# Algebraic Analysis in Klein's „Elementary Mathematics from a Higher Standpoint" [1]

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This paper will clarify the genesis and meaning of the concept of *algebraic analysis* as it is used in Felix Klein's *Elementary Mathematics From an Advanced Standpoint* (original German publication in 1908, English translation in 1932). The concept has a somewhat intricate history, and modern readers are no longer familiar with its meaning at Klein's times. We shall see that, surprisingly, it is a key concept for understanding the part on analysis in Klein's book.[2]

## 1. Algebraic Analysis at Universities and Schools in Germany

In the second half of the 19th century up to the times of Felix Klein the term *algebraic analysis* designated university courses and the related textbooks treating the elementary and preparatory parts of infinitesimal calculus (for an overview see Pringsheim & Faber 1909-21). Mathematical contents and methods of these courses dated back to Leonhard Euler's *Introductio in analysin infinitorum* (Euler 1748). This work comprised two volumes, the first one addressing 'pure analysis' as Euler called it, whereas the second treated the application of pure analysis to geometry.

Euler's and his contemporaries' understanding of 'analysis' was broader than we today are used to. In their view analysis included algebra since algebra was a key technic for 'working backwards' in the sense of the ancient Greek meaning of the term 'analysis'. This broader meaning was alive in mathematics well into the 19th century. Thus, in Euler's view the first volume of the *Introductio* treated algebra, namely, those parts of algebra he considered as preparatory to infinitesimal calculus proper. They comprised, among many others, transformations of functions, their development in infinite series, infinite products and continued fractions. The second volume contained in Euler's language the 'theory of curved lines' which, as a first approximation, we today would call analytic geometry.

In the preface Euler explained extensively the motivation of such an introductory book:

> "Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt to this more subtle art" (English translation from (Euler 1990, v)).

The *Introductio* provided algebraic techniques which prepared a student for a deeper understanding of infinitesimal analysis and which were not contained in the ordinary treatises on the elements of algebra. Moreover, the *Introductio* treated quite a few problems by which "the reader gradually and almost imperceptibly becomes acquainted with the idea of the infinite." (l. c.)

Thus, on the one hand, Euler drew a clear dividing line between the *Introductio* and the analysis of the infinite, on the other hand, he pointed to transitions and points of contact:

> "There are also many questions which are answered in this work by means of ordinary algebra, although they are usually discussed with the aid of [infinitesimal] analysis. In this way the interrelationship between the two methods [algebra vs. analysis of the infinite] becomes clear." (l.c.)

To the former he counted among others the derivations of the power series expansions of the logarithmic and the trigonometric functions which usually were treated only in the analysis of the infinite where they are obtained from the quadrature of hyperbola and circle. In volume II of the *Introductio,* among many other things, tangents, normals and curvatures of curves are calculated and Euler maintained that he achieved also this by purely algebraic methods:

> "Thus I have explained a method for defining tangents to curves, their normals, and curvature … Although all of these nowadays are ordinarily accomplished by means of differential calculus, nevertheless, I have here presented them using only ordinary algebra, in order that the transition from finite analysis to analysis of the infinite might be rendered easier." (l. c., vii)

We shall see below that and how Klein took an opposite position to Euler's in regard to the best way of introducing the logarithmic/exponential and trigonometric functions as well as the determination of tangents and curvatures in beginner's courses.

In the second half of the 18$^{th}$ century further textbooks appeared which established, in succession to the *Introductio*, *analysis of the finite* as separated from *analysis of the infinite*. The underlying intuitive idea was that *analysis of the finite* treats finite quantities whereas infinitely small quantities like differentials are the subject of *analysis of the infinite*. As we have seen *analysis of the finite* included infinite processes like power series expansions, and in the *Introductio* Euler used even infinitely small quantities for deriving results. Thus, what for mathematicians well into the 19$^{th}$ century was a clear-cut distinction between two fields is to a modern reader no longer plausible since to him infinite series and products are an integral part of infinitesimal analysis as are derivatives and integrals. To him all these concepts are based on the concept of limit.

A contemporary mathematical encyclopedia listed the following topics for *analysis of the finite* (we have shortened this list)

I. The theory of functions or of the forms of quantities
II. Theory of series
III. Combinatorial Analysis
IV. Binomial and Polynomial Theorem
V. Logarithmic and trigonometric Functions
VI. Analysis of curved lines
VII. Calculus of finite differences
VIII. Connection between the Analysis of the Finite and the Differential Calculus (among others Taylor's theorem, determination of maxima and minima)

The binomial theorem refers to the series expansion of the power function, discovered by Newton,

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} \cdot x^k ,$$

where *m* can be an arbitrary real (or even complex) number. This formula was a corner stone of *analysis of the finite* which served for numerically calculating roots as well as for developing algebraic and elementary transcendent functions into power series. Combinatorial analysis refers to a special German development into which we do not enter.

The survey shows that the determination of tangents and extrema as well as Taylor's theorem were situated at the borderline between *analysis of the finite* and the *differential calculus.* They were considered as topics in the former as well as in the latter.

The very word *algebraic analysis* as designation of a mathematical field is not used in the literature of the 18th and beginning 19th century, and, thus, it was, ironically, the landmark textbook

> Augustin Louis Cauchy (1821). Cour d'analyse de l'école royale polytechnique. Ire Partie. **Analyse algébrique**. Paris: Debure frères.

which made this term popular. As is well known this book opened the way to 19th century rigorous analysis and, because of its verdict against the use of divergent series, has contributed more than any other work to the final destruction of the Eulerian tradition.

What made this field so attractive to many mathematicians is clearly expressed in H. Burkhardt's *Algebraische Analysis* which appeared in 1903. In the preface we can read that at the University of Zürich courses on *algebraic analysis* were not only usual but even prescribed by law. Burkhardt then points to a French textbook by Jules Tannery (1886) saying that one should agree with the author's regret that the simple methods by which Euler had derived the elementary trancendent functions are completely vanished from university courses in France as well as in Germany.

> "This is caused by the fact that [EULER's] original methods are not rigorous whereas their remodelling by CAUCHY misses simplicity. But when one is not afraid to introduce already in the beginning certain concepts which belong to the modern development as especially that of uniform convergence it is possible as the example of TANNERY shows to arrive at an account which satisfies fair requirements in both regards." (Burkhardt 1903, v; translation by the author)

When in the beginning of the 19th century Prussia formalised and reformed his school system the original plans for the teaching of mathematics at gymnasium ('syllabus of Süvern' 1816) were based on *analysis of the finite*. As illustration we give the contents of a mathematically ambitious textbook (Tellkampf 1829): basic arithmetical operations, calculating with letters, number systems (especially base 10), negative numbers, polynomials, calculations with powers, 1st and 2nd degree equations, diophantine equations, continued fractions, irrational and imaginary numbers, cubic and higher equations, progressions and sequences, combinatorics, binomial and polynomial theorem, also for rational and negative exponents, infinite series and analytic operations with infinite series, among others exponential and logarithmic functions, Euclidean synthetic geometry, analytic geometry, especially conic sections.

The further development was complicated. In the 1830s at gymnasium the average number of weekly hours for mathematics was reduced from 5 to 4. Consequently, there had to be serious cuts. Analytic geometry was removed from the syllabus and stronger emphasis was laid on synthetic Euclidean geometry, the binomial theorem was limited to natural exponents and infinite series were removed. But when around 1860 the system of realistic schools which provided a full course of 9 years and led to the 'Abitur', like 'Oberrealschule' and 'Realgymnasium', was formalised, the number of weekly hours for mathematics was enhanced, analytic geometry was reintroduced and the binomial theorem extended to rational and negative exponents thus opening again the door to infinite series. In 1882, analytic geometry became obligatory also at the classic gymnasium.

In the second half of the 19th century it became usual to designate the whole field from the elementary arithmetic operations up to the binomial theorem and the development of the elementary transcendent functions in series simply as *arithmetic*, a use of language also Klein pointed at. In section 3 we shall understand the reasons of this use of language.

This entire field was considered a coherent unity. In 1827 the above mentioned teacher A. Tellkampf wrote somewhat emphatically that, as everybody knows, *analysis of the finite* with the binomial series and the expansions of the logarithm and the trigonometric functions are the fundament of the higher calculus and therefore provide the most worthy aim of preparatory teaching [at Gymnasium]. The binomial theorem "raises the student to a standpoint from which the particular topics of the whole field appear to him in their true coherence." (Tellkampf 1827, Sp. 708; translation by the author). At the end of the century teacher and teacher trainer Max Simon (see below) wrote in a similar vein: "Elementary arithmetic starting from simple counting up to the binomial theorem with arbitrary exponents is the only example of an in itself coherent science which is accessible to school teaching; only ignorance can withhold it from school." (Simon 1908, 82)

All in all, such was the situation Felix Klein was confronted with when in 1908 he published the first volume of his *Elementary Mathematics from an Advanced Standpoint* to which we now turn.

## 2. The Literature quoted in the Introduction

In the introduction Klein presented to his readers four works which in his view are of particular importance. The first one is a book by himself on the teaching of mathematics at secondary schools he had published the year before (Klein 1907).

The second one is the *Encyklopädie der Elementarmathematik* (=*encyclopedy of elementary mathematics)*, vol. 1, on *"elementary algebra and analysis"* by H. Weber (1903). Heinrich Weber (1842-1913) was a versatile mathematician whose main area was algebra, but who also made substantial contributions to analysis and mathematical physics. No wonder then, he became one of the editors of Riemann's works. From 1892 to 1895 he was a colleague of Klein's at the university of Göttingen and then moved to Strasbourg. Already in Göttingen he had given a course on *Encyclopedy of elementary mathematics* and, thus, had contributed to establishing there a tradition of courses especially designed for future teachers in which also Felix Klein's *Elementary Mathematics From an Advanced Standpoint* is to be

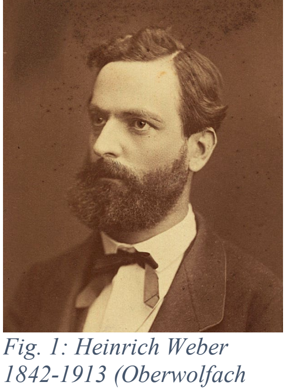

*Fig. 1: Heinrich Weber 1842-1913 (Oberwolfach Photo Collection)*

seen. The title *Elementary algebra and analysis* of Weber's book suggests clearly that he followed Euler's broader notion of analysis. Contrary to the expectations of a modern reader the part on analysis did not contain the differential and integral calculus, but confined itself to a general theory of infinite series including the binomial series and the expansions of the trigonometric and logarithmic functions. In the preface he said explicitly that one should not try to guide better students [at gymnasium] as far as possible into higher analysis. Such an attempt would hinder more than further future thorough studies in mathematics. More fruitful be the deepening of the elementary teaching within the "old confines". Referring to Klein's initiatives he added in the second edition of 1905 a short chapter on "functions, differentials and integrals" which comprised only 40 pages compared to 93 on the traditional subjects of algebraic analysis.

Klein commented on this book by saying on the one hand that it follows intentions similar to his own but sticks on the other hand to the old confines of school mathematics and concludes that Weber is "conservative" whereas he himself is "progressive".

The third book Klein presented to his readers was *Didaktik und Methodik des Rechnens und der Mathematik* (= didactics and method of elementary arithmetic and of mathematics) (Simon 1908). Max Simon (1844 – 1918) had awarded a PhD in mathematics with Weierstrass in 1867 and since 1872 worked as a teacher of mathematics at the Lycée in Strasbourg. In 1891 he became an honorary professor at the university there and since then delivered also courses on didactics of mathematics. In a certain way Simon was at the same time an ally and an opponent of Klein's. Both propagated a stronger awareness of the concept of function at school. Nevertheless, Simon's ideas were quite different from Klein's, since he thought in terms of Euler's *algebraic analysis*. To say the least, Klein was ambiguous in regard to Simon. He calls Simon's *Methodik* a "very stimulating book", but points to his "very subjective, temperamental personality" who "often clothes" his "contrasting views in vivid words." Klein's further remarks on Simon show that the word 'vivid' was a polite circumscription of the word 'polemic', and, in fact, Simon *was* a polemic writer. There is good reason for the assumption that Simon had in mind among others Klein when he used in the quotation above the word 'ignorance'. Without any further comment Klein mentioned also another small booklet by Simon, his *Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis* of 1906.

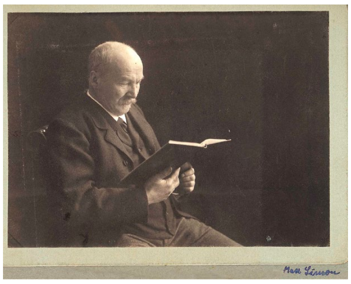

*Fig. 2: Max Simon 1844-1918 (Math. Ges. Hamburg)*

## 3. A fundamental point of view: Klein on the development and structure of mathematics

In an intermediate chapter between the parts on arithmetic and algebra Klein gives a general outlook on the development and structure of mathematics. It provides the reader with the perspective under which, in the following, he will treat algebra and analysis. I take these ten pages as a decisive key to the whole book and the only elaboration of what Klein had in mind when he spoke of an 'advanced' or 'higher' standpoint. In fact, he described not only his own perspective, but also a second one, namely that of contemporary school mathematics.

Klein distinguished between two different ‘directions of development’ (in German: ‘Entwicklungsreihen’, Klein (1832) translated this word by ‘plan’, I prefer ‘direction) which he, for lack of adequate concepts, called ‘direction A’ and ‘direction B’. Direction A is that of contemporary school mathematics, direction B his own alternative conception. Later on he mentioned also a third ‘algorithmic’ direction into which we do not enter. The elementary chapters of the system of analysis are treated under these two different perspectives as follows (we give a shortened version):

*Direction A:*
1. At the head stands the *formal theory of equations.*
2. The systematic pursuit of the *concept of power* and its inverses yields *logarithms*.
3. Whereas (up to this point) the analytic development is kept quite *separate* from geometry, one now borrows from this field, which yields the *definitions of the trigonometric functions.*
4. Then follows the so called ‘*algebraic analysis’*, which teaches the development of the simplest (algebraic and transcendent) functions into infinite series.
5. The consistent continuation of this structure, beyond school, is the *Weierstrass’ theory of functions of a complex variable*, which begins with the properties of power series.

*Direction B:*
1. The idea of *analytic geometry* which aims at a *fusion of the intuitions of number and space* leads to the *graphical representations of the simplest functions*, the zeros of the polynomials and the approximate numerical solution of equations.
2. The *geometric picture of the curve* supplies naturally the intuitive *source both for the idea of the differential quotient and that of the integral*.
3. In all those cases in which the *integration process* (or the process of quadrature, in the proper sense of that word) cannot be carried out explicitly with rational and algebraic functions, the process itself gives *rise to new functions*, namely the *logarithm* (quadrature of the hyperbola) and its inverse, the exponential function, as well as the inverses of the *trigonometric functions* (quadrature of the circle).
4. The development into infinite power series of the functions thus introduced is obtained by means of a uniform principle, namely Taylor's theorem.
5. This method carried higher, yields the *Cauchy-Riemann theory of analytic functions of a complex variable*, which is built upon the Cauchy-Riemann differential equations and the Cauchy integral theorem.

Direction A gives a fair sketch of school mathematics for the upper grades of German secondary schools at Klein’s times. Additional subjects were synthetic plane and solid geometry, spherical trigonometry with applications to positional astronomy.

Klein then continues with a very general remark on the different philosophies behind these two opposite directions. “If we try to put the *result of this survey* into definite words, we might say that *Plan A is based upon a more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part for itself, with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields.* Its ideal is to *crystallize out each of the partial fields into a logically closed system.* On the contrary*, the supporter of Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon* *another. He prefers, therefore, the methods which open for him an understanding of several fields under a uniform point of view.* His ideal is *the comprehension of the sum total of mathematical science as a great connected whole*."

The terms 'particularistic conception of science' vs. 'organic combination' and 'stimulation' show clearly that his sympathies lay in direction B. According to Klein the historical development of mathematics consisted in an interplay between directions A and B, and mathematics can only progress when this interplay works. Because, however, school mathematics suffers since a long time from a one-sided dominance of direction A Klein argued that a reform of the teaching of mathematics has to press for more emphasis on direction B (Klein (1832), 85).

According to Klein the 'advanced standpoint' for direction B was the Cauchy-Riemann theory of analytic functions of a complex variable whereas Weierstrass' approach based on power series was the scientific background theory and aim of contemporary algebraic analysis. Thus, Klein related the contrast between contemporary school mathematics and his own opposing views to the two different schools on the theory of complex functions which had emerged during the 19th century. Indeed, Klein as well as Weierstrass saw the difference of these two approaches not as a matter of pragmatic evaluation, but as rivals living from competing visions on the philosophy of mathematics.

In the introduction to his classic *Die Idee der Riemannschen Fläche* (1913) H. Weyl pointed out that it was Felix Klein who had decisively developed Riemann's ideas to transparent clarity and especially shown that Riemann surfaces are not a mere means for visualizing multi-valued functions but the 'fundament' and the 'topsoil' of the whole theory (l.c.). Nevertheless, in contrast to Klein, Weyl insisted on the opinion that only both approaches in their interplay provide an adequate notion of complex function theory.

On the other hand, also Weierstrass defended his own position as the only valid approach. In 1884 he delivered a talk at the Mathematical Seminar of the University of Berlin which was published only in 1924. Here he explained the principles of his view of complex function theory. The fundament was elementary arithmetic, and after generalizing to negative, rational, reel and complex numbers concrete functions defined by the four basic arithmetical operations can be considered. Only then sums, products etc. of infinitely many numbers are introduced which leads to the concept of power series and finally to that of an analytic function. Thus, for Weierstrass the theory of analytic functions was a natural continuation of elementary arithmetic. He intended building up complex function theory as a rigorous theory exclusively based on the notion of natural number (Richenhagen 1985, 9-43).

A rigorous construction of such a theoretical edifice required in his eyes the application of two maxims, namely (1) to start with the simplest elements (the natural numbers) and (2) not to prove elementary relations by 'higher means' but exclusively by elementary methods. Therefore, it should not be allowed to prove fundamental algebraic theorems using 'transcendental' tools. Consequently, Weierstrass vigorously rejected Cauchy's procedure to base the development of analytic functions into power series by using complex integration ('Cauchy integral'). As we have seen above (Direction B, item 5.) Klein took exactly the opposite position. It is also very telling that Klein in his characterisation of the 'particularistic conception of science' used nearly the same words as Weierstrass in his lecture of 1884.

To sum up, Weierstrass' power series approach to complex functions was a continuation of Euler's *Introductio* with other means. These other means were essentially the concepts of *uniform convergence* and, for the construction of multi-valued functions, of *analytic continuation* (see the above quotation from H. Burckhardt). In its elementary parts from arithmetic to the simple transcendent functions Weierstass' approach parallels the compositum of arithmetic, algebra and algebraic analysis at secondary schools as it had evolved in Germany during the 19th century. Between 1860 and 1890 Weierstrass' views dominated mathematics teaching at German universities. Consequently, teachers of mathematics who had studied at these times were influenced by these views. To them it must have been completely plausible to consider algebraic analysis as a legitimate didactical simplification.

In the first section we have hinted at the use of language at German secondary schools to call the whole domain of arithmetic, algebra and algebraic analysis as 'arithmetic'. We can now understand why this was the case. It was an outcome of Weierstrass' views.

In 1884, the year of Weierstrass' Berlin lecture, Max Simon published a book entitled *Die Elemente der Arithmetik als Vorbereitung auf die Funktionentheorie* (‚The Elements of Arithmetic as Preparation to the Theory of [complex] Functions'). Klein, like many others, misunderstood this title as a demand to teach the theory of complex functions at the upper grades of secondary schools (Klein 1932, 162). However, Simon had something very different in mind. He argued that the teaching of mathematics suffers from the fact that teachers frequently do not know the perspective under which arithmetic is seen in contemporary mathematics. Therefore, he found it necessary that the theory of complex functions (à la Weierstrass) becomes an obligatory component of the professional knowledge of teachers of mathematics at the upper grades. This was quite analogous to Klein who of course did not have the idea to introduce complex function theory (à la Cauchy-Riemann) into school teaching but, nevertheless, found it important that teachers of mathematics do know the 'higher' university perspective on the subjects they are teaching. Twenty years before Klein Simon had written a book on 'elementary mathematics from an advanced standpoint'.

## 4. The elementary transcendent functions: logarithm/exponential and trigonometric functions

Klein dedicated 63 pages of the part on analysis to a detailed discussion of the logarithmic/exponential and trigonometric functions "since they play an important part in school instruction" (Klein 1932, 144) whereas the section on 'infinitesimal analysis proper' comprised only 30 pages. It was in the discussion on the elementary transcendent functions where Klein mathematically and didactically elaborated his arguments against algebraic analysis in school teaching. His approach to these functions was not confined to a narrow discussion of the pros and cons of this or that way of introducing and handling them. For example, in the section on the trigonometric functions he provided, beyond these questions, a broad discussion of spherical trigonometry (by the way, a topic of school mathematics at that time), of small oscillations, especially the pendulum, and of Fourier series. In the following we confine ourselves to a sketch of Klein's general line of argument against algebraic analysis and to some special observations.

As an indispensable tool for numerical calculations students at gymnasium and realistic schools came across logarithms already at the middle grades, they were trained in the use of tables and formed an idea of what logarithms are. Later on, logarithms were formally introduced. Klein discussed the usual treatment of logarithms and his criticism under the heading 'Systematic Account of Algebraic Analysis' (Klein 1932, 144 ff). The stepwise extension of the meaning of powers

$$x = b^y$$

from natural to negative, fractional and irrational numbers led to the power function, and its inverse gave the logarithm

$$y = log_{(b)}x$$

Such a stepwise extension of the meaning of a function from natural to real or even complex numbers was a standard procedure of 18th century analysis.

According to Klein's criticism, this procedure requires a number of definitions and restrictions which cannot be explained to the learner and, therefore, must appear as arbitrary "authoritative conventions" (Klein 1932, 145). He listed five items, namely (1) only positive numbers *b* are admitted, (2) the logarithm is only defined for positive *x*, (3) for rational $y = \frac{m}{n}$ and *n* even the positive value of *x* (the 'principal value') has to be taken, (4) the definition of the Eulerian number *e* by the usual limit is simply prescribed and (5) the power series expansion of the logarithm is derived by formal calculations without questioning its existence. Especially, the restrictions (1) to (3) are by no means self-evident, and, as he later stated, in the last regard only understandable from the viewpoint of complex function theory.

To motivate his own proposal Klein entered into a long historical digression (l.c., 146-154) intending to show that the concrete calculations of logarithms by Bürgi and Napier (at around 1600) led in a natural way to the discovery that the quadrature of the hyperbola $x \cdot y = 1$ has the same additive property as the earlier logarithms (1647). From this followed for Klein that the "simple and natural way" for introducing logarithms at school is its definition as the integral of the hyperbola $y = 1 / \xi$ between the ordinates $\xi = 1$ und $\xi = x$. He then repeated the principle he had already mentioned in the "intermediary chapter" (see section 3 above).

> *"The first principle is that the proper source from which to bring in new functions is the quadrature of known curves. This corresponds, as I have shown, not only to the historical situation but also to the procedure in the higher fields of mathematics, e. g., in elliptic functions." (l.c., 156)*

Considering Klein's proposal in the light of his five critical items one must say that (1) to (3) are solved by "circumvention". The questions simply don't arise. Strangely, Klein's motivation for the definition of Euler's number *e* is not related to the quadrature of the hyperbola though, of course, this can easily be done (4). The calculation of the power series of the natural logarithm (5) is postponed to the time when Taylor's series is at the disposal of the students, for the moment the question is again circumvented.

Beyond that, adherents of the traditional approach of algebraic analysis could argue with good reasons that the stepwise extension of the power function to reel arguments contains a considerable didactical potential even if it is laborious and cannot be done completely rigorously at school. Additionally, to define objects in a safe domain without being able to answer all questions on its confines must not necessarily be didactically disadvantageous. Klein's 'general principle' operates top-down and tends to level off mathematical distinctions and details which could provide important opportunities of learning.

Klein applied his 'general principle' also to defining the trigonometric functions sin and cos. In a unit circle belongs to an angle $\varphi$ measured by its arc length a sector of magnitude $\varphi/2$. The functions sin and cos are then defined as the coordinates of the point *P* marking off the area of a sector of magnitude $\varphi/2$ on the circle (l.c., 163) (Fig. 3). To stay consistent with the usual conventions Klein wrote nevertheless $x = cos\varphi$ and $y = sin\varphi$. Thus, sin and cos are defined as functions which assign numbers to magnitudes of areas.

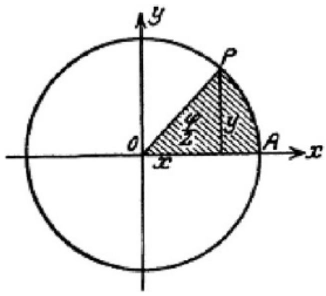


*Fig. 3: Sector of a unit circle (Klein 1932, 163)*

Klein was very explicit in that he was speaking about how to introduce these functions at school. Therefore, a remark is in order. Basically, his definitions were and still are usual at school when we confine ourselves to angles (or arc lengths) as domain of definition. Klein, however, chose the areas of circle sectors instead of them using the proportionality of angles and areas of sectors. This seems rather artificial. When a student wants to form an idea about how to determine a concrete value of, say, cos (1,2) the question would arise how one can find the point *P* marking off an area of magnitude 2,4. It is difficult to imagine how to do this without considering angles. Another question would concern Klein's 'general principle'. A beginning student will have formed a concrete idea of it by means of the example of the logarithm. In that case a measure of an area was assigned to an abscissa. The present case, however, is exactly the otheer way round since here areas are the domain of the independent variable. That means, when we follow the example of the logarithm and assign an area to an abscissa we define the inverse functions arccos and arcsin. In fact, in the intermediary chapter, he had noted the integral $\int_0^x \frac{dx}{\sqrt{1-x^2}} = \arcsin x$ for introducing the trigonometric functions.

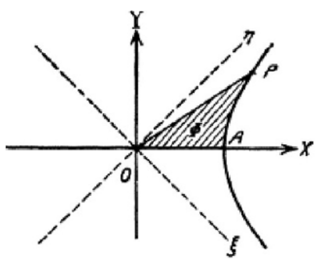


*Fig. 4: Sector of an equilateral hyperbola with half-axis 1 (Klein 1932, 165)*

Of course, Klein must have been conscious of these problems (l.c.), though he did not seriously discuss them. He was more interested in showing the reader that considering the areas of sectors leads to a wonderful analogy between cos/sin and their hyperbolic counterparts cosh and sinh. In fact, the latter can be defined as functions of the area *Φ* of a sector of an equilateral hyperbola with half axis 1 (Fig. 4). This leads to

$$x = \cosh \Phi = \frac{e^{\Phi}+e^{-\Phi}}{2} \text{ and } y = \sinh \phi = \frac{e^{\Phi}-e^{-\Phi}}{2}.$$

Using complex numbers one can derive on the other hand

$$x = \cos \varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2} \text{ and } y = \sin \varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i},$$

making perfect the analogy between the circle functions cos and sin and the hyperbolic functions cosh and sinh.

> *"If prominence is thus given, from the start, to the analogy between the circular and the hyperbolic functions, the great discovery of Euler that [that $e^{i\varphi} = \cos\varphi + i\sin\varphi$] is divested of the mystery that usually attaches to it." (l.c., 166)*

All in all, the main message of sections I and II of the part on analysis was Klein's conviction that the elementary transcendent functions should be introduced by way of the differential and integral calculus whereas, following the model of Euler's *Introductio,* 19th century algebraic analysis used more elementary methods. This was also Weierstrass' procedure in his famous lectures on complex function theory. Klein rightly pointed at mathematical and didactical problems of algebraic analysis, but also his own proposals do not seem to be fully developed.

## 5. The double discontinuity

In a didactical résumé to the part on the logarithm Klein stated:

> "It is remarkable that this modern development [infinitesimal analysis] has passed over the schools without having, for the most part, the slightest effect on the instruction, an evil to which I have often alluded. The teacher manages to get along still with the cumbersome algebraic analysis, in spite of its difficulties and imperfections, and avoids the smooth infinitesimal calculus, … The reason for this probably lies in the fact that mathematical instruction in the schools and the onward march of investigation lost all touch with each other after the beginning of the nineteenth century. … I called attention in the preface to this discontinuity, which was of long standing, and which resisted every reform of the school tradition … In a word, Euler remained the standard for the schools." (Klein 1932, 155)

These sentences clarify that Klein's phrase of a 'double discontinuity' did not only hint at missing links between school and university mathematics caused by the cognitive distance between research and elementary mathematics or by institutional boundaries. Beyond that the term designated in Klein's and his contemporaries' view a difference of mathematical conception. As Klein said, the traditional school mathematics of his time was determined by Euler's views whereas he intended to introduce into school the ideas of contemporary modern mathematics. To use Thomas Kuhn's term there was a *difference of paradigms* between Klein's conception and that of the school mathematics of his time. (Biermann & Jahnke 2013, 5/6)

## 6. Résumé

The above analysis has shown that in a large amount Klein's *Elementary Mathematics* is determined by a critical discussion of contemporary school mathematics. Klein did not leave any doubt on this and readers can scarcely ignore it. What may be a new insight is the fact that pre-Kleinian school mathematics was not a mere conglomerate of mathematical topics made up by eclectic considerations of tradition and utility, but represented in the eyes of many teachers and mathematicians a coherent whole, that means an in itself consistent paradigm. Thus, it was not by chance that in the introduction to his book Klein presented to his readers three works of this older paradigm.

All the way, Klein was debating about basic principles as he made clear in the 'intermediary chapter'. There he connected the alternative between 'conservatives' and 'progressives' with the two opposing directions in the theory of complex functions which had emerged in the course of the 19th century. This adds an interesting detail to historiography of mathematics. Much has been written about the 'Berlin-Göttingen-rivalry' (cf. Rowe (1989) and his recent (2018) as well as the recent Klein biography Tobies (2019)). At the Reichsschulkonferenz ('conference on schools of the (German) Reich') of 1890 only representatives of the 'old' paradigm spoke for mathematics. Afterwards in the 1890s the elder generation of Berlin mathematics passed away, Kronecker in 1891, Kummer in 1893, Weierstrass in 1897, and the grand old man of algebraic analysis and famous teacher trainer Karl Heinrich Schellbach in 1892. Thus, the way was open for Klein to be invited to the Reichsschulkonferenz of 1900 (very late, indeed; see Schubring 2000). Only after 1902 Klein started publicly pleading for the introduction of infinitesimal calculus at schools. It is scarcely imaginable that Klein would have been able to compose all these commissions for reforming the teaching of mathematics in the way he did when this elder generation of Berlin mathematicians still would have been alive.

In the introduction to the *Elementary Mathematics* Klein stressed that he will not address himself to beginners, but that he is going to deliver a "*comprehensive lecture… My task will always be to show you the mutual connection between problems in the various fields, a thing which is not brought out sufficiently in the usual lecture courses, and more, especially, to emphasize the relation of these problems to those of school mathematics*." (Klein 1932, 1/2) Thus, his basic intention was to provide an overview and show connections between different fields and problems of mathematics. As he explained in the intermediary chapter it was his main critic of the Weierstrassian approach that it will lead to a 'particularistic isolation' of the different fields. He was surely right with this criticism, but, as we have seen, he sometimes made recourse to 'general principles' whose application in a top-down manner led to inconsistencies in details. Klein himself knew very well that only the interplay of direction A and direction B will be fruitful, in research as well as in teaching. The necessary compromise has ever anew to be found.

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[1] In regard to the title we follow the translation in Klein (2016). However, both translations Klein (1932) and (2016) agree word-for-word in all the other quotations we consider in the following. Therefore, it seems, intellectual honesty requires to refer to the earlier Klein (1932).

[2] The present paper is an abbreviated version of Jahnke (2018). Time and again I have written on the nearly forgotten tradition of algebraic analysis as it emerged in German school and university mathematics during the 19th century. For the use of this term in modern historiography of mathematics see f. e. Fraser (1989). In part C of my book *Mathematik und Bildung in der Humboldtschen Reform* (*Mathematics and Education in the Humboldtian Reforms*) I studied school mathematics in Germany

in the times *before* Klein and found that algebraic analysis as the term was used by Klein was most characteristic for the whole domain of arithmetic-algebra-analysis. Jahnke (1993) contains a general outlook on the post-Eulerian analytic traditions, especially the so-called Combinatorial School, including their influence within philosophy, pedagogy and culture of mathematics. Jahnke (1996) shows that, in a sense, algebraic analysis was a 'complete paradigm' of school mathematics insofar as problems could be treated by its means which later on became core topics of infinitesimal analysis at school. Biermann & Jahnke (2013) adds to this concrete information about the teaching of mathematics at a specific gymnasium, the *Ratsgymnasium* in Bielefeld.