Lecture of Awardee 2

The Interplay between Construction of Knowledge by Individuals and Collective Mathematical Progress in Inquiry-Oriented Classrooms[1]

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ABSTRACT In a long-term research collaboration, the author and colleagues analyzed learning processes in inquiry-oriented classrooms in a comprehensive manner. In such classrooms, mathematical progress typically occurs in complex interplay between individuals, small groups, and the whole class. We analyzed this interplay by coordinating two theoretical frameworks, Abstraction in Context and Documenting Collective Activity, and their respective methodologies. In the present chapter, I briefly review our research.

Keywords: Classroom-based research, Inquiry-oriented classroom, Abstraction in Context, Documenting Collective Activity, Coordination.

1. Introduction

Learning in inquiry-oriented classrooms (Laursen & Rasmussen, 2019) is becoming more and more common at all levels of schooling. There is evidence from meta-analyses that student-centered approaches have advantages for learning outcomes (e.g., Theobald et al., 2020). It is important to understand why and how these advantages come about. Yet, qualitative research on classroom-based mathematics learning in inquiry-oriented classrooms is scarce. One reason for this may be the complex interplay between learning processes occurring at the different scales of social settings in such classrooms: individuals, small groups, and the class as a whole, learning processes that may strongly depend on each other. This complexity brings with it methodological questions of how to deal with learning processes at different scales. The need arises to use different theoretical frameworks, and to link between them.

I present an overview of a collaboration between Chris Rasmussen, Michal Tabach, Rina Hershkowitz, Naneh Apkarian and myself, whose aim it is to develop a methodology based on two theoretical frameworks that is designed to deal with the above complexity by networking the two previously independent frameworks. I will give an overview over the

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collaboration and some of its achievement. I will refer to other publications for detailed reports on specific research studies.

2. Background

The aim of our collaboration is to investigate learning in intact inquiry-oriented classrooms. All of us had earlier experience researching aspects of mathematical progress in such classrooms, but without obtaining a comprehensive picture of such progress. Some of us have researched the construction of knowledge by individuals and small groups learning in such classrooms (see Dreyfus et al., 2015, and references therein). This research has been based on the Abstraction in Context (AiC) theoretical framework (Hershkowitz et al., 2001). This methodology did not allow for the investigation of mathematical progress of the classroom as a whole. Others have researched knowledge becoming normative and functioning-as-if-shared in the classroom as a whole (Rasmussen & Stephan, 2008). This research has been based on the Documenting Collective Activity (DCA) framework (Stephan & Rasmussen, 2002). This methodology did not allow for investigations of the constructing of mathematical knowledge at the individual and small group level. The question thus arose whether the two frameworks and their associated methodologies could be combined so as to provide a comprehensive picture of the learning processes at the different scales in a classroom and the interactions between these learning processes. In parallel, the question arose how to network the two frameworks in order to achieve this aim, and with which networking strategy (Bikner-Ahsbahs & Prediger, 2014).

After collaborating for more than a decade, we are in a position to give positive answers to both these questions. The two frameworks, AiC and DCA, can be networked, specifically they can be coordinated, and their methodologies can be used in tandem to achieve a detailed qualitative analysis of learning in inquiry-oriented classrooms. In this chapter, we give an overview of several research studies to substantiate this claim.

3. The Studies

In this main section of the chapter, I present brief reviews of four research studies. The first three studies are empirical and illustrate the coordination of the AiC and DCA frameworks to study mathematical progress in intact inquiry-oriented classrooms; the fourth study is theoretical and explains why the coordination between AiC and DCA has succeeded.

3.1. Study 1 — knowledge shifts

We analyzed mathematical progress in an early lesson of an inquiry-oriented differential equations course (Rasmussen at al., 2018). We used Abstraction in Context for analyzing the construction of knowledge by individuals and small groups; we used the Documenting Collective Activity approach for analyzing whole class discussions. We found that students in different groups seemed to be going through similar but not identical processes of knowledge construction. Indeed, it is on this basis that participants can communicate across small groups yet still have differences to debate. Our analysis provides further evidence of the importance of whole class discussions, which partially emerged from the small group work, but also were occasions for participants to develop ideas beyond what was constructed in the small groups.

In order to coordinate between the analysis of knowledge constructed by individual students and small groups of students (using the AiC methodology) and the analysis of the development of mathematical classroom knowledge and reasoning for the whole classroom community (using the DCA methodology), we looked at mathematical progress in the classroom through the two lenses in parallel. The combination of the two methodologies allowed us to follow the evolution of ideas as they flow between individuals, small groups, and the whole class. We identified the links between the constructs that emerged and the ways of reasoning that became normative. These links revealed shifts of knowledge in the classroom: up-loading ideas from a small group to the whole class, and down-loading ideas raised at the whole class level into the work of a small group.

The links also focused our attention on the students who initiated the up- and down-loading and thus assumed the role of knowledge agents. A knowledge agent is a student who initiates an idea which is later taken up by others within the classroom community. We found knowledge agents initiating up-loading of ideas, others initiating down-loading of ideas, and still others initiating new ideas within the whole class discussion itself. The instructor has a double role with respect to knowledge shifts and knowledge agents. One role is to create opportunities which afford the activity of knowledge agents in the class. The second is to help other students benefit from the actions of knowledge agents for constructing their own knowledge. We consider the notions of knowledge agent, up-loading and down-loading to be a crucial product of the methodological coordination between AiC and DCA. A detailed report on this research study has been published elsewhere (Tabach et al, 2014).

3.2. Study 2 — teacher role

In this study, we analyzed a sequence of two lessons of a probability course for eighth grade students. The students worked on purposefully designed sequences of tasks intended to afford the emergence of abstract mathematical thinking in discussion. Our overall goal was to illuminate the role played by individuals and groups in the class as well as by the class as a whole and by the teacher in the knowledge constructing process, and to learn more about shifts of knowledge between the different social settings in a mathematics classroom during the knowledge constructing process.

We used the same approach as in Study 1 — DCA, for analyzing whole class discussions, and AiC, for analyzing group work — and we found that this approach is significant in that it offers a novel methodological tool by which to document the evolution and constitution of mathematical ideas in the classroom and the processes by which these ideas move between individuals, small groups, and the whole class under the facilitation of the teacher.

The two theoretical frameworks and associated methodologies describe different but closely related aspects of the classroom learning process. The AiC analysis has a particular focus on cognitive constructing while the DCA analysis examines how ideas function at the collective classroom level. These are complementary foci and their coordination allows for tracing the growth of ideas from small groups to classroom community and vice versa. The coordination of the findings emerging from the analyses according to the two methodologies allowed us to study knowledge shifts in the classroom as one continuum, and to trace students who have a crucial role in knowledge shifts in the classroom.

The analysis also showed that the teacher adopted the role of an orchestrator by balancing between the whole class and group work in terms of time and tasks. She kept an equilibrium between the need to teach certain content on one hand, and the strategy of affording opportunities for students to construct their knowledge on the other. She assumed responsibility to provide a learning environment that affords argumentation and interaction. This enables normative ways of reasoning to be established and enables students to be active and become knowledge agents. A detailed report on this research has been published by elsewhere (Hershkowitz et al., 2014).

3.3. Study 3 — complexity

The setting for this research is a lesson from a mathematics education master's level course on Chaos and Fractals; the topic of the lesson was the area and perimeter of the Sierpiński triangle and the apparent paradox stemming from an infinite perimeter enclosing a region without area. Our focus in this study is on an enhancement of our methodology and on the insights into the complexity of mathematical progress in inquiry-oriented classrooms that the methodological enhancement reveals. The methodology of coordinating the AiC and DCA theoretical frameworks used in studies 1 and 2 was enhanced in this study as follows: not only did we analyze small group work using AiC and whole class discussions using DCA as in the previous two studies, but we also carried out DCA analyses of the small group work and AiC analyses of the whole class discussions. This enhancement allowed us to layer the explanations for the collective and individual mathematical progress in a manner that fully integrates the collective with the individual mathematical progress, thus exhibiting the complexity of the interplay between collective and individual mathematical progress in inquiry-oriented classrooms. Thus, it enabled us to make sense of the lesson as a whole.

Our analysis showed a multiplicity of ways in which knowledge developed and mathematical progress was achieved. We showed that knowledge was not only constructed by a few students in a small group but also by a large group of students collaborating and arguing together in a whole class setting. In parallel, ideas were shown to function-as-if-shared in different situations: some without a preceding constructing process, others in proximity to a relevant, prior or almost simultaneous, constructing process. In other words, our enhanced methodology allowed us to show that in an inquiry-oriented classroom, knowledge that is new to the students may be constructed not only in small groups but also in whole class discussions. This conclusion should be seen in tandem with the conclusion that ideas may begin to function-as-if-shared not only in whole class discussions but also in small group work. We believe that this complexity is not exceptional but that similarly complex interactions between processes of knowledge construction and processes of ideas becoming normative are typical for inquiry-oriented classrooms. A detailed report on this research is available from the authors (Dreyfus et al., 2022).

3.4. Study 4 — argumentative grammar

From the previous three studies, we conclude that to make sense of learning processes in inquiry-oriented classrooms, networking two or more methodologies with somewhat different foci and grain sizes is insightful. On the other hand, researchers' experience with networking (Bikner-Ahsbahs & Prediger, 2014) has shown that there are conditions for networking to succeed. In the fourth study to be summarized here, we asked ourselves what underlies the success of the coordination between AiC and DCA in the previous three studies.

In this study we identified commonalities between the two frameworks that contribute to the productivity of their networking. To start with, there are environmental commonalities: Both frameworks require classrooms in which students are routinely explaining their thinking, listening to and indicating agreement or disagreement with each other's reasoning; such classrooms typically interweave collaborative work in both small group work and whole class discussions, where the teacher adopts a role that encourages argumentation and inquiry. Both frameworks also require the intentional use of a coherent sequence of tasks that is purposefully designed to offer students opportunities for constructing new knowledge by engaging them in problem solving and reflective activities. The tasks should be designed to afford inquiry and the emergence of new constructs by vertical mathematization (Treffers, 1987) from previous constructs. While vertical mathematization appears here as component of an environmental commonality, it is also a theoretical commonality. The methodologies associated with both frameworks are based on the premise that vertical mathematization is core to mathematical progress.

The theoretical relationship between the frameworks goes much beyond vertical mathematization. The analyses of empirical data allowed us to establish a net of internal-theoretical commonalities in the form of a correspondence between the analytical constructs of AiC and those of DCA. For example, a constructing action in AiC corresponds to an argument as a whole in DCA. Another theoretical link relates to the centrality of shared knowledge. The definition of shared knowledge used in AiC relates to cognitive aspects. We find its counterpart in sociological terms in the phrase 'function-as-if-shared' used by the DCA approach. What is common between the two constructs is that each construct operationalizes when particular ideas or ways of reasoning are, from a researcher's viewpoint, "shared" or "accepted" by the participants.

These and other environmental and theoretical commonalities make AiC and DCA highly compatible. We made use of this compatibility and at the same time raised it to a higher level by articulating an argumentative grammar for our networking; that is, we were explicit about the rationale and theoretical foundation upon which the networking is based. A simplified description of our argumentative grammar is that we employ the relationships between the analytical constructs of AiC and of DCA to explicate the theoretical commonalities of the two frameworks in the empirical data we have from the classroom. The fact that the coordination of the two frameworks has succeeded in the three different studies described in the previous subsections validates this argumentative grammar. A detailed report on this research has been published elsewhere (Tabach et al., 2020).

4. Conclusion

In the research collaboration represented by the four studies reviewed in the previous section, we coordinated two theoretical frameworks, AiC and DCA, to achieve a comprehensive and at the same time detailed qualitative analysis of learning processes in intact inquiry-oriented classrooms. We explained why these two specific frameworks are compatible and allow coordination, and we described an argumentative grammar for such coordination. We found that knowledge agents play an important role in the interplay between different social settings in such classrooms, and we analyzed this role. We found that one path of mathematical progress in such classrooms is the construction of knowledge in small groups followed by a teacher-led whole class discussion in which the knowledge constructed is institutionalized. But we also found that this is but one path, and that mathematical progress in inquiry-oriented classrooms is complex and may be made in a variety of alternative ways.

We see what we achieved as just a beginning, with much research still needed in the future. For example, studies at different grade levels are in order: So far, we have neither a study at the senior high school level nor at the elementary school level. Studies in which all groups in a classroom are observed (video-recorded) would be desirable but of course present methodological challenges to deal with the large amounts of qualitative data collected. Studies with large classes, say of 30 or more students, are likely to present additional methodological demands.

In view of the promise of student-centered instruction at all levels of learning mathematics, we hope that other research teams will take up the challenge of research that aims at investigating learning in student-centered, especially inquiry-oriented classrooms in a comprehensive manner.

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