13th International Congress on Mathematical Education
Hamburg, 24-31 July 2016

ICME-13 Survey Team (ST):

Geometry (including technology)

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This survey on the theme of Geometry Education (including new technologies) focuses chiefly on the time span since 2008. Based on our review of the literature published during this time span (in refereed journal articles, conference proceedings and edited books), we have jointly identified **seven** major threads of contributions as these relate to the early years of learning (pre-school and primary school) through to post-compulsory education and to the issue of mathematics teacher education for geometry.

Developments in and trends in the use of theories

The development and refinement of theories of teaching and learning is one of the key aims of research in education. This focus on theory includes the developing and refining of theories that are specifically about the teaching and learning of geometry, as well as the application of more general theories to the specifics of geometry education.

Examples of theories specifically about the teaching and learning of geometry include the van Hiele model of geometrical thinking (van Hiele 1986), the theory of figural concepts (Fischbein 1993, Mariotti & Fischbein 1997), the theory of figural apprehension (Duval 1998), and the theory of geometric work (c.f., Kuzniak 2014). Examples of more general theories applied to the specifics of geometry education, include the use of prototype theory (c.f., Hershkowitz 1990), semiotic bundle (c.f., Arzarello, 2006) and semiotic mediation (Bartolini-Bussi & Mariotti 2008), the theory of variation (c.f., Gu, Huang. & Marton 2004), the cK¢ (conception, knowing, concept) model (c.f., Balacheff & Margolinas 2005; Balacheff 2013), as well as more recent use of discursive, embodied, ecocultural and material perspectives (c.f., Ng & Sinclair 2015a,b; Owens 2014, 2015).

Advances in the understanding of spatial reasoning

Over the past decade, increased attention has focused on visuospatial reasoning (Healy & Powell 2013; Lowrie, Logan & Scriven 2012; Owens 2015). Others refer to this reasoning as visualisation and visualising (Clements 2012), spatial thinking (Newcombe & Stieff 2012), spatial reasoning (Davis & Spatial Reasoning Study Group 2015), visuospatial thinking (Shah & Miyake 2005), and visual reasoning (

Rivera, 2011) to name a few. Here, we use the term visuospatial reasoning to emphasise the spatial, visualising (imagistic and as representations that others can see), and reasoning aspects of the visuospatial. While visuospatial reasoning is arguably relevant in all areas of mathematics, it has particular significance in the teaching and learning of geometry. In this section, we provide an overview of research undertaken both in mathematics education and in cognitive sciences, as well as in research that is focused on sociocultural aspects of visuospatial reasoning. Across these research domains, there is converging agreement on the importance and malleability of visuospatial reasoning.

The use and role of diagrams and gestures

The research described in this section has largely emerged out of recent emphases on the semiotic and embodied nature of geometry thinking and learning. In what follows we consider historical-cultural perspectives that highlight the role of semiotic processes and artefacts in geometry teaching and learning. In this section, we consider embodiment perspectives that have highlights the roles of gestures and diagrams in geometry teaching and learning.

Researchers have also begun to study the role of gestures and diagrams in the work of professional mathematicians, both as researchers (Menz, 2015) and as lecturers (Barany & MacKenzie 2014; Hare & Sinclair 2015). This work corroborates some of Châtelet's claims, while also providing more detailed and real-time evidence of the meanings that gesturing and diagramming help to create, even in highly advanced mathematics. This work, combined with the studies summarised before, provide a clear indication of the importance of encouraging learners to engage in more gesturing and diagramming. Existing research suggests that the more teachers gesture, the more students will, but future work could provide insight into the types of gestures that might be helpful and the modalities in which students are invited to gesture as well.

Advances in the understanding of the role of technologies

Despite the fact that the role of technologies has not been understood completely or has been explored in enough detail since the introduction of DGEs *The Geometer's Sketchpad* and *Cabri-Géomètre* in the early 1990's, we have seen new technological developments over the past decade that lead to new challenges in the use of technology in the teaching and learning geometry. This demonstrates the importance of three areas of research: (1) the introduction and design of new technology, both hardware and software, (2) theory and methodology for a better understanding of the role of existing and emerging technology, and (3) empirical studies on the use of technology in teaching and learning. In terms of empirical studies, research on the conjecturing and proving processes when using technology is covered in section 6.

Technology in geometry education has become mainstream, but there is still not enough research into its specific effects. This is in part due to the way that some technologies, such as DGEs, change quite significantly geometric representations and discourse, as compared with paper-and-pencil approaches—something that can make articulate in the classroom, with textbooks, physical manipulatives and—especially—assessment (Venturini, 2015). The role of technology is just beginning to be understood, while, at the same time, it continues to evolve and rapidly change the world around us and in the classroom. Students and teachers are using digital tools throughout the day, and it is necessary to better understand how they can be used effectively for teaching and learning.

Advances in the understanding of the teaching and learning of definitions

The importance of definitions is reflected in the research literature, with many studies on this theme appearing over the past decade. In the overview reported below, we consider research that focused on the following themes: understanding the process of defining and the need for definitions; and, understanding of triangle and quadrilateral definitions. We close with some comments on areas of research that have been not been adequately addressed.

It appears that the fundamental issue of understanding the need for accepting some statements as definitions to avoid circularity has been largely under-researched in the mathematics education community. Another under-researched area seems to be exploring the existence of a *mathematical choice* between defining (and classifying) the quadrilaterals hierarchically or in partitions (compare De Villiers 1994; Usiskin et al. 2008). A specific research question in this regard might be to investigate to what extent students and teachers understand (or how can this understanding be developed?) that choosing a hierarchical definition over a partition one, for example, leads to a more economical (shorter) definition, more concise formulation of some theorems, simplifies the deductive structure substantially by decreasing the number of proofs required, assists in problem solving, etc.

Advances in the understanding of the teaching and learning of the proving process

Much research over the past decade has focused on studying the teaching and learning of the proving process, particularly in light of the increasing use of educational technology. Researchers have turned their attention to the following questions, many of them of perennial interest: What is and what constitutes a mathematical proof? How to interpret proof an explanation that convinces others, and what makes something convincing? What kind of pedagogy and pedagogical tools are conducive to the construction of proof?

Moving beyond traditional Euclidean approaches.

In this section, we first focus on research related to the teaching and learning of 3D geometry, then research on the teaching and learning of non-Euclidean geometries.

The seven themes that we have identified in our survey reflect both traditional research interests in the teaching and learning of geometry as well as new areas of growth. During the past decade, there has been increased focus on embodied and discursive theories in research on the teaching and learning of geometry, with a concomitant research emphasis on visuospatial reasoning, on the use of gestures and diagrams and on digital tools. The effectiveness of certain digital tools, such as DGES, as well as their increased availability, has also affected researched on topics that span the k-16 geometry curriculum (from early experiences with dynamic triangles to later explorations in spherical geometry) as well as major areas of research such as the proving process and the use and role of definitions. There has also been a broadening of the traditional scope of geometry, both in terms of cultural perspectives and also in terms of concepts and activities that do not follow the typical Euclidean development—including the Euclidean approach to definitions. We expect to see continued growth in these areas and also hope to see increased research interest in the teaching and learning of geometry since it is a topic whose significance has decreased in many countries because of an increased emphasis on number and algebra. We close with a quote by the mathematician Jean Dieudonné who, in the early 1980s was responding to the claim that geometry was no longer a vibrant par of mathematics:

> "[a]nd if anybody speaks of 'the death of Geometry' he [sic] merely testifies to the fact that he is utterly unaware of 90% of what mathematicians are doing today" (p. 231).

A valuable focus of future research might be to investigate how geometric ways of thinking, including spatial reasoning and diagramming, may serve not only to improve geometric understanding, but also mathematical understanding more generally, and may even broaden the range of learners who might become interested in and excel in mathematics.

Keywords: geometry, technology, diagrams, definitions, gestures, proving, digital technology, visuospatial reasoning