Invited Lecture

Automatic Reports to Support Students with Inquiry Learning: Initial Steps in the Development of Content Specific Learning Analytics

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ABSTRACT When students are asked to examine their understanding individually or in small groups, information can become part of a feedback process that supports students' learning. As designers of technology to support learning, we are interested in supporting such feedback processes in the context of guided inquiry instruction. This paper explores the potential of automatically associating mathematical descriptions with student submissions created with interactive diagrams. The paper focuses on the feedback processes that occur when students use the descriptions provided by the technology as resources for reflection and learning. We discuss the design of personal feedback processes where students reflect on and communicate their own learning, utilizing individually-reported multi-dimensional automatic analysis of their submissions in response to example-eliciting tasks. While there is much research and development work to be done, we consider mathematical descriptions of student work as an important contribution to broader developments in learning analytics.

Keywords: Inquiry learning; Feedback; Learning analytics; Technological supports for learning; Example eliciting tasks.

1. Introduction

In this chapter, we suggest that automated descriptions of student work are a new strategy that can be designed into technology for providing students with support for inquiry learning. We illustrate this strategy with a set of tasks designed in the STEP platform (Olsher et al., 2016) that are intended to support students in developing conjectures about the intersection points of perpendicular bisectors to the sides of quadrilaterals. We argue that these tasks have the potential to support reflection on commonly used classifications of geometric shapes for which a conjecture holds.

The chapter is organized in five sections. We begin with two sections that examine key aspects of relevant literature about using technology to support student inquiry

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learning in mathematics and about feedback processes in mathematics classrooms. The next two sections focus on the strategy of describing students' mathematical work in words as a way to provide students with feedback on their work, and then illustrate this new strategy in the context of a particular set of tasks on perpendicular bisectors of the sides of quadrilaterals. In this illustration, we describe the kinds of characteristics that are used to support feedback processes, as well as how the reporting on those characteristics is organized. We conclude with thoughts about how describing students' work mathematically can contribute to the growing field of learning analytics and about directions towards the design and use of reports in other types of learning settings and at other stages of inquiry learning.

2. Coordination of Examples and Concepts in the Context of Inquiry Learning

We view inquiry learning in mathematics as centrally involving the coordination of examples and of concepts, where the definition of a concept identifies criteria for classifying instances as examples or non-examples of that concept. In taking this stance, we follow Hershkowitz's theory of fundamental concepts with its focus on definitions and examples (Hershkowitz, 1990; Tall and Vinner, 1981; Vinner, 1991). Therefore, by concept, we refer to a combination of characteristics; by concept definition, to the minimal combination of critical characteristics (necessary and sufficient) to define the concept; and, by concept image, to the collection of examples and the derived properties reflected in students' work. Naftaliev and Hershkowitz (2021) stress as an important implication of their study of concept construction, that a learning trajectory, in which learners should focus and examine the relations among definition, examples, and critical characteristics (which these authors refer to as attributes) will reduce the use of prototypical examples (Presmeg, 1992) and strengthen the coherence between concept images and concepts definitions held by learners.

Given this perspective, example-eliciting-tasks (EETs) are an important component of our approach to guided inquiry with a digital environment (Yerushalmy et al., 2017). Having students construct examples of a concept, or having students interact with carefully designed repertoires of examples, are important kinds of activity for the development of mathematical ideas and concepts and can provide a window into the nature of learners' understanding (Zaslavsky and Zodik, 2014). These scholars further suggest that example generation can also be a catalyst for enhancing students' understanding and expanding students' concept image. However, each particular example is limited in what it may manifest about students' understandings. To overcome this obstacle and to gain insight into the breadth of students' concept image and the nature of their concept definition, as is illustrated in Section 4, tasks can prompt students to produce multiple submissions that are as different as possible from one another.

The discussion in this section has not yet focused on learning goals and on how learning goals shape inquiry learning in school environments. As suggested by the didactical contract (Brousseau, 1997), meaningful interactions that are part of learning in school with technology-based tools involve meeting learning goals. When technology is designed for experimentation, exemplifying, conjecturing, and arguing, it can be an important component in creating environments where situations of inquiry may occur, but there are other layers to consider, including the nature of the feedback processes involving teacher, student, and tool, that are necessary to create educative experiences.

3. Feedback in Technology Supported Mathematics Instruction

The traditional view of feedback in instruction emphasizes the role of verbal feedback by a teacher at a stage when the learner has already finished a task, or a part of the task, and the information provided assists the learner toward the next learning goal. This feedback usually focuses on evaluation, and it is often followed by verbal explanations. When students perform a task well, there is not much to the feedback process, beyond the acknowledgment of a job well done.

This type of attitude is apparent in early forms of technology supported assessment that were concerned with the need for efficiency. Yet, warning flags were raised early on. The case of Benny by Erlwanger (1973) describes a student using the Individually Prescribed Instruction (IPI) mathematics curriculum, which provided students with automated adaptive feedback about their individual progress. This case study has become a classic example of the possible effects of programmed automated feedback. Erlwanger and the many reflections and studies on independent programmed learning have pointed to the possible wrong conceptions students can develop as a result of automated feedback that later might have damaging effects on what students think about the concept and about the logic of mathematics. A main concern is with the assumption that assessment of a sequence of performances focused on evaluation without elaboration could be sufficient as constructive feedback for learners. Students may focus on reaching a correct answer without understanding why it is correct, or worse, try to get what is "correct for the teacher," which is not necessarily aligned with what the student thinks is mathematically correct.

Despite these potential pitfalls, automated assessment remains commonly used in association with multiple-choice questions (MCQ). When well-constructed, MCQs present distractors based on experience or research about student learning of the topic. Nevertheless, students might use answering strategies that do not require actually solving the MCQ, such as guessing or validating the various possible answers, moving away from what was meant to be assessed (Sangwin, 2013). Our information-rich world is moving toward extended types of mathematical competence, which require assessment that does not rely on proxies, such as multiple-choice questions, but assesses student competence directly (Stacey and William, 2012).

In the literature on learning with technology, feedback is often used to describe the information that technology presents regarding aspects of a learner's performance or understanding. These aspects may include corrective information, an alternative strategy, information to clarify ideas, encouragement, or simply the correct answer. Generalizing the types of information provided, Sadler (1989) noted that feedback needs to provide information specifically relating to the task or process of learning that fills a gap between what is understood and what is aimed to be understood. Hattie and Timperley (2007) conceptualized feedback to be "[I]nformation provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (p. 81).

Shifting from teaching-centered processes to learning-centered inquiry has implications, in particular, for the meaning and quality of feedback processes. In an environment in which self-reflections, dialogic interactions, and whole-class discussions are the means for learning, we need to consider the quality of the whole process of feedback, including the quality of the teacher's contributions, the role of technology and the active role of students. Considering the feedback process as involving only information or comments given by the teacher is not helpful. Carless (2015) described feedback in the classroom as "a dialogic process in which learners make sense of information from varied sources and use it to enhance the quality of their work or learning strategies" (p. 192). We are interested in conceptualizing feedback in a similar way, as an ongoing process. In the literature we cite, the term "feedback" is used both to describe processes in the classroom and to identify the information that technology can present for use as part of such processes. Although we conceptualize feedback as a process, to be true to our citations, we use 'feedback' in these two different ways.

We focus on personal feedback processes, exploring the uses of technological design to support students' reflection on their mathematical understandings, and addressing the key question" *What makes information part of an effective feedback process*?" to guide our thinking about automatically analyzed student work as a resource for learning.

A common practice in the study of feedback processes is to examine whether and how feedback helps students close the gap between their current and expected performance. The literature assumes that feedback is a process that "counts" only when it makes a difference to what students do, and that the information communicated to learners is intended to modify their thinking or actions for the purpose of improving learning (Shute, 2008). Yet, studies agree that the perceived effectiveness of feedback is inextricably dependent on the goals of instruction, which may remain tacit. Thus, the effectiveness of feedback is highly contextual. Researchers have reached conclusions similar to Sadler's (2010), that "feedback is capable of making a difference to learning, but the mere provision of feedback does not necessarily lead to improvement… the general picture is that the relationship between its form, timing and effectiveness is complex and variable, with no magic formulas" (p. 536).

4. Automatic Mathematical Characterizations: A Different Sort of Intellectual Mirroring Strategy

In this section, we discuss the potential affordances for students' learning of technology that automatically associates textual descriptions of mathematical characteristics with students' submissions. We do so by reflecting on how the identification of predetermined characteristics provided as linguistic resources has the potential to make a new contribution to supporting individual student learning in the context of student inquiry. We consider the potential and pitfalls of including automatic characterization of student examples in words as a way for students to interact with points of view that may be different from their point of view while engaging in inquiry. We argue that through reports constructed for individual learners, such communication could be useful, enabling students to further their learning by reflecting on their understanding. We now describe the strategy for providing students with information about their work: describing students' submissions using mathematical properties.

Our illustrations of this strategy involve the use of STEP (Olsher et al., 2016), an online platform designed to support teachers' work in assessing various open-ended example-eliciting tasks, and to support technology-enhanced didactical situations that involve learning processes with non-judgmental individual feedback. The platform was designed to support evidence-based formative assessment practices (Mislevy, 2017) that go beyond whether or not students' work is correct. The tasks include interactive diagrams (applets) in Geogebra (Hohenwarter et al., 2009), and usually prompt students to submit examples and non-examples to support or contradict a mathematical claim, or to create examples under given constraints (Yerushalmy, 2020). In this section, we focus on how STEP is designed to provide students information that describes their work back to them. The theory of conceptions of fundamental concepts (Tall and Vinner,1981; Hershkowitz, 1990) is at the core of the way in which STEP carries out automatic analysis of students' submissions.

With the STEP platform, potential characterizations of student work are programmed into tasks and then are automatically associated with student work by the machine. Task designers can make these linguistic descriptions of the mathematical characteristics of student examples available to learners throughout their work process with an interactive diagram (Naftaliev and Yerushalmy, 2017), as well as a report after they have submitted their work. Students must learn through interaction with the words used to describe their work, to appreciate the information provided to them as a resource for their inquiry efforts. In particular, in the context of conjecturing, students can explore the implications of having shared sets of characteristics in the examples that they submit; when used in this way, characteristics can be considered as conditions that a set of examples meet and that may influence results. We have been exploring this strategy through design research studies focused on particular characteristics for particular tasks (For a description of this process in the context of a particular task, see Harel and Yerushalmy, 2021).

We conceptualize the strategy of automatically characterizing student work as a departure from current technology supported feedback practices in two ways. First, providing mathematical characterizations of student work is an alternative to the information more commonly provided to learners in standard online learning platforms: evaluation of the correctness of their responses.

Second, in the context of technology supported inquiry learning, information is often presented through multiple linked representations where a user's action in one representation is reflected in another. The phrase "intellectual mirror," coined by Schwartz (1989), articulates the essence of this strategy for supporting technology-based inquiry: linked representations support a process of self-reflection where the implications of actions taken in one representation are reflected back to the user as in a mirror.

The information offered by STEP can operate like the intellectual mirror described by Schwartz in that a user receives feedback on actions that they take and can reflect on what STEP offers. On the other hand, the kind of mirroring that STEP provides is different from the mirroring provided by multiple linked representations (MLR). While MLRs are mute and do not speak, STEP uses words to characterize, rather than evaluate, student work. In this sense, STEP, like the mirror in the Snow-White tale, speaks. This sort of mirror has a perspective that a designer has developed, and offers that perspective on learners' submissions (whether or not students like what they hear).

When information about how students' work relates to a task is provided to learners while they are working, there can be a feedback process that is like providing hints or clues. The information provided by STEP is not evaluative and does not attempt to bring the student closer to a predetermined solution, but enhances students' potential interaction with key characteristics of the task and in that way can support a more compatible solution based on what is required in the task.

However, STEP offers an additional perspective or voice, the mathematical perspective of the designer, that may be in conflict with students' own understandings of the words that STEP offers (Yerushalmy et al., 2022). When STEP offers its characterizations, there can be conflicts between the usage of the student and the usage of the software; and as a result, a student may feel that the software's characterization is mistaken (Rezat et al., 2021). Thus, feedback processes with STEP may involve reflection that is generated by commognitive conflicts of the sort Sfard (2007) identifies between classroom participants.

While STEP does not evaluate student work, once students have completed their work, the information provided by STEP can also be used for evaluative processes by communicating the degree to which the student's work meets requirements of a task. Although this is a well-defined function for closed tasks, it is not always easy to define an algorithm when tasks are more open. Of particular interest is when a task asks that students submit more than one example and that those examples be different from another. Such tasks are meant to support students' development of broad personal example spaces by asking that there be a variety of examples in their submission for an example-eliciting task. To support student work with such tasks, the characteristics that STEP speaks of should enable students to compare and distinguish between their examples, suggesting mathematical descriptions or contextual descriptions of mathematical phenomena. Mathematical characteristics that allow students to distinguish between their examples define a relevant domain to start a meaningful inquiry process that goes beyond mere trial and error.

5. Illustration of How the Design of STEP Supports Students in Feedback Process

To illustrate how mathematical characterization of student work can be a strategy for supporting inquiry learning, we now turn to an activity in the STEP platform in which students use an interactive diagram to submit examples of quadrilaterals whose perpendicular bisectors meet in a point, as well as examples of quadrilaterals whose perpendicular bisectors do not meet in a single point. To organize this illustration of mathematical characterization of student work, we present the following subsections:

- We begin with a pedagogical challenge related to conjecturing and provide a rationale for an inquiry-based set of tasks about the shapes formed by the intersections of perpendicular bisectors of a quadrilateral.
- Next, in the context of these learning goals, we explore how to characterize students' submissions mathematically in an effort to support students in their inquiry learning.
- We then present three STEP tasks and provide a rationale for the design of the tasks. These tasks depend on the same interactive diagram which provides students with some initial automatically provided information for their immediate use.
- Next, we illustrate how automatic characterization can be used by STEP to support student inquiry after students have submitted responses to these tasks. Each student receives a report on their submission that can be used for subsequent classroom activity.
- Finally, we describe how classroom activity can build on the information provided to students in the reports.

5.1. A pedagogical challenge when designing to support conjecturing

A traditional high school geometry problem to prove reads: The four perpendicular bisectors in any quadrilateral where the sum of opposite angles is 180 intersect at a point. Alternatively, the problem to prove can be stated as: *A convex quadrilateral is cyclic if and only if the four perpendicular bisectors to the sides are concurrent.* Soldano and Luz (2018) studied this version of the problem as part of their approach of using dynamic construction (of a dynamic quadrilateral that can only be dragged at a point while the other three points are circumscribed), as the basis for generating sets of examples and non-examples that eventually lead to conjectures about cyclic quadrilaterals and why their perpendicular bisectors meet at a point. Marrades and Gutierrez (2000), also using a dynamic environment that allows to drag a single vertex of a quadrilateral, used this task to study types of justifications that high-school students offer when working on a given construction and the statement "A, B, and C are three fixed points. *What conditions have to be satisfied by point D for the perpendicular bisectors to the sides of ABCD to meet in a single point?"*

The results of these two studies suggest a more general challenge related to conjecturing: Students often face challenges understanding that the geometric shapes for which a conjecture holds need not always be a set of shapes that has a commonly used name. In addition, the names used for sets of shapes do not always identify the critical characteristics that are relevant to a particular conjecture. This has implications for students' efforts to prove and argue about the truth of a claim; students may not identify the critical conditions that are relevant to the desired conclusions (Haj-Yayha, 2020).

With the tasks and feedback processes that we illustrate in this section, we do not seek to help students develop a proof. Rather, we attempt to organize a meaningful autonomous inquiry experience for students that help them prepare for a teacher-guided whole class (or a group) discussion and will help them come to that discussion with an inkling that the set of quadrilaterals for which the perpendicular bisectors meet at a point (the "if" part of the conjecture) is not one of the standardly named sets of quadrilateral (e.g. trapezoids, parallelograms, …).

We accomplish this goal by stating the task as an example-eliciting-task for exploration. We ask students to create different examples, in which the perpendicular bisectors to the sides of ABCD meet in a single point. The exploration is based on dragging the four vertices of a quadrilateral of the construction in a Geogebra-based interactive diagram which is unconstrained (any vertex of a quadrilateral can be dragged anywhere).

5.2. Characteristics for describing the conjecture

As part of authoring a STEP task, designers provide the platform with direction about the characteristics of student submissions to be checked and associated with examples. When a conjecture is focal, there are two types of characteristics to consider. One set of characteristics of the interactive diagrams are associated with the *if* part, and the other with the *then* part of a conjecture.

In the case of the tasks described here, the desired outcome for the interactive diagram, the then part of the conjecture, is that all 4 of the perpendicular bisectors of the side meet in a point. One could make other choices for the *then* part of the conjecture, like for example describing the geometric shape created by the points of intersection of the angle bisectors.

The second set of characteristics to consider is the characteristics of the quadrilateral in the interactive diagram on whose sides the perpendicular bisectors are constructed, an element of the *if* part of the statement of a conjecture. For use as support for having students consider the *if* component of the conjecture, we considered three competing sets of characteristics from which to choose, one focused on angles and relationships between angles, another on the lengths of sides and one on prototypical types of quadrilaterals. These three choices each involve linguistic terms that may help students to formulate conjectures about the quadrilaterals for which the perpendicular bisectors of the sides meet in one point. To summarize the choice, we faced as designers of this activity, we developed the following list of characteristics:

Tab.1. Three sets of characteristics for describing the quadrilaterals in student submissions

Types of quadrilaterals	Angle relationships	Side relationships
Parallelogram	All angles equal	All sides equal
Trapezoid	No angles equal	Two pairs of equal sides
Kite	Two pairs of equal angles	One pair of equal side
Rectangle	Sum of adjacent is 180	One pair of opposite equal sides
Square	Sum of opposite is 180	No sides are congruent.
Rhombus	More than two angles equal	

These characteristics are candidates for use to support the identification and classification of examples and non-examples, not only by referring to types of quadrilaterals, but by identifying the characteristics of quadrilaterals for which the perpendicular bisectors intersect, using side and angle relations. These relations may guide the exploration of quadrilaterals to focus on the use of critical characteristics as the if part of the conjecture statement.

5.3. Three STEP tasks using one interactive diagram

Task 1 is stated as follows: "A, B, C and D make quadrilateral ABCD (see Fig. 1). They are all dynamic and can be dragged. If it is possible, create 3 examples that are as different as possible from each other, in which the perpendicular bisectors to the sides of ABCD to meet in a single point."

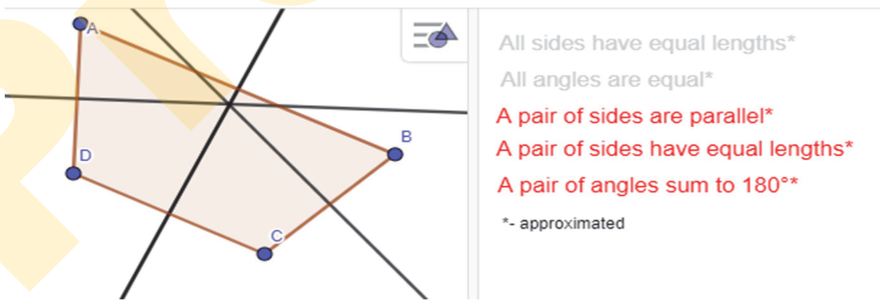

Fig. 1: An example for task 1 constructed with an interactive diagram in STEP

The main reason that this exploration in this task did not include measurements was to enable the students to identify qualitatively the possible variation between submitted examples. Students were encouraged to explore various situations before they decided which state of the interactive diagram they would like to include in their submission. STEP allows them to change their decisions before submitting.

Task 1 was for the students to create different examples and explore possible shapes that meet the condition. On the right-hand side of the screen, the Geogebra interactive diagram shown in Figure 1 provides characterization in words of the quadrilateral ABCD.

Of the 11 characteristics of sides and angles we listed in Table 1, we decided to provide three characteristics related to sides and two related to angles. Our decision was influenced by four pedagogically grounded considerations. First, we wanted to allow students to be able to explore familiar quadrilaterals. These characteristics allow students to recognize quadrilaterals by relations between the measures of their sides and of their angles. Second, the interactive list does not include the shapes names themselves, so recognizing a shape requires attending to the properties that characterize the shape (e.g. watching the diagram one may realize that there is a single intersection point when the shape looks like a rectangle, but one recognizes this by focusing on characteristics of pairs of sides and angles). Third, the characteristics we chose include those that may appear in examples but are not critical ones (e.g. 4 equal angles). And fourth, none of the five directly lead to the correct general answer (e.g., a pair of angles sum to 180 could refer to adjacent angles and thus is not sufficient nor necessary for the claim to be true.)

Task 2, using the same interactive diagram, engages students in creating a personal example space of examples and non-examples. As opposed to Task 1, Task 2 has students begin to think about non-examples and how to distinguish examples and non-examples focusing on the critical characteristics. It directs students' attention to the characteristics provided in the interactive diagram and asks students to think about subsets of those conditions and whether or not there are examples and non-examples for states of the interactive diagram that fit the same subset of the list of characteristics.

The task is an existential EET that requires finding a subset for which they can find an example and a non-example. In Fig. 2 (on the next page), the student has chosen a subset that includes three characteristics; the quadrilateral on the left is an isosceles trapezoid where two opposite angles sum to a 180 and it exemplifies the claim. The quadrilateral on the right is a right angle trapezoid and the lines intersect in 4 points. This state of the interactive diagram is a non-example as it does not fulfill the *then* part of the claim.

Task 3 requires an answer to a universal question that would require convincing arguments to why all quadrilaterals with those characteristics will demonstrate the truth of the claim: *Choose another subset from the same 5 characteristics, a subset for which you believe there can only be examples. Submit two examples and explain why you think that any quadrilaterals with these characteristics must have perpendicular bisectors that meet at a point.*

Fig. 3 (on the next page) offers two examples that are different in their position, but both look like rectangles (approximately). Any quadrilateral that is characterized by the condition "all angles are equal" will also be characterized by having a pair of parallel sides, a pair of equal length sides and a pair of angles that sum to 180 degrees. Under these conditions the four lines will meet at a point.

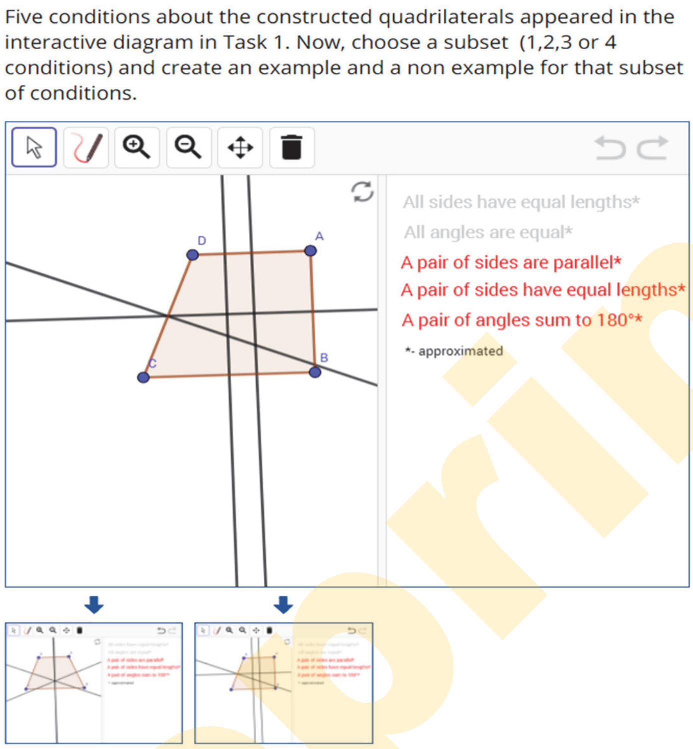

Fig. 2: An example and non-example in Task 2 for a set of three characteristics

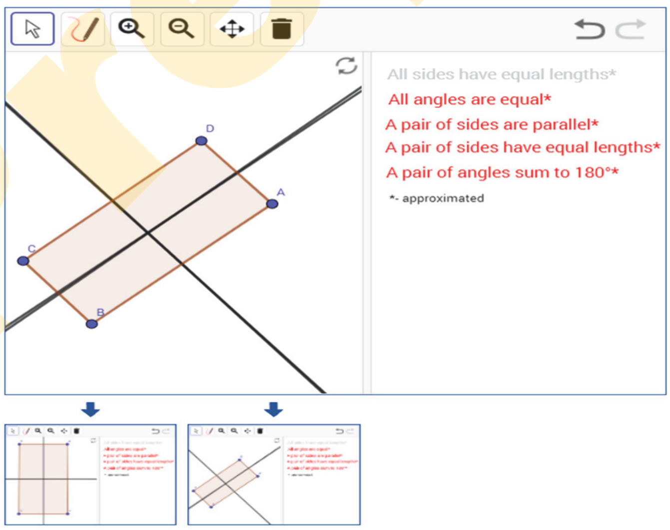

Fig. 3. Two examples in Task 3 that meet the same 4 characteristics

During the exploration phase, whenever an element of the quadrilateral has a particular characteristic, Geogebra highlights the relevant text on the list. In this way, students can begin to think about characteristics of the quadrilaterals they are creating and whether particular conditions or sets of conditions consistently only produce examples (e.g., Fig. 3 seems to suggest that the perpendicular bisectors of all rectangles meet in one point).

5.4. Supporting inquiry by presenting post-submission mathematical characterizations to students

To provide students with support for understanding their personal example space, part of the task design process in STEP includes *a priori* definitions of mathematical characteristics of submissions that appear in the students' submitted examples and could provide useful feedback to them. Thus, in addition to information provided during exploration, students' submissions are stored and automatically analyzed by STEP (on a larger set of conditions than what appears in the interactive diagram) to produce post-submission individual reports for the students (hereinafter, post-submission report) (Olsher et al., 2016). Figure 4 presents a sample post-submission report.

The post-submission report includes three parts: Part 1 addresses the relationship between a student submission and the requirements of the task. For example, in the student submission for Task 3 (as represented in Fig. 3), the top part of the report addresses two questions: Do the 4 perpendicular bisectors intersect at a point for both examples and are both examples characterized by the same subset of characteristics? Information about which subset of conditions is held in common is found in the submitted states of the interactive diagram. Part 1 helps the student to know whether their submission meets the task requirements and will thus help indicate how to interpret the remaining parts of the report.

Part 2 of the report supports a comparative view across submissions. It includes the submitted states of the interactive diagram with the characteristics found for each (multiple examples as in Task 3 or examples and non-examples as in Task 2).

The characteristics in Part 3 (appearing below each of the submitted diagrams) offer another set of characteristics that has not been introduced to the student so far. The use of the set of shapes that have common names is designed to challenge the common attention to prototypical quadrilaterals. Such information goes beyond whether students' submissions are right or wrong, have the potential to challenge the students' current perspectives, and can be used to describe where in the space of possible subsets of characteristics and the space of examples and non-examples the submission resides.

In Fig. 4, both states of the interactive diagram fulfill the task requirements of Task 2 as (i) they constrain by the same chosen subset of 3 characteristics and (ii) one is an example where under these characteristics the perpendicular bisectors meet at a point and the second is a non-example (There are four points of intersection). Finally, the quadrilaterals submitted with these characteristics are both examples of the same shape: trapezoids.

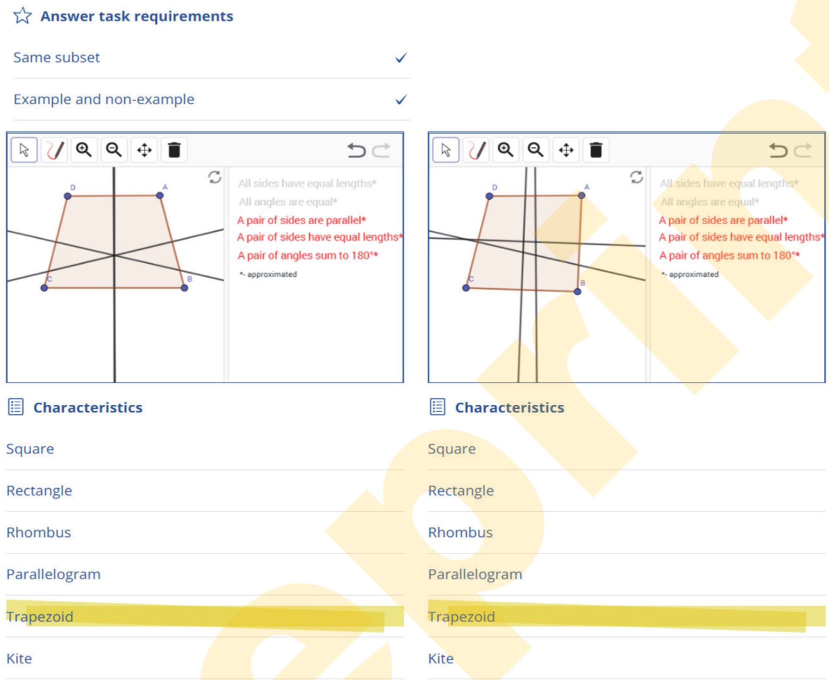

Fig. 4: Post submission information reported to the student's submissions to task 2

5.5. Working with a class when each student has a post-submission report

Group discussion, in their small groups or whole group settings, is at the core of inquiry based learning. Such discussions, done in small groups following the personal submission and getting the personal information, have potential to advance learning. Olsher (2022) describes the construction of a dialogic space by pairs of students who were working on the same task in STEP. Whether working as pairs or small size groups participants may discuss the various choices of subsets of characteristics and the resulting shapes. This would lead to discussions of critical and non-critical characteristics of the quadrilaterals in this task and the effects of the choice of the subset of characteristics on the space of examples and non-examples (similar to the group discussions described by Naftaliev and Hershkowitz, 2021).

A whole class discussion can be based on the information that both the teacher and the students receive from STEP. The teacher can already be familiar with students' submissions — answers, frequent mistakes, characteristics that dominate the choice of subset etc. The students arriving prepared for the lesson with ideas and possible conjectures to discuss based on their examples and personal or group reflection on their post submission reports. The discussion then takes the form of a meta-feedback process based on the rich information that each of the students and the teacher have. For illustrative purposes, ideas that might be raised in such discussions might include:

1. What do the two sets of characteristics appearing in the report tell us? What are the ramifications of characterizing examples and non-examples of quadrilaterals by familiar names? What other information might you suggest to be reported? For what purposes?
   Such collective analysis, similar to the one demonstrated in Olsher (2019), might lead to a discussion about the limitations of thinking about quadrilaterals only by the familiar names and to analyze the distinctions between critical and non-critical characteristics.
2. Do you find any of the characteristics helpful in answering related questions? If usually the intersections of the perpendicular bisectors create a quadrilateral, what can be said about relations between the dragged quadrilateral and the one created by the intersections (Schwartz and Yerushalmy, 1987)? Why and when does the shape collapse into a single point? And, what determines where the point of intersection: When is it outside, inside, or on the perimeter of the quadrilateral?

6. Conclusion

In this chapter, we have illustrated a new strategy for feedback processes involving teacher, student, and tool during inquiry learning. We illustrated this new strategy in the context of a particular task related to the learning of geometry that has both specific learning goals related to the content of this task (the intersection of the perpendicular bisectors of a quadrilateral) and to what it means to develop a conjecture (that the set of mathematical objects for which a conjecture holds need not be a specific named set of objects, like square, but instead can be described by their characteristics, quadrilaterals that have an opposite pair of angles that sum to 180 degrees).

As this illustration suggests, the strategy of characterizing student submissions with mathematical descriptions points the field of learning analytics in two potentially useful directions. First, the illustration suggests that the data used in learning analytics can be content-specific and directly related to learning goals (to complement other sorts of measures like time on task, correctness, and more). Second, we also are intrigued by the potential of engaging students in analysis of their own learning. We have illustrated how information can be shared directly with students and can help shape their learning both in the midst of doing their work, as well as during reflection on the work they have submitted.

We also think there is still important development work to do as well as we learn to carry out this strategy. As designers, we continue to be interested in improving the nature of the feedback that STEP provides to support student inquiry. We close by outlining three future directions we have begun exploring and implementing in STEP. We are interested in providing other kinds of reports for students at a different stage of the inquiry process, and we are eager to explore the use of other types of learning settings and tools that can support students' learning with the reported information through meta-cognitive self-reflection processes.

One direction we intend to follow is to design means for learning through an activity that includes different tasks. To do so, we are designing an activity report. This report attempts to articulate and characterize changes across related tasks and in this way help students deepen their mathematical discourse about the activity. We envision the activity reports as tools that will help students communicate the progress they have made in developing their reasoning skills and strategies of inquiry beyond working on one task, but rather on a series of tasks. The report would include more explicit indications of changes with regard to the key aspects of the problem (e.g., "more directions as activity progressed", "fewer speed changes as activity progressed"). We will study whether it is helpful for students to follow new elements that were incorporated as they moved along in the activity. Another characterization we are using now on teacher reports and analytics is descriptive statistics (e.g., the number of students who had a certain characteristic in their submissions, Abu-Raya and Olsher, 2021). These numeric results can also be communicated interactively upon the student's request by providing ways to interact with the report, using the characteristics as filter conditions. The reports could in turn introduce students to descriptions of the progress of their work, providing a broader view than is achieved when analyzing individual tasks.

Another path that might be integrated as part of enhancing the opportunities for meaningful student mathematical inquiry in classrooms is working in small groups that create, share, and reflect on their collective example space. As described in Abdu et al. (2022), we grouped students according to the analysis of their respective examples with a content-specific objective to foster students' development of their personal example spaces. This diversity of personal example spaces indicates the potential of example-eliciting tasks and associated analytics in deploying content-specific grouping of students (Olsher, 2022). We are studying ways to provide such groups of students' collective information about their work before the grouping and in the course of the group work. This will give students opportunities to interact with the descriptions that led to their grouping and provide them with additional means to describe mathematically and distinguish between their respective submissions, and to find commonalities in their work.

Finally, deepening the meaning of feedback processes in inquiry learning can combine the meta-cognitive practice of self-assessment with automated assessment. *Self-reflection* or *self-assessment by students* is important for supporting their meta-cognitive skills, and places learners in a key position where they can develop responsibility and ownership for their learning (Ruchniewicz and Barzel, 2019). We continue to study how automated online information reports can become part of the meta-cognitive feedback process (Kadan-Tabaja and Yerushalmy, 2022.)

As these examples illustrate, we have only just begun to identify the potential of automatically characterized student examples as part of feedback processes, but there is much room for continued innovation in the design of such feedback.

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