Invited Lecture

The Roles of Learning Trajectory in Teaching Mathematics Using RME Approach.

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ABSTRACT This paper discusses the critical role of a learning trajectory in teaching mathematics using realistic mathematics education (RME) approach. The first part will present a brief history of RME and how RME was adopted into the Indonesian context. It is followed by a discussion of some of the principles and characteristics of RME. Furthermore, it is explained the learning trajectory, hypothetical learning trajectory (HLT), and how the principles and characteristics of RME are integrated inti HLT in learning mathematics. In the next section, the development of HLT through design research is discussed, and in the last section, some examples of HLT are given and their impact on students' mathematical abilities.

Keywords: Learning trajectory; HLT; RME; PMRI; Design research.

1. Introduction

Mathematics learning at schools in Indonesia generally tends to take place mechanistically (teachers convey information and methods, give examples, and then ask students to do exercises such as examples (Widjaja et al., 2010; Turmudi, 2010; Fauzan et al., 2013). Surprisingly, most teachers believe it is the best way to learn mathematics (Webb et al., 2011; Fauzan, 2013 and 2015; Rangkuti, 2015). Students are less interested and unmotivated to learn mathematics (Fauzan, 2013; Fauzan and Yezita, 2016). Several results of the TIMSS and PISA studies show that the mathematical ability of Indonesian students tends to be low (http://nces.ed.gov/timss/), (Stacey, 2011; Fauzan, 2013; OECD, 2013 and 2015).

One of the causes of the above conditions is because teachers tend to follow the sequences of material presented in the textbooks (starting with definitions, then followed by several examples). In addition, the more frequently used teaching method is chalk and talk (Fauzan, 2002; Fauzan et al., 2013). As a result, most students think that the mathematical concepts they learn are complicated to understand because they are unrelated to their daily lives. It contradicts Freudenthal's idea that mathematics is

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a human activity and that learning mathematics means doing mathematics (de Lange, 1987; Gravemeijer, 1994).

Observing the problems that occurred, it is argued that the Realistic Mathematics Education (RME) approach has the potential to overcome these problems. It is because RME aims to change mathematics education so that most children can do and enjoy mathematics, solve math problems, and develop mathematical knowledge and skills (Sembiring et al., 2010). Learning mathematics with the RME approach focuses on how mathematics is taught and how students learn mathematics in class in a meaningful way. To realize the principles and characteristics of RME in teaching a mathematics topic, it is necessary to design or to develop a learning trajectory (LT).

1.1. RME in the Netherlands

RME is a didactic approach or a domain-specific instruction theory for teaching mathematics that wa*s developed in the Netherlands*. RME has its roots in Freudenthal's interpretation of mathematics as a human activity (Freudenthal, 1991; Van den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen and Drijvers, 2014; Treffers, 1987; Jupri, 1998; Gravemeijer and Cobb, 2013, Van den Heuvel-Panhuizen and Wijers, 2005). The beginning of the emergence of RME theory was since the reform of the mechanistic approach to mathematics learning which had previously been used in the Netherlands (Van den Heuvel-Panhuizen and Drijvers, 2014). It resulted in mathematics as rigid knowledge that is reproductive. As an alternative to this mechanistic approach, modern mathematics, which was currently trending in the world, almost affected the Netherlands.

RME theory is heavily influenced by Freudenthal's ideas. He says that the main idea of RME theory is that mathematics is introduced as meaningful knowledge for students, and mathematics is a human activity. Therefore, in the learning process, mathematics is not studied as a closed system but must be studied as an activity to mathematize reality and mathematics itself. RME is a term for realistic, which comes from the Dutch term zich REALISEren, which is meaningful to imagine. The term realistic does not mean that it is directly related to the real world but rather to the use of problems that students can imagine. The emphasis is on making things honest in mind. It means that RME does not always have to use real-life problems. However, abstract mathematical problems can be made real in students' minds, so the mathematics material taught needs to be accurate for students. It is what underlies so-called Realistic Mathematics Education.

Freudenthal suggests that mathematics education has to be organized as a process of guided reinvention, where students can experience a similar process to the process in which mathematicians invented mathematics. Gravemeijer (1999) sees the guided reinvention principle as long-term learning process in which the reinvention process evolves as one of gradual changes. The stages always have to be viewed in a long-term perspective, not as goals in themselves, and the focus has to be given on guided exploration. To realize this view, a learning route (learning trajectory (LT)) has to be mapped out (by a developer or instructional designer) that allows the students to find the intended mathematics by themselves.

1.2. RME in Indonesia

Indonesian Realistic Mathematics Education (PMRI) was initiated by a group of mathematics educators in Indonesia. The initial motivation was to find a replacement for modern mathematics, which was abandoned in the early 1990s. The birth of RME was applied in general in public schools. Approximately 30 years later, RME entered Indonesia under Indonesian Realistic Mathematics Education (PMRI).

The history of PMRI started with the efforts to reform mathematics education carried out by the PMRI Team (initiated by Prof. Sembiring et al.), which was officially implemented in 1998. Furthermore, the initial trials of PMRI started in late 2001 in eight elementary and four Islamic elementary schools. Then, PMRI began to be applied simultaneously from first grade in Surabaya, Bandung, and Yogyakarta. The number of schools involved, in this case, called LPTK partner schools, is not less than 1000 schools (Sembiring, 2010).

2. RME's Principles and Characteristics

2.1. RME's principles for instructional design

There are three key principles of RME for instructional design (Gravemeijer, 1994 and 1997), namely *guided reinvention through progressive mathematization, didactical phenomenology,* and *emerging models.* In the guided reinvention principle, the students should be given the opportunity to experience a process similar to that by which mathematics was invented (Gravemeijer 1994 and 1999). For this purpose, a learning route has to be designed by a developer or instructional designer to allow the students to find the intended mathematics by themselves.

Through the didactical phenomenology principle, the developer or instructional designer has to provide students with contextual problems taken from phenomena that are real and meaningful for them (Gravemeijer, 1994). The real phenomena will facilitate students to experience the process of horizontal and vertical mathematization.

Related to emerging models, developer or instructional designer have to give the opportunity to the students to use and develop their own models when they are solving the contextual problems. At the beginning the students will develop an informal model which is familiar to them. After the process of generalizing and formalizing, the model gradually becomes an entity on its own. Gravemeijer (1994) calls this process a transition from *model-of* to *model-for*.

2.2. RME's characteristics

There are at least five RME's characteristics of RME which are summarized from Treffers (1987), Zubainur et al. (2020), Wewe and Juliawan (2019), Paredes et al.

(2020), Sampoerno and Meliasari (2019). Teaching and learning using RME approach are required to apply these characteristics:

a. Using contextual problems (the use of context); learning begins by using contextual problems, not starting from the formal mathematics concepts. Contextual problems raised as the initial topic of learning must be simple problems that students recognize. It means that students are given realistic problems and begin developing their thoughts to find solutions to the problems
b. Using models (use models, bridging by vertical instruments); students pass through the levels of mathematical understanding: from understanding that is informal, semi-formal, to formal stages. To bridge it, it is necessary to model related to situational and mathematical models developed by students. Students solve problems using mathematical models (tables, graphs, pictures, equations). Solutions at this stage can be informal or formal. Teacher guides and directs students to solve the problem so that students rediscover ideas or concepts and can form a structured mathematical model of the material in their way.
c. Using student contributions (student contribution); significant contribution to the teaching and learning process is expected to come from students, meaning that all students' thoughts (construction and production) are considered. Students also compare and discuss answers in study groups which are used to train students' courage to express their opinions even if they differ from the results or income of friends or teachers.
d. Using interaction (interactivity); in the learning process, students actively discuss, and express ideas both in-class activities and group activities, so that there is an interaction between students and teachers, students and students. Interactions such as negotiations, explanations, justifications, approvals, questions or reflections are used to achieve forms of formal mathematical knowledge from informal forms of mathematical knowledge discovered by students. The teacher optimizes students' competence in the use of ideas. To optimize the interaction between students and students or with the teacher as a facilitator, students find formal mathematics to solve the problems given.
e. Using intertwinement; mathematical structures and concepts are interrelated. Mathematical problems facilitate students, and linking mathematical topics such as numbers, algebra, and geometry are not seen as separate but as interrelated and integrated topics.

2.3. The process of mathematization in RME

Linguistically, the word mathematization comes from the mathematization of mathematization. The words mathematization and mathematization are nouns from the verb mathematize or mathematize, meaning math. Treffers (1987) distinguishes mathematization into two types: horizontal and vertical. According to Van den Heuvel-Panhuizen and Drijvers (2014), the idea of horizontal and vertical mathematization in the mathematical process which was originally initiated by Treffers was taken over and refined by Freudenthal (2002), Kabael and Deniz (2016), Araújo and De Lima (2020), Amala and Ekawati (2020), Jupri et al. (2021), Widada et al. (2020). In horizontal mathematization, students use mathematics to transform realistic problem situations into mathematical situations in the form of mathematical models. In vertical mathematics, students work in the world of symbolic mathematics by reorganizing the model until a problem is found.

Freudenthal defines horizontal mathematization as an activity to convert contextual problems into mathematical problems (symbols). Horizontal mathematization is related to the generalizing process. The horizontal mathematization process begins with identifying mathematical concepts based on regularities and relationships found through visualization and schematization. While vertical mathematization is a form of a formalizing process where the mathematical model obtained in horizontal mathematization becomes the basis for developing more formal mathematical concepts, the vertical mathematization process is Interrelated. The horizontal and vertical mathematization process cannot be directly separated into two large parts sequentially. Namely, the vertical mathematization process takes place after the entire horizontal mathematization process occurs. Horizontal and Vertical Mathematization Processes. However, the two processes of mathematization can be formed like steps, often occurring alternately gradually. Fig. 1 shows the process of horizontal and vertical mathematization.

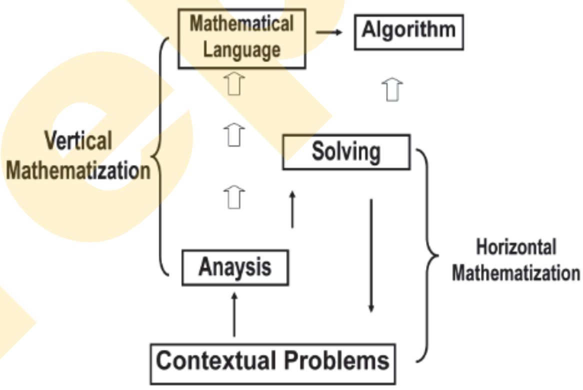

Fig. 1. The process of horizontal and vertical mathematization (Source: Gravemeijer, 1999)

3. Learning Trajectory

Learning trajectory (LT) is the sequences of activities or tasks to guide students to achieve a specific instructional goal. LT are based on "hypothetical learning trajectories" (HLT), a concept proposed by Simon (1995) from a constructivist point of view. Simon uses this concept to describe a teacher's prediction of the trajectory that can be taken in the learning process, defining it as the teacher's predicted trajectory for student learning (Simon, 2004; Daro et al. 2011; Akdeniz and Argun, 2021). It can be seen from this description that HLT includes learning and teaching practice, which is focused on the instructions given by the teacher (Clements, 2011; Sarama and Clements, 2019). The LT concept identifies student developments hierarchically and moves hypothetically with implementation and validation through empirical study.

> "…descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain."

Experts define the concept as consisting of three components: the developmental process, instructional activities, and instructional goals, which are also the main ideas of mathematics. The learning trajectory describes students' thinking levels, mental ideas and actions required, thought processes to express those ideas and actions, and instructional activities and strategies based on these processes (Clements and Sarama, 2014; Daro et al., 2011). It is supported by Fauzan (2016), which states that the learning trajectory is a series of learning activities (activities to solve contextual problems) that will facilitate students to reinvent (reinvention) formal mathematical concepts by optimizing their informal knowledge.

3.1. Hypothetical learning trajectory (HLT)

Hypothetical Learning Trajectory (HLT) is defined as three essential components, namely goals for meaningful learning (goals for meaningful learning), a series of tasks to achieve these goals (learning activities), and hypotheses about student thinking and learning (hypothetical learning process) (Simon, 2004; Cárcamo Bahamonde et al., 2017). A learning goal is a goal to be achieved in the learning process. While learning activities are a design of learning trajectories that students will pass in achieving the learning objectives set. These activities are given as mathematical tasks (mathematics tasks) in the form of contextual questions. The final, hypothetical learning process, namely predictions or assumptions about the understanding and reasoning of students that will develop in the learning process, and there is anticipation or feedback from the teacher to help students achieve learning goals. (Simon and Tzur, 2004; Sumirattana et al., 2017; Dickinson, 2020; Gravemeijer, 2020).

In its implementation, HLT does not aim to want the results obtained by students but how the process of these students rediscovering mathematical concepts. The process of rediscovery is divided into several levels of models that will appear in the learning process (Gravemeijer, 1999). In this principle, the model is understood as to how students produce each observable activity. (Zandieh and Rasmussen, 2010). In designing the learning trajectory in HLT, four activity levels will appear situation, model level, model level for, and formal knowledge. The situational level is the basic level that gives rise to situational knowledge and strategies used in conjunction with the context of the situation. The referential level is the use of models and strategies at this level to show the situation described in the problem. The referential level is also known as the model-of level.

Meanwhile, at the general level, the for-model appears in mathematical knowledge, focusing on strategy dominating the reference to the problem context. Finally, the formal level requires that reasoning with conventional symbolization does not last long to support the for-model mathematical activity. These four activity levels aim to enable students to solve problems with an informal approach and gradually construct their knowledge into formal knowledge (Gravemeijer, 2020). Meanwhile, at the general level, the for-model appears in the form of mathematical knowledge with a focus on strategy dominating the reference context of the problem. Finally, the formal level requires that reasoning with conventional symbolization does not last long to support the for-model mathematical activity. These four activity levels aim to enable students to solve problems with an informal approach and gradually construct their knowledge into formal knowledge (Gravemeijer, 2020). Meanwhile, at the general level, the for-model appears in mathematical knowledge, focusing on strategy dominating the reference to the problem context. Finally, the formal level requires that reasoning with conventional symbolization does not last long to support the for-model mathematical activity. These four activity levels aim to enable students to solve problems with an informal approach and gradually construct their knowledge into formal knowledge (Gravemeijer, 2020).

In addition, Hans Freudenthal emphasized that a realistic mathematical approach, "mathematics should be connected to the reality," must be included in every divided into several learning activities in HLT (Van den Heuvel-Panhuizen and Drijvers, 2014; Dickinson, et al., 2020). Learning activities involve the process of horizontal mathematization and vertical mathematization. In designing the HLT, one must also pay attention to the key principles of RME, namely 1) Use of context, 2) Use of models for progressive mathematization, 3) Utilization of student construction results, 4) Interactivity, 5) Linkage (Swidan, 2020; Fessakis, et al., 2017).

3.2. Local instructional theory (LIT)

Local instructional theory (LIT) is a theory in a learning process that tells a learning flow about a complete learning topic in a set of activities that support it (Gravemeijer, 2009; Andrews-Larson et al., 2017). According to Simon (2018), calling it a (domain-specific) theory, the theory only discusses a specific domain or topic of learning. By using instructional design in designing HLT to find strategies and ways of thinking, students anticipate formal concepts; identify design principles for learning activities that can be used to generate these strategies and ways of thinking; identifying design principles for learning activities can be used to leverage these strategies and ways of thinking to support formal concept development. In the end, a Local Instructional Theory will be obtained.

3.3. Developing LT through design research

As mentioned in the previous part, at the beginning, a LT is designed in form of an HLT. After a series of the try out in the classroom and revisions, the HLT become a LT. The process of designing, try outs, and revisions of HLT suit to the idea of design research proposed by Gravemeijer and Cobb (2013) which characterized by a cyclic process of preparing for the experiment, conducting the experiment, and retrospective analysis. The cyclic process is described in Fig. 2.

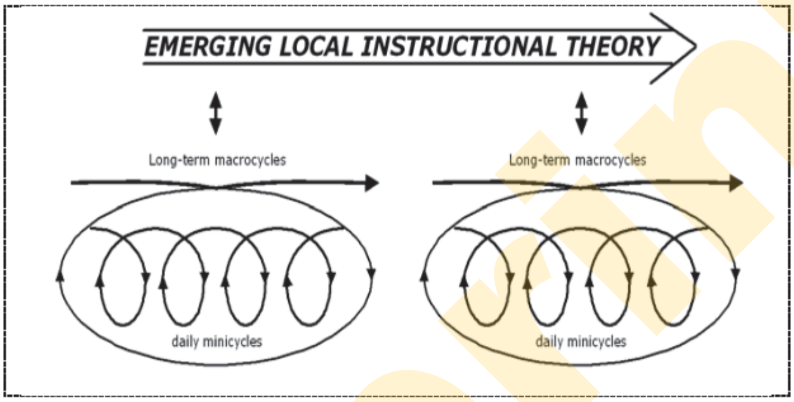

Fig. 2. The cyclic process of design research (Source: Gravemeijer and Cobb, 2013)

In preparing for the experiment, it is determined the end points of the instructions, followed by designing the series of activities of solving contextual problems. These activities are designed so that they would facilitate students to do horizontal and vertical mathematization as well as stimulate students' thinking and reasoning. In addition, we also formulated the predictions of students' thinking and solutions, and the anticipations. In the experimental phase, the HLT is tried out in the classroom. After the retrospective analysis and re-design processes, we will get a LT for teaching a mathematics topic using RME approach. Some examples of LTs that have been developed through design research will be mentioned in the next part.

3.4. Examples LT from the previous research

The local instructional theory has been successfully applied in recent years at the university level in several fields of science, such as abstract algebra material (Larsen, SP, 2013), in calculus (Bos et al., 2020; Swidan, 2020; Gilboa et al., 2019), in statistical material (Syafriandi et al., 2020), in linear algebra (Andrews-Larson et al., 2017; Cárcamo Bahamonde et al., 2017) and ordinary differential equations (Yarman et al., 2020). These studies show that students can construct informal knowledge into formal knowledge with a series of activities provided by the lecturer.

Similar research was also successfully applied to junior high school students in measurement and geometry materials (Sumirattana et al., 2017; Fauzan et al., 2020), in mathematical literacy (Arnawa and Fauzan, 2020). Other studies were also applied to high school children on the material of sequences and series (Azizah et al., 2021).

Several other researchers have also applied local instructional theory in elementary schools to the material for adding whole numbers to dyscalculia students (Fauzan et al., 2022), material abstract algebra is the use of variables for children aged 3-7 years (Ventura et al., 2021), in social arithmetic material in grade 3 students (Stemn, 2017), in measurement and geometry materials (Möhring et al., 2021; Akdeniz et al., 2013), in geometry material using mobile computing device technology (Fessakis et al., 2017)

References

D. G. Akdeniz and Z. Argün. (2021). Learning trajectory of a student with learning disabilities for the concept of length: A teaching experiment. *The Journal of Mathematical Behavior*, 6(4), 100915.

M. A. Amala and R. Ekawati. (2020). The Profile of Horizontal and Vertical Mathematization Process of Junior High School Student with High Mathematical Ability in Solving Contextual Problem of Fraction. *Jurnal Riset Pendidikan dan Inovasi Pembelajaran Matematika (JRPIPM),* 3(2). https://doi.org/10.26740/jrpipm.v3n2. p52–60

C. Andrews-Larson, M. Wawro, and M. Zandieh. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology*, 48(6), 809–829. https://doi.org/10.1080/0020739X.2016.1276225

L. J. De. Araújo and F. H. De Lima. (2020). The mathematization process as object-oriented actions of a modelling activity system. *Bolema-Mathematics Education Bulletin*, 64(68). https://doi.org/10.1590/1980-4415v34n68a01

I. M. Arnawa and A. Fauzan. (2020). The Practicality of Mathematics Learning Model Based on RME and Literacy in Junior High School. In *2nd International Conference Innovation in Education (ICoIE 2020)*. Atlantis Press. https://dx.doi.org/10.2991/assehr.k.201209.238

N. Azizah, A. Fauzan, and M. Arnawa. (2021). Developing Learning Model Based on the PMR Approach to Improve Student's Knowledge in Mathemathics Learning. *In Journal of Physics: Conference Series*. IOP Publishing. https://iopscience.iop.org/article/10.1088/1742-6596/1779/1/012045/meta

R. Bos, M. Doorman, and M. Piroi. (2020). Emergent models in a reinvention activity for learning the slope of a curve. *The Journal of Mathematical Behavior*, 59, 100773.

A. D. Cárcamo Bahamonde, J. M. Fortuny Aymemí and J. V. Gómez i Urgellés. (2017). Mathematical modelling and the learning trajectory: tools to support the teaching of linear algebra. *International Journal of Mathematical Education in Science and Technology*, 48(3), 338–352. https://doi.org/10.1080/0020739X.2016.1241436

D. H. Clements, D. C. Wilson, and J. Sarama. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning*, 6(2), 163–184. https://doi.org/10.1207/s15327833mtl0602_5

D. H. Clements. (2011). Learning trajectories: Foundations for effective, research based education. In L. R. Wiest, and T. Lamberg (Eds.), *Proceedings of the 33rd Annual*

Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

D. H. Clements and J. Sarama. (2012). Learning trajectories in mathematics education. In *Hypothetical Learning Trajectories* (pp. 81–90). Routledge.

P. Cobb, J. Confrey, A. DiSessa, R. Lehrer, and L. Schauble. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9–13.

P. Daro, F. A. Mosher and T. B. Corcoran. (2011). Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction. *CPRE Research Report Series*. Retrieved from CPRE database. https://doi.org/10.12698/cpre.2011.rr68

J. De Lange. (1987). *Mathematics, Insight and Meaning*. Utrect, Holland: Rijksuniversiteit.

P. Dickinson, F. Eade, S. Gough, S. Hough, and Y. Solomon. (2020). Intervening with Realistic Mathematics Education in England and the Cayman Islands — The Challenge of Clashing Educational Ideologies. *International Reflections on the Netherlands Didactics of Mathematics*, 341.

A. Fauzan, F. Tasman and R. Fitriza (2020). Exploration of Ethnomathematics at Rumah Gadang Minangkabau to Design Mathematics Learning Based on RME in Junior High Schools. In *2nd International Conference Innovation in Education (ICoIE 2020)*. Atlantis Press. https://dx.doi.org/10.2991/assehr.k.201209.234

A. Fauzan, C. D. Andita, G. Rada, A. Zafirah, and A. H. bin Abdullah. (2022). Developing RME-Based Learning Trajectory for Teaching Addition to A Dyscalculia Student in Elementary School. *Jurnal Didaktik Matematika*, 9(1), 39–58.

A. Fauzan, T. Plomp and K. Gravemeijer. (2013). The development of an RME-based geometry course for Indonesian primary schools. In T. Plomp and N. Nieveen (eds). *Educational Design Research Part B: Illustrative cases* (pp. 159–178). Enschede, the Netherlands: SLO.

A. Fauzan, E. Musdi, and J. Afriadi. (2018). Developing learning trajectory for teaching statistics at junior high school using RME approach. *The 6th South East Asia Design Research International Conference (6th SEA-DR IC)*. IOP Publishing. DOI:10.1088/1742-6596/1088/1/012040.

A. Fauzan, A. Armiati, and C. Ceria. (2018). A learning trajectory for teaching social arithmetic using RME approach. *The 2nd International Conference on Mathematics, Science, Education and Technology*. IOP Publishing. DOI:10.1088/1757-899X/335/1/012121.

A. Fauzan and Y. Yerizon. (2013). Pengaruh Pendekatan RME dan Kemandirian Belajar Terhadap Kemamampuan Matematis Siswa. Prosiding SEMIRATA 2013, 1(1).

A. Fauzan (2015). Analyzing Teachers' Difficulties in Understanding Mathematical Basic Concepts. *Prosiding ICOMSET UNP*.

A. Fauzan and E. Yezita. (2016). Pengembangan Alur Belajar Topik Perbandingan dengan Pendekatan RME. *Prosiding Konaspi VIII di Jakarta*.

A. Fauzan (2002). Applying Realistic Mathematics Education (RME) in Teaching *Geometry in Indonesian Primary Schools*. University of Twente.

G. Fessakis, P. Karta and K. Kozas. (2017). The math trail as a learning activity model for m-learning enhanced realistic mathematics education: A case study in primary education. In *International Conference on Interactive Collaborative Learning*. Springer, Cham. doi:10.1007/978-3-319-73210-7_39

H. Freudenthal (1991). *Revisiting Mathematics Education. China Lectures*. Kluwer Academic Publishers.

H. Freudenthal (2002). *Didactical Phenomenology of Mathematical Structures*. Kluwer Academic Publishers.

N. Gilboa, I. Kidron, and T. Dreyfus. (2019). Constructing a mathematical definition: The case of the tangent. *International Journal of Mathematical Education in Science and Technology*, 50(3), 421–446. https://doi.org/10.1080/0020739X.2018.1516824.

K. Gravemeijer (2020). Emergent Modeling: an RME Design Heuristic Elaborated in a Series of Examples. *J. Int. Soc. Des. Dev. Educ.*, vol. 4, no. 13, pp. 1–31.

K. Gravemeijer and P. Cobb (2013). Design research from the learning design perspective. In Plomp T., and N. Nieveen (Eds.), *Educational Design research. Part A: Illustrative cases* pp. 72–113. Enschede, The Netherlands: SLO.

K. Gravemeijer and D. Van Eerde (2009). Design Reseach as a Means for Building Knowledge Base for Teaching in Mathematics Education. *The elementary School Journal*, 109(5).

K. Gravemeijer (1994). *Developing Realistic Mathematics Education*. Utrecht: Freudenthal Institute.

K. Gravemeijer (1999). How emergent models may foster the constitution of formal mathematics. *Journal of Mathematical Thinking and Learning*: 1(155).

M. Van den Heuvel-Panhuizen and Drijvers (2014). Realistic Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*: pp. 521–525. Dordrecht, the Netherlands: Springer. https://doi.org/10.1007/978-94-007-4978-8

M. Van den Heuvel-Panhuizen and M. Wijers. (2005). Mathematics standards and curricula in the Netherlands. ZDM — *International Journal on Mathematics Education,* 37(4). https://doi.org/10.1007/BF02655816

M. Van den Heuvel-Panhuizen (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, 54 (1), 9–35.

A. Jupri (1998). Pendidikan Matematika Realistik: Sejarah, Teori, dan Implementasinya. *In Encyclopedia of Immunology* (Issue 1).

A. Jupri, D. Usdiyana, and R. Sispiyati (2021). Teaching and learning process for mathematization activities: The case of solving maximum and minimum problems. *JRAMathEdu (Journal of Research and Advances in Mathematics Education),* 6(2). https://doi.org/10.23917/jramathedu.v6i2.13263

T. Kabael and Ö. Deniz. (2016). Students' Mathematization Process of the Concept of Slope within the Realistic Mathematics Education. *Hacettepe University Journal of Education*. https://doi.org/10.16986/huje.2016018796

K. Khairuddin at al. (2020). Developing hypothetical learning trajectory for Green's theorem. *International Journal of Advance Science and Technology*. 29(108).

S. P. Larsen (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, 32(4), 712–725. https://doi.org/10.1016/j.jmathb.2013.04.006.

Y. Liljekvist, E. Mellroth, J. Olsson, and J. Boesen. (2017). Conceptualizing a local instruction theory in design research: report from a symposium. *In MADIF10, The tenth research seminar of the Swedish Society for Research in Mathematics Education, Karlstad.* Svensk förening för MatematikDidaktisk Forskning-SMDF.

W. Möhring, A. D. Ribner, R. Segerer, M. E. Libertus, T. Kahl, L. M. Troesch, and A. Grob. (2021). Developmental trajectories of children's spatial skills: Influencing variables and associations with later mathematical thinking. *Learning and Instruction*, 75, 101515. https://doi.org/10.1016/j.learninstruc.2021.101515

OECD. 2013. *PISA 2009 Results Vol. I - V*. OECD: Paris

OECD. 2015. *PISA 2009 Results Vol. I - V.* OECD: Paris

S. Paredes, M. J. Cáceres, J. M. Diego-Mantecón, T. F. Blanco, and J. M. Chamoso. (2020). Creating realistic mathematics tasks involving authenticity, cognitive domains, and openness characteristics: A study with pre-service teachers. *Sustainability (Switzerland),* 12(22). https://doi.org/10.3390/su12229656

A. N. Rangkuti (2015). *Pengembangan Alur Pembelajaran Topik Pecahan di Sekolah Dasar dengan Pendekatan Pendidikan Matematika Realistik.* Padang: Program Pascasarjana UNP.

J. Sarama and D. H. Clements (2019). *Learning Trajectories in Early Mathematics Education. Researching and Using Progressions (Trajectories) in Mathematics*. Brill Sense.

P. D. Sampoerno and M. Meiliasari (2019). Analysis of the mathematical learning materials with the characteristics of realistic mathematics education in the design research pre-service teachers' theses in Indonesia*. The 4th Annual Applied Science and Engineering Conference*. IOP Publishing. https://doi.org/10.1088/1742-6596/ 1402/7/077105

R. Sembiring (2010). Pendidikan Matematika Realistik Indonesia (PMRI): Perkembangan dan Tantangannya. *IndoMS. J.M.E*, 1(1).

R. Sembiring, K. Hoogland, and M. Dolk (2010). *A Decade of PMRI in Indonesia*. Utrecht: TenBrink.

M. A. Simon and R.Tzur (2004). Explicating the Role of Mathematical Tasks in Conceptual Learning: An Elaboration of the Hypothetical Learning Trajectory. *Math. Think. Learn.*, 6(2). pp 91–104, , doi: 10.1207/s15327833mtl0602_2.

M. A. Simon, M. Kara, N. Placa, and A. Avitzur (2018). Towards an integrated theory of mathematics conceptual learning and instructional design: The Learning Through Activity theoretical framework. *The Journal of Mathematical Behavior, 52, 95–112*. doi: 10.1016/j.jmathb.2018.04.002.

M. A. Simon (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 114–145. https://doi. org/10.2307/749205

K. Stacey (2011). The PISA view of mathematical literacy in Indonesia. *Journal on Mathematics Education,* 2(2), 95–126.

B. S. Stemn (2017). Rethinking mathematics teaching in Liberia: Realistic mathematics education. *Childhood Education*, 93(5), 388–393. doi:10.1080/00094056.2017.13672

S. Sumirattana, A. Makanong, and S. Thipkong (2017). Using realistic mathematics education and the DAPIC problem-solving process to enhance secondary school students' mathematical literacy. *Kasetsart Journal of Social Sciences*, 38(3), 307–315.

O. Swidan (2020). A learning trajectory for the fundamental theorem of calculus using digital tools. *International Journal of Mathematical Education in Science and Technology*, 51(4), 542–562.

S. Syafriandi, A. Fauzan, L. Lufri, and A. Armiati (2020). Designing hypothetical learning trajectory for learning the importance of hypothesis testing. *International Conference on Mathematics and Mathematics Education.* IOP Publishing. https:// iopscience.iop.org/article/10.1088/1742-6596/1554/1/012045/meta

A. Treffers (1987). Three Dimensions A Model of Goal and Theory Description in Mathematics Instruction — The Wiskobas Project. *In SpringerBriefs in Applied Sciences and Technology.*

T. Turmudi (2010). *Pembelajaran Matematika Kini dan Kecenderungan Masa Mendatang*. Bandung: JICA FPMIPA UPI.

A. C. Ventura, B. M. Brizuela, M. Blanton, K. Sawrey, A. M. Gardiner, and A. Newman-Owens (2021). A learning trajectory in Kindergarten and first grade students' thinking of variable and use of variable notation to represent indeterminate quantities. *The Journal of Mathematical Behavior*, 62, 100866.

M. Wewe and I. W. Juliawan (2019). Developing Mathematical Devices with Characteristics Realistic Mathematics Education. *Al-Jabar: Jurnal Pendidikan Matematika*, 10(1). https://doi.org/10.24042/ajpm.v10i1.3884

D. C. Webb, H. Van der Kooij, and M. R. Geist (2011). Design research in the Netherlands: Introducing logarithms using realistic mathematics education. *Journal of Mathematics Education at Teachers College*, 2(1).

W. Widada, D. Herawaty, Y. Beka, R. M. Sari, R. Riyani, and K. Umam Zaid Nugroho (2020). The mathematization process of students to understand the concept of vectors through learning realistic mathematics and ethnomathematics. *The 7th South East Asia Design Research International Conference (SEADRIC 2019)*. IOP Publishing. https://doi.org/10.1088/1742-6596/1470/1/012071

W. Widjaja, M. Dolk, and A. Fauzan (2010). The role of contexts and teacher's questioning to enhance students' thinking. *Journal of Science and Mathematics Education in Southeast Asia*, 33(2), 168–186.

Y. Yarman, A. Fauzan, A. Armiati, and L. Lufri (2020). Hypothetical Learning Trajectory for First-Order Ordinary Differential Equations. In *2nd International Conference Innovation in Education (ICoIE 2020)*. Atlantis Press. https://dx.doi.org/10.2991/assehr.k.201209.245

C. M. Zubainur, R. Johar, R. Hayati, and M. Ikhsan (2020). Teachers' understanding about the characteristics of realistic mathematics education. *Journal of Education and Learning (EduLearn)*, 14(3). https://doi.org/10.11591/edulearn.v14i3.8458