Invited Lecture

Argumentation towards Educational Change in Mathematics

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ABSTRACT We refer to four general theories of argumentation that provide insights on innovative current approaches in mathematics education. Through several examples of tasks, we show the richness of argumentative practices in the learning and teaching of mathematics that have some bonds with these general theories of argumentation. We show, however, that these theories do not capture the specific processes and the complexities of argumentation in the learning and teaching of mathematics according to innovative pedagogies. We pledge for new advances in mathematics education based on design-based research that fosters deliberative, epistemological, rhetorical, and structural aspects of argumentation.

Keywords: Argumentation; Argumentative designs; Model for argumentation.

1. Introduction

Argumentation is as old as the history of civilization. Van Eemeren and Grootendorst (2016) trace its origin in the birth of democracy in Ancient Greece. Cities were ruled by kings that used their power to take advantage of their citizens. Some literate people decided to represent dispossessed citizens in tribunals against them. These literate people, later on called Sophists, developed techniques to convince judges of the rightfulness of the claims of their defendants. They first acted for political reasons — to challenge laws decreed by dictators and Gods and to defend citizens against unjust claims, but gradually found in their techniques sources of income and developed sophisticated techniques to fool judges. Argumentative techniques became tricksy. Socrates, who featured in Platonician philosophy, used argumentative techniques to oppose and defeat Sophists on their own ground and to reach eternal truths. Therefore, from its inception, argumentation was polysemic — it was a rhetorical, deceiving, and epistemological tool at the same time. Interestingly, Plato excluded mathematics from its argumentative epistemology and reserved argumentation to reach the truth in social domains. As exemplified in Meno, for Plato, mathematical proofs are reached through logical moves only.

The Platonician view of mathematics has dominated the scene for more than 2000 years. Although, as Netz (1998) noticed, Greek mathematicians used geometrical figures as rhetorical tools for convincing their audience of the correctness of their

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proofs, the apodictic presentation of mathematical knowledge suggested that any hesitation in quest of a mathematical truth uncovers the limits of the human mind. No room is left for discussion or argumentation when mathematical truths are at stake.

The polemic about the intuitionists and the formalists constitutes one of the first cracks in this epistemological view. Hilbert's formalism contrasted with Brouwer and Poincaré's intuitionism as fundamental for the foundation of mathematics. These giants in mathematics did not articulate additional details on the elaboration of mathematical knowledge. Polya's How to solve it? (1945/2004) and Lakatos' Proofs and Refutations (1976/2015) are landmarks in the epistemology of mathematics. Although there is no unanimity on their contributions among mathematicians, their contribution to the realm of education is enormous. Polya's heuristics and Lakatos' refutations point at crucial moments in mathematical activity during which the epistemological status of statements is at stake. Argumentation is in the air.

Philosophers of mathematics such as Rav (1999) bridged between these two perspectives but stressed insight and meaning of mathematical actions over formal logical structure when describing proof as a sequence of claims, where "the passage from one claim to another is based on drawing consequences on the basis of meanings or through accepted symbol manipulation, not by citing rules of predicate logic" (p. 13).

The formalism-intuitionism controversy is echoed in two pedagogical approaches to proofs in mathematics education. In traditional education, proving activities tend to be substantiated by a formalistic approach. Proving is often disconnected from conjecturing (Aaron and Herbst, 2019), a great emphasis is given to proper proof inscription (e.g., Dimmel and Herbst, 2020), proof comes without conviction and explanation (Hanna, 2000) and is presented as merely devoid of human agency (Morgan, 2016). A formalistic approach to proofs, in the educational context, implies that while the presenter's sole responsibility is to state true statements considered as proofs by expert mathematicians, the responsibility of their readers is "to convince themselves" of their correctness. Proof presenters are not expected to convince their audience; their presentation is monologic and often does not contain informal arguments like diagrams or specific numeric examples (Fukawa-Connelly et al., 2016). This formalist approach to the teaching of proofs is repeatedly criticized by contemporary thinkers in mathematics education as causing students to be excluded from the "mathematician society" and to feel that if they do not understand proofs presented, "there must be something wrong with me."

In contrast, among educators in mathematics that promote novel pedagogies, the practices of mathematicians that stress the non-formalism of mathematics are models for educational practices. These novelties are particularly salient in the domain of mathematical proofs. In pioneering efforts, Mejia-Ramos and Inglis (2009) surveyed argumentative and proving activities in mathematics education in published research journals. They relied on De-Villier's (1990) model of proof functions, which is based on sub-activities whose nature is argumentative: proof construction, proof comprehension, and proof presentation. Construction activities are divided threefold into the exploration of a problem (related to the discovery function), estimation of the truth of a conjecture (referring to the verification function), and the justification of a statement estimated to be true (related to the explanation and systemization functions). The comprehension proof activity includes understanding a given argument and evaluating an argument concerning a given set of criteria. As for proof presentation, Mejia-Ramos and Inglis differentiate between two types of proof presentations: First, to explain the argument as a claim to a given audience and convince them that this claim is true. Second, demonstrate to an expert one's understanding of the given argument. In their review of research on mathematical proofs in education, Mejia-Ramos and Inglis (2009) found that most of them focused on proof construction, a minority involved proof comprehension, and none examined proof presentation. Proof presentation, therefore, is considerably understudied.

Mejia-Ramos and Inglis' observations are not neutral. They convey a deep concern about the teaching of proofs in mathematics classrooms. Indeed, many scholars have reported a superficial preoccupation with technicalities and refraining from giving students an open space to explore through dialogue. A recent example can be found in Dimmel and Herbst's (2020) report on what they call "proof transcription", a prevalent American proof-related activity in which teachers require "mark-for-mark reproductions of written proofs that students would copy to the board from a note sheet" (p. 72). The researchers worry that this activity involves an obsession with notational details that reduces the opportunities for students to develop other mathematical communication skills and does not foster a sense of discovery and the gaining of mathematical insights (de-Villiers, 2020; Dimmel and Herbst, 2020). These concerns partly explain the decline of proof-related activities in mathematics classrooms, which also originates from the typical, non-dialogic educational strategies that do not engage students meaningfully (de Villiers, 2010; Herbst and Brach, 2006). In proving activities that do not emphasize the discovery function of proofs through dialogic processes, students are more inclined to perceive proofs as a tedious chore to satisfy the teacher instead of an exciting task to satisfy their own curiosity (Lavie et al., 2019).

In this worrying context, several researchers have invested efforts in promoting new tasks on mathematical proofs. We do not review these efforts. We refer to our own line of research (Schwarz et al., 2010), which stresses the ubiquity of argumentation in mathematical practices related to the elaboration of proofs that model the practices of mathematicians. Schwarz and colleagues have identified three different argumentative activities: (1) Enquiring — an initial probing stage that concerns conjecturing solutions. It includes preliminary actions for making sense of a problem and setting a tentative plan for the solution process. (2) Proving — activity aims to find logical consequences to turn conjectures into proofs. (3) Inscribing proofs involves translating and rearranging the proof as a chain of logical inferences in a formal way. In their model of argumentative activities in mathematics, proof plays a central role both as a process and as an artifact that is a product of the argumentative activity.

In their efforts to convey the ubiquity of argumentation in authentic proof activities, Mejia-Ramos and Inglis, as well as Schwarz and colleagues, may aspire to achieve the same pedagogical ideal. However, their use of the term argumentation is not exactly the same. Indeed, many researchers who relate to argumentation in their studies use different definitions, which may lead their results to be misinterpreted. Hence, in the next section, we will discuss four general theoretical models for argumentation. Each model emphasizes different aspects and functions of argumentation. We will show that these models provide a "grammar" for argumentative activities in mathematics.

2. Succinct Considerations about the Theories of Argumentation

Many general theories of argumentation have been developed in the last 70 years. These theories were developed by philosophers and logicians who were not acquainted with the world of education and the world of learning. As noted by Schwarz and Baker (2017), this fact suggests that a general theory of argumentation for learning is necessary, to which they contribute. However, we claim that the general theories of argumentation are relevant to argumentation in mathematics. We review the four leading theories succinctly. Two monologic theories were developed by Perlman (Perelman and Olbrechts-Tyteca, 1958/2012) in his New Rhetoric and Toulmin (1958/2003) in his The Uses of Argument, which are, respectively, discursive and structural. Both Perelman and Toulmin see argumentation as a technique for structuring discourse in order to lead the auditory to accept it; the second perspective sees it as a complex and differentiated structure of interrelated statements. Both theories are monologic. Both are highly relevant to education. For example, Perelman's New Rhetoric merges Aristotelian dialectic and persuasive discursive techniques that may help the audience (the learners) become convinced of the correctness of the argument. Toulmin's argument schemes provide a language for specifying the roles of various types of statements in argumentative discourse.

The two other general theories of argumentation are dialogical. Van Eemeren and Groothendorst (2016) have developed a pragma-dialectic model of argumentation, which is modeled as a critical discussion. This critical discussion is discursive. It is conceived as a multiparty game, with a starting position, allowable and obligatory "moves" (speech acts), and rules for deciding who won or lost. This relates to a constructivist theory of truth, according to which what is true is not correspondence with facts or states of affairs but rather what has emerged as the "winner" from a societal debate. It is also based on dialogical logic (Barth and Krabbe, 1982/2010). The theory is intended to be both descriptive and normative — deciding what a reasonable way to discuss, for which set of rules governs the dialogue game. Argumentative discussions go through several stages: confrontation, opening, argumentation, and concluding. Plantin's (2005) argumentation dialogue arises once the discourse of one person is not accepted (or is called into doubt, questioned) by another person, who then produces a counter-discourse concerning it. Argumentation dialogue is a confrontation of discourses, from which emerges a question to be debated, to which discourse and counter-discourse are justifications for the answers either "Yes" or "No".

We suggest that all four theories of argumentation help understand pedagogical novelties used to promote mathematical ideas through argumentative processes.

Interestingly, Toulmin (1958/2003) thought that his model of argumentation was applicable for many contents but excluded mathematics from the realm of application of this theory. Ironically, the Toulmin model is the predominant model used by researchers in mathematics education, probably because research in mathematics education has shown that learners generally rely on informal considerations to elaborate mathematical claims.

3. New Directions in Argumentative Activities in Mathematics Education — Theoretical Examination

This section presents several examples of activities designed to encourage argumentation in mathematics. Most of them have been implemented in Israeli schools. Our focus on this particular context does not point at provinciality but at the importance of knowing the exact circumstances that afford the deployment of argumentation. We show that these examples refer to some extent to the general models of argumentation we just reviewed. We show that this reference sheds light on the argumentative nature of these activities. However, we show that the general theories fall short in capturing some other critical aspects, such as the role of resources and the role of dialogic norms. More generally, we show the decisive role of educational design in affording various aspects of mathematical argumentation. We then stress the importance of theorizing several aspects of mathematical argumentation (epistemological, dialogical, rhetorical, structural) and show that it characterizes novelty in mathematics education.

3.1. Critical discussions in mathematical tasks

We begin our review of innovative tasks in mathematics education with the pragma-dialectic model that van Eemeren and Groothendorst (2004) developed — a model of critical discussion. It requests different reasoned arguments as a starting point of the discussion. The six-cards task (Schwarz et al., 2000) is presented in Fig. 1. Different

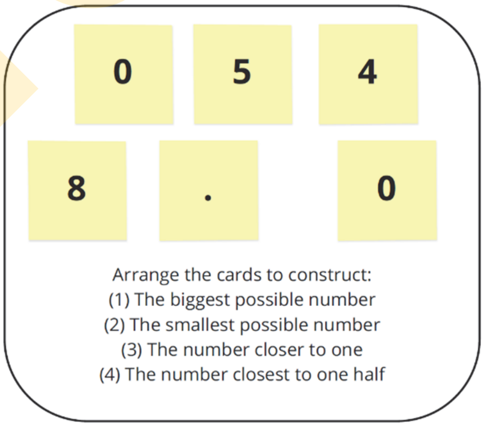


Fig. 1. The six-card task

preconceptions about decimal numbers (also called conceptual bugs) could be detected when the students solved the task alone. Examples of preconceptions are (a) identifying 4.3 and 4.03 as being the same as "0" does not count, or (b) claiming that 4.7 is less than 4.3 because dividing a whole into seven parts leaves less for one part than when dividing the whole into three parts. They were then arranged in dyads. The students were encouraged to discuss their solutions, and in the case of disagreement, to check their hypothesis with a calculator. Schwarz and colleagues identified the 'two "wrongs" make a right if they argue together' phenomenon through the six-cards task. The students criticized each other and were able to fix the bugs of their mates mutually. The six cards task led to a conceptual change with respect to the understanding of decimal numbers. This change was shown to be triggered by the deployment of argumentation in interactions among dyads. This argumentation can be referred to as a critical discussion (van Eemeren and Grootendorst, 2004). Each student had a firm preconception that led to a productive interaction among disagreeing peers. However, we doubt that van Eemeren and Groothendorst would envision such a kind of argumentation. Schwarz and colleagues showed that the discussion was nurtured by incessant hypothesis testing undertaken by the students. Explanations were convincing only when they followed the testing of conjectures with the calculator.

The two wrongs that make a right phenomenon is interesting but rare, though. More generally, critical discussions do not easily emerge in mathematical tasks. The argumentation in the case of the six cards task was productive because the different preconceptions did not relate to different levels in mathematics. Students with different preconceptions adopted wrong strategies but, at the same time, had comparable levels. This situation led them to criticize each other in a constructive way, co-elaborate on a right answer, and achieve a conceptual change through argumentation. In another experiment, Schwarz and Linchevski (2007) designed the Blocks task (see Fig. 2). Students solved the task alone. They were then arranged in dyads and were provided a balance to test their hypothesis. In this case, too, their interactions led to conceptual change (in proportional reasoning), and some examples of argumentative processes that led to this change could be detected (Schwarz and Linchevski, 2007). However, a fine-grained analysis of the talk of dyads showed that this was generally not the case (Asterhan et al., 2014). Rather, when students whose strategies were additively interacted with students whose strategies were multiplicative, and both failed to solve the Blocks task alone, the students with multiplicative strategies dominated the talk, and conceptual change happened through explanations rather than through argumentation. This experiment suggests that, in contrast with other disciplines (like civic education or history) for which argumentation among students can be easily designed, the emergence of argumentation as a critical discussion in mathematics relies on a meticulous design that ensures some symmetry between the members of the group. The two wrongs-may-make-a-right phenomenon is then correctly labeled through if they argue together since engagement in a critical discussion hardly happens in mathematics when students have different levels. We attribute this specificity of

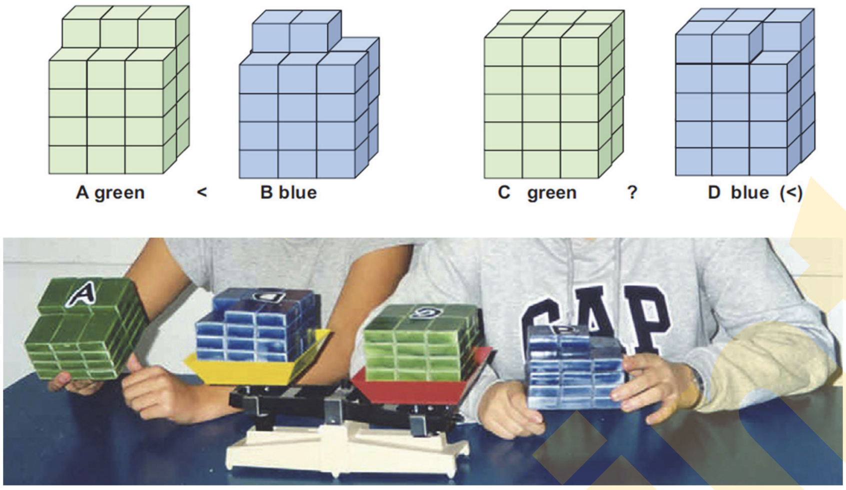


Fig. 2. Top: An example of a blocks task (the correct answer in parenthesis).
Bottom: The balance used as hypotheses checking device

argumentation in mathematics because mathematical levels confer power to stronger students upon weaker students — a fact that avoids the deployment of argumentation.

Like the six-cards task, the blocks task relies on a sophisticated learning environment designed to foster a critical discussion towards conceptual change. The design included the provision of a hypothesis testing device and carefully pre-chosen minimal guidance interventions introduced by the experimenter in case students needed help to make progress. Conceptual change indeed occurred (progress was observed in proportional reasoning three weeks after the experiment), but argumentation in a critical discussion was rare.

3.2. Examples of tasks encouraging proofs and refutations

We have stressed the importance of refutations in the structural model proposed by Toulmin. The elaboration of reasoned arguments is not the only part of this model — a fact that is often ignored by educators that refer to it. The realization of the Toulmin model in mathematical tasks is not easy. Hadas et al. (2002) used several tasks that confronted students with contradiction (or uncertainty for the very least) between initial conjectures/predictions and findings/conclusions after an investigation in a Dynamic Geometry software. They form an activity in which students are encouraged to establish an initial argument/conjecture and then to gradually abandon this argument for a more elaborated and informed one that results from their own inquiry. In fact, they refute their initial arguments and feel the necessity to prove their final argument. Fig. 3, left, shows one of the tasks Hadas and colleagues developed — the three angles task, in which the students are asked to determine the relationship between the three angles denoted in the diagram. The right part of Fig. 3 shows the map of the

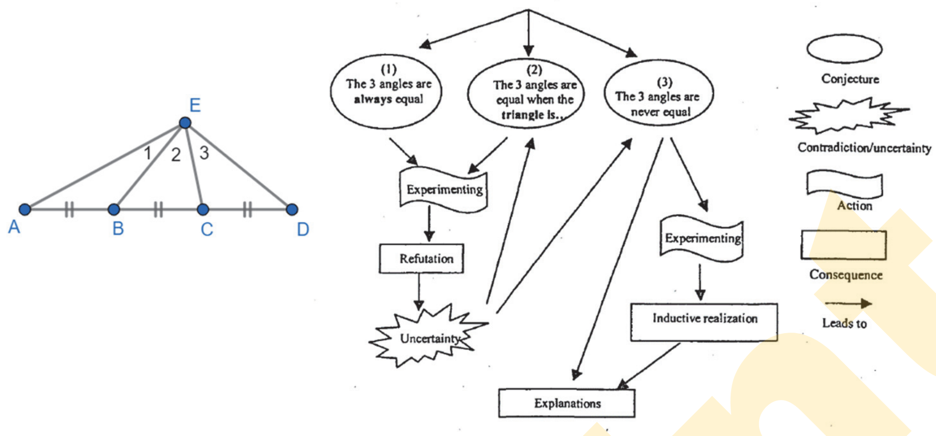


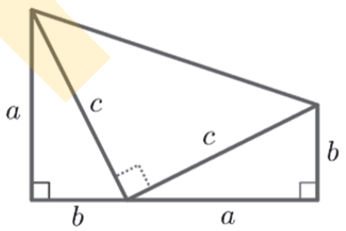
Fig. 3. The three angles task (left) and a map of its epistemological resolution (right)

epistemology of the resolution of the task (Hadas et al., 2002). The map shows that junior high schoolers are almost inevitably led to claim first that the three angles are always equal, and Dynamic Geometry manipulations refute this claim. The alternative claim that the three angles are equal in some cases is again refuted through DG manipulations, a fact which invites them to claim that the angles are never equal. As Hadas et al. (2002) showed, the students are not sure that their claim is correct since it is too surprising. They feel the necessity to prove this claim and succeed in this endeavor. Their argumentation at that stage is aimed both at elaborating an argument (a Toulmin-based argumentation) and a self-conviction (a Perelman-based argumentation). The task invites students to bring forward conjectures that are refuted through manipulations of Dynamic Geometry software. The refutations are informal – undertaken by creating displays that constitute counter-examples of a conjecture and lead students arranged in small groups to construct an argument as a mathematical proof. However, we should say that the presence of resources such as Dynamic Geometry software is also crucial. Argumentation accompanies an inquiry process mediated by technologies.

Although the design of tasks that afford the elaboration of a Toulmin argumentative structure is challenging, this design has been successfully undertaken in several instances in elementary, secondary, and higher education (e.g., Prusak et al., 2013). Toulmin's theory is often used to model the guidance of teachers in the elaboration of mathematical arguments. It is useful to describe how teachers can coordinate students' contributions to co-construct mathematical arguments. This description often reflects traditional teacher-centered guidance, but it sometimes describes a more subtle kind of guidance. For example, Conner (2022) exemplified such a description. This co-construction was also made possible through multiple resources — diagrams, video clips, micro-worlds, and inscriptions on the board, which the teacher used in this co-construction.

3.3. *Examples of tasks that encourage the identification of problems*

Proof-Without-Words (PWW; Nelsen, 1993) are mathematical texts that allude implicitly to theorems known or unknown. Fig. 4 displays a PWW that alludes to the proof of the Pythagorean Theorem. The reader of such a PWW is expected to fill in the gaps and complete the proof based on the diagram's limited information. In order to fill in the gaps and construct a proof based on the clues given by the diagram, one must identify the proposition to be proved, identify the different components of the diagram and the relations between them and the proposition, realize the dependencies and (in)equalities of different terms in the diagram and justify them based on prior knowledge, determine the order of constructions and phases of the proof and, finally, understand how and to what extent the idea shown in the concrete diagram can be generalized (Marco and Schwarz, 2019). Marco et al. (2021) suggested the gap-filling framework for analyzing students' argumentation when working collaboratively to develop a proof based on a PWW. The theory of gap-filling is a reader-oriented theory taken from literary criticism (Perry and Sternberg, 1986), whose fundamental premise is that any text contains a limited amount of information and that the reader constantly adds information to the text to construct meaning and make sense of it. The fact that students independently identify gaps in a PWW and fill these gaps based on their prior knowledge makes this activity befit the Plantin model, which emphasizes problematization as the departure point for argumentation. Even before the student presents her argumentation to peers or the teacher, she develops her mathematical argumentation in front of a diagrammatical text while interacting with it. This subterranean layer of argumentation does not seem to be better understood by Plantin's model or any other of the models we mentioned. However, it can probably be more productively studied using theories such as Herbs's (2004) conceptual framework of modes of interaction with diagrams. Marco et al. (2022) used the notion of gap-filling to redesign the PWW artifacts striving to enhance students' interactions with them and improve their proof constructions.



$$\frac{(a+b)^2}{2} = \frac{c^2}{2} + 2\,\frac{ab}{2}$$

$$a^2 + b^2 = c^2$$

Fig. 4. A PWW for the Pythagorean Theorem

The same kind of text — a PWW, provided to groups of students, may invite students to discover the problem to be inquired about and proved. This is the case in Fig. 5, which diagrammatically hints at a proof for the Vivianni Theorem. If students are not familiar with the theorem, ask groups of students to look at the picture and conjecture a mathematical claim and then prove it, further point at the Plantin model of argumentation in which the identification of the problem is crucial.

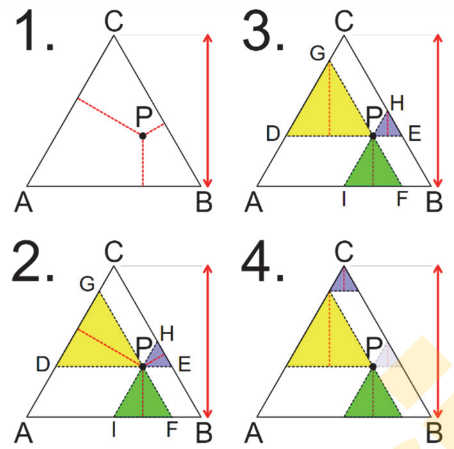


Fig. 5. Viviani's theorem — will students be able to understand both the proposition and its proof?

3.4. Examples of tasks that help teachers convince their students and students to convince each other

To show another example of the usefulness of the different theories of argumentation to describe novelty in mathematical practices, let us consider the use of the Viviani PWW in another setting: teachers may use this mathematical text as an artifact in a whole-class collective argumentation to convince the students of the correctness of a theorem. The visualization is a powerful device that teachers can exploit in explanations. In this case, the Perelman model is adequate, as it helps students adhere to what the teacher explains. The use of several PWWs by the teacher may strengthen the students' adherence to the truth of the Pythagorean Theorem (Marco et al., 2022).

Let us consider other activities labeled as "Who-Is-Right" (WIR) tasks. Fig. 6 displays a circle passing through three points and not passing through a fourth one. Two claims about the (im-)possibility that a circle would (not) pass through four points

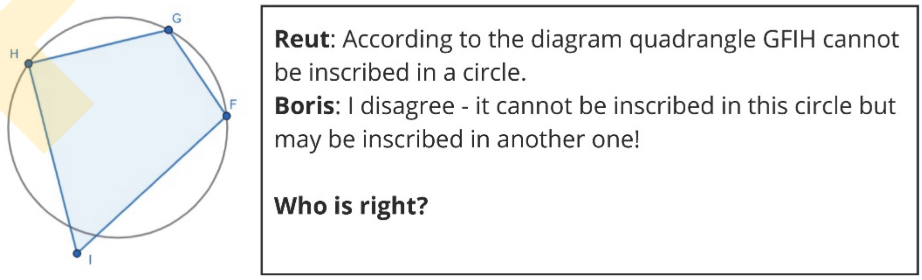


Fig. 6. Who-Is-Right task

are suggested. Such mathematical texts were developed by Koichu et al. (2021) and provided to small groups of students. Such a setting may encourage the development of a critical discussion since the opinions suggested in the text reflect common opinions held by junior-high-school students (Koichu et al., 2021). However, the opinions stated in the WIR are not necessarily the opinions held by the discussants. The activity is then more a way for the group to be convinced that one of the opinions is correct. Koichu et al. (2021) have found that enactment of WIR tasks increases students' engagement in looking-back strategies. These are reflective post-solution dialogical moves that include "queries on verification of the obtained solution(s), comparative consideration of alternative solutions, and formulation of implications for future problem-solving." (p. 831). They argue that considering the question 'why is the other solution wrong?', is different from addressing the question 'why is the chosen solution right?' Answering the former requires the students to use various argumentative practices. Aside from the reported advantages of WIR tasks in promoting looking bake strategies, we see their potential for advancing argumentation skills. In a typical problem, the students encounter a problematic situation and should produce a solution. In the WIR context, the students are confronted with an erroneous solution and a correct one and should decide which one is more persuasive and uphold their decision. The task itself contains a text with a discussion that prompts 'discussion on discussion'. This simple, but productive, argumentative design which is most suitable to the Perelman rhetorical model, is also suitable for van Eemeren and Groothendorst's model, as the students need to decide which of the two interlocutors is more persuasive.

4. Discussion

The examples of activities designed to trigger argumentative activities and the successes we reported on this design suggest that the design of argumentative activities is at the heart of educational change in mathematics. We confess that such examples do not represent very frequent kinds of activities in mathematics classrooms. Rather, activities in mathematics classes generally consist of the engagement in exercises that lead to the skillful resolution of problems and prepare students for exams in which similar problems are posed. Why is the link between mathematics education and theories of argumentation so weak? We suggest that the weakness of this link is not fortuitous and that it points at weaknesses in mathematics education that innovators aim to palliate. To begin with, students are often requested to solve problems in which the question is given. The curiosity of the students is not aroused. The inadequacy of Plantin's theory of argumentation dialogue points at the lack of care in the progressive identification of problems and questions in mathematics. Secondly, the pragma-dialectical model (van Eemeren and Groothendorst, 2004) fits a situation in which several standpoints have a priori comparable epistemic statuses. The fact that in mathematics, solutions are generally either "right" or "wrong" makes critical discussions difficult to happen. We have stressed the difficulties of designing activities in which critical discussions occur. A promising venue in this direction that we cannot develop here because of length limitations is to promote interdisciplinarity. Thirdly, Perelman's rhetorical model refers to persuasion (rather than conviction), led by the teacher.

Persuasion has not a good press in mathematics. The teacher is expected to present clear and logical statements and not to persuade her students (see an exception in Gabel and Dreyfus, 2022). The only model that seems to be relevant is the Toulmin model. However, this model is monologic. Many innovators in mathematics education pledge for dialogic teaching and situate the Toulmin model in dialogue. Mathematics educators should be cautious, though. Pseudo-dialogues during which the teacher leads students to an inexorable conclusion are frequent in dialogic education (Alexander, 2005). Finding the balance between the attainment of rigorous mathematical ideas and the attentiveness to students' voices in mathematical classrooms is a huge challenge.

Besides the weaknesses of mathematics education that the inadequacy of general theories of argumentation uncovered, the innovative examples we presented show that the scope of these general theories is limited. For example, we have shown the importance of texts in innovative activities in mathematics. The general theories do not clearly relate to such texts (written texts, videos, pictures, or diagrams). We described peer discussions around texts, but the role of the text in the argumentation is not addressed and covered by the models. More generally, a learning environment was presented based on a meticulous design for each of the examples we presented. Abundant literature on design for disciplinary engagement (Engle and Conant, 2002), or argumentative design (Andriessen and Schwarz, 2009) provides design principles for argumentative activities in mathematics, such as the provision of resources (for example, for raising hypotheses and checking them), conferring authority to students (e.g., through collaborative settings), the problematization of tasks, the creation of sociocognitive conflicts, and providing ground rules for high-quality talk (Accountable Talk, Explanatory Talk, etc.). We should stress the surprisingly untapped research direction in the role of texts in mathematics education in general and in particular in argumentative activities.

The specificity of mathematical argumentation is especially salient in the potentiality of tools for checking hypotheses/conjectures toward elaborating proofs. In two of the examples we presented, these tools equipped students with an inquiry channel through which they could feed argumentation and, by such, could enhance certainty in their claims towards conceptual change and other learning gains.

There is always a breach between evolving theories in education and actual classroom practices. One of the research roles is to narrow this gap and enrich relations between theory and practice. However, theories may become too popular and hinder actual classroom practice development. Some of the most popular theories in argumentation used in mathematics education are 70 years old. Their authors did not imagine mathematics as a domain of application of their theories. Toulmin even declared that his theory is not adequate for mathematics. We believe that his image of mathematical activity was flawed — he probably believed that mathematicians' thinking is solely based on logical inferences. The structural but experimental model he suggested is adequate for certain aspects of mathematical activity. The popularity of his model does not reflect these aspects, though. We suggest that this adoption often strengthens conservative models of mathematical education in which the teacher dominates the elaboration of mathematical ideas. The Toulmin model provides more clarity to this kind of teaching but does not revolutionize mathematics education.

We hope that we succeeded in showing that argumentative theories may inspire designers to initiate considerable changes in practice in mathematical education. The theories provide the general grammar of argumentation, but the educational design should be meticulous. While deliberative, rhetorical, epistemological, or structural aspects may inspire designers, argumentation in the mathematical class involves instruments, hypothesis-testing devices, texts, and technologies that theorists of argumentation did not envision. With such resources, identification of problems, critical discussions, elaboration of arguments/proofs, or their presentation is interwoven with inquiry processes. Texts such as "Who Is Right?" tasks or PWWs may help students identify problems before discussing and solving them, as conveyed by Plantin's model of argumentation dialogue. If the theorems conveyed by PWWs are familiar to students, they may help students reconstruct the argument that proves the correctness of these theorems. Alternatively, teachers may use PWWs to convince students of the correctness of theorems they are familiar with. The three examples of tasks that encourage critical discussions show the challenges that their design involves. In a nutshell, the interactions between different argumentation theories and mathematical practices and advances in educational design are rich grounds for educational changes in mathematics education.

References

W. R. Aaron and P. G. Herbst (2019). The teacher's perspective on the separation between conjecturing and proving in high school geometry classrooms. *Journal of Mathematics Teacher Education,* 22(3), 231–256.

R. J. Alexander (2005). *Teaching through Dialogue: The First Year*. London Borough of Barking and Dagenham.

J. Andriessen and B. Schwarz (2009). Argumentative design. In N. Muller-Mirza and A. N. Perret-Clermont (Eds.), *Argumentation and Education*. Springer, pp. 145–174.

F. Arzarello, D. Paola, O. Robutti, and C. Sabena (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics,* 70(2), 97–109.

C. S. Asterhan, B. B. Schwarz, and N. Cohen-Eliyahu (2014). Outcome feedback during collaborative learning: Contingencies between feedback and dyad composition. *Learning and Instruction,* 34, 1–10.

E. M. Barth and E. C. Krabbe (2010). *From Axiom to Dialogue: A Philosophical Study of Logics and Argumentation.* Walter de Gruyter.

A. Conner (2022). Adaptive instruction that supports collective argumentation. *The Journal of Mathematical Behavior,* 66, 100969.

M. de Villiers (1990) The role and function of proof in mathematics. *Pythagoras* 24, 17–24.

M. de Villiers (2010). Experimentation and proof in mathematics. In Hanna, G., Jahnke, H., Pulte, H. (Eds). *Explanation and Proof in Mathematics*. Springer, pp. 205–221.

M. de Villiers (2020). Proof as a means of discovery. *International Journal of Mathematical Education in Science and Technology,* 51(3), 451–455.

J. K. Dimmel and P. G. Herbst (2020). Proof transcription in high school geometry: a study of what teachers recognize as normative when students present proofs at the board. *Educational Studies in Mathematics,* 105(1), 71–89.

R. A. Engle and F. R. Conant (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and instruction,* 20(4), 399–483.

T. Fukawa-Connelly, E. Johnson, and R. Keller (2016). Can math education research improve the teaching of abstract algebra. *Notices of the AMS,* 63(3), 276–281.

M. Gabel and T. Dreyfus (2022). Rhetorical aspects of the flow of a proof — A shared basis of agreement between lecturer and students. *The Journal of Mathematical Behavior,* 66, 100971.

N. Hadas, R. Hershkowitz, and B. B. Schwarz (2002). Analyses of activity design in geometry in the light of student actions. *Canadian Journal of Math, Science & Technology Education,* 2(4), 529–552.

G. Hanna (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics,* 44(1), 5–23.

P. Herbst (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. *ZDM — Mathematics Education,* 36(5), 129–139. http://dx.doi.org/10.1007/BF02655665

P. Herbst and C. Brach (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students?. *Cognition and Instruction,* 24(1), 73–122.

B. Koichu, R. Parasha, and M. Tabach (2021). Who-Is-Right tasks as a means for supporting collective looking-back practices. *ZDM — Mathematics Education,* 53(4), 831–846.

I. Lakatos (2015). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge University Press.

I. Lavie, A. Steiner, and A. Sfard (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics,* 101(2), 153–176.

N. Marco, A. Palatnik, and B. B. Schwarz (2021). Mind the gaps — gap-filling in proving activities. *For the Learning of Mathematics,* 41(2), 21–25.

N. Marco, A. Palatnik, and B. B. Schwarz (2022). When less is more? Investigating gap-filling in proofs without words activities. *Educational Studies in Mathematics.* 111(2), 271–297. https://doi.org/10.1007/s10649-022-10159-z

N. Marco and B. B. Schwarz (2019). Proof without words as artifacts for supporting the learning of mathematical proofs. In M. Graven, H. Venkat, A. Essien, and P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of the Mathematics Education(Vol. 4).* PME, pp. 68.

J. P. Mejía-Ramos and M. Inglis (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna, and M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2). National Taiwan Normal University, pp. 8–93.

C. Morgan (2016). Studying the role of human agency in school mathematics. *Research in Mathematics Education,* 18(2), 120–141.

R. B. Nelsen (1993). *Proofs without Words: Exercises in Visual Thinking* (No. 1). MAA.

R. Netz (1998). Greek mathematical diagrams: Their use and their meaning. *For the Learning of Mathematics,* 18(3), 33–39.

C. Perelman (2012). *The New Rhetoric and the Humanities: Essays on Rhetoric and Its Applications* (Vol. 140). Springer Science & Business Media.

C. Plantin (2005). *L'argumentation — Histoire, Théories, Perspectives*. [Argumentation – History, theories, perspectives]. Presses Universitaires de France.

G. Polya (2004). *How to Solve It: A New Aspect of Mathematical Method*. Princeton University Press.

N. Prusak, R. Hershkowitz, and B. B. Schwarz (2013). Conceptual learning in a principled design problem solving environment. *Research in Mathematics Education,* 15(3), 266–285.

Y. Rav (1999). Why do we prove theorems?. *Philosophia mathematica,* 7(1), 5–41.

B. B. Schwarz and M. J. Baker (2017). *Dialogue, Argumentation and Education: History, Theory and Practice.* Cambridge University Press.

B. B. Schwarz, R. Hershkowitz, and N. Prusak (2010). Argumentation and Mathematics. In C. Howe and K. Littleton (Eds.), *Educational Dialogues: Understanding and Promoting Productive Interaction*. Routledge, pp. 115–141.

B. B. Schwarz and L. Linchevski (2007). The role of task design and argumentation in cognitive development during peer interaction: The case of proportional reasoning. *Learning and Instruction,* 17(5), 510–531.

B. B. Schwarz, Y. Neuman, and S. Biezuner (2000). Two wrongs may make a right... if they argue together!. *Cognition and Instruction,* 18(4), 461–494.

S. E. Toulmin (2003). *The Uses of Argument* (updated ed.). Cambridge University Press. https://doi.org/10.1017/CBO9780511840005.

F. H. Van Eemeren and R. Grootendorst (2004). *A Systematic Theory of Argumentation: The Pragma-Dialectical Approach.* Cambridge University Press.

F. H. Van Eemeren and R. Grootendorst (2016). *Argumentation, Communication, and Fallacies: A Pragma-Dialectical Perspective*. Routledge.