GERMAN-SPEAKING TRADITIONS IN MATHEMATICS EDUCATION RESEARCH

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In 1976 ICME 3 has taken place in Germany at the Technical University of Karlsruhe. Now, after forty years, ICME is coming back to Germany. In the course of the two-hours event of the afternoon a team of fifteen German mathematics educators supplemented by a number of scientists from abroad will undertake to answer the questions of which developments have taken place in German mathematics education research during these forty years, which developments and ideas were characteristic for the discussion, and in which regards Germany was influenced by and interacted with the international community. The presentations will be historically informed and, on some characteristic points take a look at a more remote past.

Theme **"Subject-matter didactics (German: 'Stoffdidaktik')** (Lisa Hefendehl-Hebeker, Rudolf vom Hofe)

In the development of didactics of mathematics as a professional field in Germany subject-related approaches play an important role. Felix Klein created a model being referred to for a long time. Its goal was to develop approaches for representing mathematical concepts and knowledge in a way that corresponded to the cognitive abilities of the students without disturbing the mathematical substance. A fundamental claim was that simplifications of mathematical material should be "intellectually honest" and "upwardly compatible" (A. Kirsch). In the 1980s views upon the nature of learning as well as objects and methods of research in mathematics education changed and the perspective was widened and opened towards new directions. This shift of view issued new challenges to subject-related considerations that are enhanced by the recent discussion about professional mathematical knowledge for teaching.

Theme **"Design science"** (Marcus Nührenbörger, Bettina Rösken-Winter)

Within the German speaking tradition, considering 'mathematics education as a design science' primarily draws on the work by E. C. Wittmann. Core of this approach is the development of substantial learning environments (SLE) which are seen under a twofold perspective. First, designing SLEs should be based on substantial mathematics, meaning that students can be immersed in mathematical processes such as mathematizing, exploring, reasoning and communicating. Second, investigating SLEs should be the essential starting point of mathematics education research. Bringing these two intentions together allows for essentially bridging theory and practice.

Theme **"Modelling"** (Gilbert Greefrath, Kathrin Vorhölter)

German work on modelling in mathematical education started in the 1980ies. The presentations give an overview of present definitions, pedagogical aims, typical modelling cycles and key examples of the German debate and report about pragmatic and specific approaches. In addition, current developments concerning educational standards, promoting of modelling competencies,

comparative studies and final exams and the role of technology in mathematical modelling are addressed.

Theme **"Allgemeinbildung and Mathematical Literacy"** (Rolf Biehler, Hans Niels Jahnke)

In Germany, the idea that mathematics has to be a constitutive component for the cultivation of human beings and, thus, an indispensable part of 'Allgemeinbildung' dates back to Wilhelm von Humboldt (1810). This constituted a tradition of pedagogical thinking which is still influential in modern conceptions about the contribution of mathematics to general education (H. Winter 1995). In the 1990ies the literacy concept of PISA got a strong influence in Germany and caused an orientation towards 'competencies' which today is a central feature of curriculum reforms in Germany. Whether this orientation towards competencies is helpful or counter-productive is still an open problem and an important part of the current debate.

Theme **"Theory traditions in German speaking countries"** (Angelika Bikner-Ahsbahs, Andreas Vohn)

How far has didactics of mathematics been developed as a scientific discipline? This question was intensely discussed in the 1980s with both affirmative and critical reference to Kuhn and Masterman. In 1984, Hans-Georg Steiner inaugurated a series of international conferences on "Theories of Mathematics Education" (TME), pursuing a scientific program that aimed at founding and developing didactics of mathematics as a scientific discipline. Today, a more bottom-up meta-theoretical approach is investigated in the networking of theories. The audience is invited to take up this line of thought. For that purpose, two theory strands from German speaking countries are presented: the semiotic approaches of Willibald Dörfler and Michael Otte, and a contribution to activity-theory in the work of Joachim Lompscher.

Theme **"Classroom Studies"** (Uwe Gellert, Götz Krummheuer)

The specific aspects of Classroom Studies, as a focus within the German Speaking Traditions in Mathematics Education Research, rest on the fundamental sociological orientation on mathematics lessons. Initiated by the works of Heinrich Bauersfeld, the first sociological perspective unfolds its power of description by reconstructing social processes regarding the negotiation of meaning and the social constitution of shared knowledge through collective argumentation in the daily practice of mathematic lessons. A second sociological perspective aims at the reconstruction of the conditions and the structure surrounding the construction of performance and success in mathematics lessons.

Theme **"Educational Research on Learning and Teaching of Mathematics"** (Timo Leuders, Andreas Schulz)

Educational Research on Learning and Teaching of Mathematics aims at generating knowledge on processes of teaching and learning. To achieve this goal the approaches of many projects which fall into this category can be characterized by (some of the) following three aspects: (1) a narrow focus on distinct phenomena within learning and teaching mathematics, (2) a multi-step approach that develops theory in a series of consecutive studies (often an area of interest is pursued over many years) and (3) a mixed-method approach that integrates different methodological practices.

Several research projects from the last two decades that represent the mentioned characteristics will be described.

Theme **"Large-Scale Studies"** (Andreas Obersteiner, Kristina Reiss)

In this strand the focus is on large-scale assessments, that is, studies that aim at assessing mathematical competence in large samples. They often compare mathematical competence between groups of individuals within or between countries. The development of sophisticated statistical methods in recent years has encouraged collaborations between researchers from mathematics education on the one hand and from statistics or psychology on the other. This development has also allowed verifying empirically theoretical models of mathematical competence and competence development.

The Development of Didactics of Mathematics in Germany – An Introduction[1]

Uwe Gellert, Lisa Hefendehl-Hebeker, Hans Niels Jahnke, Timo Leuders

In Germany, didactics of mathematics as a university discipline is a comparably new achievement. Its establishment started about 50 years ago, mainly by creating professorships and opportunities of graduation at universities (Burscheid 2003). However, already in the 19th century, beginning with the educational reforms of W. v. Humboldt, a broad body of literature on problems of the teaching of mathematics emerged. School programs, textbooks and journals for the educated public provided opportunities for presenting new didactical ideas and reflections. In the second half of the 19th century specialized journals on the teaching of mathematics and science were founded, and books on didactics of mathematics began to appear. At the turn of the 20th century F. Klein launched his seminal project of reforming school mathematics, and other university professors of mathematics offered specialized university courses for future teachers at gymnasia (German type of secondary school) as well. However, with a few exceptions didactics of mathematics remained a sub-academic field of inquiry and knowledge well until the 1960s when, in the wake of the Sputnik shock and New Math, didactics of mathematics step by step became a university discipline.

1. Early Developments

1.1. The Century of Humboldt

In 1810, in the course of the Humboldtian educational reforms mathematics became a full-fledged university discipline and at the same time a school subject. The teaching of mathematics at schools was no longer to be confined to imparting practically useful arithmetic skills as had been the case in the 18th century. Rather, Humboldt and other reformers of that period considered mathematics as a constitutive part of 'Allgemeinbildung', which is the formation of a person's personality. The intellectual activity of an educated human being was not supposed to be determined by prescribed rules. Rather, an autonomous person was expected to be able to guide him- or herself by autonomous ideas. Accordingly, the reformers were strongly convinced that pupils should learn to understand things by their own means and from within themselves ("organic thinking"). In this regard, mathematics was considered an especially important subject.

What resulted from this general attitude was a pronounced *anti-utilitarianism*. This becomes clear at some places in Humboldt's educational writings where he clearly expressed an attitude against everyday applications and a preference for pure mathematics. According to him, education was to be developed to ensure "that understanding, knowledge and intellectual creativity become fascinating not by external circumstances, but rather by its internal precision, harmony and beauty. It is primarily mathematics that must be used for this purpose, starting with the very first exercises of the faculty of thinking." ["...dass das Verstehen, Wissen und geistige Schaffen nicht durch äussere Umstände, sondern durch seine innere Präcision, Harmonie und Schönheit Reiz gewinnt. Dazu und zur Vorübung des Kopfes zur reinen Wissenschaft muss vorzüglich die Mathematik und zwar von den ersten Uebungen des Denkvermögens an gebraucht werden."] (v. Humboldt 1810, p. 261). At another place, he expressed himself against the tendency "... of digressing from the possibility of future scientific activity and considering only everyday life. ... Why, for example, should mathematics be taught according to Wirth, and not according to Euclides, Lorenz or another rigorous mathematician? Any suitable mind, and most are suitable, is able to exercise mathematical rigour, even without extensive education; and if, because of a lack of specialized schools, it is considered necessary to integrate more applications into general education, this can be done particularly toward the end of schooling. However, the pure should be left pure. Even in the field of numbers, I do not favour too many applications to Carolins, Ducats, and the like." ["...sich selbst von der Möglichkeit künftiger Wissenschaft zu entfernen, und aufs naheliegende Leben zu denken. Warum soll z. B. Mathematik nach Wirth und nicht nach Euclides, Lorenz oder einem andern strengen Mathematiker gelehrt werden? Mathematischer

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Strenge ist jeder an sich dazu geeignete Kopf, und die meisten sind es, auch ohne vielseitige Bildung fähig, und will man in Ermangelung von Specialschulen aus Noth mehr Anwendungen in den allgemeinen Unterricht mischen, so kann man es gegen das Ende besonders tun. Nur das Reine lasse man rein. Selbst bei den Zahlverhältnissen liebe ich nicht zu häufige Anwendungen auf Carolinen, Ducaten und so fort."] (v. Humboldt, 1809/1964, p. 194, our translation).

In the beginning of the 19th century Prussia established a school system consisting of elementary schools („Volksschulen") for all children, and of secondary schools ("Gymnasium") in order to prepare students for university studies, civil service careers, outstanding positions in commerce, or industry. During the entire 19th century, the (male) students attending the higher schools comprised only 7% of the school age cohort (Jahnke 1990, 7). Teachers at these higher schools were trained at universities and had to engage in purely scientific studies, without any didactical elements. Teachers at the elementary schools were trained at newly founded seminars of teacher training ("Lehrerbildungsseminare"). In the 1860s a separate system of schools with a stronger emphasis on science and mathematics ("Realschulen") was established in addition to the gymnasium.

Following the Swiss pedagogue J. H. Pestalozzi (1746-1827), outstanding educators as E. Tillich (1780 - 1807), F. A. W. Diesterweg (1790 - 1866) and W. Harnisch (1787 - 1864) advocated didactical ideas according to which pupils at elementary schools should reach a profound and reflected understanding of arithmetics and geometry, and not only be trained by drill and practice. Harnisch coined the concise phrase that pupils should "calculate by thinking and think by calculating" ["denkend rechnen und rechnend denken"] (Simon 1908, 22).

At the secondary schools (gymnasia and Realschulen), on the other hand, in accordance with Humboldt's ideas, the teaching of mathematics had a stronger scientific character; and teachers considered themselves as scientists. Within the framework of the general regulations by the state, teachers of mathematics developed multifarious didactical ideas. These were published and discussed in textbooks, school programs, journals, and at meetings. Thus, a rich culture of reflection about the teaching of mathematics evolved.

As an important guideline for structuring the mathematical curriculum, the principle of the stepwise extension of number domains from the natural numbers through negative and rational to the real numbers emerged. The underlying principle of adjoining the "new" numbers to the respective "old" domain while maintaining the laws of operation was called "principle of permanence" and received its final formulation in 1867 by mathematician H. Hankel.

In the middle of the 19th century, in the wake of the revolutionary events of 1848, the Prussian government became convinced that too much education of the lower classes was politically dangerous and, thus, officially cut down the teaching of arithmetic and geometry at the elementary schools to a simple training of elementary skills (Stiehl's regulatives 1854). On the other hand, the teaching of mathematics at the gymnasia experienced a cultural and mathematical loss of meaning which called for new ideas. Also, the emerging system of "Realschulen" changed the role of mathematics education. Thus, after 1860 a climate in favour of reforming education developed. Specialized journals on the teaching of mathematics and science were founded, and books on didactics of mathematics began to appear. This development finally paved the way for Felix Klein's initiatives in regard to the teaching of mathematics.

1.2 „Reformpädagogik" and Its Influence on the Teaching of Mathematics

„Reformpädagogik" was an educational theory favouring the promotion of the child's creativity. It originated at the beginning of the 20th century and resulted in a new attitude towards the child as well as a new understanding of "Allgemeinbildung", which in part continued Humboldtian ideas. From its very beginnings, this reform movement was international. The Swedish pedagogue E. Key proclaimed the *century of the child*, the Italian pedagogue M. Montessori considered the child as a *constructor of its self*. Teaching was to be based and focused on the child and promoted its productive powers, thus letting the child evolve its individual personality.

The new attitude towards the child was furthered by the emergence of cognitive psychology. Gestalt psychology (M. Wertheimer) and later, from the 1920s onward, the research of J. Piaget and L. Wygotski considered the thinking of children a quality of its own.

Also, the reform movement influenced the teaching of arithmetics at elementary schools. Mathematics educators like J. Kühnel (1869–1928) and J. Wittmann (1885-1960) created concepts that replaced the predominant style based on passive reception and drill by teaching environments in which students could independently work with adequate material and discover number relations by themselves.

1.3 New ideas on the teaching of mathematics at the gymnasia

By the end of the 19th and the beginning of the 20th century, some professors of mathematics began to offer special courses on school mathematics for future teachers at gymnasia. Among them were R. Baltzer (1818-1887), H. Weber (1842-1913) and F. Meyer (1856-1925). Above all, the activities of Felix Klein (1849-1925)

had a strong influence, which persists until today: He proposed the concept of function as a new guiding principle of school mathematics, and engaged himself in the introduction of infinitesimal analysis into school mathematics. In 1911, Klein made possible the habilitation of the mathematics teacher R. Schimmack which was the first habilitation in the field "Didactics of mathematics" in Germany. This was due mainly to Klein's influence. Nevertheless habilitations in didactics of mathematics at a university remained controversial up to the 1980s. Beyond this, Klein was also influential in developing international relations between people interested in the teaching of mathematics which, among others, resulted in the foundation of ICMI. These international initiatives, however, were severely set back by the two world wars and the political development in Germany during the fascist period.

In the 1920s and in the tradition of F. Klein, the mathematicians O. Toeplitz (1881-1940) and H. Behnke (1898-1970) engaged in improving the teaching of mathematics at gymnasia. Toeplitz worked on a concept of genetic teaching (see (Toeplitz 1927), translated into English in (Fried & Jahnke 2015)); Behnke founded a seminar for furthering the cooperation between universities and schools. In 1932, the two of them founded the journal "Mathematisch-physikalische Semesterberichte".

2. The Establishment of Didactics of Mathematics as an Academic Discipline

With the expansion of the university system about fifty years ago didactics of mathematics gradually became a university discipline. Different factors were involved in this development.

Particularly, the launching of the first artificial satellite caused a discussion in the Western countries about a supposed technological deficit in regard to the the USSR (the so-called Sputnik shock) and mobilized political efforts towards reforming the educational system. They were based on the assumption that a scientific modernization of the subject, especially on the secondary level, could contribute to enhance the competitive ability of a nation. This development encouraged activities in many states of the Federal Republic of Germany to offer university courses in mathematics education, also for teachers at gymnasia, and to establish corresponding professorships for didactics of mathematics at universities. Most of the institutions for the training of elementary teachers ("Pädagogische Hochschulen") were integrated into the universities. Those institutions which remained autonomous gradually developed into universities of education with corresponding research and graduation opportunities.

Thus, research in mathematics education included different types of schools and linked different institutions. Furthermore, international cooperation became possible. The works of J. Bruner, J. Piaget, and L. S. Wygotski initially defined a common frame of reference for the international discussion. In this context, the Institute for Didactics of Mathematics (IDM) in Bielefeld, which was founded in 1973, provided essential stimuli, some of which are described below.

The further development of didactics of mathematics in Germany was determined by an interaction between efforts of educational policies as well as internal impulses. Within the scientific community of didactics of mathematics research approaches from other academic disciplines such as psychology, sociology, history and philosophy of mathematics played an important part. They led to a progressive widening of perspective upon all determinants which influenced the process of teaching and learning, including the individual mental state of the actors.

3. The New Math Reform and its Consequences

In the middle of the 1960s, the New Math movement led to a comprehensive conceptual system of school mathematics with a high level of formalization.

The "modernization of mathematics teaching" was significantly influenced by Bourbaki, a French group of mathematicians who, as is well known, pursued to reorganize the entire field of mathematical knowledge on a structural-logical basis. In Bourbaki's approach the concepts of set, relation, and composition became fundamental. Some mathematics educators and influential mathematicians adopted this approach to school teaching and elements of logic and set theory were implemented in mathematics teaching from the primary level onwards. This process took place in many countries.

An important motive of the reform was the idea that mathematics education should be science-oriented from the very beginning. Thus, at the secondary schools the treatment of the number domains was consequently organized according to the principle of permanence, and enriched by structural aspects and concepts (for example the con-

cept of the "ordered field of rational numbers"). There was a clear succession which more or less persists until today: natural numbers in grade 5, fractions in grade 6, negative numbers in grade 7, irrational numbers in grade 9. Algebra and functions were integrated into this setting. The algebra of equations experienced a rigorous extension and reorganization by concepts of logic and set theory, variables were conceived as placeholders.

Basic elements of logic and set theory were also implemented in the primary schools, which entailed vigorous public discussions. Thus, in the beginning 1970s the guidelines for primary schools were revised by the Standing Conference of the Ministers of Education and Cultural Affairs (Kultusministerkonferenz - KMK). The focus of the teaching of mathematics at primary schools shifted. The considerable influence of H. Winter (1975) strengthened the awareness that mathematics teaching should meet different requirements: the science of mathematics, the demands of society, the dispositions of the learners, and their right of self-realization. With respect to content, weights were shifted: set theory was no longer considered as an appropriate foundation for the development of number concept and number operations at school, rather the number concept was considered and taught in its entire complexity (Müller & Wittmann 1984, S. 154). The predominance of arithmetic was restored, however, in another shape than in traditional teaching, and enriched by further subjects (combinatorial counting, algebraic and number theoretical considerations). Also geometry as an extended study of spatial phenomena, such as multiple applications (magnitudes, applied arithmetic, stochastic situations), played an important role.

The newly established concept of mathematics teaching at the secondary schools initially was mainly subject-oriented. "Didactically oriented content analysis" was developed as a tool for research in didactics of mathematics, resulting from the ambition for solid foundations and conducted with the aim to present the contents in a way that is compatible with the standards of the field and at the same time appropriate to the learners and the requirements of teaching (Griesel 1974).

In a first period the emphasis was on the lower secondary school level, especially in the domain of algebra and arithmetic, complemented by an analysis of the concept of function (Vollrath 1974). Thus, different concepts of fractions were discussed (Griesel 1978); A. Kirsch presented far-reaching analyses of the foundations of proportional reasoning as well as of linear and exponential growth (Kirsch 1969, 1976a). A special problem was the attempt to find exact foundations for the relations between numbers and quantities in primary and secondary teaching (Kirsch 1970, Griesel 1997). The focus in geometry was determined by the ambition to organize geometry by a unifying concept corresponding to the concept of function in arithmetic and algebra (Struve 2015). Thus, didactically oriented content analysis in geometry was mostly centred on transformation geometry (see for example Holland 1974/1977, Schupp 1968).

Later on, the didactically oriented content analyses were extended to domains of upper secondary school teaching. Here the contents already had a solid scientific foundation and the problems were mostly opposite to those with respect to the lower stages. The question was how mathematical theories and concepts could be simplified and made accessible without falsifying the central mathematical content. W. Blum and A. Kirsch suggested more intuitive approaches (at least for basic courses) with the original naïve ideas of function and limit and sequential steps of exactitude, which could be achieved according to the capacity of the learners (Blum & Kirsch 1979, Kirsch 1976b).

A general goal was to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, while simultaneously simplifying mathematical material without distorting it from its original form, with the aim of making it accessible for learners (Kirsch 1977). The simplifications introduced into mathematical subjects should be "intellectually honest" and "upwardly compatible" (Kirsch 1987). That is, concepts and explanations should be taught to students with sufficient mathematical rigour in a manner that connects with and expands their knowledge of the subject.

4. Widening the Perspective

The didactically oriented content analysis as a research method was strongly related to the teaching methods of the 1970s and 1980s, which were primarily based upon instruction and supported by the implicit belief that, in an appropriate ready-made setting, knowledge could be handed over or transmitted from the teacher to the learners – an attitude which was entitled by its critics as "broadcast metaphor" (Seeger & Steinbring 1992). Among others, this criticism was supported by carefully documented teaching experiences showing that even sophisticated concepts of teaching frequently led to unsatisfactory results on the side of the pupils (Hefendehl-Hebeker 1988 and 1991).

The analyses of interpretative research studies (Maier & Voigt 1991) sharpened the awareness that knowledge cannot be transferred in a simple manner, but is developed within the social interaction between a learner, the teacher and other learners of the group (Steinbring 2009). This awareness was accompanied by the conviction

that a good structure of an optimal representation of ready-made mathematics does not automatically provide a good structure for teaching, but that the learning process should be oriented towards natural conditions of knowledge acquirement.

H. Freudenthal (1973) had given an influential impetus through the idea that learning of mathematics should become an active process in the construction of knowledge and that substantial mathematics, geared to the pre-conditions of learners, should be made the foundation of teaching. Following this idea, the development and research project "mathe 2000" (Müller et al. 1997, Wittmann 2002) was founded in Dortmund, accompanied by a textbook that is based upon the concepts of "productive exercise" (Winter 1984) and "substantial learning environment" (Wittmann 2002). It promises potential also for advanced levels of teaching and learning. E. C. Wittmann characterized this approach to didactics of mathematics as "design science" with a focus on "mathematics education *emerging from the subject*" (Wittmann 2012). In this concept mathematics curricula are organized around "*substantial learning environments*", where children can gain mathematical experience, recognize patterns and solve problems. To construct such environments requires a "structure-genetic empirical analysis" (ibid), which comprises content-related analyses of the traditional type, but in addition to it the analysis of the cognitive preconditions of the learners, mathematical practices of exploring patterns and the objectives of teaching.

Parallel to this development, the didactics of mathematics in the Federal Republic of Germany experienced an increasing international orientation which was supported by the foundation of the Institute of Didactics of Mathematics (IDM) in Bielefeld (1973). An indication for this internationalization is the *Third International Congress on Mathematical Education* (ICME-3) in 1976, during which more than 2000 interested educators from all over the world gathered in Karlsruhe. Subsequently, in the 1980s and the early 1990s, several international research groups including German researchers emerged.

4.1 Theoretical Foundation of Didactics of Mathematics

As any other scientific discipline didactics of mathematics needs a reflection on its theoretical foundations as well as its paradigmatic problems and its basic methods. Such meta-theoretical considerations were partly developed, and at the same time gained recognition. Works stemming from the IDM had a high influence on this process, especially those of M. Otte and H.-G. Steiner. Otte (1993) worked on the philosophy of mathematics by focusing on the teaching of mathematics in regard to its historical, cultural, social, technical, and ecological status. Steiner's (1985) main interest lay in systematising fundamental theories and methodologies in mathematics education with respect to research, development, and educational practice. Furthermore, he aimed at identifying or drawing interdisciplinary connections to other disciplines as well as analysing its relationship of theory and practical doing. In addition to that, Steiner initiated the foundation of an international research group, *Theory of Mathematics Education* (TME), which got together five times in the years between 1984 and 1991. This early research on meta-theoretical questions about the didactics of mathematics still has an impact on today's research. Especially the broad variety of research areas of didactics of mathematics can be traced back to these early beginnings. The works and findings of the TME are regarded as basic foundation of working groups at meetings such as CERME and ICME.

4.2 Socio-political Perspectives on Mathematics and Mathematics Education

As one consequence of the internationalisation of didactics of mathematics, it became common practice to see it well distinguished from its former tight and traditional link to teaching and learning processes at schools. From this emancipated point of view, parameters such as the cultural relevance, political guidelines of governmental institutions, the socio-historical perspective on the education of mathematics, and the mathematically based technical progressing of society were put into the focus of attention. Attention to these parameters became manifest in 1988 at ICME 6 in the topic *Mathematics, Education, and Society*. The respective report was published by UNESCO and edited by C. Keitel and P. Damerow, together with A. Bishop and P. Gerdes (1989). Internationally, the focus on this research program was, among others, integrated into the conference set *Mathematics, Education, and Society* (MES), which was established in 1998; eight meetings have taken place so far.

4.3 Applications and Modelling

The basic idea to understand mathematical modelling not only as a specific scientific research area, but also as a part of the school curriculum, and thereby acknowledging its didactical value, originated from Great Britain. Since the beginning 1990s math educators from Germany also contributed substantially to the internationalisation, development and spreading of the idea to focus the teaching of mathematics on applications. Under the strong influence of W. Blum and G. Kaiser, a new curricular concept was developed. Summarised by the keyword 'modelling', it was defined as one of the basic mathematical competences and integrated into the educational standards ("Bildungsstandards") and curricula ("Rahmenlehrpläne"). Many of the so far 17 conferences of the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), an *ICMI-*

affiliated Study Group, which were organised from 1983 onwards, or the *ICMI Study 14: Applications and Modelling in Mathematical Education* were under the guidance of German colleagues (see for example the cinference proceedings Blum et al. 1989). Furthermore, in 1991, a study group (ISTRON) was founded. This study group entailed members from universities as well as from schools in order to connect theoretical research with practical observations and findings. Its main goal was to discuss and develop suggestions for integrating modelling as well as fitting references to reality into the teaching of mathematics at schools.

4.4 Construction of Meaning Within Classroom Interaction

Even though the cooperation of a study group in Bielefeld around H. Bauersfeld and colleagues with a study group around P. Cobb at the Purdue University in Indiana did not reach an institutionalised status comparable to one of the above mentioned *Study Groups*, their influence on the development of the research on the didactics of mathematics is still of considerable importance. Due to this cooperation, the practical ongoings of the teaching of mathematics were considered a non-negotiable part of the research on teaching-learning processes. Furthermore, it systematically interconnected socio-cultural as well as individual-based psychological perspectives of learning mathematics to their theoretical foundation. Cobb and colleagues started with describing the learning of mathematics as an individual process of constructing a mental concept. The group of researchers in Bielefeld, on the other hand, focused on the social and interactional processes of learning mathematics within the school environment (Krummheuer 1995, Voigt 1995). As a result of this cooperation, learning mathematics was from then on considered a socially situated process of emergence based on an individual's interpretation and construction (Cobb & Bauersfeld 1995).

5. Recent stimuli: international large scale assessments, education reform and interdisciplinary research initiatives

In the international comparisons delivered by large scale studies (particularly TIMSS 1997 and PISA 2000) German students unexpectedly showed only average achievements. This triggered repercussions in society and politics that can be compared to the Sputnik shock (see above). Several activities in education policy and education research ensued that were mutually interconnected. Education research experienced a boost in political awareness and a demand for "evidence based policy". This also led to increased funding in educational research and had an impact especially on those branches of research in mathematics education that used quantitative approaches.

Seen as one of the core subjects in school, roughly beginning in 2000, mathematics was subjected to a governance system which relied on ‚outcome standards' rather than ‚content standards' (Klieme et al. 2004). For this purpose a system of continuous, centrally administered assessment procedures and examinations were developed. These developments resulted also in newly accentuated goals in mathematics teaching (although this shift is only gradually recognisable in the classroom): Referring to the concept of „mathematical literacy" (Niss 2003, Neubrand 2003), the use of mathematics in everyday and professional life received more attention, strengthening competence areas like modelling or using data, while classical topics like plane geometry receded. A debate on the alleged overemphasis of aspects of usefulness of mathematics (incorrectly attributed to Heymann's analysis of the educational purposes of mathematics 1996/2003) quickly subsided and scholars compromised on H. Winters suggestion of balancing mathematics as useful tool in describing reality, as deductively structured domain and as opportunity for intellectual and heuristic activity (Winter 1996). Not until 2003 German education standards throughout the 16 states began to require competences related to mathematical processes (such as modelling, reasoning and problem solving) on the same level as competences for the dimensions of content.

The stimuli that influenced education policy also had an impact on the reasearch in mathematics education in several areas. It was characteristic for this research that it was conducted cooperatively in interdisciplinary groups connecting their respective research interests and domains of expertise. Some outstanding examples are the following:

- The video studies conducted within the TIMSS-assessment (Stigler et al. 1997) inspired researchers from Germany and Switzerland to investigate the quality of teaching and its impact on learning by means of video analysis of mathematics lessons as encountered in classrooms (Rakoczy et al. 2007). The relation between general and subject specific features of mathematics classroom are still object of interest and debate.

- A federal priority research initiative on ‚Educational quality in schools' (BiQUA) initiated many fruitful cooperations between educational psychology and mathematics educational research. Researchers connected their expertise to investigate questions of mathematics learning and teaching, such as how to learn proof by

heuristic worked examples (Reiss & Renkl 2002), or how to foster problem solving and self regulation (Bruder & Schmitz, cf. Perels et al. 2007).

- The interest of education policy in a continuous monitoring of the outcome led to a broad development of assessment measures (predominantly based on the measurement approach used by PISA), coordinated by a centrally founded „Institute for Educational Quality Improvement " (IQB). Researchers in mathematics education were regularly consulted as domain experts.

- Serveral research inititiaves were launched to develop innovative approaches to educational assessment, e.g. in the federally funded priority programme ‚Competence Models for Assessing Individual Learning Outcomes and Evaluating Educational Processes' (Koeppen et al. 2008) or the international study of the IEA on teacher competences (Blömeke et al. 2013). Many projects were based on the cooperations between researchers from educational psychology and mathematics education (e.g. Leuders et al. 2016)

A similar development can also be found in education policy and education research with respect to other domains (e.g. Fischer & Kauertz 2010) . The impact is especially strong in mathematics, in science education and in the domain of reading. Furthermore the mentioned impulses and cooperations are accompanied by a certain emphasis on quantitative research strategies used in educational psychology. Other research strategies, which are highly relevant in mathematics education, such as the development of local (topic specific) learning theories by means of qualitative analysis of case studies or the cyclical development of learning environments in a design research approach receive almost no funding by the mentioned programmes and initiatives. This may bear the risk of splitting up the research community in opposing factions.

6. Institutional structure of mathematic education in Germans

The development described above led to the situation of mathematics education in Germany that we encounter today (as of 2016). Pre-service mathematics teacher education in Germany for all school types is located at universities (and universities of education, „Pädagogische Hochschulen"), each endowed with one up to six full professorships and further research and teaching personnel. In most universities mathematics education is located within a faculty of mathematics, sometimes within a faculty of education. In Switzerland and Austria there are also teacher training institutions without university status (also „Pädagogische Hochschulen").

Newly appointed professors usually have accomplished a PhD-thesis in mathematics education, have done further research and have a certain amount of practical teaching experience in schools. During the last decade the group of young researchers has grown considerably, since more universities offer funded doctoral programs. Therefore nowadays less researchers have experience in mathematics research and more researchers have experience in educational research as in former times.

German mathematics educators meet on a regular basis on national conferences and special interest groups. A large part of their publications is in the German language, some of them can be found in two peer-reviewed journals (Journal für Mathematikdidaktik, mathematica didactica). The proportion of international publications is steadily increasing.

In conclusion it can be stated that mathematics education in Germany has grown into a research area that bears all features of a scientific discipline.

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