CHAPTER 6

Objects of mathematical discourse: What mathematizing is all about

> I close my eyes and see a flock of birds. The vision lasts a second, or perhaps less; I am not sure how many birds I saw. Was the number of birds definite or indefinite? The problem involves the existence of God. If God exists, the number is definite, because God knows how many birds I saw. If God does not exist, then the number is indefinite, because no one can have counted. In this case I saw fewer than ten birds (let us say) and more than one, but did not see nine, eight, seven, six, five, four, three or two birds. I saw a number between ten and one, which was not nine, eight, seven, six, five, etc. That integer--not-nine, not-eight, not-seven, not-six, not-five, etc,--is inconceivable. Ergo, God exists.
>
> Luis Jorge Borges[1]

> I remember as a child, in fifth grade, coming to the amazing (for me) realization that the answer to 134 divided by 29 is 134/29 (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a tedious chore, while 134/29 was an object with no implicit work. I went excitedly to my father to explain my discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him, it was just a small variation in notation.
>
> William Thurston[2]

> The 'content' of mathematics does not exist in the material world; it is created by the activity of mathematics itself and consists of ideal objects like numbers, square roots and triangles.
>
> Michael A. K. Halliday[3]

Mathematicians and philosophers have been grappling with the idea of a mathematical object for ages, always recognizing its inherent blurriness, but never considering the option of simply giving it up. After all, if there is no such thing as mathematical reality, why should one bother to engage in mathematical investigations? In their most extreme forms, the claims about the nature of mathematics implied that mathematical objects have an independent existence of sorts. Those who objected, have been reproached by their Platonically minded colleagues:

> Everything considered, mathematicians should have courage of their most profound convictions and thus affirm that mathematical forms indeed have an existence that is independent of the mind considering them…. [4]

If I opt for operationalizing the time-honored idea of mathematical object rather than trying to do without it, it is only partly out of reverence to its long history, and certainly not because of any Platonic leanings on my part. My main reason is the hope that this special notion, with its deep metaphorical roots, will help us in understanding the developmental connection between mathematical discourses and discourses on material reality.

1. Mathematical objects

1.1 Discursive objects

While mathematizing, we are in the incessant chase after the objects of our activity. True, in this “object hunt” we proceed from one tangible entity to another, but I called these latter entities "realizations" rather than "mathematical objects." There is a number of reasons for this lexical restrain. First, realizations are characterized by being perceptually accessible – a property which one does not expect to find in a genuine mathematical object. Second, one signifier would usually have many visual realizations and it would be difficult to tell which of them deserves being singled out as "the" object. Finally, as was already mentioned, the distinction between signifier and realization is relative. Symbolic artifacts are often exchangeable in these two roles. For example, one can use a table of function values as a signifier and realize it in a formula, and vice versa – the formula may be realized in a table. Thus, whether a word, algebraic symbol or icon should count as a signifier or as a realization of a signifier is a matter of use, not of any intrinsic property of these artifacts.

Basically, therefore, almost any mathematical realization may be used as a signifier and then realized even further. From here it follows that any signifier can be seen as a “root” of a “tree” of realizations. In this tree, each node fulfills the double role of a realization of the node just above it and of a signifier realized by the nodes just beneath it. Figure 6.1 presents a schematic beginning of a possible realization tree for the signifier "Solution of 7x+4=4x+8." The nodes featuring "2" and "3" can be unpacked even further, showing that each of these signifiers may be realized, for example, as equipotent sets of objects. On the basis of what we saw in Episode 5.6 it is justified to claim that at least the middle sub-tree reflects Jas's realizing capacities. I supplemented the scheme with the other two sub-trees so as to present my own realizations that corresponds to "solution of 7x+4=4x+8."[5] The notion of a realization tree will now help me to define discursive object.

Definition: *The (discursive) object signified by S* (or simply *object S*) in a given discourse on S is the realization tree of S within this discourse.[6]

A few remarks about thus defined mathematical objects are now in order. First, realization trees, and hence mathematical objects, are *personal constructs,* even though they originate in public discourses that support only certain versions of such trees. As researchers, we may try to map personal realization trees and present them in diagrams such as the one in Figure 6.1. Inclusion of a specific realization – the graph of a function, for instance – in the tree would mean that in certain situations the person has been observed implementing this realization.

Second, the realization trees are a source of valuable information about the given person's discourse. Making skillful transitions from one realization to another is the gist of mathematical problem solving. In addition, a person's tendency to apply mathematical discourse in solving practical problems depends on her ability to decompose signifiers into chains of realizations long enough to reach beyond the discourse, to familiar real-life objects and experiences. Hence, one method to gauge the quality of one's discourse about, say, function, would be to assess the richness, the depth, and the cross-situational stability of the person's realization tree for the signifier "function".

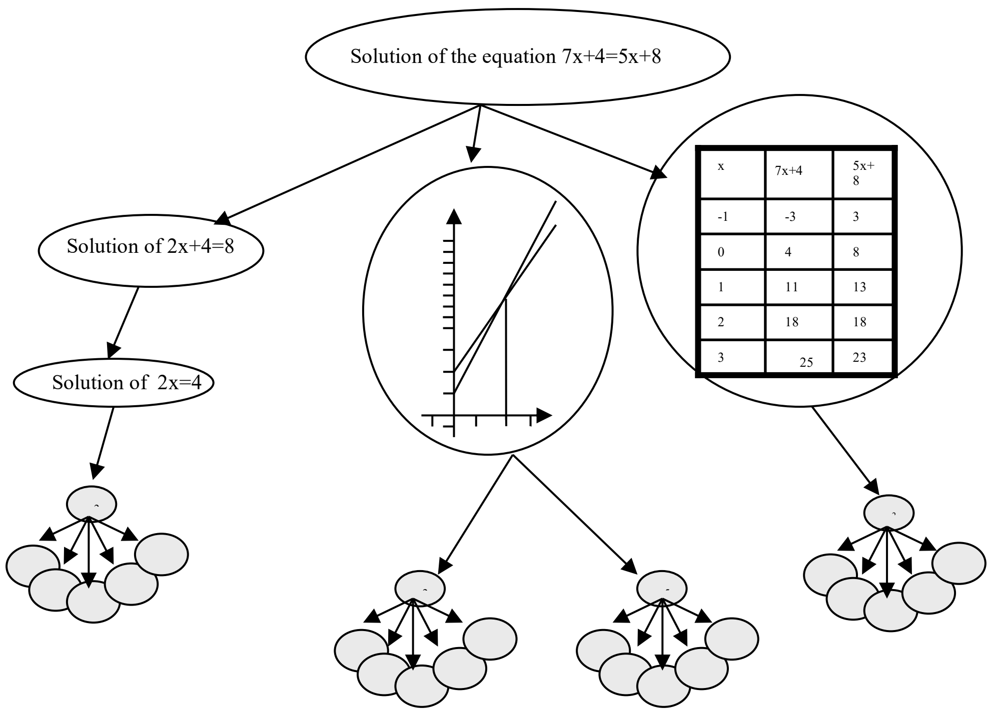

Figure 6.1 A realization tree of the signifier "solution of 7x+4=5x+8"

This last statement leads me to the third point. While analyzing transcripts of conversations in the attempt to map discursive objects, one needs to remember that *these personal constructs may be highly situated* and, in particular, can be easily influenced by co-interlocutors and by other specifics of the given interaction. For example, some realizations, although well known to the person and likely to be used in a skillful manner whenever such use is initiated by others, may never be evoked by the person on her own accord (in this case, we may say that the person did not fully individualize the use of the given signifier – the fact that escapes our eyes if we never have the opportunity to observe the person trying to solve problems on her own). Realization trees of an individual, as mapped by a researcher on the basis of a finite number of observations, may thus change from one set of observations to another. In particular, as shown time and again in cross-cultural and cross-situational studies, processes of realizations of a given signifier, say "four times thirty five," evoked by a person in school or in a research interview may be quite different from those that arise spontaneously while the same person is implementing everyday activities.[7]

Table 6.1: Object analysis of Ari's talk in Episode 5.7

Utterances	Object-signifier	Realizing procedure	Realizations	Object [I]
[1a], [1b], [11a], [1c]	"the slope" "the intercept" "the zero"	**in table** **1.** Find the zero in the left column of the table **2.** In the right column of the table, find the number b corresponding to that zero	**Written:** -5 **Announced:** *minus five*	The intercept
Writes: 5x+-5		**in algebraic formula** Locate the free coefficient in the formula 5x+-5	**Written:** -5 **Announced:** *minus five*	The intercept
[1d], [13], [15], [19]	"slope"	**In table[II]** **1.** In the left column, check the difference Δx between successive numbers, x_1 and x_2 **2.** In the right column, check the difference Δy between the corresponding numbers, y_1 and y **3.** Find the ratio a = Δx/Δy	**Written:** 5 **Announced:** *five*	The slope
[3], [5]	"slope"	**In algebraic formula[II]** Locate the coefficient of x in the formula 5x+-5	**Written:** 5 **Announced:** *five*	The slope

[I] This is, of course, my interpretation of Ari's signified object (in this case, interpretation is the signifier which the interpreter uses exchangeably with the singifier that is being interpreted). Since Ari's use of the signifiers 'slope' and 'intercept' was rather confusing (see his hesitation between these two words in [1] and [11] in Episode 5.7), I needed to attend to his realizing procedure and the resulting realizations before I could come up with my interpretation of his signified object. In the first row, for example, I concluded, that his object is the one I myself evoke when I use the signifier 'slope.' My interpretation, like any other, is tentative, and I will regard it as the best available hypothesis as long as no contradicting evidence is found.

[II] This table-based procedure can also be described in structural terms as "Locating the right-column counterpart of the left-column zero".

Finally, *different interlocutors may realize the same signifier in different ways*. Unacknowledged differences between personal realizations harm the effectiveness of communication and may even lead to a breach. The conversations between Ari and Gur, between Noa and her teacher, and between Roni and her father are good illustrations of this claim. In each of these cases realization trees of the two interlocutors differed not only in the amount of components and their mutual arrangement, but also in the nature of these components. Identifying individuals' discursive objects may thus help in assessing the effectiveness of interpersonal communication. While analyzing Ari and Gur's conversation about the slope I tried to do exactly this: I scrutinized the conversation for accessible parts of their realization trees "growing" from the root (signifier) "slope of g". According to my interpretation, Gur's realization tree was practically non-existent, even though the boy did try to create it ad hoc. The result of my analysis of Ari's discourse appears in the rightmost column of Table 6.1 under the heading 'object.' I assessed Ari's realizations for 'slope' as equivalent to my own. Of course, in stating this equivalence I relied on the absence of negative evidence not any less than on the presence of the positive. In mapping people's mathematical objects one needs to remember that trying to specify *all* the elements of one's realization tree is not a viable research task. In our analyses, rather than asking whether interlocutors' objects are "the same," we should be trying to see whether there is a reason to suspect that they might be different.

1.2 How discursive objects come into being

Let me now turn to the processes of construction of discursive objects. The task is, in a sense, the reverse of what was done above. So far, we have been looking at processes of realization employed by the mathematyst in the attempt to interpret familiar signifiers. I am now going to look at realization trees in the reverse direction: rather than "unpacking" them from their roots I will now proceed from the "leaves" of the threes to their roots. This is the direction in which realization trees, and thus discursive objects, are being constructed in the first place. The main question I will be asking may be formulated as follows: Why and how does a signifier of an existing object become a realization of another signifier? Or, to put it in a somewhat different way, what is it that makes people collapse a number of dissimilar things into one – into realizations of a new signifier? To translate it into a concrete example, how do entities as different as the canonic parabola, table of numbers paired with their squares, and the formula x^2 come to be seen, one day, as similar enough to become realizations of the same signifier, 'the basic quadratic function'?

Let me begin by dividing all the objects into *primary* and *discursive,* or *p-objects* and *d-objects*, for short. Discursive objects have already been defined, and one way to define the term *primary object* is to say simply that it refers to an object – a perceptually accessible entity – that cannot be called discursive. More specifically,

Definition: The term *primary object* (*p-object*) refers to any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds).[8]

In other words, primary object is a real-life tangible thing that has not yet been signified and thus did not become an object of communication. The process of construction of discursive objects may now be described recursively, as follows: *Discursive object* (*d-object*) arises by assigning a new signifier to a number of p-objects or formerly constructed d-objects (note that once this pairing is performed, the component p-objects and the signifiers of the component d-objects become realizations of S). From now on, the new signifier will be used in certain well-defined ways in the talk about certain special aspects of the signified objects. Thus, the effect of such assigning is creation of a whole new discourse, with its own objects. To explain how to assign, let me first define the simplest, 'atomic' d-objects and then show how compound d-objects are built from those that have been constructed before.

Simple (atomic) discursive objects arise in the process of *proper naming* (*baptizing*): assigning a noun or other noun-like symbolic artifact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. For example, assigning my dog with the noun 'Rexie' (or with the words 'my dog', for that matter) is an act of creation of the discursive object Rexie (my dog).

Compound discursive objects arise by according a noun or pronoun to extant objects – either discursive or primary in one of the following ways:

- by *saming*, that is, by assigning one signifier (giving one name) to a number of things that, so far, have not been considered as in any way 'the same'

- by *encapsulating*, that is, assigning a signifier to a set of objects and using this signifier in singular when talking about a property of all of the set members taken together; and
- by *reifying*, that is, by introducing a noun or pronoun with the help of which narratives about processes on some objects can now be told as "timeless" stories about relations between objects.[9]

Let me elaborate on each one of these constructions.

The process of *saming* can be seen as the act of calling different things the same name. Thus, we create a new d-object when we assign the signifier 'finger' to all the elongated objects growing from human palms, when we pair the signifier 'fraction' with all the symbols of the form ${}^{a}/_{b}$ where a and b are sequences of digits (numerals), or when we use the expression 'basic square function' in communicating both about parabola and the expression x^2 . Saming is thus the act of associating one signifier with many realizations. The necessary basis for such saming is the fact that whatever is said with the common signifier (e.g., 'basic quadratic function'), and turns out to be endorsable when translated into a narrative about any of this signifier's realizations (the parabola), will be endorsable also when translated into a narrative about the other realization (the expression x^2). To put it simpler, the basis for calling two objects the same name is the fact that a certain closed subset of endorsed narratives about one of these objects is isomorphic to a certain closed subset of endorsed narratives about the other object (a set of narratives is called *closed* if it contains all the narratives that can be logically derived from those already in the set). While describing mathematics as "the art of calling different things the same name" Henri Poincaré[6] stressed the fact that although the process of saming-with-names is not unique to mathematical discourses, it plays a particularly prominent role in this discourse. The range and depth of the resulting realization trees is much greater than in any other discourse.

Encapsulation is the act of assigning a noun or pronoun (signifier) to *a specific set* of extant primary or discursive objects, so that some of the stories about the members of this set that have, so far, been told in plural may now be told in singular. Encapsulation, therefore, is the creation of the pair <noun, specific *set of* objects> which turns a number of objects into a single entity for any communicative purpose. For example, when we speak about the *Addams family*, we may continue and say "the Addams family *is* rich", and this is discursively equivalent to saying, in plural, "members of Addams family, when taken together, *are* rich". Similarly, when we say "Three-quarters *is* bigger than two-thirds" (rather than saying that the three-quarters *are* bigger, as seems to be suggested by the plural form of the 'three-quarters') we encapsulate the set of three parts, each of them called 'quarter'. Finally, when we speak about 'basic quadratic function,' we encapsulate the set of ordered pairs of numbers such as (1, 1), (2, 4), (3, 9).

It is notable that the above number-pairs are, in themselves a product of *reification* of the squaring operation. Much has already been said already about this latter type of process, so let me add just a brief reminder. Basically, reification involves replacement of talk about processes with talk about objects. This is what happens, for instance, when the signifier $^5/_7$ is introduced and the utterance "I divided the whole by 7 and took 5 of the parts" turns into "I have 5/7 of the whole." Or to use another example, reifying the operation of squaring 2 leads to the ordered pair <2, 4> which can also be realized as a point in the Cartesian plane. Combined with encapsulation of all such pairs while their first element ranges over all possible numerical values, the reification leads to the discursive object called "basic quadratic function." To give another example, the object we use to refer to as 'number five' arises from sets of objects which, when counted, lead to the final number word 'five.' This happens in two steps. First, the term "five fingers" is used to reify the process of counting the fingers of one's hand, the phrase "five apples" comes to replace the discursive process of counting apples up to five, etc. This assignment reifies the process of counting in that the noun phrase "five apples" replaces the processual description which says, "When I count these apples, I invariably end with the word 'five'." At a later point, the discursive object 'number five' arises when we decide to use the common name *five* to same all the instances of "five *somethings*".[10]

Note that all three constructions which create a new object S – saming, encapsulating and reifying – turn the component p-objects and the signifiers of the component d-objects into realizations of S. Indeed, according to the definition of these three constructions, whatever endorsed narratives is now created on S, this narrative is a translation of a narrative on its component sub-object. Such translation is performed according to well defined rules, the exact nature of which depends on whether the new object was created in the act of saming, encapsulating or reifying. The discourse on S is thus isomorphic to certain closed sub-discourses about component objects.

1.3 Mathematical objects as abstract d-objects

Let me now revisit the time-honored dichotomy between concrete and abstract objects. The term *concrete objects* can be defined as including all primary objects and all those discursive objects that arise through saming or encapsulating familiar primary objects. The realization trees of concrete objects are thus free from reifications. In contrast, abstract objects may be defined as d-objects that originate, among others, in reification of discursive processes.[11] According to this definition, a good example of a concrete d-object is animal, which is the product of saming fish, bird, mammal, etc., with each of these component objects being a product of saming of concrete d-objects. The d-objects signified by 'number' or '5' are abstract. The relations between the different types of objects, primary and discursive, concrete and abstract are presented in Figure 6.2.

Having made all these distinctions, I may now say that *mathematical objects are abstract discursive objects with distinctly mathematical signifiers,* that is, signifiers regarded as mathematical. The claim made in the beginning of this chapter about the importance of perceptual elements in mathematical discourse can now be put in an even stronger form. Mathematical objects are not any less 'material' than the primary objects, except that rather than being a single tangible entity that predates the discourse, they are complex hierarchical systems of partially exchangeable symbolic artifacts. A number of practical implications immediately follows.

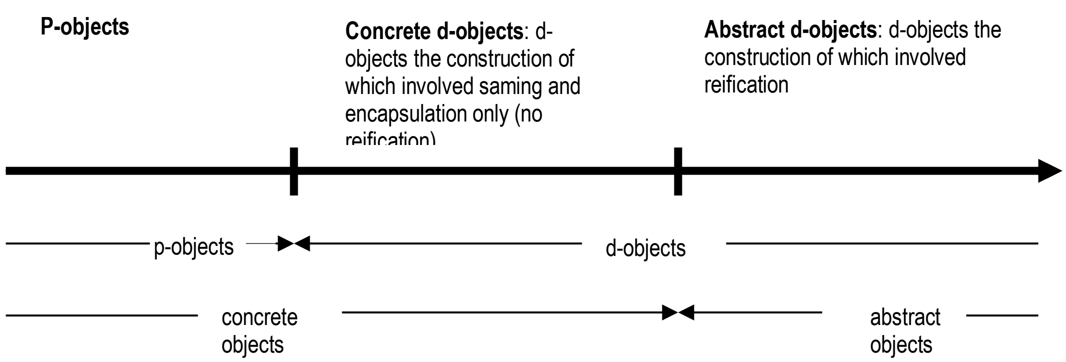

Figure 6.2. **Mutual Relations Between Categories of Objects**

First, the need for teaching "mathematical formalism" in schools has always been a moot point. The objectors of "formalization" clearly assume that one can separate between 'mathematical objects' and their 'representations'. This dualism of content and form or of object and tool-of-description is made quite explicitly by mathematician Alain Connes:

> The mathematician fashions what may be called *thought tools* [symbolic artifacts] for the purpose of investigating mathematical reality. These are not to be confused with mathematical reality itself.[12]

And yet, I have just argued that symbolic artifacts, far from being but 'earthly incarnations' of the inherently intangible entities called mathematical objects are, in fact, the fabric of which these objects are made.

Another issue worth attention is the current tendency to engage school children in the activity of inventing their own symbolic systems. While this is certainly a highly educative type of task, it does not remove the child's need for getting acquainted with commonly endorsed realizations of generally adopted signifiers. Once again, far from being just optional proxies of the 'real thing', the consensual, publicly endorsed signifiers and their realizations are the very thing that is being learned. To communicate with others and build on their ideas, one needs to use the same means as those endorsed by his or her interlocutors.

Finally, because mathematical communication does not differ from any other in its reliance on the senses, impairments of one's vision, hearing or bodily movement may stand in the person's way to becoming a fluent mathematyst.

2. Historical development of mathematical objects

Having stated that mathematical discourse is an autopoietic system which creates its own objects, and having defined the latter type of objects as those that originate in discursive *processes* on concrete objects rather than in the objects as such, I am now in the position to address yet another related question. *Historically speaking, what is it that spurred the emergence of different mathematical objects as well as their further evolution?* The issue is of great interest to philosophers of mathematics who wish to fathom the nature of mathematical discourse, and it is crucially important to students of human development and to educators who care about processes of learning – of individualization of mathematical discourse. The topic requires theoretical as well as empirical studies, and many monographs would have to be written if one wanted to deal with it properly. Here, I will limit myself to a brief outline of the history of one mathematical object and to some general reflection on processes of individual object-making that constitutes a part and parcel of mathematical learning.

Creation of the discourse on function was the act of exogenous compression in which at least three different discourses were brought together and subsumed in a new one. These discourses were, respectively, about algebraic formulas, about curves in the Cartesian plane, and about physical processes, such as the movement of falling bodies or of vibrating string. The focal object called *function* was thus a product of saming of three types of discursive objects. An opportunity for this type of saming arises when mathematicians become aware of an isomorphism between different, seemingly unrelated sets of endorsed narratives. Identification of two such isomorphisms was crucial to the emergence of the discourse on functions. The first step has been made in the seventeenth century by René Descartes (1596-1650), the founder of analytic geometry, who is credited with the idea of matching curves in the plane with newly introduced algebraic symbols (formulas). This invention was grounded in his awareness of one-to-one relations-preserving correspondence between sets of algebraic and geometric narratives. A few decades later, in the work of Johann Bernoulli (1667-1748), Isaac Newton (1643-1727), Gottfried Leibniz (1646-1716) and many others, algebraic narratives have also been associated with physical processes.

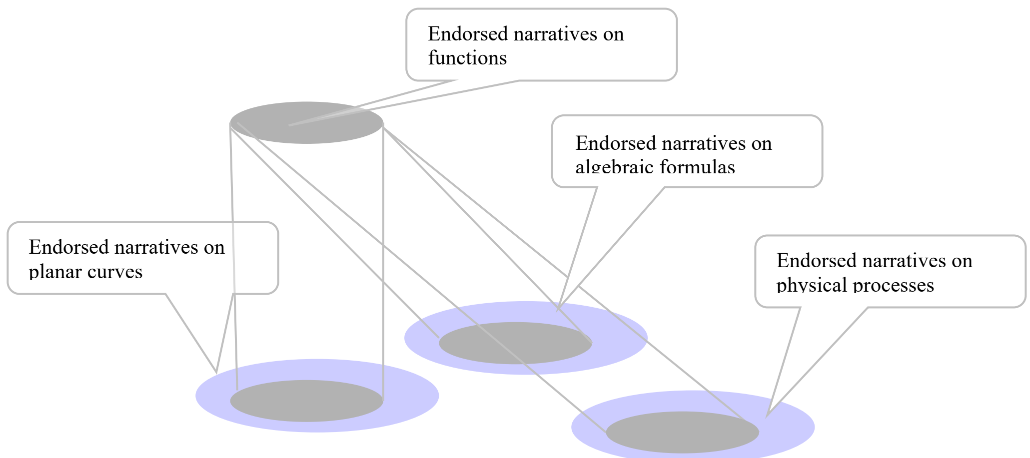

Figure 6.3: Discourse on function subsumes discourses on algebraic formulas, on curves and on physical processes

Of course, the set of all the generally endorsable stories one can tell about function is more restricted than the set of generally endorsable stories about any of its visual realizations – graphs, algebraic formulas, tables, etc. One can say that the stories about function that we endorse are those narratives that are true about all three of its realizations – the formulas, the curves, and physical processes, and they clearly do not exhaust all the true things that can be said about any of the latter objects (see Figure 6.3). Thus, some endorsed narratives about algebraic formulas – for instance, those that regard syntactic rules for constructing formulas – do not have an isomorphic equivalent in the discourse about curves, whereas some narratives about curves do not correspond to endorsed narratives about formulas (think about a curve representing the change of temperature over a period of time). Some general truths about formulas will not make it to the new subsuming discourse on functions, and the same may be said in the case of curves or physical processes.[13] What is lost in the amount of endorsable narratives is gained, however, in the remaining narratives' expressive power. One such narrative reveals 'the truth' about more aspects of reality than the corresponding narratives on any of its realizations.

The above explanation also implies that the discourse on functions is inherently unrealizable in just one mode, symbolic, iconic or concrete. After all, if there were just one type of realization, say formula, mathematicians would have no incentive to introduce a new signifier – we would simply speak about formulas.[14] This is why the first definitions of function which associated the new signifier exclusively with a combination of "variables and constants" or with "analytic expressions" (see Figure 6.5) were short lived: they failed to capture the subsuming aspect of the new discourse (after all, even this early idea of function was already a response to the awareness of isomorphic correspondence between narratives on "analytic expressions" and on curves.) The insufficiency of the definition that identified function exclusively with what we now view as its algebraic realizations became obvious when, following his famous debate with Jean-le-Rond d'Alembert about the problem of vibrating string,[15] Leonhard Euler became aware that his original rendering excluded the possibility to view certain types of physical movement as realizations of functions. These movements were not describable by a single formula but rather required what we now call 'split-domain" function. Following this observation, Euler proposed a new definition of function, one that made no explicit reference to any specific visual realization. from now on, he said, "a quantity should be called function only if it depends on another quantity in such a way that if the latter is changed, the former undergoes change itself."[16] He went on to formulate a new definition: "If... x denotes a variable quantity then all the quantities which depend on x in any manner whatever, or are determined by it, are called its functions."[17] This time, rather than being a mark on paper, function presented itself as a disembodied abstract entity, existing independently of its perceptually accessible "avatars". This formulation made it clear that functions could not be identified with any specific primary object, but at the same time it blurred the fact that they were complex composition of such objects.

The benefits of the definition that made no reference to visual realizations showed themselves when also the hegemony of iconic realizations (curves) ended as a result of the attempts at accommodating additional types of mathematical objects among those recognized as functions.[18] Generalizing from this example, we may say that the inherent indispensability of multiple visual mediation is one of the defining characteristics of mathematical discourse.

Jean Bernoulli, 1718

One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants.

Leonard Euler, 1748

A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities.

Figure 6.5. **Early definitions of function**[19]

3. Individualization of mathematical objects

You have never heard the word *krasnal* before, but you have just read the sentence:

A krasnal woke up and got up from his bed.

This sentence does not tell you what *krasnal* is (it is not a definition), but after you read it you may still be able to answer many questions about *krasnals*. Just try the following:

1. Which of the following syntactically-correct propositions seem to you to be meaningful sentences about krasnals, and which of them do not?
   - *Yesterday, a krasnal went to a supermarket.*
   - *A krasnal was divided by three and then squared.*
   - *Some of the krasnals were cheerful, some of them sang.*
   - *This krasnal is raised by public subscription.*
   - *A krasnal begins at 5:30 pm.*
   - *This krasnal is younger than this one.*
2. Now, can you complete the following sentence in a meaningful way?
   - *Krasnal* A is cheerful, whereas *krasnal B* is.....
3. Finally, try to construct a possibly meaningful sentence about *krasnals* yourself. Build one you believe cannot be meaningful.
4. And now, reflect on what you did and try to tell what made you able to
   - disqualify the utterance about "squaring *krasnals*" as senseless
   - complete the sentence about *krasnals* in a sensible way
   - create a new sentence about *krasnals*

Figure 6.5: **The mechanism of template recycling for interpreting new signifiers**

While being induced to mathematical discourse, one is faced with other people's objectified uses of words or symbols. The order of things in the processes of discourse individualization is thus different than in historical processes of object-creation. Think, for example, about such signifiers as number-words in the case of Roni and Eynat or "the slope of g" in the case of Gur. Initially, neither of these expressions signifies much for the young learners; however, the need to communicate with those for whom the signifiers are but tips of rich realization trees will fuel the children's interpretive efforts. Their unwritten aim will be to connect the new with the old – to find a way to realize the novel signifiers in possibly unusual combinations of discursive constructs with which they are already familiar.

Only rarely will the realization effort be a mere guessing game. To begin with, the learners' attempts may be guided by examples and explicit definitions offered by more experienced interlocutors. Indeed, exemplifying and defining is what mathematics teachers usually do while introducing a new signifier. As straightforward and promising as this strategy may appear, however, it may not be the first priority of the newcomer. More often than not, the learner would opt for a gradual immersion in the new mathematical discourse – the process in which she may be able to take advantage of learning techniques that have been working for her in colloquial discourses. One such technique is based on the mechanism of metaphor, that is, of inserting the new signifier into familiar discursive templates. To see how it works, you are invited to pause for a moment and implement the sequence of tasks presented in Figure 6.5. This self-experiment will give you the opportunity to learn from first-hand experience that a single, very brief exposure to the use of a word would often be enough to turn a person into a beginning participant of a new discourse. It is thanks to the spontaneous metaphorical projections that we manage to break the inherent circularity of the process of object-creation and engage in the new type of talk while still unable to realize the new signifier in any way. The workings of metaphor is pretty straightforward. The familiar discursive form into which the unfamiliar signifier has been inserted brought an association with other familiar forms and evoked an awareness of what may be proper or improper as an utterance about the new object which, in itself, is yet to be built.

Repetition of what was done before in new situations that, for one reason or another, seem to invite a similar sequence of actions is the very gist of learning. Such repetitions may be quite crude - they may be too indiscriminate or out of place altogether. Be they as rough as they might, however, these first awkward word-uses are the indispensable beginning. They will be fine-tuned in further interactions with more experienced mathematysts.

These and other processes have certainly contributed to the changes that we were able to notice in the numerical discourse of Roni and Eynat when we returned to them after a seven month long break with the same battery of comparison tasks as the one used in the first series of interviews. This time, the children's use of number words and words of numerical comparisons was not so different from that of the grownups as it was the first time round. To begin with, Roni and Eynat were now using number words in full sentences, such as "Six is less than eight."[20] This is a considerable step forward toward a more variegated, more flexible use of these words. Having said this, I should also stress that the girls still displayed a preference for adverbs *less* and *more* over the adjectives *smaller* and *bigger*, and this indicated that they used number words mainly as descriptors of sets, and not as signifiers of self-sustained objects. Another thing to note is that they were now using the generic word *number* – the word that has never appeared in their former utterances and which, in their earlier conversations, Roni's mother seemed to deliberately avoid[21] so as not to expose the children to terms with which she did not expect them to be able to cope. Thus, for example, after having counted the contents of a box, Eynat pointed to that box and said "Look at the *number* that it gave me," thereby urging Roni's mother to check for herself that the number she found was correct. On another occasion, while faced with an empty box, Roni declared, "There is no number". Even if rather non-standard, both these utterances belong to the category of objectified uses of the word *number*.

As the conversation proceeded, the children also became able to use the word *number* in conjunction with the expression *the same*. To be sure, they did not seem to be capable of such use when the new meeting began. Their enduring resistance to the term "the same" in the numerical context is readily visible in the following exchange which took place after the children discovered two marbles in each of the two boxes.

Episode 6.1a: *The same* – seven months later

125. Mother:	If there is 2 here and 2 here, in which is there more?	
126. Roni:	In none.	Shows 2 with her fingers.
127. Mother:	And where is there less?	
128. Roni:	In none	
……	And this is… more or less?	
132. Roni:	It is not more and not less	
133. Mother:	Neither more nor less? So what?	
134. Roni:	In the middle.	

In the view of all the advances made during the seven months that passed since the first meeting, we found this persistent confusion quite striking. The puzzlement was aggravated by the fact that the girls were using the words *the same* in other contexts. For example, Roni declared on a number of occasions that she and Eynat "did the same thing". It was thus extremely interesting to see the sudden breakthrough that happened just moments after Episode 6.1a. Frustrated with the children's persistent inability to say what she considered as obvious, Roni's mother eventually decided to make her intentions explicit and said:

147.	Mother:	Roni, so what does it say about the number of marbles? That it is…. *the same*?

Let me remind that also in Episode 5.2, which took place seven months earlier, the words *the same* were offered to the girls explicitly in a similar context (see Roni's father's utterance [56] in Episode 5.2). At that time, however, this offering had no effect on the girls' discourse. Now, the result was immediate. The children's next task was to compare boxes with 2 and 4 marbles, respectively. The following exchange took place after they successfully completed the assignment:

Episode 6.1b: The same – seven months later

288.	Mother:	Can you do it so that there will be the same amount of marbles in the two boxes?	
289.	Roni:	Yes	
290.	Mother:	How?	
291.	Roni:	(a) One moment.	(a) Empties both boxes
		(b) It is *the same* number now	

Later, Roni was also able to implement her mother's request "to make the same amount" ([288]) differently, by distributing the marbles evenly (3 and 3) between the two boxes.

Although the sudden jump in discursive possibilities is certainly impressive, it might be premature to see it as the ultimate evidence of reification of counting processes and of the emergence of a new discursive object. Rather, we have been witnessing the creation of a bond between the words *the same* and a certain type of situation, namely a situation in which counting marbles ends with the same number word in both boxes. Thus, the only thing that can safely be claimed at this point is that the expression *the same* has been successfully assciated to the procedure of evenly distributing marbles in boxes.

To sum up, Roni and Eynat are in the midst of creating their first mathematical signifier-realization pair. For all the advances already made, they have still a considerable way to go. Let me venture a general hypothesis about how their word uses are likely to change in the process of individualization. In the first phase, while not yet able to use a word in her own talk, the child may nevertheless be capable of certain routine re-actions to other people's utterances containing the given word. This is the case, for example, when she does not yet incorporate the word *number* in her sentences but would begin counting upon hearing the question “What is the *number* of marbles in this box”? With respect to the word *number*, Roni and Eynat were at this stage of *passive use* when we met the for the first time. Seven months later they are already beyond it; they are now actually uttering the word. This active use, however, is made only in a restricted number of specific routines, as a part of constant discursive sequences. This type of use can be called *routine-driven*. The next step in the development of word use will be witnessed when words become linked with constant phrases rather than with whole routines. At this stage, which can be called *phrase-driven*, the entire phrases rather than the word as such constitute the basic building blocks of child’s utterances. In the case of words such as *number*, the process of individualization is completed when the words “gets a life of its own” as a noun. One can now insert this word in any proposition in which there is a slot for this particular grammatical category. It is at this stage that the word becomes linked to a unique realization tree that remains relatively stable across contexts. Another characteristic phenomenon is the *transparency of the signifier*. Wwhen used, the signifier evokes immediate association with its realizations so that realizations rather then the signifier itself become a focus of attention.[22] The use of the word is now guided by the signified object – by the user’s awareness of the availability and contextual appropriateness of different realizations of the word. We may thus start talking about *object-driven* use of the word. The development of word use is schematically summarized in Figure 6.6.

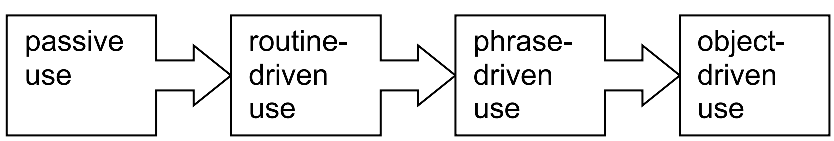

Figure 6.6: Four-stage model of the development of word use

4. Challenges of object construction

The hypothetical four-stage model presented above constitutes only a top-level description of the development of word use. This development may also be described in terms of object construction. As was explained before, this construction would usually involve many inter-related acts of reification, saming and encapsulation, each of which faces the learner with its own challenges. Let me survey some of these challenges briefly.

4.1 The challenge of reification and the anxiety of unrealized signifiers

The very same process-object duality of algebraic symbolism which constitutes the source of its special advantages may put many students off simply because of its being at odds with universally endorsed narratives about things in the world. Just to remind, composite symbolic expressions, such as $4+5$, $^{134}/_{29}$ or $2x+1$ may be used both as prescriptions for processes, and as the products of these processes.[23] Thus, we may treat $^{134}/_{29}$ as an *operation* of division or as the *result* of the division. This latter interpretation is involved when, for instance, one incorporates the expression $^{134}/_{29}$ into other symbolic expressions, such as, say, $(^{134}/_{29})^2 + 7$, thus treating $^{134}/_{29}$ as a realized object, ready to be operated upon. And yet, in the extra-discursive world, the notion of process that also serves as its own product sounds as implausible as the idea of eating the recipe for a cake instead of the cake itself. The confusion a participant of algebraic discourse may experience while having to deal with what looks like a prescription for action but needs to be treated as the result of this action is well instantiated in the following conversation with Guy, a 15 year old student with nearly two years of algebra behind him, who tries to solve for x the parametric equation kx-x = -2. Guy's momentary bafflement clearly stems from the difficulty to treat *k-1* as its own realization.

Episode 6.2: Guy solves *kx-x* = -2

[1] Guy:	There is a multiplication here, so what can I do?	*points to kx*
[2] Interviewer:	And if I wrote *3x-x*, would you be able to proceed?	
[3] Guy:	*3x-x*? It's *2x*.	
[4] Interviewer:	So? Isn't *kx-x* similar?	
[5] Guy:	But this… but this doesn't work… I don't know what k is.	
[6] Interviewer:	What have you done here to get *2x*? What did you do to the 3?	*points to 3x-x*
[7] Guy:	I subtracted 1… So what? Shall I subtract 1 here? I don't know… If I subtract 1 from k I will be left with the same mess… see, I don't know how to write it.. How I subtract 1 from k? How do I write it? *k-1*?	

The "anxiety of unrealized d-objects", instantiated in this episode, may be explained in yet another way. As was argued earlier, the main advantage of realizations is that they bring with them new endorsed narratives. This added value is made possible by the naturally occurring merge of the present discourse with the more familiar one, from which the realization was taken. Thus, when the *promenade*, which was initially but a bunch of intricately interconnected algebraic symbols, gets realized as a 5×5 lattice (see Figures 5.4

and 5.5), one starts capitalizing on her ability to create and endorse narratives about geometric shapes just by scanning these shapes visually and without any symbolic manipulation. This example may suffice to conclude that to be truly helpful, processes of realization need to yield forms different from the original signifier. One can hardly see how anything new can be asserted about *k-1* without its being first worked out into a familiar object, comparable with another familiar object of the same kind.[24]

A close look at the history of mathematics reveals that the worry about unrealized algebraic expressions did not pass over mathematicians. This difficulty could well be the reason why computational discourse was much slower to develop than the geometrical. It was probably the reason why the 3rd century Greek mathematician Diophant, who was among the first to use combinations of letters and numerals while dealing with computational procedures, did not nevertheless enter the history as the father of the symbolic algebra.[25] For Diophant the idea of using a prescription-for-a-process as the ready-made result of this process must have been as foreign as it was for Guy, who balked at the sight of the 'unrealized' expression *k-1*. A similar difficulty might have prompted Newton's declaration that "algebra is the analysis of bunglers in mathematics".[26] The autobiographical testimony of the mathematician William Thurston that appears as a motto to this chapter is a rare case of retroactive documentation of the experience of coming to terms with the process-object duality of mathematical symbols.

At this point it is natural to ask how one can help students who had not yet reconciled themselves with unrealized expressions. One such method would be to replace compound symbolic expressions with simple ones, thus according them the appearance of an accomplished, full-fledged "thing". This ploy, however, cannot be truly effective. Historically speaking, this is what was done by mathematicians in the case of negative and complex numbers when expressions such as *3-8* or *½ - 1* were replaced with –*5* and *- ½* , respectively, and when *√(-1)* was substituted with *i*. This mere renaming did not result in any real breakthrough, though. One explanation may be that the new symbols did not bring with them any new discursive possibilities, the way icons and concrete objects usually do. In both cases, it was discourse-enriching iconic mediation that eventually did the job. For negative numbers, the discourse-enhancing realization was the number line extended infinitely to the left of the origin; for complex numbers, it was the complex plane organized by 'real' and 'imaginary' axes.[27] Iconic and concrete realizations seem thus indispensable also in mathematics classrooms.

Another issue to consider in the present context is the remedial potential of practice. That practicing a discourse on, say, negative and complex numbers may, indeed, help in getting used to the counterintuitive duality of algebraic symbols has been repeatedly noted by mathematicians. For instance, Girolamo Cardan (1501 – 1576) who could see the usefulness, although not “the inner logic” of “unrealizable” formulas such as 3-8 or $\sqrt{(-1)}$, urged his fellow mathematicians to persist in using these expressions while “putting aside mental tortures involved.”[28] A few centuries later, the French historian and philosopher of mathematics Philip E.B. Jourdain justified this advice as one that, in hindsight, obviously proved itself:

> “For centuries mathematicians used ‘negative’ and ‘positive’ numbers, and identified ‘positive’ numbers with signless numbers like 1, 2, and 3, without any scruple, just as they used fractionary and irrational ‘numbers’. And when logically-minded men objected to these wrong statements, mathematicians simply ignored them and said: “Go on; faith will come to you”. And the mathematicians were right….”[29]

4.2 Challenges of saming

The first challenge facing those who wish to create a subsuming discourse by saming hitherto unrelated objects with the help of a single signifier is the resulting loss of certain deeply entrenched endorsed narratives. This difficulty is particularly acute when the saming signifier comes from one of the subsumed discourses. As a result of the new saming, a considerable change will occur in this signifier’s use. The amount of its realizations will grow whereas the amount of relevant endorsed narratives will go down. Consider, for example, one’s first encounter with the discourse on rational numbers. This discourse subsumes two seemingly unrelated forms of talk: the discourse on objects such as *one*, *two*, *three* etc., and the discourse on objects called *ratios*, such as 1:2, 3:5, etc. So far, the word *number* has been reserved for the first type of objects, but in the subsuming discourse the ratios will also count as its realizations, that is, as numbers (see the schematic presentation of this transition in Figure 4.3.) Following this growth of the realization tree, a narrative such as “Multiplication makes bigger” (or, more precisely, “A product of two numbers is bigger than any of the numbers”), so readily and obviously endorsable as long as the word *number* is reserved for *one*, *two*, *three*, etc., will have to be given up. The concession may not be easy to make. The lingering of old discursive endorsements and their reappearance in discourses in which they are bound to lead to contradictions is the well known phenomenon that gave rise to the theory of misconceptions discussed in Chapter 1.

Another challenge that comes with first attempts to use a common signifier for objects that did not count, so far, as in any way “the same” is the counter-intuitive nature of this process. The primary source of saming is in visually performed routines: we speak about differently looking things as “the same” if we can transform one of the images into the other one in a continuous manner. It is such transformability that underlies our claim that the person who speaks to us now is *the same person* as the one who was talking to us a minute ago, even though the image before our eyes has changed. Indeed, either we actually witnessed this continuous transformation of the former image into the present one or, based on our previous experience, we are aware that such a transformation must have taken place.[30]

In abstract discourses, the mechanism of saming is different. Consider such an endorsed narrative as *5 + 3(x+2) = 3x+11 for all x*, according to which the two component expressions, *5 + 3(x+2)* and *3x+11*, are equivalent (and thus, in a sense, “the same”.) As in the case of concrete objects, one way to substantiate this narrative is to show a certain kind of transformability. For instance, we may manipulate the first expression by applying the distributive law and then grouping similar terms. And yet, this operation is quite unlike the one that allows us to transform one image of a person into another. First, the chain of operations performed according to the laws of algebra does not result in a visible *continuous* transformation of the image we see. What we get is a discrete sequence of intermediary images (e.g, 5 + (3x +6), and then 3x + (5 + 6)), none of which resembles its predecessor in an immediately obvious way. Second, identity-preserving transformations of concrete images, such as an image of a person, do not leave behind them the visible “history” of the transformation, the way symbolic artifacts do. Indeed, when we transform a formula, all the intermediary expressions can be seen simultaneously written on a page one next to the other. Algebraic saming may thus be seen as contradicting the experiences underlying our sense of sameness in the case of concrete objects. After all, we cannot see a person simultaneously as she is now and as she was three minutes ago (unless helped by a camera, of course). If we did see these two images together, we would have said we were facing two different people.

For reasons of mathematical consistency and elegance, which I will not discuss now, transformability is not the preferred textbook substantiation of algebraic equivalence. Rather, textbook authors would explain that two formulas such as *5 + 3(x+2)* and *3x+11* count as *equal* or *equivalent* because they may be realized with the help of the same table of values or the same graph. This kind of substantiation, as elegant and desirable as it is in the eyes of the mathematician, may have little appeal for the student. Data from several studies have shown that although the symbolic transformations deviate considerably from the transformations of concrete objects, the argument of transformability may still appear more acceptable than the claim about shared graphs or tables. Thus, for example, in the Montreal Algebra Project the students were introduced to the notion of equivalent expressions after they discovered that differently looking linear expressions may have the same graph. A few days passed during which the class engaged in solving problems such as "Among the given expressions, which are equivalent to *3x+11*"? Following are excerpts from the classroom conversation that took place some time later.

Episode 6.3: Equivalence of algebraic expressions

[1] Teacher:	What does it mean that two expressions are equivalent? …. If two expressions are called equivalent, what do you, what does that mean? Sam…
[2] Sam:	That they *equal the same*.
[3] Teacher:	What do you mean when you say that?
……	…….
[7] Sam:	They *are the same*.
……………….	
………………….	
[35] Jas:	They, *they're basically the same thing*, but they look different.

The debate went on for a long time, but the excerpts above convey the gist of things. It is remarkable that the existence of a common table or a common graph, which had been discussed in the class as the defining feature of equivalent expressions, was never brought up in this conversation, and that the students spoke in terms of *sameness* rather than equivalence ("equal the same" [2], "are the same" [7], "they are… the same thing" [35]). The language of sameness is yet another indication of their preference for transformability as the required defining property. Indeed, this language imposes itself whenever the present image appears as a transformation of what was seen before. This is clearly how one tends to think when a new formula is connected to the former one with the equality symbol.

Resistance to the loss of endorsed narratives and the preference for the criterion of transformability can be a hurdle to mathematical saming. These may well be the reasons why beginning mathematysts would often be unable to see as the same what grownups cannot see as different. Based on what has been learned from cross-cultural and cross-situational research, saming may be most problematic when it is supposed to bring together colloquial and literate discourses. If such cross-discursive saming does not occur, the two discourses would function as mutually exclusive rather than exchangeable. In particular, everyday situations would evoke only colloquial forms of mathematical talk, whereas institutionalized educational settings would be dominated by literate discourses. In the traditional language, this phenomenon would be described as the "lack of transfer". The case of the Brazilian street vendor, M, who did not associate the school signifier "4·35" with the money transaction that he implemented so skillfully just few days earlier is a good example. In this context, I also recall a successful psychology graduate, Rinat, who, when asked to recount her story as a mathematics student, wrote: '[in elementary school] I could not understand why they told us to solve "¼ of 5" as "¼ · 5".' In the conversation that followed she explained: 'I was perfectly able to find a quarter of five cups of flour, and I could multiply ¼ by 5; what I didn't know was what made these two operations in any way "the same".' M's and Rinat's literate and colloquial realization trees were fully disjoint: the signifiers "4·35" and "¼ · 5" failed to work for them as the "kingpins of sameness" through which two realization trees combine into one.

An important thing to remember is that the ability to see sameness in differently looking things may be highly situated. A person who realized a signifier in a given way in one context may be incapable of the same association in another context. To put it in a metaphorical way, some paths down or up one's realization tree may be open in some situations and blocked in others. Once again, this phenomenon is most common for those links that connect colloquial and literate realizations of mathematical signifiers. In our interviews we often saw students who seemed unable to realize mathematical signifiers in colloquial ways until explicitly ensured that it would be "perfectly ok" to do so. This is what happened in the case of Mira who had no difficulty realizing literate signifiers such as 7·16 via icons and concrete objects, but who would not reveal this ability without a great deal of probing on the part of the interviewer. Clearly, the link between the literate signifier and the colloquial realizations remained blocked as long as she interpreted the interview as a classroom situation where such realization would often be deemed improper.

Finally, one needs to remember that different people may use the same signifier while saming across different sets of objects. Roni's and Eynat's inability to see as "the same" the things that the grownups could not see as different is one manifestation of this phenomenon. A similar example comes from Lewis Carol's famous character Humpty Dumpty, who could only see as the same what most people could see as different:

> I shouldn't know you again if we did meet," Humpty Dumpty replied in a discontented tone, giving [Alice] one of his fingers to shake: "you are so exactly like other people." "The face is what one goes by, generally," Alice remarked in a thoughtful tone. "That's just what I complain of," said Dumpty Humpty. "Your face is the same as everybody has—the two eyes… nose in the middle, mouth under. It's always the same."[31]

4.3 Challenges of encapsulation

Encapsulation – replacing the plural form with the singular when referring to a collection of objects – faces the learner with challenges of its own. The mere grammatical change may be not enough to bring about the consolidation of a collection into a single entity. Some students would thus continue referring to individual elements even when asked about the set as a whole. In a study on school students' discourse on infinity, the interviewees were asked to "tell which of the two sets, the set of odd numbers or the set of even numbers, [was] bigger".[32] The following excerpt is a representative of solutions offered by a sizable proportion of interviewees:

Episode 6.4: Which set is larger?

[1] Interviewer:	Given the set of all the even numbers and the set of all the odd numbers, which set is bigger?
[2] Rona:	The evens.
[3] Interviewer:	The evens is bigger? *[note the teacher's use of the singular in spite of the plural form of the subject]*
[4] Rona:	Because.. one… one and… one is odd and two is even. And so it goes.

The "so it goes" in utterance [4] seems to say that for each subsequent odd number the corresponding even number is bigger. This latter inequality is translated into the relation between "all the odds" and "all the evens". Thus, rather than trying to compare the numerosity of the two sets by constructing one-to-one mapping from one of the sets to the other, as could be expected from an experienced mathematyst, the interviewee compared single elements with respect to their numerical values.

Another related phenomenon was observed in a study in which a class was just introduced to the set-theoretical operation of *unifying* sets.[33] In the problem-solving activities that followed, the most common students' error was the confusion between connectives *and* and *or* (conjunction and disjunction) in presenting the defining conditions of the unification of two sets. Thus, for example, the student would write:

$$\{x: x < 3\} \cap \{x: x > 5\} = \{x: x < 3 \textbf{ and } x > 5\}$$

or even in the "simplified form"

$$\{x: x < 3\} \cap \{x: x > 5\} = \{x: 5 < x < 3\}$$

instead of the required

$$\{x: x < 3\} \cap \{x: x > 5\} = \{x: x < 3 \textbf{ or } x > 5\}.$$

This common confusion seems, indeed, indicative of the difficulty with the transition from plural to singular (or, in this case, from the talk about numerous objects to the talk about a single representative that epitomizes them all): the connector *and*, which would have been appropriate if the condition was put in plural ("the union contains all the elements of A *and* all the elements of B") becomes inadequate when applied to a single element of the set.

4.4 Pedagogical remark

The upshot of what has been said above is that those who wish to come to terms with new signifiers face many challenges. The obvious question is how a novice mathematyst can be helped in the task of object construction. This query merits its own studies, and my colleagues and I are already engaged in such research. For now, let me mention just one general principle.

All the hurdles of object-construction mentioned above contribute to, and are in turn aggravated by, the self-generating (autopoietic) nature of mathematical discourse and by the resulting inherent circularity of construction processes. The fundamental question, therefore, is how the circle of discourse-building can be broken. The principle "reflective practice makes meaningful," previously mentioned as a possible cure for the anxiety of unrealized objects may be of help also in this more general case. This principle is certainly in tune with the teachings of Wittgenstein, for whom the meaning of a word (or mediator) was no other than this word's use in discourse, and who, in fact, endorsed this maxim openly while offering the following 'instructional' advice: "Let the use teach you the meaning."[34]

Earlier, I have remarked that unlike the historical process of signification, the processes of individualization are grounded mainly in attempts to realize new signifiers to which one is exposed while particiapting in the discourse with more expereinced interlocutors. Metaphorically, we can thus say that the historical and individual developments stress opposite directions: the former are predominantly upward oriented, that is, aim at creating ever higher realization-trees; the latter is mainly an attempt to connect a new signifier to familiar objects. Such linking, if successful, will turn the new signfier into a top of a new realization-tree, with the familiar objects constituting this tree's lower layers. This said, let me stress that neither historical creations nor the processes of individualization are unidirectional. Indeed, both types of construction involve up and down zigzagging along the "branches" of realiztion trees, from one layer of mathematical objects to another. In the processes of learning, the proportions of significations – of the upward movement from existing objects to new ones – and of realizations – of the downward movement from a new signifier to its realizations in the existing objects – are a matter of pedagogical philosophy of the teacher.[35]

5. Objects of mathematical discourse – in the nutshell

While trying to pinpoint the gist of famously impalpable mathematical objects one is likely to feel as if she chased a phantom. In this chapter, after having shown that perception – the sense of sight, of touch, and of hearing – play as fundamental a role in mathematics as in any other discourse, I engaged in the project of operationalizing this elusive idea. To implement the task, I focused on the question of how signifier-realization pairs come into being in the first place.

The first thing to note in this context was that more often than not, realizations can also serve as signifiers and they can thus lead to their own realizations. If the process of ‘unpacking’ of a given signifier is reiterated, its *tree of realizations* results. The signifier S together with its realization-tree is called *discursive object* or *d-object*, for short; this, as opposed to *primary objects* (*p-objects*) which are unnamed perceptually accessible things. To put it recursively, S is a d-object if S is an atomic d-object of the form <proper name, specific primary object>; or S is a compound object created through the processes of *saming*, *encapsulating* or *reifying* of other d-objects with the help of S. Saming is attained by giving one name to many different objects. This can be done whenever the samed object share a closed set of endorsed narratives (that is, every narrative about one of the objects has an isomorphic counterpart in the form of an endorsed narrative about the other object). Reification, as explained before, consists in associating a noun with a discursive process. Encapsulation is the act of replacing a talk about numerous objects, in plural, with the talk in singular, in which one signifier refers to all these former objects taken together as one entity.

Discursive object is called *concrete* if it is either a p-object or a d-object constructed by saming or encapsulating primary objects. *Abstract objects* are d-objects originating in reified processes on p-objects. *Mathematical objects* are abstract objects with distinctly mathematical signifiers. These objects are personal constructions and different mathematysts may associate different objects with the same signifier. If they do, their ability to communicate is impaired.

A number of conclusions about mathematical objects immediately follow. First, although regarded as inaccessible to senses, mathematical objects are in fact complex combinations of visible realizations. Second, a special property of literate mathematical discourses that sets them apart from many others is that no one type of visual mediation – symbolic, iconic or concrete – would suffice to realize this discourse in its entirety. Metaphorically, one can say that mathematics resides in relations between visual realizations, not in the realizations as such. Third, mathematical communication apparently reverses the developmental order known from colloquial discourses: whereas these latter discourses are created for the sake of communication about physical reality, in mathematical discourse objects are created for the sake of communication. True, also mathematical communication is supposed, eventually, to mediate practical activities, and thus to pertain, in one way or another to the world of primary objects that predate the discourse. However, this fact may easily escape one's attention. The realization trees of mathematical signifiers, although likely to have primary objects or processes on such objects at their basis, may be too rich and complex to be embraced at a glance. Leaving the concrete foundations of such trees out of sight may thus be the condition for the proficiency of mathematical communication.

Processes of individualization of the use of mathematical nouns are of particular interest to those who seek pedagogical applications of research on human development. A model has been suggested according to which learners proceed from the *passive* use of such signifiers to *routine-driven*, to *phrase-driven*, and eventually to *object-driven* use. As one advances through these stages, the use of the word becomes broader and more flexible. In this process, the increasingly skillful "peripheral participant" overcomes multiple hurdles, inherent in the processes of saming, reifying and encapsulating. First, creation of subsuming discourses involves loss of some of the previously endorsed generalizing narratives. Second, saming processes in mathematical discourses may often appear counter-intuitive, as they do not match our everyday experience. Two properties make them quite different from the identity-preserving transformations of concrete discourses: the discreteness of the symbolic operations that transform one realization into an equivalent one, and the fact that they leave behind them a trace of visible intermediary forms. With relation to reification, the learner may suffer from the anxiety of unrealized signifiers and be baffled by the counter-intuitiveness of process-object duality. The action of encapsulation faces the learners with yet another type of difficulty, one that finds its expression in their frequently observed inability to translate the properties of elements into properties of the set, and vice versa

On the top of all these obstacles, there is the already mentioned inherent circularity of the process of individualization: participation in mathematical discourse is both a result and a precondition for our ability to construct mathematical objects. This dilemma is yet to be dealt with in a detailed way. In the meantime, the principle "practice makes meaningful," consonant with Wittgenstein's theory of meaning as word's use in discourse, has been put forward as an alternative to the idea of "meaning before practice".

Although by operationalizing the notion of mathematical object I seem to have answered the question of what mathematical discourse is all about, many important queries are yet to be tackled. One of them is how mathematical objects mediate our practical actions. We shall deal with this issue in Chapter 8. In the meantime, in the next chapter, we will take a closer look at how mathematysts perform their discursive actions and how they decide when to perform them.

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[1] Borges, *Argumentum ornitologicum*
[2] Thurston, 1990, p. 847
[3] Halliday, 2003, p. 140
[4] Thom, R. (1971). Modern mathematics: An educational and philosophical error? American Scientist, 59, 695-699 (quoted in Davis & Hersh, 1981, p. 319)
[5] The term *tree of realization* is reminiscent of the notion *chains signification.* This latter notion was introduced and extensively dealt with (under differing names) by the prominent semioticians Peirce and Lacan; recently, it was explained and instantiated in Walkerdine (1988) and in Cobb, Gravemeijer, Yackel, McClain, & Whitenack (1997). If I am reluctant to use this term, it is because the word chain implies linearity rather than a complex hierarchical structure, which is better captured by the term tree.
[6] To put it recursively, discursive object signified by S is this the S itself together with all the objects signified by its realizations.
[7] This example has been elaborated by Lave (1988), who observed a person finding two thirds of three quarters of cup of cottage cheese simply by spreading the cheese evenly on a plate, dividing it with two cuts into four equal parts, taking three of them and then removing one.
[8] One can even go further and say that the primary object is a set of images or sounds that are associated one with another and recognized as "the same" (the criterion for the "saming" is that a person reacts to the different images or sounds in the same way) primary to (independently of) naming. According to Piaget, people are not born with even this most basic form of saming (objects) but rather construct the primary objects – learn to treat the different images as the images "of the same thing" - in the first few months of their lives. It is this initial saming that Piaget had in mind while speaking about children acquiring the principle of "permanence of objects".
[9] Some writers (e.g. ,Dubinsky, 1991) use the word *encapsulation* as exchangeable with the word *reification*. In this book the two are used in distinct ways, as described above.
[10] To put it still differently, the relation between the noun 'five' and its realizations in the form of specific sets of five objects is that of reification: endorsed narratives that feature the noun 'five' can be translated into endorsed narratives about the process of counting of the members of these sets.

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[11] This definition corresponds to Jean Piaget's claim that what he called concrete and abstract thinking develop, respectively, through empirical and reflective abstraction. Lev Vygotsky's would speak, in this context, about empirical and theoretical conceptualization. Of course, one needs to remember that neither Piaget nor Vygotsky regarded concept construction as a discursive process.
[12] Jean-Pierre Changeaux and Alain Connes (1995, p. 13).
[13] This claim, however, stays in force even if we consider numerical functions only. The function that assigns to each natural number a value between 1 and 6 obtained by throwing a dice is an example of function without a formula. Dirichlet function, defined later in this text, is a function without a graph (or, at least, its graph is not a line in the Cartesian plane).
[14] Many people, especially students, do express themselves in this way. One needs, however, to distinguish, between the cases when this expression is just a convenient abbreviation, and the cases when a person cannot see the difference between the two formulation.
[15] Kleiner, 1989
[16] Ruthing, 1984, pp. 72-73
[17] *Ibid*
[18] For example, Johann Dirichlet (1805 –1859) offered the definition which is satisfied even by a construct as strange as the mapping that assigns the value 1 to every rational number and 0 to every irrational number.
[19] After Kleiner, 1989
[20] Because of idiosyncrasies of Hebrew, the literal translation should be "Six is more-little than eight," with the 'more little' not entirely standard but easily understandable as equivalent to "less".
[21] For example, here is how she formulated the request to make the contents of two boxes equal: "Can it be done so that there will be the same [thing]? That there be the same marbles in both boxes [….] the same amount of marbles in the tw…". Obviously, without the word *number* (or the word *amount*, for that matter, which the mother eventually did utter) the efforts could not be very successful.
[22] Because of idiosyncrasies of Hebrew, the literal translation should be "Six is more-little than eight," with the 'more little' not entirely standard but easily understandable as equivalent to "less."
[23] The process-object duality is not unique to algebraic symbolism. Many colloquial words, such as *solution* may be used in two roles – as signifiers of processes (in the present case, the process of solving), and of objects (the result obtained at the end of the solution process). However, the dualism of algebraic symbols is more difficult to accept, since, first, these are compound symbols that *read* as descriptions of processes; and second, whereas the word *solution* is realizable in two distinct separate forms, as a process (the description of a procedure) and as an object (the product of the solving procedure), this is not the case with algebraic expressions such as $2x+1$ (there is no separate product of multiplying x by 2 and adding 1).
[24] The related well known phenomenon that expresses itself in students' tendency to "simplify" expressions such as $3x+2$ as $5x$ or even just 5 is known in literature as "the need for closure" (Chalouh & Herscovics, 1988.) One can view this type of action as yet another evidence for students' inability to use what appears as prescriptions for a sequence of operation as if they were realized objects.
[25] Diophant's mathematical discourse that involved a mixture of verbal and symbolic expressions is known as syncopated algebra. Until his times, and for more than millennium after him, algebra was mainly rhetoric, that is, practiced in words only, without any symbolic mediation. The algebraic symbolism, as we know it, was introduced only in the end of the 16th century, and although proposed in one form or another by many individuals simultaneously, it is mainly credited to the French mathematicians, François Viète (1540 –1603) and René Descartes (1596 – 1650).
[26] Kline, 1980, p. 124
[27] This iconic mediator is also known as *Argand plane* and is sometimes called *Gauss plane*. The real number line was introduced in the 18th century and the complex plane in the 19th. (Kline, 1980; Boyer & Mertzbach, 1989)
[28] Kline, 1980, p. 116
[29] Jourdain, 1956, p. 27 [Jourdain. P. E. B. (1956). The nature of mathematics. In J. R. Newman (Ed.), *The world of mathematics*. New York: Simon & Schuster.]
[30] That even this ability of objectifying continuously changing images into permanent objects is not our inborn property but rather develops throughout a person's life experience has been stated and documented by many researchers, beginning with Piaget.
[31] Carroll, 1968, p. 229. [Carroll, L. (1968). *Alice's adventures in Wonderland & Through the looking glass.* New York: Lancer Books.]

[32] Caduri, 2005 [Caduri, Galit (2005). *The development of discourse on infinity*. Unpublished masters thesis. Haifa: The University of Haifa. In Hebrew]

[33] Sfard, 1987 [Sfard, Anna (1987). *Teaching the theory of algorithms in high school*. Unpublished PhD dissertation. Jerusalem: The Hebrew University of Jerusalem. In Hebrew]

[34] Wittgenstein, 1953, p. ?

[35] These days, the tendency is to keep processes of learning close to those of historical invention. In such processes the element of signification – of inventing new signifiers and creating one's own mathematical objects as a prelude to being introduced to those taken from existing public discourses – is strongly recommended (e.g., NCTM, 200)