Invited Lecture

The Power of Mathematical Tasks for Teacher Training: The Case of Suma y Sigue

Salomé Martínez[1], Farzaneh Saadati[2], Paulina Araya[3], Eugenio Chandía[4], and Daniela Rojas[5]

ABSTRACT The need to improve teachers' preparation to teach mathematics is shared by many countries. E-learning professional development (PD) programs appear as an attractive option due to their flexibility and availability. Suma y Sigue is an e-learning PD program for Chilean teachers that focuses on the development of Mathematical Knowledge for Teaching (MKT). The program is characterized based on a constructivist perspective of learning by using a contextualized problem-based approach. This article describes the instructional design of the program learning activities that demonstrate how mathematical tasks centered on the construction of MKT are articulated and implemented. The learning performance of the participants in a specific course within the program is analyzed. The findings show empirical evidence of improvement in teachers' knowledge. The detailed description of the course and participants' performance can aid PD developers to design principles and the use of different instructional strategies, especially when the course focuses on MKT development.

Keywords: E-learning; Instructional design; Professional development; Mathematical knowledge for teaching.

1. Introduction

Improving teachers' knowledge and skill to teach mathematics is a need in many places and contexts. Ball and Bass (2000) believe that teachers' mathematical knowledge should be strong enough to allow them to deal flexibly with the complexity of teaching mathematics to different students. Ball (2003) emphasizes the importance of "designing courses in mathematical knowledge for teaching, helping instructors and professional developers teach them well, and doing so at scale" (p.38). There are recommendations for Professional Development (PD) programs for teachers, which

[1] Departamento de Ingeniería Matemática, Centro de Modelamiento Matemático, CNRS IRL 2807, Universidad de Chile. Santiago, 8320000, Chile. E-mail: samartin@dim.uchile.cl
[2] Centro de Investigación Avanzada en Educación, Universidad de Chile. Santiago, 8320000, Chile. E-mail: farzaneh.saadati@ciae.uchile.cl
[3] Centro de Modelamiento Matemático CNRS IRL 2807, Universidad de Chile. Santiago, 8320000, Chile. E-mail: pauaraya@dim.uchile.cl
[4] Departamento de Curriculum e Instrucción, Universidad de Concepción. Concepción, 4070386, Chile. E-mail: echandia@udec.cl
[5] Centro de Modelamiento Matemático CNRS IRL 2807, Universidad de Chile. Santiago, 8320000, Chile. E-mail: daniela.rojas.b@uchile.cl

Preprint

acknowledge the need for a clear focus on Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008) especially to impact teachers' work with students in the classroom (Campbell and Lee, 2017). Many efforts have been made to understand the role of strategies used to deliver PDs activities and bring learning opportunities for teachers to improve their MKT (Copur-Gencturk et al., 2019). Borba and Llinares (2012) identify knowledge-building practices in technology-mediated workgroup interactions among several key topics that require further research in online mathematics teacher education. Yet, it is not evident the types of learning activities that might help teachers to develop such knowledge.

Suma y Sigue is a PD program designed based on e-learning modalities (fully online or through blended learning) for Chilean teachers, which is focused on the development of teachers' MKT. The program is based on a constructivist perspective of learning by using a contextualized problem-based approach. Martínez et al. (2020) studied teachers' satisfaction by participating in the blended model of the Suma y Sigue PD program. Starting in 2020, the program changed its format by adopting an e-learning approach. This was planned due to the necessity to reach teachers in remote locations and was also precipitated by the Covid-19 pandemic. In this study, we describe the instructional design of a virtual learning environment that allows teachers to develop their MKT through an instructional model with a high autonomous learning component. We also emphasize how the learning sequencing of this model promotes teachers' engagement in mathematical tasks, and how it guides a user in both the reinvention of elementary mathematics and the development of professional mathematical knowledge. In this paper, we aim to explain the Suma y Sigue design structure in online model by focusing on the activity design and the learning sequence of a particular course called "Working with multiplication and division". Moreover, we analyze the learning results of participating teachers in this course, discussing the relationship between course activities and change on teachers' MKT.

2. The Importance of Professional Development Programs and Teachers' Knowledge

For the past three decades, efforts to improve the competencies of math teachers and thus the quality of math education have been a constant feature of educational policies around the world. To this end, policymakers and educational organizations have implemented various plans and programs to reform education. In the field of reform, teacher preparation plays a key role in enhancing teachers' professional competencies (Barber and Mourshed, 2008). By focusing on in-service teachers' education, PD programs have been known as an important area of research in promoting mathematics education and also as a goal of various governments and research communities during the last decade (Koellner et al., 2011; Martin and Mulvihill, 2020).

Preprint

Chile has begun extensive reforms to improve the quality of education (Santiago et al., 2017; Toledo and Wittenberg, 2014), particularly in mathematics (Saadati et al., 2021; Martínez et al., 2020). For example, through the new national Teacher Professional Development System in 2016, teachers have been encouraged and granted the right to participate in PD programs. Thus, it is expected that teachers will have access to free and relevant education in order to further develop their professional careers and improve their knowledge and professional competencies. That system brings opportunities to develop different types of PDs in Chile to support educational reform.

Considering the reforms in mathematics education, efforts have been focused on improving the knowledge of teachers so that they are able to carry out teaching in a way that allows them to help students understand mathematics conceptually. In fact, teaching mathematics is a serious and demanding arena of work and teachers need to be prepared to handle it (Ball, 2003; Ball et al., 2005). The quality of teaching depends on the knowledge and teaching skills of the teachers, so focusing on improving MKT is a key strategy to address this need (Campbell and Lee, 2017).

Copur-Gencturk and colleagues (2019) examined the successful characteristics of a PD in improving teachers' MKT and learning, such as PD tasks, materials, and agendas. They showed that a focus on curricular content knowledge and reviewing student work seems to be important to improve teachers' content knowledge for teaching. Garet et al. (2001) proposed two groups of features for effective teacher PD programs. The first one is structural features such as the form of activity, its duration and collective participation, and the second group refers to the core features which include content knowledge, active learning, and coherence. Considering these two groups of features is a must to design an effective PD program to improve teachers' MKT.

In general, providing opportunities for in-service teachers to develop MKT requires constructing and implementing a specific type of task that should: (a) create opportunities to unpack, make explicit, and develop a flexible understanding of mathematical ideas; (b) provoke a stumble due to a superficial understanding of an idea; (c) help to make connections among mathematical ideas; (d) lend themselves to constructing multiple representations and solutions methods; and (e) provide opportunities to engage in different mathematical practices (Suzuka et al., 2009).

2.1. Contextualized problem-based learning

According to the Realistic Mathematics Education (RME) theory (Freudenthal, 2012), contextualized problems are inextricably linked to mathematics learning, with "realistic" contexts serving both as a source for initiating the development of mathematical ideas and as settings to later apply mathematical knowledge. Context problems function as anchoring points for a guided reinvention of mathematics by the

Preprint

students, helping to bridge the gap between informal and formal mathematical knowledge (Gravemeijer and Doorman, 1999).

Even though mathematical literacy is defined in terms of an individual's capacity to solve problems in a variety of real-world contexts (OECD, 2018), most problems which students face in mathematics classrooms can be solved by a simple and straightforward application of one or a combination of the four basic arithmetic operations and are not closely related to students' experiential worlds (Depaepe et al., 2010). Moreover, teachers and students, mostly focus on the mathematical structure of a problem, ignoring the realistic aspects that could help them to make sense of its structure (Depaepe et al., 2010; Peled and Balacheff, 2011). Thus, it is necessary that teacher education addresses the need to improve teachers' understanding of the value of real-life problems to mathematics learning (Peled and Balacheff, 2011). Considering contextualized problem-based learning in professional development programs may aid teachers to use more realistic situations as a starting point for mathematical activities in the classroom.

Using teaching situations as the context for problem solving activities may also impact teacher competencies and pedagogical knowledge. Zaslavsky and Sullivan (2011) indicate that worthwhile teacher education tasks are those that are motivated by the desire to foster the orientation in prospective teachers to the study of practice. Case-based teaching can be used for creating meaningful settings for teacher learning (Putnam and Borko, 2000). This approach allows enacting tasks which are idiosyncratic to teacher education and explores the richness and complexity of genuine pedagogical problems (Putnam and Borko, 2000).

2.2. Virtual learning and Mathematics teacher education

The idea of virtual learning has been underlined in the mathematics learning literature and recently has been translated into mathematics teacher training (Borba and Llinares, 2012; Goos et al., 2020; Martínez et al., 2020). In this shift, constructivism helps educators design virtual courses and learning environments, in which learners build their own knowledge. Constructivists believe that learners construct knowledge (rather than acquiring it) individually through their interactions with the environment (including other learners) from their authentic experience, mental structures, and beliefs, which are themselves mediated by the prior knowledge (Ernest, 1996; Simon, 1995; Thompson 2014).

Before the widespread use of the Internet, mathematics knowledge belonged to teachers and textbooks, and mathematics teaching happened in formal classrooms or teacher-centered settings with a mandated curriculum (Borba et al., 2012). After the availability of the Internet and the use of new technologies, the perspective on learning has changed. Learning can be happening face to face or virtually, synchronous or asynchronous, in a classroom or through the Internet (as e-learning), especially in a large and distributed community.

Preprint

With the use of the Internet or a shift to e-learning in mathematics education, three fundamental foci within mathematics education can undergo a radical change; mathematics knowledge, teaching and the context of classrooms (Borba et al., 2012). In fact, thoughtful replacement of face-to-face education with online learning includes three main features: a fundamental change in course design to optimize learners' interaction with the learning environment as well as with other learners; the ability to restructure and replace traditional class contact hours with the flexibility to choose the learning time; the flexibility to choose different learning activities according to the needs of learners, including the content of educational materials in various forms of documents, videos, animations, simulations.

Research showed that teachers often resist participating in intensive long-term PD programs, especially when the commitment involves traveling from their school to another location (McConnel et al., 2013). Moreover, a lesson we learned from the Covid-19 pandemic and the closure of schools, is the importance of online learning. Therefore, given the key features of e-learning, a PD program in a virtual modality can be a solution for continuing teacher education. It brings several benefits for teachers such as: reducing the attendance time of in-service teachers in face-to-face or synchronous workshops, which is an inconvenient factor for participants due to conflict with their work schedule (Eroğlu and Kaya, 2021), arranging courses for a longer period which is recommended to increase the efficiency of a PD program (Garet et al., 2001), and offering interactive online learning plans, which help teachers to have rich and flexible knowledge about the subject they teach (Borko, 2004).

In line with the importance of online learning in teacher education, Borba and Llinares (2012) suggest that online activities can transform teacher collaboration and cause individual development. However, designing a virtual PD course needs specific attention.

Goos et al. (2020) described the effectiveness of a blended learning PD program to address the lack of mathematics content knowledge and mathematics pedagogical competencies among in-service mathematics teachers. In Martínez et al. (2020), the design of the PD program aimed at improving teachers' MKT is discussed in detail. This work presents how to use different instructional strategies in e-learning modalities to develop teachers' MKT. Although there are significant benefits for teachers, constructing virtual learning environments to improve teachers' MKT is a difficult task for PD developers. To help developers, it is vital to present samples of PDs with a full description of the principles of design and its materialization.

3. The case of Suma y Sigue: an e-learning PD program

The Suma y Sigue program was developed at the Center for Mathematical Modeling (CMM), a research institution of the University of Chile. Its development included the joint work of several teams, including content development, graphic design, and programming. The content team was made up of teachers, mathematicians, and experts

Preprint

in mathematics education. This composition was essential to focus the contents of the course on the professional knowledge involved in teaching mathematics and the design of learning activities that were relevant to that effort. The creation of a single course took around 8 months. The development and implementation of the program was supported by collaboration agreements between the CMM and the Chilean Ministry of Education (MINEDUC).

3.1. Design Principles

The instructional model of "Suma y Sigue" aims to improve teachers' MKT, through contextualized learning activities focused on the deep analysis of elementary mathematics. For this, three fundamental principles are established, which are discussed in more detail in Martínez et al. (2020). The first principle is based on the constructivist perspective of learning, that is, it is considered that knowledge is not passively received by learners but actively constructed by them using their previous knowledge (Thompson, 2014; Ernest, 1996). For this reason, in the program, the contents emerge as the teachers solve mathematical and didactic problems that force them to use their previous knowledge and restructure it to find a solution. A second principle considers the importance of using contextualized problems in realistic situations. This materializes through the use of mathematical problems placed in contextualized situations that allow solvers to make sense of formal mathematics (Freudenthal, 2012; Gravemeijer and Doorman, 1999), and also, with the use of didactic problems that put teachers in plausible classroom situations (Putnam and Borko, 2000; Zaslavsky and Sullivan, 2011). Finally, the third principle is articulated around the MKT model (Ball et al. 2005, 2008), which proposes that the teaching of mathematics requires specific knowledge, which can be distinguished, characterized and developed. Some of the courses focus heavily on the subject knowledge components of the MKT model, that is, on common and specialized knowledge, while other courses incorporate the pedagogical knowledge components to a greater extent.

3.2. Materializing the principles in design

The program's online activities are built around a mathematical story which allows articulating the different types of tasks necessary to develop MKT. As the story unfolds, new conflicts arise, which makes it possible to modify the didactic variables of a task or change the type of task addressed. For instance, a discussion among the characters of the story can help to produce a conflict around a mathematical idea, triggering a questioning process that leads users to unpack mathematical concepts. Technology allows the design of dynamic learning scenarios in which tasks and content are progressively displayed, facilitating not only addressing different types of knowledge in an integrated way but also a scaffolded learning process.

Preprint

3.3. Structure of the program

The "Suma y Sigue" program offers thirteen courses dedicated to teachers who teach mathematics at different primary and secondary school levels. Each course addresses topics specific to a curricular domain (Numbers and Operations, Geometry and Measurement, Algebra and Patterns, Probability, and Data). The courses are organized in two modules, each module is made up of two or three asynchronous virtual workshops followed by a synchronous workshop (see Fig. 1). At the end of each module, participants answer an online test whose items address the mathematical and pedagogical content of the course. The courses last approximately 34 hours distributed over 10 weeks. About 25 hours are of asynchronous work and 9 synchronous work.

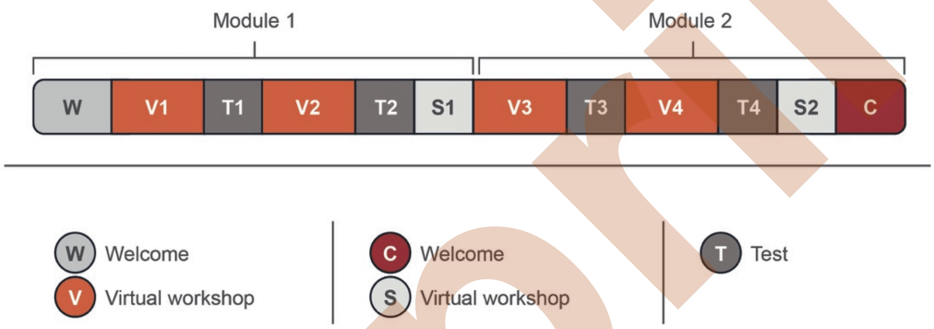

Fig. 1. Course structure

3.3.1. Virtual workshops

The learning unit of a workshop is known as a virtual activity. Each activity has a specific objective defined according to the mathematical content addressed. A contextual situation is proposed for each activity, acting as a frame for arranging different tasks. The resolution of these tasks requires building specific mathematical knowledge to address the situations. Both the context and the learning tasks are dynamically integrated, in the sense that they become more complex throughout the activity, allowing participants to consider a variety of aspects of MKT. Each workshop is made up of three to six virtual activities and ends with a systematization section that summarizes all the contents seen in the activities.

The virtual activities are designed to trigger autonomous learning which requires involving and keeping the teacher focused on the construction of knowledge. Each activity has three phases: engagement, construction, and systematization. In the engagement phase, a contextualized situation is introduced, usually in daily life or a classroom setting, using a variety of learning resources such as cartoon stories, dialogues between characters, and animations. Depending on the complexity of the knowledge involved, the construction phase is structured in an activation-

Preprint

institutionalization-practice cycle, which is repeated as new MKT elements are incorporated. In the systematization phase, the contents addressed through the activity are summarized. We refer to Martínez et al. (2020) for a more detailed description. In Section 4 we provide an example, describing the different phases of activity of the first virtual workshop of the course called "Working with multiplication and division".

3.3.2. Synchronous workshops

At the end of each module, an asynchronous workshop is held. The objective of these workshops is to discuss pedagogical issues related to the mathematical contents of the course. These workshops are organized in three stages: the first one called "Activation" occurs before the synchronous session. Teachers must reflect individually on a mathematical or classroom situation. In the second stage called "Synchronous discussion," the participants meet by zoom to discuss the situation, which is connected with a reflection on the teaching of the content. Collaboration is promoted here through small group discussions, the conclusions of which are then brought back to the full group. The last stage is called "Discussion projections" in which participants reflect on the learnings achieved in the discussion and consider new deepening questions. The participants must analyze a document that systematizes the main ideas of the synchronous virtual discussion and introduces new questions. The discussion happens in a virtual forum moderated by the course tutor.

3.4. Assessment

The evaluation process of the "Suma y Sigue" courses fulfill two purposes: on the one hand, the learning of the participants is qualified to certify the approval of the course (summative assessment) and, on the other hand, feedback is given to the participants on their performance (formative assessment). As for the summative evaluation, teachers take 4 tests throughout the course. These tests are applied at the end of each workshop on the fixed dates scheduled before. Each test has a total of 7 items in multiple-choice, true and false, and open response format. These items assess the mathematical and didactic content reviewed in each workshop. Regarding formative evaluation, the platform offers constant feedback as teachers progress through the course. Depending on the complexity of a question, after the participant submits their answer, a feedback narrative capsule (Narciss, 2008), named "Exploring a possible solution", is displayed to her/him containing feedback and justification of correct and incorrect answers. This allows participants to reflect about the knowledge required to answer the question. These feedback capsules have different levels of complexity involving written explanations, pictorial representations and animations.

The course approval criteria include achieving at least 60% correct answers in the tests, having completed at least 80% of the platform activities and attending the two synchronous workshops.

Preprint

4. A Course Description

The "Working with Multiplication and Division" course is aimed at teachers who teach mathematics from 2nd to 4th grade. This course has four asynchronous virtual workshops. The first one, "Multiplicative situations", is devoted to analyzing different types of problems and representations associated with multiplication and division, providing different interpretations of these operations with whole numbers, addressing the transit between them, and establishing some basic properties. In the second virtual workshop, "Multiplication", the justification of properties of multiplication will be addressed using generic examples. Also, the construction of the multiplication tables and different calculation strategies, including the standard algorithm, will be justified in this workshop. For the third virtual workshop, "Division", various contextualized problems are proposed to address the properties of division in connection with calculation strategies. In the last virtual workshop, called "problem solving", a series of stories showing students engaging in problem solving activity are proposed to analyze the relevant aspect of this type of activity in the classroom, such as the use of representations, understanding different solutions strategies, recognizing errors as an opportunity to enrich learning, and correctly interpreting mathematical results.

The two content-focused synchronous workshops are taught after teachers have finished the virtual workshops 2 and 4 respectively. In the first one, which will be described in more depth below, the teaching of the multiplicative situations of grouping and combination type, their models and representations are discussed. The second synchronous workshop is focused on the teaching of different division strategies. For that, a video clip showing a classroom situation in which students are using different strategies for solving a multiplication task.

4.1. The "Multiplicative situations" virtual workshop

In this workshop, three types of multiplicative situations are introduced to provide meaning and connections between multiplication and division. For that, a story involving the discussions of two characters that work at a restaurant is constructed. This context is useful to motivate teachers to think about different types of situations, such as the distribution of pastries among diners, the combinations of dishes for a dinner, and the assignment of waiters to rooms. This story allows teachers to build connections between possible interpretations and representations of multiplication/division, understanding the role of the numbers involved accordingly. This workshop aims to help teachers develop specialized mathematical knowledge that they need to teach these operations with students from 2nd to 4th grade, according to the Chilean curriculum.

Four activities comprise the "Multiplicative situations" virtual workshop: "Grouping and arranging pastries" which introduces multiplications as a solution to grouping and array-type problems; "Choosing the menu" in which multiplication is the

Preprint

solution to combination problems; “Sharing in the restaurant” where division appears as a solution to grouping problems associated with the question “How many are there in each group?”; “Setting groups for events”, that connects division with grouping problems associated to the question “How many groups are there?”. We describe the first activity in more detail showing its different construction cycles.

4.1.1. Grouping and arranging pastries

This activity starts by introducing Anaís, a cook, who drew the following diagram to represent the number of pastries that she should bake for each table, as shown in Fig. 2 below:

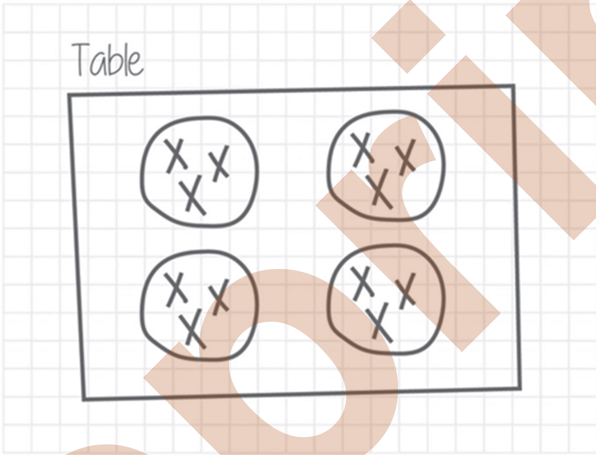

Fig. 2. Pastries’ arrangement

In the first question, teachers are asked to recognize how Anaís is organizing the pastries, distinguishing the number of groups and the number of elements of each group, as well as the addition that corresponds to this organization (Fig. 3).

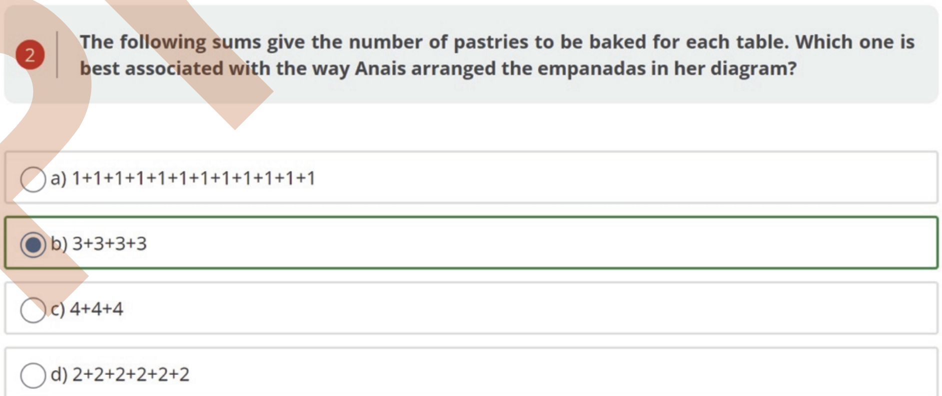

Fig. 3. Question example

Preprint

After a teacher submits her/his answer a narrative feedback capsule is displayed showing the transit between both representations (pictorial and symbolic), and justification to understand the reason behind representations and the transit (Fig. 4).

The connection between this situation and the multiplication of 4 × 3 is established in a content capsule that defines the multiplication of a × b as an iterated addition (Fig. 5), ending the "engagement phase" of the activity.

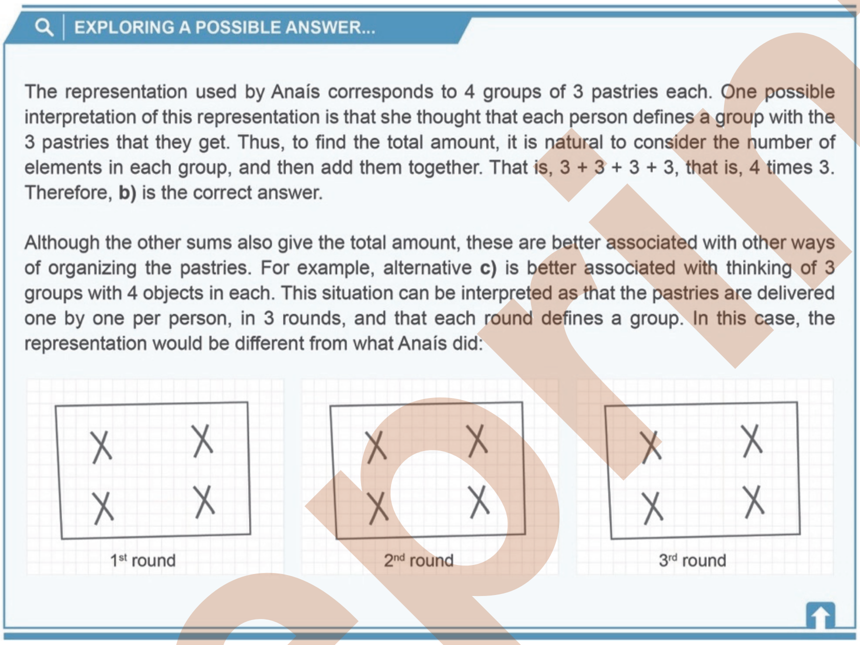

Fig. 4. An example of a feedback capsule

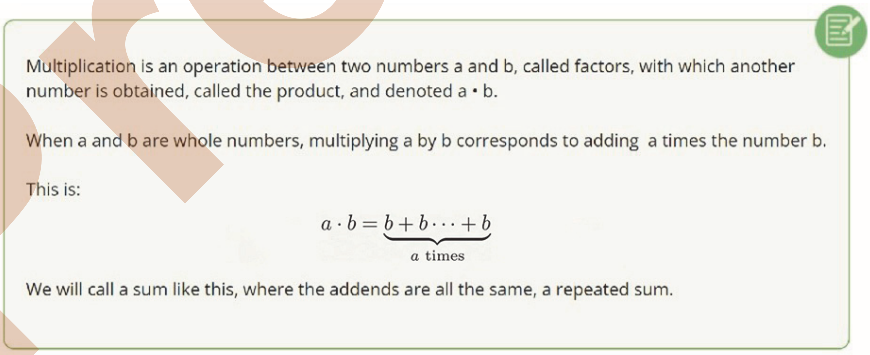

Fig. 5. A multiplication definition

The first "construction phase" starts with a question, in which teachers must decide whether a situation is described by 2 × 5 or 5 × 2 according to the definition given

Preprint

above. Then, a question regarding plates of pastries with different types of elements is introduced, to show that even though the number of pastries is 12, it is not connected to a multiplication. After deducting that having groups with an equal number of elements is necessary to relate grouping problems with multiplication, the connection is established in a content capsule (Fig. 6).

The product $a \times b$ between two whole numbers a and b can be interpreted as the number of objects in a groups, each with b elements.

Problems involving groups with the same number of objects each will be called **grouping problems**. As we saw, multiplication allows us to obtain the total number of objects when the number of groups and the number of elements in each of them are known.

Fig. 6. A content capsule that addresses grouping problems and multiplication

The next construction phase starts with Juan Pablo, a cook, who has placed some pastries in a tray as follows (Fig.7). A conversation between Anaís and Juan Pablo unfolds (Fig. 8).

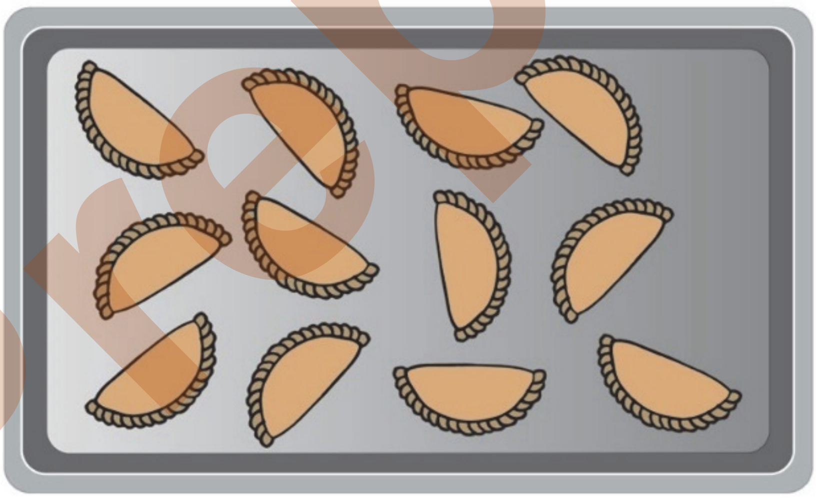

Fig. 7. Pastries in a tray

Teachers are asked to recognize the Juan Pablo's counting of pastries with the multiplication, and then to connect the grouping problem with the array as shown in the dropdown question below (Fig. 9):

Preprint

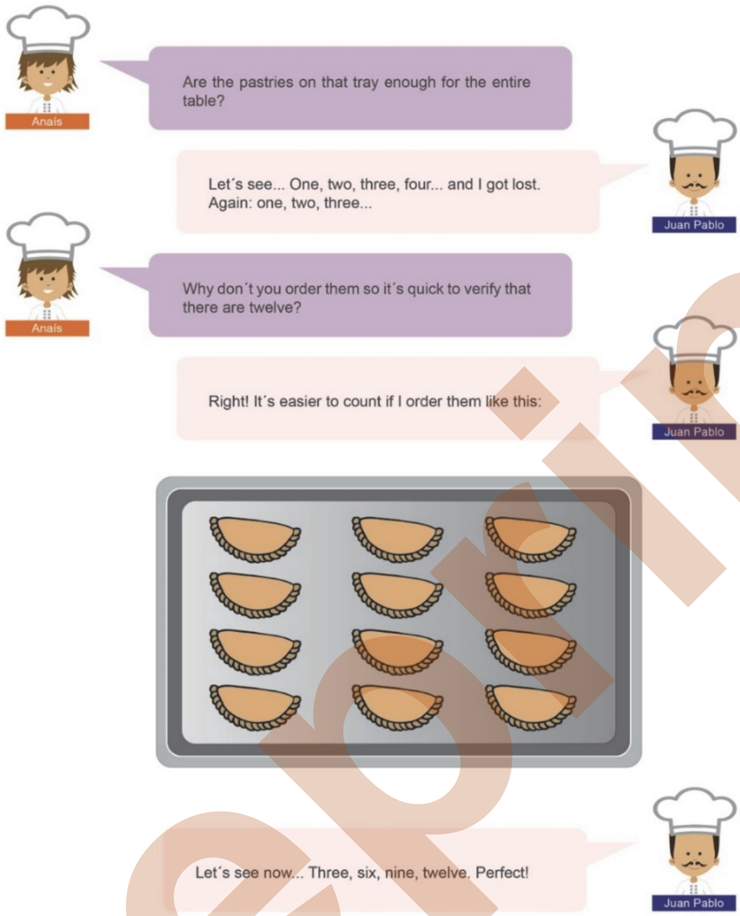

Fig. 8. A dialogue leading to an array of pastries

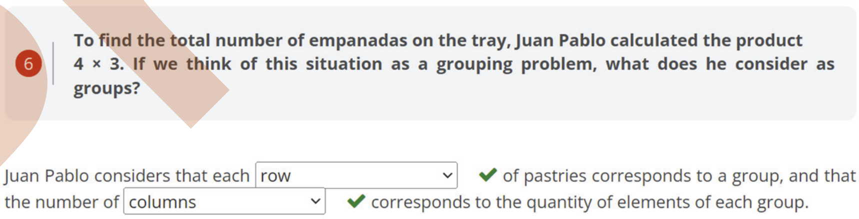

Fig. 9. A question connecting a grouping problem with an array representation

After this, some characteristics of array problems are addressed through questioning, so that teachers deduce that, given groups with the same number of elements, it is always possible to organize their elements in an array, but that not all array diagrams correspond to grouping problems. Thus, it is established that under

Preprint

certain conditions, both types of problems, grouping and array provide two possible representations for one scenario. This conclusion ends the second construction cycle.

The last construction cycle of this activity is focused on the deduction of the commutative property using an array. It starts with Juan Pablo assembling the pastries and placing them on a table, as shown in Fig. 10:

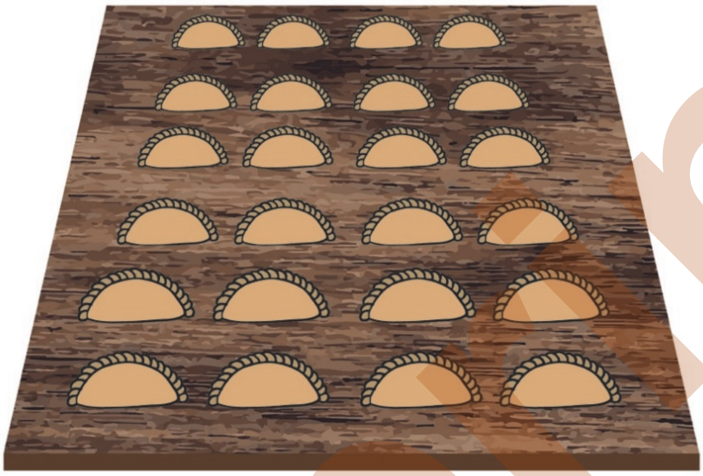

Fig. 10. An array of pastries

Then, a conversation between Juan Pablo and Anaís follows:

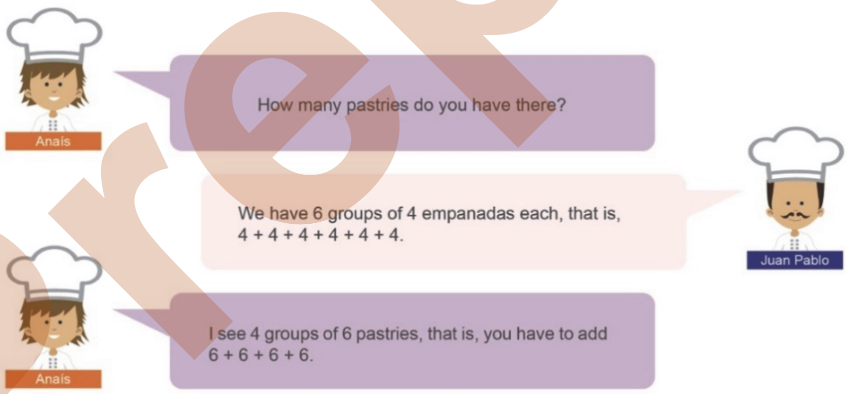

Fig. 11. Two iterative addition expressions for the number of pastries.

This dialogue presents two different expressions to compute the number of pastries. Teachers are asked to recognize them and connect them with the array representation, showing that the two different iterative additions are indeed equal. This cycle ends with a discussion regarding the use of generic examples to justify properties, in this case, the commutative property of multiplication.

Preprint

4.2. Description of a synchronous workshop

The first synchronous workshop of this course delves into didactical aspects relevant to teaching of multiplicative situations, and it is held at the end of the second virtual workshop. In the first stage “activation”, an activity involving the analysis of a classroom video is proposed, in which 4th grade students solve a problem. Teachers are asked to recognize the type of problem proposed, the role of the quantities involved, and then to describe and interpret the different answers given by the students to the problem proposed. In the main activity of the “synchronous discussion” stage the problem shown in Fig. 12 is introduced for discussion among participants:

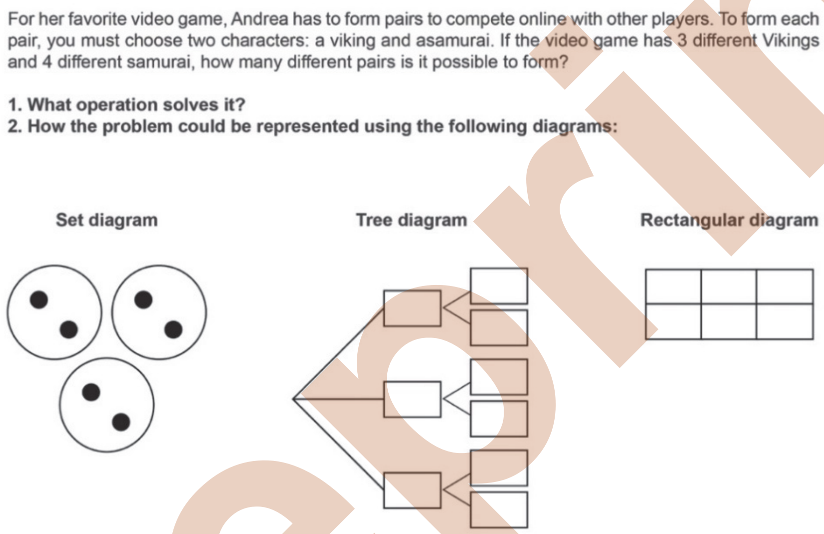

Fig. 12. A problem for discussion

In the group discussion that ensues, teachers are expected to recognize that there are representations that are more suitable than others to model a particular situation, and that the representations themselves are helpful to understand why the operation that models the situation is a multiplication. Later, teachers are asked if the video game problem above can be modeled through with the phrase in Fig. 13 below, and to recognize differences between the grouping and combination situations.

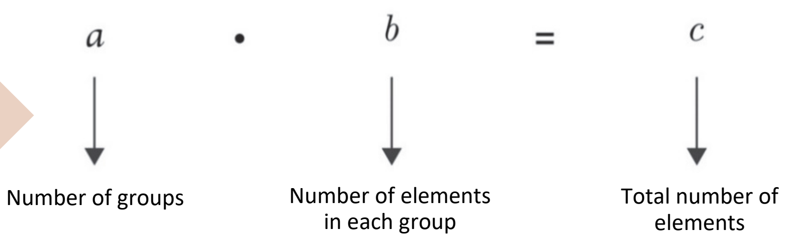

Fig. 13. Symbolic representation

Preprint

In the last stage "discussion projections" participants are presented with a document in which a classification of multiplicative situations is proposed. Then, starting with a set of problems, they are asked to propose a teaching sequence for them, justifying the reasons that support such progression.

5. Methodology

A quasi-experimental design with pre- and post-tests was adopted, applying two analysis techniques to describe how the teachers participating in the course can be distinguished or classified based on their MKT before and after participating in the course.

5.1. Participants

The study involved 124 primary school teachers from different regions of Chile, who enrolled in the program voluntarily. However, only 91 who answered all the items of the instrument in the pre and post-test were considered in the analysis section. This is because of the limitation of the statistical analysis technique (the latent class analysis or LCA) that applies here. It is suggested to eliminate those teachers who do not participate or answer all items in order to reduce the classification bias (Rose et al., 2017). The LCA as a multivariate technique classifies teachers according to the latent construct arranged in the observed variables (teachers' responses as their selection of all distractors or correct responses) obtained through the multiple-choice items in the instrument (Vermunt and Magidson, 2002). This assumes the evocation of the intended knowledge when interpreting the stimuli arranged in the items.

5.2. The Instrument

The instrument used in this study included 10 items of multiple-choice, true and false and open-ended responses. It was requested to be answered by participating teachers before and after the course at their convenient time. The analysis was done only considering the 5 multiple-choice items, due to their psychometric properties, which presented good indices of discrimination, difficulty, and reliability (Mean_discr = 0.2; Mean_dif = 0.5, $\alpha = .68$).

Tab. 1. Domains of mathematical knowledge to teach that evaluates each item

Item number	Assessed domain
1	Specialized mathematical knowledge (posing multiplicative problems).
2	Common mathematical knowledge (solving multiplicative problems).
3	Specialized mathematical knowledge (identifying properties that justify a calculation procedure).
4	Specialized mathematical knowledge (identifying properties that justify a calculation procedure).
5	Common mathematical knowledge (estimation strategies).

Preprint

The instrument was designed to evaluate the teachers' knowledge of teaching multiplication and division, including common knowledge of the content and specialized knowledge of the content. Tab. 1 presents the details of each specific domain of knowledge measured by the instrument. Fig. 14 shows two items (1 and 5) used to explain the learning results of participating teachers in this course.

Ignacio wants to give his students a problem involving a grouping situation in which he asks the number of groups that it is possible to form.

Which of the following problems corresponds to the type that Ignacio is looking for?

1 Which of the following problems corresponds to the type that Ignacio is looking for?

- a) Alberto must place 15 brochures in each folder. If he has 120 brochures, how many folders can he complete?
- b) Juan must put the same number of brochures in each of the 12 folders he has. If he only has 100 booklets, what is the maximum number of booklets he can put in each folder?
- c) María has 13 folders and wants to put 10 brochures in each one. How many brochures does she need to fill all 13 folders?
- d) Esther distributes 15 brochures in 3 folders. How many brochures are in each?

A stadium, which has a total capacity of 102,963 spectators, is divided into 8 areas with equal capacity. For an upcoming match, only one area of the stadium has been allocated for the visiting team's spectators. Juan, a fan of the visiting team, wants to know the approximate capacity of each zone.

5 Which of the following strategies gives a better approximation for the capacity of each zone?

- a) 812,000 = 96,000 and 813,000 = 104,000, 96,000 < 102,963 < 104,000. Therefore, the capacity of each area is between 12,000 and 13,000 spectators.
- b) 812,000 = 96,000 and 813,000 = 104,000, 96,000 < 102,963 < 104,000. Since 102,963 is further from 104,000 than from 96,000, the capacity of each zone can be estimated at a little over 12,000 spectators.
- c) 812,000 = 96,000 and 813,000 = 104,000, 96,000 < 102,963 < 104,000. Since 102,963 is closer to 104,000 than 96,000, the capacity of each zone can be estimated at just under 13,000 spectators.
- d) Assuming there were 10 zones, the capacity of each zone would be approximately 10,296 spectators. Since 8 is less than 10, dividing by 8 increases the quotient. Therefore, the capacity of each zone is a little more than 10,296 spectators.

Fig. 14. Items 1 and 5

Preprint

5.3. Data analysis method

As we mentioned earlier, two methods of analysis are conducted. First, due to the non-normal distribution of the sample, McNemar's proportion comparison test (McCrum-Gardner, 2008) was used to assess the progress of the participants' MKT in each item of the instrument. Then, the LCA is used to describe how the teachers participating in the course are classified based on the probabilities of answering the items which describe their MKT. Thus, the MKT of the participants is modeled and characterized by using variable indicators distributed in the items of the instrument through this technique. Thus, latent classes were determined separately for the pre-test and then for the post-test. Consequently, the LCA is used to explain the movement of the teachers or change the class membership between these latent classes associated with the pre-test and post-test. This change describes the modification of MKT among participants.

The fit of the classification or psychometric properties was examined using the Akaike Information Criterion (AIC) (Akaike, 1998) and Bayesian Information Criterion (BIC) (Schwarz, 1978) as suggested by the literature. As a reference, the smaller the AIC and BIC values, the better the model fit (Vrieze, 2012).

6. Results

6.1. Difference between pre- and post-test

As shown in Tab. 2, teachers show significant progress in their achievement in solving items 1 and 4 from the pre- to the post-test. Item 1 involves specialized knowledge related to the identification of multiplicative grouping problems and Item 4 requires the identification of properties underlying a multiplicative calculus procedure (see Tab. 1). Significant progress in teachers' performance is also observed in Item 5, which involves some common mathematical knowledge for quantity estimation.

Tab. 2. Percentage of correct answers per item

Item number	Pre-test (%)	Post-test (%)	Δ_{diff} (%)	χ^2 (p-value)
1	45 (49%)	74 (80%)	29 (32%)	27.034 (0.000)
2	42 (46%)	47 (51%)	5 (5%)	0.761 (0.382)
3	27 (29%)	32 (35%)	5 (5%)	0.516 (0.472)
4	38 (41%)	54 (59%)	16 (17%)	5.113 (0.023)
5	46 (50%)	63 (68%)	17 (18%)	6.918 (0.008)

Note: pre= number of correct answers in the pre-test; post= number of correct answers in the post-test; the absolute difference between the correct answers of the post and the pretest; = McNemar's chi-square.

Preprint

6.2. *The LCA: Classification of the participants based on their MKT*

6.2.1. Before the course

From the LCA applied to the pretest responses, teachers are classified into two classes according to their performance in the MKT pre-test (AIC = 1138.48; BIC = 1216.32; G^2 = 298.14; χ^2 = 1168.59).

Class 1 includes about 22% of participating teachers (20 teachers), while the rest of teachers (71 teachers) belong to Class 2. As can be seen in Tab. 3, what distinguishes the classes is the performance on Items 2, 3 and 5. Items 2 and 5 require common knowledge of multiplication and division for solving problems. Item 3 requires identifying the properties that underlie a computational procedure.

Tab.3. Probabilities of answering correctly an item

Item number	Pre-test		Post-test	
	Class 1 (.22)	Class 2 (.78)	Class 1 (.11)	Class 2 (.89)
1	.52	.49	.27	.88
2	.30	.50	.13	.57
3	.16	.33	.00	.40
4	.45	.41	.58	.60
5	.00	.63	.27	.74

6.2.2. After the course

The classification of participants based on their performance after terminating the course, confirms that the teachers are still classified into two classes or groups according to their MKT post-test. The data shows a goodness fit index for the classification (AIC=928.26; BIC=1006.09; G^2=174.91; χ^2=499.17).

Tab. 3 shows that the teachers of Class 2 have a significantly better performance than those of Class 1 in Items 1, 2, 3 and 5. Moreover, 89% of the teachers (81 teachers) belong to this class (Class 2) with the best performance. In Item 4, which involves the identification of properties that underlie a mental calculation strategy, both groups show a similar performance, having probabilities of around 60% of answering the item correctly.

6.3. *Teacher movement between classes according to their performance*

Tab. 4 illustrates the movement of teachers according to their performance in the evaluations carried out before and after taking the course (pre-test and post-test). As we discussed earlier, Class 1 included those teachers who show a lower performance compared to teachers in Class 2 during the pre-test. The analysis shows that 90% (18 out of 20) of the teachers in this class (Class 1), move to Class 2 according to the post-test. Class 2 in the post-test included the teachers with better achievement in post-test.

Preprint

Only 11% (8 out of 71) of the teachers in Class 2 from the pre-test, move to Class 1 from the post-test.

Tab. 4. Number of teachers in each class

		Post-test		Total
		Class 1	Class 2	
Pre-test	Class 1	2	18	20
	Class 2	8	63	71
Total		10	81	91

When comparing class assignments by the LCA before and after the course using the non-parametric Mann-Whitney test, a significant difference between both moments is observed (W = 3685.5, p = .046).

7. Discussion and Conclusion

This article describes the instructional design of an e-learning PD program with a high degree of autonomous asynchronous work that aims to improve teachers' MKT. The instructional design of the learning activities is shown for a particular course, highlighting the instructional strategies and program's features that promote active learning focused in the different domains of MKT. In addition, for this course, a pre- and post-test design was used to see how the course brings change on teachers' MKT.

Developing an e-learning program that includes a high level of autonomous work has benefits for teachers, such as providing flexibility in scheduling. Also, the program's virtual asynchronous activities are less demanding of high-speed broadband internet than synchronous activities that rely on software like Zoom or Meet. Indeed, accessibility to a high-speed internet connection is a limitation for most people in Chile (Sepulveda-Escobar and Morrison, 2020), particularly for teachers living in rural areas. So, this type of instructional design that privileges autonomous work on a platform that facilitates interaction between the participant and the content, can be a successful alternative to provide access to PD with territorial equity. Even though designing the asynchronous virtual activities of the program was costly and time consuming, the dissemination of the program was more robust and less dependent on highly qualified instructors, which is critical to maintaining the quality of a large-scale PD program (Carney et al., 2019; Roesken-Winter et al., 2015).

We highlight the characteristics of the course activities from two different perspectives. By focusing on the relationship between activities and the construction of MKT we can point out how activities design materializes the principles declared by Suzuka et al. (2009). The course activities are built on the basis of construction cycles helping to progressively unpack elementary mathematical knowledge required to analyze a situation. For example, in the activity of "Grouping and arranging pastries" the concept of multiplication from grouping situations is gradually built and developed.

Preprint

In the first construction cycle, a type of pictorial representation is connected with a symbolic representation (the numerical phrase) emphasizing in both the meaning of numbers involved. In the second cycle, the situation is extended by presenting a second pictorial representation, which allows reinterpretation of the factors of a multiplication, thus extending the meaning of this operation. In the third cycle, the commutative property is argued through a generic example and the use of an adequate representation. On the other hand, the use of dialogues allows proposing questions that provoke a stumble due to a superficial understanding of an idea, in this case, they can point to considering groups with different numbers of elements in a grouping problem, or representing combination situations with set diagrams.

Moreover, we can characterize the nature of the course activities by considering the strategies that encourage inquiry among learners (Lim, 2001). Several of these strategies are evidenced in the course description: (1) Designing problems from a simplest version (required less cognition) to the most complicated version of the problem which mainly required a high level of cognition; (2) Providing a different representation of one scenario, that can help learners to have a better understanding of the problem; (3) Using different technological tools with colors, animations, illustrations for providing a better visualization for learners that can also capture the attention; (4) Making time for reflection by providing feedback constantly during an activity, which leads learners to reach a correct reasoning and answer; (5) Designing a task with a problematic scenario, which starts with what learners already know and continues to a situation where they become curious about knowing a new concept.

The empirical finding regarding learning outcomes of the course "Working with multiplication and division" showed progress in teachers' common and specialized knowledge for teaching according to the MKT model (Ball et al., 2008). Knowing about the movement of teachers between classes from the beginning to the end of the course, by using the LCA technique, provides evidence about the change on their MKT. In the case of our study, a majority of teachers (89%) classified in a class characterized by having a good performance in most of the items at the end of the course. This is in contrast to the initial situation of the participants, in which the majority of teachers had at most 0.55 probabilities of correctly answering just 2 of the 5 items. This result is, in fact, evidence of the change that the course promises to bring to participants' MKT (Martínez et al., 2020).

Regarding the relationship between the characteristics of the course and the progress observed in the items, we consider that the instructional design and characteristics mentioned above contributed to the teachers' progress in their knowledge. For instance, the activity described in Section 4.1 and the participants' achievements for item 1 shown in Tab. 2. Although this study did not aim to find a causal relationship between instructional design and learning outcomes, research shows that a curriculum-based PD leads to greater effectiveness in teachers' MKT (Copur-Gencturk et al., 2019).

Preprint

In conclusion, we would like to emphasize the importance of discussing the design of e-learning-based activities in the field of math teacher programs. Indeed, the type of activities and the instructional design that are beneficial to develop MKT in teachers have not been studied well. In addition, it is rare to find works that detail the type of activities carried out so that other PD developers can learn from these experiences. In this sense, the present work proposes to advance in this subject, showing principles of design, its materialization by using different instructional strategies that allow focusing the course on the development of MKT. Although in this article we report teacher learning according to a pre- and post-test, we do not have a good understanding of how teachers interact with the content or how different elements may affect the learning process. Future studies are needed to establish relationships between different features of course instructional design and teachers' learning on specific MKT domains.

Acknowledgments

This research was supported by FB210005 and FB0003 Basal funds for Centers of Excellence ANID-Chile, UNESCO Chair "Preparing teachers to teach math in the 21st century", collaboration agreements CMM-MINEDUC, Projects FONDECYT 3201094 and FONDECYT 3220465.

References

H. Akaike (1998). A Bayesian analysis of the minimum AIC procedure. In *Selected Papers of Hirotugu Akaike*, pp. 275–280. New York, NY: Springer.

D. L. Ball (2003). What mathematical knowledge is needed for teaching mathematics. Secretary's Summit on Mathematics. *US Department of Education*, Vol. 3, No. 3, pp. 30–40.

D. L. Ball and H. Bass (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (ed.), *Multiple Perspectives on the Teaching and Learning of Mathematics*, pp. 83–104. Westport, CT: Ablex.

D. Ball, M. Thames, and G. Phelps (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, Vol. 59, No. 5, pp. 389–407.

Ball, D. L., Hill, H. C., and Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, Vol. 29, No. 1, pp. 14–17, 20–22, 43–46.

M. Barber and M. Mourshed (2008). Cómo hicieron los sistemas educativos con mejor desempeño del mundo para alcanzar sus objetivos [*How the World'S Best-Performing Education Systems Achieved Their Goals*], No. 371, e. B234c. Santiago, CL: PREAL.

M. D. C. Borba and S. Llinares (2012). Online mathematics teacher education: overview of an emergent field of research. *ZDM — Mathematics Education*, Vol. 44, No. 6, pp. 697–704.

M. C. Borba, P. Clarkson, and G. Gadanidis (2012). Learning with the use of the Internet. In *Third International Handbook of Mathematics Education*, pp. 691–720. New York, NY: Springer.

H. Borko (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, Vol. 33, No. 8, pp. 3–15.

Preprint

M. P. Campbell and H. S. Lee (2017). Examining secondary mathematics teachers' opportunities to develop mathematically in professional learning communities. *School Science and Mathematics*, Vol. 117, No. 3–4, pp. 115–126.

M. B Carney, J. L. Brendefur, G. Hughes, K. Thiede, A. R. Crawford, D. Jesse, and W. S. Brandie (2019). Scaling professional development for mathematics teacher educators. *Teaching and Teacher Education*, Vol. 80, pp. 205–217.

Y. Copur-Gencturk, D. Plowmanand, and H. Bai (2019). Mathematics teachers' learning: Identifying key learning opportunities linked to teachers' knowledge growth. *American Educational Research Journal*, Vol. 56, No. 5, pp. 1590–1628.

F. Depaepe, E. De Corte, and L. Verschaffel (2010). Teachers' approaches towards word problem solving: Elaborating or restricting the problem context. *Teaching and Teacher education*, Vol. 26, No. 2, pp. 152–160.

P. Ernest (1996). Varieties of constructivism: A framework for comparison. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, and B. Greer (eds.), *Theories of Mathematical Learning*. Mahwah, NJ: Lawrence Erlbaum.

M. Erroğlu and V. D. Kaya (2021). Professional development barriers of teachers: a qualitative research: Professional development barriers of teachers. *International Journal of Curriculum and Instruction*, Vol. 13, No. 2, pp. 1896–1922.

H. Freudenthal (2012). *Mathematics as an Educational Task*. Dordrecht: Springer.

M. A. GaretPorter, L. Desimone, B. Birman, and K. S. Yoon (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, Vol. 38, No. 4, pp. 915–945.

M. Goos, J. O'Donoghue, M. Ní Ríordáin, F. Faulkner, T. Hall, and N. O'Meara (2020). Designing a national blended learning program for "out-of-field" mathematics teacher professional development. *ZDM — Mathematics Education*, Vol. 52, No. 5, pp. 893–905.

K. Gravemeijer and M. Doorman (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, Vol. 39, No. 1–3, pp. 111–129.

K. Koellner, J. Jacobsand, and H. Borko (2011). Mathematics Professional Development: Critical Features for Developing Leadership Skills and Building Teachers' Capacity. *Mathematics Teacher Education and Development*, Vol. 13, No. 1, pp. 115–136.

B. R. Lim (2001). Guidelines for designing inquiry-based learning on the web: Online professional development of educators. Doctoral dissertation, Indiana University.

L. E. Martin and T. Mulvihill (2020). Voices in Education: Professional Development Schools (PDS): in the Rear-View Mirror or Still a Promising Model? *The Teacher Educator*, Vol. 55, No. 3, pp. 239–247.

S. Martínez, F. Guíñez, R. Zamora, S. Bustos, and B. Rodríguez (2020). On the instructional model of a blended learning program for developing mathematical knowledge for teaching. *ZDM — Mathematics Education*, Vol. 52, No. 5, pp. 877–891.

T. J. McConnell, J. M. Parker, J. Eberhardt, M.J. Koehler, and M. A. Lundeberg (2013). Virtual professional learning communities: Teachers' perceptions of virtual versus face-to-face professional development. *Journal of Science Education and Technology*, Vol. 22, No. 3, pp. 267–277.

E. McCrum-Gardner (2008). Which Is the Correct Statistical Test to Use? *British Journal of Oral and Maxillofacial Surgery*, Vol. 46, No. 1, pp. 38–41.

S. Narciss (2008). Feedback strategies for interactive learning tasks. *Handbook of Research on Educational Communications and Technology*, Vol. 3, pp.125–144.

OECD (2018). PISA 2022, Mathematics framework (Draft). Retrieved from https://pisa2022maths.oecd.org/files/PISA%202022%20Mathematics%20Framework%20Draft.pdf

Preprint

I. Peled and N. Balacheff (2011). Beyond realistic considerations: Modeling conceptions and controls in task examples with simple word problems. *ZDM — Mathematics Education*, Vol. 43, No. 2, pp. 307–315.

R. Putnam and H. Borko (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Research*, Vol. 29, No. 1, pp. 4–15.

N. Rose, M. von Davier, and B. Nagengast (2017). Modeling omitted and not-reached items in IRT models. *Psychometrika*, 82(3), pp. 795–819.

B. Roesken-Winter, C. Hoyles, and S. Blömeke (2015). Evidence-based CPD: Scaling up sustainable interventions. *ZDM — Mathematics Education*, Vol. 47, No. 1, pp.1–12.

F. Saadati, E. Chandia, G. Cerda, and P. Felmer (2021). Self-efficacy, practices, and their relationships; the impact of a professional development program for mathematics teachers. *Journal of Mathematics Teacher Education*, pp.1–22.

P. Santiago, A. Fiszbein, S. G. Jaramillo, and T. Radinger (2017). OCDE Revisiones de recursos escolares: Chile 2017 [School Resource Reviews: Chile 2017]. OECD: Organisation for Economic Co-operation and Development.

G. Schwarz (1978). Estimating the dimension of a model. *The Annals of Statistics*, Vol. 6, No. 2, pp. 461–464.

P. Sepulveda-Escobar and A. Morrison (2020). Online teaching placement during the COVID-19 pandemic in Chile: challenges and opportunities, *European Journal of Teacher Education*, Vol. 43, No. 4, pp. 587–607. https://doi.org/10.1080/02619768.2020.1820981

M. A. Simon (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal For Research in Mathematics Education*, Vol. 26, No. 2, pp. 114–145.

K. Suzuka, L. Sleep, D. L. Ball, H. Bass, J. Lewis, and M. Thames (2009). Designing and using tasks to teach mathematical knowledge for teaching. In D. S. Mewborn and H. S. Lee (eds.), AMTE Monograph Series, 6. *Scholarly Practices and Inquiry in the Preparation of Mathematics Teachers*, pp. 7–24. San Diego, CA: Association of Mathematics Teacher Educators.

P. W. Thompson (2014). Constructivism in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education*, pp. 96–102. Dordrecht: Springer.

D. Toledo and D. Wittenberg (2014). Re-shaping Teacher Careers in Chile: Selected International Evidence. OECD Publishing. Retrieved from: http://www.oecd.org/chile/Re-shaping-teacher-careers-in-Chile.pdf.

J. K. Vermunt and J. Magidson (2002). Latent Class Cluster Analysis. In J. A. Hagenaars and A. L. McCutcheon (eds.), *Applied Latent Class Analysis*, pp. 89–106. Cambridge: Cambridge University Press

S. I. Vrieze (2012). Model selection and psychological theory: a discussion of the differences between the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). *Psychological Methods*, Vol. 17, No 2, pp. 228–243.

O. Zaslavsky and P. Sullivan, eds. (2011). *Constructing Knowledge for Teaching Secondary Mathematics: Tasks to Enhance Prospective and Practicing Teacher Learning*. Vol. 6. Dordrecht: Springer.

Preprint