguities and, at the same time, create a misleading impression of clarity; second, some metaphorical entailments may induce decisions that harm the learners of mathematics rather than helping them; and third, the unacknowledged ambiguities may lead to faulty reasoning and to logical fallacies. Let me elaborate on each of these claims.

Ambiguity is the inherent property of metaphors, which, although greatly appreciated by poets, makes scientists suspicious. The researcher's wariness is fully understandable: when used without necessary precautions, the non-operationalized metaphors may lead to controversies that appear to be disagreements about *facts*, while being in reality the result of differing *uses of words*. For instance, the long-standing controversies around Piaget's famous number-conservation task may be explained by the fact that the central idea of "having the concept of number" has never been defined in operational terms and that different interpreters probably did not mean the same thing while talking about children's numerical thinking (Sfard, 2008).

To illustrate the dangers that come with uncontrolled metaphorical entailments, let me use the example of the talk about *learning disability*. We are tempted to use these words whenever we face a child who has a long history of poor scholarly performance. Succumbing to the urge for objectification, we begin speaking in nouns and adjectives that indicate a property of the learner (*learning disability*, LD); this, as opposed to using verbs and adverbs that would make us focus on properties of the learners' actions (*performs poorly*). Without realizing, we also start being guided by the implicit message of the objectified talk: properties of a person, unlike those of her actions, are more likely to be given by nature than shaped by people; are general rather than context-dependent; and are permanent rather than transient. This view of the student's difficulty may to lead to consequential decisions: we are likely to direct those with "learning disability" to a separate life trajectory where there is little chance for them to further their mathematical education. In this way, our talk about LD becomes self-fulfilling prophecy: we create the reality rather than just reacting to it.

The last noteworthy consequence of objectifying is faulty reasoning expressing itself, among others, in our phony explanations of the observed phenomena. Think about the way we diagnose dyscalculia. To do so, we observe the student's performance on certain numerical tasks. If the performance is disappointing, we often conclude that the child "has" discalculia. From now on, we will use this "fact" to *explain* her inadequate numerical skills. At a closer look, however, this explanation is circular. While diagnosing, we stated: "poor performance, therefore dyscalculia". Later, while "explaining", we said "dyscalculia, therefore poor performance". The cause and effect exchanged their roles. In today's reality, where dyscalculia does not have a clear definition and is describable only in terms of how the learner *acts*, we do not add any information while saying that his poor performance is "due to dyscalculia". Since our diagnoses may impact the learners' lives, this kind of reasoning is a luxury we cannot afford.

4. Some implications for educational research and practice

Implications for research in mathematics education

The ramifications for educational research are quite obvious. First, we should look at past studies critically, staying alert to the danger of being misled by the MofOs with which the resulting literature is replete. Second, in any new research, we should eschew objectifications as much as we can. And if our discourse cannot be hermetically closed to new objects, we need to take precautions whenever such object is introduced. This means, above all, that we need to define the new notion as clearly and operationally as we can, while also adding an explicit disclaimer with regard to those metaphorical entailments and interpretations that we wish to exclude.

Implications for the teaching and learning of mathematics

In this paper, a closer look at the process of objectification brought an insight about arguably the greatest challenge for the learner of mathematics: turning processes into objects, which is an inherently circular procedure, with its different conditions constituting prerequisites for each other. The question of how to teach to overcome this difficulty requires much thinking. This said, one thing is pretty clear: the teacher may greatly increase the effectiveness of instruction just by staying aware of those special junctures at which the learner may be trapped in the "vicious circle" of reification. Another possibly helpful thing to do is supporting students in developing proper attitudes and realistic expectations. This can be done by exposing the learners' to expert discourse on new mathematical objects before they are capable of full-fledged participation, and accompanying this early experience with the slogan that, according to the historian of mathematics Jourdain (1956), has been assisting mathematicians of the past in similar situations: "Go on, faith will come" (p. 27)

While deliberating on how to define natural numbers, the founders of the axiomatic set theory Zermelo and Fraenkel could not escape the conclusion that these numbers originate in literally nothing: in the empty set. It is by turning the empty set into a set of which it is the only element, and then creating a new set composed of this latter set and the empty set, and so on, that natural numbers could be properly "ojectified". Indeed, metaphor of object seems to be a magic wand for turning nothing into something. Considering how much can be done with mathematical objects such as natural numbers – after all, our world would not be the same without them – one cannot but agree that metaphors may be among the most important sources of our creative powers.

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