Invited Lecture

For Human Flourishing, Build Mathematical Virtues, Not Just Skills

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ABSTRACT A great mathematical education should build mathematical virtues, not just mathematical skills. Virtues are what make mathematical experiences enriching and they serve one well no matter what one does in life. They enable human beings to flourish.

Keywords: Mathematical practices; Human flourishing.

1. Introduction

During a job interview at a high-powered company, a student of mine was asked this question:

> How many gallons of water are flushed in New York City during a commercial break of the Super Bowl?

Although the Super Bowl is an annual football game watched by millions of people across the United States, this company was not asking the question because they had any particular interest in American sports or New York City infrastructure. Rather, they were interested in seeing how my student, a mathematics major, could think and reason her way to answer a question of this nature — a question with some inherent uncertainty and ambiguity, with many strategies for determining an answer. Such questions arise often in the work of this company. As she talked out a solution strategy, the interviewer would probe her thinking and suggest ways to circumscribe her answer.

Notice that the company wasn't trying to assess the job candidate's *skills*, such as whether she could apply the quadratic formula or could calculate an integral. Rather they were interested in *virtues*, such as her ability to strategize (in planning a path to a solution), her resourcefulness (in suggesting potential sources of data to draw on), and her creativity in putting that information together. They were also assessing her collaborative abilities — how well she could talk through a mathematical problem with others.

If virtues are what employers are looking for when they hire a math major, why does a mathematics education mostly focus on skills? Of course, skills are important, but to this employer, virtues were even more important than skills.

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2. The Difference between Skills and Virtues

In this article I'll use the term *skills* to refer to mathematical content knowledge such as facts, algorithms, and formulas and fluency with them. By contrast, *virtues* are human qualities rather than specific items of knowledge. Most people equate virtues with moral virtues such as truthfulness and honesty, but there are many other character qualities that can be considered virtues. For instance, in the Aristotelian philosophical tradition (Aristotle, Ross, and Brown, 2009), courage and wisdom are virtues. However, I want to enlarge our conception of virtue even further.

One way to define virtue is: *an excellence of character that leads to excellence of conduct*. So there's a "being" aspect and a "doing" aspect. The "being" aspect attends to shaping one's character so that the "doing" aspect (good action) naturally follows. In a mathematical context, persistence is one of the many intellectual virtues that can be cultivated by a great mathematics education. Shaping one's character to be persistent in mathematical problem-solving leads to a lifetime of fulfilling choices to grind away at tough problems.

Examples of Mathematical Skills

Fluency with facts, algorithms, formulas
Factoring polynomials
Taking a derivative
Computing things

Examples of Mathematical Virtues

Persistence
Curiosity
Creativity
Thirst for Deep Understanding
Habits of Generalization
Independent Thinking
Capacities to: Define, Quantify, Abstract, Visualize, Strategize, Collaborate

Virtues are important for both practical as well as personal reasons.

On a practical level, virtues are more important to employers than skills. Often that's because the highly technical skills an employer needs are often learned on the job, rather than taught in school. Also, skills are easily replaceable, while human virtues are not. Computers can take derivatives and factor polynomials quicker and more accurately than humans can. (That doesn't necessarily mean that we shouldn't teach those subjects, but it should cause us to consider that the purpose of teaching certain skills is often for the underlying virtues they build, such as developing a deep understanding.)

On the other hand, computers and even artificial intelligence are not (yet) able to be creative and to generalize in the same way that humans can. Moreover, the skills needed in any profession may change over time — just think of all the technological gizmos today that use ideas which didn't exist twenty years ago. But the virtues needed from a math education will not. Employers will always need employees who are persistent problem-solvers, who are collaborative but are also able to independently learn new things.

On a personal level, virtues are more important to one's life than skills. At the end of our lives, are we more likely to appreciate that we knew how to bisect an angle, or to appreciate the ways that being curious and creative enriched our lives?

Virtues provide a better answer to the question "Why do I need to know this stuff?" than the ones teachers often give: "Because you'll need this stuff later." That answer is unsatisfying to students, and it's often not even true. Furthermore, it doesn't help students see that doing mathematics can be relevant to their lives *right now*. A better answer would be: "Because mathematics enables you to marvel at the hidden structures of the world and solve problems you've never seen before, and thinking mathematically builds aspects of character in you that enable you to flourish."

Unlike mathematical skills, which are usually useful only if you end up in a mathematical profession, mathematical virtues will benefit you no matter what you do in life. Think about how much the average person needs to factor a quadratic. Then compare that to how being a creative solver will help you in any profession, or how being able to visualize makes your life richer.

3. Virtues are more than Mathematical Practices

Some attention to the role of virtues in math education can be found in the Common Core Standards for Mathematics (NGA Center and CCSSO 2010), an initiative in the U.S. to establish consistent standards across the states for what students should know. These standards include not just standards for mathematical content in each grade level, but also include "Standards for Mathematical Practice" that cut across grade levels. For instance, some of them are:

- Reason abstractly and quantitatively.
- Make sense of problems and persevere in solving them.
- Look for and make use of structure.

These practices describe attitudes built by a great math education that are indeed virtues; they are useful in any profession or life circumstance. Embedded within them, you can see some of the other virtues I've called out: persistence, thirst for deep understanding, habits of generalization. The Mathematical Practices are valuable, but they do not come close to capturing all the virtues that a math education fosters. There are many other intellectual virtues we exhibit when we think well.

And we should not forget the *affective* virtues that are also part of a great mathematical experience. These are what the classroom or workplace a place of wonder, delight, and joy.

Examples of affective mathematical virtues

An Expectation of Enchantment
Disposition towards Beauty
Hospitality in Welcoming Others to Mathematics
Hopefulness
Self-Confidence
Affection for Mathematics

A great math education also builds all these virtues too. Such virtues make one's life richer, and enable one to flourish.

4. Building Virtues by Attending to Basic Human Desires

At ICME-14, I spoke about the purpose of mathematics in promoting human flourishing, a vision that was also described in detail in my recent book (Su 2020a). Virtues are at the heart of what it means to be human and to flourish in this world. Here, I want to build on that presentation by outlining some practical ways to cultivate virtue in the classroom by attending to basic human desires that all our students have. None of these strategies are earthshakingly new, but their framing around human desires may offer some perspective about why they are successful.

For instance, beauty is a basic human desire that all human beings share. We all long to behold beautiful things, and we are curious to understand what others find beautiful. Our mathematics classrooms can and should tap into this desire. Some ways to do this include:

- Making your classroom a place of sensory beauty, such as through artwork or music.
- Highlighting wondrous beauty (the beauty of ideas) and insightful beauty (the beauty of reasoning) and helping students see how they contribute to an appreciation of life.
- Providing space for students to reflect on beauty.

Attending to a desire for beauty cultivates in our students' affective virtues, such as *reflection* and *joyful gratitude*, which contribute to positive identities in mathematics activities. It can provide experiences of *transcendent awe*, when we see the connectedness of mathematical ideas to each other and to explaining the world. It also promotes *habits of generalization*, because we are trained to look for beautiful overarching patterns where we might not expect them.

Another basic human desire is exploration. From our earliest moments as infants, we are eager to explore our environment and learn about the world. How do we nurture that kind of exploration in the classroom? Some strategies include:

- Changing dull computational problems to exploratory problems (one where creative thinking is required and many solutions are possible);
- Praising good questions, not just good answers;

- Showcasing enchanting ideas (and making it a professional goal to learn enchanting ideas).

Cultivating an attitude of exploration can help our students build virtues such as *imagination* and *curiosity*. And as we showcase beautiful and enchanting ideas, students develop the *expectation of enchantment*. In the most rewarding math experiences, budding mathematicians learn to expect this enchantment. It's what keeps them coming back for more. So we should also make it a professional goal as teachers to be continually learning enchanting ideas in mathematics by regularly doing reading about mathematical ideas.

Play is another basic human desire. Humans enjoy fiddling with things, focusing our attention on fun diversions, interacting with others in activities (like games) that intermix structure with freedom, and marveling at the surprises that result. Mathematics teaching should also make mathematical experiences feel playful. Some ways that this can be done:

- Giving students space and time to play, such as open-ended questions that aren't answered right away.
- Lowering the stakes of "right answers."
- Adopting "rough draft thinking" (Jansen, 2020): ideas don't have to be perfect the first time they emerge, and they can always be improved through revision.
- Viewing each mathematical idea from multiple perspectives (just like one does in a game, viewing a strategic situation from the viewpoints of others).

Such practices cultivate in our students' virtues, such as concentration and perseverance, because play can be an intensely pleasurable focus that shuns the other distractions of daily life. They build hopefulness, because when you tinker with a problem long enough, you are exercising hope that you will eventually solve it. And they train in us the ability to change perspectives, a virtue that serves us well in solving math problems but also in understanding the life experiences of others.

These are just a few examples of ways that attending to basic human desires can make our classrooms places of human delight and connection, and cultivate virtues that will enrich our students' lives no matter what they do in the future. I explore several other human desires and their attendant virtues in Su (2020a).

5. Assessing Virtue

This begs the question: if virtues are just as (if not more) important than skills, why do most math educational experiences focus primarily on skills? One reason — in my opinion, a primary reason — is that skills are easy to assess, but virtues are not. It's easy to grade skills — just make a worksheet, have students compute twenty multiplication problems, and check for correct answers. It's harder to assess virtues, since that feels messier — we have to find ways to elicit student thinking, and decide how to assess whether they've been persistent or curious. That feels subjective, doesn't it? And yet, even worksheets — if you consider the choices made about what kinds of problems to include — are subjective assessments. Thus, the fact that assessing virtues is messy and possibly subjective doesn't mean we shouldn't try.

I understand that assessing virtues is hard to do, and I hope math educators will continue to work on methods and rubrics to assess some of the ones I've mentioned in this article. But let me close by suggesting a few ideas about how one might begin.

For several years now, and especially during the pandemic, I've been including reflection questions on my assignments that attempt to get at virtue development. See Su (2020b). For instance, here's a question that I use to evaluate persistence:

> Take one homework problem you have worked on this semester that you struggled to understand and solve, and explain how the struggle itself was valuable. In the context of that problem, describe the struggle and how you overcame that struggle. You might also discuss how struggling built aspects of character in you (endurance, self-confidence, competence to solve new problems) and how these virtues might benefit you in later ventures. Thoughtful answers receive at least 9 out of 10 points.

Having students reflect on what they've gained from struggle is more formative than summative, since I don't try too hard to make find distinctions between answers — I give nearly full credit for any thoughtful answer. If you try it, I encourage you not to sweat fine distinctions. It's much better to make the questions low stakes, and let students know in advance that any thoughtful answer will receive full or nearly full marks. What's crucially important is to notice that by the process of reflection, students are engaging in building the virtue you are hoping to assess. Asking reflective questions sends a strong message to my students that I care about more than just skills. Of course, that message should be reinforced by all that you do and say in the course, not just delivered in homework.

I also like to see how my students' curiosity has grown with questions like this one:

> What mathematical ideas are you curious to know more about as a result of taking this class? Give one example of a question about the material that you'd like to explore further, and describe why that is an interesting question to you. A thoughtful question will receive 9 points, and especially insightful question will receive 10 points.

It's been remarkable to me to see the kinds of questions students ask, and it makes me reflect on how I can encourage more of these questions during our in-person class time.

Finally, here's a question that I use to evaluate a disposition toward beauty.

Consider one mathematical idea from this course that you have found beautiful, and explain why it is beautiful to you. Your answer should: (1) explain the idea in a way that could be understood by a classmate who has taken classes X and Y but has not yet taken this class and (2) address how this beauty is similar to or different from other kinds of beauty that humans encounter. Thoughtful and correct answers will receive at least 8 points, and especially insightful answers will receive 9 or 10 points.

I know some readers will be skeptical that such a question could be meaningful, since it seems far-fetched that one could tell whether a student could have developed such a disposition. And yet I find that it is not hard at all to tell. Moreover, I've learned useful ways of thinking about my own subject matter from students!

For example, here are a couple of responses I received on this question in a course about symmetries of polynomials (Galois Theory).

[In speaking about straightedge-and-compass constructions:]

> Similar to the Mona Lisa or other timeless classic art, these constructions can be appreciated by anyone, regardless of upbringing or ability. In the same way that everyone finds a flower beautiful, these constructions can be appreciated as well… I find the depth of constructions similar to the relationship between traditional and modern art. The construction of the equilateral triangle is easily appreciated, just like a traditional painting. Both are accessible. On the other hand, the construction of the 17-gon is difficult to understand, and even more difficult to attempt to replicate. When constructions are taken to their limit, they are obtuse. I feel like modern art is in a similar position. In my art history class last semester, my classmates and I were often left puzzled after the professor discussed a banana taped to a wall or a toilet made of gold. We often had to ask "what is the meaning of this piece?" Someone seeing the construction of a 17-gon for the first time might ask the same question. Ultimately, I believe the accessibility and universal nature of these constructions make them beautiful.

[In speaking about roots of polynomials that always appear together:]

> This kind of beauty is reminiscent of the idea of a soul mate: someone who in some sense matches you perfectly and who you always want to be together with. Roots of the same minimal polynomial have the same "nature" in some way, and never occur without each other. The idea of two separate things being connected on some deep level also occurs in a lot of other ideas of dualism (you can't have shadow without light, you can't have cold without warmth, etc) which have been present in religions and human beliefs for longer than recorded history, and must appeal to us on some deep level. I guess it is beautiful to see that his idea that is so deeply ingrained in us as has equivalent which can be proved even in the world of mathematics.

Did these students possess a disposition toward beauty? Most certainly yes, and they were encouraged by asking a formative question. One benefit of asking such questions that I did not anticipate was how much it would help me as a teacher — by showing me what they are thinking and informing the examples and analogies I would use when I teach the subject again. And most importantly, I actually *enjoy* reading their reflections, much more than grading any other question.

6. Conclusion

In summary, a great mathematics education should build much more than content knowledge and fluency with skills. A great math education should build intellectual virtues, like curiosity, creativity, and a thirst for deep understanding. It should also build affective virtues, like hopefulness and an expectation of enchantment. It should even build communal virtues, like collaboration and hospitality in welcoming others to mathematics. These are virtues that build healthy math identities, serve us well both practically and personally, and make our lives richer. They also foster more equitable mathematics experiences. Through the development of virtues, mathematics education can enable human flourishing.

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