3rd International Congress
on Mathematical Education

ICME

Karlsruhe 16–21 August 1976

Proceedings of the Third International Congress on Mathematical Education

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Edited by
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Editors' Preface

The aim of the organisers of the 3rd ICME 1976 was to present the situation at the moment and the world-wide tendencies toward future developments in mathematics education in its entire breadth and in its great variety. The result was a very extensive scholarly Congress programme, which had to be structured by putting the accent on the most important programme components: main lectures in plenary sessions, section meetings, working groups and projects.

We have seen it as our task as editors to make this programme structure as well as the results obtained within this framework visible in these Proceedings. Of course it was not possible to present everything that was accomplished in the Congress lectures, discussions or workshops in this book. But it is nevertheless a more or less complete documentation that recounts the essential activities, for the most part in original articles.

As was to be expected, the nearly 70 individual articles we collected were rather different in type and in length. We exercised no influence at all on the contents, but occasionally a bit on the outer form of the articles. An important unifying factor is the language: all the articles have been written in or translated into English. It is our hope that these Proceedings thereby offer the reader not only a mosaic, but also a synopsis of the Congress.

In preparing this book, we have had invaluable help from many people. We would especially like to thank Professor H.G. Steiner, Vice-President of ICMI, and J. Mohrhardt, Secretary of the 3rd ICME, for their contributions in the planning and putting together of this volume. Miss G. Neumaier typed the manuscript; the figures were drawn by S. Bumm. We had help on the translations from Mrs. V. Teichmann (Karlsruhe) as well as from the staff of the Institute for Mathematics Education in Bielefeld, and in proof-reading from the staff of the Mathematical Insti-

tute of the University of Karlsruhe. And we would also like to thank G. König of the editorial staff of the 'Zentralblatt der Mathematik' and Ernst Klett Publishers in Stuttgart for having the Proceedings printed.

H. Athen H. Kunle

Part 1

The Planning and the Programme of the Congress

1.1 Planning

Heinz Kunle
Chairman of the West German Sub-Commission of ICMI

1. The International Commission on Mathematical Instruction (ICMI) will celebrate its 70th anniversary in the near future: it was founded in 1908 at the International Congress of Mathematicians in Rome. The unforgettable Felix Klein of Göttingen was among the original initiators and founders, and also became a member of the first board of ICMI, along with Sir George Greenhill (London) and Henri Fehr (Geneva). F. Klein was also the originator of a great many ideas and impulses which have had a lasting influence on mathematics education up to the present day; his suggestions were visible in the first papers and recommendations of ICMI as well, for example, the idea of treating infinitesimal calculus at the school level, a revolutionary idea in those days, or of including descriptive geometry and projective geometry in school curricula.

German mathematicians and mathematics educators have continued to play an active part in the further development of ICMI, most recently especially in the person of Heinrich Behnke of Münster, who was Secretary of ICMI from 1952 to 1954, President from 1955 to 1958 and who is still Honorary Chairman of the West German Sub-Commission.

In view of these traditionally close ties to ICMI, the West German Sub-Commission (WGSC), while preparing for the 2nd Congress in Exeter, began considerations to invite ICMI to hold the 3rd International Congress on Mathematical Education (ICME) in the Federal Republic of Germany. The WGSC was well aware of the importance of the task and of the great responsibility involved in organising such a congress, but at the same time, the Sub-Commission was convinced that the Congress would have a most productive effect on the further development of the didactics of mathematics in the Federal Republic.

The official invitation to ICMI to hold the 3rd ICME in the

Federal Republic was extended then by the Chairman of the WGSC before the General Assembly in Exeter on September 1st, 1972.

2. In the weeks and months following Exeter, the WGSC began to shape its concrete ideas about the planning and organisation of the 3rd ICME. To this end, it was found most useful to analyse and evaluate the experiences and results of the preceding Congresses in Lyons and Exeter, each of which had been impressive in its own way.

Two competing basic points of view already arose in these preliminary discussions, both of which had to be taken into consideration and adjusted to each other as far as possible: on the one hand, an international congress on mathematics education is supposed to offer its participants an opportunity to gain a wide spectrum of information and a survey of the latest developments on an international level, e.g. through survey lectures. But on the other hand, it is just as important that the participants take an active part in the work of the congress and set their own priorities, e.g. through short communications or in small discussion groups outside the plenary sessions. This is even more true for a congress on mathematics education than for a congress of mathematicians.

H. Freudenthal expressed this problem in a somewhat different connection at the opening of the Lyons Congress in his typically precise way: 'Voilà le grand problème de l'enseignement mathématique: d'unir les forces de la mathématique faite qu'on apprend, et de la mathématique à faire qu'on doit créer lui-même.' (Educ. Stud. in Math. Vol. 2, 1969, p. 136.)

The WGSC decided to take a middle road and to encourage more individual activity on the part of the participants than at Lyons, at the same time providing for a greater concentration than at Exeter with its many working groups, by setting up relatively few sections. Consequently, the Congress Programme was designed to include three main components:

- several *plenary lectures* with invited speakers;
- work on clearly defined topics in several *sections*, with survey reports by invited speakers, short communications by individual

participants and free discussions (in sub-sections, if necessary);
- *further activities*, for example panel discussions, presentations of selected projects, exhibitions, films, etc.

These three main components were absolutely not to take place at conflicting times, but nevertheless, parallel events within the second or within the third component were unavoidable and even desired. And finally, the scholarly Congress Programme was to be supplemented by a varied Social Activities Programme, to give the participants a chance to relax and to get better acquainted with each other.

The WGSC then summarized these preliminary ideas on the programme structure and organisation of the 3rd ICME in its final application, which was delivered to the President of ICMI, Sir James Lighthill, in February 1973. Karlsruhe was chosen as the place for the Congress, since this city and its university seemed to be the best suited for various reasons. The acceptance of the invitation was then confirmed by the Executive Committee of ICMI in July 1973.

3. Now - three years before the opening of the Congress - the real planning and preparation could begin. The WGSC and the ICMI were designated as joint organisers, and the preparatory work was delegated primarily to two sub-committees:

The *International Programme Committee* (Chairman: H.G. Steiner, Bielefeld) developed the programme structure further as it had been worked out by the WGSC in outline, and gave it concrete form. There were a great many problems and details, especially regarding the plenary lectures, the sections and survey reports, the projects, the working groups, etc., which had to be clarified and decided upon; this was done in cooperation with the WGSC and the Local Organising Committee. A more detailed report is included in Part 1.2 of these Proceedings. However, it should be noted here that the International Programme Committee consisted of experts from several countries and made a special effort in the course of its varied activities to establish and maintain contact with as many national Sub-Commissions of ICMI as possible. In this way, the international character of the Congress, a fundamental aspect in the planning of any ICME, was taken into account.

The *Local Organising Committee* (Chairman: H. Kunle, Karlsruhe) was responsible for realising the programme that had been set up as far as the details of organisation, time and space were concerned and consequently for finally carrying out the Congress. At first, the Secretary of the Committee was E.F. an Huef, who died unexpectedly in April 1975 at the age of 37; J. Mohrhardt (also of Karlsruhe) was named his successor.

This is not the place to go into the many tasks and problems in planning, personnel, and even finances. A few remarks may suffice: without the helpful cooperation of the University and the City of Karlsruhe, the realisation and accomodation of such a large congress with so many participants would have been impossible. A variety of technical preparations in the lecture halls was necessary for the projects, exhibitions, films, etc. A team of interpreters from the European Council in Strasbourg agreed to do the simultaneous interpreting in the 'Schwarzwaldhalle'. The Congress Bureau was in operation for over two years and took care of mailing the three Congress Announcements, of preparing the four parts of the Programme as well as corresponding with many thousands of participants and people interested in the Congress. And even the Social Activities Programme had to be planned down to the last detail. The staff included up to 80 people at times.

Finally, the problems of financing such a congress must not go unmentioned. We would like to express our thanks once again to the German Research Council (Deutsche Forschungsgemeinschaft), the State of Baden-Württemberg, and all the contributors for the considerable funds necessary for an international congress of this size and this importance. Last, but by far not least, the success of the Congress was determined by the many participants from all over the world, who made the effort and, in some cases, the sacrifice to attend the Congress in spite of the world-wide economic recession and the shifts in exchange rates accompanying it.

1.2 The Programme

Hans-Georg Steiner
Chairman of the International Programme Committee

In developing the programme for the 3rd ICME, the *International Programme Committee (IPC)* was able to use the experiences gained from the two previous congresses. The 1969 Congress at Lyons was substantially based on 20 papers by invited speakers who had individually chosen their topics. In addition there were book exhibitions, project presentations, workshops, mathematical classrooms, and a small number of short communications. Six panel discussions were arranged in a more or less ad hoc manner. At Exeter, the number of plenary lectures was reduced to seven, the Presidential Address included. The short communications by congress members were abandoned entirely. A significant feature was given to the 2nd ICME by establishing a totality of 38 working groups which by their themes covered a broad spectrum of mathematics education. The work in these groups was based on individual papers submitted before the congress for limited distribution among group members and discussions. Book exhibitions, project presentations, and workshops were organized as National Presentations.

There are clearly various ways of designing a programme for an international congress. In a field with relatively recent research components, with an exponential growth of development, and with rapidly changing orientations, like mathematics education, the pattern of the programme must change according to changing needs.

The first congress was a new beginning. Before 1969 the International Commission on Mathematical Instruction (ICMI) had organized fully international conferences only as sections of the International Congresses of Mathematicians. As a consequence of this link, mathematical content in school teaching and teacher training and matters of organization in the national training systems played a dominant role. At Lyons the way was opened for a broader view and several experts from various areas of mathematics education were among the main speakers. The programme was heavily centred on these contributions by invited experts which guaranteed quality and standards for the new kind of congress.

The Programme Committee for the Exeter Congress took another standpoint. They agreed that the Congress should take into account the great range of conditions in which mathematical education takes place, should attempt to cater for all interests, and put much emphasis on the active participation of the members themselves. It therefore provided a great variety of small groups devoted to different aspects of the field which could be joined according to individual and specialist interests.

The IPC for the 3rd ICME in agreement with the West-German subcommission of ICMI started out from the following analysis:

Nearly all invited papers at Lyons were highly interesting and substantial. However, the themes were individually chosen by the speakers, and by their topics and treatment the papers did not sufficiently cover the range of important areas and problems of mathematical education as it was seen now some years after. Several areas of research and development needed a critical review of recent progress, problems and failures, as well as present trends. Such areas should be identified and experts be asked to prepare survey-trend reports.

As with the Exeter programme the number of *main lectures* should be limited to about 6. In contrast to the intended reports these lectures should be based on individual views held by outstanding scholars in mathematics, education, and other fields relevant to mathematics education.

The high degree of specialization as represented by the 38 Exeter working groups was accepted in principle. However, there were doubts whether such diversity should be a pattern for continuous work throughout the congress. At Exeter many participants felt in some way locked in for too long in the two working groups they had to choose. Also only about one third of the groups seemed to have been really successful in planning, and performance, and in laying ground for further work.

The intended *survey-trend reports* seemed to be a vehicle to overcome these difficulties, if the areas for the survey-trend reports, whose number should be limited to about 12, could also become the themes for the *congress sections*. The work in the sections could then consist of the report, followed by a discussion, some selected short communications, and further discussions in groups.

On the other hand it was agreed that all *Exeter working groups* should be given an opportunity for a follow up meeting at Karlsruhe, provided that a programme for this meeting was prepared before the congress. The principle of survival of the fittest finally showed that 7 of these groups met again and had very fruitful sessions.

There were doubts, whether the organization of project presentations, exhibitions, workshops etc. as National Presentations was the right way of focussing on substantial work of international interest. Projects should at least be considered by the criterion of what they had contributed to fundamental problems of mathematical education of super-national importance. It was therefore decided that on the basis of a careful analysis by the IPC only *selected projects* should be invited for presentation.

An activity similar to developmental projects is that of *study groups*. Such groups consist of experts who are collaborating in research and analytical studies devoted to special problems of mathematics education. Outstanding groups of this nature should be invited to present their work and organize programmes for group activities during the Congress.

The problem of handling orally presented *short communications* at big congresses is well known. However, the abandonment of short communications seemed to be too strong a restriction on the variety of contributions and ideas which might be spread by them. An excellent solution seemed to be the organization of *poster-sessions*. Since a large number of short communications was to be expected, only a very small part could be selected for oral presentation during the sections' work. The main part would have to be assigned to one of the sections and presented in writing in the section's area with its author standing by during the poster-session. Nearby exhibitions of related material would also allow small projects to present themselves within the poster-sessions.

The shaping and final planning of the programme was an extensive enterprise with many people from various countries involved. In addition to the two preparatory committees mentioned (see page 13-14), in some countries national preparatory committees were established and collaborated with the IPC and the Karlsruhe Organizing Committee. Through letters addressed to their chairmen the national sub-commissions of ICMI were asked to make suggestions for the programme and to provide information. Many research and development centers were consulted. Members of the IPC on their travels talked with hundreds of people in many countries to exchange views and collect information relevant to the congress programme.

An extremely valuable cooperation between ICMI and UNESCO was established at a consultative meeting at the UNESCO headquarters at Paris in November 1974. At this meeting 13 themes for the survey-trend reports were identified and an agreement was made that the final versions of the reports should be the chapters of Volume IV of the UNESCO-series 'New Trends in Mathematics Teaching'. UNESCO would provide financial support for authors' meetings and participation in the congress.

During the Paris meeting an extensive system for collecting data for the reporters and for receiving reactions and feedback to the various versions of the reports was designed. Each reporter was to have an internationally composed advisory group of about 15 members who were to provide him with information on developments in the various regions of the world, on research findings, literature, centers, experts etc. In addition the reporters were to extend their information system by directly contacting other experts, centers, national committees, and whatever sources they found useful. On the basis of a large amount of information the reporters and the members of the 13 advisory groups were selected by the IPC in agreement with other committees.

By means of financial support from the VW-Foundation (Stiftung Volkswagenwerk) it was made possible to hold a meeting of the 13 reporters, the 13 West-German coordinators from each advisory group and some members of the IPC at the Mathematics Research Institute, Oberwolfach, Black Forest, in December 1975. In addition the two reporters from the Institute for Didactics of Mathematics (IDM) of the University of Bielefeld, FRG were each able to organize a conference with members of their advisory groups and other international experts at Bielefeld in the spring of 1976.

During the Congress the members of the advisory groups who were able to come to Karlsruhe formed panels for each section. They participated in a panel discussion which followed the report and met several times during the course of the congress. The sessions of the sections and the meetings were chaired by internationally chosen chairpersons who reported on the work of their sections in a joint meeting of the Executive Committee and the members-at-large of ICMI, the IPC and the West-German Sub-Commission.

There were many suggestions for themes and speakers for the plenary sessions. Reports of a study of *The Interaction between Mathematics and Society* conducted by the Royal Society were received. In his capacity as an outstanding applied mathematician, acquainted with these studies and familiar with problems of mathematics education in particular by his former presidency of ICMI, Professor Sir James Lighthill was invited to speak on the above theme.

Interpretations of the intent of pure mathematics and its contemporary developments have significantly affected the orientation of mathematics teaching in the past. A continuous dialogue and interaction between mathematics research and mathematics education is indispensable. Therefore Professor Michael Atiyah, a world famous mathematician with the ability to reflect upon mathematics, what mathematicians do, and what implications this may have on the learning and teaching of mathematics was invited to talk about *Trends in Pure Mathematics*.

Professor Georges Guilbaud of the Ecole Pratique des Hautes Etudes, Paris had been a speaker at regional ICMI conferences.

On the basis of a profound knowledge of pure and applied mathematics and with a philosophical mind he had analysed specific aspects of mathematics relevant for education. Being asked to speak at the Congress he chose as his topic *Mathêmatiques et Approximations*.

Professor Peter Hilton combines a deep involvement in mathematical research with a strong engagement and competency in mathematics education. He is a consultant of several US curriculum projects and has played a leading role at conferences and in committees devoted to mathematics education. With respect to controversial issues he has always taken independent and clear position. When he was asked to talk at Karlsruhe about the dangers of behaviorism and managerialism in mathematics education he accepted, but wanted to treat this problem as part of the broader theme *Education in Mathematics and Science Today: The Spread of False Dichotomies*.

It was highly recommended by non-German members of the IPC to have as one of the invited plenary speakers a representative of the specific work in the didactics of mathematics which has been done in recent years in the Federal Republic of Germany. One of the most prominent representatives of this work is Arnold Kirsch. Professor Kirsch chose as title of his lecture: *Aspekte des Vereinfachens im Mathematikunterricht*.

Though there was a section devoted to algorithms and computers it was felt that this topic might also deserve a discussion in a plenary meeting. The time for a sixth main lecture was therefore used for a Panel Discussion chaired by Professor Hans Freudenthal on *What May Computers and Calculators Mean in Mathematical Education in the Future*.

Globally speaking the programme for the 3rd ICME tried to combine variety and concentration. The working programme components were: plenary lectures and a panel discussion at a plenary session, the work in the sections and poster-sessions, project presentations, workshops, follow up meetings of Exeter working groups, study groups, exhibitions, film presentations, and committee meetings. A basic guideline for the IPC was the professionalization of work in mathematics education.

1.3 The Congress in Action

1. The Congress was opened on Monday, the 16th August, 1976 at four o'clock in the afternoon in the 'Schwarzwaldhalle' in Karlsruhe by the Chairman of the Congress, H. Kunle (University of Karlsruhe), the President of ICMI, S. Iyanaga, and the Vice-President of ICMI, B. Christiansen. The Minister of Education of the State of Baden-Württemberg, the Lord Mayor of the City of Karlsruhe, the Rector of the University of Karlsruhe, and the Dean of the Faculty of Mathematics also addressed the Congress participants with their words of greeting.

1831 Congress participants (Full Members) from 76 countries came to Karlsruhe:

1281 from Western and Eastern Europe (including 610 from the Federal Republic of Germany, 193 from France, 175 from Great Britain),

318 from North, Middle and South America (including 210 from the USA, 51 from Canada, 21 from Brazil),

123 from Asian countries (including 70 from Japan),

68 from African countries, and

41 from Australia and New Zealand.

In addition, there were 237 Associate Members.

2. After the opening addresses on Monday afternoon and an introduction by the Hon. Chairman of the Congress, H. Behnke (University of Münster), Sir James Lighthill (University of Cambridge) gave the first main lecture.

The other four *Main Lectures* were held on Wednesday and Thursday mornings in the 'Schwarzwaldhalle'. The complete texts of these lectures are printed in Part 2 of these Proceedings.

The six *A-Sections* met on Tuesday and Friday mornings, the seven *B-Sections* on Tuesday and Friday afternoons (until 4 or 5 o'clock) in the 7 zones of the University campus. The abstracts of the introductory survey reports have been included again in the Proceedings, as well as short summaries of the work done in the sections (see Part 3.1). Part 3.2 consists of a list of names of the almost 200 participants who sent in *Short Communications* to the Congress. These Short Communications were displayed on poster boards and discussed during the Poster-Sessions on Thursday afternoon; some of them were also presented orally in the Sections.

A great variety of activities took place during the afternoon hours available after 4 or 5 o'clock: the 7 *Exeter Working Groups*, which met again in Karlsruhe; the 4 new *Study Groups;* the 4 *Workshops;* and the 15 invited *Projects* from 6 countries. In addition, *Films* on mathematics education were presented evenings, and *Exhibitions* from scientific institutions and from 40 publishers and firms were open all the time. These activities are documented in Part 4 of the Proceedings.

The *General Assembly of ICMI* met on Thursday from 5.30 to 7.15 p.m. The reports may be found in the ICMI Bulletin Nr. 8 (1976) and in Parts 5.1 and 5.2 of these Proceedings. The Chairmen of the National Teachers' Associations also held a meeting on Thursday at 4.30 p.m. to exchange information (see Part 5.3).

An important factor in any international congress - perhaps the most important factor altogether - is the opportunity for personal conversations, for scholarly and social contacts between the Congress participants. These activities could also take place during the scholarly parts of the programme of course, but flourished especially in the *Social Activities Programme:* receptions by the City and the University of Karlsruhe on Tuesday evening, a concert by the Radio Symphony Orchestra of Stuttgart on Thursday evening, excursions on 13 different routes with a total of 27 busses on Wednesday afternoon, and finally, the many sightseeing tours during the entire week of the Congress. These were designed to give the participants a chance to get to know Karlsruhe and its surroundings.

The Congress ended on Saturday, the 21st August, with a plenary session in the morning. After a panel discussion on the role of computers and calculators in mathematics education chaired by Professor H. Freudenthal (see Part 4.1) and closing addresses, the President of ICMI declared the Congress closed.

3. The events of the 3rd ICME and the results so far are documented in the *Congress Programme*, which appeared in four parts (Part I: Time Schedule and General Survey; Part II: Abstracts of the Survey Reports in the Sections; Part III: Abstracts of the Short Communications; Part IV: List of the Full Members) and in these *Proceedings*. In addition, Volume 4 of the UNESCO-series 'New Trends in Mathematics Teaching' will be published in the near future.

It is not our task, as editors of the Proceedings, to pass final judgement on the Congress. But we have tried to prepare a basis for an evaluation of the Congress, by collecting about 70 individual articles and piecing them together to a mosaic synopsis of the Congress. Now it is up to the participants and the readers to judge and evaluate for themselves, each from his own standpoint.

Were some fields emphasized in the programme and in the course of the Congress? We think that the entire area of mathematics education was covered by the various Congress activities, in such a way that the participants could set their own priorities. At least that was the hope of the organisers. Perhaps the programme put a particular accent on the topic 'Computers and Calculators in Mathematical Education' by stressing it in Section B 5 and B 7 and through the closing panel discussion.

With an eye to the organisation and realisation of future Congresses, the following remarks may be useful. As did the Karlsruhe Congress, so will any future ICME have to deal with the dilemma of two controversial points of view (see 1.1, page 12): a wide range of information through survey lectures and consequently the 'oligarchy' of a few active speakers over a great many passive participants - or, on the other hand, individual activities by as many participants as possible and the resulting 'anarchy' of free-for-all discussion? The Karlsruhe Congress tried to take the middle road. In any case, the Poster-Sessions certainly proved themselves invaluable and will undoubtably return in this or another form at later Congresses.

Last but not least, we would like to touch upon the difficult problem of language: the organisers of future international congresses on mathematics education will have to give it some thought again and again, so that large linguistic minorities are not limited in their active participation in the Congress. This is particularly true for discussions in sections and in working groups, less so for main or survey lectures.

Part 2

The Main Lectures

2.1 The Interaction between Mathematics and Society

Sir James Lighthill

1. Reasons Why Teachers Need Information on the Newer Uses of Mathematics

It is, I must confess, a relief that on the first day of this 3rd International Congress of Mathematical Education I do not have to give a Presidential Address, as I did in Exeter at the 2nd ICME in 1972; an Address where I had to try to mention every aspect of the Congress and therefore could only *briefly* discuss a personal theme 'Integrated Pure and Applied Mathematics'. Today I can concentrate entirely on a particular theme: 'Interaction Between Mathematics and Society'; though obviously both these themes have something in common, and are related to my experiences as a Professor of Applied Mathematics.

Applied mathematics has two aspects: the interaction between mathematics and other subjects; and the interaction between mathematics and the external world. In my Exeter Presidential Address I emphasized the importance of viewing mathematical education within the context of the total education of the individual; including *all* the learning situations to which the student is exposed, in school and out of school. I spoke of the teacher seeking to communicate how mathematics interacts with other subjects, and with the external world. I spoke of the teacher seeking to convey that mathematics has an important *contribution* to make to solving the world's most pressing problems.

At the Exeter Congress, two different Working Groups were concerned with these two aspects of applied mathematics[1]: the Working Group 'Links with Other Subjects' (organised by Professors Bell and Matthews) and the Working Group 'Application of Mathematics' (organised by Doctors Pollak and Rayner). Similarly, here at the Karlsruhe Congress, there is a Section B6 under Dr Pollak's direction on 'Interaction between Mathematics and Other School Subjects', and there is the present Plenary Lecture on

'The Interaction between Mathematics and Society.'

I might mention, too, that at Exeter there was also[1] a Working Group 'Mathematics in Developing Countries' (organised by Messrs Cundy and Wilson) which strongly influenced the International Commission on Mathematical Instruction to hold subsequently two Regional Conferences, in Kenya (September 1974) and India (December 1975), both of which I attended. Of particular relevance to the subject of this Lecture was the Indian Conference on the theme 'Integrated Curriculum for the Developing Countries of Asia'. The stated aim of this Conference was to move, in each society represented there, towards a mathematics curriculum relevant to that society's goals. It was noted that, especially in developing countries, *practical* goals predominate; and that these may need to be allowed to exert a powerful influence on curriculum development in those countries.

All the Working Groups, and the Conferences, emphasized the value of *two things*. The first of these was the building up of a great store of *concrete examples* to assist in establishing abstract concepts, and in their reinforcement; and to help in *answering* (or, still better, forestalling) *the question* 'What is the use of it?' with regard to a mathematical concept (or even with regard to the whole subject!).

A second major need was noted especially in the report[1] of Dr Pollak's Working Group on 'Application of Mathematics'. This emphasized the great benefits to students of *project work in depth* on 'real-world practical problems'. To experience work on one or two projects of this kind (there can never be time for more than a few) gives students *motivation*; it adds *interest* to their mathematical studies; it gives them *insight* into the process of building a bridge between the abstract world of mathematics and the real world; it gives them insight into the formulation of mathematical models. Above all, it gives students *confidence*; it gives them confidence in mathematics as something that they can use in society. The Working Group recommended that participation in such a project should be taken to the stage where the model yields numerical results, *and* on to the next stage where those results are carefully and critically interpreted.

At Karlsruhe there is also a Section B3, under the direction of Dr d'Ambrosio, on the 'Goals and Objectives of Mathematics Teaching'. It seems inevitable that, whether in developing or in developed societies, one important influence on these objectives will be the general goals and aims of the society. Obviously, these are not the only influence; it seems inevitable, however, that out of the *various main influences* on the goals and objectives of mathematics education, *one* will be the general goals and aims of the society which pays for that mathematical education.

Now I come to the axiom on which this Lecture is founded.

Axiom: *The planning of mathematics education needs to take into account the interaction between mathematics and society.*

All my earlier remarks may, I believe, contribute to explaining why this (fairly weak!) axiom seems to me a sound basis for the treatment of my subject. Possibly an even clearer way of stating the axiom is to say: 'In each society, the development of mathematics curricula needs to take into account the interaction between mathematics and that society'.

An immediate consequence of the axiom is my main

Theorem: *Information about that interaction* (between mathematics and society) *is urgently required.*

Lecturing in Germany about a theorem which follows so clearly (unmittelbar!) from the axiom, I feel justified in adding:

Beweis: klar.

Evidently, the information is urgently required for the planning of mathematics education; that is, for curriculum development in mathematics. The information in question must also, as we saw above, be of great value to teachers. It can help to build up their store of concrete examples for establishing concepts and reinforcing them. It can help teachers to give their students motivation and interest and *confidence in mathematics as something they can use in society.*

I must point out here that the interaction between mathematics and society is *twofold:* (i) indirect, and (ii) direct. By (i) the indirect interaction of mathematics with society, I mean an interaction 'at second hand', through the interaction of mathematics with other subjects; for example, with mechanics, physics and chemistry; and, through those, with different branches of engineering (such as mechanical, electrical and chemical engineering). Mathematical statistics, again, has a major interaction with biology and economics; an interaction which has had important influences in medicine and agriculture, and also in financial organisation.

All this *indirect* interaction between mathematics and society is generally familiar to educators and *has* influenced curricula; in which, for example, mechanical and statistical applications of mathematics have often played an important role. The only point about these familiar areas of application which I would like to mention here is that the mathematics is used *not only* for 'analysis' of problems, but also for 'synthesis', as in engineering design; or as in the statistical design of experiments. Here we see an important optimization function exercised by mathematics in its application to different kinds of design.

However, the rest of this Lecture is about the *newer uses of mathematics;* that is, about (ii) the direct interaction between mathematics and society.It follows from my Theorem that we *should* ask: 'Outside the applications to pure science, and beyond the *familiar* practical uses of mathematics (through engineering and statistics) what are the *newer uses of mathematics*?'.

It is helpful to seek answers to this question through the activities of societies such as SIAM (Society for Industrial and Applied Mathematics) in America, or GAMM (Gesellschaft für angewandte Mathematik und Mechanik) in Germany, or IMA (Institute of Mathematics and its Applications) in Britain. A major function of these bodies has been the organisation of meetings concerned with describing all these newer uses of mathematics.

Another way of seeking answers to the question is through the collaboration of *several* persons with different experiences in the practical application of mathematics. This happened when I worked with five other authors (Doctors Hiorns and Hollingdale and Professors Potts, Beard and Rivett) under the auspices of

IMA for a year (1975) to produce a 'popular' book[2] entitled 'Newer Uses of Mathematics'. This is to be published in 1977 by Penguin Books Ltd.

This book is in six chapters (one by each author). It is written for several groups of readers, including students in their middle 'teens with some interest in mathematics, and including *also* their teachers. We have seen how material of this kind can help teachers to give their students motivation, and insight, and confidence in mathematics as something they can use in society. Another important group of readers may be those students' parents, who need to know why their children's interest in mathematics may be worth encouraging.

The book's object is to explain an important new phenomenon of the 3rd quarter of the 20th century: a phenomenon which the book describes as follows. 'Recently, mathematics has been more and more used to help with the things that really matter to us most.

'We care about the air we breathe, and its freshness and the climate that it brings us. We care about the water that quenches our thirst and cleans us and irrigates our crops. We care about the earth as a source of food and of raw materials..... All those things are what people mean when they speak about the ENVIRONMENT. In the fight to improve it, mathematics is much used.

'Again, we care about jobs. That means that we need our industries to be competitive and to avoid wasteful use of resources. Prosperity for all of us depends on work being organised economically. We need to communicate with each other, so we want good telephone systems and good transport systems; indeed, we want NETWORKS distributing to our homes and factories those essential services and many others such as power. There's a direct benefit to standards of living from EFFICIENCY in all of these areas. In the fight to improve it, mathematics is much used.

'We depend also upon trade. We all want shops where we can find what we need, and money which shopkeepers will accept for goods we buy and which people will pay us for services we can give. We care about our industries earning enough from the goods they sell to be able to pay their employees and buy their raw materials and pay for re-equipment ..... In commerce and industry, then, and indeed also on the national scale, we need sound FINANCE and PLANNING. In the fight to improve *them*, mathematics is much used.'

The book divides the newer uses of mathematics, then, into three main areas: (a) the mathematics of the ENVIRONMENT; (b) the mathematics of EFFICIENCY; (c) the mathematics of PLANNING. The book devotes two chapters to *each* of these three main areas.

The area (a), of mathematics applied to the ENVIRONMENT, is concerned with the mathematics of water pollution, of air pollution, of weather forecasting, of flood prediction, of exploration for natural resources, and so on. The mathematics of the environment is especially concerned with formulating and using mathematical models of environmental processes; and with developing effective computer algorithms to implement those models.

It is concerned also with *stability*: stability of the environmental processes described, *and* stability of the algorithms used to represent them. It is concerned, often, with *extrapolation*: either temporal extrapolation as in weather forecasting or flood prediction; or *spatial* extrapolation as in exploration. This latter may be exemplified by a survey of the gravity field over the earth's surface which is used to extrapolate below the surface to determine the density anomalies producing it. In this Lecture, however, I concentrate on just two of these environmental applications: the mathematics of water pollution (section 2) and the mathematics of weather forecasting (section 3).

The area (b), of mathematics applied to industrial EFFICIENCY, is concerned to *improve the utilisation of resources* through the application of a wide variety of mathematical techniques. In this Lecture, however, I shall concentrate on the mathematics of NETWORKS; particularly, because I want to avoid giving the impression that only the numerical aspects of mathematics are important. We shall see, in fact (section 4), that *applied network analysis* works through an intimate blend of the topology of networks with more analytical techniques.

We shall be concerned with telecommunications networks and transport networks and fluid-flow networks (such as hydraulic networks and networks of natural-gas pipelines). We shall find that the more familiar techniques of the mathematics of industrial efficiency (such as linear programming or nonlinear optimisation) are often *combined with* the topology of networks; that is, with graph theory. We shall also see that all these matters are concerned with DESIGN; actually, design in the 'tactical' sense (design of ways of improving resource utilisation on a day-to-day basis).

The area (c), of mathematics applied to PLANNING, is concerned with design in the 'strategic' sense. Often, this is fundamentally a matter of getting the dimensions of a system right. In the mathematics of telecommunications traffic we speak of the process of 'dimensioning' the elements in a switching network. In other applications, we may be concerned with getting the 'time dimension' right for a major construction programme, by Critical Path Analysis; or getting the 'human dimension' right as in manpower planning .....

2. The Mathematics of Water Pollution

My first example of environmental mathematics, then, is from the mathematics of water pollution; illustrated by the problems of Britain's capital city (London) situated on the estuary of the River Thames.

Around the coasts of the British Isles there are rather big tidal movements: vertical distances of around 4 m between high tide and low tide are common; furthermore, values twice as much as that are typical within an estuary (where a river meets the sea) owing to local *amplification* of the tidal oscillations. Thus, there is a rapid rush of water into such an estuary and out again as the tide rises and falls.

Actually, much of Britain's population lives near estuaries, and this strong tidal flow into and out of them is a great national asset; which allows rapid and convenient disposal of the domestic waste generated by that big population, and its mixing and dilution into the billion cubic kilometres of the oceans. But the more important a national asset may be, the more necessary it is to quantify it; in this case, to quantify the amounts of pollutant which the flow is able to disperse. For the Thames Estuary, around which live ten million people (the population of Greater London), this quantification was achieved by the famous Thames Estuary Mathematical Model[3], developed during the years 1948-63 (before pollution researches became fashionable!) and since then imitated in many of the world's other densely populated estuaries[4,5].

'The Thames was dead, and is alive once more' is a brief statement of the results of using this mathematical model. More precisely, anaerobic conditions in the estuary have been eliminated since 1964.

The sea, of course, contains not only dissolved salts but also dissolved oxygen (O_2). Dissolved oxygen is essential to the 'life' of a body of water. Fishes absorb dissolved oxygen through their gills, and in the absence of dissolved oxygen (that is, in anaerobic conditions) no fish or other animal life is possible. Furthermore, an anaerobic body of water smells offensively of sulphide because certain bacteria, when they cannot obtain oxygen directly, derive it indirectly by reducing sulphates to sulphides.

In the nineteen-fifties, no fish could be found in the upper 70km of the Thames Estuary. However, anaerobic conditions were permanently eliminated from 1964, and by 1970 fish of over 50 species were living in all parts of the estuary[4]. An essential pre-requisite of this change was the development of a mathematical model of the dissolved-oxygen content. Only by this means was it possible to determine the modifications that would be needed to achieve a permanently aerobic condition.

Mathematics, of course, is an outward-looking body of knowledge. A mathematical model makes use of observed data; it makes use of laboratory research; then, the mathematical model *combines together* all that information.

'Counting' may be the first mathematics that a child learns, and 'adding up' is often the next. The operations involved in making a mathematical model are extensions of these two processes; collecting the data can be regarded as an extension of 'counting', and combining them all into a model as an extension of 'adding up'.

Fortunately, the voluminous data on the Thames make it one of the best documented estuaries in the world. Also, excellent laboratory research was done during the investigation.

Figure 1 shows the Thames Estuary, which is tidal up to Teddington. The tidal estuary is 100km long; the numbers on the map give distances down-river from Teddington, increasing to 100 km where the estuary meets the sea at Southend. The average tidal movement at London Bridge (that is, at the City of London) takes the water 14 km up the river and back. The chief pollutant inputs occur at the Main Sewage Works (*Cloacus Maximus*!) situated at Beckton, Crossness and Mogden; although there are also certain pollutant inputs from various small tributaries.

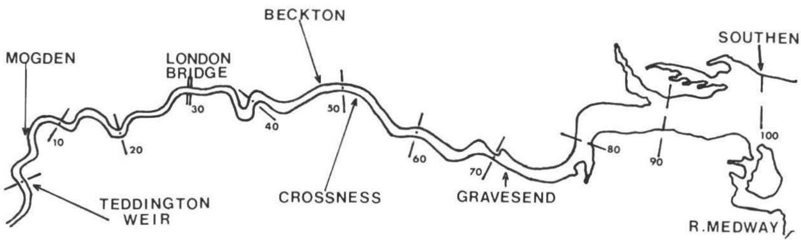

Figure 1. The Thames Estuary. Figures denote distance in kilometres seaward from Teddington Weir. There are Sewage Works at Beckton, Crossness and Mogden.

The volume flow of fresh river water over Teddington Weir averages 71 $m^3\ s^{-1}$. However, it is very variable: average values over different *years* range from 20 to 123 $m^3\ s^{-1}$, while averages over different *months* range from 2 to 360 $m^3\ s^{-1}$. Accordingly, this fresh-water inflow represents one of the most important *variable inputs* into the mathematical model. Another important variable input is temperature, because all the various processes by which bacteria modify sewage take place faster at higher temperatures.

Further variable inputs, of course, are represented by the amounts of pollutant discharge at the various input points. One important discovery made during the investigation was that two variables are needed to represent the effect of any pollutant input: the quantities of

(a) oxidisable organic matter;

(b) oxidisable nitrogen.

Dissolved oxygen is depleted relatively *fast* by (a) as bacteria bring about the oxidation of that organic matter. The corresponding action on (b) generates ammonia (NH_3) whose subsequent relatively *slow* oxidation is an oxygen-depleting process with a much longer time constant than (a).

The maxim 'Divide and Rule!' (*Divide et impera*), applied to politics by Louis XI, holds with equal force in the formulation of mathematical models. The Thames Estuary model *divides the estuary* into many short stretches of equal length. Each of these contains *a known volume of water at half-tide*; that is, at a water-level halfway between high tide and low tide.

In each such short stretch, the model calculates the changes in composition during one whole tidal period; that is, while the water moves up-river from its half-tide position, then right down to its low-tide position, and back to half-tide. By the 'composition' of the water I mean its dissolved salt concentration, dissolved oxygen concentration, amounts of (a) oxidisable organic matter and (b) oxidisable nitrogen, and also its content of ammonia, oxides of nitrogen, etc.

The changes during one whole tidal period taken into account by the model[3] are due to 8 effects, as follows.

(1) The push towards the sea, due to the total volume of the inflows upstream of any stretch, causes a bodily movement downstream of exactly that volume of water from this stretch into the next stretch.

(2) A definite amount of mixing between the contents of two adjacent stretches occurs as a consequence of the back-and-forth tidal movement. The reason why this amount of mixing can be accurately estimated is because (1) and (2) alone fix the *dissolved-salt* distribution, which has been measured accurately under a wide range of conditions.

(3) The input of each constituent is specified at each input point.

(4) A further input that the model needs to specify is the input of oxygen from the air. Theory indicates that this must be proportional to the defect of dissolved-oxygen concentration below the so-called 'equilibrium' concentration; that is, the concentration *in equilibrium with air* (at the given temperature and dissolved-salt concentration); finally, the constant of proportionality was obtained by direct experiment.

(5) The model next takes into account (see (a) above) the oxidation of organic matter produced by bacterial action during one whole tidal period. A definite fraction of the oxidisable organic matter is removed, along with a corresponding reduction in dissolved oxygen.

(6) Simultaneously, the processes acting (see (b) above) on oxidisable nitrogen generate ammonia; furthermore, during a tidal period a certain *small* fraction of the ammonia is oxidised, leading to a correspondingly small reduction in dissolved oxygen.

(7) More briefly, I note that the model allows for the bacterial action on sulphates which I mentioned before.

(8) Finally, their action on the oxides of nitrogen is allowed for, and this turns out to be of importance for the *stability* of the model. The reason is that certain abundant bacteria cease to absorb dissolved oxygen directly at concentrations below about 5 % of the equilibrium concentration and then obtain their oxygen instead by breaking up the nitrate ion (NO_3^-) into O_2 and N_2. This gives the model what is called 'robustness': when the model predicts aerobic conditions, any relatively small errors in the assumptions will not vitiate the prediction, because the activities of these bacteria 'cushion' the river against a sudden 'bump' down to anaerobic conditions.

The mathematical model whose equations I have described verbally under these eight headings might have been used in various ways, but it turned out that the best use of the model was to find *steady states for different input vectors.* By 'input vector' I mean the combination of freshwater inflow, temperature, and the different pollutant inputs. By 'steady state' I mean a state where all changes of composition during one whole tidal period are zero. Finding such a steady state involves the solution of a large set of simultaneous equations; a job that computers are particularly good at!

This led to a comprehensive set of calculations (involving a big ensemble of different input vectors). Finally, those calculations were used to determine the level of modification of the pollutant input that would be needed to eliminate anaerobic conditions altogether. The model predicted correctly that a modification programme due to be carried out in 1955 would *not* be sufficient; and also specified two further modification programmes, actually carried out in 1960 and 1963, which would be sufficient. The predictions were borne out: anaerobic conditions were eliminated from 1964, with enormous benefit to those living near the estuary[4]. 'The Thames was dead and is alive once more'.

3. The Mathematics of Weather Forecasting

Now I want to describe a much bigger exercise in 'counting' and 'adding up'; namely, the mathematics of weather forecasting. Here, I am regarding as an extension of 'counting' those meteorological observations that are made twice daily at a great number of weather stations all over the world, and the collection and reduction of that data; while an extension of 'adding up' is the process of step-by-step forward extrapolation in time from that initial data.

Weather forecasting is a splended example of the power of mathematics to 'add up' the effects of environmental processes so extensive in space that they are going on over a great part of the Earth's surface[6,7]. It is of course well known from weather maps that the weather patterns with which the mathematical model is dealing extend over many thousands of kilometres. Yet the model is concerned, not with anything remote from human beings (or from school children), but with the weather: a physical process of tremendous significance to us all (as well as to many major industries on which we depend), but operating on a vastly *bigger-than-human* scale.

Once more the key to mastery of this massive problem is the Louis XI maxim DIVIDE ET IMPERA: 'Divide and Rule!' The mathematical model[2] aims to forecast the weather 1,2 or 3 days ahead; but it *divides* the day into many equal time-intervals of a few minutes each, and seeks to make a forecast by *adding up* the effects of changes in a succession of such equal short time-intervals.

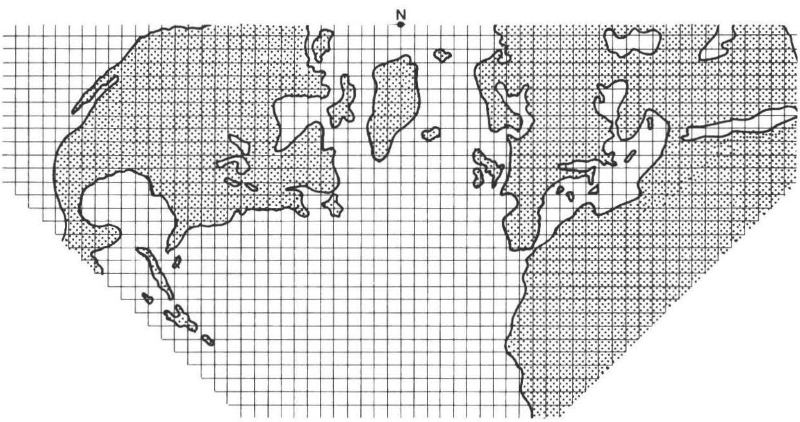

Figure 2. The British Meteorological Office uses a map of the northern hemisphere on a 'stereographic' projection and applies its main computational prediction process to an eight-sided area of the map centred on the north pole N. This eight-sided area (of which exactly half is shown here) comprises nearly the whole northern hemisphere. It is divided into a grid of equal squares, as drawn here, for the half illustrated, over a very rough sketch of the main land areas in it.
Note: this is the so-called 'coarse mesh'. Also, a 'fine mesh' (with grid-points three times closer together) is used over a smaller area centred on the British Isles for more detailed local predictions.

In addition, the model divides a map of the northern hemisphere into many equal squares. (At present, an effective extrapolation forward in time is feasible mainly for the northern hemisphere; in the southern hemisphere there is still a serious shortage of weather stations to provide the necessary initial data.) A map on a stereographic projection is used, of which just half is displayed in figure 2 (the complete octagonal grid of squares practically covers the whole northern hemisphere).

Furthermore, the model divides the air above each square into *ten equal 'slices'* (that is, layers); *equal* in the sense that they include equal weights of air. The weight of each, per unit area, is 1 decibar. A familiar fact is that a typical pressure exerted by the weight of the air above us is 1 bar, so we take the ten slices as each having the exact weight per unit area 1/10 bar; that is, 1 decibar or $10^4 Nm^{-2}$.

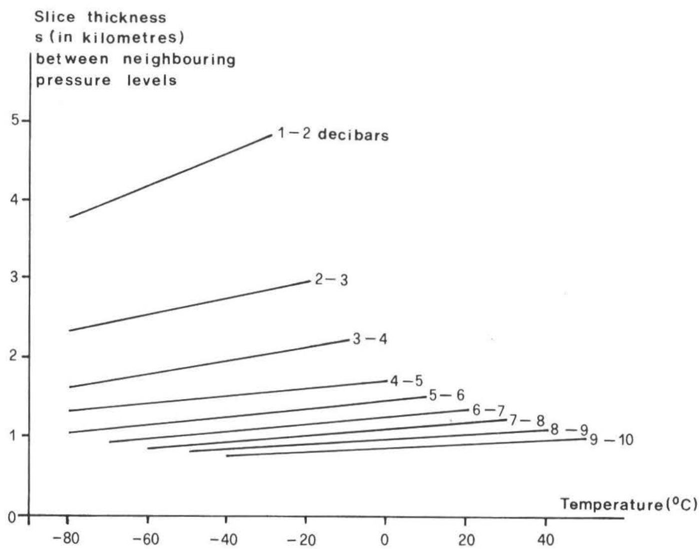

Figure 3. The temperature of each slice fixes its thickness as shown here.

Of course, the upper slices have a greater *thickness* (that is, the total depth of the slice from top to bottom) than do the lower slices, which take up less volume because they have a bigger weight of air pressing down on them. Figure 3 shows that the thickness of the slice between the pressure levels 1 and 2 decibars is over 4 km, but that the slice between the 9- and 10-decibar levels is just under 1 km thick. The thickness of each slice increases with temperature (because air expands when it is heated); and, in the mathematical model, this slice thickness acts as a measure of the temperature at the level in question.

I shall rather briefly describe the mathematical model that has been used since 1974, twice daily, by the British Meteorological Office at Bracknell: a 10-level model[8] for the northern hemisphere implemented on an IBM 360/195 computer carrying out ten million operations per second. This model generates forecasts *three days ahead* for aviation (which of course is interested in the winds at all the heights I've been talking about), for shipping, for the general public, and for many industries (farming, construction, power, etc).

Now I want to describe the basic algorithm, using 'algorithm' in the sense of a process that will be repeated *many* times within the computer. The basic algorithm takes the model forward by one *double* time-interval, Dt. For example, with 6-minute time-intervals, the algorithm takes the model forwards by 12 minutes. Then this needs to be repeated 120 times to take the model forward a whole day, and that in fact uses up just 12 minutes of computer time. (The three-day forecast is completed in 36 minutes).

Here, this double time interval Dt is of course a *finite difference*. It is true of computer models generally that they work with finite differences (even though their authors *may* have used partial differential equations while developing them). Accordingly, such models can be explained to students without knowledge of infinitesimal calculus. In explanations for educational purposes, it may be advisable to avoid the usual slightly repelling Greek-letter notation for finite differences. That is why we use a simple Roman capital D for the DOUBLE time-interval Dt.

Next, a popular account needs to explain the numbering system for a grid of squares (figure 4). This is simply the use of cartesian coordinates, and will be familiar to most readers.

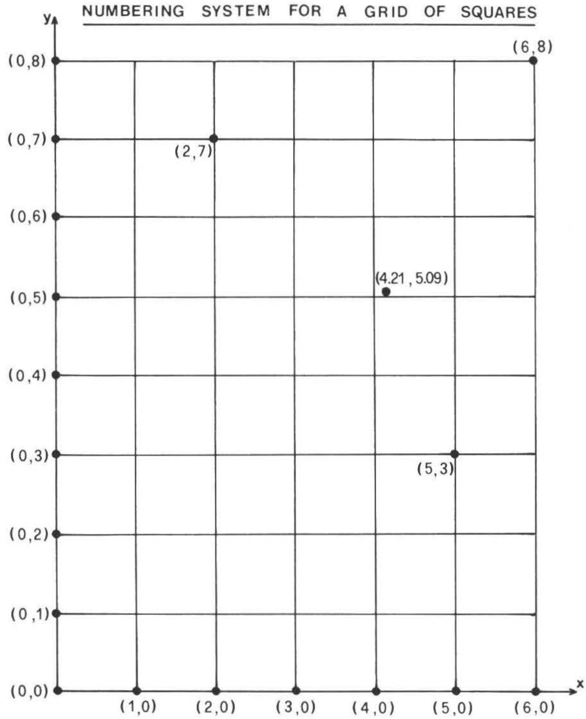

Figure 4. Introducing the map coordinates x,y both when they are whole numbers (and thus represent grid-points), and when they are not whole numbers.

In a finite-difference scheme, speed at one instant can be taken to mean the distance travelled between the preceding instant and the subsequent instant, divided by the length Dt of that double time-interval. This is a centred-difference definition of speed.

But we are interested not only in the speed but in the direction of the wind. The idea of *components* of velocity along coordinate axes will not be familiar to general readers of the book 'Newer Uses of Mathematics'. Therefore, we use Dx and Dy (figure 5):

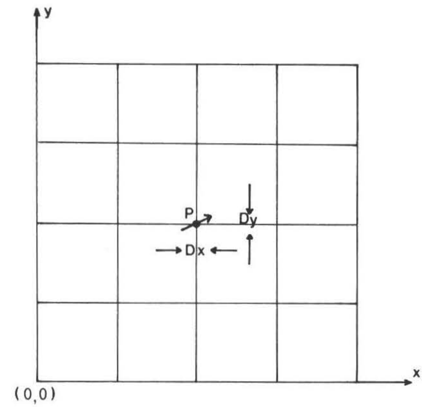

Figure 5. Knowing wind speed and direction fixes the length of the arrow in this diagram and, therefore, fixes Dx and Dy. Alternatively, knowing Dx and Dy fixes the position of this arrow centred on P and, therefore, fixes wind speed and direction.

the change in the x-coordinate of the air between the preceding instant and the subsequent instant, and the change in the y-coordinate of the air during the same double time-interval. Knowing both Dx and Dy is equivalent to knowing both the speed and direction of the wind at the centre of that double time-interval.

Now we come to one of the important meteorological ideas: that of *divergence;* the idea that the winds can blow so that the *total amount of air above us* is changing. Under what circumstances is there a *net escape of air* from a slice?

Figure 6 gives the net escape from a slice of air during a double time-interval (a) when all winds blow to the right; (b) when all winds blow 'up the page'. In case (a), there can be net escape when the displacement Dx is greater on the right (at R) than it is on the left (at L). Again we use a capital letter, this time X, for this centred difference; thus, X(Dx) stands for the difference between the values of Dx at R (where x is greater by 1) and at L (where x is less by 1). A positive value of X(Dx) produces net escape of air, and similarly with Y(Dy) for displacements 'up the page', and it turns out that for general displacements the divergence (proportional loss of air from the slice during the double time-interval) is the average of the two:

$$\frac{X(Dx) + Y(Dy)}{2} = \text{DIVERGENCE} \qquad (1)$$

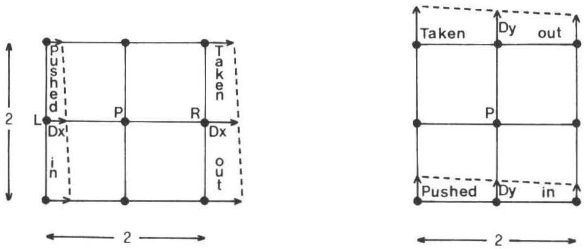

Figure 6. Illustrations to introduce equation (1) (the definition of divergence).

Now, any loss (or gain) of air in a slice alters the pressure with which it is weighing down on slices of air below it. One slice exerts a pressure of 1 decibar, so that the proportional loss of air (1) represents the reduction of pressure exerted, in decibars. The air in any lower slice, then, feels a pressure change Dp (in decibars) equal to

$$Dp = - \text{(DIVERGENCE added up for all slices above).} \qquad (2)$$

Also, the mathematical model uses an input from *laboratory* knowledge (actually, quite classical knowledge derived in the seventeenth and nineteenth centuries). However, there is a rather interesting general difficulty in applying the results of laboratory researches to make a step forward in time in a mathematical model of the environment. Laboratory researches tell us how a definite mass of air will change its properties (for example, how its volume will respond to pressure changes). The mathematical model, however, must calculate quantities *at grid-points:* the storage capacity of the computer allows values of quantities only at those grid-points to be stored. The difficulty, then, is that laboratory research tells how a definite mass of air behaves in the time-interval Dt, during which however the winds blow it *off* the grid-point.

There is a standard mathematical 'way round' this rather general difficulty. For any measurable quantity q, laboratory research may indicate the value of what I will continue to call Dq: the change in q for a definite mass of air during the double time-interval Dt. For the algorithm, however, we want the change in *q at a grid-point*. I shall use a capital letter for this centred difference, too, and write Tq for the difference between the values of q at the grid-point (that is, for fixed x and y, and p) at the next instant of time and at the preceding instant.

What effects produce a departure of Dq (the change in q for a definite mass of air) from Tq (the change in q for fixed values of x and y and p)? Obviously, that departure includes the effects of Dx and Dy (winds blowing the air off the grid-point); and we have seen that there is also a change Dp.

The effect of the change in x can be written down in terms of our centred-difference notation X. Here, Xq stands for the change in q when x increases by 2 (from L to R in figure 6); but *we want* the change when x increases by Dx, which is evidently a *proportion* (1/2)(Dx) *of* Xq. Adding on the effects of the changes in y and in p, we obtain the basic stepping-forward formula

$$\mathrm{D}q-\mathrm{T}q=\left|(1/2)(\mathrm{D}x)(\mathrm{X}q)\right| + \left|(1/2)(\mathrm{D}y)(\mathrm{Y}q)\right| + \left|(1/2)(\mathrm{D}p)(\mathrm{P}q)\right|. \qquad (3)$$

Now I can write down the whole algorithm, in the case of what I call 'dry' mathematics. This is the form that the algorithm takes in any areas where there is no condensation of water vapour (that is, condensation into rain, snow, cloud, fog, etc.) Although in our book we explain also the algorithm for the case of 'wet' mathematics, I can give here a sufficient flavour of the subject if I describe 'dry' mathematics (which I believe you may find none the less interesting!).

The algorithm consists of the three equations (1), (2) and (3) above together with the following additional equations derived from classical laboratory knowledge. Firstly, the equations

$$\begin{aligned}\mathrm{D}(\mathrm{D}x) &= -\,k(\mathrm{X}h) + f(\mathrm{D}y),\\ \mathrm{D}(\mathrm{D}y) &= -\,k(\mathrm{Y}h) - f(\mathrm{D}x),\end{aligned} \qquad (4)$$

represent the changes in the wind due to pressure gradient and to the rotation of the earth. Here, h is the height of a pressure-level, and a *sloping* pressure-level (represented by Xh and Yh) allows the pressure gradient to have horizontal components which can speed up or slow down the wind. The coefficients k and f are proportional, respectively, to the acceleration due to gravity and to the speed of rotation of the Earth.

Next, the slice thickness s (its total depth from top to bottom) changes by an amount

$$\mathrm{D}s = (-0.7\ \frac{\mathrm{D}p}{p})\,s \qquad (5_{\mathrm{dry}})$$

owing to the change in pressure Dp. The coefficient would be 1.0 if the change took place at constant temperature, from Boyle's Law. However, the effect of adiabatic cooling under a drop in pressure reduces this coefficient to about 0.7. (I omit details of modifications to both (4) and (5) in the bottommost layer associated with interaction of the air with the Earth's surface.)

The object of the algorithm is that given Dx, Dy, and h at grid-points (the wind speed and direction, and the height of each pressure level) we must be able to obtain T(Dx), T(Dy), and Th (the changes in those quantities *at grid-points*). The algorithm works as follows.

First, equations (1) and (2) give Dp, as we have seen. Next, equations (4) give both D(Dx) and D(Dy). Then, the general stepping forward formula (3) (applied with q = Dx or Dy) gives T(Dx) and T(Dy). Similarly, (5) gives Ds and then (3) (applied with q = s) gives Ts. Finally, given the slice-thickness changes Ts we obtain the changes Th in heights of pressure levels from an equation

$$\mathrm{T}h = (\text{added up T}s\text{ for all slices below}) + \mathrm{T}h_{\mathrm{bottom}} \qquad (6)$$

and we obtain the change Th_{bottom} in the height of the bottom (10-decibar) pressure-level from an equation

$$\mathrm{D}(h_{\mathrm{bottom}} - h_{\mathrm{ground}}) = 0, \qquad (7)$$

which simply states that winds blowing near the ground follow the contours of the ground.

Thus, the algorithm goes from Dx, Dy, h to the 'steps forward' in time (at each fixed grid-point) $T(Dx)$, $T(Dy)$ and Th. Actually, in 'wet' mathematics just one of the equations, (5_{dry}), is modified by an additional term, representing expansion of the air through the release of the latent heat of condensation of water vapour; and there is quite a complicated algorithm determining the amount of condensation during the interval Dt (or, alternatively, the amount of evaporation of liquid water) within each slice. This is described in some detail in our book[2].

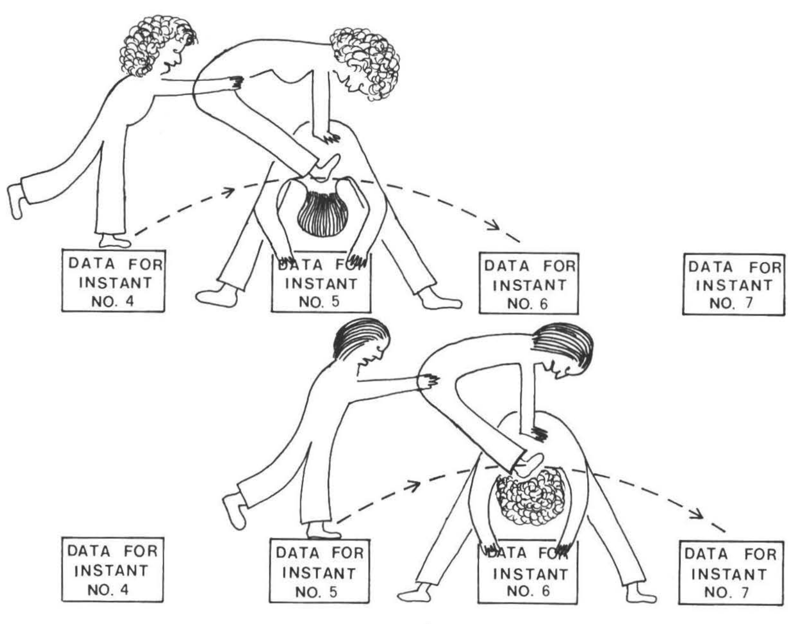

Figure 7. 'Leapfrogging' using centred differences is impossible to 'keep straight'.

We also have an extended section entitled 'Keeping Straight'. A very young child may have learned to make a *step forward*, and then another and then many more, and yet be unable to use the process to get anywhere because he *can't keep straight!* In other words, stability is necessary too.

We need *stable algorithms*; *not* methods of 'adding up' which combine many small errors to give very big errors. Actually, any process involving repeated 'steps forward' is particularly difficult to 'keep straight'. Finding stable ways of combining them is an important branch of the mathematician's art, usually described as the avoidance of 'numerical instability'.

To indicate briefly one aspect of that subject I show two diagrams from our book. Figure 7 shows a boy and a girl playing 'Leapfrog' and suggests what may happen if centred differences are used in a naive way so that values of quantities (h,Dx and Dy) at one central instant are used to obtain the changes in those quantities between the *preceding* instant and the *subsequent* instant. Then, we can obtain two populations of values (as it were, one male and one female) which get increasingly out of step and lead to unacceptably large errors.

But if the girl is self-reliant (figure 8) and uses as a jumping-block best-estimate values of the data at an intermediate instant, calculated from the data where she is now, she can make the double jump forward all by herself, in a manner which can be proved to be stable. (This is a highly simplified description of the important discovery[6,7,8] made by the celebrated mathematician P.D. Lax.)

The account of the mathematics of weather forecasting in our book[2] ends with a section called 'Starting and Finishing'. This goes into some detail on the process by which the observations made twice daily at weather stations all over the northern hemisphere (see figure 9 which shows those in *half* of the area, including the weather *ships* represented by large blobs) are reduced. An interesting application of least-squares techniques is made to derive from those the initial data at grid-points

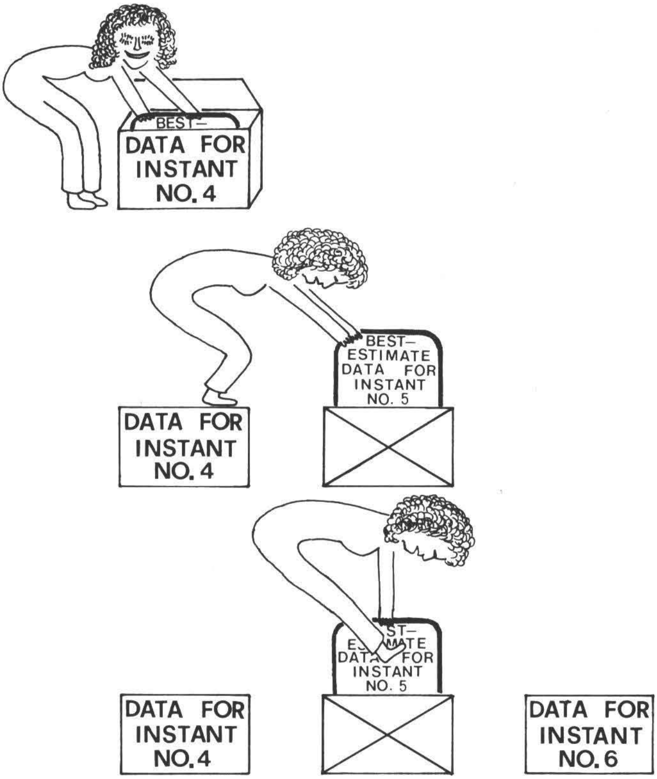

Figure 8. Double step forward from data known only at a single even-numbered instant, no. 4. This method, which has good accuracy and is able to 'keep straight', has been much used by the Meteorological Office, Bracknell. Quite recently, it was superseded by a more complicated alternative which has the same advantages and produces its results faster, yielding the result quoted in section 3 (a 24-hour forecast produced in 12 minutes).

from which the mathematical model can make its forward extrapolation in time.

It is important to impress upon future computer users the fundamental principle of computer applications which has been force-

Figure 9. This shows the same half as in figure 2 of the area to which the British Meteorological Office applies its main mathematical forecasting process. Against the background of the same very rough sketch map as in figure 2, the small dots indicate the rough position of each station feeding regular radiosonde-balloon data into the Main Trunk Circuit for meteorological communications. In the whole northern hemisphere there are 615 such stations on land areas (including small islands). The large blobs represent weather ships, of which there are 7 in the whole northern hemisphere.

fully expressed as GARBAGE IN, GARBAGE OUT! This is why an intricate and carefully designed automatic system for generating good initial grid-point data from the observations needed to be devised, and why it may be educationally important to draw attention to this. By being careful in those respects, we avoid putting 'garbage in' to the computer - and may, therefore, hope to avoid getting garbage out!

At regular intervals (for example, when the mathematical model has reached noon or midnight Greenwich Mean Time) the computed data are copied on to a separate data store before being used to continue the prediction process. That output is then used to generate weather forecasts in many different forms for different users ....

4. Applied Network Analysis

The rest of this Lecture is on applied network analysis. This forms an important branch of the *mathematics of industrial efficiency*[9]; the branch, furthermore, which I have chosen to discuss here because I want to emphasize that it is *not only* numerical mathematics which is useful! The topology of networks, usually called graph theory[10], has very many important applications.Often, the famous numerical techniques in the mathematics of industrial efficiency[11,12,13] (such as linear programming or 'hillclimbing') need to be combined with graph theory as we shall see.

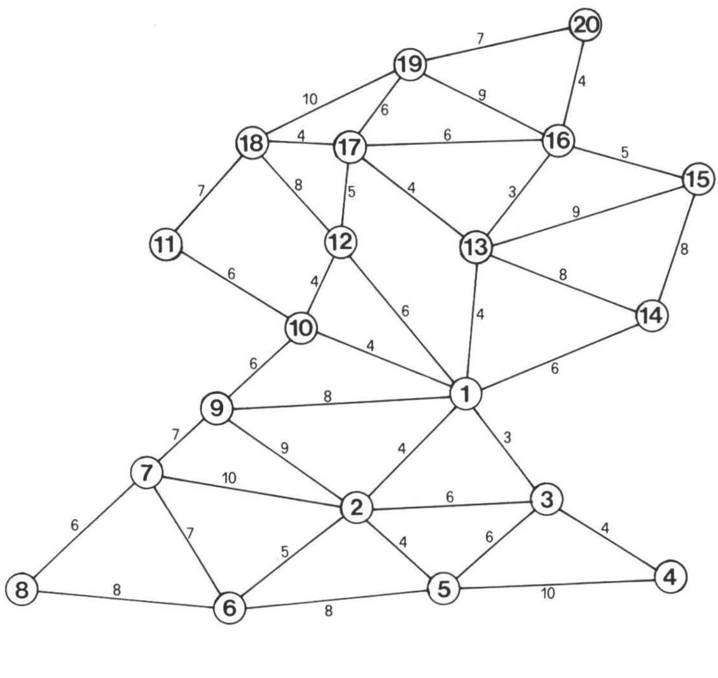

Figure 10. An undirected network, representing the main trunk roads of a country. The numbered circles are the 'vertices' of the network, and the lines joining them are the 'edges'. The number attached to each edge represents the 'cost' to a driver using that road.

I begin with *undirected networks*. For example, figure 10 represents the main trunk roads of a country, where traffic along each of them in either direction is permitted[2,14]. In a computer, such a network is defined by specifying the set of vertices (numbered by integers) and the set of edges (which are *unordered* pairs of integers).

Often, one is interested in paths through such a network which minimise the *distance* travelled, or else minimise the *time* of the journey; or, perhaps, minimise the *difficulty* of the journey. More commonly, a driver will choose his route taking all those factors into account. He behaves as if he were seeking a path with minimum 'cost' where path 'costs' may *combine* these aspects (distance, time and difficulty). This gives a 'costed' network as in figure 10, where (as often happens) many roads stretching radially out from the capital city (vertex 1) are in some sense 'better' and so have lower costs relative to distance travelled than do most circumferential roads.

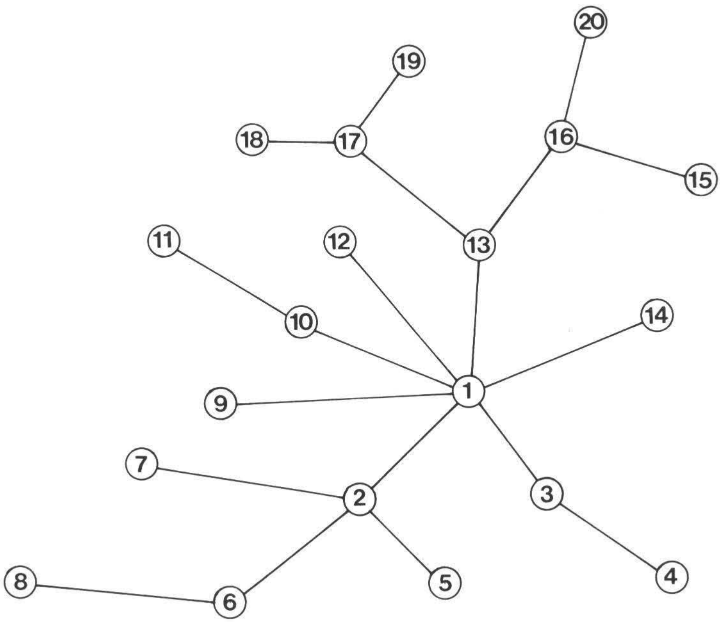

Figure 11. A tree, representing the 'cheapest' routes to and from vertex 1, for all other vertices in the network of figure 10.

One of the classical algorithms for costed networks is the Cheapest Path Algorithm by which a computer automatically derives the set of cheapest paths from a particular vertex to every other vertex. For example, figure 11 shows the cheapest paths from vertex 1 of figure 10 to every other vertex. A set of cheapest paths forms a *tree;* that is, a network without cycles. (For example, there are many closed loops in figure 10, but none in figure 11.)

For an effective Cheapest Path Algorithm we need to determine *for each vertex* i the *preceding vertex* $P(i)$ on the cheapest path from vertex 1 to vertex i, and to determine the minimum cost $K(i)$. The algorithm starts by fixing $K = 0$ at $i = 1$ (it costs nothing to make no journey) and arbitrarily fixes $P = 0$ at $i = 1$ (there is of course no preceding vertex to vertex 1 itself). These values $K = 0$ and $P = 0$ at $i = 1$ are then *fixed.*

Next, the algorithm *temporarily* puts $P = 0$ and $K = \infty$ at all other vertices. Remember that an algorithm is a process which will be repeated many times within the computer. This particular algorithm is such that at the end of each step the values of P and K at just *one more* vertex are fixed. Thus, the algorithm needs to be repeated a number of times equal to the number of vertices (or, strictly speaking, one less than that).

The algorithm is as follows:

(i) For *every edge* joining a vertex i with P and K *fixed* to a vertex j with P and K not yet fixed, test whether

$$K(i) + \text{edge cost} < K(j).$$

If so, make the left-hand side the new value of $K(j)$, and make i the new value of $P(j)$. *If not,* do nothing.

(ii) When instruction (i) has been completed for every such edge, *fix* P and K at just one new vertex: the one*) with P and K not fixed where K is *least.*

Traffic assignment[14] is an important procedure that often makes use of the Cheapest Path Algorithm, as follows. Given the

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*) If there is more than one with the same least value of K, an arbitrary choice among them is made (e.g. that specified by the smallest integer). Provided that this is done, then the 'set of cheapest paths' does form a tree.

traffic flow (vehicles per hour) *between each pair of vertices*, as determined perhaps by a traffic census, we assign that to the cheapest path between that pair of vertices. Then for each edge (that is, for each road in the network) we 'add up' all the flows using that edge to obtain the total flow along that road.

But a problem remains: is the *capacity* of that road really enough to take this calculated flow? If not, then vehicles will experience unexpected delays on that road, so that over a period their preferred paths may become changed.

These considerations suggest an approach involving a blend of network algorithms with linear programming, known as the theory of *capacitated networks*. Here, the procedure of traffic assignment allows for road capacity (an upper limit on the traffic flow along each road). The flows must then satisfy certain inequalities, and of course they must all be positive, while the total cost must be minimised; all exactly as in Linear Programming. Accordingly, the Cheapest Path Algorithm and the well known Simplex Algorithm of Linear Programming need to be combined in order to obtain a solution[14].

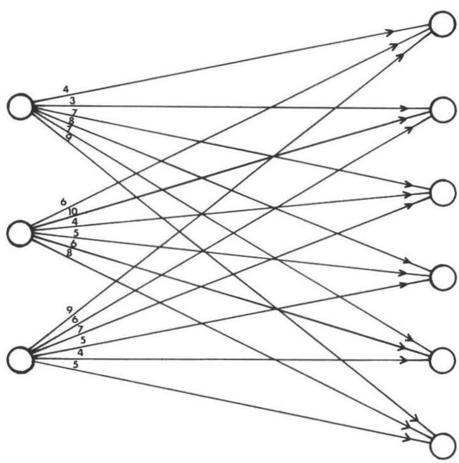


Figure 12. A bipartite network, defining the Hitchcock Transportation Problem. Figures on each directed edge indicate the transportation cost per unit of goods transported by that route.

Another problem where such a blend is appropriate is the famous Hitchcock Transportation Problem[12]. It relates to a *bipartite* network; that is, one where the vertices are in two disjoint sets and every edge is directed from a vertex in one set to a vertex in the other (see figure 12). These edges are all costed.

The Hitchcock Problem is as follows: given only the flow *from* each source and the flow *into* each sink, find the cheapest flow assignment (where again, of course, all flows must be positive). The fine algorithms that have been developed for solving this problem are among those most widely and beneficially used for improving industrial efficiency. (For example, the sources may be factories and the sinks warehouses; or the sources oil production centres and the sinks refineries.)

A remarkable general development of recent years has been that so-called 'hillclimbing' (that is, *nonlinear* optimisation in spaces of very many dimensions) has become a rapid process[13]. This has come about through the improvement of 'gradient algorithms' (in cases where the derivative of the function to be maximised *can* be calculated) and of 'search algorithms' (in cases where it cannot). These 'hillclimbing' techniques can also be used for flow assignment on a network with *nonlinear* costs; that is, edge costs varying nonlinearly with the flow along the edge.

Interestingly, this includes the case of fluid flow networks, where the *energy dissipation rate* varies nonlinearly with the flow. (Often, it varies as the *cube* of the flow.) The fluid flow in such networks, which may be hydraulic networks or natural-gas networks[15], adjusts itself so that the total energy dissipation rate is a minimum; accordingly, it can be calculated by a *blend* of network algorithms and gradient algorithms.

Actually, it was in telephone network problems that *queue theory*[16] was first developed (by Erlang); and the blend of queue theory and network analysis remains the most characteristic feature of telecommunications traffic theory[17,18]. But that theory too may involve the analysis of capacitated networks;and, indeed, some pure network analysis, as in the idea of the Nonblocking Switching Network.

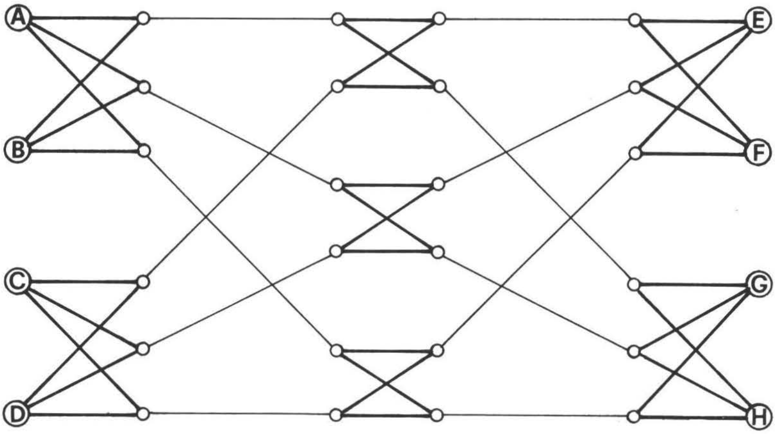


Figure 13. A three-stage Non-Blocking Switching Network.

Figure 13 shows a nonblocking switching network with three groups of switches: the first-stage switches have 2 inlets and 3 outlets, the second-stage switches 2 inlets and 2 outlets, and the third-stage switches 3 inlets and 2 outlets. It is nonblocking in the sense that if, for example, A is connected to H and C to E and D to F this does not stop B from being connected to G. A famous theorem states that such a three-stage switching network is non-blocking if and only if each first-stage switch has n inlets and m outlets with $m \geq 2n - 1$. (In figure 13, for example $n = 2$ and $m = 3$.)

Now I come at last to the mathematics of *planning*; that is, of design at the strategic level. A large area of important and interesting work for mathematicians using models is in the planning of telecommunications networks[18], and especially in their '*dimensioning*'. This means providing the right sizes of exchanges and the right number of junctions between them; 'right' in the sense of the cost being as low as possible for a desired *grade of service*; for example, less than 1 % of calls failing to be connected (or less than 0.1 %, or something in between) with the teletraffic at the levels observed, or forecast. This is an area where network analysis can be of great importance in planning.

That is true also in *activity planning*[9,19], as when a construction programme is planned. This introduces the time dimension. A 'dimensioning' of the construction programme with respect to time is achieved when the minimum time for the total activity is determined through Critical Path Analysis.

The Activity Networks used in Critical Path Analysis are *directed networks without directed cycles*. This is because each edge of an Activity Network (figure 14) is directed *from* a particular *stage* of the total activity *towards* another stage which can be reached *only* after the first has been completed. If the arrows were absent in figure 14 there would be many cycles, but with the arrows present there are no cycles.

An important Theorem states that such a network can always have its vertices numbered so that, for each directed edge $(i,j), i<j$. The algorithm by which this numbering is achieved is as follows (the algorithm, in fact, amounts essentially to a proof of the Theorem):

(a) Number consecutively (in any order) vertices with no incoming edges.

(b) Delete numbered vertices and their outgoing edges.

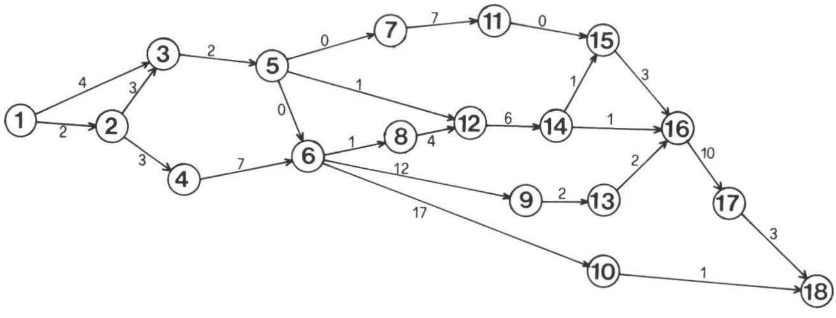


Figure 14. A directed network without directed cycles (Activity Network). The number attached to each edge represents the time taken (in working days) for the activity which the edge represents.

Figure 14 has been numbered by using this algorithm: in the first step vertex 1 was numbered; in the second, vertex 2; in the third, vertices 3 and 4; next, vertices 5 and 6; next, 7,8,9 and 10; next, 11,12 and 13; next, only 14; next, only 15; next, only 16; next, only 17; and, finally, 18.

To each *edge* of an Activity Network such as figure 14, a number is attached representing the time required (often, measured in *working days*) to move along that edge. It is worth noting that the edge usually does represent an activity needing a definite amount of time to complete; a few edges, however, may be numbered zero because no significant time is required: they simply point towards some stage from another stage which has to precede it. For each directed edge (i,j), then, we know $t(i,j)$, a nonnegative time needed to move from vertex i to vertex j.

The theorem I just stated has an interesting consequence; namely, that the Critical Path Algorithm can be simply stated as a *definition by mathematical induction*. For each vertex i we want to know $T(i)$, the minimum time required to reach vertex i; and, also, $P(i)$, the preceding vertex on the critical path to i. We obtain these by the following inductive definition: - First, put $T(1) = 0$ and $P(1) = 0$; then, for all $j > 1$, define $T(j)$ as the upper bound of $T(i) + t(i,j)$ for all edges (i,j) incident on j (where necessarily $i < j$, by the Theorem); finally, define $P(j)$ as an i for which the upper bound is attained.

By this algorithm, we find that the minimum time required for the total activity represented in figure 14 is 41 working days. The Critical Path, in fact, goes through vertices 1,2,4,6,9,13, 16,17 and 18.

But at this stage, Mr Chairman, my own time dimension has run out!- and this prevents me from saying anything about planning involving either the *money* dimension or the *human* dimension; that is, financial planning[20] or manpower planning[21]. What I have tried to depict in this Lecture is the intensity of the interaction between mathematics and society; an interaction embracing the older uses of mathematics and the newer uses, so as to make it *far more widespread* than is appreciated by most students of mathematics; or even, perhaps, by many of their teachers. In the future, such information regarding the intense interaction between mathematics and society needs to be gathered together still more systematically than in the past, and disseminated still more widely, so that it can exercise the *influence on curriculum development*, and the influence on mathematical education as a whole, which, axiomatically, it should.

References

1. *Developments in Mathematical Education* (Proceedings of the 2nd International Congress of Mathematical Education), edited by A.G. Howson. Cambridge University Press, 1973.

2. *Newer Uses of Mathematics* by M.J. Lighthill, R.W. Hiorns, S.H. Hollingdale, R.B. Potts, R.E. Beard and B.H.P. Rivett. London: Penguin Books, 1977.

3. *Effects of Polluting Discharges on the Thames Estuary.* Water Pollution Research Technical Paper No. 11. London: Her Majesty's Stationery Office, 1964.

4. *Mathematical and Hydraulic Modelling of Estuarine Pollution.* Water Pollution Research Technical Paper No. 13. London: Her Majesty's Stationery Office, 1972.

5. *Advances in Water Pollution Research*, edited by S.H. Jenkins. Oxford: Pergamon Press, 1972.

6. *Numerical Weather Analysis and Prediction* by P.D. Thompson. New York: Macmillan, 1961.

7. *Numerical Weather Prediction* by G.J. Holtiner. New York: John Wiley, 1971.

8. *The Bushby-Timpson 10-level Model on a Fine Mesh* by G.R.R. Benwell, A.J. Gadd, J.K. Kears, M.S. Timpson and P.W. White. Meteorological Office Scientific Paper No. 32. London: Her Majesty's Stationery Office, 1971.

9. *Management and Mathematics* by A.Fletcher and G. Clarke (2nd Ed., revised by C.W. Lowe). London: Business Publications,1972.

10. *Graph Theory*, by F. Harary. New York: Addison-Wesley, 1969.

11. *Mathematical Programming*, by S. Vajda. New York: Addison-Wesley, 1961.

12. *Linear Programming and Extensions*, by G.B. Dantzig. Princeton University Press, 1963.

13. *Introduction to Optimization Methods*, by P.R. Adby and M.A.H. Dempster. London: Chapman and Hall, 1974.

14. *Flows in Transportation Networks*, by R.B. Potts and R.M. Oliver. New York: Academic Press, 1972.

15. 'Network Analysis' by H. Frank and I.T. Frisch. *Scientific American* 223, 94-103 (1970).

16. *Queues* by D.R. Cox and W.L. Smith. London: Methuen, 1961.

17. Proceedings of the IMA Symposium on the Mathematics of Telecommunications Traffic. *Bull. Inst. Math. and Applics.* 11,84-107 (1975).

18. *Communication System Engineering Handbook*, edited by D.H. Hamsher. New York: McGraw-Hill, 1967.

19. *Network Analysis for Planning and Scheduling*, by A. Battersby. London: Macmillan, 1971.

20. *Linear Programming in Financial Planning*, by G. Salkin and J. Kornbluth. Accounting Age Books, 1973.

21. *A Survey of Mathematical Methods of Estimating Supply of and Demand for Manpower*, by R.E. Laslett. Engineering Industry Training Board Occasional Paper No. 1. London: Her Majesty's Stationery Office, 1972.

2.2 Trends in Pure Mathematics

Michael F. Atiyah

1. Introduction

What I propose to do in this talk is to describe the development of mathematics over a fairly long period. It is only the long-term trends that have any relevance to the problems of mathematical education, which is the main concern of this congress. Research in mathematics, like that in so many other fields, moves nowadays at an ever-increasing pace. Subjects change radically over a short space of time, topics come in and out of fashion with alarming rapidity and it would be quite impossible for mathematical education to try and reflect these oscillations on the frontiers of the subject. In fact much damage can be done by trying to move too quickly with the times and to impose on the educational world some enthusiasm of the moment. This is not to argue that all change in education is bad and that Euclid should remain the basis of all mathematics teaching. I merely want to emphasize that it is only the broad aspects of mathematical development which should influence the teaching of the subject.

I would like therefore to look at mathematics over the past hundred years and to describe in broad outline the main changes that have taken place. Naturally there will be an element of subjective bias in my description: I cannot do equal justice to all parts of mathematics, and I apologize in advance for the many omissions and over-simplifications in what I shall say.

It is frequently asserted that modern mathematics is entirely different, both in spirit and in content, from traditional mathematics. I will examine this point in some detail and for the sake of simplicity I will take the dividing line around the beginning of this century. This is of course very crude, and whereas historians may be impressed with nice round centuries mathematicians should know better. Certainly the new ideas, characteristic of modern mathematics, have their roots far into the past and much classical mathematics continues to thrive at the present day, but the years since 1900 have seen a general drift in the centre of gravity of mathematics.

2. Classical Mathematics

Let me begin by summarizing very briefly the problems and achievements of mathematicians before the end of the nineteenth century. Broadly speaking we can say that mathematics arose from the practical problems associated with counting, geometry and physics in that order. The development of an adequate concept of number and a proper understanding of its relation to geometry and physics was the first major pre-occupation of mathematicians over the centuries. It is sobering to recall that the problems which troubled the Greeks, such as Zeno's Paradox and Euclid's Parallel Postulate were only resolved in the nineteenth century. At this time the real numbers acquired a foundation independent of geometry and geometry in its turn, as a result of the work of Bolyai and Lobatchevsky became emancipated from physics.

In the early stages we can say that the typical mathematical problem would be concerned with providing the value of some unknown quantity x. In its most obvious form the problem might reduce to finding the root of a polynomial equation $p(x) = 0$ and of course this problem occupied a central rôle for several centuries. The nature of the problem only became understood with the work of Galois and Abel early in the nineteenth century.

A second level of mathematical problem, which played an increasing role from the time of Descartes and Newton, was that of finding an unknown *function* $f(x)$. Geometrically the graph $y = f(x)$ might be a curve with desired properties or in physical terms it might be the trajectory of a particle, moving under given forces. The unknown function f would typically satisfy a differential equation, and the study of differential equations has remained the single most profound application of mathematics to the physical world. Unfortunately it is also the most difficult and the problems are still with us.

The great achievement of classical mathematics, culminating in the work of the great analysts of the nineteenth century, was in getting to grips with the problems of infinity which lie at the basis of our notion of a real variable x and of functions $f(x)$. With these foundations properly laid, mathematicians could hopefully go on to greater things.

3. Several Variables

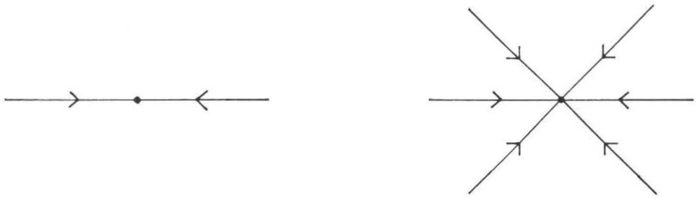
In what direction would these greater things lie? In my view the answer is simple, mundane but far-reaching: the direction to be followed was that of several variables $x_1, \ldots, x_n$ and functions $f(x_1, \ldots, x_n)$ of several variables. If one is looking for the *single major factor* distinguishing mathematics of the 19th and 20th centuries it lies I believe in the increasing importance of the study of functions of several variables.

It is of course not surprising that, with the problems of one variable under some sort of control, mathematicians would become more ambitious and take on several variables. Certainly the mathematical problems arising from the physical world usually involve at least three dimensions! It is clear that the need for such a development is there, but what is not so clear is why the jump from one to more variables should produce any essential difficulties or why it should necessitate the introduction of so many new ideas and techniques.

The explanation is that the difference between one variable x and several variables $(x_1, \ldots, x_n)$ is essentially *geometrical*. In fact one could say that geometry on a line is trivial (because the real numbers are *ordered*) and that it requires at least two or three dimensions for geometrical concepts to become significant. Let me illustrate this in three different ways.

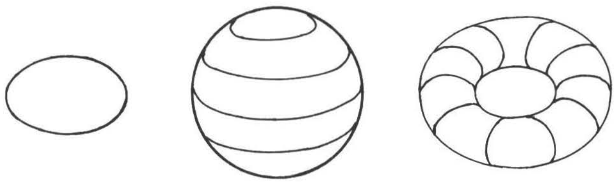
(i) *Local*

There are just two directions at the origin on a line but infinitely many directions at the origin on a plane.



(ii) *Global*

A straight line can be 'closed up' in only one way: into a circle, but a plane can be closed up into many different kinds of surfaces, e.g. a sphere (by stereographic projection) or a torus (by identifying opposite sides of a square).



(iii) *Algebraic*

There are no rigid rotations on a line. There are infinitely many rotations in 2 or 3 dimensions. In 2 dimensions these are measured by an angle and *commute* amongst themselves, but in 3 dimensions they do not always commute.

Example (iii) shows that some new phenomena can appear when we pass from 2 to 3 dimensions, and in general all problems get significantly harder as the dimension increases. There is 'more room' and new phenomena can unexpectedly appear. For example in (ii) all possible ways of 'closing up' the plane can be easily described, but in 3 dimensions the problem is unsolved.

Given that these new difficulties appear what general principles are there which might lead to simplifications? There are at least two which are easy to explain and which have been the driving force behind much of 20th century mathematics. These are:

(i) *Symmetry*

If a problem has certain symmetrical features then this can be used to simplify it. For example, if a function $f(x_1, x_2, x_3)$ in three dimensions is known to be spherically symmetric about the origin, then we can write it as a function of the single radial function

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The Mathematical formulation of symmetry is in the Theory of Groups and this permeates the whole of modern mathematics (and much of mathematical physics).

(ii) *Qualitative Analysis*

When a problem is too complicated to admit of a simple explicit solution we may well settle for a qualitative description of the overall behaviour of our unknown function. The mathematical formulation of this notion is Topology. For example, the distinction between a sphere and a torus is the simplest example of an interesting topological phenomenon.

To exploit these two principles an appropriate apparatus has had to be developed. In addition more standard techniques have been built up both in Algebraic Geometry (the study of polynomials in several variables) and in Partial Differential Equations (differentiating with respect to several variables).

4. Abstraction - The Axiomatic Era

As I have just indicated a major difference in content between 20th and 19th century mathematics lies in the degree of complication, measured in terms of the number of variables involved. This difference in content has led to a difference in approach with more emphasis on general principles, and an extensive use of the axiomatic method. It is frequently asserted that the characteristic feature of modern mathematics is precisely the freedom with which it has built up axiomatic systems, the implication being that we keep inventing new rules for ourselves and playing games that are unrelated to the traditional problems of mathematics. I believe this point of view to be mistaken so let me describe the role of the axiomatic approach as I see it.

The oldest and most influential axiomatic treatment in mathematics is of course that of Euclid, but the axioms of Euclidean geometry were meant to be self-evident truths derived from physical experience. It took a long time for mathematicians to recognize that alternative geometries (such as hyperbolic geometry) were possible and that they might even provide better descriptions of physical space. It eventually dawned on them that Euclidean geometry was simply one of several possible models for the geometry of our universe.

You will note that I have used the word 'model' and 'axioms' almost interchangeably. In fact the former is used by applied mathematicians and physicists while the latter is preferred by pure mathematicians, but essentially both words describe the same process. When an applied mathematician sets up a theoretical model for a physical process what he is doing is deciding which physical factors are important and which can be ignored (at least in the first instance). He then works out (if he can!) the theoretical consequences of his simplifying assumptions and compares these hopefully with the physical reality.

Faced with increasingly complicated mathematical problems pure mathematicians reacted in essentially the same way. They concentrated on various aspects of their problems, put these into their axioms and then studied their implications. As in the case of physics, recognizing what basic features common to many different problems deserved to be abstracted out and axiomatised was a matter of experience and judgement. Ultimately the acid test would be whether new light was cast on the original mathematical problems.

There is no doubt that this century has seen the development of many new abstract branches of mathematics, but these should be viewed not as independent self-contained bodies. Rather they should be viewed as convenient administrative divisions, each of which has been set up to perfect and develop certain types of tool, and all of which are needed in the natural problems of a mathematical nature.

The first of these new abstract branches to develop was Algebra. This is not surprising since Algebra has from its earliest beginnings been an abstraction, with symbols standing for numbers or other more complicated quantities. Abstract or modern algebra is one stage more abstract: the symbols do not stand for anything, only their rules of combination and mutual relationship are significant. One might say that Algebra is a machine for solving certain types of problem but that abstract Algebra is a machine for making machines.

Roughly speaking modern algebra can be divided into two parts: commutative and non-commutative. In the former one studies polynomials in several variables while the latter primarily centres round Group Theory, the abstract study of symmetry. The importance of both kinds in higher dimensional problems should by now be clear.

The second major abstraction of the present century is Topology which can be described as the abstract study of continuity. As such it is not surprising that topology lurks in the background of most areas of mathematics.

Finally I should mention another important abstract branch which goes under the heading of Functional or Linear Analysis. In this the typical basic object is Hilbert space which is a Euclidean space of infinitely many dimensions. The points of this space are usually representing functions f, but our geometrical intuition is so powerful that it has proved helpful to use a geometrical terminology. The difference between algebraic equations for an unknown x and differential equations for an unknown function f now disappears, at the expense of having infinitely many dimensions to our space. All geometric difficulties concerning n dimensions are of course present in even greater force when $n = \infty$, which helps to explain why differential equations present enormous difficulties.

5. The Second Stage - Interaction

While it would be hard to put an exact date on the foundation of the various abstract branches I have just enumerated, it is not far wrong to say they got going in the early years of this century and have been well-established for about 50 years. In each field, great progress has been made in developing techniques and building up elaborate structures. While difficult technical problems remain one can say that the major features are clear and that the consequences of the initial axioms have been thoroughly worked out.

Once each subject had been through a period of introspection and internal development the stage was set for a period of external interaction. This has taken place on a number of fronts and for a variety of reasons.

The frontier between Algebra and Topology has been particularly active. The impact of Algebra on Topology was so strong that Topology split off an identifiable and vigorous offshoot called *Algebraic Topology*, in which the qualitative topological information is put into algebraic form. Let me illustrate this by considering the examples of the plane or 3-dimensional space 'closed up at ∞' as indicated in 3. Recall that there are several different ways of 'closing up' and the problem is how to distinguish between the different possibilities in some algebraic fashion. The answer involves looking at loops on the space which we may roughly describe as closed paths going to ∞ and back again. We do not distinguish between two loops if one can be continuously deformed into the other. In dimension one, where our space is the circle, a loop is entirely determined by the number of times we go round and this may be represented by an integer (negative integers correspond to going backwards). In general we can follow one loop by another to get a composite loop and this makes the loops into a *group*, in the sense of abstract algebra. In dimension one our group is the integers under addition. In higher dimensions we get different groups (usually not commutative) and this fundamental group, as it is called,is a very significant feature of our space. It was introduced in the early years of the 20th century by Poincaré and it has provided a very fruitful link between Algebra and Topology.

Now groups are supposed to be the abstraction of symmetry, so where is the symmetry in these examples? Consider first the one-dimensional case of the circle. We can 'unwind' a circle into an infinite straight line, and we can consider the symmetries of the line given by integer shifts up and down. The circle is then obtained from the line by identifying a point with all its shifts (in the usual symbols, $\theta \pm 2n\pi$). The higher dimensional case is analogous: we can 'unwind' our space to form what is called the universal covering space and the fundamental group is a group of symmetries of this new space.

In addition to Algebraic Topology another hybrid has more recently appeared under the name of *Homological Algebra*. Here the ideas developed in topology have turned out to be highly relevant to the study of various parts of algebra and in particular to polynomials in several variables. I will take a few minutes to try to indicate its essential features. Let me return first to the spaces previously considered and describe one general technique used in Algebraic Topology. This consists in triangulating the space, dividing it into triangles (or tetrahedra) and then spelling out the way in which these triangles fit together along their edges. Clearly the space is determined by such a combinatorial scheme. The trouble is that there are lots of different ways of triangulating the space, in other words our combinatorial scheme contains redundant information. The problem is to extract information which is independent of the triangulation. The fundamental group is one piece of such information but there are others, depending not only on edges but on triangles (and higher dimensional analogues) which go under the name of Homology. Roughly speaking Homology counts the number of holes in a space.

Now the techniques developed in Homology Theory turn out to have numerous applications in pure algebra - hence the name Homological Algebra - and in particular to polynomial equations. To see how this comes about consider a system of polynomial equations

$$f_j(z_1,\ldots,z_n) = 0, \qquad j = 1,\ldots,m.$$

We are interested in the simultaneous solutions of these equations - briefly referred to as the zeros. The trouble is that there are many systems of polynomials f_j which lead to the same zeros (except when $n = 1$ in which case f is essentially unique: thus we have a new phenomenon when $n > 1$). Moreover the f_j are not necessarily independent; there may be relations between them, that is an identity of the form

$$(\ast) \qquad \sum_{j=1}^{m} g_j f_j \equiv 0$$

where each g_j is some polynomial in $(z_1,\ldots,z_n)$.

Note: If any g_j is a non-vanishing constant then we can divide $(\ast)$ by it and hence solve for the corresponding f_j. For a general polynomial we cannot divide and still end up with polynomials. There are many such relations but the famous Hilbert basis theorem asserts that all relations are consequences of a finite number. If we let g_j^k $(j = 1,\ldots,m;\ k = 1,\ldots,s)$ denote the coefficients of these basic relations, then there may be identities between them of the form

$$\sum_{k=1}^{s} h_k\, g_j^k \equiv 0, \qquad j = 1,\ldots,m.$$

We can go on in this way introducing all these super-relations and we end up with a complicated algebraic scheme of polynomials $\{f, g, h, \ldots\}$ which can be thought of as analogous to the triangulation of a space. The f's correspond to vertices, the g's to edges, the h's to triangles and so on. The main problem again is to extract information about the zeros which is independent of the collection of polynomials and relations used to describe them. It turns out that the notions of Homology provide an answer to this problem.

These examples from Algebra and Topology are cases in which a technique developed in isolation in one area has turned out to be applicable in a quite different context. A different and more predictable development has arisen from a return to the original mathematical problems, which preceded the abstract era. In many of these problems the situation is more complicated than in the simplified initial abstractions and more elaborate branches have developed to deal with these. Typically both algebraic, topological and analytic aspects are present in these fused theories. For instance, the symmetries that arise in geometry, such as the rotation group in 3-space, involve continuity as well - they are examples of topological (or continuous) groups. As another example, combining this time both Algebra and Linear Analysis, consider not just one differential equation but several. The algebraic relations between these equations are now important and this leads to a subject which might be called Algebraic Analysis (but is usually referred to as Algebras of Operators). Quantum Mechanics furnishes the most important application of these ideas. I recall that the famous Heisenberg commutation relations take the form

$$PQ - QP = 1$$

where Q, P are operators that correspond to position and momentum. A standard representation is given by taking $P = \frac{d}{dx}$ and Q to be multiplication by x: both P, Q being regarded as operating on the space of (square-integrable) functions of x.

Yet another development occurs when a particular abstract theory turns out to provide a good model for some new class of problems. An example is furnished by Probability Theory, which in its modern form is a special branch of Linear Analysis. Despite its importance I shall say no more on this, since I am not really competent in this area.

Returning once more to Group Theory we can see all types of interaction at work. We might represent the situation diagrammatically as follows:

$$\begin{tikzcd}
 & \text{Groups} \arrow[dl] \arrow[dr] & & \\
\text{Continuous Groups} \arrow[dr] & & \text{Groups in Euclidean Space} \arrow[dl] \arrow[r] & \text{Crystallography} \\
 & \text{Cont. Groups in Euclidean Space} \arrow[d] & & \\
 & \text{Cont. Groups in Hilbert Space} \arrow[r] & \text{Elementary Particle Physics} &
\end{tikzcd}$$

The symmetry groups of Lattices in Euclidean space have a well-known importance and application to crystallography. More recently it has become clear that symmetries of the Lorentz group in a Hilbert space (i.e. a Euclidean space with infinitely many dimensions) is fundamental in the quantum theory of elementary particles. Spurred on by the physicists, mathematicians have been working hard on the symmetries of Hilbert space - a topic which is a thorough fusion of Algebra, Topology and Linear Analysis.

Another remarkable interaction has taken place between Topology and the classical theory of functions of a complex variable. In attempting to extend this beautiful theory to functions of several complex variables, mathematicians discovered some entirely new phenomena. For example, in one variable an analytic function can have an isolated singularity at one point. For several variables this cannot happen, or rather the singularity is always removable. Much progress was made on this in the 1930's but many difficulties remained. Then in the 1950's the new methods invented in topology turned out to provide precisely the necessary tools to clear up the whole theory of several complex variables. This was a very impressive achievement. Interestingly enough, and very unexpectedly, it seems that this topological theory of complex variables is just the right language in which to study certain models of general relativity which are currently in fashion. It is still too early to tell if this new application of Geometry to physics will have the same kind of impact as Riemannian Geometry had on Einstein's theory of relativity.

I would like to conclude this brief survey of interaction between different fields with a few general observations.

(1) The development of different specialities, each built on a limited set of axioms, and turning out precisely finished technical tools need not inhibit fruitful interaction between different fields. On the contrary, by leading to a better organization it may actually help the potential users. It's rather like going into a large modern departmental store where all items are properly classified. Finding what you want is then comparatively easy.

(2) It is quite surprising how far you can get with a small set of axioms (as in Group Theory or Topology) but a time comes when it is more fruitful to interact with other fields.

(3) In any given field of mathematics there are always some very fine points which present great technical challenges to the specialist but are not usually of interest to the general mathematician. To take an analogy, if you want to buy some hand-made pottery you are not usually interested in whether the potter used two hands or only one, though the potter would no doubt take great pride in his single-handed effort. In general the mathematical results that have the widest impact are not technically the most difficult.

6. Lessons for Education

As you will have gathered by now my main theme is that modern mathematics is not as divorced from traditional mathematics as is sometimes implied. Mathematicians have regrouped their forces and spread out in different directions but the basic objectives are still much the same. The difference is more in the manner than the substance and if Newton or Gauss were to reappear in our midst only a short refresher course would be necessary before they could understand the problems being tackled by the present generation of mathematicians.

If you accept this point of view; what are its implications for mathematical education? Let me, with some diffidence, make a few suggestions.

(1) It is, I believe, a mistake to lay too much emphasis on the formal structures of mathematics. For example, I am not convinced that it is desirable to introduce primary school children to sets, commutative laws and distributive laws.

(2) Abstraction only makes sense on a firm basis of experience. Moreover, when abstract ideas are introduced their utility should be demonstrated on concrete problems.

(3) The best aspect of modern mathematics is its emphasis on a

few basic ideas such as symmetry, continuity and linearity which have very wide applications. This should be reflected in teaching whenever possible.

(4) Finally, I would like to say a few words on the subject of Geometry. From being the great source of mathematical raw material and the pillar of school mathematics over the centuries it has now been relegated to a back seat: Euclid has been dethroned. The great battles of the nineteenth century resulted in the triumph of algebra and analysis, and this has eventually led to the virtual demise of Geometry in schools and universities. I regard this as most unfortunate for a variety of reasons. In the first place, it stems from a misunderstanding of the changes that have taken place in mathematics. As I have tried to indicate the mathematics of the present century is to a great extent grappling with difficulties which are essentially geometrical in character, that is, they arise from the study of higher-dimensional problems. Of course Euclidean geometry is too narrow a framework for this more general view of geometry but what has frequently happened is that Euclidean geometry has been ejected and nothing has been put in its place. My second reason for regretting the diminishing role of geometry is that geometrical intuition remains the most powerful channel for mathematical understanding, and it should therefore be encouraged and cultivated. I should make it clear that I am not pressing for the inclusion of any specific geometrical topic. I only wish to make a plea for the widest possible use of geometrical thinking at all levels.

It may be that I am preaching to the converted and that my information about what goes in the educational world is inadequate. In any case I have given you my personal views on how mathematics is developing and I hope that this may prove helpful to some of you by providing a general background, however impressionistic, to your work.

In conclusion let me say again that what is happening on the frontiers of mathematics should only be allowed to have a distant influence on education. I am fully aware of the many practical, social and pedagogical considerations which have quite properly a much more direct bearing on what is, or should be, taught.

2.3 Education in Mathematics and Science Today: The Spread of False Dichotomies

Peter Hilton

0. Introduction

The theory and practice of education are today in a state of ferment, in the United States and in many other countries. Over the whole spectrum of educational level, from kindergarten to graduate school, traditional ideas are being challenged. Traditional modes of instruction are being questioned, particularly by those who favor greater informality and a more active involvement of the student. Traditional norms of education are questioned by those who assert that the criteria adopted unduly favor academic man at the expense of practical man, and that they introduce an unhealthy bias into our social system - in brief, education creates snobbism. Further, the traditional content is under attack by those who claim that it is irrelevant to the needs and concerns of present-day society; and that important and exciting new areas of human concern are neglected in our educational system simply because they formed no part of that system when it was first formulated. The immense cost of education sensitizes taxpayers and legislatures alike, and causes many groups of people to give expression to their discontent at the low success rate of the educational process, and to propose remedies. Further, and for the same reason, these same groups insist on having available criteria of success of virtually instant application, so that failures can be quickly detected and erroneous procedures, with regard to the individual student or an entire curriculum, can be rapidly corrected and not allowed to contaminate the entire educational process. It is perceived by many in the United States that education has failed to meet the needs of special groups within the community. It is cogently argued that the system must adapt itself to the needs of the underprivileged, of those whose mother tongue is not that of the dominant group within the society and of members of other minority groups within the community.*) It is further argued that the experience of the student during his period of fulltime education should not be differentiated, as sharply as in the traditional model, from his experiences when he emerges into adulthood. Conclusions are drawn from this thesis, relating both to the question of appropriate content of courses and to that of the social context in which the education takes place. A further consequence of this particular concern is the emergence of a major thrust in the direction designated as 'career education'.

Amidst all the theorizing and experimenting which is taking place, it is natural and indeed healthy that many widely different views are expressed. There are those who claim, sometimes with messianic fervor, to have detected the defects and shortcomings of the prevailing system, and to be in a position to recommend the universal recipe for successful education. Unfortunately, many serious errors, both theoretical and practical, are committed in the undoubtedly worthy attempt to improve the quality of education. It is an unfortunate fact that education is an area in which errors are particularly likely, and in which, if implemented, they have serious and longterm consequences.

In this article, an attempt is made to analyze some of these errors using the schema of the *false dichotomy*. For it appears that the human mind is particularly prone, when searching for panaceas for the profound ills of society, to indulge in dichotomies, arguing that a given system or principle is wrong, and that, consequently, to improve the situation it is necessary to replace that system or principle by its opposite.**) It will be argued that many of the prevailing dichotomies are false, that is to say that the two concepts which are set in opposition to each

*)The phrase 'minority group' here enjoys its American technical meaning of a disadvantaged component of the complement of the set of 'male wasps'. A wasp is a White Anglo-Saxon Protestant.
**)The prevalence of false dichotomies was already noted by the author in his invited address at the Sixth Annual Meeting of the Open Court Editorial Advisory Board (May, 1974). See also Recommendation 1, p. 136, of Overview and Analysis of School Mathematics, Grades K-12, National Advisory Committee on Mathematical Education (1975). This publication is generally referred to as the NACOME report.

other do not form part of an either/or situation; that while the two concepts under scrutiny are different, they have an essential overlap, and that, when properly understood and applied, they can in fact mutually reinforce each other. This is, to be sure, the point of view adopted by the authors of the valuable NACOME report referred to in the footnote (p. 76).

No attempt will be made to obtain a complete list of such false dichotomies; in fact the reader is encouraged to invent some for himself. Moreover not all the dichotomies listed will be discussed here in depth. Nevertheless, by giving serious attention to these dichotomies, certain educational principles should emerge which should help us to improve the quality of the teaching and learning which take place in our schools and colleges.

Since I am a mathematician, and since I am speaking at a congress devoted to problems of mathematical education, I prefer to take my examples from the teaching of mathematics. Again, the reader is invited to generalize to other disciplines. However, I am old-fashioned enough to believe it to be proper, when undertaking a better than superficial study of an important problem, to concentrate on those aspects of the problem in which one can claim to have some modest expertise. I certainly believe that, if specialists in different fields come together and compare their experiences, then the big broad general truths can emerge. I do not believe, on the other hand, that it is proper or helpful for any individual to attempt, on his own and without consulting colleagues, to draw on their hypothetical experience in reaching his own judgments. I have indeed been appalled by the willingness of experts in particular disciplines to pontificate in matters of education affecting a variety of disciplines virtually disjoint from their own. Thus I make no apology for confining myself to a discussion of mathematics and science, while at the same time I express the hope that my readers will, themselves, investigate the relevance of my observations to other disciplines and the extent to which my analysis would require modification in those other contexts.*)

*) I will also naturally be drawing on my experience of mathematical education in the United States. This bias I will often be content to leave implicit.

Thus, in enunciating a list of dichotomies, I will give some which are clearly of general applicability, and others in which the very formulation of the dichotomy itself derives from the context of education in mathematics and science. However, I repeat that in my discussion of any of these dichotomies, I will confine myself very largely to these two areas of education, however the dichotomies may be formulated.

Here then is an incomplete list of false dichotomies currently very prevalent in education in mathematics and science.

Old Mathematics vs. New Mathematics
Education vs. Training
Skill vs. Understanding
Useful Education vs. Diverting Education
Elitism vs. Egalitarianism
Structure Building vs. Problem Solving
Axiomatics vs. Constructiveness
Art vs. Science
Pure Mathematics vs. Applied Mathematics

My intention is to discuss these nine dichotomies. Certainly their themes are not disjoint, so there will be overlap in our discussions. Indeed, in some cases the themes are so close that the dichotomies will be discussed together. In the course of these discussions there should emerge, as I have said, certain positive recommendations for curriculum and pedagogy.

I will suggest, moreover, that these recommendations do in fact conform with the principles underlying certain experimental pre-college curricula in the United States today; for example, the CSMP elementary program; the USMES program; the BUMP junior high school program; and the experimental elementary program in mathematics and science being developed under the auspices of the Open Court Publishing Company. However, again in accordance with my old-fashioned principle of preferring to speak of those things of which I have closest knowledge, I will feel free to draw my examples not only from pre-college mathematics education, but also from within mathematics at the undergraduate and graduate levels. The reader is, of course, encouraged to seek examples at the level of mathematics education with which he or she is most familiar.

Formally, a dichotomy is a partitioning of a set S into two mutually disjoint subsets P,Q; thus,

$P \cup Q = S$ 'the union of the sets P and Q is the set S'

$P \cap Q = \phi$ 'the intersection of the sets P and Q is the empty set ϕ'.

(For example, the set S of human beings is the disjoint union of the subset P of females and the subset Q of males.) Thus to show that a dichotomy is false, one must either show that $P \cup Q$ is strictly smaller than S, that is, that the possibilities stated are not exhaustive (this is, selfevidently, the case with the first dichotomy on our list!); or that $P \cap Q$ is not empty, that is, that the possibilities stated are not mutually exclusive. It will be by the latter procedure that we will demonstrate falsity. For it is always possible to overcome any objection to the exhaustiveness of the possibilities by *defining* S to be the union of P and Q. No such device can meet the objection that P and Q have a significant common part.

1. Old Mathematics vs. New Mathematics

There has been a very strong movement, particularly in the United States, away from the new mathematics. This movement has found support in many different quarters; among the mathematicians there have figured conspicuously the attack on the new mathematics by Professor Morris Kline, in particular his book *Why Johnny Can't Add* (St. Martin's Press, 1973), and the sharp criticism of certain aspects of modern mathematics by Professor René Thom, for example, in his address to the Exeter Congress on Mathematical Education and in his article *Modern Mathematics: An Educational and Philosophical Error?* (American Scientist 59 (1971), 695-699). But the attack has also come from less well-informed quarters. The new mathematics made many enemies because it imposed a new language of mathematical discourse, a language unfamiliar to parents and legislators. Also, the introduction of the new mathematics coincided with a decline in the computational skills of children, and was held by many to be responsible for this decline. Figures were quoted of the results of standardized tests administered by the National Assessment of Educational Progress and the Educational Testing Service which caused great alarm in the community in the United States; but many overlooked the fact that the decline in computational skill matched a general decline in the effectiveness of basic education. The decline in the scores, for example, for reading comprehension were just as sensational, just as alarming.

Thus, while it is perfectly reasonable to make criticisms of the new mathematics, the failure of mathematical education cannot properly be laid at its door. Nor does the failure of the new mathematics begin to justify a return to the old - a point made very trenchantly by Dieudonné in his reply to Thom. In fact, of course, the new mathematics was never intended to be in any sense an alternative to the old; it was intended to enrich the old mathematics and to replace certain archaic features of the old mathematics by material more relevant to the attitudes and needs of today. In practice, the new mathematics cannot be said to have achieved those objectives. There are many explanations for this, and it would certainly be very well worth while going into this question with great care and in great detail. However the point to be made here is that the old mathematics and the new mathematics have a very substantial intersection. We should be taking the best of the old and the best of the new and adding to their union material which was never an integral part of either; in particular, I have in mind the necessity to educate our students in the techniques of approximation and estimation and in the uses of the hand calculator. Moreover, this last device has rendered virtually useless the traditional algorithms of long multiplication and long division. That there *are* algorithms is a very important lesson to be learnt in mathematics education; but their role is as 'stand-by', and the student must proceed to understand how such tedious algorithms can be avoided.

It would be irresponsible to leave this topic without referring to a strong movement which aims to restore what it sees to be best in the old mathematics. This movement styles itself 'Back to Basics'. Whatever its intentions,*) this movement is ill-conceived and misguided. For much of what it sees as basic is no longer basic to anything of importance either in mathematics or in its application to the world around us; and what it fails to see as basic includes so much which is essential to the educated citizen. This movement exemplifies precisely the danger to which we have already referred, the tendency of enthusiasts to endeavor to replace a system observed to have certain unsatisfactory features by its precise opposite. It is virtually never the case that such a wholesale abandonment of a system is the optimal strategy. If we stopped thinking in terms of dichotomies and started thinking instead in terms of mutual complementarity, then I believe that the risk of such primitively reactionary tendencies playing an important role in our thinking about mathematical education would be virtually eliminated.

2. Education vs. Training; Skill vs. Understanding

These two alleged dichotomies featured prominently in a talk I gave fairly recently at a gathering of educators.**) However, in that talk, I was principally concerned to show that education differed from training and that skill differed from understanding. I now realize that it is just as necessary to show that there is a significant common part to education and training, as to skill and understanding. This I also attempted in that talk, but it was an aspect which did not feature very prominently. However, it is being said today that we have to make our education more meaningful for the student, and so for example we are told that we should institute, at all levels, a form of 'career education'. The device of referring to vocational training as career education is not sufficient to disguise the pau-

*) In this bicentennial year it is not surprising that there are those traditionalists who would wish to restore the style and content of mathematics education in vogue in 1776.

**) Some Problems of Contemporary Education, Papers on Educational Reform, Volume IV, Open Court Publishing Company, 1974, 77-104.

city of the intellectual content present in courses, persisting at least through junior high school and high school, and designed simply to prepare students for particular dollar-earning occupations once they have completed their education. The rapidly changing technology which is characteristic of our time makes it extremely difficult to anticipate the precise knowledge which a young person would need to have in order to be able to contribute effectively and productively as an adult. Thus the responsibility of the educational system, insofar as the student's subsequent career prospects are concerned, is to provide the student with the necessary flexibility and suppleness which will enable him to adapt to change, and, further, to recognize progress and welcome it with a positive, confident, and optimistic spirit. Thus it would seem that the needs of our time emphasize the importance of education in the more classical sense precisely from the point of view of rendering the student more capable of benefiting from and contributing to the social system in which he will live. Similarly, the mere acquisition of traditional skills is quite inadequate from even the most practical point of view, since those skills are likely to become obsolescent very rapidly. However the capacity to acquire a skill will never become obsolete, and that capacity must depend in its turn on the student acquiring through his education an ability to really understand the content of his courses rather than merely being able to memorize certain features of that content and to reproduce them in formal tests. To provide examples of these principles, I would ask you to consider the standard way in which the differential calculus is taught to undergraduates. Calculus is made to appear as a set of skills - one might almost say tricks - and the student learns those skills and, once he is trained in them, his study of the calculus is deemed to have reached the requisite level of maturity. Lip service is also paid to the need to render his learning of calculus relevant by offering him examples which are tailor-made; that is to say, the examples are designed simply to illustrate the particular skill which the student is currently endeavoring to acquire. A genuinely relevant course would have to take into account the whole question of how one chooses a mathematical model to tackle a problem from the real world, how one reasons within the mathematical model, how one checks the results of one's reasoning against the original problem in order to verify the appropriateness of the model, and how one modifies the model in the light of an unsatisfactory fit between theory and practice. All these aspects of genuine applied mathematics involve the education of the student in a very broad sense, and his acquisition of genuine understanding of mathematics and its role.

I will have this example in mind, later, in discussing other dichotomies on the list. However, at this point, it would be useful to point out that the need to give a student a genuine education is nowhere more urgent than at the elementary level, nor is the student more receptive to the great natural attractiveness of education, and more delighted by his capacity to understand than when he is a young child still unspoiled by misguided attempts of adults to reform him in their own image. At the elementary stage, the child has a beautiful free-ranging imagination which can be fed and enriched by the teacher offering him a variety of experiences drawing on the child's natural curiosity and on the environment in which he lives. An enlightened program at this stage would certainly present mathematics as a means of organizing those experiences. Thus a coordinated mathematics and science program at the elementary stage emerges as the natural format for a successful educational experience. However it undoubtedly calls for a level of understanding, on the part of the teacher, of mathematics and of science which it is not realistic to count upon today in view of the training and background of most elementary school teachers.

Perhaps also one should add a word of warning. One hears much these days of *integrated* mathematics and science programs. I would like to make it clear that, on the contrary, what I am advocating may be described as a *coordinated* mathematics and science program. The distinction is, I believe, crucial. An integrated program, as I understand it and as it has been recommended by many scientists, would blur the distinction between mathematical and scientific activity. I believe that it is essential to preserve that distinction. That is to say, the student should know when he is doing mathematics. He should understand that mathematics can be an art in itself and that it can also be employed in the service of knowledge. He should appreciate that there are mathematical activities - for example, mathematical proofs - which have no counterpart in any other human activity. But here I trespass on other false dichotomies!

Let me then sum up this section by saying that, in the modern world, training and skill will be acquired as by-products of good education and real understanding. To try to inculcate skill or to give training which will be genuinely useful would be an exercise in futility were it unaccompanied by a real education of the entire personality of the student. Thus the education which has always been the proper function of our schools and colleges can be justified not only because every person in an enlightened society has the right to a real education, but also on the severely practical grounds that this is the best way to prepare him to be able to make a contribution to society - and to be able to earn a living in the process!

3. Useful Education vs. Diverting Education

I should begin by explaining what I mean by diverting education or 'fun' education. At the elementary level, I have in mind thinking stories of the kind which feature prominently in the Open Court program to which I have already referred. In these stories, a situation is presented which is almost certainly unrealistic. Nevertheless it is a situation which is meaningful to the child and which amuses and diverts him, and lures him into consideration of the problems posed by the story. At a more advanced level, one might think in terms of the sort of problems posed by Lewis Carroll or by Caliban and those which one finds very often in British Sunday newspapers. But I would wish to broaden the concept of diverting education even further. Not only do I have in mind, then, the use of some fictitious illustrations of mathematics, but also the power of mathematics itself to excite and delight the student and seduce him into a state of mind in which he positively demands the satisfaction of his intellectual curiosity. The contrast here then is not the same as that between pure and applied mathematics. The apparent dichotomy is between the belief that the child's attention should be exclusively directed toward useful skills and techniques and useful applications of what he is learning, and the belief that the child's natural taste for fantasy and his natural free-ranging imagination should, in fact, be encouraged to lead him into areas of thought not directly related to actual experience.

The dichotomy is false; for plainly a problem posed from whatever source, provided it really captures the imagination of the child, will lead him into a genuine educational experience. And that educational experience can then be further exploited in the interest of advancing his knowledge of the real world. The thinking stories of Open Court are, in my view, excellent and would constitute, even on their own, an argument for regarding the program as superior to almost all others. But let me take a somewhat different example to illustrate the fact that useful and diverting educations are not mutually exclusive concepts. The example, itself, also reflects a feature of the Open Court program.

One of the most important areas of applied mathematics, precisely because of its ubiquity, is that of probability and statistics. For it is through these two mathematical disciplines that we are able to react intelligently to our environment and take rational decisions. How should one teach elementary probability and statistics? I believe that the obvious way is through games. The natural interest of the child in games - and even his natural desire to win - can be exploited in order to generate a desire to understand the elementary principles of probability and statistics. Thus through games the skills will be acquired and the intellectual curiosity reinforced. These skills will then be applied to situations of incomplete knowledge which abound in the world around us. There are obvious reasons for choosing this procedure for inculcating an understanding of these areas of mathematics; indeed the reasons are so obvious that it would be superfluous to adumbrate them here. But it should perhaps be emphasized that an overriding argument in favor of this procedure, over and above the arguments already given or implicit, is that we will finally wish to apply our mathematics to situations involving scientific experiment, in which we do not know the outcome, and that we would have wished to test the validity of our method under control conditions such as are provided by games.

There is also a somewhat different fallacy underlying this dichotomy which is, in my experience, peculiar to the United States, although it may very well be that it would also be found in certain socialist countries. I refer to the puritanical belief that the most worthy activities of man should not be accompanied by sensations of intense enjoyment. This leads to the view that nothing really important can be going on in the educational process, or in any other activity, if the individuals concerned are greatly enjoying what they are doing. It is most regrettable that this belief informs our approach to earning a living. It is doubly regrettable if it informs our approach to acquiring an education. Enlightened opinion recognizes that elementary education should often be fun. But there is the unfortunate belief that, as one advances through the educational system, it is necessary to become increasingly serious-minded *in* the classroom, reserving one's relaxation for extracurricular activities. The effect of this is just what one should avoid and what many people claim they are trying to avoid - it is to separate classroom experience from the rest of life. The effect on the student's subsequent life and career is unfortunate indeed - to encourage and institutionalize the false dichotomy of work and play.

Much of education should be fun*); at a deeper level, it should be intensely enjoyable; and, far from enjoyment of education being in conflict with its ultimate utility, it is an absolutely essential prerequisite if the education is to exercise a positive influence on the life of the future adult. One certainly would

*) It will be argued that there is great fun, great enjoyment to be had out of studying real-life situations in the classroom. Of course! It would be absurd to suggest that fun is only to be had in fantasy - yet another false dichotomy!

not wish an education in mathematics to deflect the student from concern for all that is important in life outside mathematics. However, only good can result from mathematics being associated with enjoyment and with the full play of the imagination and the capacity for fantasy.

4. Elitism vs. Egalitarianism

The United States is a country which rightly prides itself on always seeking to provide equality of opportunity for its citizens, whatever their background. One must speak here of a worthy attempt rather than an achievement, since it is well known that there are still serious shortcomings in the educational opportunities provided to many American citizens. Nevertheless, it is a magnificent principle that every child should have an equal opportunity to obtain a good education. However the principle leads to absurdity in practice when it is held that education should be so designed as to ensure that it bestow no advantages on those with privileged backgrounds. It is even held by some to be élitist to provide an education of such a kind that some will visibly profit more than others! And courses of dubious educational value are inserted into the curriculum in order that students with cultural and economic disadvantages should have equal chance of scoring well. One cannot eliminate fundamental disadvantages by pretending they do not exist.

It has to be said that education by its very nature produces differences; if it does not, it is unsuccessful. It will always be the case, even if socio-economic inequalities were somehow eliminated, that one student will benefit more than another from his education. This simple fact cannot be masked by any device however well-intentioned. If education is successful, then as many students as possible benefit from it, but the benefit they derive will be unequal from student to student. Moreover, where the student enters a course or program with disadvantages in his background, one must expect that he will be handicapped with respect to the benefit he will derive from the course. We must always be seeking, as responsible citizens, to eliminate or, at least, constantly reduce the differences in the socio-economic backgrounds of our students. But even in a completely egalitarian society, we will not achieve equal response to educational opportunity. Thus we can and should be motivated by the principle of egalitarianism in devising our entire educational system; but this principle certainly does not justify an across-the-board dilution of the curriculum. We must expect and welcome the emergence of students from the educational system who, by virtue of their ability to profit from their educational opportunities, deserve to become members of an élite. Membership of this élite should confer no social advantages; but it surely confers duties and responsibilities which we would hope that these outstanding citizens would shoulder willingly. Among the most important of those responsibilities is that of ensuring the best possible educational opportunities to the disadvantaged.

5. Structure Building vs. Problem Solving; Axiomatics vs. Constructiveness

These two false antitheses are closely related to each other. The charge has been made against the 'new mathematics' that it emphasizes too strongly the importance of mathematical structure, and that it overlooks the necessity to educate the child to be able to solve problems. It could well be that this is a valid criticism of much of the teaching which has taken place under the banner of the new mathematics; but, if so, then this is due to a misunderstanding of the purpose of the reform movement in the teaching of mathematics, and certainly does not correspond to the intention of those responsible for the educational philosophy of that movement.

Certainly the student of mathematics must be able to solve problems. But the problems are presumably to be those which require mathematics in their solution, and consequently the mathematics must be well understood if it is to be effectively applied. It will be readily acknowledged that it is an important educational experience to be faced with a problem and then to attempt to solve it. But to deny oneself access to the available theory in attempting the solution is to place oneself under a very grave handicap and enormously to diminish the probability of obtaining a good solution in a reasonable time. Just as exclusive concentration on the building of mathematical structure is a distortion of the principles underlying the new mathematics, so is exclusive concentration on the solving of problems a distortion of the principles underlying the discovery method in education. Problems are most efficiently solved by the application of the appropriate theory; and the taste for the theory is most likely to be developed in response to the desire to solve interesting problems. Thus the two activities of structure-building and problem-solving are highly complementary to each other.

Within mathematics itself, the two sides of this coin are called *axiomatics* and *constructive methods*. The detractors of the axiomatic method envisage it as the unmotivated and soulless enunciation of axioms for an abstract system with no reference whatsoever to possible applications. For example, Professor Feynman, the Nobel Prize-winning physicist, in his stimulating Messenger Lectures at Cornell University in 1964, portrayed the mathematician as starting from a rigid but arbitrarily prescribed set of axioms and coldbloodedly grinding out all the consequences of those axioms. He contrasted this dehumanized activity with that of the physicist, stepping lightly from context to context with delicate imaginative leaps, and every now and again penetrating the unknown with bright exciting shafts of light! In fact, of course, Feynman's description of the natural *modus operandi* of the physicist precisely describes that of the good mathematician. However what we are concerned with here is the false dichotomy made between the axiomatic method in mathematics and the constructive method in which one actually executes 'concrete' operations on 'concrete' entities and derives consequences in this way.

That there are the two methods is undoubtedly true. It is also true that there is a school of philosophical thought - intuitionism - which holds that only constructive proofs in mathematics are really valid. However this point of view is taken by an extremely small number of mathematicians; nor does this point of view exclude the totality of axiomatics - it simply denies the validity of proof by contradiction. The axiomatic method was employed by Euclid to establish elementary geometric properties, and undoubtedly marked a tremendous advance in human thought. Unfortunately, in the teaching of geometry today, it so very often happens that the geometric results being proved are uninteresting and obvious, and the proofs smack unmistakably of pedantry. However if geometric proofs were used to generate discovery, as indeed they might and should be used, then the argument against axiomatics would surely disappear.

The founder of the modern method of axiomatics in mathematics was David Hilbert. With Hilbert one finds for the first time the clear statement that the axiomatic system carries no ontological commitment. The axioms are postulates about undefined entities, and not self-evident truths about actual objects. Hilbert employed the axiomatic method with tremendous success. One of his earliest triumphs was his solution of the problem of finding a finite basis for the system of invariants of algebraic forms. The German mathematician Gordan had proved that there existed a finite basis for binary forms by a constructive argument. In 1888 Hilbert found a new non-computational proof of Gordan's theorem, and in September of the same year he announced the complete solution. In attacking this very 'concrete' problem of mathematics, Hilbert had essentially invented the genuine axiomatic method. For Hilbert did not construct a finite basis for the invariants, but he proved that the system of invariants must have a finite basis. The reaction of Hilbert's contemporaries is very interesting in the light of the prejudices which abound today. Lindemann, one of the great founders of modern number theory, described the argument as 'unheimlich' (weird). Gordan himself said 'Das ist nicht Mathematik, das ist Theologie', (that is not mathematics, that is theology). Kronecker, who can be said to have been the precursor of intuitionism and who was Cantor's bitterest opponent in the latter's development of modern set theory, rejected Hilbert's argument out of hand. It is interesting that in 1890 Hilbert obtained a method of constructing the finite basis whose existence he had proved.But he wrote 'to prohibit existence statements ... is tantamount to relinquishing the science of mathematics altogether'. Where the axiomatic method is employed in the solution of a problem, it has great value and utility. The mere building of mathematical structure, with no thought to the problems to which that mathematical structure can be applied, is not, by any conceivable standards, a worthy mathematical activity, but that is a caricature of the axiomatic method.

In more recent times, René Thom, the great French mathematician, constructed a new mathematical theory called cobordism, and received a Fields Medal for this invention. Thom's new concept served to unify and to systematize wide areas of topology - and also to solve important problems. The enormous amount of literature which has followed Thom's pioneering work is adequate testimony to its importance.

To disabuse those who believe that the building of mathematical structure and axiomatic reasoning are simply the preserves of a self-serving clique of pure mathematicians it should not be necessary to do more than quote the remarks of Richard Courant, one of the greatest applied mathematicians of this century, in his centenary tribute to Hilbert. Courant said, in 1962, 'Living mathematics rests on the fluctuation between the antithetical powers of intuition and logic, the individuality of grounded problems and the generality of far-reaching abstractions. We, ourselves, must prevent the development being forced to only one pole of the life-giving antithesis. Mathematics must be cherished and strengthened as a unified vital branch in the broad river of science. It dares not trickle away in the sand. Hilbert has shown us ... that there is no gap between pure and applied mathematics, and that between mathematics and science as a whole, a fruitful community can be established'.

The axiomatic method has established itself as an enormously powerful tool of abstract reasoning. If this tool is to be acquired by our students, then it is essential that in the schools the role of structure in mathematics should be understood by teacher and student. It is disturbing that, in the new round of curriculum reform in mathematics, so much emphasis is placed on problem-solving as if this were an alternative to the development of mathematics itself. Far from being an alternative, it is an essential part of the development of the understanding of mathematics and cannot take place effectively except in the context of a rich mathematical education. The desire to solve problems is the motivating force for the creation of mathematics, and so it is the motivating force for the understanding and acquisition of mathematics in the educational process. To try to solve problems which are appropriate to mathematical treatment, and to remain in ignorance of the appropriate mathematics is absurd and self-defeating.

Morris Kline, in his provocative book *Why Johnny Can't Add*, has attacked the use of the axiomatic method as a basis for elementary mathematics education. There can be no dispute with Kline's point of view, only with his claim that he has identified the main reason for our lack of success in trying to give a good mathematical education to our children. The axiomatic method, as an explicit formulation, should only occur relatively late in the educational process. But the study of the actual structure of the number system which we use is surely appropriate at the elementary level. Here it should always be assumed that the study is not made for its own sake, but in order the better to use mathematics as a tool in our understanding of the world around us. Thus one finds, in the programs to which I have referred, due concern for the practical importance of computational skill allied to an awareness of the fact that the number system does present certain structural features which can very much simplify the resulting arithmetic, and which enable the student to understand better the natural development of mathematics itself. Another example of an important structural concept is that of linearity. It is silly to spend hours, days, weeks solving simple equations. But it is of immense educational value to understand properly the advantages of the linear model. The objective of any good curricular program must be the education of the future citizen, whatever his special field. That education must include an understanding of the nature of the mathematical method. To reject the study of mathematical structure for its own sake - and we should reject that for the majority of our students - is not at all to deny the importance of structure in reasoning about the real world.

6. Art vs. Science; Pure Mathematics vs. Applied Mathematics

A contrast between art and science is often felt to be a basic feature of education and, indeed, of the entire way of life of an individual. Currently the sciences are in retreat before the attack of those who maintain that scientists have shown a lack of concern, amounting even to contempt, for the welfare of the human race, and have pursued their science in total disregard of its social, political and moral consequences. Identifying science with technology, critics have pointed to the spread of pollution, and the proliferation of ever more sophisticated weaponry, as evidence of the lack of any humanity in the scientist, and have deduced from this a lack of any humanism in his education. As an expression of this revulsion against science, students are enrolling in courses in business studies and social sciences at the expense of courses in the natural sciences, which are therefore undersubscribed in our universities - and bright young people are even turning their backs on university education altogether, feeling the university to be part of an establishment characterized by indifference to moral issues.

It is ironical that this attack on science takes place at the same time as the attack on mathematics for being too pure, too abstract, and for neglecting applications. Of course the attack does not come from the same people. But the fact that those attacks are taking place simultaneously illustrates the important principle that one should never seek to please all the people all the time.

It is not my purpose in this paper to enter into the broad sociological issues raised by the attack on science.*) Here my limited purpose is to show, first, that there is no true dicho-

*)An interesting treatment of these issues, furnished with a comprehensive bibliography, is to be found in the article The Soluble in Pawn to the Possible, by R. Williams, in Encounter, January,1974.

tomy between pure mathematics and applied mathematics, and, second, that much of the art of mathematics is concerned with applying mathematics to the world around us.

With regard to the first thesis, I would like to begin by discussing the question of what should be understood by the term 'applied mathematics'. In my view, it would be better that this term should be used only in its literal sense to refer to mathematics which has been applied to a non-mathematical problem, and that we should talk of 'applicable mathematics' and 'applications of mathematics' where these terms are self-evidently appropriate. Certainly the term 'applied mathematics' should not be confined to mathematics which was devised with a view to its application, and even more certainly it should not be confined to those parts of mathematics which have appeared in traditional applied mathematics courses. Today, when mathematics is being applied in so many new areas, particularly in the social sciences, it is clearly not possible to prescribe in advance what parts of mathematics will prove to be useful in application. For example, the theory of error-correcting codes, which is plainly of great importance in automata theory and computer science, depends on a study of positive definite quadratic forms over the field of two elements, and numbers among its theorems the fact that any function defined on a finite field with values in that field is polynomial. Certainly the theory of quadratic forms and the theory of finite fields were developed with no thought to applications at all, but have now been appropriated by applied mathematics because applications have been discovered.

Thus it is clear that the distinction between pure and applied mathematics is incidental. Far better simply to talk of mathematics and, where appropriate, of applications of mathematics. Since all of mathematics is potentially susceptible of application , we must reject the heresy which preaches that there are parts of mathematics - algebra, geometry, topology, number theory, for example - which can safely be neglected by the future applied mathematician. We should also reject the view that a sharp distinction should be made at the undergraduate level, or even perhaps earlier, as some misguided people might claim, between those who propose to use mathematics as a tool, and those who propose to study it for its own sake. All students of mathematics should learn mathematics, and should also learn something of the way in which mathematics is applied. For while there is a distinction between the pure and applied mathematician with respect to his characteristic motivations, the methodologies overlap substantially. In both, there is a very strong experimental component which tends to be overlooked in the more rigid deductive treatments which one finds in many textbooks. Both involve the judicious selection of problems suitable for mathematical attack. These problems may fall within or without the domain of mathematics itself; and the question of whether a given individual finds the inspiration for his mathematics from within mathematics or from outside mathematics is the most conspicuous discriminant between the pure and applied mathematician.

In analyzing a schema for doing applied mathematics, for example, the schema elaborated by Henry Pollak, one finds key places where art plays a crucial role. For example, there is the question, already referred to, of selecting a suitable problem. This is unmistakably an art, since it is impossible to render it algorithmic. There is then the matter of choosing a suitable mathematical model. Here the criteria are again complex. They must involve the area of competence of the mathematician, his judgment as to the extent to which he is willing to approximate to the problem situation by the model, and various other constraints such as the expected expenditure, in terms of time, effort and even money, in gathering the necessary numerical data in order to develop and test his solution. If he succeeds in reasoning within the model and doing the appropriate calculations and then, on testing his solution against the original non-mathematical situation, finds that the fit is not good, he then has the problem of modifying the mathematical model, or perhaps of collecting further data. Here again, judgment is essential - no systematic procedural algorithm can be adequate. Thus there are many examples of the art of applied mathematics, and we see that it is correct to speak of the science of mathematics, and of the art of applied mathematics. For, since mathematics incorporates a systematic body of knowledge and involves cumulative reasoning and understanding, it is to that extent a science. And since applied mathematics involves choices which must be made on the basis of experience, intuition, and even inspiration, it partakes of the quality of art. Thus in mathematics there are certainly to be found both art and science, and there is science in both pure and applied mathematics, as there is art. We have demolished both false dichotomies at one blow!

The rest of science is informed by mathematics - this is, I believe, a very fundamental criterion of the maturity of a science. Mathematics without science is viable but it is bad pedagogical practice; science without mathematics is not even viable.*) A good educational program must be based on a full realization of these facts. For the science program should stress the importance of intelligent observation; and many opportunities should be provided for converting mere seeing, and mere awareness of the individual constituents of one's environment, into a source of speculation and enlightened curiosity. The incorporation of the results of intelligent observation and experiment into a science involve mathematics. Feynman has said that 'nature speaks to us in the language of mathematics'. Peter Medawar, the Nobel Prize-winning biologist, has described science**) as 'the art of the soluble'. We solve problems by applying symbolic reasoning, and that is the domain of mathematics.

Conclusion

We have been concerned in this article to dispose of certain false dichotomies. The prevalence of these false dichotomies has been a barrier to clear thinking about educational problems. In the case of all of these sterile and misleading antitheses, those treated in the article and many others, we need and we can have both sides of the artificial opposition. We will not get both sides, and are indeed in danger of getting neither side, if we

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*) I do not speak of science without mathematicians! The claim that this would not be viable is more contentious.

**) Americans would replace 'soluble' by 'solvable'.

listen to special interest groups advertising their wares and trumpeting their pet panaceas. Good education involves a kaleidoscope of qualities and attributes; its success depends on a many-faceted approach and therefore on the cooperation of all those genuinely concerned - content experts, educators, psychologists, parents, administrators. I salute, in particular, the Open Court elementary program for putting this principle into full practice, and I hope that others will find it a model to follow.

2.4 Aspects of Simplification in Mathematics Teaching

Arnold Kirsch

0. Introduction: Simplification as a Way of Making Accessible

In the discussion about curriculum, the main problem is *choosing* and *justifying* the content to be taught. Also of importance, and closely tied to it, is the question of *dressing* the content. Everyone expects the teacher to 'simplify' and 'elementarize'. In what follows I shall consider *simplification as the process of making accessible*. I shall not touch on simplification in the sense of 'pruning'[1] or 'stepping down to a lower level'.

Simplification in the above sense has been discussed at length in the pedagogic-psychological literature, for example as 're-structuring' (Umstrukturieren). However, in practice it takes place without much further thought. G. Becker has made a study of 'elementarization' in a specifically mathematical context [3], and I wish to use this as a starting point[2].

The term simplification is commonplace: this should not lead one to believe that in a concrete situation there is necessarily a-greement as to whether the matter at hand has been simplified or not. A mathematician talks of 'simplifying' a proof if he can eliminate extraneous concepts and machinery - which often means that the arguments become more refined and less obvious, and thus *more difficult* for the student. Or else the didactician tries to make equations 'easier' to understand by explaining the concepts term, variable, expression - which the classroom teacher may see as *complicating* the matter. The author of a pedagogical best-seller recently 'simplified' the concept of a null sequence

1) cf. [18, p. 117], [35].

2) I am grateful to many colleagues for discussions and sugges-tions, in particular to Mrs. E. Schildkamp-Kündiger and Messrs. B. Artmann, G. Becker, W. Blum, H. Meißner and S. Seyfferth.

of positive numbers by defining it as a monotone decreasing sequence ('every number is smaller than the preceeding one'[1]), which is simply *false*[2].

Who is to decide whether something has really been made easier to understand? One would like to leave this to the judgement of the *teacher*; yet he often finds something difficult only because it is new to him. What about the *pupil*? But how is he to judge whether what he has been taught is the real thing - whether something essential hasn't been 'removed from his way' and thus withheld from him.

These questions illustrate the methodological difficulties in developing general and sufficiently precise concepts in a science of mathematics education. I shall not try to give a general theory of simplification which aims at being widely acceptable. I shall use simplification in the sense of making accessible, as a *guiding line*, along which to organize some didactic developments, and in order to report on some concrete suggestions. To do this I shall classify certain *specifically mathematical aspects of making accessible*. Perhaps they can be seen as part of more general questions in learning theory, but I do not think they can be inferred from the latter.

The first aspect is

1. Making Accessible by Concentration on the Mathematical Heart of the Matter

This aspect corresponds to the view that mathematics in its most mature form, i.e. *mathematics in the narrowest sense of the word*, stripped of all genetic elements and connections with reality, is the simplest mathematics[3]. Working out the central concepts,

1) Fr. Vester: Denken, Lernen, Vergessen. Stuttgart 1975 (p.169).

2) The (old) dichotomy simplification-falsification is discussed in [35] and specially with reference to mathematics teaching in [45, p. 11].

3) This was the point of view taken already by A. Pringsheim in his controversy with Felix Klein; cf. Jahresberichte der DMV 1898, p.73-83.

generalizing, and emphasizing the fundamental structures is a way of making mathematics more accessible.

This corresponds without doubt to certain approaches in learning theory[1]. In his 'principle of progressive differentiation', D.P. Ausubel [1, p.167/168] suggests that one should first present the general, inclusive ideas of a discipline, as he calls them. These should make access to special problems easier by playing a 'subsuming role'. Here one may also think of Z.P. Dienes' well-known deep-end principle.

As a first example of this, take the approach to *word problems in primary school* which uses letters as variables before numerical computation. H. Freudenthal [13] reported recently on the suggestions and experience of V.V. Davydov and his school in the USSR in this direction. Abstraction and generality are supposed to be achieved here directly rather than from a large number of special cases, and to make access to concrete problems easier.

On the other hand, it is possible to make a concept more difficult by refusing to take a sufficiently general point of view. This I had a chance to observe recently in a class of 15-year-olds. A full hour was used to explain the invariance of parallelism under shear mappings; the proof was split up into many cases requiring special geometrical arguments which even ran over into homework exercises. If one had thought back to the *heart of the matter* here - that a shear is a (line-preserving) permutation of the plane - then the question would have become clear immediately.

And no one will deny that *Fermat's little theorem* can only really be understood and appreciated in its abstract group-theoretic form. But the following observation makes one pause and think. Repeated efforts to lead student teachers to the group concept via permutation groups[2] found little resonance: they objected

1) Cf. also H. Lenné [31,p.82] and the references given there.

2) With permutation groups it is natural first to consider the inversion of permutations and then to look for the identity element, so that one can compose permutations; with abstract groups one must first introduce the identity and then inverse elements.

that it was 'much easier' to begin straightaway with 'real groups'.

In vector geometry in schools one still finds the fossil expressions 'collinear' and 'coplanar', the introduction of which creates many difficulties, especially with the zero vector. These ideas become much better accessible if *'linear dependance'*, which is what one is aiming for, is introduced immediately. A suitable definition might be: 'Vectors are linearly dependant if at least one of them can be written as a linear combination of the others'[1]. It is not the notion itself which is difficult for pupils, but the definition which mathematicians usually give in order to simplify the deductions which follow it.

Here one sees that making accessible is not the same as simplifying the deductive structure. In school mathematics, which is what I wish to restrict myself to here, it is the understanding of the meaning of important concepts and the ability to sensefully manipulate them that counts. For this it is not enough to establish deductive interrelations[2].

There is no doubt that a proper choice of definitions and axioms makes an area of mathematics much more accessible; but a simplification of the deductive structure cannot be the only criterion of choice here[3]. Above and beyond internal mathematical considerations, the didactician and teacher must show imagination, and take into account the pupils' background knowledge. He must be *sensitive to alternatives* just where a mathematician would see no difference. (For example, are vectors as classes of arrows 'the same as translations' or not?)

The following examples illustrate what I would call a (structural) simplification by *putting the cart before the horse*[4], which only serves to make access more difficult:

'defining' the product of natural numbers via the cartesian product of sets;

1) It is natural at first not to explain linear dependance for one vector. This follows later as an extension (not a revision!) of the given definition.
2) Cf. H. Freudenthal [12.p.78], E. Spanier [36].
3) R. Fischer also emphasizes this in [11].
4) This is related to what H. Freudenthal calls "antididactic inversion" in [12,p.100].

'defining' the sum of fractions as $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ [1];

'defining' a square as a quadrilateral with four axes of symmetry;

'defining' a rotation as a product of reflections;

'defining' a plane in three-dimensional space via a linear equation;

'defining' a convex function via the first or even the second derivative;

'defining' the area of a trapezoid with curved sides using an integral (defined via primitive functions);

'defining' the logarithm as the integral of $\frac{1}{x}$;

'defining' the sine and cosine as solutions of a system of functional equations.

Here and in many similar situations one uses properties as definitions which can and must be explained if the essence of the concept is to become clear[2]. From a scientific point of view this is of course legitimate; I do not want to make a value judgement, but only to pinpoint a possible complication in the learning process. Of course what the essence of a concept is, may be hard to decide. But it is by no means merely a matter of taste.

Thus the phenomenon of 'putting the cart before the horse' illustrates a *specifically mathematical problem* in making something accessible: structural simplification can make access more difficult.

The next aspect is in a sense complementary to the proceeding one.

2. Making Accessible by Including the 'Surroundings' of Mathematics

One has always tried to make mathematics more accessible to pupils by introducing mathematical objects in a less abrupt fashion and by taking a *broader view of mathematics* which includes

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1) Interpret the fractions as lengths, say. The given equation then follows from the properties of a domain of quantities (cf. [21, p. 123]).

2) This is true for many axioms which have been suggested for didactic reasons.

the origin of concepts and their relations to reality. This approach too is supported by arguments from learning theory, in particular those concerned with the problem of motivation. It is supported by methodological experience as well.

A controversial example from primary school mathematics is the introduction of powers. Instead of attacking the heart of the matter (iterated multiplication) immediately, some masters of method try to make the concept easier by talking about iterated 'bundling' of real objects, and thus to relate it to the surroundings of mathematics[1].

A particularly important example where the origins of an idea have been exploited is Dienes' treatment of *groups*. He was able to make the group concept accessible even to younger pupils by distinguishing between 'states' and 'operators' and in particular using Cayley graphs (thus going back to the prehistory of group theory)[2].

What lies behind this, from the mathematician's point of view, is a *'duplication' of the group structure*, in the sense expressed in Cayley's theorem. At this level, a group is a sharply transitive group of operators acting on a set of states. This avoids many difficulties: operators appear as concrete objects, and one 'knows' an operator as soon as *one* initial state and the corresponding target state are known (!). But just here arises the danger that pupils (and teachers) will not reach mathematical maturity.

Such a duplication of the group structure also lies at the base of the traditional *arithmetic of fractions*. Unlike group theory, this is a substantially meaningful topic[3] for the majority of

1) Originally I talked of the 'premathematical area' here. But Z. Semadeni (Warsaw) kindly pointed out to me that the notion of 'premathematics' has been used since 1973 with a well-defined meaning (Z. Semadeni, The concept of premathematics as a theoretical background for primary mathematics teaching. Institute of Mathematics, Polish Academy of Sci., 1973), so I have avoided this term.

2) Of course, what concerns Dienes may not be making the group concept accessible, but rather the more general goal of stimulating the cognitive development of younger pupils.

3) D. Kahle discusses 'substantially meaningful - suitable for more general goals' as alternative criteria in selecting teaching material.

pupils. In treating the multiplicative group $\mathbb{Q}^{+,\cdot}$, one differentiates between 'concrete' numbers and 'pure' numbers, or in the current terminology: quantities and operators. Exhaustive analyses, especially by H. Griesel, have cleared up this area and have succeeded in justifying several traditional methods of presentation. The importance of the concept of a 'domain of quantities' for teaching which is oriented toward genetic development and applications, has also been recognized[1].

As a result, the operator method in the arithmetic of fractions, originally published by P. Braunfeld [5] and G. Pickert [33], has been steadily developed[2] and is widely used today. One gives increasing weight thereby to decimal fractions and the fact that these, considered as operators, can be realized directly with pocket calculators.

Critics of current mathematics teaching see in the distinction between states and operators an unnecessary *complication*. It is indeed true that the distinction has been 'overworked'. The main problem has turned out to be that background analyses meant for the teacher have often been mistaken for proposed classroom material. Here is an example of this 'making implicit content explicit' which R. Thom [40] criticizes. Notice by the way that many mathematicians introduce the same complication (duplication of the basis group structure) at a higher level when they treat affine spaces parallel to vector spaces - despite J. Dieudonné's categorical condemnation of any such concession to genetic aspects.

The set of states of a Cayley graph, domains of quantities, and affine spaces are outlying objects in relation to the central concepts of mathematics. They are added to mathematics (in the strict sense) to make it more accessible. Another example of this, which seems to me to typify the efforts of mathematical didactics in Germany to take into account the surroundings of mathematics, is the use of metaconcepts in introducing the *language of algebra*, i.e. in teaching how to cope with variables.

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1) See for example [17], [21], [23], [39].

2) Cf. "Der Mathematikunterricht, 1970, vol. 2 and 1975, vol. 1

After this approach had first been clarified, particularly by H.-G. Steiner [37] and H. Wäsche [42], there followed a period where it was taught with an exaggerated perfectionism that led to dismaying complexities. One is now beginning to find a mean and to reap the fruits of experience[1]. There is no doubt that this area has become more accessible thanks to its having been demystified. The number of pupils has been reduced whose helplessness in the face of 'statements' like $a = b$ impeded their mathematical progress. In return, one must accept a certain complexity caused by the use of such metaconcepts.

How does this fit in with our first aspect? All experience indicates that a certain *complexity does not usually bother pupils and teachers so much as excessive abstractness*. This too is a specifically mathematical observation about simplification which probably cannot be deduced from general learning theory.

The aspect just discussed can be put under the label *enrichment*[2]. The next one, on the other hand, is a form of *reducing the content*[2]. However, what concerns me here is not a systematic theory, but rather certain aspects which I consider especially important in the present state of mathematics teaching. The following aspect is formulated deliberately in a positive way (not as a reduction):

3. Making Accessible by Recognizing and Activating Pre-Existing Knowledge

In emphasizing this aspect, we are setting ourselves against the widespread tendency to develop mathematics 'ab ovo', or to go right back to the beginning and start without assuming anything. This tendency can be found not only with systematizing mathematicians (when for example they tell their students to forget what they learned in school), but also in genetically oriented didacticians like A.I. Wittenberg [44,p.67] and even in the pri-

1) I would draw attention particularly to H.J. Vollrath's book [41].

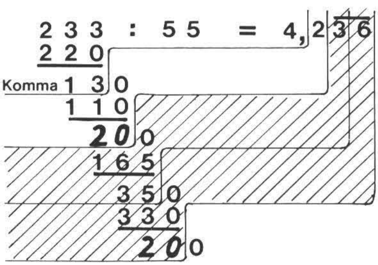
2) Cf. G. Becker [3].

mary school methodologist when he ignores children's previous experience with numbers[1].

We, on the other hand, want to see pupils encouraged to make use of pre-existing knowledge, also from outside of mathematics. This approach takes into account realities like Sesame Street, and saves time and effort on the part of the teacher. Above all, we want to avoid the frustration which can be caused by denigrating what pupils already know.

Learning theory also encourages a linking-up to existing knowledge. In particular, it belongs, according to D.P. Ausubel [1,p.53,153], to the conditions for 'meaningful learning'. At the same time, it is in accordance with didactic principles like E. Wittmann's integration principle [45], or more generally with the genetic principle[2].

Here is an example: one should recognize and utilize the pupil's knowledge of the *division algorithm* when explaining periodicity in decimal fractions or when introducing nested intervals. Because of his experience with this algorithm, he will see that the repetition in the calculation in the shaded part of the figure below corresponds to a period in the decimal expansion. This insight will only be made hazy if one tries to explain the algorithm or to translate everything into 'mathematical' language. And he will immediately see that the inequalities on the right hold. But he would not understand a formal proof of them using the theorem on division with remainder.



$$
\begin{array}{llcl}
4 & < \frac{233}{55} & < & 5 \\
4,2 & < & < & 4,3 \\
4,23 & < & < & 4,24 \\
4,236 & < & < & 4,237
\end{array}
$$

1) This holds particularly for the teacher in the 5th school year who tries to 'explain' the natural numbers via one-to-one correspondence between sets; cf. [25].
2) Cf. [46,p.231].

One should also accept that the familiar notion of a decimal fraction as a sequence of digits[1)] is a perfectly good basis for introducing the real numbers. One doesn't have to begin with a precise *definition of a sequence*.

In analysis too, this naive idea of a sequence is sufficient for a long time. Defining sequences as mappings with domain $\mathbb{N}$ too early in fact inhibits the development of a creative understanding of what a sequence is and makes it impossible to operate freely with subsequences. Fortunately one 'forgets' this definition soon: for example, when one explains monotony (already defined for functions) in the form $a_1 \leq a_2 \leq a_3 \leq \ldots$.

Pupil's experience with the *decimal representation* of natural numbers makes possible a proof of the insolvability of the equation $\left(\frac{m}{n}\right)^2 = 2$ $(m,n \in \mathbb{N})$ which experience shows to be particularly accessible to them: one simply compares the last digits in the expansions of m^2 and $2n^2$.

Children's acquaintance with finite sequences as *words* can also be built upon in elementary combinatorics[2)]. One can already explain to younger pupils what a word with s letters in a given alphabet of n letters is. What they have learned in spelling enables them to recognize when two words are the same and when they are different. In this way, they understand immediately that there are precisely n^s such words. And they can apply this in real situations by using the words as 'code-words' (to describe escape routes for example). For such purposes it is superfluous and indeed harmful to define a 'word', something children are familiar with, as a mapping, a concept which in early stages isn't even available. This doesn't mean that we are for a return to the misleading language of combinatorics as it was traditionally taught. We are simply arguing for the recognition of what pupils have already learned where it can be made use of.

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1) Here is a striking (though artificial) example of how fruitful the naive idea of 'infinite decimal fractions' can be:
Claim: There exists an injective probability measure P on the power set of $\mathbb{N}$ (as sample space).

Proof: for $A \subset \mathbb{N}$, let $P(A) = 9 \cdot 0.z_1 z_2 z_3 \ldots$ where $z_i = 1$ if $i \in A$, $z_i = 0$ otherwise.

2) Cf. [26].

In the classroom situation, this requires that *the teacher can judge* what knowledge of the pupils can be built upon. It is too much to expect a general consensus on this. By no means do we wish to exclude that this already existent knowledge be later questioned or deepened. In fact it *should* later be delved into and made dispensable, step by step. But one shouldn't throw it out straightaway. This should be kept in mind especially in the following examples.

Pupils have considerable experience which can be made use of in elementary geometry[1]. In particular, I am thinking of their familiarity with the existence and properties of the *elementary measures* of length, angle and area. This familiarity comes from outside the mathematics class, even from outside of school, which we may regard as a particularly fortunate situation.

In the classroom today, one doesn't insist on developing elementary geometry completely rigorously[2], and usually makes use of these measures without saying anything. But even at a more demanding level, the use of these measures as indefined basic concepts can lead to a desirable simplification. This was originally brought out by the book of Birkhoff and Beatley. In recent years it has been further developed and put successfully into practice in several courses[3].

By the way, one is carrying the recognition of existing knowledge even further if one bases elementary geometry on vector spaces as J. Dieudonné. For this means that one assumes, de facto, previous knowledge of the similarity theorems. This does *not* seem justified, either in classroom teaching or in the instruction of student teachers.

What we are concerned about becomes especially clear if you think of how *trigonometric functions* are sometimes introduced: it is a

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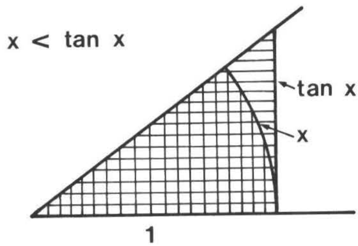
1) Like J. Dieudonné, we understand this as 'a kind of physics of space', though not as a purely experimental discipline the way he does.

2) Cf. [19].

3) DIFF Studienbrief III, 1 'Elementargeometrie', 1. Teil, Tübingen 1974; G. Holland, Geometrie für Lehrer und Studenten, Band 1, Hannover 1974.

waste of time and only frustrates pupils if one runs down their existing knowledge of angle or of arc length. It is quite possible to build on this knowledge even in teacher training programs. An independent development of the trigonometric functions is on the whole only necessary as background knowledge for teaching the final school years, and here it *is* necessary.

In the same way, it is legitimate in school mathematics to make use of the formula for the area of a sector (in terms of arc length) in order to prove[1] $\lim_{x \to 0} \frac{\sin x}{x} = 1$. The basic inequality $x < \tan x$ $(0 < x < \frac{\pi}{2})$ then follows by comparing the corresponding areas [2].



On the other hand, it is not justifiable to obtain the inequality by simply comparing the arc with the tangent segment, without any arguing, as is commonly done. Here lies the dividing line between legitimate simplification and a falsification which does not get past critical pupils.

The *area* of plane figures is a concept where intuition is particularly reliable. It seldom leads to false conclusions (unlike one's intuitive idea of arc length, where zig-zag lines can already cause difficulties). The demonstration that there are unmeasurable figures in the plane is refined and non-constructive. Imagine what it means for a pupil if his notion of area is questioned: the formula for the area of a triangle no longer appears as a theorem one can prove, but rather as a definition (another example of putting the cart before the horse). And Hippocrates' discovery about the 'little moons' is degraded to a hypothetical statement.

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1) Cf. [30, p. 193] and the references given there.

2) H. Freund [15, p. 49] also proves the inequality $\sin x > x - \frac{x^3}{6}$ by considerations of area.

Intuitive ideas of area can even be used as a foundation for the *concept of integral.*[1] This is useful for pupils in courses with a minimum of mathematics, who at present tend to have no contact at all with integrals. We simply define (following E. Artin[2]) the integral of a function as the area under its graph (taken positively above the x-axis and negatively below). Then we can devote ourselves straightaway to the central problem of calculating integrals[3].

We thus avoid questions of definition and existence proofs which cause insurmountable difficulties for the weaker pupils and which they forget later anyway. With this geometric definition, one can also *prove* the fundamental theorem of calculus; one doesn't have to degrade it to a definition as in the approach via primitive functions.

In the proof that all primitive functions (of a function defined on an interval) differ from one another by a constant one should accept that the pupils know from experience that $f' = 0$ implies f constant. This knowledge can come from watching a speedometer, given a little imagination. What is important, first of all, is that they can make the little step from this statement to that about all primitive functions, and understand its meaning. Only then is it didactically justified and desirable to examine its plausibility and to provide a proof.

In this connection, let us formulate a general *principle for constructing courses:* one should construct them so that things one has assumed that the pupils are familiar with can be isolated and examined later more closely, or even removed altogether, without the whole structure collapsing. This principle is an example of how to sensibly isolate difficulties and is also in accord with J.S. Bruner's spiral principle [6].

An illustration of this occurs in what is called in Germany

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1) Cf. [27].

2) A Freshman Honors course in Calculus and Analytic Geometry, Princeton University, Charlottesville, Virginia, 1957.

3) W. Blum [4] has pointed out that this is really the main problem for beginners.

'*mapping geometry*' (*Abbildungsgeometrie*)[1]. Analyses have shown that 'mapping proofs', when carried out in detail, are usually more difficult than the traditional proofs using congruence theorems. But they don't fall to pieces if part of the proof is removed and replaced by what pupils know about congruences - knowledge which actually comes from outside of mathematics lessons, from their everyday experience of rigid bodies.

Some topics which up to now have been quite inaccessible to school mathematics can be made so by recognizing existing knowledge and indeed by *cultivating* it. An impressive example of this is the fundamental theorem of algebra. In a didactic analysis, H.G. Steiner [38] has shown how to make the well-known topological proof accessible for the classroom. Making the topological elements in the proof precise would by no means give the simplest proof of the theorem. The simplification here arises because one can isolate them in the way explained above.

How does this aspect of recognizing pre-existing knowledge fit in with the tendency, widespread today, to introduce concepts *axiomatically* when it is too laborious to define or construct them rigorously? One need only think of the axiomatic definitions of area, of the real numbers, or of the exponential function[2]. This is not 'exactly what we mean'. Here too one should develop a certain sensitivity to differences even if at first glance the two are the same. For example: recognizing pre-existing knowledge of order properties in geometry does not mean that one makes explicit axioms out of them, but rather that one doesn't even talk about them[3].

In general, I would like to emphasize: if introducing a concept axiomatically simply formulates explicitly what one can reasonably assume is known to the pupils, then it corresponds to what we have in mind. (To be more precise, the axiomatic formulation is a higher form of what we have in mind.) But if it

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1) See for example W. Breidenbach in [9, p. 182].

2) Incidentally, in all these cases it is a question of categorical system of axioms, where the real importance of the axiomatic method does not yet become clear.

3) Cf. [24, p. 142].

is a case of 'putting the cart before the horse', if the axioms 'come out of the blue', then it is not at all what we intend.

If we view the recognition of pre-existing knowledge as a *reduction in content*, then it is nonetheless a relatively slight reduction. One is only doing without explanations for which there is no need anyway. A teacher often has to undertake much *more extensive reductions*, even to assume results which are by no means known to his pupils[1]. But I do not want to discuss this problem here. Instead, I would like to turn to an aspect which perhaps can also be seen as a form of reduction, although this does not necessarily mean sacrificing any mathematical substance:

4. Making Accessible by Changing the Mode of Representation

One has always tried to make mathematical concepts more accessible by illustration, more generally by changing the mode of representation[2]. Following J.S. Bruner [7], one differentiates nowadays between *enactive* representation, *iconical* representation, and representation *by symbolical means* (through language as well as symbols in the narrower sense). In didactic principles like the 'prefiguration principle', one encourages the use of the presymbolic modes[3].

We cannot develop a theory, or even the phenomenology, of the ways of representing mathematical ideas here[4]. This is a problem which the psychologist is not in a position to solve, and which doesn't interest the mathematician. So it falls to the specialist in didactics. Bruner's E-I-S scheme must be modified or refined before it can be applied to mathematics. One can't simply fit the usual ways of representing mathematical concepts into

1) It is common in all the sciences to assume results in this fashion, especially from other disciplines. In mathematics, too, it has its place, and pupils should get a taste of it.Cf.[31,p.68].

2) Cf. G. Becker [3, p. 16].

3) See for example [45, p. 73] and the references given there.

4) [29] is a first try at this.

this scheme. (How, for example, is one to categorize the representation of structural algebraic concepts through models inside of mathematics?)

In what follows, I shall only recall briefly how one can represent mathematical ideas enactively or iconically to make them more accessible.

First of all, an example of *enactive representation:* we explain the rule for divisibility by 9 on a primitive abacus. Let the given number n be represented by n beads in the ones column. We perform the following *action* as often as possible: 'Take 10 beads, place 1 in the column to the left and set the other 9 aside'. The beads remaining on the abacus give the decimal representation of the number n. The number of beads remaining is the sum of the digits in the decimal representation. Now we have put aside groups of 9 beads each time, so the number n is 'just as well' divisible by 9 as the sum of its digits is.

There is no doubt that the essential point has been adequately presented here, and it has been made accessible to pupils for whom the rule might otherwise only be a recipe. Translating the argument into a more sophisticated form is by comparison of secondary importance.

It should be part of the professional expertise of the teacher to know such possibilities for representing ideas and how to employ them fruitfully. This is not always easy: the transition to the enactive level has its pitfalls. As is well known, representing permutations by 'games' with real objects can lead to considerable complications[1].

I would now like to make two remarks on *iconical representations*. The first concerns *arrow diagrams*, which have become widespread thanks to the efforts of G. Papy. Apparently, some pupils can only understand concepts like 'injective' or 'surjective' (regardless of whether one already uses these terms or not) via arrow diagrams, in other words in the form 'in no point do two or more arrows end' or 'in every point at least one arrow ends'.

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1) See [12. p. 360], [34. p. 49] as well as [29].

One asks oneself whether it is adequate to understand concepts in this form. The answer is 'yes', if one then argues with them at this level in a sound way. This is the case e.g. when one proves, using arrow diagrams, that if the mapping $g \circ f$ is injective then so is f, or if $g \circ f$ is surjective then so is g.

The next remark concerns the representation of finite algebraic system via *composition tables*. That these representations are twodimensional is definitely an iconical feature. Experience in the classroom has shown the following: concepts like the regularity of a composition (cancellation rule) or deductions such as 'regularity implies that every equation $ax = b$ is solvable (in the finite case)' become accessible to many pupils only when they are interpreted in an imagined table, i.e.: 'In every row and in every column each element occurs only once, so it must occur at least once.'

In one class I taught pupils were led (independently) through their work with tables to make the conjecture: 'Every proper subgroup of a finite group G has at most half as many elements as G'[1]. (Lagrange's theorem was not known to them.) And the pupils could give good reasons why this should be so. Thus they worked creatively at a lower level of representation. This is doubtless better than their being largely passive at a higher level.

The same holds for the many possible gradations in *verbal and symbolic representations*. For example, when one can see, in teaching proportions, how a suitable verbal explanation[2] of the basic equality enables pupils at all levels to work with them on their own, whereas symbolic formulations are largely lost on them.

There are new results in this direction on the *functional equation of the exponential function*. A. Engel [10] reports: when they discuss real growth processes, pupils suggest formulations

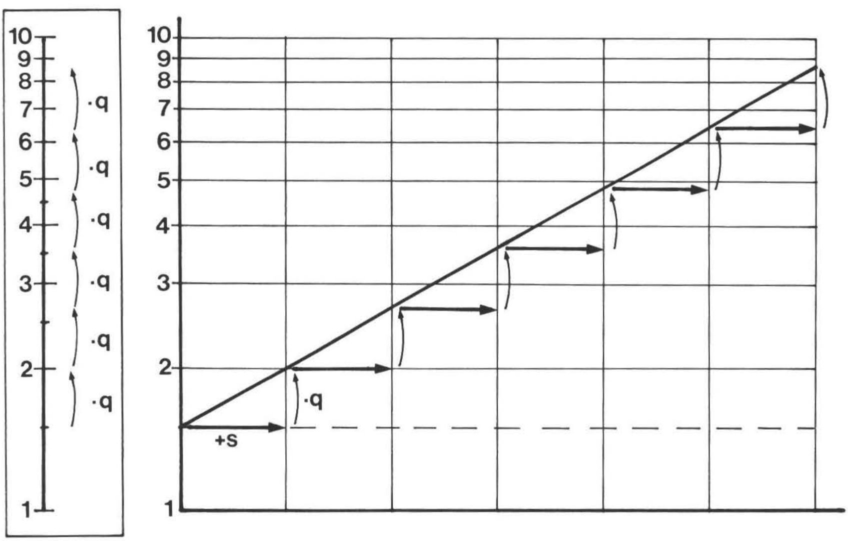
1) This statement and its explanation using the table hold true already for (finite) groupoids instead of groups.

2) See [23]. R. Fischer [11] emphasizes the importance of verbal explanations.

like '*equal lengths of time always give the same growth factor*'. This form of the equation can be used to deduce non-trivial consequences and applications which become in this way accessible to large groups of pupils.

An example (from [28]): let us follow a growth process on the scale of a slide rule. Equal factors mean moving equal lengths along the scale. So to equal lengths of time correspond steps of equal length on the scale. In other words, we travel at a constant speed along the scale. Now pupils can make logarithmic graph paper themselves, simply by marking off the slide rule scale on the left and right hand sides of the paper and connecting the corresponding points.

Thus anyone who has ever drawn a time-table graph will see: exponential growth appears on logarithmic paper as a straight line (see the drawing below). In this way - and by using enactive and iconical modes as well - one makes an important aid in practical mathematics understandable for all pupils. Teachers have assured me that this is 'too easy' for advanced pupils. However, at present the majority of pupils do not become acquainted with logarithmic graph paper, nor do they even acquire the capacity to learn about it on their own.



I cannot go into questions of symbolic representation in the more restricted sense here. Let me only say: one can make mathematics accessible using different modes of representation. Here 'anything goes', be it 'abus de langage' or comics, as used by P. Braunfeld. I have experienced how Braunfeld had to defend himself against the accusation that this wasn't mathematics. However, 'it is not justified', as R. Fischer [11] says, 'not to call something mathematics simply because it is communicated in a different jargon'. Doubtless many simple mathematical ideas remain inaccessible at present to our pupils only because they are presented in a formulation or symbolism which brings difficulties having nothing to do with the mathematical essence.

By emphasizing the validity of the more 'primitive' modes of representation, we may seem to be neglecting the question of *intermodal transfer:* H. Bauersfeld [2, p. 244] in particular has pointed out how important this is.

Naturally we should exercise pupils in the various modes and especially in the various levels of language. H. Freudenthal [14, pp. 16, 17] has illustrated this in a particularly impressive example. Nonetheless, we can do perfectly good mathematics already at the simplest level, and open doors for our pupils without misleading them, or falsifying anything. Perhaps a better formulation of our fourth aspect would be: making accessible by *beginning* at the appropriate level of representation.

5. Concluding Remarks

I must finish here. Doubtless I have left out important aspects of making mathematics accessible, such as through *illuminating examples*. These are certainly of great importance for learning mathematics[1]. But I haven't tried to cover the whole question anyway.

I wish to stress especially: all the ideas sketched here are

1) H. Griesel has shown how important it is particularly in primary school to continuously provide examples [16].

only aids to make access easier for the learner, they cannot spare him the *basic act of understanding*. Not every pupil manages this in every case.

Long experience seems to me to suggest that mathematical induction is a 'counterexample' in this sense. It has driven teachers to remarkable efforts and stimulated impressive didactic creativity, beginning on the 'enactive level' with falling dominoes. But so far there doesn't seem to be any widespread success in teaching it[1].

In closing, a word about the *method* in this paper. The aspects sketched are meant mainly to be a *means* of organizing some experience and thoughts on teaching mathematics, rather than the *objects* in their own right of theoretical studies. Nevertheless, perhaps they can serve to organize, to set standards for, or even to stimulate didactic researches.

References

[1] Ausubel, D.P. / Robinson, F.G.: School Learning. New York 1969.

[2] Bauersfeld, H.: Einige Bemerkungen zum Frankfurter Projekt und zum 'alef'-Programm. In: Materialien zum Mathematikunterricht in der Grundschule. Frankfurt 1972.

[3] Becker, G.: Möglichkeiten und Probleme des Elementarisierens im mathematischen Unterricht. In: Die Schulwarte 1974, no. 10/11, p. 10-19.

[4] Blum, W.: Ein Grundkurs in Analysis. In: Didaktik der Mathematik 3 (1975), no. 3, p. 163- 184.

[5] Braunfeld, P. / Dilley, C. / Rucker, W.: A new UICSM approach to fractions for the junior high school. In:The Math. Teacher LX, no. 3, p. 215 - 221.

[6] Bruner, J.S.: The Process of Education. New York 1960.

[7] Bruner, J.S. / Olver, R.R. / Greenfield, P.M.: Studies in

1) See [32].

Cognitive Growth. New York 1966.

[8] Dieudonné, J.: Moderne Mathematik und Unterricht auf der Höheren Schule. In: Math.-Phys. Semesterberichte 8 (1962), p. 166 - 178.

[9] Drenckhahn, F. (Ed.): Der mathematische Unterricht für die sechs- bis fünfzehnjährige Jugend in der Bundesrepublik Deutschland. Göttingen 1958.

[10] Engel, A.: Anwendungen der Analysis zur Konstruktion mathematischer Modelle.In: Der Mathematikunterricht 17 (1971), no. 3, p. 5-56.

[11] Fischer, R.: Fundamentale Ideen bei den reellen Funktionen. In: Zentralblatt für Didaktik der Mathematik 8(1976),p.185-192

[12] Freudenthal, H.: Mathematik als pädagogische Aufgabe, Vol. 1 and Vol. 2. Stuttgart 1973.

[13] - : Soviet Research on Teaching Algebra at the Lower Grades of the Elementary School. In: Educ. Studies in Math. 5 (1974) p. 391 - 412.

[14] - : Kennst Du Deinen Vater? In: Der Mathematikunterricht 20 (1974), no. 5, p. 5-18.

[15] Freund, H.: Die Gewinnung von Steigungswerten durch analytisch-geometrische Betrachtungen. In: Der Mathematikunterricht 6 (1960), no. 2, p. 22 - 51.

[16] Griesel, H.: Die Neue Mathematik für Lehrer und Studenten, Vol. 1 and Vol. 2. Hannover 1971, 1973.

[17] - : Der wissenschaftliche Hintergrund der Bruchrechnung. In: Der Mathematikunterricht 16 (1970), no. 2, p. 5 - 29.

[18] - : Überlegungen zur Didaktik der Mathematik als Wissenschaft. In: Zentralblatt für Didaktik der Mathematik 6 (1974), p. 115 - 119.

[19] Holland, G.: Vorschläge zur Entwicklung eines Curriculums für den Geometrieunterricht in der Sekundarstufe I. In: Beiträge zum Mathematikunterricht 1972, part 1, p. 103-118.

[20] Kahle, D.: Gesichtspunkte zur Stoffauswahl und zur Lernziel-

problematik.In: Beiträge zum Mathematikunterricht 1976, p. 103 - 106.

[21] Kirsch, A.: Elementare Zahlen- und Größenbereiche. Göttingen 1970.

[22] - : Endliche Gruppen als Gegenstand für axiomatische Übungen. In: Der Mathematikunterricht 9 (1963), no. 4, p. 88- 100.

[23] - : An Analysis of Commercial Arithmetic. In: Educ. Studies in Math. 1 (1969), p.300 - 311.

[24] - : Ein didaktisch orientiertes Axiomensystem der Elementargeometrie. In: Der mathm. und naturw. Unterricht 25 (1972), p. 139- 145.

[25] - : Eineindeutige Zuordnungen im 5. Schuljahr: Begründung des Zahlbegriffs oder Förderung der Kombinationsfähigkeit? In: Beiträge zum Mathematikunterricht 1973, p. 143 - 149.

[26] - : Eine moderne und einprägsame Fassung der kombinatorischen Grundaufgaben. In: Didaktik der Mathematik 1 (1975), no. 2, p. 113 - 130.

[27] - : Eine 'intellektuell ehrliche' Einführung des Integralbegriffs in Grundkursen. In: Didaktik der Mathematik 4 (1976), no. 2, p. 87 - 105.

[28] - : Vorschläge zur Behandlung von Wachstumsprozessen und Exponentialfunktionen im Mittelstufenunterricht. In: Didaktik der Mathematik 4(1976), no. 4, p. 257 - 284.

[29] - : Über die 'enaktive' Repräsentation von Abbildungen, insbesondere Permutationen. Comes out in: Didaktik der Mathematik.

[30] Laugwitz, D.: Ist Differentialrechnung ohne Grenzwertbegriff möglich? In: Math.-Phys. Semesterberichte 20 (1973), p. 189-201.

[31] Lenné, H. (edited by W. Jung): Analyse der Mathematikdidaktik in Deutschland. Stuttgart 1969.

[32] Leppig, M.: Leistungen von Studienbewerbern hinsichtlich des Beweisprinzips 'vollständige Induktion'. In: Beiträge zum Mathematikunterricht 1976, p. 120- 124.

[33] Pickert, G.: Die Bruchrechnung als Operieren mit Abbildungen. In: Math.-Phys. Semesterberichte 15 (1968), p. 32 - 47.

[34] - : Fachwissenschaftliche Grundlagen, Lehrerbegleitheft W. Mathematik S-2 (Ed. B. Andelfinger). Freiburg i.Br. 1975.

[35] Rumpf, H.: Zum Problem der didaktischen Vereinfachung. In: Schulwissen (edited by H. Rumpf). Göttingen 1971, p.68-82.

[36] Spanier, E.: The Undergraduate Program in Mathematics. In: Amer. Math. Monthly 77 (1970), p. 752 - 755.

[37] Steiner, H.-G.: Logische Probleme im Mathematikunterricht: Die Gleichungslehre. In: Math.-Phys. Semesterberichte 7 (1961), p. 178 - 207.

[38] - : Elementare Beweise zum Fundamentalsatz der Algebra. In: Der Mathematikunterricht 10 (1964), no. 2, p. 60 - 93.

[39] - : Magnitudes and Rational Numbers - a Didactical Analysis. In: Educ. Studies in Math. 2 (1969), p. 371 - 392.

[40] Thom, R.: Modern mathematics: does it exist? In: Developments in Mathematical Education (edited by A.G. Howson). Cambridge 1973, S. 194 - 209.

[41] Vollrath, H.-J.: Didaktik der Algebra. Stuttgart 1974.

[42] Wäsche, H.: Logische Probleme der Lehre von den Gleichungen und Ungleichungen. In: Der Mathematikunterricht 7 (1961), no. 1, p. 7 - 37.

[43] Winter, H.: Vorstellungen zur Entwicklung von Curricula für den Mathematikunterricht in der Gesamtschule. In: Beiträge zum Lernzielproblem, no. 16, Ratingen 1972, p. 67 - 95.

[44] Wittenberg, A.I.: Bildung und Mathematik. Stuttgart 1963.

[45] Wittmann, E.: Grundfragen des Mathematikunterrichts. Braunschweig 1974.

[46] - : Zur Rolle von Prinzipien in der Mathematikdidaktik. In: Beiträge zum Mathematikunterricht 1975, p. 226 - 235.

2.5 Mathematics and Approximation

Georges Th. Guilbaud

The practice of mathematics rules out the vague and the approximate. This is, I think, a very common opinion. Some would even say that mathematical practice is a good remedy for this unfortunate tendency of the human mind, which quite often contents itself with approximate and fuzzy ideas. For it is accepted among scientists that the vague and the approximate are undesirable. The popular scheme is:

- at the bottom of the scale: the vague and the approximate, which are mediocre;
- in the middle: the language of the sciences, aptly called 'exact';
- at the top: the splendid 'rigour' of mathematics.

For several centuries, mathematics instruction (which is our concern) has been consolidating this idea of opposition between the *exact* and the *approximate*. It seems to me that certain attitudes of rejection towards mathematics, certain obstacles met in mathematics education, arise from this. I myself have experienced it quite often. Let me give a simplified view of these highly varied experiences (I am even going to do a little caricaturing).

It is my interlocutor who is speaking:

'Of course, you mathematicians live in certainty and security. For you, two and two always make four. All calculations must be *exact*! Alas, in real life, it is quite different! Everything changes and is uncertain. And your splendid algebras are too beautiful and too rigorous as well, for our human practice, which is bogged down in the approximate'.

It is that statement that I shall answer.

First I would like to simply say that if mathematics speaks of approximations, and that is often the case, it does so *rigorously*.

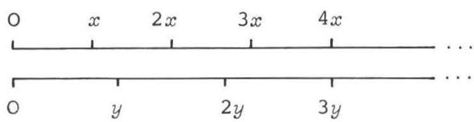
This has been the task of mathematics since its inception, and this has been its greatest driving force in all its history.

'To speak rigorously of the approximate': this seems to be a paradoxical phrase. It is as a matter of fact a kind of *challenge* to man's intelligent activity. On one hand there is this requirement of certainty and rigour, on the other hand, the inaccessibility of this perfection.

* * *

The first clear expression of an awareness of this challenge was made 24 or 25 centuries ago. Fragmentary evidence of this may be found in Greek texts. Amongst these texts, in the somewhat disparate collection called 'Euclid's Elements', the wonderful 'Fifth Book' in which experts discern the thought of the mysterious *Eudoxos* should be singled out and appreciated. I have neither the time nor the necessary competency for doing so. But from it I shall retain one single idea, one suggestion, and beg you to excuse the simplifications and anachronisms.

A figure will help shorten my explanations.



In order to compare two quantities (represented here by line segments), one considers all multiples of both. One could as well say: x and y being elements of an ordered Abelian group, one studies all linear combinations with integers as coefficients:

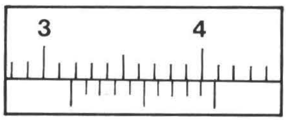
$$u = ax - by;\ a,b \in \mathbb{Z}\ ;\ x,y \in \mathbf{G}$$

An important phenomenon then appears: the *entanglement* of the multiples of x and of y can be analysed and finally put in order. If there is an *exact* coincidence, then x and y are commensurable quantities; but if this does not happen, one can say that the difference u between two multiples can be made to be as *small* as desired; the reasoning leading to this conclusion may be put into different forms. For those who discover it for the first time, it is useful to examine several of them, more or less rigorous.

Of course, nothing could prevent us from being carried away by a tendency which still exists in mathematics, to prefer the forms which are the most abstract and the most remote from common usage. But without looking down on this rhetoric, it is useful to examine those forms of speech in which the very *practical* idea of approximation is not concealed.

I shall mention only two topics:

The first being the measuring instrument, still in use today, which was invented, it is said, around 1630 by Pierre *Vernier*.



This very convenient instrument, the vernier scale[1], is known to numerous technicians of different trades. Just bring a technician and a mathematics teacher face to face with the instrument between them, and you may very well be surprised, as I was, by the communication difficulties. The one knows how to use it and is confident; the other wants to 'demonstrate', as he says. The second topic I would like to relate to the 'Eudoxian Vernier' is also a very old thought, which, as legend has it, goes back to our blessed father Pythagoras.

This is once again a kind of vernier, but here the group is multiplicative and not additive. One need not be a scholar to undertake an experiment.

Let us write in two parallel lines the series of the powers of *two* and of the powers of *three:*

---

1)also called 'nonius' in some countries after a **Portuguese** inventor.

1 2 4 8 16 32 64 128 256...

1 3 9 27 81 243...

Let us seek the approximations (literally speaking the approaches):

8 is close to 9, but it is mediocre;

243 is close to 256, it is better (error of about 5%).

If we contented ourselves with that, we could say that the fifth power of $\frac{3}{2}$ is nearly equal to the third power of two. Or, speaking like musicians, five fifths are nearly one octave.

For a better approximation, we have to go on:

$$2^{19} = 524288; \; 3^{12} = 531441$$

$$2^{19} \simeq 3^{12},$$

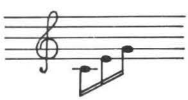
$$\left(\frac{3}{2}\right)^{12} \simeq 2^7; \qquad \frac{3}{2} \simeq 2^{\frac{7}{12}} .$$

Twelve fifths are nearly equal to seven octaves; or the fifth will be located near the seventh degree of a division of the octave into 12 equal parts. This is the starting point for the construction of all the scales in western music.

Of course what I have said is just a sketch. The study can and must be extented. Leonhard *Euler* devoted some interesting writings to this topic (adding in particular the comparison of the powers of five, which is full of surprises).

But the only point I wanted to make is this:

We will never understand the construction of musical scales as long as we do not perceive that it is a matter of seeking approximate numerical solutions of an exponential equation



$$\frac{3}{2} \simeq 2^{\frac{7}{12}}$$

$$A^x = B \Longrightarrow x \simeq ?$$

Note in particular that this search is capable of a very general mathematical formalization and that it may serve as an introduction to the learning of approximations. It is not necessary that it lead to 'continued fractions', but it is possible.

* * *

I have said that the systematic search for approximations could serve as a thread for going through the whole history of mathematics. Of course, this is not the right place to try our luck - nor to admit that in saying that, I have exaggerated or simplified a little.

I would just like to make a passing remark.

The historical point of view is often praised in mathematics. True, one should not forget that mathematics is a social phenomenon evolving in time and space. But 'to do history' is a different thing. What kind of history? It has happened too often, especially in education, that the historical presentation is reduced to biographies of great men and to the account of famous battles, or just of great discoveries. Without ceasing to pay Leibniz, Newton and Euler the tribute they deserve, I must confess that I am not content with this piety. I hope for a real history of the ways of thinking. More than in the recognized geniuses, I am interested in the host of the unknown: the scholars and teachers of all times and their students, and the men applying knowledge in the various trades.

There is enough material. Research will take a long time; it has already started. But in the meantime we can just as well consider our own era.

Let us leave history now. We are in 1976. What about approximation in our present culture? Is it an essential ingredient of our ways of thinking? And in what relations does it stand to mathematical science? We must, I think, distinguish several levels and start with everyday life.
We may, for instance, have a look at the media: newspapers, radio, TV. The extent of this fact is astonishing as soon as our

attention is drawn to it: nobody can speak, it seems, without constantly referring to 'nearly' or 'about':

presque
almost
fast
environ
ungefähr
ziemlich
nearly
à-peu-près
about
en gros
etwa
...
...

Just try yourself: take a newspaper. They are full of figures today: most of the figures are given implicitly or explicitly, with a qualification as to the degree of certainty. The population of a town, an industrial output, a distance, a temperature, a duration, a speed, a percentage etc. How are we to find our way about? An approximate value, as common sense says, is that which *is not* exact. Is it a lie then? I would not deny that newspapers sometimes contain lies. But there is not always so much malice. No! Talking and thinking by means of 'about', 'nearly' is a *necessity*. The enquiry can be made more systematic by choosing special areas of human activity. What is the accuracy used in sports timing? How are unavoidable uncertainties expressed in geodesy? What is the accuracy of financial calculations? And in demography? In economics? With a little imagination, one can then prepare numerous study trips for mathematics education, because it is essential to know what the different trades do with the numerical categories they have instituted.

One should not believe that this approximate language is abandoned in scientific discussion. This is not only so in social sciences. Phrases like 'by first approximation', 'practically verified' and 'negligible' ... may be found everywhere.

And what did I do myself as I wrote:

$$\frac{3}{2} \approx 2^{\frac{7}{12}}$$

What is then the logical status of the predicate 'approximately equal' $\approx$?

I shall call what has been mentioned so far the *first level* of approximate thinking: one says something, knowing that it is not '*totally*' true. It would be very interesting to study the words and signs used in the diverse modes of speech (and in the different languages) to mark this modality. But I am neither a linguist nor a semiotics specialist. We must first recognize that this is a universal practice, which plays an important part in the life of human societies. How do mathematicians react when confronted with this state of affairs? A first reaction is well known: it is condemnation. These manners of speaking are called vulgar, crude, obscene and reprehensible. One rejects them for oneself and condemns them in others; one forbids them if one can. One erects a barrier between vulgar practice and mathematical practice. This segregationist policy, so the legend, goes back to *Plato*: 'Nobody shall enter who does not speak as a geometrician'. It even sounds like a police order. But can we do without police?

But there is another procedure which, although less strict, is perhaps more efficient. If I placed the former under Plato's patronage, I will place the latter under Saint Augustine's, who said, thinking of the barbarians: 'Do not shout, do not cry, do not become indignant, but try to understand'.
Try to understand what your interlocutor *meant*.

In the Third Announcement on the International Congress on Mathematical Education I read: 'It will take you *about* twenty minutes to go from Karlsruhe Central Station to the University'. You see that even at a Congress mathematicians must express themselves like anybody else. But what did the organizer mean?

To reach the level of exact thought at which mathematics operates, we have to add some additional information. This can be done in many ways: the easiest way, and the oldest too, is to add an indication of uncertainty to the uncertain figure. Instead of '*about* twenty minutes' we will then say: 'twenty minutes to within *five* minutes' (for instance), i.e. 'between 15 minutes and 25 minutes'. The schema is well-known; with symbols:

$$t \simeq 20; \qquad 15 \leq t \leq 25;$$
$$t \simeq t_o; \qquad t_1 \leq t \leq t_2.$$

We then reach the *second level* of the language of approximations.

A unique number is replaced by an interval. This is also called bounding. I said second level because it is not the last one. Going from the first level over to the second seems quite natural today. It would be interesting to know how this change took place, what resistance it met. I note only two facts pointed out to me by professional historians. First, when a government has decided to mint gold coins, we realize that the weight of each coin cannot always conform exactly to the authority's decision. It is then decided to accept a small difference, provided it remains below a threshold. *Remedium* was the Latin word to signify this permissible variation. This usage has lasted and has been extended to all kinds of mechanical products. The standardized *tolerance systems* are well-known to all apprentices in the fitting trade. The mathematics teacher ought to have a look at this and have the why and wherefore of it explained to him.

The second fact is related to astronomical calculations. When Aristarchos of Samos was seeking the dimensions of the solar system, he used data such as the following:

$$\frac{1}{20} < \sin 3^{\circ} < \frac{1}{18}$$

Once we have accepted this procedure to signify numbers by means of intervals, we must know how to compute.

The great classical work in this field, which was read until the 17th century, is Archimedes' work on the 'Measurement of the Circle', which, unfortunately, has reached us in a rather bad state. The final outcome is rightly famous.

Αρχιμεδησ:

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7} .$$

But a great novelty appears here:

Without being told explicitly (what a beautiful lesson in pedagogy!), we understand very well that once we have studied the 6-gons, 12-gons, 24-gons, 48-gons and 96-gons, nothing stops us from going on, as long as we wish.

The *third level* then emerges in a flash: the germ from which infinitesimal calculus began its development, not without difficult battles.
Let us sum up the situation:

First level: the circle is not much longer than three times its diameter,

second level: it is between 3.14 and 3.15,

third level: tell me what accuracy you need (considering what you want to do), and I shall give you the suitable figures.

At first glance, the idea of successive approximations should raise no psychological or metaphysical problems, since common sense has no doubt as to the possibility of representing the area of a disk by a number - it shows how this number can be approached as desired. And why should it be a rational number, or even an algebraic one?

But the scandal is elsewhere. We know, for instance, how to construct two polyhedra having the same volume, the equivalence of which, however, cannot be established by a cutting procedure (in a finite number of pieces). Hence successive approximations are unavoidable to establish a rigorous formula giving the volume of any pyramid.

A strange reversal indeed that the conquest of accuracy has to take place via approximations!

I think that after twenty-two or twenty-three centuries we still have not reconciled ourselves to this need for approximation procedures. It is not a certain Greek philosophy which is being questioned here, but something far more profound and permanent. Since my early childhood I have felt a certain deference to *exact* calculations, to closed formulae. I feel a bit ashamed. But what should I do?

Let us rattle off the litany of the ancient superstitions: differential equations solvable by squaring; or algebraic equations by simplifying; or even geometrical constructions with ruler and compass; or, above all, the exorbitant privilege of rational numbers still existing in our elementary education.

It is not enough to expose a prejudice in order to be free from it. Even the well-educated mathematician remains more or less attached to the common way of thinking:

> 'exact' is perfection;
> 'approximate' is by far not as good, a deterioration, a makeshift.

On the other hand through Archimedes, Pascal, Riemann, Lebesgue etc., integral calculus should induce us to a reversal of values. To be obliged to approximate and to know how to do it properly is not a weak point of mathematics but its strength and its glory.

Instead of thinking like everybody:

> 'An approximation is something not exact',

we should say, reversing the terms:

> 'What is an exact calculation? - A calculation for which no approximation is needed'.

And we could say, speaking with the linguists:

> 'Exact *calculation* is the "*zero degree* of approximation"'.

I mentioned the three phases of 'approximate' thinking in connection with real numbers. What is given is:

1) an approximate value,
2) an interval,
3) an indefinite sequence of intervals.

Of course, this is but a special instance, probably the first; but we should not dwell on it.

It is already clear that when we *calculate* the area of a disk (circular), it is the whole circle which is approximated through a polygon. And for the integrals that Archimedes, Pascal or Fermat could calculate, it is a function as a whole which is approximated by a step function. Geometrical images are of great help as the topological structure involved is, so to speak, given in our intuition. But one should be careful. Each time we introduce approximation procedures for a category of objects, we must specify to which kind of topology we are going to appeal. In a great number of problems originating in geometry the topology seems to be prescribed; but the time will come when we realize that we must *choose* a topology and that the mathematician then enjoys a certain degree of liberty.

I shall not dwell on this subject. However, I cannot help recalling a memory from my youth: to be allowed to teach mathematics, I had to take examinations. I had been warned that it was forbidden to speak the words 'uniform convergence', as there had to be one convergence only, only one way of 'approaching the limit', as we said. This was, I think, a great pedagogical error, but I did not know it. There is no lack of difficulties even if we have the help of geometric intuition. This may surprise us today, but we had to wait until the 19th century for an approximation of a curve or a surface.

In order to approximate the length of a curve it seemed natural to use inscribed polygonal lines. Why not do the same with curved surfaces using inscribed polyhedric surfaces! But there is an astonishing phenomenon (paradoxical: i.e. running counter to current prejudices), made known, so it seems, by *Schwarz* and *Peano*. True, one may say that an inscribed polyhedron is an approximation of the surface; one may even add that one increasingly *approaches* the surface by adequately increasing the faces of the polyhedron. But one may not conclude that the area of the polyhedron is an approximation of the area of the surface. It is even very simple to calculate this for quite simple figures such as cylinders, cones etc.

It should always be kept in mind that we always seek an approximation for a specific purpose. This finality may be symbolized by the following schema:

$$\begin{array}{c} A \simeq B \\ \Downarrow \\ f(A) \simeq f(B) \end{array}$$

We will then answer: the function f must then be continuous. Well! This is the definition of continuity. But the problem is precisely to know which topologies *to choose for* f to be continuous. For a function is not continuous as such. Since the beginning of the world, a continuous function has been a mapping and two topologies, one at each end.

Let us come back to the curves and surfaces. The difficulty was that we had unwittingly given our preference to the *inscribed* polygons (or polyhedra). For even for a curve it is easy to understand that two curves can be 'very close' and have very different lengths (as close as may be desired, the lengths being as different as may be desired).

For two curves to have close lengths it is necessary to have not only a proximity for all points but also for all tangents.

This should not shock even an unpolished intuition: Henri *Lebesgue*, who liked observing artisans, advised us to observe a land-surveyor measuring the length of a path: not only does he take into account the position of the pickets but also of the *direction* of sighting from one to another.

What has been done for a single real number must be done again for a vector. For real numbers, we will say that a is an approximation of x when the difference $(x-a)$ is small enough (in absolute value)

$$x \simeq a, \quad |x-a| \leq \epsilon.$$

We may say that the control of approximations is carried out by means of the *norm* given by the absolute value. The emergence of this norm is a very natural thing and has, in practice, come even before mathematical formalization. It is quite a different thing with vectors. The difference between two vectors (of the same vector space) is another vector of the same kind. It will then be a matter of being able to tell what is a 'rather small', a 'negligible' vector. We will also choose a *norm* here

$$A \simeq V$$

$$\|A - V\| \leq \epsilon.$$

But we then meet with the plurality of norms. If we reject the questionable hegemony of euclidian geometry, if we refer, amongst other things, to statistical practice, we will then have to consider the use of at least three norms as very elementary and useful:

$$V = (a_1, a_2, \ldots, a_n);$$

$$\|V\|_0 = \max_i |a_i|;$$

$$\|V\|_1 = \sum_i |a_i|;$$

$$\|V\|_{\frac{1}{2}} = \left[\sum_i |a_i|^2\right]^{\frac{1}{2}}.$$

They provide the same topology for a vector space, but in the practice of approximations one must know how to handle them all. Each of them can claim to be 'natural'; the skilful choice of an illustrative context will do.

What is a norm? A *numerical* indicator for the significance of the vector. If the norm is small enough, the vector will be negligible. We then understand the meaning of the 'triangular inequality':

$$\|A + B\| \leq \|A\| + \|B\|.$$

It only indicates that we can control the approximation of a sum if we know how to control the approximation of its terms. In other words: it makes sure that addition is continuous. This is the purpose it has to serve, and in spite of its name, it no longer has much to do with the geometry of the triangle. But it is not enough to know how to norm a vector space. Vector spaces live in society, they play with each other; to speak in a pedantic way, we need a *category*. And the coherence of the norms has to be ensured. What does this mean?

We start with a remark, very useful to every calculator and very simple, concerning the (scalar) product of two vectors:

$$|x_1y_1+\ldots+ x_ny_n| \le (\max_i|x_i|) \cdot (\sum_i |y_i|)$$

$$|xy| \le \|x\|_0 \cdot \|y\|_1 .$$

One then goes on with the famous inequality of Cauchy, Buniakovski and Schwarz:

$$\underset{\text{C.-B.-S.}}{|xy| \le \|x\|_{\frac{1}{2}} \|y\|_{\frac{1}{2}}}$$

and one becomes aware of the necessity of having the different operations of multiplication become continuous:

$$\|PQ\| \le \|P\|\cdot\|Q\| .$$

What must be normed are no longer vectors but matrices; and, of course, we should not be too generous in observing the inequalities: they must become equalities in not too exceptional circumstances. This results in a great diversity of the matrix norms, which is very well known to specialists in numerical analysis.

It is a little complicated indeed. But is it not unwise to teach matrix calculus as a pure algebra, pretending to believe that all calculations in this algebra are carried out with accuracy, and to relegate all means of control of the approximation an inferior position? Personally, I think that the teaching of linear calculus must be concerned with approximations from the beginning. This is simultaneously the foundation of analysis and of all its uses.

Before leaving analysis we should note a significant fact: mathematical analysis has not had any hesitations about making use of common language: it also uses words like 'almost'. Thus it is now accepted to say 'almost everywhere' to signify: 'up to a set of measure 0'.

If the sequence $u(n)$ has limit L, we say that *almost* $u(n)$ are *almost equal* to L. 'Almost all' means 'all except a finite number of them'. 'Almost equal' means that the absolute value of the difference is smaller than an arbitrary number.

I do not exactly know when the expression 'almost everywhere' was first used in analysis. But in the 17th century *Fermat* and others already said 'almost equal' in the Latin form 'ad-aequalis'.

I have tried to describe three successive levels of behaviour and language. But this is not the end of the enterprise. Let us consider now the determination of a unique number. Giving an interval 'x lies between $a-\epsilon$ and $a+\epsilon$' is good, but not always sufficient. It is excellent when the margin of uncertainty (2ϵ) is to become as small as we like. But does every curve have a tangent everywhere? Can it have a length without having a tangent? Has every domain a volume? We often imagine that it is the perverse imagination of mathematicians that creates frightening monsters. Should this mountaineer's phrase be adapted to mathematicians: 'This world which is not ours'? I do not think so.

True, properly speaking *Peano's* and *von Koch's* curves do not exist in nature; but neither do real numbers and they have long since been domesticated all the same. But in nature we find starting points, invitations to reflection, 'situations', as educators nowadays say.

Is it necessary to recall the famous fable, which I would like to read in toto some day: 'The length of a jagged coast'. One could just as well think of the 'length of a river'. As long as there is no need for great accuracy, this is not problematical; but as soon as one gradually comes to the minute details, imagination is lost and reason gets out of breath. We then need a *model*. Should we then take the probabilistic models such as those which served to represent the Brownian motion?

This path is becoming too learned, you will say. I agree, and I shall take another approach, which, however banal, may be more useful.

How much time does it take to get to the station? - About twenty minutes. - What do you mean? What can you guarantee me? - I cannot assure you of anything. - Tell me the distance then. - It must be about two kilometers. - Could you be a little more accurate. Is it more than 1800 m, less than 2300 m?

You can guess that the dialogue is going to become troublesome. Probably one of the two partners in this dialogue is going to lose his temper. At least if the one doing the asking only thinks of the bounding schema: x lies between a and b. He will promptly realize that the interval between a and b will be too large if he insists on perfect accuracy. Why don't we try to represent the state of our knowlegde at a given time by means of a new schema: There is such and such probability that x lies in such and such interval?

Insurance companies (when estimating the duration of a contract) and gunners (when finding out the range of a shot) were probably the first to exploit this model. Shortly after, or shortly before, I am not sure which, astronomers did the same.

If you need to know the speed of light in a vacuum the 'Union Radioscientifique Internationale' will advise you to introduce the following value in your calculation:

$$c = 299792{,}5 \text{ km/sec}$$

But they will not try to delude you into thinking that this figure is *exact*. They will tell you that the 'possible error' would be 'of the order of a few hundred meters' (first level of information). Should you insist, they would deny the second level, which would consist in giving clearly marked boundaries. Instead of a *tolerance* interval, as in industrial production, it will be a *confidence* interval: i.e., roughly speaking, the probability is $\frac{99}{100}$ that the number c will lie between such and such values. This I will call the *fourth level* of information. It is very widespread today. It opens a new chapter in mathematics.

As for classical analysis, I shall say 'next to nothing'; I shall, however, only mention a thought on computing by means of the simple, too simple, example of addition. If we know several real

numbers, we know how to add them (exact computing). If we know an estimated value for both of them (first level), we will have an estimated value of the sum, but without any control of accuracy. If we know an interval for each of them (second level), then the uncertainties add up and the final interval may be ridiculously large. This is a practical difficulty which is strongly felt by the users and has even had far-reaching repercussions in mathematical thinking. I remember those vertiginous demonstrations in which one starts augmenting something by epsilon over 24 to obtain a fair accuracy at the end of a long calculation.
At the third level, the limit of the sum is also perfectly known if all the terms of the sum are defined as a limit.

The fourth level, the probabilistic one, is the marvellous level of the addition of random variables - and of the rather important emergence of the notion of independence. Even, and especially, for beginners, the addition of several numerical chances is a very instructive exercise.

Shall I dare draw a pessimistic conclusion? Since the year of grace 1654 the theory of probability has developed in diverse directions, which have become vigorous trees in the mathematical forest. Famous names can be mentioned. But what about the host of users? What about teaching? This has not made much progress. When I hear people confusing the usual convergence in real numbers with various laws of large numbers (and even with that of Bernoulli); when I hear people who dare say: 'probability *is* the limit of a frequency' - Oh, I don't lose my temper, I don't even feel like demanding that the offenders be thrown out, as did Plato.

However, I think that the ways of thinking evolve slowly. How many centuries were necessary from Archimedes to Lebesgue to develop the adequate techniques, languages and pedagogies (dispute about the indivisibles, crisis of the fundations etc.). Seen from this point of view 1654 to the present time is a short span of time. See you here again in a few centuries!

Part 3

The Sections and Poster-Sessions

3.1 The A- and B-Sections

The work of the Congress was focussed on the thirteen sections. Each section was opened with a *survey report*, followed by a discussion by the International Advisory Group and by the full body of the section.

Furthermore, *short papers* were given in each section, which had been selected by the Programme Committee among the short communications submitted by participants. A complete list of all *short communications* is to be found under Poster-Sessions (cf. p. 276).

Further procedures of the work in each section were determined by the Chairman and the full body of the section (for example initiation of sub-sections, or additional short papers selected from the Poster-Sessions).

Short versions of the survey reports (Survey-Abstracts) and summaries of the activities and results in the section (Discussion-Summaries) are reproduced below. The final versions of the survey reports will appear in volume 4 of the series 'New Trends in Mathematics Teaching', published by UNESCO.

A 1
Mathematics Education at Pre-School and Primary Level (Ages 4-12)

Reporter: F. Colmez, France
Chairman: Bakary Traore, Mali
Coordinator: H. Winter, FRG/H. Besuden, FRG
Short Papers: G. Lister, GB; P. Nesher, Israel; J. Schwartz, USA; P. Srinivasan, Nigeria

A 1.1
SURVEY-ABSTRACT BY THE REPORTER

Children get their first contact with mathematics at the level of elementary instruction.

During the sixties, the initial thrust of reforms accented new contents, with the goal of raising the comprehension level of each child through the reorganization of instruction according to main mathematical structures.

During the seventies, the emphasis has been more on studying and improving the learning processes of children and on training teachers in new teaching methods.

1. Goals

1.1 *Mathematics or Arithmetic?* The new mathematics has often appeared to the public to be competing for first place with or even to be replacing traditional arithmetic. This is a mistake; it is more correct to say that the goals have been broadened and that the present aim is to insure children a correct approach to and a real comprehension of the techniques related to arithmetic.

1.2 *The Place of Acquired Knowledge.* By spending a great deal of time on new subjects like sets, functions, etc. without always seeing them as a means in the service of arithmetic, some teachers gave the impression at the beginning of the reform period of no longer attaching importance to acquired arithmetical knowledge. The present tendency is to integrate this acquired knowledge into a greater context of learning processes, while bringing in simultaneously competencies, know-how and knowledge.

1.3 *Development of a Research Attitude.* Based on the fundamental idea that there is no difference between the nature of a child's thinking and that of a mathematician, a tendency is slowly but surely establishing itself; this tendency is to replace the learning of mechanisms and their applications to standard problems by activities in which the child demonstrates research and inventiveness, favouring in this way a dialectical structure of knowledge (appealing to children's wanting to understand, letting them develop their own research strategy and thus experience the pleasure of solving a problem, mobilizing their knowledge and previous competencies and inviting them to propose new questions).

1.4 *The Intellectualization of Elementary Instruction.* A new tendency, published but as yet little practiced in the classroom, is to lead children to better structure their knowledge by dialectic steps between the different levels of action and of thinking, progressively integrating new areas; the children are then encouraged to look for recognizable patterns and to construct from this knowledge theories allowing economic thought processes. This step can be taken by inventing an adequate and evolving mathematical language.

1.5 *Mathematics as a Collective Creation.* Another concept of elementary mathematics instruction is full of promise, but as yet not wide-spread due to the fact that the precise conditions of its application are little known and require research; this is the construction of mathematics as a collective creation of the class. This concept allows the children to construct their language, to express their hypotheses and to validate them themselves. The teacher must force himself to play less of a role as a transmitter of knowledge, but must be all the more competent as an organizer.

1.6 *Pre-Elementary Instruction.* One notes at the pre-elementary level a tendency to introduce activities with a mathematical component with the aim of encouraging the intellectual growth of the children. This is done by helping them to recognize objects pertinent to thinking and to arrange their mental schemes and structures by activities which they might not have a chance to perform in their family environment. This tendency is curbed by the very character of pre-elementary instruction and by the teachers whose mathematical training is often very weak and who attach more importance to the emotional problems of the children. This tendency is more a matter of research than of practice.

2. The Contents

In spite of appearances, which are all too often superficially analyzed, arithmetic remains the central subject of elementary instruction.

2.1 *Operative Techniques.* One finds in elementary instruction a tendency to aim for a better comprehension of operation and a better mastery of operative techniques by substituting functional methods for learning by parcelled conditioning (on the one hand the importance and the use of the operations, on the other algorithms and tables to memorize); these functional methods favour global learning touching both the areas of the importance and of the operative technique. The immoderate use of different foundations of counting are tending to decrease, since they have shown themselves to be of little use and even harmful in their complication without bringing any real gain to the task of the pupils.

2.2 *Natural Numbers.* The construction of cardinal numbers by comparison of sets leads to new pupil automatisms when the teacher attaches too much importance to sets and presents them in a stereotyped manner (Venn diagrams), thus confusing the symbol and the object itself ('signifiant' and 'signifiê'). This confusion of level has gradual repurcussions and requires 'hand counting' with the help of materials.

The present tendency is to react against these errors with a progression in which the cardinal and ordinal aspects of numbers and of counting are brought in at their respective levels of abstraction, by enlarging in stages the field of numbers mastered by the children.

2.3 *Arithmetic and the Extensions of Numbers.* The attempt has been made to transform the instruction of arithmetic in a structural manner by the introduction of activities based on the ideas of transformations, relations, groups, etc. For example, to prepare the introduction of $(\mathbb{Z},+)$ or of $(\mathbb{Q}^+,\cdot)$, the teacher suggests to the children situations which may be described mathematically by a group of bijections, this permits the introduction of suitable notation (arrows, tables). With this same notation, the functions in $\mathbb{N}$ - translations and homothetic transformations - $(+a,-a,\cdot a,:a)$ are then introduced; but these functions do not form a group in respect to composition, which reduces the usefulness of this method for the desired goal and leads to disappointments. (In practice, one either has to ignore the difficulties, which ruins previous attempts at rigouressness, or to work in a formal way at the level of the language).

Due to these difficulties, the systematic character of this tendency is weak; but it does better demonstrate and allow better use of the algebraic properties of numbers, and thus, together with combinatory activities, allows children to better think through their computations.

The introduction of natural numbers and of certain of their properties remains fundamental to elementary instruction. In some countries, negative whole numbers are introduced. As far as rational numbers are concerned, the tendency is to place the accent exclusively on decimals, in accordance with their importance in modern life and in the sciences; this tendency cannot be but reinforced by the wide-spread use of small pocket calculators.

2.4 *Geometry.* One can distinguish two tendencies:

A structural tendency in which a simplified model of affine plane geometry is studied with the help of quadrilateral nets. And an exploratory and functional tendency which uses classifications and problems in the construction of geometric objects in order to develop in the child a better knowledge of his spatial environment.

These two tendencies are not incompatible.

For measuring, the activities are structured around the idea of associating to each object a number (approximated if necessary). After enough measuring activities, the systems of legitimate units are introduced.

2.5 *Probability and Statistics.* Two tendencies are becoming clear: A rather wide-spread tendency to limit oneself to the teaching of statistics based simultaneously on the givens furnished by the environment and on experiments in the classroom, with the aim of letting the children experience situations where chance plays a role and of introducing a useful and precise vocabulary for making statements about them.

Another tendency, often still in the experimental stage, adds to these goals the construction of probability models and even of theoretical beginnings, allowing the description and solution of certain categories of problems. One thus uses the relations of numbers and combinatorial analysis as tools to aim toward the introduction of certain concepts of probability. A great deal of instruction material has been conceived with this goal.

2.6 *The Influence of Computer Science.* The influence of computer science may be felt in the following ways:

- The introduction of symbols taking into account the dynamics of thinking (arrows, diagrams, etc.).

- The use of diagrams and flow diagrams to solve and to generalize certain problems or to describe algorithms.

- The use of a fictive computer for which one has to write programmes; this forces the children to be precise (one thus has them practice a kind of axiomatic method).

2.7 *Language and Logic.* The introduction of new contents has involved changes and doubtless a slight increase in vocabulary. But, contrary to what one might have hoped, there have not been great improvements in the functioning of this vocabulary. One always finds many more words for designating or describing the symbols (the representations) than for the objects themselves (the concepts). As in discussions held in class, it is not always possible to strike precisely the mathematical object; one is often led to suggest several different representations in the attempt to eliminate undesirable connotations.

The same mistake in the realm of logic leads to the creation of an artificial language which retains the main ambiguities (the logical connections) of the natural language. At the moment, the attempt is being made to remedy these mistakes by approaching the problem of teaching logic through situations that can be described by a natural language as well as by mathematical tools; in this way, one tries to clarify the natural language by means of mathematics, not the other way round.

3. Methods and Means of Instruction

At present the instruction of mathematics in elementary school offers a growing diversity of activities proposed to the pupils.

3.1 *The Choice of Situations and of Materials.* With the functional aim of showing children different aspects of mathematical activity, the scope of activities offered has been greatly expanded in the direction of the creation of mathematical ideas as well as in the direction of applications of mathematics.

While the role of the tool of mathematics was somewhat neglected at the beginning of the reform period, there is a tendency now to suggest to the children activities that are more anchored in reality than was formerly the case.

To show the nature of mathematics, one uses either artificial situations or materials structured in advance to allow the teacher to guide the work of the pupils toward the precise mathematical objectives. But using these materials is often a touchy matter; used well, they help in mathematical structuring, but used as concrete examples of mathematical concepts, they hinder this very structuring.

3.2 *Technological Aids.* Means of communication, whether written or drawn (felt markers, overhead projectors, etc.) facilitate group work and collective discussions.

Many school materials, recently commercialized, have been mistakenly called indispensable for the modernization of instruction. The media (television, films, photos, etc.) are often used to present activities to the children, but more often for teacher training.

The problem of using small pocket calculators in elementary instruction has come up.

Larger aids (closed circuit television, computer terminals, etc.) have remained almost exclusively reserved to didactic research.

3.3 *Books and Scholarly Publications.*Teacher's handbooks are often the main source of information for teachers; their conception is varied and their presentation is generally more attractive; their quality is very uneven. However, most of them present research exercises and situations for mathematical description, in addition to explanations and traditional exercises.

There are collections of individual cards of the programmed learning type which allow the pupil to work at his own speed. They constitute a certain danger if the teacher omits group sessions, which are indispensable. Likewise in using exercises with gaps ('exercices à trou') only for checking the acquired knowledge of the pupils, the teacher runs the risk of misjudging their real competence.

3.4 *Changes in Method from the Point of View of the Pupil.* Very often, the intelligence of the children is better encouraged and the self-motivated activities proposed to them more varied from the point of view of the methods, of the subjects and of the relationships with other subjects. In classes that are too large to allow such a change, one witnesses a lowering of the instruction standards; (if one treats something as important, with multiple repetitions, which is really just nomenclature or a means of communication, one burdens the memories of the children unnecessarily and intensifies the unfortunate effects of dogmatic instruction).

3.5 *Changes in Method from the Point of View of the Teacher.* Traditionally, instruction is based on the idea that there is a body of knowledge to be transmitted and that each learning unit is divided into a series of elementary difficulties; for each of them, the teacher shows how to solve the problem. A change in content has not necessarily meant a change in attitude toward this knowledge or a change in method on the part of the teachers; many continue to be the only person in the classroom with the right to present knowledge and to judge.

A tendency, as yet rather modest, conceives of mathematics instruction as a collective creation of the class and requires more differentiated methods; for each subject, these methods arrange moments of action, of formulation of the discoveries and hypotheses, and of their validation.

Most teachers are somewhere between these two extremes and try to leave somewhat more initiative to the pupils.

3.6 *Links between the New Methods in Mathematics and Other Subjects.* The new objectives, which have the character of general education (creativity, initiative, exchanges and cooperation between the pupils, etc.) involve changes in the other subjects similar to those in mathematics.

3.7 *Evaluation.* Checking the acquired knowledge of the pupil poses a problem which has not yet been resolved, if one wishes to take into account the new objectives, since these are difficult to translate into performance terms. Teachers are more and more occupied with evaluating the efficiency of their on-going instruction.

3.8 *Slow Learners and Handicapped Children.* Many teachers feel pre-occupied with slow learners. It seems that the new contents and methods allow the limitation of learning blocks and delays somewhat better than before. Physically or mentally handicapped children have been little touched by the recent changes; some trial approaches are rather promising.

Intensified research in the two fields would be desirable.

3.9 *Mathematics and Language.* Mathematics instruction tends to be at least a support for the mastery of written language and can even help in its instruction. Moreover, it calls for different levels of spoken and non-verbal language.

4. The Teacher

The reforms can only succeed if the teachers understand their objectives, master the new contents and are capable of modifying their methods of instruction. This has been put into practice to various degrees at the moment.

4.1 *The Freedom of the Teacher in his Instruction.* The disparity between mathematics instruction on one class and in another has increased with the reform and with it the teacher's freedom. This freedom is necessary if the teacher is to use the new methods; but it is also a temptation for some teachers to change nothing in their instruction, and for some, it causes uncertainty about what to do.

4.2 *The Teacher in the Classroom Situation.* Many teachers are alarmed by the breadth of the changes demanded of them and consider going about them little by little, but insufficient change leads to a basic alteration in the nature of the new methods. It is absolutely necessary for the teachers to have an opportunity to experiment themselves with the new behaviour patterns and to observe the pupil's reactions to these changes. For this, structures of pedagogical motivation are necessary, which offer an opportunity for group discussion.

Many teachers fear that they do not have the competence required to direct the children in an open manner while simultaneously picking up the promising ideas the children develop, and they also fear losing time.

4.3 *The Teacher's Multiple Skills.* Elementary school teachers are not mathematical specialists in most countries, there is no reason that mathematics should interest them particularly. Moreover, they are often taken up with the changes taking place in the instruction of other subjects and haven't a chance to give enough attention to mathematics.

4.4 *Teacher Training.* For reasons of convenience of thought, of habits and of the priority of needs, contents and methods are usually separated in teacher training, the accent falling especially on the former (particularly in continuing education). A tendency is becoming visible to link these two components more closely by epistemological and didactic reflection and analysis, but teacher training institutions suffer from a large inertia.

Some countries have specialized teachers at the level of elementary instruction. Others are considering the possibility of such specialization; this is a very thorny problem.

In general, teachers at the pre-elementary level know very little about mathematics and do not concern themselves very much with it.

5. The Sociological Components

Changes in the course of mathematics instruction, particularly in elementary school, have caused a shock wave which explains many of the passionate reactions.

5.1 *The Projects.* The main characteristic of the present movement is the important role played by professional mathematicians in the dynamics of change and the collaboration between instructors of all categories, however limited as yet.

The beginning was marked by the ideas of several people; the interest shown by a growing number of instructors of all categories in this area has permitted a diversification of the tendencies. At the moment, one can note a structural tendency, an arithmetic tendency and an empirical tendency.

5.2 *The Various Effects.* The change in contents is the primary justification for all the other changes in the eyes of the public and of the authorities. But the hope that this change would also involve changes in teaching methods has proved to be too great. Publications - books for the public and manuals - have been the main sources of information for teachers; their quality often leaves something to be desired and they serve to accentuate the divergency between the projects themselves and putting these projects into practice.

The teachers, who are under strain to live up to expectations, are not always encouraged and assisted in this effort. They demand more and more training aids and practical research.

Parents oscillate between interest, anxiety and apathy. Each time a position against modern mathematics is taken, the difficulties of the teacher are accentuated.

5.3 *The Difficulties.* The needs specific to mathematics in implementing the desired changes were not shown clearly enough, and the teacher is often divided between a concern for immediate effectiveness and a general educational project, while having the impression that the two are incompatible. The parents share the same anxiety.

A notable improvement in the situation can only come about through a change in attitude toward mathematics (personal or institutional attitude which one adopts by imitation and repetition). The division of mathematics into the language aspect and the knowledge aspect has rendered this transformation more difficult.

The continuing education of instructors is often, to a great extent, the well-meant work of associations of mathematics teachers; official training has not always been particularly suitable; assistance plans for the classroom have rarely been effectively implemented.

Television is not very effective in teacher training if it is not used within a framework of group reception and in the context of a dialogue with the broadcast planners.

Knowledge about the precise conditions for effective teacher training is lacking.

The various problems that have arisen have now been more clearly articulated.

6. Research and Problems

The convergence between psychological knowledge about the operative stages of the child and the reorganization of mathematics into larger structures is the point of departure for the new didactics of arithmetic at the elementary level.

But the spread of these studies has clashed with the behaviourist attitude toward learning by conditioning (and, rather, often, has itself been undertaken according to this principle).

6.1 *Research in Progress*. There is at the moment an increase in and diversification of research. One may note:

- inquiries and evaluations,
- spot studies resulting in texts suited to short didactic sequences,
- research bearing on a limited area with the idea that the knowledge thus acquired by the child can be later integrated into a body of larger structures.

Research is often conceived bureaucratically, thus reinforcing the dependence of the teacher and the importance of the hierarchy.

There are attempts to establish the didactics of mathematics as an independent fundamental discipline with its own objects of study and its own methods.

6.2 *Some Problems*. Here are some of the most important problems at present:

- Research into situations favourable to fundamental learning processes and into conditions for the reproductability of organized didactic sequences in progression.

- Research into functional, non-structural ways of communicating mathematical or didactic knowledge.

- Research on the role of the analogy (a method of discovery or of knowledge?).

- Study of the importance of personal discovery by the child.

- Research on the difficulties of slow learners and handicapped children (are they best helped by pedagogy or medicine?).

- Development of didactic theories taking into account the topological aspects of the contents.

Taking up the study of these problems raises a number of political and institutional questions.

A 1.2
DISCUSSION-SUMMARY BY THE COORDINATOR

In section A1 there were two main presentations for a big audience, the first session starting with the survey report, the second meeting having six short communications selected for oral presentations. Besides that 19 poster sessions were presented and discussed. The international panel group had a special conference and at different times the reporter-chairman-coordinator triplet met to discuss details.

At the first session, attended by approximately 300 participants, F. Colmez presented his survey report on the present status and trends of mathematics education at pre-school and primary level (ages 4 to 12): Goals, contents, methods of and aids for instruction, teachers, sociological components, research, and other related problems. The lecture was given in French and translated after each chapter into English. Such an eager interest was soon shown in exchanging ideas that the chairman, after a short break, opened the discussion to the audience. As to the last part of the survey report, not orally presented, the audience referred to the abstract printed in four languages.

In general, the audience-discussion resulted in a wide consensus about the situation of math education on the primary level, refusing one-sidedness or the exaggeration of the 'New Math' in the late sixties, turning to meaningful mathematics in schools, and pointing to new methods that stress reasoning and show relations. There was a definite agreement to Colmez characterizing the trends in his report. The remarks given concerned confirmations, interpretations, and suggestions concerning some accentuations.

In particular the following problems were pointed to:

Great difficulties are still observed in teacher education especially in-service teacher education (Australia). New Math was brought to teachers as a subject matter instead of showing its importance for education (France). Too often we still try to stamp on the children our way of thinking (USA). The language children use needs as much attention as the discrimination between objects and symbols (F.R. Germany). On the other hand: Children can get help by introducing logic and artificial language (Belgium). But isn't 'New Math' dead? Or should we still work with math programs of that kind in developing countries? (Nigeria). Set theory and non-decimal numeration systems do not help children in, let's say, Tansania. We all need mathematics for everyday life. We made mistakes and should admit them (Denmark). New educational TV programs in the US stress useful mathematics. Developing countries should not copy but adjust experiments to their local needs (USA). It is not only the idea of structure that should determine our mathematics curriculum (France). We need the psychological, pedagogical, and sociological point of view. Mathematics is part of general education. It is not only mathematically 'right' or 'false' that matter (Australia). Nevertheless there are many unsolved problems in mathematics itself (GDR). What we need in school is: Insight and reasoning, starting with manipulative materials and leading to accuracy (Austria).

To get information from those who couldn't participate orally, because time was limited, the Chairman finally collected brief notes concerning problems of the survey report to add to the final version.

The second session was dedicated to the four selected short communications. 10 minutes were scheduled for oral presentation followed by 20 minutes of discussion.

(1) *George Lister*, GB, his topic being Logic and Number, showed aids which are used to combine both number activities and training with Attribute Blocks or any other kind of developing logical thinking. He referred to I.B. Biggs and Z.P. Dienes.

Some participants in the discussion reproached the speaker for advertising commercial aids. Only little time was left for ventilating the question of how to go from sets to numbers and to numerals.

(2) *Pearla Nesher*, Israel, made remarks on the notation of mathemathical readiness: The conventional 'readiness-programs' do not make a preschool child ready for mathematics. It is merely language training when using the popular terms like big - small, long - short, wide - narrow, deep - shallow, thick - thin, fast - slow, etc. Another approach that includes the introduction of measurement helps children much better. Practical experience would incorporate the dimension-concept, establishing a measurement-unit, the relationship between a chosen unit and a number system, and the approximation of any measurement.

During the discussion, the importance of relational terms in mathematical-physical language was recognized: bigger - smaller, longer - shorter, thicker - thinner, etc. Picking up the ideas of Piaget, there was a demand for active experience versus verbal training.

(3) *P.K. Srinivasan*, Nigeria, reported on teacher resistance vs. learner resistance in modern primary school mathematics teaching. Learner resistance could be reduced by intrinsic motivation using effective methods like discovery through exploration; teachers should become more familiar with the rationale underlying the changes. The reporter gave interesting illustrations of special

problems and how to solve them.

The audience gave general assent, but some were doubtful about whether some topics mentioned are really necessary in primary schools: the empty set and the one element set, the laws of commutativity, associativity, and distributivity, non-decimal systems and numeration to different bases.

(4) *Judah L. Schwartz*, USA, introduced a new TV series on mathematics and problem-solving, 'Infinity Factory', developed in Cambridge, Mass., and broadcast on a pilot basis in Boston and Los Angeles. The program begins nationwide broadcast in the fall of 1976. The primary target audience is minority children ages 8 to 11. Essentials of the program: It starts with the abilities that children already have, refers to well known facts instead of introducing sensational news.

The idea of the program was well received by the group; doubts were expressed generally, however, toward the increasing use of television programs in schools because they cannot replace personal experiences. On the other hand TV has a great influence in life and we should make good use of it in school.

After these four short communications two groups were formed ad hoc which concentrated on two other topics: Dynamic Labyrinths, a preliminary preparation for algorithms and computers at primary level (E. Cohors-Fresenborg, F.R. Germany) and Mathematical Issues in pre-primary education in England (Julia Matthews, GB).

Besides these activities there was chance to discuss special problems of mathematics education with experts in 19 poster sessions. They dealt with new topics and methods in elementary math education, aids and materials for primary school children, advice for teacher education, and new mathematics in developing countries. All materials shown and information offered were deeply appreciated and actively discussed by the congress participants.

On Friday night a final meeting of the international panel group took place. There ideas and impressions of the various activities were exchanged, reactions to the survey report were taken into account, and a final version of it was prepared.

A 2
Mathematics Education at Upper Primary and Junior High School Level (Ages 10-16)

Reporter: A.Z. Krygovska, Poland
Chairman: C.G. Maslova, USSR
Coordinator: H.J. Vollrath, FRG
Short Papers: U. Haber-Schaim, USA; L. Henkin, USA; Y. Hindam, Doha-Qatar

A 2.1
SURVEY-ABSTRACT BY THE REPORTER

1. General Sources of Present Tendencies and Problems

1.1 *Rapid Growth in the Number of Pupils, Changes in the Social Structure of Secondary School Classes.* In many countries, mathematics education at the post-elementary level has become mass mathematics education. In other countries, the number of pupils whose schooling continues beyond the primary level is increasing or may increase in the more or less near future, according to government planning. Mathematics instruction at this stage of mass education poses many difficult problems, and the need is making itself felt for a clear conception of mathematical culture for all, which could and ought to be shaped at school, independent of the future studies and professions of the pupils and profoundly integrated into their general culture.

1.2 *Criticism of Certain Reforms of the 'First Wave'.* In the course of the past ten years, reforms with the goal of modernizing elementary mathematics instruction have been subjected in some countries to first analyses and to first evaluations, which have had considerable influence on the present tendencies and research and point to the second wave of reforms.

1.3 *The Influence of Society.* Cultural, structural and economic changes put great pressure on school education in general and on mathematics education in particular.

An in depth analysis of these factors is indispensable to an understanding of the present situation in mass mathematics education.

1.4 *School Structure.* The diversity of school structures reflects on the one hand differences in the concept of national education, and is on the other hand the source of tendencies and of problems, specific to various countries. But certain tendencies and certain problems which come up everywhere, independent of school structure, merit special attention.

2. Tendencies, Problems

2.1 *Goals of Mathematics Instruction at the Post-Elementary Level up to 16 Years of Age.* The new social situation of post-elementary education requires an attempt to define precisely the aims and goals of mass mathematics education. A study of various texts reveals tendencies aiming above all at the development of mental activities through mathematics education and at the acquiring of intellectual qualifications by most pupils, rather than at the assimilation of extensive knowledge. Growing opposition may be noted to exaggerated pragmatism and to the behaviourist method of defining the goals of mathematics education by lists of narrow competencies.

2.2 *Content, its Organization, Programmes.* Outside the rather limited common ground, there are great differences in the content of mathematics instruction at the level discussed here in different countries. Statistics, probability and finite structures are becoming more and more part of mathematics for the masses. Disagreement may always be found concerning the role of geometry in the whole of mathematics education and the concept of its teaching at the level under consideration, but it is evident that the concept of deductive geometry is losing its importance.

A study of the programmes and handbooks reveals that there are three positions regarding the place and role of abstract structures in instruction at the level that interests us here, namely: 1) the introduction of fragments of the respective theories, 2) their use only as elements of a universal language, 3) the total elimination of the terminology involved.

The programmes are overloaded on all sides, which hinders the modernization of teaching methods.

The attempt to integrate mathematical material is undertaken according to different models, namely: 1) integration on the basis of general structures, linear organization and global deduction; 2) integration around problems; freer organization, often concentric or spiral; localized deduction; 3) integration by 'transverse bridges' constructed between certain areas that have been developed independently.

Difficulties have been observed in the correlation and in the coordination of mathematics instruction with other school subjects. Applications are still always too traditional. Some progress has been made on the border between mathematics and computer science.

2.3 *Teaching Methods and Teaching Aids, Evaluation.* The need has made itself felt for an analysis of the function and of the role of handbook, of work cards, of programmes and for the establishment of evaluation criteria for their usefulness.

Problems of differentiation, of homogeneous or heterogeneous classes, of the integration of the pupil's independent work into group work and of the coordination of different teaching and learning methods are at the center of pedagogical interest at the moment. The evaluation of the progress made by the pupils poses particularly difficult problems. A certain crisis may be observed in regard to confidence in the diagnostic and prognostic value of tests, as well as opposition to the quantity and the role of examinations and to selection on the basis of mathematics. The conditions for passing children from the primary to the post-elementary level arouse anxiety regarding methods of premature selection.

2.4 *Teacher Training.* The initial training of mathematics teachers for the level considered here, however differentiated in various countries, has been criticized from many standpoints. On the other hand, some progress has been made in continuing education and in the help offered teachers in practice. Nevertheless, the overburdening of teachers and their social and economic status limits them in taking advantage of all these possibilities.

2.5 *Perspectivies on Research.* There is a keenly felt need for in depth knowledge of how mathematics is learned at the level under consideration (processes, pupils' difficulties in mathematics, their natural thought paths, methodological errors, etc). A collection of problems that deserve research on a world-wide cooperative basis should be worked out.

A 2.2
DISCUSSION-SUMMARY BY THE COORDINATOR

In the panel discussions and in the short communications following the main report, the trends indicated were corroborated in the main. Additional statements were related to the following problems:

(1) Mathematical education is a part of mass education on the primary level in both developed and developing countries; however, it is only an education for an elite in many developing countries for the age group 11-16.

(2) There is a very strong interdependence between the process of mathematical education and the social development in many developing countries.

(3) It is still very difficult to get a valid appraisal about the situation in mathematical education in developing countries because of the lack of information. In some reports critical comment is lacking, sometimes creating the wrong impression that

modern mathematical instruction is without any difficulties.

(4) There is some influence of the changing role of women in society in mathematical instruction. It is conceivable that the type of problems, examples, and applications in mathematical instruction can help to establish an improved understanding of the role of women in society.

(5) It is necessary to point out the importance of geometry for children's understanding of their environment. Children's intuition should be encouraged to get a basis for a better perception of the complex reality.

(6) Mathematicians organize mathematical knowledge into deductive theories, and educators organize it into graded curricula. Discussing the wide variety of ways of organizing mathematical contents, one can get possible ways of developing adequate curricula at several levels.

(7) Encouragement is a very important factor in teachers' helping their students to overcome difficulties in learning. Therefore, more attention should be paid to psychological and educational aspects in mathematical instruction.

(8) Various fundamental principles of instruction should be taken into account in teaching, such as reduction of the number of treated items, integration of induction and deduction, developing skills and abilities slowly, and a balanced use of introductions and exercises.

(9) The development of linguistic abilities should be carefully cultivated. It is necessary to balance the proportion of colloquial speech and mathematical terminology in instruction.

(10) There is a trend to be content with minimal goals for mathematical instruction. The danger was pointed out that one would not only restrict oneself to minimal skills, but also to minimal intellectual abilities and attitudes.

(11) In comparison to the research on teaching mathematics at the primary level, there is still a large deficit in research for the age 11-16 on teaching concepts, formal languages, proof, algorithms, and structures. The psychological investigations on

learning mathematics are related to a limited extent only to these problems. There is a need for more research related to this age group.

A 3
Mathematics Education at Senior High School, College and University Transition (Ages 15-20)

Reporter: D.A. Quadling, GB
Chairman: R. Fischer, Austria
Coordinator: B. Winkelmann, FRG
Short Papers: D. Ambrose, Lesotho; A. Blakers, Australia;
H. Möller, FRG; C. Ormell, GB

A 3.1
SURVEY-ABSTRACT BY THE REPORTER

1. Introduction

At this stage there is a marked differentiation between the courses which different groups of students choose to follow. They may have the choice whether to continue in education at all (full-time, or part-time), whether to continue with mathematics, and how much and what kind of mathematics to study.

The report identifies four types of courses at this level:

(1) Academic courses, emphasising theoretical principles and logical coherence.

(2) General courses, emphasising aspects of relevance to the ordinary citizen.

(3) Technical courses, with emphasis on application.

(4) Skill courses, providing basic skills needed for particular crafts.

Typically an academic or technical course might occupy 5 hours a week over 2 or 3 years.

2. Goals

For academic courses, these may refer to: precise and critical thought, logical deduction, precise verbal expression, independent intellectual activity, creativity and intuition, insight into subjectmatter and method, knowledge and abilities required for other subjects and real-world applications, objectivity and self-criticism, perseverance, philosophical depth, mental agility, problem-solving skills, ability to learn from books, development of the collective spirit; more specifically, knowledge of axiomatic method, training in proof and capacity for abstraction, familiartity with language and symbolic expression, fluency with manipulative procedures, treatment of problems from science technology and economics, training in geometrical sense, mathematics as a basis of civilisation, judgement in formulating and interpreting mathematical models, feeling for the power of mathematics, encouragement of good attitudes.

These goals reflect the status of mathematics in society. However, few are likely to be achieved in any measurable degree by more than a small proportion of students. For prestige reasons, academic courses are often followed by students who are far from fluent in the language and symbolism of abstract mathematics; therefore many students and teachers work to more limited goals related to specific knowledge, skills and applications.

Attention is drawn to the conflict between the goals stressed by educators and those valued by employers and teachers of other subjects, which is leading to increasingly vocal criticism of mathematical education in some quarters.

Goals for technical courses stress exact, logical and critical thought, ability to apply mathematics to problems in the particular field, and reliability in using techniques. An important factor for change in technical courses has been the increased availability of computers.

In general courses there is some trend away from abstract mathematics toward more immediately usable knowledge.

3. Mathematics Curriculum and Content

3.1 *Mathematics within the total curriculum.* The report describes the curricular patterns in various countries, and the factors which influence students to make choices relating to mathematics. There is a converging trend towards a curriculum of moderate breadth in which students can exercise some element of choice.

3.2 *Variety of mathematics courses.* In technical courses a balance has to be struck between specialist mathematics courses relating to particular technologies, which are organisationally extravagant, and general mathematics courses which attract less student motivation.

Academic courses typically contain a core of mathematics which all students are expected to follow (calculus, algebra, trigonometry), together with optional courses in subjects such as statistics, linear algebra, abstract algebra, numerical analysis, informatics - possibly determined by the requirements of other subjects in the students' curriculum. In small schools, however, choice may be limited; and if too much variety is offered, problems arise at the next stage of education. The trend is therefore away from the unrestrained proliferation of options. This point is illustrated by reference to developments in the International Baccalaureate.

In general courses students are often able to choose between courses stressing particular areas of application - business technical, statistics, design, consumer interests, etc.

3.3 *Provision for mathematically gifted students.* Various means of providing motivation for gifted students are described:

(1) Extra time within the curriculum

(2) Advanced placement

(3) Olympiads and other contests

(4) Lectures from distinguished mathematicians and outstanding teachers

(5) Mathematical journals directed at student readership

(6) School mathematics clubs

3.4 *Style of the mathematics course.* This topic is discussed under three heads: the degree of abstraction, organisation of the teaching material, mathematical formality.

Abstraction offers a short-cut to the frontiers of mathematics, and is at the root of its applicability. However, with immature students an over-abstract presentation of the subject can deteriorate to mere booklearning and to one-way instruction. In order that the abstract concepts may be successfully formed, students must be involved in the process of abstraction. It may be damaging to treat students at this level as if they are fully-fashioned adult mathematicians, rather than as being at a crucial phase of development. There is now a trend towards a more concrete, process-oriented programme until the final year, when what has been learnt is summarised and interpreted from a more abstract point of view.

In some countries this programme is organised into separate branches of mathematics, which facilitates student choice and simplifies problems of students who move from one institution to another. Elsewhere the programme is an integrated one, in which topics from different parts of mathematics with a common conceptual basis are linked. The former approach aids the identification of definitions and axioms; in the latter case the deductive aspects are likely to be local rather than global.

There is, however, a trend towards a more intuitive introduction to mathematical ideas, with formalism deferred until students have some experience of applying the concepts and using the language.

3.5 *Topics covered in the course*. After a period of radical innovation in many countries in the 1960s, a period of relative stability appears to have been reached. Typically probability, statistics, linear algebra, vector space theory and algebraic structures have been introduced, at the expense of solid and spherical geometry, conic sections, commercial applications and some trigonometry. There has also, from pressure of time, been a reduction in the performance of exercises and in problem-solving.

In most recent revisions some of the new work has moved down into the lower secondary school, and some of the most abstract work on algebraic structures has been discarded.
In technical courses there has been a trend to introduce ideas

of set and function to support conventional arithmetic, algebra and calculus; and to include some statistics, probability, Boolean algebra, vectors and informatics. The availability of computers has resulted in a more numerical and less analytical emphasis.

3.6 *The balance of the mathematics curriculum*. It is suggested that the academic curriculum must strike a balance between the increasing abstraction and logical precision of modern pure mathematics, the process skills and knowledge of specific applications required by traditional users of mathematics in engineering and physical science, and the more open interest in the general applicability of mathematics shown by newer users in the biological, social and environmental sciences.

There appears to be little attempt through curriculum committees to ensure that the mathematics taught in schools is in tune with the needs of society at large.

3.7 *Integration of mathematics with other subjects*. With the exception of a few technical courses, there is little experience of teaching mathematics through its application in other disciplines. The difficulty of ensuring a logical development of the mathematical content, the fact that teachers are trained in a single discipline, and the dangers of 'mathematical imperialism' are arguments used to oppose this development.

A few experimental developments on these lines are described.

4. Teaching Methods

4.1 *Treatment of the course content*. Various new approaches to teaching the content of the mathematics course are described. These include the use of computers and calculators, emphasis on common underlying concepts such as function vector and group, teaching mathematics in the context of applications, use of case studies in teaching statistics.

The practice of mathematics teaching is constrasted with the declared goals.

4.2 *Teaching mathematics through exploratory methods.* The place of open-ended 'personal mathematics', supplementing conventional instruction, is discussed. Computing and statistics are identified as two areas in which this is especially appropriate, and some experiments are described. Problems of time, evaluation and teacher training are barriers to the development of this type of work.

4.3 *Individualised learning programmes.* There is growing interest at this level, especially in technical courses, in the designation of behavioural objectives and the construction of programmes of individual study directed to the attainment of these objectives, using suitably chosen media. It has yet to be demonstrated that this technique can take the student beyond limited goals at the cognitive level.

5. Evaluation

Student performance is usually assessed by examination, but occasionally also by assignments carried out during the course, oral examination or project work. Students and teachers put excessive emphasis on those goals which are reflected in the form of evaluation; we are a long way from finding valid and reliable tests covering the whole range of declared goals.

Special problems arise where universities or industry set separate tests not directly related to the school curriculum. It is desirable that those responsible for examinations should also be involved in planning the curriculum.

The importance of the final grade at this level has had an inhibiting effect on experiment in examination procedures. Some new components of the mathematics curriculum are not well suited to examination by conventional methods.

6. Research Issues

Little reliable evidence has emerged from the implementation of experimental programmes in recent years. The major questions needing research at this level are the identification of realistic and relevant goals, problems of language, the role of geometrical insights, the development of problem-solving strategies and the relation between assessment and curricular goals.

A 3.2
DISCUSSION-SUMMARY BY THE COORDINATOR

The sessions were attended by an audience of about 350 persons. The panel consisted of Fischer, Austria (Chairman); Winkelmann, FRG (Coordinator); Lichtenberg, Denmark; Souza-Dantas, Brazil; Arora, India; Turnau, Poland (only on Tuesday); Råde, Sweden; Fey, USA; Reisz, France; Neill, GB. Parts of the panel and the reporter Mr. Quadling had met half an hour before the beginning of the first session to discuss questions of structuring the sessions and to come into contact with each other, but time was far too short for fundamental preparations. This was a great handicap for the sessions, especially for the one on Tuesday.

The second handicap of this group was the language-problem. Since there was a rather large group of people who understood and spoke only French but no English, a French translation of all contributions not abstracted in the Program had to be given. On Tuesday a special request for a German translation was given, too, which was abandonned on Friday. These translations shortened the time available for discussion by about one half or even more.

The report of Mr. Quadling was given on Tuesday in two parts, each followed by some contributions of the panel or other members of the audience. According to a request of the reporter for more information about the gifted-student-problem (3.3) some speakers gave details on the handling of this matter in their countries. As the report in general had described the various possibilities in fair detail, the individual national conditions shall not be reported here. It was said that there are not enough competitions and that special curricula were needed for the gifted, not only the covering of 'normal' courses in much shorter time. Another contribution was given about the International Baccalaureate, again dealt with in sufficient detail in the Report.

The report given by Mr. Quadling was generally found to describe the existing trends and realities in the world very well and in a balanced manner, but an urgent need was felt for the trends to be evaluated and judged by the audience. The problems raised by these trends and facts should be identified, as the report did in some instances, and hints for solutions or recommendations for needed research should be given. One of these problems was raised by the extension of mass education to this age group. Should mathematics education in this situation be qualificatory, as many university mathematicians claim in light of the future mathematicians' needs at school, or should it be educational in transmitting mathematical literacy (as for instance described in Sir James Lighthill's speech and consisting of general knowledge about mathematics, its applications and its role in society) to a broader range of educated citizens?

Other problems regarded the balance between the amount of knowledge and skills taught to the students and the development of their abilities to solve problems, to form concepts on their own and the development of their emotional stability in believing in their own abilities in such fields.

For the session of the A3-group on Friday, the panel had identified some general problem fields, which were written on the blackboard, namely:

Goals: how to invent them? Empirical research on skills, how to use resources, general knowledge about mathematics.

Reasons for discrepancies between existing goals and students' achievements. Axiomatics, rigour.

Gifted students. How can all students get fair treatment?

Integrated courses.

Role of universities.

The discussion was to center on these problems and perhaps on some questions raised by the short presentations.

The Friday-session began with four short presentations singled out by the Program Committee of ICME for A3.

(1) *D.P. Ambrose: 'Mathematics Education in Developing Countries with Rapidly Expanding Secondary School Systems'*, gave an impressive overview over the somewhat catastrophic situation in small developing countries.

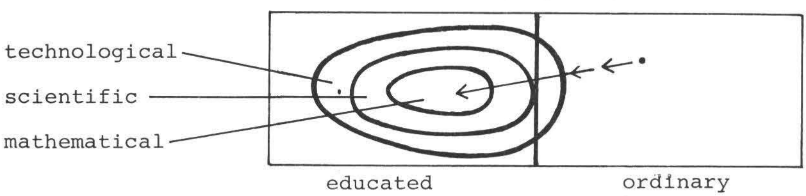
(2) *A.L. Blakers: 'Fostering Mathematical Talent'*, gave some general remarks on the need for also meeting the potential of the gifted students and gave some details on Australian activities.

(3) *H. Möller: 'Vereinfachte Analysis für den Schulunterricht' (Simplified Analysis for Schools)*, gave some details on how to change some fundamental mathematical concepts of analysis in order to make the subject more accessible for most students.

It was remarked that this kind of reducing difficulties by restricting concepts has a 20-year-tradition in Hungary, but great care has been taken to teach the unrestricted concepts before students leave school.

(4) *C. Ormell: 'On Piercean or Projective Applicability'*, did not follow his abstract, but after some general remarks on the changing image of mathematics in western societies he gave an example of projective applicability worked on in Congress practicum on parking problems and then concentrated on the meaning and on the use of various types of languages in mathematical education. The child, being in the area of non-technological ordinary language, has to be led to mathematical language, passing the stages of technological and then scientific language, thereby reaching the level of educated language (see Language map).

Language map



In each stage there must be a local area in which the pupil can move with full understanding.

The rest of the time was given to some personal contributions on some of the previously mentioned problems written on the blackboard. There were no general recommendations on certain problems by the plenum.

Discrepancies between goals and achievements: One possible solution for overcoming this difficulty was seen in the development of abilities for and attitudes toward the intelligent use of mathematical resources, which implies some skills, understanding and overview and, unlike general goals, seems to be teachable.

Integrated courses: There was a short report on the reading of ancient and medieval mathematical literature in Greek and Latin courses.

Gifted students: The claim was made for concentration on algebra, geometry, trigonometry and inequalities in the general curriculum to give a sound basis for the gifted students as well, so that they could solve the problems in the mathematical competitions.

The course of the discussion showed clearly, that there was no common level among the participants either on the plain of problem identification or on the plain of language and abstraction. In some cases, contributions of a higher theoretical level were apparently not quite understood by part of the audience. It is regrettable, that there was no room in the session itself for spontaneous discussion which could have overcome this difficulty. So the discussion in the A3-sessions could contribute only a little to the process of professionalization of mathematical educators.

A 4
Mathematics Education at University Level (Excluding Teacher Training)

Reporter: J.H. van Lint, Netherlands
Chairman: L. Iliev, Bulgaria/J.M. Howie, GB
Coordinator: K. Kirchgäßner, FRG
Short Papers: J.T. Fey, USA; M.L. Fuller, Australia; L. Kudryavtsev, USSR; D. Roether, FRG

A 4.1
SURVEY-ABSTRACT BY THE REPORTER

The report has four parts: I Curriculum (content and objectives), II Structure of program (i.e. educational methods), III Mathematics as minor subject, IV The position of the teachers.

The report deals primarily with the education of mathematicians, although section III discusses some trends related to disciplines which need mathematics as a tool. There is nothing about incidental mathematics courses which students in many different areas may take as options. The report was written on the basis of answers received from 50 universities throughout the world which had received an outline of this report and a set of questions.

1. Curriculum Trends

The education of mathematicians at bachelor's level and higher has in many countries and for a long time been based on the assumption that they would become research mathematicians (even if this were not so). A recent trend is that the objectives of mathematical education at universities are much more diverse than ever before. Terminal courses of a few years, opposed to earlier systems where the first few years were preparatory for further education, are becoming more frequent. A problem worth discussing (though it may be too hard to solve) is structuring the university education in stages, each of which is a good terminal education and a good preparation for the next stage.

1.1 *Early years (leading to a bachelor's degree e.g.).* Since about 20 years there has been a process of increasing *abstraction* and increasing stress on *rigor* in the first few years. E.g. abstract integration theory, general topology, metric spaces etc. replaced 'old-fashioned' calculus courses, and geometry has practically disappeared. In many countries this has clearly gone too far and there is a trend to reverse the process. It is desirable to discuss how to achieve a good balance; which new things should stay (e.g. linear algebra, metric spaces) and which topics should not have disappeared (e.g. geometric insight).

Practically everywhere some *new topics* have become part of the standard (usually compulsory) curriculum. Probability, statistics and topics from computer science are the most frequent. There is a trend in some countries to introduce more discrete mathematics in early years. It has the advantage that students can discover a lot for themselves and it is certainly growing in importance as a tool in other areas. The possibility of computer oriented calculus courses is worth discussing.

It is a universal trend that *students entering university* have much less manipulative skill than before (in fact far too little) and practically no geometric insight. Remedial teaching is becoming an element of freshman education in many countries. Furthermore a certain unwillingness to learn on the part of the students is apparently a recent trend. Although a majority of correspondents considered the 'new math' in secondary education a failure it is not my intention to start an argument on this point. However, it should be discussed how universities should keep in contact with developments in secondary education in order to ensure that students enter university with a better preparation than is presently the case.

A general trend is that the number of compulsory physics courses in the curriculum is decreasing. In some countries it has reached the 0-level. This seems most undesirable. In fact it is a trend that in a few countries the process is already being reversed. It is worth discussing which other disciplines use enough interesting mathematics to deserve a place in the curriculum at this level.

Many other *non-mathematical* elements are possible in the curriculum.How much of this is desirable (of course depending on whether this phase of education is preparatory or not)?

Many universities are introducing courses in *mathematical models*. Both content and pedagogy of such courses are of interest. Generally the courses emphasize the principles behind the construction of mathematical models, and how a model should contribute to the understanding of the real situation it is intended to describe. It is difficult to find suitable problems since both judgement of the practical situation and the necessary mathematical knowledge to solve the problem resulting from the model will not be present in most students (for most of the really interesting practical problems). A method for pooling the knowledge of the design of these courses should be discussed.

It is desirable that in every country at least one university, preferably a technological university, offers an education in *mathematical engineering*. Very few countries know what is meant by this term. A possible program is described in Vol. II of the CUPM Recommendations Compendium. Such a program has existed in the Netherlands for about 10 years. It is quite different from traditional university programs and it is definitely a useful example to follow for the developing countries. The main principle behind the term mathematical engineer is that one must learn to understand the problems of the nonprofessional mathematician, find a solution and then explain this to the customer. If such a program is established it is important that secondary school teachers and counsellors know the difference between the programs a prospective student can choose from.

It seems a trend that in those countries where the bachelor's degree is often a terminal degree the amount of *choice* which the students have in making up their program has become too large. This is leading to unbalanced programs (studying to become a dilletante in three subjects simultaneously), to a tendency to replace courses in hard analysis, etc. by more popular subjects, and finally sometimes to overspecialization. The problem is not present yet at institutes where the first few years are preparatory. Here, however, there is a trend which should be discussed. It happens more frequently recently that young students think they know better what they will need later in their career than their educators. Demands for all kinds of options are quite common. It should be discussed how far one can go in this direction without being irresponsible.

1.2 *Later years (master's degree, e.g.).* Except for vocational directions such as statistics and computing there is still very little *specialisation* at this level in many countries. Usually it depends on the presence of a Ph.D. program which makes postponement of specialisation desirable. Probably the employment situation will soon make it advisable to produce mathematicians with a broad education.

Traditionally physics has been the subject where students of mathematics could study the *interaction* of mathematics with other sciences. Here, high level mathematics is used and quite often it was even developed for this purpose. The decreasing knowledge of physics among students is making this element of education more difficult. Both biology and economics are beginning to play an important role in this area. Other topics such as control theory are becoming more popular. A problem to be discussed here is how smaller departments, (e.g. in developing countries) which do not have an expert in biology or economics, etc. can introduce these elements into the program. There is very little literature or other pooled knowledge.

Very little is done during the master's phase of mathematical education to prepare a student for a *specific function* in society (again with the exception of statisticians and computer scientists). Of course the mathematical engineering program mentioned earlier does do this. Clearly, if there is great diversity in future careers of the students, one cannot do more than provide a good general mathematics education. For the developing countries it seems more important to spend some time preparing the student for a position where he can be most useful to his country. The role of the university in the education of future mathematics teachers at the highest secondary level should be discussed in this respect.

1.3 *(Post-)Graduate (doctorates and later).* There are a few interesting trends concerning the doctorate. One of these is the possibility to get the degree on the basis of a number of reprints of papers, published in a period of not more than 5 years, to which an introduction is written. In a number of countries the requirements have been lowered slightly. A difficult problem related to requirements is the increasing number of foreign students. It is desirable that some method is found to provide a Ph.D Program which meets the standards of the university from which the student comes and to let this university award the degree. Standardizing the level of all students of the world is impossible anyway.

An important trend is the growth of the American Doctor of Arts degree program. More than 100 institutions in the U.S.A. offer something comparable to this degree. The degree is intended as a difficult course of study for which succes is reasonably predictable for students which are suitably qualified. This in contrast to the Ph.D. where a lack of creative talent is prohibitive. The program was designed to provide exceptionally strong preperation for teaching mathematics to undergraduates. The thesis can be a good survey, research in recent history of mathematics, a didactical essay, etc. The program merits a serious study by countries with a shortage of highly qualified teachers.
Besides some occasional refresher courses for teachers at secondary level there is still hardly any contribution of universities to *continuing education*. However the need is felt and has been acknowledged by many governments. It is time to start preparing for this task.

2. Structure of Program (Educational Methods)

When considering objectives of structure the question is not which mathematics one is teaching or why but what one is trying to achieve through the special form of the program. For instance a series of lectures can have as objective the complete teaching of a subject or only the preparation of the students for reading the literature. We look into objectives of structure.

2.1 There are no significant trends concerning types of *courses*. An idea with which some experimentation has been going on is the 'self-paced course'. The purpose is to cope with the problem of diversity of background of freshmen, e.g. The method is said to be suitable for courses with a high manipulative content.

2.2 *The method of apprentice-type education* is not used much yet but is worth more attention. The idea is to let the student work for a short period of time in one of the research groups of the institute. Depending on what is going on at the time he will need to read up on background knowledge, etc. Credit is given for the period.

2.3 *Problem solving* is in bad shape everywhere. Except for the standard (and always rather dull) problem sessions attached to courses, very little is done to stimulate problem solving as a mathematical activity and to incorporate it into the educational program. A revival should be encouraged.

2.4 Except for the Ph.D. part of education there is a much too small role for *research* in the program. It is significant that those universities that require a thesis of some kind for the master's degree consider this one of the most essential elements. The idea of research projects at undergraduate level has been used very successfully at a few universities. Especially combinatorics is a good area to let the students experience the joy of independent discovery and learn to pose their own problems. Student seminars usually seem to be a failure. It would be interesting to find a way to make the useful experience of talking about a subject less disastrous for the audience.

2.5 The most significant trend in *examination methods* is an increase in continual assessment, assessment by observation and

more personal contact. A second trend is haste; too many students take exams right after completion of the course, taking no time to digest the material or to acquire some skill.
Multiple choice examinations are luckily disappearing.

3. Mathematics as a Minor Subject

There are several trends in this area. Students taking mathematics because they need it for their own major subject have less trust in the teachers. They have to be motivated continuously and keep asking what the 'use' is of the subjects taught. If there were enough time to do this and simultaneously teach them the required manipulative skills, all would be fine. However, this is not the case and the situation is becoming quite a problem.

A more interesting trend is that more disciplines now need mathematics as a tool. Service courses in mathematics for economics, biology, the social sciences, etc. are popping up in many places. A method to pool information on these new courses is an important subject for discussion. A dangerous trend and one which should be discussed seriously is the increasing number of mathematics courses for non-mathematicians being offered by other departments. Very often these courses are extremely poor and that alone is cause for concern. Besides this, a trend in this direction is bad for the employment situation. The first thing to be done is to examine our own mistakes. Bad arguments for teaching such courses can be attacked but if the argument is that the mathematicians are not sufficiently interested in the needs of the students of other departments and their difficulties, then our position is weakened.

4. The Position of the Teachers

Trends that were identified are increased personal contact with students and an increased amount of time devoted to assistance, correction of papers, supervision, etc. This has resulted in less time for research. In some universities there are full-time teaching positions which enable others to spend more time on research.

Is this a good system in the long run?

Another trend was an increase in paper work, administration, committees, etc. Much of this is unnecessary and often caused by poor organisational systems. Is anybody doing anything about this problem (except complaining)?

A 4.2
DISCUSSION-SUMMARY BY THE COORDINATOR

1. Technical Remarks

The two sessions on Tuesday and on Friday morning were attended by about 150 participants (each time), a considerable portion of which took an active part in the discussion, following the general report of Prof. van Lint (The Netherlands). These discussions were chaired by Prof. Iliev (Bulgaria) on Tuesday, and by Prof. Howie (Great Britain) on Friday. A poster session (9 presentations) and 4 short communications selected from those completed the program. On Friday afternoon the general trend report and the coordinator's summary were discussed by panel members.

2. The General Report

Prof. van Lint's general trend report on 'Mathematics Education at University Level' was generally acknowledged as a well balanced, thoroughly researched view of the current state and of possible future trends. In particular, the discussion confirmed the relative completeness of the picture given. Most participants stressed major points of the report, occasionally adding valuable - even quantitative - information. Very few issues were mentioned which were felt to have been given insufficient emphasis. A disappointing fact was the almost complete absence of information and data from developing countries.

Since the general report will appear in print elsewhere, I prefer to concentrate on those issues which played a central role in the discussion rather than to give an unbiased abstract of the report. Let me group these issues into three subclasses:

(1) Tendencies determined by the standard of the average student entering the university.

(2) Tendencies caused by changing professional expectations.

(3) Tendencies in teaching non-mathematicians.

Ad (1): The general report deplores quite explicitly the universally observed decline of manipulative skill, geometric insight and intuition, and it discusses to some extent its causes and its implications for undergraduate curricula. In addition, the amount of choice in mathematical and non-mathematical content on different levels of education is mentioned.

Ad (2): The rapidly decreasing amount of physics known by an average mathematics student and the conscious orientation towards the professional life of a mathematician had and will have a strong influence upon the curricula. The report mentions courses in mathematical modelling, program of mathematical engineering and, on a smaller scale algorithmic oriented calculus courses, problem-solving seminars and practica in industry. The program of Dr. of Arts established at some US universities should also be mentioned in this connection.

Ad (3): The teaching of mathematics to students in physics, biology, engineering etc. has long been a neglected responsibility of mathematical departments all over the world. Although there are no new trends in teaching methods discernible, the report discovers a growing awareness of the importance of this subject. It emphasizes the danger which lies in the increasing number of mathematical courses being taught by engineers, social scientists, etc.

3. The Discussions

There seems to be broad consent about the observation that the manipulative skill of students entering the universities has declined steadily during the last decade. In spite of the general confirmation of the phenomenon there are different opinions about its causes. Many blame the introduction of abstract concepts into secondary schools, others a general diminishing interest for technical and scientific problems, and some attribute it to a general change in social structure and attitude towards responsibility. Occasionally it is observed that our present students are more eager and better prepared to grasp abstract concepts.

The amount of choice in the undergraduate curricula differs - as described in the general report - considerably from country to country and even from one university to another. However, the discussion shows a universal trend towards an increase in choice at higher levels of the education. It was even noted that at some universities the amount of choice has reached a point where totally unsuitable curricula are possible.

Experiments with algorithmically oriented calculus courses are reported from Bulgaria, Israel and Sweden.

In most developed countries the professional situation of mathematicians is characterized by a considerable shortage of open positions and in some countries by a dramatic drop in enrollment. A quantitative picture is given in the short communication of *J.T. Fey:* 'Patterns of Course Offering and Enrollments in Mathematical Sciences of US Universities' where data are presented, covering the situation until fall 75. Some participants pointed out that, taking into account the enrollment in computer science and statistics could very well round out the picture.

A nondisputed phenomenon is the trend in mathematical education towards applications. Courses in Mathematical Modelling, in Mathematical Engineering, curricula in Industrial Mathematics are symptoms of this process. A debate whether applications of mathematics in other fields should be taught by mathematicians or by specialists of the other field found the audience divided. An advocate of the latter point of view is *Prof. Kudryavtsev* in his short communication 'On Mathematical Education at Higher Technical Schools', whereas the first opinion is supported - among others - by Sir James Lighthill. Here the program for Dr. of Arts is to be mentioned which has been introduced at several US universities. This program is intended for students who are brilliant in their capacity to conceive but not gifted for doing research. Prof. Young spoke shortly about this program but seems to be quite hesitant to recommend it plainly to other universities. As he put it, the program should be introduced only after serious and comprehensive thoughts beforehand.

One of the main issues of the Friday morning session was the teaching of mathematics to physics-, engineering- or business students. It is a generally deplored fact that an increasing number of engineering and business faculties offers mathematics courses by their own staff members. Causes of this development were discussed; strategies to avoid such a process were not yet present. The problem was discussed in detail by *M.L. Fuller* from Australia who reported about a research project investigating the mathematical education of business/management undergraduate students.

The last short communication by *Dr. Roether*, FRG, concerned a project of computer aided teaching at the Fachhochschule at Furtwangen.

A 5

Adult and Continuing Education in Mathematics (with Reference to Correspondence Studies and Television)

Reporter: R. M. Pengelly, GB

Chairman: T.J. Fletcher, GB

Coordinator: W. Böddeker, FRG

Short Papers: E. Bartley, USA; H. Brönstrup, FRG; J. Mason, GB; N. Shelley, Australia

A 5.1
SURVEY-ABSTRACT BY THE REPORTER

For the first time, Adult and Continuing Education in Mathematics is emerging as an independent topic for discussion at an International Congress on Mathematics Education. Creating an adequate basis for a discussion on this new theme presents a major challenge because -

(1) the current state of the literature in the field is so poor. While a great deal has been written on the subject of Adult and Continuing Education from the point of view of the design of distance teaching systems little if anything appears to have been written on the teaching of particular subjects;

(2) continuing education in mathematics for adult students effectively spans the whole range of issues covered by themes A1 to A4 and A6, and brings into practical use almost all of the aspects of mathematics education covered by themes B1 to B6;

(3) the variety of activities currently being undertaken in the field of Adult and Continuing Education in Mathematics is substantial - varying from a brief series of broadcast talks to a full length postgraduate programme.

To try and meet this challenge the theme report has been written to take the broadest possible view field in order to show the way in which Mathematics Education comes into Adult and Continuing Education. In particular the report focus attention on major new developments which are currently going on in Adult Education which have led to the development of multi-media distance study systems.

The *first part* of the theme report serves largely to set the scene by describing Adult and Continuing Education in terms of

- the range of activities that are going on;
- goals and legislation;
- social and cultural influences;
- development of teacher/student relationships.

The *second part* of the report describes how Mathematics Education comes into the field of Adult and Continuing Education, reviews the factors which influence the scope and content of Mathematics programmes in Adult Education and presents a factual report on a number of current projects which involve the teaching of Mathematics to Adult.

The *third part* of the report reviews some of the novel features of existing projects and trys to identify those current trends which are likely to influence developments in the areas covered by the report.

The *fourth part* of the report describes some of the problems encountered in the production of adult education programmes in Mathematics and tries to give an indication of the direction in which the solution to some of these problems may lie.

The *last two sections* of the report are concerned largely with the features and problems of Adult Education in general because there is currently so little information available which is specific to the teaching of Mathematics. For this reason the report concludes with a series of questions, which those contributing to the discussion on this theme are invited to try to answer, and a proposal that those concerned with developments in areas covered by the report work for the creation of an Information Centre to gather, analyse, record and disseminate information on activities in the field of Adult and Continuing Education in Mathematics. The intention is to stimulate participants in the present Congress to take what steps they can to ensure that

- the final report provides a useful source of reference;
- a foundation is laid for the treatment of this theme at a future congress;
- all those working in the area covered by the report are encouraged to publish papers describing their activities.

Each part of the report provides information on a whole range of issues and, where possible, examines the issues from several points of view. The principle issues covered in the report are :

(1) the revolution in home study courses that has led to traditional correspondence course being enhanced by the introduction of a variety of audio-visual teaching aids and new forms of contact between teacher and student;

(2) the economic, social, cultural and educational pressures which serve to determine the goals of Adult Education and the way in which legislation and social and cultural factors influence the development of education programmes to meet these goals;

(3) the development of student and teacher roles and relationships in Adult Education and the impact of distance teaching techniques on these roles and relationships;

(4) the factors which influence the scope and content of Mathematics programmes in Adult Education and their relationship to the general factors influencing Adult Education as a whole;

(5) programmes designed to teach Mathematics at Elementary Level in which the aim is to either provide the student with basic mathematical skills which he can apply later in his education, or provide skills he can use directly in his trade or profession;

(6) programmes designed to teach Mathematics at Advanced Level in which Mathematics is taught as a component (perhaps the only component) in a professional development programme;

(7) the factors which are influencing the development of teaching methods and the design of systems of adult education;

(8) the use being made of various educational media and resources in Adult Education programmes;

(9) the problems of designing and producing adult education programmes and particularly the problems involved in the design and production of multi-media distance teaching systems;

(10) the difficulties involved in determining the effectiveness of adult education programmes including a discussion of how feed-back may be used to identify the strengths and weaknesses of a teaching system;

(11) a discussion of the possibility of using feed-back information to design and produce an improved educational system and a review of the factors that influence attempts to produce improved systems.

Throughout the report every effort has been made to relate the discussion of general issues to specific issues affecting the development of Adult and Continuing Education in Mathematics. An effort has also been made to identify aspects of the issues discussed which may be treated differently in developing countries. It is hoped that input from the Panel-Discussions on Theme A5 will serve to highlight these last two aspects of the report.

A 5.2
DISCUSSION-SUMMARY BY THE COORDINATOR

R.M. Pengelly's trend report was followed by the panel discussion. First *M. Barner* raised the fundamental question as to what specific significance mathematics had for the problems under discussion. He said that nearly everything that had been said about continuing training, adult education and distance courses applied to all subjects and contained nothing specific to the learning of mathematics. In his view, what is specific to mathematics should play a more important role in such discussions among mathematicians on account of the significance of the field. The question arises as to what distinguishes continuing education and adult education in mathematics from education in other subjects.

D.W. Sida remarked that the focus on two aspects only, i.e. 'distance education' and 'mass education', left out many of the very important facets as far as mathematics is concerned. At his university, in the capital city of Ottawa, there is the problem of continuing education of persons highly qualified in other fields, for example in administration, finance, economics, and who, to a certain extent, need a knowledge of mathematics, which they do not possess. One is thus faced with the problem of people who attend very high level courses in certain fields which require mathematics for which they are insufficiently prepared. The answer to the question of how to bring persons who can hardly count to university level in maths in a short time is a very thorny problem.

Then *D.W. Sida* hinted at the significance of in-service teacher training, though the reporter had left it out. There are two kinds of courses for teachers: those which enrich their teaching practice and those which expand their knowledge and understanding of the contents to be taught. The latter are an important point in our discussion.

Thirdly, *D.W. Sida* pointed out that the report made no reference whatsoever to the specific problems of continuing education and adult education in mathematics for women as a special group. The normal approach to mathematics, devised by men for men, might not be the best one.

In contrast *J.H. Hlavaty* remarked that, as he had been able to gather from his own experience, there were no significant differences in the learning of mathematics by men and women.

Starting from the problem of understanding in mathematics, *J.H. van der Merwe* referred to the fundamental problem posed by any 'distance courses' in mathematics that proceed from written mathematics. The problem is, in his view, how to bring the student to work and understand by himself from a book or a letter without having someone at hand for giving him explanations.

J.H. Hlavaty considered the question of the legal foundation for continuing education and a more precise description of the group of people who expect such education to be very important.

In this connection he distinguished three groups:

(1) people who were unsuccessful in their conventional school career and would like to try it again later on;

(2) people who want 'continuing education' for reasons other than for acquiring a professional qualification, as is the case for women who come back to school when their children no longer need their care at home;

(3) elderly people - in the Western world, surely - who increasingly want to learn mathematics out of a general cultural interest.

A. Bouvier underscored the significance of the social and legislative aspects of continuing education. Before 1971, continuing education in France was addressed to a small number of people only. The law passed on July 16, 1971, which made it compulsory for any firm employing more than 10 workers to make at least 1% of the overall working time available to its employees for continuing education ushered in a fundamental change, which is made apparent by the following facts:

(1) About one out of four employed persons has already taken part in some kind of continuing education.

(2) 2,5 million persons took a course in 1974.

(3) The number of participants in continuing education has gone up by 60% from 1972 through 1974.

These data show that this is an important social problem.

W. Böddeker commented that in the FRG the problem of the general leave for continuing education is still an unsolved question. Coming back to the issue touched upon by M. Barner, Pengelly's report, in his view, raised the question of the extent to which the basic reflections on the general problem of continuing adult education contributed something towards the special problems of continuing education in mathematics. He sees the danger that a general 'metapedagogy' of adult education may be worked out which, however, would be of no consequence for the practical problems of continuing education in mathematics. Therefore, we should try to find out and analyse what is thought to be of importance for the specific problems of continuing education in maths.

Secondly, in his opinion, the special problem of in-service education for teachers definitely belongs here, for significant reasons - nearly everything that is done in the FRG in the 'distance education' field relates to teachers, mathematics teachers, either to allow them to achieve a higher qualification or to enrich their present activities. This has however been assigned to section A6 by the congress organization.

With respect to general continuing education in the professional field there is a great number of highly varied activities which are mainly organized by regional or community institutions ('Volkshochschulen'). There is a tendency to provide a legal basis for these varied activities and to make them more homogeneous, to lay down general regulations, coupled with the possibility of obtaining diplomas and qualifications recognized by the state.

Finally, *A. Bouvier* described a tendency which, in his view, has become apparent in France since the 1971 law, namely, that correspondence courses are loosing ground to learning in small groups, in schools or at the place of work.

The plenary discussion followed immediately. The points which emerged from the discussion may be summed up as follows:

(1) the influence of trade unions and of the professional environment in all its variety;

(2) the 'right' use of the intended aids, among others of letters, books and learning sequences, i.e. the self-sufficient and effective working of the learner with these materials;

(3) the problem of the feed-back and of contacts with the learner, especially where time and space difficulties arise.

R.M. Pengelly first called upon the participants to contribute further information on their experience and then summarized the major points of the panel discussion as follows:

(1) Referring to the question 'What distinguishes continuing

education in mathematics from continuing education in general?', he remarked that problems of adult education in mathematics cannot be understood before one is well informed about general problems of adult education.

(2) The restriction to 'distance-education' and 'mass-education' is not meant as an absolute restriction; it is only intended to provide a basis for the selection of interesting and wide-spread conditions. He wanted to leave out the advanced courses for Civil Servants, for instance, since they appear too numerous and varied for a general outlook. He dealt in any case with those courses which start off without prerequisites as this represents a general problem of adult education.

(3) How can we make these problems seem worthy of note to all colleagues?

(4) How students can be led to work more independently and self-sufficiently is a key question, especially for distance courses. Important cues can be provided here by observing the attitudes and expectations of teachers and the attitudes and expectations of students as to what is intended.

(5) Finally, the importance of the legal basis for adult education was noted, and it was stressed that it was often the precondition for making adult education possible at all.

The following *short reports* were made at the second session:

E. Bartley: Adult and Continuing Education as Conducted by Correspondence by International Correspondence Schools (ICS).

J. Mason: Tutor-Student Interaction in Universities: Comments Generated by Experience with the Open University.

H. Brönstrup: Computer Aided Distance Courses.

N. Shelley: Mathematical Illiteracy and some of its Implications.

Particularly the last report was followed by a detailed discussion. The questions and problems raised were finally summed up:

(1) What kind of mathematics should be taught?

(2) How should it be taught? Individually or in groups?

(3) Should special or general curricula be developed? What about certificates and recognition in the framework of general school achievements?

(4) Of what kind should the necessary teaching aids (in the broadest sense) be?

(5) What about the feed-back? What special part does it play in adult education and in distance courses? How does it help the student? How does it help the teacher?

The formulation of the results in the form of questions makes clear to what extent further experiences and investigations are needed.

A 6
The Training and Professional Life of Mathematics Teachers

Reporter: M. Otte, FRG
Chairman: J. van Dormolen, Netherlands
Coordinator: H. Griesel, FRG
Short Papers: W. Böddeker, FRG; C. Gaulin, Canada; H. Shuard, GB; G. Wain, GB

A 6.1
SURVEY-ABSTRACT BY THE REPORTER

This paper is the first to deal with this topic; in previous papers on new trends, the training and the professional situation of the mathematics teacher were not treated as main topics in themselves, but were mentioned marginally and unsystematically in various papers on other topics. This fact demonstrates that there has been a shift of emphasis in the didactics of mathematics within the last few years: a shift from the problems of learning to those of teaching, from the problems of teaching materials to those of the teacher, from curricular developments to teacher training and continuing education.

The training and professional life of the mathematics teacher are heavily determined on the one hand by the field itself; they take place however within the larger framework of the educational system. These general aspects seemed so important to us that we divided the report into two parts. In the first section, 'The Training and Professional Activity of the Mathematics Teacher', those problems are treated which involve this social and institutional context. In the second section, 'Mathematics and Teacher Training', those questions having to do more directly with the field itself are discussed. We are aware of the fact that social and institutional problems cannot be separated from content problems in the didactics of mathematics. This hypothesis, which is the basis of the whole paper, closely relates the two parts to each other.

Teacher training and educational reforms are inextricably intertwined: for example, undertaking curricular reforms without taking simultaneous or previous measures for the corresponding training of teachers, has not had the desired success. Increased attention is being given to the importance of teacher training and continuing education as an impetus to changes in the school system, especially in times when, in many countries, cost-benefit considerations and efficiency criteria determine the prerequisites for reform attempts. Nevertheless, teacher training has not played a leading role in the past, but has had some difficulties in adapting itself to rapidly changing school conditions. The latitude in decision-making of the individual mathematics teacher is dependent on *exterior constraints* which cannot be determined by his training. Freedom of decision on a day-to-day classroom level is partially determined by organizational principles of the individual school, partially by such external influences as examination standards and teaching materials. The school systems themselves suffer grave deficiencies in this field. Teacher training cannot influence these deficiencies, but could prepare the teacher to deal with them. *Practical relevance* has become increasingly important in all the discussions about the reform of mathematics teacher training within the last few years. But since 'practical relevance' is still little more than a fashionable expression used to describe a wide variety of needs, and since there are neither adequate descriptions of the practice of mathematics teaching nor scientifically rigorous qualifying standards, the tendencies within this field are not uniform.

The professionalization of the mathematics teacher is a decisive prerequisite for the improvement of mathematics instruction. This requires a rise in scholarly standards in all the relevant areas, as seen in the present worldwide trend toward ever higher university training for teaching at the primary level, as well as in the establishment of a common pool of experience between mathematics teachers of all school levels. At the same time, problems in the integration of theory and practice and of the individual areas of mathematics instruction are increasing. This is most clearly visible in the increasing demands made on the faculty, which are difficult to meet, and also in the growing need for didactic media about content and organization.

By international consensus, *the continuing education of the mathematics teacher* is at the center of the discussion of training reforms. However, the resources and programs of the individual countries are inadequate and are characterized by extreme variety and fractioning. Difficulties in identifying needs, difficulties in continuity and in institutional support on the one hand, and tensions between theory and practice on the other are typical of the present situation.

The relationship between the scholarly field of mathematics and mathematics instruction in universal education is a key problem in the training of mathematics teachers. To state the problem bluntly and somewhat oversimplified, there are two tendencies which oppose each other. The school of thought which holds that the teacher should really be a research mathematician (primarily applicable to the secondary level) can hardly be reconciled with the opinion that he need understand only the material he teaches. Traditionally, there has been a 'natural' relationship of the scholarly field to university teaching and to school instruction by means of mutual cross-fertilization between teaching and scholarly research. In spite of its unquestionable importance, this relationship cannot offer orientation for universal educational instruction. The scholarly fields can express views on content or method, but can offer only limited insight into their own social role. This social dimension of the relationship between scholarly mathematics and school mathematics is gaining increasing importance for the behaviour of the mathematics teacher and his instruction methods. First steps towards an understanding of this dimension are visible in the didactics of mathematics in such key words as 'problem oriented' or 'application oriented' mathematics instruction, in the emphasis of problems of motivation, in the discussion of the connection between general and professionally oriented instruction and so on. The somewhat optimistic belief in the connection between scholarly mathematics and school mathematics, expressed in a logically reduced and abstract form of the 'structure of the discipline', which characterized the methodology of the field by 'elementarizing and fundamentalizing' in the fifties and sixties, is on the way out in almost all countries today. In the course of this shift in orientation, a latent hostility toward rigorous scholarship in the field of education has become evident.

In order to deal with the basic problem of the relationship between a dynamic scholarly field and mathematics instruction for universal education, the teacher need a kind of mathematically specialized comprehensive set of tools for understanding alongside his pedagogical and psychological knowledge and his mastery of the field. In order to cope intelligently with curricula and to grasp their implications, knowledge about the inner structure and the logical connections of those curricula, as well as knowledge about the relationship to curricula of other school subjects, about their applicability and their relevance to the needs of the pupils are necessary. Such considerations are being increasingly taken into account in mathematics teacher training.

Didactics of the individual field is just starting to come into its own as a discipline within teacher training. It dealt at first with relatively narrow methodological and curricular questions, while the concept of pedagogical education remained almost completely general. The consequence was often a lack of coordination between the narrowness of the didactic perspective and a 'pragmatic' dependency of the instructional process on chance influences. The didactic discussion is often still governed by such pseudoalternatives as 'child-oriented vs. field-oriented', premature solutions and value judgements are serious hurdles on the road towards a coordinated identification of the real problems of mathematics instruction and mathematics teacher training. Today, changes in the scholarly field and in school systems demand new concepts of the contents and methods of mathematical didactics for teacher training. Only on the basis of such new concepts can teacher training meet the urgent demands for the integration of theory and practice, of scholarship and education.

A 6.2

DISCUSSION-SUMMARY BY THE REPORTER

Issues relating to the training of teachers have become a major focal point of attention by educators for it has been widely acknowledged that the success of new curricula and other educational programmes depends, in the ultimate analysis, on their effective implementation by teachers. It was for this reason, among others, that the 'Training and Professional Life of Mathematics Teachers' was given a prominent place on the Congress programme as a whole. The fact that some two hundred Congress members participated in the Section meeting dealing with this theme, amply demonstrates the importance which is widely attributed to issues of teacher training.

Two working sessions were held. During the first, M. Otte (Bielefeld, Germany) presented a trend paper in which he sought to identify and analyse major factors relating to and influencing the initial and in-service training of mathematics teachers. The main issues raised may be summarized under the following headings:

(1) The Education and Professional Life of Mathematics Teachers as affected by

a) the relation between teacher education and school reform and/or curriculum innovation;

b) environmental conditions of teaching;

c) the professional orientations represented in mathematics teacher education programmes;

d) problems relating to the professionalization of mathematics teachers;

e) specific problems concerning the establishment and implementation of in-service teacher education programmes.

(2) Mathematics in the context of Mathematics Teacher Education, with reference to

a) the nature of mathematics as a science discipline and as a school subject;

b) the structure of mathematics as a knowledge domain and the learning/teaching process;

c) pedagogic aspects of mathematics education and their place in teacher education programmes.

A concise summary of Otte's lecture is given in the preceding abstract (A6.1); the full report will appear in the forthcoming 'Trends in Mathematics Education' volume, published by UNESCO.

In keeping with the intentions of an international meeting, Otte's report did not consider specific teacher education problems relating to particular countries. However, during the first discussion session, immediately following the report, delegates took the opportunity of amplifying and relating many of Otte's points to the situation in their own countries. From this emerged a list of 'high priority' items for discussion in the second Section meeting.

The first section meeting was concluded with short contributions from *H.B. Shuard* (Cambridge, England) and *G.T. Wain* (Leeds, England) on mathematics teacher education in the United Kingdom and on the Mathematics Teacher Education Project (MTEP) recently launched in England, respectively.

For the purpose of the discussions in the second group meeting, four subgroups were formed. Each subgroup developed its own discussion programme, but reported back through its chairman. The following presents a summary of the main points expressed by Congress members on issues concerning the training and professionalization of mathematics teachers.

(1) It was generally agreed that, so far, no adequate overall analysis had been made of the academic and paedagogical skills required by mathematics teachers in their professional activities. Without a comprehensive analysis to identify skills and abilities needed, the design of mathematics teacher education programmes would continue to be based on pragmatic decisions and expediencies.

The identification and clear description of such skills and abilities could lead to the development of a better theoretical framework for teacher education than is currently available and, in particular, allow round decisions to be made about what paedagogical and academic concepts should be given emphasis in mathematics teacher education programmes.

(2) The view was expressed that demands for the reform or modification of mathematics teacher education programmes often implied that existing programmes were 'failing' in some respects. No real consideration had been given, however, to the question of what constitutes 'success' and 'failure' in relation to initial teacher training programmes. It was pointed out that although criteria of success would be useful, they could relate only to short-term aims of mathematics teacher education programmes. It would not be possible, in relation to long-term aims, to separate initial teacher training from school-based induction and probationary periods and from in-service training programmes.

(3) It was generally regretted that in none of the countries represented an adequate structure for the in-service training of mathematics teachers had been evolved. In-service training was seen not only as a method of retraining already qualified teachers, but also - and in particular - as a desirable institution to enhance the professional status and competences of teachers already in the profession. Attention was drawn to the desirability of involving teachers in school-based development work and in research activities relating to mathematics education.

(4) Major concern was expressed about the training, at both the initial and in-service levels, of non-specialist teachers for the teaching of mathematics at the lower end of the age spectrum. A high proportion of teachers teaching mathematics at the primary school level had only minimal mathematical qualifications. Initial training courses had introduced them to mathematics in the context of primary school teaching, but this had not enhanced their mathematical competence as such. It was suggested that more mathematics were required for work in the primary school sector.

(5) The question of what mathematical training a future mathematics teacher should receive produced divergent views. There were those who advocated 'mathematics for teachers' with emphasis

on those aspects of mathematics which featured in school courses; others pointed to the position of mathematics teachers as members of the mathematical community and felt that two different types of mathematics do not exist. This is likely to remain an unresolved question, but one that required continuous examination.

Several other points were touched upon during the discussion, among them the important problem of retaining highly qualified mathematics teachers in actual teaching service. Frequent reference was made to the change of such persons into high level administrative posts, leaving the teaching institution all the poorer. It was felt that the career structure of the teaching profession needed re-examination with respect to this phenomenon.

Generally the discussion contributions reflected a very high level of constructive and analytical thinking on the part of Congress participants about problems concerning the training and professionalization of mathematics teachers in the contemporary context of mathematics education.

It was recognised that many of the issues identified in the Trend paper and during the discussion sessions, whilst relevant to the situation in almost every country, could be debated and solved only within national contexts. Therefore, the discussion groups of the Section meetings felt that it would be inappropriate for specific recommendations to emerge from this section of the Congress.

There was unanimous agreement to recommend to the Executive Committee of ICMI that

'the training and professional life of mathematics teachers' should be accepted as a permanent theme for future congresses.

B 1
A Critical Analysis of Curriculum Development in Mathematics Education

Reporter: A.G. Howson, GB
Chairman: B.H. Neumann, Australia
Coordinator: B. Andelfinger, FRG
Short Papers: B. Andelfinger, FRG; F. Haq and B. Vogeli, Afghanistan; K.J. Travers, USA

B 1.1
SURVEY-ABSTRACT BY THE REPORTER

(1) The last twenty years have witnessed a vast expenditure of money and effort on curriculum development, with end results that have often been very disappointing. Now educational systems in many countries are having to exist on pared budgets and money for curriculum development is no longer so easily obtained. How in a time of financial stringency and amidst an air of disillusionment, is curriculum development to proceed? What lessons have been learned in the last two decades; what transferable principles have emerged?

(2) Without doubt the most significant fact to emerge is that the teacher's role is crucial. No matter how outstanding a project's team or materials, the success of its work ultimately hinges on the receptiveness and adaptability of the classroom teacher. Curriculum development is a process which involves people, their likes and dislikes, their strengths and weaknesses, and not only mathematics.

(3) The curriculum must mean more than merely the syllabus and texts. Content cannot be viewed in isolation and neither can method. It is unrealistic to believe that the problems of mathematics teaching will be solved solely by the introduction of new materials or on the other hand by the introduction of

'discovery methods', 'individualised learning', 'a systems approach', 'computer assisted instruction', etc. The best plans will founder unless examination systems are revised to test desired objectives, and to encourage the attainment of educational and mathematical goals rather than to militate against them.

(4) The most popular pattern for curriculum development in the 1960s was the research-development-dissemination (RDD) model. Almost invariably 'dissemination' has failed and the message has reached the classroom in a garbled form. This failure sprang from underestimating the provisions needed for in-service education. It is now becoming clearer that curriculum development and in-service education must be closely integrated. They cannot be seen as chronologically distinct operations.

(5) Frequently, however, objectives and materials were misconceived. Projects did not consult widely enough and insufficient consideration was given to how changes would affect other levels of education, users of mathematics and employers of students. Even where such factors were considered the need to explain changes to those likely to be affected by them was often ignored.

(6) The form that curriculum development can take in any country is governed by that country's educational system and its historical development: whether or not it is centralised, the degree of autonomy granted to its teachers, the role of commercial publishers, etc. Nevertheless, in many countries it is being realised that 'local' projects in which all schools can be involved in course design have a great part to play in curriculum development. There is also a need to foster the professional growth and autonomy of the individual teacher and to encourage the individual innovator. Subject associations of teachers can still be a potent influence.

(7) 'Local' projects, in which teachers prepare materials cooperatively, still present major problems and do not allow us to sidestep the crucial questions faced by 'large-scale' projects such as 'Who initiates such developments?' 'How are writing groups to be constituted?' 'Who is to determine the subject-matter, aims and style of the materials?' 'Who is to ensure that a reasonable professional mathematical standard is attained?' There

is also an evident need to coordinate 'large-scale' with 'local' activities.

(8) A teacher's reaction to curriculum changes - both initially and when problems arise later - will be affected by the dissеminatory strategy used. Four common strategies are the 'power-coercive' ('from 1st October, 19-- all schools shall'), the 'pressure-coercive' ('if you were up-to-date, you would be using'), the 'rational-empirical' ('pupils learn to take responsibility to a greater extent with IMU'), and the 're-educative' (in-service education and curriculum development combined). To some extent this serves to define the teacher's role in the eyes of the administrator: as a servant of the system, as a semi-autonomous being to be manipulated or urged along, as a professional capable of making sensible decisions.

(9) For the innovator, change is exciting and can bring with it professional and financial rewards. One must, however, consider the incentives for the ordinary teacher who is asked to change his practices. Whereas the former has 'all to gain and nothing to lose', the latter may well believe he has 'all to lose and nothing to gain'.

(10) Although curriculum theory has a long history and has itself 'boomed' in recent years, in most countries developers have chosen to ignore the theorists and have perpetuated the traditional division in education between theory and practice. The establishment of central agencies for curriculum development with staff drawn from many disciplines may well lead to a reconciliation.

(11) Educational research has prompted several curricular activities, yet it has had very little effect on many others. There is a need for researchers to disseminate their results more widely and in a more comprehensible manner. Researchers must see it as a major responsibility that their work should have impact on society and, in particular, on curricula.

(12) The problems of the 'comparative' evaluation of curriculum development are so great that there is a danger that evaluators will concentrate on technical measurement problems with a possible overemphasis on cognitive goals. The multivariate nature of

mathematical achievement coupled with the apparent impossibility of unravelling the effects of mathematical changes from those arising from other social and educational causes present evaluators with an apparently impossible task. Even when data provided by evaluators appear in an 'objective' form they must still be interpreted and acted upon subjectively.

(13) The dominating position of textbooks has been challenged by workcards, topic books and by non-printed aids such as films, television recordings, tape cassettes etc. - often locally produced. Teachers still, however, appear to opt for pre-packaged materials which can be obtained from the publisher and transplanted immediately into the classroom. Pupils' materials exert a stronger influence than do guides or handbooks for teachers. There is a need for guides (which go far beyond supplying answers) to supplement pupil's materials, but effecting change by means of teachers' guides alone has proved an impossible task.

(14) Even in those countries where projects have not been a major feature of curriculum development it has been realised that group authorship of texts has much to offer. The advantages of using practising teachers as authors have been frequently demonstrated.

(15) During the 1960s project materials were translated and transferred between countries with insufficient thought given to the social, educational and cultural differences involved. Such transfer problems can also exist within a single country where distinct cultures may have developed side by side, or an alien culture has been imposed on an indigenous one.

(16) *Trends in Curriculum Development.* Any attempt to delineate trends is bound to have a subjective element - in many cases the 'trend' may be more wishful thinking than actuality. Yet surveying the international scene it would appear that the following general statements can be made with a fair degree of confidence. (It must be realised here that even though one can provide counterexamples to each suggested trend, that, unlike proof where one counterexample is sufficient to disprove a hypothesis, in this instance a single example cannot by itself invalidate a

trend.)

It is becoming accepted that:

a) The provision of materials is not a sufficient condition for the attainment of curriculum development. The provision of new materials is a prelude to successful curriculum development - the hard work lies in their successful dissemination.

b) In-service education and curriculum development must be more closely integrated. The realisation of this has led to greater emphasis on local rather than large-scale projects.

c) There is a need to design suitable mathematics curricula not only for the academic élite, but for all ability ranges and for the physically and socially handicapped.

d) Where national educational and social pressures make it necessary, there is a need to produce materials and to develop teaching methods that can be used over a wide ability range (mixed-ability classes).

e) Although individualisation of work via work cards, programmed modules, the computer, etc. has much to offer, it is not by itself the solution to our problems. More attention must be given to investigating the proper role of individualised programmes in the classroom.

f) Content and method must be considered together (reaction to previous over-emphasis would appear to be generating new biases towards method in secondary schools and towards content in the primary schools).

g) Increased emphasis must be placed on affective goals and on demonstrating the usefulness of mathematics.

h) The rethinking of examination procedures and objectives, within a total view of learning and teaching, is an essential ingredient of successful curriculum development.

i) The role of the teacher is vital. He should become personally involved in curriculum development and should not be subject to unrealistic demands which would only have a disillusioning and dispiriting effect on him. The individual innovator should be cherished.

j) Pre-service education must prepare teachers for participating in on-going curriculum development. In-service education must aim at fostering the professional growth and autonomy of teachers.

k) In the design of mathematics curricula, we must not ignore general social and educational objectives and, in particular, the demands of non-mathematicians, such as science teachers, and future employers. Particular attention must be paid to the problems likely to arise at the various 'interfaces'.

l) There is a need to consider more closely the processes of innovation and to base curriculum development upon better elaborated theories of teaching and learning.

m) Curriculum development must become a gradual cumulative process, rather than a frantic pendulum-swinging exercise.

n) Curriculum development is the art of the possible, not a yearning for the unattainable.

B 1.2
DISCUSSION-SUMMARY BY THE COORDINATOR

(1) The opening address of Section B1 by B. Neumann on Tuesday, 17-8-76, was followed by the lecture of A.G. Howson. This was an elaboration of the abstract published in the congress materials.

In presenting the report he identified various aspects of curriculum development and emphasized the great importance to be attached to the processes of initiation, to the part to be played by teachers, and to the need for a continuous dialogue between developer and teacher. He drew attention to the move away from 'large-scale' national projects towards more 'local' activities based on the contributions of practising teachers.

These latter projects were, however, unlikely to prove a solution to our problems unless they were subject to more stringent criticism and evaluation and were given greater professional backing. Ideally, curriculum development would be based on coordinated 'large-scale' and 'local' developments in which the in-service education of teachers would be an integral part. Here the example of the IREMs was quoted. Emphasis was also placed on a full discussion of aims in which the views of industry, mathematicians at other levels of education, and users of mathematics are taken into account.

Most of the panel members then took the opportunity to comment further on points arising from the report. Some keypoints made included:

Curriculum development develops curriculum developers. This can turn out well or badly (Isaacs).

The RDD-model has often failed, yet there must be a place for external institutions in our model (Wilson).

When determining aims, one must take into account the relation between curriculum and society and consider the needs of society as a whole, not merely those of industry and other users (Keitel).

The textbook is not the curriculum. The curriculum is defined by a set of goals and mathematical concepts behind the textbook. The teachers should be helped to keep in mind these goals and concepts (Kieren).

Scarcely any curriculum development project in the USA has achieved the aims of its participants.

There is a shortage of informal materials which emphasise processes and the solving of problems.

Dissemination is not the same as 'passing out' from a curriculum development center; it also includes evaluation.

The successful intregation of supply materials in teaching is of great importance (Jones).

In curriculum development there is no process without content. Local processes are important.

Mathematics curricula should not be prepared with only future mathematicians in mind (Allotey).

These general observations were then illustrated more concretely in two reports on curriculum projects:

SMP (Hersee) realised that changes in the 11-16 curriculum must be made not only because of feed-back and evaluation of the texts produced in the 1960s but also because of changes in other factors; for example, technical (calculators), social (employment) and political/educational (comprehensive schools).

Like SMP, the *IOWO (Schoemaker)* saw curriculum development as a continuous process: a human activity, not solely a mathematical one. Materials and aims must be revised in the light of classroom experience.

The participants of Section B1 expressed agreement with the trends and views set out in Howson's report. Discussion between the section-members centred on such topics as:

- the variety of components in the crucial disseminatory activities;

- the need for teacher-involvement and for the coordination of in-service education and curriculum development;

- the need for the curriculum to be rooted in the culture of a country (yet there is still a need for international colloboration and assistance);

- the growing knowledge of the way in which curriculum development operates, and the hope that this will prevent us from repeating earlier errors.

(2) The B1 members reassembled on Friday, 20-8-76. Two short papers were presented in plenary session, after which the participants divided into three working groups.

First, *Andelfinger* described a curriculum development project in Germany (Kollegschule Nordrhein-Westfalen) for pupils aged 16-19. He proposed special terms for describing curriculum development projects:

A local project is a '1-stage-model'; a '2-stage-model' is a more complex system of local and national components, a '3-stage-model' includes, in addition, external institutions which supervise and advise.

The models consist of institutions, persons and activities and so can be described with the help of relation-diagrams. The curriculum development project 'Kollegschule' is a 3-stage-model. The curriculum concept includes two steps: a fundamental system with courses on 2 levels (1-2 years), a course-package-system (2 years). The packages are fixed on vocational or activity areas, not only on mathematical topics.

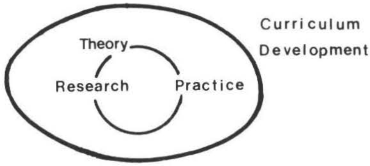
Haq and Vogeli then presented a description of a primary school curriculum development project in Afghanistan (a 'large-scale' national project).

Materials had been prepared and distributed. Now revision in the light of experience was required. Teacher training presented a great problem.

The work of B1 was concluded by sessions of 3 groups.

Group 1 worked on the topic 'Teacher Involvement'. The developing countries (but also the so-called developed countries) have most of the problems in this area. Especially the adaptation of text-books, the insufficient knowledge of the teachers and the social and cultural circumstances are a handicap for teacher involvement in curriculum development. Usually teachers listen to the curriculum developers and university teachers, but do not react themselves.

Group 2 discussed the topic 'Theory Research Development'. The most important statement of this group was the view of research and development as a whole. Curriculum development is a 'Circular Model', not a linear model:



We have a lot of (logical or scientific) 'true' curriculum theories, but not all of them are 'valid' in relation to curriculum development.

Group 3 dealt with the 'Special Problems of the Developing Countries'. The developing countries asked for a more intensive interchange of information between themselves. Help from the developed countries should be given too. But the developed countries must see the problems of adoption and adaptation faced by the developing countries. The developed countries could be asked for information and criteria for choosing materials. It will then be a matter for the developing countries to make their own decisions.

(3) Summarizing the work in B1 there seem to be some general agreements round the world:

- Curriculum development is not the task of specialist groups only;
- Curriculum development is a network of persons, institutions and processes on various fields;
- Curriculum development is also a discussion between mathematics and society;
- Curriculum development needs national and international coordination and interchange of information;
- Curriculum development has to be tolerant of all participants of its processes.

This tolerance was realized by all members of Section B1 during the days of the Karlsruhe Congress.

B 2
Methods and Results of Evaluation with Respect to Mathematics Teaching

Reporter: J. Kilpatrick, USA
Chairman: A. Bishop, GB
Coordinator: D. Lind, FRG
Short Papers: C. Bentley, GB; Y. Hashimoto/T. Sawada/S.Shimada, Japan; W. Rouse/J.B. Wesson, USA

B 2.1
SURVEY-ABSTRACT BY THE REPORTER

Educational evaluation is an interactive process, affecting those people whose accomplishments are being evaluated as well as those who do the evaluation and those on whose behalf it is done. The process is both psychological and socio-political; it does not occur in a vacuum.

Although in theory the process of evaluation is quite simple - one chooses a value dimension, gathers information relative to the dimension about the object to be evaluated, and applies a scale of value to the information obtained - in educational practice the process is never simple. The objects of educational evaluation, such as a pupil's ability or a curriculum's adequacy, are multi-dimensional; they change over time and across situations.

Many problems in educational evaluation occur because of the difficulty one faces in collecting information about the complex and dynamic qualities of the object to be evaluated and then distilling this information so that decisions can be made. An example is the issue of external examinations at the end of secondary school. A general trend has been the development of schemes in which teachers set and grade their own examinations and make informal evaluations throughout the year. These schemes have proved popular with pupils and teachers because they permit more valid assessment of student progress, but such schemes raise difficult questions of how to compare the performance of pupils who have had different teachers.

There is an inevitable tension between the 'firing line' and the 'headquarters' conceptions of evaluation. Pupils and teachers on the firing line need evaluation information that is diagnostic and descriptive of the complexity of the situation they face; administrators at headquarters need a simplified overview of the situation. Some knowledge has been gained in recent years on the technical aspects of sampling and data analysis, but the evaluator will always face questions of deciding what information to obtain, what scales of value to use, and what weights to assign scales in arriving at a final decision.

As one surveys trends and issues in evaluation with respect to mathematics education, one should strive constantly to clarify the purposes served by an evaluation. Key questions to be asked as part of any evaluation are what actions will be taken as a result and what will be the effects on the participants in the process. One should give special attention to the balance between the individual's right to develop uniquely as a student or teacher of mathematics and the society's need for citizens with mathematical competence.

An example of an issue involving evaluation is the alleged 'failure' of the 'new math'. Arguments on this issue have been heated, but firm empirical data have been scarce. Typical evaluation studies of new curriculum programs have cast the programs as 'treatments' in imitation of experimentation in medicine or agriculture. Evaluators have had to face the paradox that large numbers of teachers, and therefore classes, are required in order to separate program effects from teacher effects, but the larger the sample of teachers, the harder it is to determine whether each program is being implemented according to specifications. Few evaluation studies of 'new math' programs have had an impact on decision making, which may be just as well in view of the questionable validity of the studies.

Another issue involving evaluation concerns changes in examination systems and testing practice. With the expansion of comprehensive mass education has come a shift in the role of evaluation from *selecting* pupils for further education to *allocating* them to alternative instructional programs. It has become more important to *diagnose* a pupil's achievement in mathematics than to *compare* it with that of another pupil. Consequently, tests and examinations have become more objective, more frequent, and more closely linked to specific educational objectives. Taxonomies and matrix models for specifying objectives have become increasingly popular.

Both the evaluation of new curriculum programs as 'treatments' to be compared and the use of a matrix to classify instructional objectives and test items can be seen as expressions of the dominant metaphor today for addressing questions of educational evaluation: the metaphor of the evaluator as engineer. In this metaphor, education is seen as a production process, with inputs and outputs, and the evaluator's job is to find out what programs and techniques give maximum output for a given input.

The engineering metaphor can also be seen operating in various local, national and international programs of assessment in mathematics. Education authorities have attempted to measure the 'output' of the educational system in a given locality by administering achievement tests to samples of pupils. The analogy between a school and a factory is not complete, however. When one measures a factory's output, the measurement process has no influence on future production. The measurement process in school, on the other hand, is an interactive process: teachers and pupils alike use tests and examinations as indicators of what should be learned.

The engineering metaphor has tended to limit evaluation to a search for 'effectiveness' - the effectiveness of a curriculum, of a set of instructional materials, of a teacher - under the assumption that such a quality exists independently of situational factors. This assumption has been seriously questioned, however, and alternative conceptions of the evaluation process have been put forward in recent years. For example, evaluation has been conceived of as medical diagnosis, as literary criticism, as art criticism, as anthropological study, as criminal investigation, and as legal argument. The role of the evaluator has been described as that of clinician, critic, participant observer, passive observer, detective, juror, and attorney, among others. The point is not that any of these alternatives is better than the engineering metaphor but that only by becoming aware of alternative conceptions can one see the limitations of any single conception.

Recent criticisms of taxonomies of educational objectives, for example, have questioned the value of separating content from process and have asked whether one can classify objectives (and associated test items) apart from the instructional system in which they are embedded. Similarly, one can question the idea of evaluating a 'curriculum' in any general sense if it is seen as being manifested differently in different instructional settings. There is an important distinction to be made between evaluating a curriculum and evaluating the activities and products of a curriculum development project.

Dissatisfaction with existing tests and examinations has led to experimentation with alternative forms of assessment. The last five years have seen promising work begun on the measurement of such poorly-defined but important contructs as problem-solving ability and creativity in mathematics. Attitude scale construction is also underway, with greater attention being paid to questions of validity.

Although many techniques have been developed for analyzing teaching processes, the evaluation of mathematics teaching remains at a relatively primitive level. Teaching competence is often evaluated indirectly and unsystematically - through one's credentials, degrees, diplomas, recommendations, and performance in an interview - rather than through direct evidence of what and how much pupils are learning from one's teaching. Attempts to show that certain patterns of teaching behavior lead consistently to improved learning have generally been unsuccessful.

The bewildering variety of materials for mathematics instruction have prompted requests that evaluators compare the effectiveness of different materials. But the notion of 'effective materials' - like the notions of 'effective teacher' and 'effective curriculum'-

neglects the situational variability of instruction; it assumes that effectiveness remains constant across situations, independent of such factors as the pupils taught, the methods used, and the school setting. A reasonable request, and one that is increasingly being addressed, is that evaluators produce descriptions of instructional materials with suggestions as to situations in which they might be used.

Most of the recent attempts to create new instruments for evaluating mathematics learning, attitude toward mathematics, mathematics teaching, and materials for mathematics instruction can be seen as efforts to provide a richer description of the dynamics of the teaching-learning situation. Evaluators are increasingly attempting to seek out causes as well as effects, to describe and conjecture as well as to measure. They are beginning to break free from their heavy reliance on standardized tests and numerical scores.

Many problems of evaluation in mathematics education can be redefined, and perhaps resolved more fully, if they are seen from alternative points of view. The engineering metaphor should not be rejected entirely, for it has something to contribute, and our discipline will not progress if we always discard old concepts without building on them. Rather than searching for one best approach, we should try to become aware of the value and limitations of various alternative approaches to evaluation problems.

B 2.2
DISCUSSION-SUMMARY BY THE COORDINATOR

1. Survey of the Activities

The work in section B2 was done on three days. On the first day, there was an audience of about 100 persons in the session. *A. Bishop*, as the chairman of the section, gave an outline of the program. For about one hour, *J. Kilpatrick* presented the main topics of his trend report. This report was followed by a short panel discussion, the panel consisting of *S. Avital* (Israel), *A. Bishop* (GB), *H. Brolin* (Sweden), *P. Buisson* (France), *E. Jacobsen* (Unesco), *J. Kilpatrick* (USA), *U. Viet* (FRG), and *D. Lind* (FRG) (coordinator of the section). After the panel discussion, the discussion was opened to the audience; later it was adjourned to the third day of work.

On the second day in a so-called 'poster-session', six short communications were presented in written form. The authors of the short communications were present to discuss their papers with interested persons.

On the third day, there was an audience of about 50 persons in the session. The work of the section started with *Y. Hashimoto* reading a paper on some open-ended problems, written by *S. Shimada, T. Sawada,* and himself. *W. Rouse* and *J. Wesson* presented their proposals for classroom observations. A third one written by *Ch. Bentley* could not be presented. The discussion of the two presented papers was integrated in the final discussion of the trend report. Therefore, in the following only a summary of the discussion on the trend report will be given, followed by a short description of the poster-session and a conclusion.

2. The Trend Report and its Discussion

The following is an account of the presentation of the trend report and its discussion. The discussion on the first and the third days will be summarized as a whole:

J. Kilpatrick started his report with a general description of evaluation and evaluation problems with respect to mathematics education.

He used the example of the 'failure of New Math' to make the concept of some evaluation studies dubious. After giving a summary on changes in examination systems and testing practice, he discussed the dominant metaphor in evaluation which he called the engineering metaphor. Because there is a growing dissatisfaction among evaluators with this metaphor in USA, other points of view being discussed in USA were presented. *J. Kilpatrick* criticized concepts of curriculum evaluation which are based on the view of effectiveness. The three last parts of his report dealt with the evaluation of mathematics learning, mathematics teaching, and the evaluation of materials for mathematics instruction. A part of the final report (being published in the UNESCO-report on new trends) on assessment programs was not presented because a working group on that topic had been established at the congress.

In the panel discussion, *E. Jacobsen* pointed out the fact that for many developing countries there is only one question considered important in evaluation: 'Does this mathematics course work?' The only answer expected is either 'yes' or 'no'. *P. Buisson* mentioned a growing use of tests in France, contrary to some trends in USA. Considering criterion referenced measuring which is often supposed to be used mainly in individual evaluation, he made the remark that this type of measuring is also used for selection. *S. Avital* asked for concepts with respect to minimal achievement and demanded attempts to measure higher level abilities. *U. Viet* agreed with J. Kilpatrick in his criticism of 'typical' evaluation studies, but called attention to a danger she sees: emphasizing the difficulties of evaluation too much involves the risk that nobody will try to do evaluation in the future. The coordinator called for observation techniques and observation schemes for gathering more information in evaluation without losing too much objectivity. *H. Brolin* took a position on oral examinations and deplored their diminishing use.

After the panel discussion, *A. Bishop* opened the discussion to the audience and guided the discussion very effectively.

Some remarks dealt with changes in the meaning of 'New Math' and the fact that many attacks on the New Math have been made by the public in various countries. A broad discussion arose on individual evaluation: there was agreement in assuming that tests are being used more frequently for diagnosis, but P. Buisson's remark was also confirmed. The discussion on the engineering metaphor confirmed its usefulness for some fields in evaluation, but asked also for alternative concepts such as those mentioned by J. Kilpatrick in his report. The contributions on the evaluation of mathematics teaching dealt with the lack of proven methods.

Besides some discussion for and against evaluation done by pupils, observation methods were considered in solving the problems.

After a discussion on evaluation research, showing clearly converse trends in USA and other countries concerning the use of tests, the role of open-ended problems in assessing higher level abilities was discussed. The final point of the discussion in the audience concerned problems of observation techniques and the objectivity of observations.

On the evening of the third day, the panel and the reporter of the section had a meeting where some aspects of the trend report were discussed, including the results of the discussion in the audience.

3. The Poster-Session

On the second day of section work, six short communications were presented in written form. Interested persons had the opportunity to discuss the communications with the authors.

Ch. Bentley presented a paper on a mathematics project in the United Kingdom on applicable mathematics. *Y. Hashimoto* and *T. Sawada* reported experiences with some open-ended problems. *D. Lind* presented a statistical test which solves the so-called 'power problem' in the case of dichotomous classification. *D. Malvern* introduced a new concept for the posing of complex problems, using hints. *W. Rouse* and *J. Wesson* presented a paper dealing with the observation of learning children in activity centered classrooms. *J. Schwartz* presented the project TORQUE which uses games and validates test items against games.

4. Conclusion

The discussion in section B2 resulted the following insights: In the United States overtesting is still practiced. Therefore we find a growing dissatisfaction among evaluators with tests in the USA. Other countries had no overtesting in the past and use tests more frequently. We hope that there will be no overtesting in these countries in the future. The use of tests for selection is somewhat diminishing, but it is still present. Especially in developing countries which do not have enough capacities in secondary schools, test designers are asked for tests with specified selecting properties.

There was agreement in the section for broadening evaluation concepts to overcome narrow empirism. The assessment of higher level abilities must be considered as a main problem; observation and interview techniques may serve as a good tool.

Evaluation of mathematics teaching is still developing and also makes observation techniques necessary.

In the field of assessment and assessment programs, the question of minimal achievement should be emphasized more in the future.

B 3
Overall Goals and Objectives for Mathematics Teaching (Why Do We Teach Mathematics?)

Reporter: U.d'Ambrosio, Brazil
Chairman: W. Servais, Belgium
Coordinator: E. Wittmann, FRG
Short Papers: S. Hill, USA; G. Matthews, GB; P. Mbaeyi, Nigeria; A. Tarp, Denmark; D. Wheeler, Canada

B 3.1
SURVEY-ABSTRACT BY THE REPORTER

In the paper we discuss the overall goals and trends in mathematical education with a very significative subtitle: 'Why teach mathematics?'.

The paper, due to obvious limitations of space and time to prepare it, cannot be but an introduction to our approach to the theme. It raises issues and refers to sources where our approach finds its supporting basis, as well as to sources for a more conventional treatment of the subject. We hope the paper generates positive anguish among mathematical educators, stressing our responsibilities and confidence in the power we have, as educators, in shaping the future.

We focus the question of the subtitle from the point of view of development of society, of school systems, of teacher education, as well as of the changing views of the nature and role of mathematics, and on the nature of the teaching-learning process. We refer to the role of mathematics for the individual, keeping in mind his place in society. We stress the relation between general education goals and the goals and objectives for mathematical education, dealing with them within a general context, and placing them in different models of societies and various stages of development.

The paper has five chapters:

(1) *Introduction*

(2) *Mathematics and school*

(3) *The role and nature of mathematics*

(4) *What does society expect of mathematical education?*

(5) *Again mathematics and school*

plus a final part on *References and Bibliography*, listing about 100 items and identifying sources.

Instead of concentrating on specific objectives and goals for mathematical education, we opted for an analysis of the place of mathematics in the present and in the future, drawing many conclusions by the interpretation of the past, and its necessary implications on mathematical education.

We recognize a total interaction between science, in particular mathematics, and society, not only from the historical perspective, accepting mathematics development as a result of socio-economic factors, but also accepting that mathematics, though less apparently and immediately perceived as in the case of natural sciences, affects in an ever increasing degree the whole pattern of thought, culture and politics. We take into account a rapidly changing educational pattern all over the world, mainly due to the rapid concretization of the ideal of mass education. This is inserted in a context of pursuing the concretization of the ideal society, which though taking different forms and shapes keep an entire agreement on some of the basic issues, which include the tolerance of a minimum disparity between power and privilege. The resulting conflict between educational growth and educational disparity can be summarized into three different philosophies underlying the entire study of objectives and goals for mathematical education:

(1) the search for values;

(2) the pursuit of new knowledge;

(3) the support of a designated social structure.

It is a mere palliative to improve the existing educational model without a global analysis of school as an institution and science as a needed activity and an element in improving the quality of life.

The definition of priorities in education and in science poses major problems for developed nations and gigantic problems for developing countries. We accept education, in particular science and mathematical education, hence scientific research structure, as a socio-cultural phenomenon, anchored on cultural values; therefore its aims have to be analysed in the global context of national priorities and goals and to depend less on the intrinsic structure of science itself. On the other hand, the political and economic structure of the world, and the growing interdependence of the various countries, nations and societies, make it absolutely necessary for emerging countries to close the so-called technological gap with developed and industrialized countries, and develop a sound and capable intelligentsia. To conciliate the goals, essential for emerging countries as well as for smaller developed countries, of engaging in a dignified and honest way of relationship with bigger developed countries, and at the same time preserving traditional socio-cultural and moral values, seems to be the major problems facing educators all over the world, both from developed and developing countries. In this context we have to analyse science, in particular mathematics, and education, in particular mathematical education.

We claim mathematicians share the responsibility of bringing better days to fellow human beings. But some see in the very nature of mathematics a danger as an oppressive tool. This danger, present in developed and industrialized societies, is of concern for countries which, having achieved political independence, are now beginning to struggle for economic and cultural independence. This concern must be associated to the inadequacy of a structure of education, designed for an elite and now being applied to the mass of the populations. Probably, in understanding this fact resides the key to the preservation of mathematics as an important subject in school, specifically in reassessing its position as an independent subject in general education and inserting it into the broader concept of culture, of more value than the mere knowledge associated with it.

We see the educational process as the conjugation of global socio-economic aspects aiming at the betterment of the quality of life. In this conjugation intervene the same as in the technological process, the philosophy to which society subscribes, as well as considerations about manpower and available material resources.

We distinguish several forms of education: a purely vital education, instinctive, through which children learn how to survive, and to continue the species; a societal or behavioural (in the lay sense) education, in which children are taught basic attitudes and conduct, and acquire moral values; an artisanal or professional education; and a contemplative or speculative education. We concentrate our analysis on the last two. In fact, we identify instances in which the two forms join efforts and new knowledge is created. Mathematics is a good example of this link between the two patterns of education. The very beginnings of numbers show a strong practical component as well as reflections on the purely contemplative aspect. Historical considerations lead me to displace counting as a major component of mathematical education in the context of a contemplative or speculative scheme. Although taken for granted as part of theoretical mathematics in western culture, and somewhat misplaced in medieval schools, ordinary arithmetic as the core of mathematical studies has to be reexamined.

The development of a new economic order towards the end of the Middle Ages, with the growing importance of towns, trades and industry, forced a deviation from the economy of feudalism. At the same time, the basis for the industrial revolution was laid, heavily based on the springing up of the new science, experimental and quantified. Another important factor was the opening of the frontiers of the world through navigation and conquest, introducing new and cheap materials into an economy geared increasingly toward the processing of those materials.

In the period going from the Middle Ages through the nineteenth century, schools went through radical changes. We see the appearance of universities and at the same time the development of what we have called artisanal or professional education, resulting from a form of professional guild in a parallel structure. In what refers to mathematics, its place in education of this period is quite weak.

We thus reach the end of the nineteenth century, and enter the twentieth century with strong motivation for mathematical research, derived from many sources. Much of what dominates mathematics research today can be traced back to last century. More than significant new ideas in mathematics, we see the richness of new fields of applications, anticipating a fast and profound change resulting from the use of computers by mathematicians. In the same vein, we do not see significant changes in schooling. There is a major change, which is mass education, but the approach to education is practically the same as it was in the last century, always focused on 'how much children learn', and seeing that 'children behave according to a certain pattern'. There is a lack of emphasis on creative work.

Paradoxically, there is a profound richness of new directions which science and society are taking. No doubt we are living through a second scientific revolution, with news fields of research being opened; new tools for the understanding and control of nature, both in smaller and larger dimensions than man has ever thought, are being created. We cannot see mathematics and mathematical education unrelated to these new directions, as well as to social and economic changes occuring in the world.

The main objective of mathematical education which we advance throughout the paper is to develop the ability of individuals to identify mathematics in intellectual experience, and distinguish mathematical reasoning and method in all the situations where they are either present or can be potentially inserted. Mathematics is placed within the ample context of scientific reasoning and method, and related to other languages which try to simulate a reality. The process of mathematical creation proceeds like any form of creation, as the result of some internal force which we cannot identify while the object of mathematics is a combination of sensorial experiences with some process of abstraction and correlation of ideas. We recognize a trend towards a new relationship between mathematicians and non-mathematicians, and as a result the appearance of new foci of interest for research mathematicians. This new relationship goes both ways and the appearance of completely new scientific fields, bearing mixed characteristics of mathematics as we know it today, and of other desciplines is quite predictable. The role of computers in those new scientific disciplines will probably be a major one.

The implications of this reasoning in defining the objectives for mathematical education are of considerable magnitude. Certainly, the emphasis should be changed from what is taught, that is, from curricula, syllabi and contents, to methodology and to a new classroom environment, both geared toward creativity. The main objective is to give mathematics a new dimension that will fit a new world and societies which are closer to each other, and yet confronting each other. We need new mathematics, and all the creative power of youth to generate new forms of thought which we cannot envisage. The primary objective of mathematical education should not be the perpetuation of knowledge, or to push a little further existing knowledge. This knowledge will go on or fade away *per se*. But the primary objective of mathematical education is to foster the creation of new knowledge. At the same time, the need for making existing knowledge available to a growing number of professionals - and this affects in particular developing countries - should be provided in a parallel structure, basically not differing from what happened in the Middle Ages through the industrial revolution. A major pedagogical effort should be promoted to bring to a larger number of individuals and to be made available to developing countries, through educational methodology, using audio-visual and computarized technology, advanced mathematical theories. This cannot be achieved through pre-structure curricula. A dynamical and flexible approach to contents is needed with considerable emphasis on methodology of access to existing knowledge, in a readily accessible, understandable and applicable form. This approach responds more immediately and realistically to the needs of developing countries, as well as to the changing role of mathematical education in the changing pattern of developed and industrialized societies.

We believe a sound strategy of reform is to implement it in all levels simultaneously. Certainly, changing the attitude of teachers and moving the focus of teaching from what is taught to the learner, is an essential preamble to any improvement of education. A realistic and dynamic approach to change means, in its essence, a built-in mechanism of change, in which teachers and learners are brought into a single organic component.

With a more dynamic and unbiased attitude, we believe mathematics can play an important role in education, and contribute to the fullfilment to the primary, essential and undisputed goal of education which is to enhance the betterment of the quality of life of mankind.

B 3.2
DISCUSSION-SUMMARY BY THE COORDINATOR

The work of the section was structured as follows: A three hour session on August 17, 1976 and a two hour session on August 20, 1976 (about 300 participants), followed by a discussion between the reporter and the members of the panel.

The center of the session was *U. d'Ambrosio's* report; the above summary of this report had been circulated with the congress documents. The reporter's exposition was hinged on the view that mathematics teaching is facing a gigantic challenge both from the intellectual and the organizational points of view. In his opinion, the traditional reflections on the general goals of mathematics teaching are no longer sufficient; the problem needs to be analysed with a much greater philosophical depth than as has hitherto been the case. The speaker was especially concerned with the imbrication of mathematical education in the general context of education and the school and with demands critical of society; referring to historical facts, he also treated the interrelationships in the development of mathematics and society. The speaker demanded that mathematics teaching be made to serve the purpose of a higher quality of life. He finally sketched a conception of mathematics teaching that would rather emphasize a creativity-oriented organization of teaching and the corresponding arrangement of the classroom than curricula, syllabuses and content, and that would use modern teaching technology. The members of the panel made short statements on different points of the report. *Christiansen* discerns a trend towards a conscious concern for affective and socio-integrative goals in mathematics teaching. He considers the classification and taxonomy of goals, which had been left out by the reporter, as a first necessary step towards a comparative and critical examination of the problem. *Härtig* suggests distinguishing between possible and realizable goals and considers it dangerous to pose questions that are too general (e.g. what is mathematics? what is music?). *Higginson* considers reflections on the role of mathematics in human nature and culture as very important. With respect to the former, mathematics seems to him to be a kind of miracle; it seems to be a power with respect to the latter. *Higginson* recommended considering the relation between human culture and mathematics from the standpoint of a neo-pythagorism. Mathematics is, in his view, part of culture and need not be artificially planted. *Shirley Hill* sees a gap in the title and the subtitle of the report and considers it dangerous to treat the question of the goals of mathematics teaching solely at a philosophical level without reaching the curriculum. *Hilton* reported on the allegedly over-privileged role of research mathematicians and on the demand that mathematical research be increasingly subordinated to social necessities. He remarked that it is but natural that a mathematician does research freely and he thinks that there is no better solution. The developing countries should also further mathematical research, which would then work as an 'élan vital' in the chain 'good mathematician - good teacher - good teaching'. The president also reported on the nature of mathematics. He describes mathematics as 'rational thinking which has become operative'. Mathematics is simultaneously beautiful and useful (knowledge as power). *Suranyi* considers the attempt to describe mathematics with precision as a desperate endeavour. Only a few aspects can be made out, and, for the rest, mathematics goes its own way. Mathematics must be taught just because it is part of culture. *Touré* made an impressive report on the difficulties encountered by education and schools in his country, which, as was confirmed, are representative of the developing countries. In the developing countries mathematics teaching is still rudimentary. The big problem is the education of their own teachers.

After the statements by the members of the panel, the audience was called on to make contributions, which mostly concentrated on the problems raised by mathematics teaching in the developing countries.

The second session on August 20, 1976 was essentially devoted to short reports:

1) *Matthews* (GB): 'Why Teach Mathematics to Most Children?'

2) *Hill* (USA): 'Overview and Analyses of School Mathematics in the USA'

3) *Tarp* (Denmark: 'On Different Perceptions of Mathematics'

4) *Wheeler* (Canada): 'Towards the Humanization of Mathematics Education'

5) *Mbaeyi* (Nigeria): 'Arts vs. Technique in the Achievement of Goals and Objectives in Mathematics Education?'

The well-known Russian mathematician *A. Kolmogorov* reported on various goals of mathematics teaching in different school types (general education schools, special schools for students with a gift for mathematics, evening classes and summer classes). Before the general discussion the coordinator commented on a paper in which he had proposed a framework for treating the goal problem in mathematics teaching. He is convinced that it cannot be the business of the pedagogy of mathematics to decide on goals; it should rather prepare a basis for these decisions. But of course specialists in mathematics education should participate in making the decisions about the goals, which are basically political decisions.

In the closing discussion *Håstad* examined in great detail the important question of how the further contribution of specialists in mathematics education to the analysis of the goals of mathematics teaching should be designed. The golden 60s of pedagogy of mathematics being past, there is a need for more scientific approaches and meticulous documentation. For this purpose, the goals should be systematized and subdivided, priorities should be examined in relation to the financial possibilities, concealed goals should be identified. In this connection we are, as was said, at the beginning of a new development, and we may still be in the dark.

The question of an adequate methodology was also at the center of the closing discussion between the members of the panel and the reporter. We discussed possible ways of structuring the goal problems and suggested that particularly important research problems be identified.

B 4
Research Related to the Mathematical Learning Process

Reporter: H. Bauersfeld, FRG

Chairman: L.D. Nelson, Canada

Coordinator: H. Skowronek, FRG

Short Papers: N. Branca, USA; M. Brown a.o., GB; E. Pavelka, USA

B 4.1
SURVEY-ABSTRACT BY THE REPORTER

1. Introductory Orientation

For a long time research outcomes have influenced the reality of mathematics instruction and mathematical learning on a very small scale only. Research has followed the needs of school practice rather than hurrying on ahead. Recent indications suggest a change in this deplorable state of affairs. Within the last decade research interests have switched over from the curriculum and from the pupil to the teacher. To some extent this is a reaction to the follow-up problems of the first big wave of curriculum development. But it is as well an answer to problems of generalizing from research. Accordingly, isolated curriculum theories and learning theories have been increasingly put in doubt, while the development of more complicated teaching-learning theories has been favoured. The latter care about the teacher, and moreover, they take care of the interactions between pupil, teacher, and curriculum too.

How did the change come about? Until recently, both research and development had focused on only one of two main determinants of the learning process: the pupil or the curriculum. But they did not consider the influence of the teacher nor of the general context of instruction. Theory-related research has particularly focused on the pupil. Since the 1920s researchers have tried to identify stages of the pupil's learning and investigate the conditions of his success (experimental psychology), and to describe the developmental stages of his thinking (developmental psychology), aiming towards a general learning theory. School-related developmental work of the 1950s and 1960s, on the other hand, was engaged in the curriculum in a wide sense. Mainly by means of projects, pupil's books, learning aids, and other media were developed, as well as an analysis of the structure of the subject matter and teaching methods related to it. Both movements were characterized by a neglect of the teacher's role and of the general context of learning. Consequently both run into increasing difficulties. Neither have the big curriculum projects improved mathematics instruction and mathematics learning on a broad scale, nor has research developed a valid general learning theory. Only competing explanations for partial views of learning have been developed, as e.g. Piaget's genetic theory of epistemology, Gagné's hierarchical model, or the Gestalt theories. These positions cannot be integrated and represent generalizations of only limited value for explanation and prediction. In particular, these are not sufficient for the planning and realization of classroom practice, as school practioners have complained. More sophisticated theories of teaching and learning seem to be necessary.

The analysis of the possible causes for the poor success and the different assessments of the analysis have produced quite different reactions and new approaches within the last few years. Thus the research scene has changed noticeably. This is the genuine concern of this report. For the identification of the general characteristics of relevant research within the 1970s my main criterion was the potential relevance for changes in school practice and the conclusions derived for future research.

2. Main Trends

The first five points characterize goals and methods of research in the social sciences related to mathematics learning. The following five are concerned with the teaching-learning process.

2.1 An immense number of previous research studies tried to produce statements about the relationship between two variables from representative samples, e.g. mathematics achievement as related to sex or the comparison of two treatments. The correspondingly immense number of contradictory or non-significant outcomes has led to more differentiated questions such as the investigation of interactions: Does it make a difference in mathematics achievement when a male teacher or a female teacher teaches a male student (or a female student)? Or, is treatment A more effective with highly-motivated students and treatment B for poorly-motivated students, or vice versa? Data about such interactions give better hints for matching treatments and groups of students, though for limited situations and for relatively elementary content only. In the last few years interaction has been interpreted as social interaction, and not only as a statistical interaction among variables, and this has caused an important improvement of our insight into the teaching-learning process. At the same time the isolated view of the main determinants has ceased.

2.2 Many research outcomes drawn from the isolation of a laboratory context are not generalizable, and standardized tests are only of limited value for the interpretation of process: both of these factors have contributed to an increase in research studies of real classroom situations. Systematic observations of mathematics instruction, particularly long-term case studies of learning processes in natural environments, have produced remarkable results. Soviet psychology of education especially has concentrated on teaching experiments, by which the learning of mathematics is studied under variations in the conditions of instruction.

2.3 The focus of research interest on the teacher who is no longer recognized as an 'intervening variable' in order to have a clean investigation of the effects of the treatment or of the students' learning, has increased the participation of teachers in research. In particular, the French IREM (Institut de Recherche pour l'Enseignement des Mathématiques) which have made a principle of the teachers' participation for reasons of innovation strategies, have accomplished pioneer work in this area. Similar work is being done at the IOWO (Instituut voor Ontwikkeling van Wiskunde-Onderwijs) at Utrecht, Netherlands, and in many of the English Teacher Centres, as far as they have specialized in mathematics education.

2.4 One of the most conspicous reactions to the meager relevance of educational research for practice and to the related problems of constructing theory, has been an extension of the repertoire of research methods, which on the one hand announces a loosening of the hitherto rigid standards for the acceptability of methods, but on the other means that the integration of outcomes will become more difficult. The earlier non-acknowledgement of Piaget's informal research and the receding wave of formal follow-up studies of his statements support this. In the USA, where about 85% of the research studies related to our theme were produced, the information-processing approach with its more open methods and simulation techniques is displacing learning theory and the struggle between cognitivists and behaviorists. The methodological opening has caused increased attention to the descriptional systems, the 'philosophies' that are fundamental to the different research approaches (see 2.10).

2.5 Consequently, a strong need for theoretical orientation has emerged, whose power indicates a kind of crisis. The procedure of studying intervening influences from certain studies as main variables in subsequent studies is very common in the social sciences. But rather than integrating and concentrating, it is differentiating and ramefying our knowledge. Increasing understanding of the complexity of the learning process creates an increasing complexity of understanding, which is obviously difficult to deal with. For economic reasons alone, it is impossible to increase the number of variables in a study ad libitum, but it is equally impossible because of the interactions to keep all the other variables constant or just to neglect them. Additionally, in a rapidly changing modern society, the validity of many research outcomes is limited in time (the historicity of statements). So substantial doubts have arisen as to whether it is even possible to construct general theories along the ways favored in recent research. Thus the situation may be characterized by an intensive search for a frame of orientation - be it a theory or just a functional regulation of the research processes - a search in which many single researchers and groups at larger research institutes are engaged, e.g. at the Shell Centres for Mathematical Education in Chelsea and Nottingham, at the Georgia Center for the Study of Learning and Teaching Mathematics in Athens, and at the Institut für Didaktik der Mathematik (IDM) at Bielefeld (see also 3.3).

2.6 The learning of mathematics requires more than the availability of rote knowledge. This 'more' can be described as meaning, as understanding or insight, as adaptation to reality, etc. Research shows that this 'more' is employed in the classroom in at least three forms: as the structure of the mathematical discipline, the 'matter meant'; as the content of the teaching process shaped by the teacher's learned structure and routines, the 'matter taught'; and as the cognitive structure of the individual student, the 'matter learned'. These three forms coincide in the ideal case only. What does that mean for teaching and learning?

2.6.1 We have to abandon the image of the unchangeable subject mathematics which is passed on to the student by the teacher. On the contrary, the subject matter varies and changes in the course of the teaching process and as well in the individual learning process. By the way, this also supports the relevance of theories of social interaction for mathematics education (see 2.9 and 3.1).

2.6.2 The insight into the individuality of the structures and their development in the teacher and in the student is a strong argument against the search for *the* best treatment. Instead, the question may be put: Optimal for which students and under which accompanying conditions? Direct answers to this question are somewhat rare and are still in need of pragmatic interpretation.

2.6.3 Furthermore, the students develop interpretations and strategies on their own, which may remain hidden for a long time, although active, if not detected by means of a mistake or an unusual solution. The extent to which the teacher can adapt to such situations characterizes the quality of teaching. For improvement, a long-term diagnosis and assessment of the student's development is required, particularly since the time required for learning increases with the complexity of the subject matter structures and the rules for their generation.

2.7 Because of difficulties with the formulation and hierarchization of objectives and because of difficulties with the validity of related tests, doubts have arisen as to whether the different models for teaching content - this means the modes of representing and embodying the 'matter meant' - really transmit the identical content. There is good reason for the assumption that different models can bring the student to the construction of concepts that differ in range and content and that appear identical only because of uniform symbolical description and of normed technical language. This is shown clearly through poor generalizations and through difficulties in solving application tasks. Since the lower a student's achievement, the more he depends on the material and its connotative background, disadvantages are caused just for those students for whom embodiments are made. Particularly in pre-school and primary education this observation makes necessary a revision of so-called introductions or fundamental experiences and suggests the development of compensating aims for a more complete understanding. It also makes necessary a critique of achievement testing.

2.8 One cannot expect sound statements about the learning process without a proper elucidation of what has to be learned. Many empirical investigations give the impression that mathematics is an enduring body of knowledge, a hierarchically ordered set of fixed statements and rules about concepts, which are related to one another like knots in network. Mathematicians probably will not agree with this narrow picture of mathematics as a theory of static structures; instead, they may appreciate the more dynamic metaphor of 'mathematics as a process' (Freudenthal) or 'mathematics as a language' (Papert). No doubt such fundamentally different images of the 'matter meant' will influence the 'matter taught', this means that they will produce different forms of mathematics teaching. The problem gets more complicated if one takes into account the different fundamental philosophies that mathematicians have about their own subject matter (e.g. intuitionists vs. rationalists), which stands in opposition to popular ideas about mathematics. In the empirical sciences there is still not much reflection about these differences. Some evidence is to be found in more informal studies, from which it is rather clear that, indeed, different views of mathematics generate a different comprehension of e.g. (mathematical) generalization or abstraction, or what comprehension of a concept may mean, or what is meant by a sufficient solution to a mathematical problem.

2.9 On the other hand, it is impossible to think about research related to the learning of mathematics that is not shaped by a fundamental view of the learning process, or of the teaching-learning process. This is, at least for the learning of mathematics, not just a question of different descriptional systems which are applied to the one identical subject matter. Since at school, mathematics is a product of communication (a variation of a statement by Watzlawik on 'reality' in general), the product is deeply related to the structure of this social or communication situation. Interest in clarifying this dependency has increased very recently. Easley has presented a sophisticated study of seven modelling perspectives on teaching and learning, and Joyce has described four different families of models: 'social interaction models', 'information-processing models', 'personal models', and 'behavior modification models'. Since the models differ in their view of man, the nature of his knowledge, and his learning, they indeed define different issues of research and of education. By the way, this is an important and essential reason for the non-integrability of research outcomes on the 'same' theme.

2.10 Though the process of mathematics learning cannot be described sufficiently from the view of the learner only, there is an increasing interest in the conditions for the success of students' learning, particularly in the structure of mathematical abilities and the rules of their development. This theme comprises the relatively largest number of research studies, among which are hundreds of follow-up studies to Piagetian experiments (mainly concerned with conservation). Attempts to split up mathematics ability, which in tests is often treated as a unit, differ in their outcomes according to the approach used: a re-analysis of factor-analytical studies produces models of abilities similar to well-known models for the structure of intelligence; case studies using fixed sets of tasks with high achieving students lead to generalized mathematical processes. Hitherto the results are not detailed enough to provide a more reliable diagnosis for ability grouping in mathematics education, but they do give aid to the improvement of individualization.

On the other hand, a large variety of variables influencing mathematics learning has been investigated, such as the previous learning experiences of the student, his self-esteem and self-concept, his styles of perception and information processing, the reinforcing behavior of the teacher, and so on, to mention a few of the most important ones. Several of these variables turn out to be rather stable during school, but surprisingly, others are rather unstable, which means they are dependent on the learning situation and on the interacting persons. The results of this research support scepticism concerning the usual practice of streaming according to achievement measures, since such streaming might cause a fixation of unstable characteristics. Additionally, important consequences have to be drawn from this research for pre-service and in-service teacher training.

3. Critique and Conclusions

The critical assessment of developments during the last few years and orientations for future work deduced from it are grouped into the following three dimensions, which characterize aspects rather than strictly separate issues.

3.1 *The social dimensions of the mathematical learning process.*

Here three aspects are to be stressed, which need thorough analysis and further investigation:

3.1.1 If it is true that mathematics can be learned only through dialogue, then the rules for communication situations, and more generally for social relationships between persons, are valid for these processes also, and this presumably applies to issues specific to the subject matter. From this point of view, the learning of mathematics as the learning of meaning requires the 'negotiating' of meaning for the student. In order to prevent possible misunderstanding: the point is not that students share the decision as to what is mathematically 'true', but it is that only in communication with others can the student test the appropriateness of his construct and correct 'his' concept. To a large extent social situations are determined by the preliminary experiences, the value systems, and the expectations of the persons participating in them. Therefore, rather than perceiving objectively what a student does, the teacher sees it through the pattern of his expectancies and through the attributions of causality he makes. The same holds for the student's relationship to the teacher. Careful further investigation is needed to derive implications for the individual development of the student, for the assessment of his achievement, for ability grouping, and so on. That 'specificity to situations' is used increasingly as an explanation for contradictory findings or non-significant results in research studies confirms the importance of this point of view.

3.1.2 Without doubt the intensive discussion of the cultural bias of tests is of concern as well for the accessability of mathematical concepts. Numerous reviews of experiences with the introduction of 'new mathematics', particularly in developing countries, give details. Positive and negative interference in comprehension is caused not only by differences among languages in grammar or semantics but also by the general context of different cultures. Such insights have raised the sensitivity of researchers in developed countries about the possible interference with learning caused by different social background. This refers to the whole range of possible differences: from general patterns of behavior to the connotative meaning of concepts. Thus, a sociology of knowledge and of the relevant strategies of behavior in mathematics education has to be developed.

3.1.3 The problem of a proper mathematical development and education of the individual student will not find adequate solutions until we have careful analyses of the causes of individual differences, develop diagnostic aids for the teacher, provide corresponding preparation during the teacher's training, and draw conclusions for school organization.

3.2 *The content dimension of the mathematical learning process.* Here one must point out the deficits indicated in 2.6,7,8.It seems necessary to specialize the investigation of learning processes. This means that we have to care about how certain concepts are being learned - instead of generating general theories of concept formation. We have to find out about the function certain concepts have for certain students in sufficiently determinded situations, and about the relations and meanings that are developed from it in a long-term view. New ideas are required for the description of the different forms or developmental stages of a concept, particularly for the description of the 'matter taught' and the 'matter learned'. The same is true for the methods of investigation, which should not be restricted to a replication of available routines. In this connection it might be important as well to investigate the intuitive and usually not reflected patterns of teaching and rules of interaction that successful teachers employ.

3.3 *The theoretical dimension of research.* The precariousness of the situation in the empirical social sciences seems to be near to a crisis and has led to the assumption that, in Kuhn's terminology, a change of paradigm is going to occur. This is especially true for psychology. Research methods that mainly follow the paradigm of the natural sciences are being doubted. But the difficulties are probably not so much with the methods employed as with defining the area for their adequate application. The outcomes of an investigation are always determined in advance by the fundamental theories on which the hypothesis have been developed. The weaker the theoretical basis of empirical work, the more severe the difficulties it encounters and the less it can lead to new theories. If a change of paradigm on the way - and much suggests it - then it will be a fundamental change in the understanding of the object and aim of research: The behavioristic paradigm has run into difficulties due to its definition of man as driven by unknown forces, as reacting predominantly to external stimuli, and as an object whose behavior can be determined by control of his environment. This view, for example, admits self-reflection as a hypothetical construct only, as a mentalistic intervening variable. The opposing view is of man as a reflecting subject, using his subjective theories for the explanation of his environment and deducing his activity from these explanatory theories. In contrast to the behavioristic models, this model is applicable to itself: The non-symmetrical subject-object relationship is abandoned in favor of a principally symmetrical subject-subject relationship. The subjective explanatory theories can be criticized in terms of psychological rationality (e.g. theory in scientific terms), and vice versa. That this symmetry formed a key characteristic in the European tradition of hermeneutic psychology might be stated here with some satisfaction.

Consequences can easily be drawn but will take time to be realized: The construction of theories related to the teaching-learning process has to include the knowledge of all of the participants - teacher and student - about themselves. Theory therefore is possibly only as (meta-)theory about subjective explanatory theories. This can be understood as a main orientation for future research work.

4. Prospect

A review of the more than 3000 research studies relevant to the theme leads one to the conviction that mathematics educators - being transmitters between research and school practice - cannot restrict themselves to selecting from the rich body of knowledge produced by the relevant sciences and to interpreting these outcomes for practical use. On the contrary, by defining their own questions and problems considerable deficits appear, a shortage in the midst of abundance. Furthermore, since relevant research in other disciplines is conducted under aims and interests that are not directly aimed at the promotion of mathematics learning, and since it is difficult to get a discipline to adapt its research strategies for use in other disciplines, mathematics educators will have to develop their own approaches. Because of their transmitting function they will not be able to restrict themselves explicitly to the stringent methods of sociometrics, nor will they be allowed to get lost in the lack of control and non-replicability of pragmatic action. Research and development within a discipline of mathematics education therefore will have to develop its own standards and methods. This means that an adequate precision will have to be developed which on the one hand makes the outcomes accessible to the practitioner without further interpretation and on the other hand is open to scientific control.

But the overestimation of mathematics education as an omnipotent trouble-shooter has to be avoided. Mathematics educators and teachers have to learn:

(1)How to cope with a highly complex reality, i.e. with a variety of explanatory theories that are powerful only in certain respects, but that are not integratable or that are even contradictory.

(2) How to work with a variety of general descriptive systems (ideologies or philosophies) that allow optimal description only in certain respects in spite of their aspiration to totality, but that do not work well in other respects.

(3) How to deal with a variety of methods for research and for teaching that rarely match, but that are as liable to historicity as the descriptive systems and the explanatory theories.

What is legitimate for researchers - that is, embedding one's investigations in one and only one system (philosophy, theory, method) because scientific truth is only in such a system - and what is an unavoidable necessity for the teacher - that is, the one-sidedness of his actual descisions in a given context and under the severe pressure of time - neither is allowed for the mathematics educator in his transmitting function. He should not reduce complexity by following current modes of research or public opinion, nor should he get lost in a poorly oriented pragmatism. He has to develop his own self-concept.

B 4.2
DISCUSSION-SUMMARY BY THE COORDINATOR

The comments made in the discussion were all centered on the general assessment and trend descriptions contained in the research work presented by the reporter. Virtually no mention was made of details of content. Particularly the belief that research on mathematical learning processes has reached a turning point or is going through a fruitful crisis elicited a wide response. A first consequence was that many assessments underlined and explained the deficits of current research approaches.

(1) It was stated that the orientation to behavioristic research paradigms borrowed from the natural sciences, i.e. stressing manifest behavior, results in insufficient 'theoretical' efforts; that, paradoxically, the intelligence is prevented from reflecting upon its own activity, which is in fact the essential achievement of the construction of mathematical concepts. Numerous studies on mathematics learning are mostly statistical in character and accordingly focused on average achievements and learning results, not on learning processes, thus tending to conceal the decisive recognition that mathematical 'insight' or mathematical understanding may be achieved in *individually* highly different learning processes.

The study of cognitive structures, as done by Piaget and his research team, is relevant to the elucidation of mathematical learning processes. But so far, these studies have only proven that corresponding structures exist in a static sense, as general potentialities. However, they obviously cannot be automatically applied to concrete mathematical problems. There is then a need for a study of the conditions under which pupils learn how to productively transfer general cognitive schemata to specific mathematical objects - possibly in individually highly different learning processes.

In other words, we should *not* start fom a general and content-neutral theory of learning and derive from it a theory of mathematical learning. More efficient contributions to improving teaching are more likely to emerge if we concentrate from the start on content specific learning processes, i.e. in the teaching of mathematics as far as we are concerned.

(2) However, an elaborated theory of learning processes in mathematics cannot be expected to materialize in the near future. There must be first a broad description of individual acquisition and understanding processes. The apparently most fruitful stages are those which can be identified as 'leaps' or discontinuities, or whose purpose it is to overcome former learning patterns. Individual heuristic strategies also deserve special attention. Not until a comprehensive description of the individual acquisition processes operating in teaching has been provided, will it be possible to make the first attempts at a formulation of a comprehensive theory of mathematical learning.

This programme still presumes great patience on the part of teachers, who are demanding a kind of research they can use to improve their teaching. Teachers should at least be informed about the difficulties and risks of research if we do not want them to lose patience. From a general point of view, it is fair to say that the differentiated mediation levels and mechanisms which are necessary for a direct exploitation in the classroom are failing. It was suggested as a first step that institutions like the IREM, the Institut für Didaktik der Mathematik and the Georgia Center adapt very significant research works to make them more easily accessible to classroom practioners and allow their use in schools. A further step could be a rotation of teachers in research institutions and of researchers in schools so as to gradually further more realistic problem definitions for research and better insights into the possibilities and limitations of research. In the manner of a 'realistic utopia', we could imagine the teacher as a classroom researcher examining in concrete teaching situations the specific problems raised by the integration

- of the content structure,
- of the cognitive structure of the learner,
- of the structure of the organization of teaching.

Undoubtedly, this idea of the teacher as a researcher represents a very excessive demand on the individual teacher in view of the present state of pre- and in-service education. This can only be achieved after research has met the above demands with respect to the concrete improvement of mathematics teaching and provided that teacher training allows teachers to acquire the research and development competencies that are necessary for this double role.

(3) On a less futuristic level, we may, however, already expect and promote, even now, precise effects of research and 'theory' on teaching practice. Research provides teachers with a background of insights in learning processes in mathematics, which prevents them in the long run from acting impulsively and thoughtlessly, which urges them to make more frequent pauses, to better observe and respond with greater sensitivity to the different 'signals' of pupils. Teacher education itself should give potential teachers opportunities to get some training in this observation and reflection work guided by theory.

Moreover, direct research on teaching activities and teaching strategies of teachers of mathematics can help reveal the typical weak and strong points of the 'natural' teacher behaviour and lead to adequate adaptations and corrections of existing training schemes. The participation of teachers in the development and pragmatic critique of teaching materials, the processing of case studies of individual 'learning biographies' in mathematics and the like can also contribute efficiently to raise teaching competencies.

B 5
A Critical Analysis of the Use of Educational Technology in Mathematics Teaching

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B 5.1
SURVEY-ABSTRACT BY THE REPORTER

The interpretation of educational technology as it is employed in this report will refer to '... the media born of the communications revolution which can be used for instructional purposes alongside the teacher, textbook, and blackboard'. The rationale for this approach is that the media of interest, including films, filmstrips and slides, television, and computers have entered education independently, and still operate more in isolation than in combination.

The *book* was the major technological advance of five centuries ago, and just as it had a revolutionary effect on education as it had existed up to that time, many educators feel that recent technological developments, particularly those involving computers, have the potential for equal or greater impact on the educational enterprise. The cause for such optimism, of course, is a direct outgrowth of the pedagogical attributes of the newer technologies, and a list of most of the important ones is given below:

(1) Media can do things - can create learning situations - that cannot otherwise be accomplished (like bringing current events into the classroom).

(2) Media can be used to present information in a variety of ways, which best meet particular learning objectives.

(3) By varying the medium, information can be presented to groups of many differing sizes - from individuals to national audiences - and it can be presented simultaneously to those differing audiences.

(4) Media can make learning more effective by increasing the realism, the dynamics, and the attitudinal impact of information; it can increase the motivation to learn.

(5) Some media, such as television, can make the best teachers and learning situations available to more students than would otherwise be possible.

(6) Media can extend common limitations of the learning situation by reinforcing and expanding the experience and background of teacher and student. In addition, they can extend limitations imposed by school plant and geographical location.

(7) Media can allow for individualization of curricula by presenting the same information, or variations, to different students at different times.

(8) Media can allow students to work under many situations without teacher guidance or supervision, freeing the teacher for individual assistance.

(9) Education can be made more efficient by using media to direct information to some people in shorter-than-conventional periods of time.

(10) Some educational objectives can be realized more economically by using media rather than by conventional means.

(11) The demands on our educational system require that every possible means of upgrading education be explored. The use of media forces educators to examine their goals and objectives more closely than before.

Each individual medium (technology) has its own special pedagogical capabilities and limitations, and so the educator must be aware of them in order to effectively plan for their use, either singly or in association.

The non-computer-related technologies of special interest are films, filmstrips, cassette tapes, video recordings, television and radiovision. The pedagogical attributes and potentials of these media are somewhat different. In the case of *films*, the major advantages are the prospects for increasing motivation, for providing students with experiences which they could not otherwise enjoy, for incorporating the illusion of motion through the technique of animation, and for making use of time-lapse photography and microcinephotography.

Generally speaking, however, films do not provide *responsive* learning environments; that is they are not built to accept, or respond to, student inputs nor are they designed so as to be capable of making 'real time' adjustments based on the reactions of the learners who are viewing them. The proper incorporation of films into classroom settings, therefore, necessitates considerable planning and preparation on the part of the teacher. So, too, do the problems of setting-up and operating conventional film projection equipment, and collectively these matters have served as somewhat of a deterrent to the wider use of films for the purpose of teaching. Fortunately, the development of the 8mm cartridge projectors during the 1960's has helped ameliorate the problem.

Mathematics teachers seem to make more use of *filmstrips* than motion pictures, a circumstance that can be accounted for in part by the fact that filmstrips are cheaper and easier to store, access, and use than films. Another possible reason for this state of affairs is that many topics in mathematics must, by their very nature, be developed in a particular sequence. When a filmstrip is being shown, each frame can be viewed or discussed as long as necessary, the teacher can stop and answer questions as they arise, and the strip can be backed up, if desired, to examine previous frames. Filmstrips are flexible in that they can be used with an entire class, or they can be used on an individual basis. Some filmstrips also are coordinated with sound recordings, a combination which provides for an even richer learning environment. Individual or groups of students can employ such instructional packages in much the same way as cartridge films.

Essentially, most of what has been said about filmstrips applies to the 35mm *slide* technology. While the actual storing and handling of slides is a bit more cumbersome than filmstrips, the slide technology does have the special virtue of permitting changes and additions to teaching sequences on an ad hoc basis. Though there has been a continuing development of mathematical films and filmstrips, there is a distinct trend toward the construction of *multi-media* learning packages, and a long-term prognosis would seem to call for their increased development and use.

Fundamentally, *television* enjoys the same basic pedagogical characteristics as motion picture films. The conditions and pedagogical considerations undergirding the use of television in the classroom are somewhat different from the use of films, however. 'Film is a medium that is used by the teacher and is under his control. Television, at a national level, is essentially a medium that uses the teacher'. The point is that the teacher, or school, that wishes to make use of a television program must arrange class schedules to conform to broadcast times, and generally must build other aspects of the instructional program around the broadcasts; thus, television imposes a more rigid structure on the classroom and the teacher. One major consequence of this fact is that there is a need for considerable classroom materials in support of television broadcasts.

On the surface, the importance of the visual dimensions of learning and using mathematics would suggest that radio technology would have a very minor impact on mathematics education, but the concept of *radiovision*, radio accompanied by slides for classroom use, is on the rise, and promises to be an effective means to reach students in far-flung areas who are out-of-reach of other broadcast or basic communications media. Indeed, there are a multitude of important television and radiovision projects now underway in various parts of the world, and there is good reason to believe that these media could play important roles in the improvement of educational systems in the developing parts of the world where rapid educational progress is sought.

The computer-related technologies of concern are computer-managed instruction and computer-assisted instruction. *Computer-managed* *instruction* (CMI) is one of the most rapidly growing uses of the computer in education. As the term suggests, this type of computer support utilizes the computer as a manager; student achievement assessment, management of classroom learning, student records, and information retrieval typically are considered to fall within the sphere of CMI. It is clear, therefore, that the CMI concept is associated with two of the major current pedagogical concerns of the educational enterprise, namely *accountability* and the *individualization* of instruction. It follows that to the extent that these concerns remain in the forefront of educational thought, CMI would be expected not only to become increasingly sophisticated but also to become more and more of a potent force in the conduct of classroom instruction and learning.

The term *computer-assisted instruction* (CAI) as it is used here refers to the employment of a computer as a teaching machine, performing the functions of *tutor*, *tester* and *exerciser* (drill and practice). The pedagogical features of CAI, therefore, are essentially those attributed to programmed instruction by the futurists of twenty years ago. CAI certainly can offer the student an opportunity to proceed on an individual basis - in terms of content, in terms of pace, and in terms of learning mode. It can offer the opportunity to break down the customary monolithic educational units which start and end at specified points in time and which everyone must conform to. It can offer a highly interactive (responsive) learning environment that is capable of making 'real time' (adaptive) adjustments in conformance with individual learning outcomes. It can offer opportunities to engage in sophisticated forms of acquisition of student performance data and their analysis, and thereby remove much of the record-keeping routine that often prevents a teacher from being able to attend to the actual business of teaching. Computer-managed instruction and computer-assisted instruction constitute educational ideas that are still in their infancy, however, and thus the many analyses and prognostications that are now being put forward concerning them and their potentials for impact must be tempered by this fact.

In conclusion, there are several noticeable trends in the development and uses of media:

(1) There has been a massive and continuing improvement in the 'hardware' associated with the various technologies. A few years ago, projectors or video-tape recorders, for example, were operated by audio-visual specialists or by a few adventuresome teachers, but rarely by the learners themselves. All this has changed. Most of the hardware now being used has automatic controls and can easily be operated by students or teachers. Computers are also gradually becoming more accessible for day-to-day use in classrooms.

(2) Trends in the production of audio-visual materials have paralleled those of equipment. Films, filmstrips and slides, for example, are now being produced in great numbers to fulfill specific instructional needs.

(3) The applications of instructional television continue to grow, and it is becoming commonplace for many countries to offer instructional programs on a national basis. It is to be expected that the developing countries of the world will adopt such a practice.

(4) Methods of instruction are tending toward a greater emphasis on personalized learning, which, in turn, imposes the necessity for a much greater degree of flexibility. It is in this regard that technology seems to hold its greatest potential.

In 1972, the Carnegie Commission on Higher Education published a report entitled 'The Fourth Revolution' in which the role of technology in education is explored. The title of the report was derived from Eric Ashby's observation that four great revolutions have occurred in education. According to Ashby, the *first* revolution took place when the task of education for the young was shifted in part from parents to teachers and from home to school; the *second* revolution was the adoption of the written word as a tool of education; the *third* revolution was the invention of printing and the widespread availability of books; and the *fourth* revolution was the development of electronics, notably radio, television and the computer, *though the computer is distinctly the imperative in the fourth revolution.* This is the idea of paramount importance in any analysis of the contemporary technologies; the computer with its vast potential for impact is center stage.

B 5.2
DISCUSSION-SUMMARY BY THE COORDINATOR

About 60 participants attended the plenary sessions. The panel included the following participants:

H. Fiedler (FRG), W. Fraunholz (FRG), G. Chastenet de Géry (F), R.T. Heimer (reporter; USA), J. Hunter (GB), D.R. Lichtenberg (USA), S. Schuster (USA), H. Stever (coordinator; FRG).

R.T. Heimer's report was structured as follows:

(1) A general view of technology and its import for mathematical education.

(2) A critical analysis of the individual technologies, their pedagogical attributes and trends in their uses.

(3) A synthesis and critical analysis of the past, present and future of educational technology in mathematics teaching.

The report was first discussed in the panel. The members of the panel made short statements along with critical remarks, additions and suggestions. No essential criticism was levelled at the general lines of the report. However, according to *Fraunholz* the following fundamental problem of educational technology had been neglected: How should/could the media be used in education, and where and when will teachers be trained to teach by using media? A contribution by *D.R. Lichtenberg* was concerned with the quality of the available teaching films. He estimated that at most 50% of the audio-visual media based on films could be used for teaching purposes. The question of the quantitative aspect of the use of such media was answered by the audience with a hint at a study conducted in 1974 by the 'Institut für Film und Bild' in Munich. Drawing on his own experience, *S. Schuster* analysed the reasons for the poor pedagogical quality of educational films. In this connection *H. Stever* raised the question of finding a quality standard for evaluating films used in mathematics instruction. A further point which was unclear was the availability of studies on the differences between the pedagogical potentialities of slides, on the one hand, and of films on the other as seen from the point of view of the learner's behaviour. *S. Schuster* elucidated the potentialities of films referring to good examples, while *J. Hunter* advocated that only the positive aspects of the various media be used which could be done in an optimal way by producing multi-media learning sequences. A project headed by J. Hunter is based on this approach. Finally, *G. Chastenet de Géry* drew the attention of the participants to the problems of CAI. In his opinion, the feasibility of group work with CAI should influence the pedagogy of mathematics.

At this point, the panel discussion was terminated in order to give the general assembly an opportunity to discuss the report and the contributions by the panel members.

At the beginning of the *second session*, short communications on various problems and projects related to educational technology were submitted and discussed. *J. Hunter* reported on the 'Computer-Assisted Learning (CAL) Project in Mathematics at the University of Glasgow'. *K.-A. Keil* reported on his experience with the use of ready made computer programmes in mathematics education. *F. Nestle* analysed the reasons for the neglects of educational technology in teacher education, whereas *O. Sangiorgi* reported on experiments made in Brazil in including TV in a multi-media system for teaching mathematics. A special aspect of CAI was presented by *S.A. Sloan* in his contribution 'A Proven Use of the Computer to Improve Mathematics Skills and Teaching'.

All contributions were closely related to the report, so that the discussion of the short communications could serve as a continuation of the discussion of the first session, which, accordingly, needed not be continued separately. The discussion of the second session was also influenced by the poster session which had taken place the day before.

The main results of the discussion of R. T. Heimer's report, the short communications, and the poster-session may be summed up in the following theses, which found wide acceptance among the panel:

(1) There is a lack of subject-oriented pedagogical conceptions regarding different media. In most cases intuition serves as a substitute for a theoretical foundation. From the technological standpoint there are hardly any problems, from the pedagogical point of view there are many problems, all of them unsolved.

(2) A general criticism of educational technology in mathematics education is not possible. Only particular products can be criticized.

We cannot report here to the last detail the numerous aspects raised in the discussion sketched above. In spite of the fact that the pedagogical problems were often in danger of being overpowered by the technological aspects, there was a fruitful exchange of views on different approaches to the pedagogical use of the numerous possibilities of educational technology in mathematics teaching.

B 6
The Interaction between Mathematics and Other School Subjects (Including Integrated Courses)

Reporter: H.O. Pollak, USA
Chairman: J. Rosenmüller, FRG
Coordinator: W. Blum, FRG
Short Papers: U. Beck, FRG; P. Bhatnagar, India; B. Dudley, GB; P. Häussler, FRG; Z. Usiskin, USA

B 6.1
SURVEY-ABSTRACT BY THE REPORTER

The purpose of this report is to present an overview of the educational implications of the applications of mathematics. This means, first of all, that we must understand what actually goes on in the relationship between mathematics and other disciplines. We shall then use the systematic framework we will have developed in order to examine worldwide trends and issues in the teaching of the usefulness of mathematics.

In discussions of applied mathematics, a large amount of unnecessary difficulty is sometimes created by unspoken disagreements on the definition. We find it necessary to think in terms of *four* different definitions:

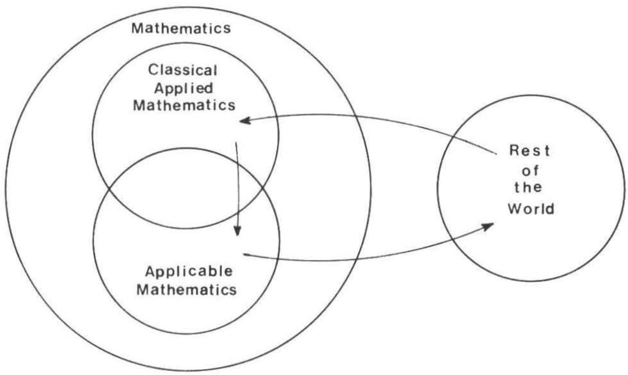
(1) Applied mathematics means classical applied mathematics, that is, the classical branches of analysis, parts of which apply to physics.

(2) Applied mathematics means all mathematics that has significant practical applications. This includes everything that has been considered world-wide for elementary and secondary school, almost everything at the tertiary level and much graduate mathematics. In this view, and in the eyes of many people, probability, statistics, linear algebra and computer science are as important as classical analysis.

(3) Applied mathematics means beginning with a situation in some other field or in real life, making a mathematical interpretation or model, doing mathematical work within that model, and applying the results to the original situation.

(4) Applied mathematics means what people who apply mathematics for a living actually do. This is like (3) but usually involves going around the loop between the rest of the world and the mathematics many times.

We may visualize these four definitions as follows:



In this picture classical applied mathematics represents definition 1, applicable mathematics definition 2, once around the loop from the rest of the world to mathematics and back again is definition 3, and many times around the loop is definition 4.

It is interesting to view our diagram in an alternate way. Applications of mathematics may consist of routine uses of mathematics, mathematical reasoning as opposed to direct use, building of small models, full mathematization of real situations, and truly large-scale applications of mathematics.

We propose to examine the diagram in considerable detail. The content of classical applied mathematics (definition 1) includes calculus, various kinds of differential and integral equations, plus perhaps the prerequisites for calculus: algebra, geometry and trigonometry in the schools. Applicable mathematics (definition 2) is too all-inclusive to list here. An important feature of it is that mathematical concepts and structures have applications, not just mathematical technique. There is an ever-changing relationship between mathematics and its applications, with a dynamic effect on mathematics itself. Areas of mathematics which many originally considered quite pure - such as, for example, entire functions - turn out to have significant practical applications; areas of mathematics which were invented for purposes of application - such as information theory - turn out to have a major influence on core mathematics.

The rest of the world to which mathematics is applied is also in the process of dynamic change. No area of human endeavor is immune from quantitative reasoning or mathematical modeling. Besides the physical sciences and engineering, the social sciences, the biological sciences, the humanities and everyday life are all sources for interactions with mathematics. This current diversity of applications of mathematics may be contrasted with the historical monolith of applications to physics. Arguments between those who stress the great variety of applications in recent years and those who feel that their total impact cannot compare to the 2,000-year accumulation of success of mathematical physics have by no means died down.

When mathematics is actually applied to a situation in some other field, there are typically a number of distinguishable steps in the process. These consist of a recognition that a situation needs understanding, an attempt to formulate the situation in precise mathematical terms, mathematical work on the derived model, (frequently) numerical work to gain further insight into the results, and an evaluation of what has been learned in terms of the original external situation. This model-building process has many interesting attributes as well as pitfalls. A good model is one which is to some extent successful in explaining, or even predicting, external reality. If it fails to explain, then, no matter how satisfying the mathematics itself, the model is not good applied mathematics and must be changed. Sometimes a mathematical model predicts too much rather than too little: It may happen that phenomena observed in the other field are indeed explained satisfactorily, but that further logical implications of the model are not acceptable. Such a situation again leads to much hard work and to changes in the model. The purposes for which a mathematical model is created are also quite varied, ranging from insight to action. The often-held view that the purpose is essentially determined by the applied field is too simplistic. It is not necessarily true, for example, that applications in the physical sciences always lead to action, and that models in the social sciences never do.

The overall picture of applications of mathematics would not be complete without a discussion of truly interdisciplinary activity. Much of the most exciting current work is in fact on the borderline between several fields, one of which being in the mathematical sciences.

We have now set the framework for examining the effects of applications of mathematics on education. Much applied mathematics in the schools, whichever definition is assumed, is found under the activities called 'problem solving'. The meaning of problem solving needs to be examined with care. Genuine applications of mathematics to everyday life and to other fields should ideally be in the character of definitions 3 and 4. Problems which use words from other disciplines, but neglect the mapping from the rest of the world to mathematics implied by definition 3, can nevertheless be honest abstraction from the rest of the world. On the other hand, the facts alleged in the statement of a problem are sometimes quite unrealistic. Sometimes problems are clothed in a mantle of external vocabulary only for amusement, and the pretended application is not meant to be taken seriously. We shall call such problems whimsical problems. Whimsical and unreal problems are not necessarily devoid of pedagogic value. For example, it can be quite effective to begin with an unsatisfactory oversimplification of a real situation, and to approach a genuine application in the sense of definition 4 through a series of increasingly realistic problems.

The increased awareness in many countries of the importance of teaching the applicability of mathematics has led to a number of very interesting attempts to collect real problems at various levels, and from various disciplines, and to make them available for teaching purposes. We shall have occasion to refer to a number of these collections and to associated efforts to incorporate them into the curriculum.

The diversity of applicable mathematics (definition 2) which has emerged in recent years has greatly complicated the task of designing curricula for elementary and secondary schools. The traditional goals of preparing students for either shopkeeping or calculus (associated with definition 1) cease to be uniquely valid when so many more areas in the mathematical sciences are of undeniable importance to so many of the world's people. As the number of reasonable choices increases, so does the difficulty of designing a curriculum. It has become clear in many countries that more than one set of new materials may be needed. Furthermore, countries need to examine the partial ordering among mathematical topics implied by the observation that, at this particular time, topic A is socially more important than topic B. It is essential to observe that these partial orderings of importance will be different in different countries, and perhaps even in different parts of the same country. Transportation of curricula from one part of the world to another was probably never desirable, but it is even more questionable now.

An appreciation for the different forms of applications of mathematics will affect not only the curriculum materials of the schools but also the pedagogy. If you examine even relatively simple uses of mathematics you find that it is necessary to understand when and how and why the mathematics works in order to apply it correctly. Thus the natural desire of mathematics teachers to emphasize understanding as well as technique is reinforced, not contradicted, by applications. More deeply, the model building process in definitions 3 and 4 requires an understanding of the situation outside of mathematics and of the process of mathematization as well as of the mathematics itself. A great weakness of some courses with titles like 'Methods of Applied Mathematics' is that no attempt is made to provide an opportunity for the student to understand the situation and the mathematization process. Some of the collections of real problems which we mentioned previously hope to overcome this difficulty.

The interaction between mathematics and other disciplines implied by definition 4 is clearly open-ended. It is thus very valuable for the student to have open-ended modeling experience in the course of education, just as it has long been recognized that discovery teaching is a very important component of pedagogy in mathematics itself. Experiments in open-ended discovery teaching of mathematical applications, or of truly interdisciplinary materials, are underway in a number of countries. So far, they appear to have been more prevalent at the elementary and the tertiary levels than the secondary.

As mathematics teaching changes in the light of the increasing applicability of the subject, so must teacher training. Teachers must become familiar with the new fields of applicable mathematics, with the process of model building, and with the associated pedagogic emphases on understanding and open-endedness. A very interesting new idea is an industrial internship for prospective or current teachers of mathematics. In this way, it is possible for the teacher to learn something of how the mathematical sciences are really applied. There are a number of noteworthy developments in this direction, for example in Great Britain and in the People's Republic of China.

A further educational effect of applications of mathematics is in vocational education. As the importance of the mathematical sciences increases for many disciplines, so does the need for workers and technicians in these disciplines to learn the most appropriate mathematical techniques. Noteworthy vocational materials in a variety of technical fields have been developed in a number of countries, for example Hungary. A different development in the same spirit is the increasing popularity of special curricula for technicians in computer science and data analysis.

Teaching which is truly multidisciplinary is easiest to achieve in the elementary school, where a single teacher normally handles most if not all subjects. Such multidisciplinary activities are satisfying for the students and help to lessen any feelings the student may have that school has nothing to do with real life. On the other hand, the time for such activities must be contributed by the various disciplines involved; also it is very helpful for teachers to have participated in multidisciplinary activities as part of their training. On the secondary level, the implications for the structure of the educational system are much more severe. If a single unit involves mathematics, science, social science and language arts all in a significant way, who is going to teach the material, who will contribute the time, how should the school be organized? These problems have not yet been solved. Team teaching is one possible answer.

The drive to make mathematics in the school more applied is worldwide. In some cases, the applied point of view is a more accurate reflection of the country's social system. In some it is part of a reaction against the 'New Math'. In some it is a recognition of the increased mathematization of many other fields. Forces that help the tendency towards more teaching of applications are many. There is the increasing realization that applications are an integral part of mathematics teaching. Many countries have found that the problem of motivating students becomes easier when applications are used as one of the possible motivations. The recognition that job opportunities for mathematics students at all levels are increased when the student knows how mathematics is applied has also been very helpful. This same argument applies to prospective mathematics teachers. Furthermore, really interesting applications of mathematics have become increasingly available, partly because of the spread of mathematics to more applied disciplines.

There are also forces that hinder applications of mathematics in teaching. Some mathematicians fundamentally believe in the purity of mathematics, and do not, in their view, wish to corrupt the beauty of mathematics through extraneous considerations. Perhaps some mathematics teachers are ignorant of other disciplines, and are even afraid of them. In the past, after all, mathematics teaching has been a possible way of earning a livelihood while having relatively little contact with the real world. Next, as we have seen, time for applied mathematical activities must be carved from the rest of mathematics or from other subjects, and there will always be those who resist such changes. There are also difficulties arising from the disciplines to which mathematics is applied. Many practitioners of these other disciplines do not themselves use much mathematics, and have not kept up with the recent analytical developments in their fields. Furthermore the way in which mathematics is actually used may differ from the way in which it has been taught in mathematics classes. Such differences, for example in notation and specific techniques, make communication difficult. Finally, in some countries there has been a marked decline of student interest in science. Such a development may deprive mathematics teaching of some of the most powerful applications of mathematics.

B 6.2
DISCUSSION-SUMMARY BY THE COORDINATOR

In the first session of section B6, the trend report on B6 was given by *Dr. Pollak*. In addition to the written version of his report (see above) Dr. Pollak stressed that we all know too little about the classroom reality concerning applications of mathematics.

Afterwards the panel members gave short comments and additional statements to this report. Obviously there are interdisciplinary approaches to bridge the gap between several disciplines; we have to distinguish between an approach by common elements (concepts or processes or languages) and an approach by real life situations. One can show three ways of looking at the problems of linking mathematics with applied fields: from the point of view of mathematics, of the applied fields, and interdisciplinarily; there are biological examples for the difficulties arising with this. Really there is another point in the list of forces that hinder applications of mathematics given by Dr. Pollak: The teachers' fear of 'open' situations.

One of the panel members spoke about the 'beauty' and the 'power' of mathematics and about some motivations that applications can provide. The Coordinator gave a brief survey on the recent literature in the German speaking countries, concerning applications of mathematics, beginning in 1970. He mentioned his classification scheme: external discipline; branch of mathematics; level of application; kind of reality; intention. Though he could find an obvious trend in the recent literature towards more applications, he doubted whether one could speak of a similar trend in the actual classroom situation.

In the subsequent *plenary discussion* many different items were discussed, but no main issues could be recognized. One of these items was the problem of whether it is just the point of view of a mathematician when beginning a report with a definition of what is 'applied mathematics'. Another item was the difference between the two guiding questions 'how best to use mathematics' and 'how best to teach mathematics'. Some classroom experiments concerning the interaction of mathematics and other disciplines were discussed, for example a linking of mathematics with music. Several possibilities for organizing the interaction of mathematics with other school subjects were discussed, for example the idea of a course with a core of pure mathematics and a variety of applications round about.

During the *Poster-Session* the following authors presented their papers:

U. Beck (FRG); J. Chabrier (France); B. Dudley (GB); W. Flemming (GB); P. Häußler (FRG); S.W. Hockey (GB); P. Holmes (GB); W. Münzinger (FRG); A.D. Turner (GB); Z. Usiskin (USA).

There were lively discussions with the authors, especially concerning concrete examples for interactions of mathematics with other subjects.

In the *second session* of section B6, *P. Häußler* spoke about 'Investigations of Mathematical Reasoning in Science Problems', concerning an investigation about operations which are used by students (age 13 - 16) in order to recognize functional relationships underlying measurement data presented to them. The author further discussed the effects of two alternative treatments aiming at a general increase of recognizing functional relationships. *Z. Usiskin* told of his ideas and experiences concerning 'Curriculum Development in the Applications of Mathematics'. To change curriculum by means of applications, Prof. Usiskin demanded that one should know which applications are important. He himself had searched for so-called *basic* applications; he mentioned some examples and discussed *powering* in detail. *U. Dudley* gave a brief report on the work in a Congress Working Group continuing EWG 6: 'Links of Mathematics with Other Subjects'. *P. Bhatnagar* spoke about 'Concept of Integrated Curriculum in Mathematics'. He started with the 'axiom', that mathematics has two aspects: beauty and power, and stated several types of integration.

Finally *U. Beck* reported on his 'Computer- und graphenorientiertes Verfahren zur Bestimmung chemischer Summenformeln'. He gave an example of cooperation between didacticians of mathematics and chemistry. The solution consisted of four steps: qualitative analysis, quantitative analysis, determination of the mass of the molecules, chemical sum formulas by means of the theory of graph methods. It was stressed that the economic aspect of mathematics could help the chemistry teacher.

B 7
The Role of Algorithms and Computers in Teaching Mathematics at School

Reporter: A. Engel, FRG
Chairman: J. Nievergelt, Switzerland
Coordinator: H. Deussen, FRG
Short Papers: G. Akkerhuis, USA; J.R. Caravella, USA; R.E. Fraser, GB; H. Meißner, FRG; W. Schäfer, FRG; W. Walter, FRG

B 7.1
SURVEY-ABSTRACT BY THE REPORTER

1. Computer Use at School: Present State and Critical Appraisal

For the US we have reliable data on computer use in public secondary schools. Let us look at some of the data for 1975.

The Language Problem. BASIC has become the predominant computer language. Many computer scientists deplore this. Indeed, BASIC is inadequate for computer science. But for mathematics this language is satisfactory. New versions of the language have the control structures *if-then-else*, *do-while*, *do-until* which are essential for structured programming. For us structured programming is not crucial since mathematical programs are short and deep. In Europe several special school languages have been developed. We do not need these. It is more important for assembly languages to disappear as quickly as possible.

Extent of Computer Use. In 1975 58% of the schools were using a computer in administration and/or instruction. However 54% of these schools used the computer *only* for administrative purposes. Thus only 27% of the schools reported some instructional use. 43% of the instructional computer courses were in mathematics. Schools using computers tended to be larger than nonuser schools. So the schools using computers for instructional purposes may well represent more than one third of the total student population.

There is a severe shortage of terminals. The *mean* number of terminals per user school is *five*, but their distribution is very lopsided. The *mode*, i.e. the most frequent number of terminals is *one*. Usually only *one* or *two* mathematics teachers per school are involved. Thus the actual number of students using the computer tends to be quite modest, and the depth of usage is mostly superficial. In short there is a lack of money, of competent teachers, of good textual material, and of information exchange. Most teachers start from scratch.

In Western Europe and Japan the amount of computing activity is considerably lower. Teachers mostly have to rely on desk-top calculators. In the US only 7% of the user schools rely solely on desk-top calculators. But this modest share is increasing as the calculators become cheaper, more powerful and versatile (e.g. WANG 2200 S, DEC's CLASSIC, HP 98-20,-25,-30, Tektronix 4051).

Type of Computers Use. There are two major uses of the computer in education with opposite philosophies: *CAI and problem solving.* The proponents of CAI use *the computer as a teacher.* The advocates of problem solving use *the computer as a pupil.* In CAI *the computer controls the student,* in problem solving *the student is in control of the computer.* Between 1960 and 1970 immense effort was put into CAI with disappointing results. But now the two big CAI-projects PLATO and TICCIT show modest promise. We do not deal with CAI since it will be covered by report B 6, on Educational Technology.

The philosophy of problem solvers is as follows: The best way to learn something is to teach it. The computer plays the role of a 'model student'. It is a very demanding student since it forces you to express the topic to be taught as an algorithm. It is claimed that this is the best way to learn a topic.

At present most uses of the computer are quite superficial. Yet there are hundreds of schools where teachers and students use the computer in a highly creative way. They have demonstrated that the computer has tremendeous potential for education. Several projects are exploring systematically creative computer uses. For example the LOGO-project directed by S. Papert at MIT, T. Dwyer's SOLO-project at the University of Pittsburgh and the XEROX Palo Alto Research Center.

The most critical problems hampering the spread of computer use are money and teacher education. Teachers at all levels must be educated in the use of computers in the classroom and especially in their own subject. The trouble is that the majority of computer scientists are not qualified to satisfy the needs of future mathematics teachers. We need high quality textbooks in computer oriented mathematics which could be used for a course. A committee of computer scientists is not able to produce such a course.

2. Algorithms at School: Past and Present

Algorithms have always played a major role in school mathematics. Yet the lack of an efficient tool for executing algorithms prevented a consciously algorithmic attitude. Design and analysis of algorithms was hardly ever practiced. Instead children were used as programmable calculators for executing a few standard algorithms which they had to memorize before they could really grasp them. For this reason algorithms fell into disrepute among mathematics educators.

Without an algorithmic attitude a teacher can only make superficial use of the computer. Hence the whole of school mathematics should be restructured from an algorithmic point of view. The pocket calculator suffices to initiate a successful reorientation toward algorithms. Algorithms are more important than the computer (the device). This explains why the main part of our report deals with algorithms and not with computers.

3. The Pocket Calculator

Pocket Calculators are so new that we can merely put down some questions and conjectures. Empirical data are not yet available.
(1) In the near future the influence of the pocket calculator

will be much greater than that of the computer.

(2) For the price of a book ($ 10) and some instruction the child acquires greater computational skill than in many years of paper and pencil drill and practice. We may be able to save hundreds of hours of arithmetic drudgery and use this time creatively by giving more attention to basic ideas and algorithms.

(3) Children still need basic numeracy. But what does it include? Can we postpone the multiplication and division algorithms as well as operation with common fractions for the algebra class? Or do we need them as preparation for algebra? Is the pocket calculator helpful or harmful in acquiring basic numeracy?

(4) Children should be able to operate with small numbers, understand place value, make rough estimates. In addition they will need more approximations, scientific notation, order of magnitude roundoff error, significant figures, floating point and fixed point notation, absolute and relative error. Decimal numbers must be introduced very early.

(5) The teacher needs quick help with ideas and curriculum material.

(6) Outdated computational techniques with logarithms and slide rule should be dropped without delay.

(7) Do we still need tables? If yes, what should they contain? A collection of basic algorithms and some high precision values of values of elementary functions to be used as test cases?

(8) For traditional mathematics instruction the pocket calculator is quite adequate. Its slight drawbacks compared to a computer are balanced by advantages such as unlimited accessibility. It also forces us to use *efficient* numerical algorithms which are more instructive. The search for efficient algorithms is a powerful motive for a deeper study of the properties of elementary functions.

(9) In elementary school the four-function calculator with at least one memory is adequate. It should probably be an algebraic machine. In junior high school we need a 7-function calculator: $+, -, \cdot, :, x^2, \sqrt{x}, 1/x$. For senior high school we require in addition exp, sin, cos, tan and their inverses.

4. School Mathematics from an Algorithmic Point of View

We propose guidelines for mathematics teaching. They are applicable worldwide and they are independent of the current state of computer technology. They can be implemented if there are some pocket calculators in the classroom. They imply a slight change in content and a major change in the point of view. The guidelines can be summarized into one sentence:

TEACH MATHEMATICS FROM AN ALGORITHMIC STANDPOINT

1908 Felix Klein initiated a reform of mathematics teaching. The reform movement adopted the slogan '*functional thinking*'. The reformers claimed that functional thinking must pervade all of mathematics teaching. What the student should have learned in his mathematics classes is thinking in terms of functions. The reform has profoundly changed mathematics teaching and by now it is accepted by teachers that mathematics is a study of functions or mappings.

The time has come for a new reform with the slogan '*algorithmic thinking*'. The algorithmic thinking should pervade all of mathematics. Teachers must realize that the major activity in the classroom should be design and analysis of algorithms. The algorithmic approach embraces the functional approach since functions are algorithms.

Algorithms are nontrivial mechanisms which simulate some process. Students must learn to design mechanisms and to analyze mechanisms which were designed by others.

5. A Mathematics Curriculum from an Algorithmic Standpoint, Topical Outline

In this final section we identify the basic algorithms occurring in school mathematics, grades 5 to 12. More than 60 complete programs are given. Most of the procedures can be executed by means of a pocket calculator. This outline is the core of the report. It shows by examples that the 'algorithmic standpoint' is not just another slogan and that the bulk of the subject matter of school mathematics indeed gains by treating it from an algorithmic point of view. Only geometry is absent. It does not seem to be an algorithmic discipline. All the algorithms in the outline with a geometric content really belong to calculus. Take for example the section with the title 'The Circle and the Hyperbola'. It contains geometric derivations of elegant and efficient algorithms for 18 functions: the trigonometric and hyperbolic functions and their inverses as well as for ln and $\exp = \ln^{-1}$.

6. Summary of the Procedures in the Outline

Program reading and design for lower grades: swap(x,y)-procedures and its application to sorting of 2,3,4 elements. Procedures for generating simple sequences: switches, oscillators, powers, factorials, squares, triangular numbers, mod n - procedures, scramblers, Fibonacci sequence, golden ratio, Fibonacci sequence mod m. Algorithms in the natural language, max, min, abs, int, ancient Egyptian multiplication procedure. Growth, decay and fast powering procedure.

Roots: square root algorithm, procedure for $\sqrt[n]{x}$ by repeated square root extractions. Convergence rate.

Logarithms, powers and random digits: Logarithms and random digits by repeated squaring or powering. Powers by repeated square root extractions. Linear congruential procedure for generating random numbers.

Procedures from elementary number theory: conversions from a-base to b-base, Euclid's algorithm and Pythagorean triples, extended Euclidean algorithm, continued fractions, period length of 1/n, prime number tables by division and by sieving, fast twin prime procedure. The circle and the hyperbola:

By means of the circle $x^2 + y^2 = 1$ and the hyperbola $xy = 1$ we give geometric and constructive definitions of all elementary transcendental functions. For instance, the procedure in Fig. 1 computes the functions ln, arcsin,

```
inp s,c
c ← √((1+c)/2)
s ← s/c
if c ≠ 1
prt s
end
```

Fig. 1

arccos, arctan, arccot, arsinh, arcosh, artanh, arcoth. A similar procedure for the inverses of these functions is given.

Roots of equations: Bisection method, regula falsi, secant method iteration, Newton's method etc.

Optimization: Maxima and minima of arrays, maximum (minimum) of a unimodal function by trisection, bisection, golden search. Search in the plane: random search, Gauss-Seidel, steepest descent etc. Numerical integration (no calculus is needed): Euler, trapezoid and midpoint rule. Simpson and Romberg procedures can be discovered by numerical experimentation.

Simulation of deterministic processes (without calculus): pursuit problems, planetary motion, spread of rumours and epidemics, predator-prey-problems. Combinatorics: Three procedures for Pascal's triangle, frequency counts (e.g. money change problems), permutation problems (e.g. generation of a random permutation, its decomposition into cycles, Josephus problem).

Probability: Birthday problems, binomial probabilities, asymptotic formulas by numerical experimentation.

Sorting: Insertion sort, selection sort, interchange sort, sorting by frequency counts, merge sort.

Simulation of random processes. Simulation with the random number generator: drawing random samples, simulate the Crap game, study runs in binary random digits, relative frequency and probability, symmetric random walk and the $\sqrt{n}$-law, diffusion and Brownian motion, data analysis, simulate radioactive decay. Simulation of Markov chains without random numbers by operating the corresponding linear system.

7. Bibliography

The bibliography comprises about 80 items.

B 7.2
DISCUSSION-SUMMARY BY THE COORDINATOR

In his comment the reporter challenged both the panel and the floor by his thesis 'Teach mathematics from an algorithmic standpoint'.

It is neither possible to record here all of the one hour vivacious discussion, nor would such a recording clearly show the various points of view that emerged. Therefore, this report tries to summarize what was contributed during the two sessions.

Many members of the audience accepted this thesis, if not verbatim then at least as far as its intention is concerned.

Others objected and claimed that computers should only serve to free teaching from tedious calculations in order to gain more time for teaching the essence of mathematics; they regretted that the use of computers and calculators had not been demonstrated under this aspect, although one participant reported on his pedagogical efforts and results in motivating limits through numerical experiments. The problem of training teachers arose; it was mentioned that merely putting flowcharts and programming in current curricula would by no means be sufficient; the selection of adequate problem classes for the algorithmical treatment was considered important: application oriented or 'real' problems as presented by Sir James Lighthill, or problems stemming from evaluating usual mathematical functions.

There was no doubt that the design and construction of algorithms cannot be taught by letting the students receive recipes, but only by letting them actively participate in the design process.

The ultimate goal in designing algorithms is to perform them by whatever means. The opinions reached from one extreme to the other: performance by hand is sufficient (purists!), or the only sufficient instrument is a computer.

Prof. Engel stressed that 'pocket calculators are sufficient to initiate a successful reorientation toward algorithms'. In addition, pocket calculators may alleviate the difficulties that certain students have with arithmetic.

In this context, algorithms were tacitly assumed to be numerical because of the predominance of calculators that treat only numerical algorithms in a reasonable way. Yet, the pedagogical value of nonnumerical problems and algorithms was touched on, although one was aware of the (financial) constraints in executing them.

For a long time, discussion centered around the programming language problem. Prof. Engel considered the question of programming language unimportant, a standpoint that was objected to by those who emphasized that the programming language should facilitate the design of algorithms. Furthermore, as an algorithm is a mathematical object that ought to be proven correct (cf. E.W. Dijkstra: 'program testing can be used to show the presence of bugs but never to show their absence'), its representation and, hence, the programming language should facilitate program proving.

If there is both an informatics course and an algorithmically oriented mathematical course in school, then both should use the same programming language.

It was pointed out that a programming language need not necessarily be implemented: the algorithms can be translated by the student himself into respective mechanical languages if he is provided with appropriate translation routines.

By this way one gains freedom in the choice of programming languages, even a stylized mother tongue could be used to formulate algorithms, thereby taking care of the thinking habits in different ages of students. Some even went so far as to suggest that, at least for younger groups, the connection between thinking habits and language as far as the formulation of algorithms is concerned should be investigated.

The questions concerning the relationship between classical and algorithmically oriented school mathematics could not be resolved. There was agreement that algorithmic mathematics should be a subject in its own right with strong interdependence with classical mathematics. But the extent to which both subjects should be taught clearly remained open and needs further clarification about and experience in the new subject. After that discussions in a larger group can be devoted to that relationship.

After the discussion short communications were given:

(1) *R.E. Fraser:* 'The role of algorithms and computers in the teaching of mathematics at school', which was a report on her experiences.

(2) *G. Akkerhuis:* 'The School Computer Project', in which decision criteria for a hardware configuration (38.500 $) and a software configuration were presented.

(3) *W. Schäfer:* 'Anwendung einer höheren Programmiersprache zur Erstellung von Maschinenprogrammen', demonstrated that translation of a high programming language into a machine language by hand serves three purposes: Knowledge of that high programming language, knowledge of principles used in translation programs, practically no errors in the generated machine programs.

(4) *W. Walter:* 'Use of programmable pocket calculators in high school', showed that with 72 program storage cells and 10 result cells, except for linear algebra, practically all numerical problems appearing in high school can be treated, even the integration of the elementary two-body-problem.

(5) *H. Meissner:* 'Working group Pocket Calculator', reported on experiences made by introducing pocket calculators in base schools; even the property of being non-skid and questions concerning symmetry and legibility of displayed digits were discussed.

(6) *J.R. Caravella:* 'Minicalculators in the classroom', again discussed the impact of this new tool in high school.

As a last topic, the chairman raised the question of whether informatics (computer science) should be introduced as a subject in its own right. He argued that sooner or later the algorithms will become more and more voluminous and that their documentation, structuring, and the property of being self-explanatory will become important. Consequently, concepts like procedures or datatypes should necessarily be introduced.

The discussion from the floor - the opinion being half and half -

went along the same line as it did with the question of programming languages.

Concluding, the coordinator should be allowed to give his personal view on this subject. Small calculators - programmable or not, carried in the pocket or not - should serve to free mathematics from the burden of calculations. Their unreflected use, if not sophisticated, can be taught in less time than needed for slide rules and logarithm tables. No time should be spent to overcome the constraints of this small tool by inventing and applying tricky procedures. A more profound knowledge of computers, their use, limits and implications goes far beyond the framework of mathematics and should be taught in special courses if at all.

3.2 The Short Communications (Poster-Sessions)

The short communications submitted by Congress participants (as well as the selected short papers for the 13 sections) were presented in the Poster-Sessions. The short communications had been grouped according to topic under one of the 13 sections (see the list below).

The authors had been requested to personally display a summary of their communication (up to four pages) on poster boards and to keep near their poster display in order to discuss their communication with interested Congress participants.

AKKERHUIS, G., USA: The 'School Computer' Project (B7).

AMBROSE, D.P., Lesotho: Mathematics Education in Developing Countries with Rapidly Expanding Secondary School Systems (A3).

ANDELFINGER, B., FRG: Entwicklung und Erprobung eines Kurssystems für die Sekundarstufe II in den Bereichen Informatik - Mathematik - Philosophie (B1).

ANDERSON, R.D., USA: Calculus for Management Science Students (B6).

BAJPAI, S.D., Nigeria: Mathematics is the Language to Express All Phenomena of Nature and Life (B3).

BAJPAI, S.D., Nigeria: New Mathematics in Developing Countries (A1).

BARRETO, A.C., Brazil: A Functional Approach to Measure for Segments and Angles (A2).

BARRETO, A.D., Brazil: Inter-disciplinary Models of Teaching Geometry (A4).

BARTLEY, E., USA: Adult and Continuing Education as Conducted by Correspondence by the International Correspondence Schools (A5).

BAUHOFF, E.P., FRG: Zwei einfache Anwendungen der Linearen Algebra in der Analysis (A3).

BEBBE-NJOH, E., Cameroon: Recherches piagetiennes en milieu camerounais (B4).

BEBBE-NJOH, E., Cameroon: La methode axiomatique et l'enseignement des mathématiques (B4).

BECK, U., FRG: Computer- und graphenorientiertes Verfahren zur Bestimmung chemischer Summenformeln und zur Strukturauflösung (B6).

BENTLEY, C., GB: Evaluation of Applicable Mathematics (B2).

BERG, D.J. van den, South Africa: Basic Considerations for Curriculum Development (B1).

BEZUSZKA, S.J., USA: Three Phases of Problem Solving (A2).

BHATNAGAR, P.L., India: Concept of Integrated Curriculum in Mathematics (B6).

BIDWELL, J.K., USA: The Development of Shape Experience Material for Young Children (A1).

BINER, H./SUAREZ, A., Switzerland: Die lineare Funktion (B4).

BLAKERS, A.L., Australia: Fostering Mathematical Talent (A3).

BÖDDEKER, W., FRG: Die Fortbildung von Lehrern für die 'neue Mathematik' in den Grundschulen in der BRD 1970-74 - Probleme der Lehrerfortbildung (A6).

BOYS, G.R.H., GB: The Mathematics Curriculum - A Critical Review (A2).

BRANCA, N., USA: Problem Solving Processes of Fifth and Sixth Grade Students (B4).

BRAUNER, R., FRG: Grundschulmathematik - Lehrmeinungen und Schulpraxis (A5).

BREUER, S./ZWAS, G./LIVNE, A., Israel: Numerical Algorithms in the Teaching of Mathematics (B7).

BRINK, F.J. van den, Netherlands: Waterland - mathematische Weltkunde auf der Grundschule (A1).

BRISSENDEN, T.H.F./DAVIES, A.J., GB: Computer Graphics in the Teaching of Science and Mathematics (B5).

BRÖNSTRUP, H., FRG: Computerunterstütztes Fernstudium (A5).

BROWN, M. a.o., GB: The Understanding of Various Mathematical Topics by Children (B4).

BRUNTON, J., GB: An Algorithm for Finding the Cube and Other Roots Using only the Normal Functions of a Pocket Calculator with a Square Root Key (B7).

CARAVELLA, J.R., USA: Minicalculators in the Classroom (B7).

CHABRIER, J., France: Premier cycle secondaire (A2).

CHABRIER, J., France: Les mathématiques et les autres disciplines (enseignements intégrés) (B6).

CLANCY, L.J., GB: Mathematics and Personality (B4).

COHORS-FRESENBORG, E., FRG: Dynamische Labyrinthe (A1).

COHORS-FRESENBORG, E., FRG: Leistungskurs 'Mathematische Grundlagen der Informatik' (A3).

CONROY, J.S., Australia: Learning a Mathematical Structure (B4).

COTTER, S., USA: An Audio-Visual Program of Instruction on the Relationship between Mathematics and the Visual Arts (B5).

DADA, S.A., Nigeria: Mathematical Education for Teachers in Nigeria (A1).

DAHNKE, P.E., USA: Mathematics in a Comprehensive High School (A3).

DAVIES, J.L., USA: The Use of Mathematical Models with Computers at the Secondary and College Levels (B5).

DAVIES, J.L., USA: A Live Demonstration of the Feasibility of Using Inexpensive Microcomputers as Aids for Mathematics Education (B5).

DAVIS, J.R., Malawi: Symbols for Physical Quantities and their Numerical Values (A3).

DELERUE, J., France: Analyse et création d'images mathématiques en milieu scolaire ou préscolaire (B4).

DESPOTOVIC, R., Yugoslavia: Sur une propriété des figures géometriques (A2).

DUDLEY, B.A.C., GB: Links of Mathematics with Other Subjects (B6).

EISEN, F.E., USA: The Inverted U Activation Function and Logical Inference Performance (B4).

EMLER, W., FRG: Anforderungen eines Integrationsmodells der Sekundarstufe II an die curriculare Entwicklung des Mathematikunterrichts (B1).

ENGELHARDT, J.M., USA: The Mathematics Learning Clinic (A6).

ERVYNCK, G., Belgium: Remarque concernant l'assimilation de nouvelles notions mathématiques (A4).

EVYATAR, A., Israel: The Next Number in a Series - Mathematics or Psychology (B4).

FELSCH, W., FRG: Symmetriegruppen von Flächenornamenten - eine anwendungsbezogene Behandlung von Gruppen in der Sekundarstufe II (A3).

FEY, J.T., USA: Patterns of Course Offerings and Enrollments in Mathematical Sciences Departments of US Universities (A4).

FISCHER, W.L., FRG: Heuristics and Logic (B4).

FLEMMING, W., GB: Creative Modelling in School Mathematics (B6).

FRASER, R.E., GB: The Role of Algorithms and Computers in the Teaching of Mathematics at School (B7).

FULLER, M.L., Australia: The Role of Mathematics Departments in the Mathematical Education of Undergraduate Students Enrolled in Business/Management Courses (A4).

GAULIN, C./LACASSE, R./PALLASCIO, R., Canada: Etude pratique à distance des relations élèves-professeur de mathématiques selon une conception organique des activités de perfectionnement (A6).

GLENCROSS, M.J., Rhodesia: A Scheme for the Preparation of Africans in Rhodesia for a Upper Secondary School Mathematics Teaching (A6).

GOLDBERG, J., USA: Some Aspects of the Soviet and American Mathematical Education (A6).

GRAENING, J., USA: A Statewide Mathematics Education Program for Secondary Teachers (A6).

GRIMM, W., FRG: Computer Aided Constructions in Descriptive Geometry (B7).

HABER-SCHAIM, U., USA: The Rationale, Goals, and Content of the Mathematics Program for Grades 7 and 8 (A2).

HAQ, F./VOGELI, B., Afghanistan: Primary Mathematics Education in Afghanistan: Procedures, Problems and Progress of Curriculum Revision in a Developing Country (B1).

HASHIMOTO, Y./SAWADA,T./SHIMADA,S., Japan: Development Study on a Method of Evaluation Student's Achievement in Higher Objectives of Mathematics Education (B2).

HÄUSSLER, P., FRG: Investigation of Mathematical Reasoning in Science Problems (B6).

HAYMAN, M., GB: Summary of the Presidential Address Given to the Mathematical Association in Britain 1975 (B1).

HENKIN, L., USA: Logical and Pedagogical Foundations of the Theory of Positive Rational Numbers (A2).

HIGGINSON, W., Canada: Frames: A Divergent, Non-linear Introduction to Basic Number Concepts (A1).

HILL, S., USA: Overview and Analysis of School Mathematics in the United States (B3).

HILL, W., USA: Data Collection and Problem-solving Strategies (B1).

HINDAM, Y., Doha-Qatar: The Effect of Teaching a Course in 'Informal Logic' on the Scholastic Achievement of the Beginners in Geometry (A2).

HINDAM, Y., Doha-Qatar: The Effect of Studying the Technique of Logical Proof on the Pupil's Ability to Solve Mathematical and Life Problems (A2).

HIROKAWA, K., Japan: The Development of Creative Thinking in Mathematics in Lower Secondary Schools (A2).

HOFFELNER, K., FRG: Vorführung eines Dialogauskunftssystems für Lehrpersonen im Mathematikunterricht (B5).

HOLMES, P., GB: On the Schools Council Project on Statistical Education (A2).

HUGHES, B., USA: Learning to Abstract (A3).

HUMAN, P., South Africa: On Goals and Objectives for the Teaching of Mathematics at School Level (B3).

HUNTER, J., GB: The Computer-Assisted Learning (CAL) Project in Mathematics at the University of Glasgow (B5).

HUTTER, R., USA: The Role of Mathematics in the Process of Becoming a 'Competent' Citizen (B5).

IGBOKO, P.M., Nigeria: The Potential for Teaching Modern Mathematics in the Secondary Schools and Teacher Training Colleges of the Imo and Anambra States of Nigeria (A6).

ITO, T., Japan: On the Discovery Teaching of the Concept of Function in Elementary School (A1).

JANVIER, C., Canada: La composante psychopédagogique des programmes de formation et de perfectionnement des enseignants de mathématiques (A6).

JOHNSON, C., USA: In Math: Individualizing at the Secondary Level (A3).

KALIN, R., USA: The Preparation of Prospective Elementary School Teachers for the Teaching of Mathematics - a Summary of a New Program at the Florida State University (A6).

KAPADIA, R., GB: Deduction in Mathematical Education (B3).

KARPLUS, R., USA: Proportional Reasoning in Seven Countries (A2).

KEIL, K.-A., FRG: Arbeiten mit gespeicherten Programmen im Mathematikunterricht (B5).

KENNEY, M., USA: Problem Solving Models in the Senior High School (A3).

KEPNER, H.S., USA: The Impact of the Minicalculator on the Curriculum (B7).

KESON, J., Denmark: Cartoons in the Classroom - Producing Your Own Animated Films (B5).

KLINGEN, L., FRG: Modulares Problemlösen in der Schulmathematik durch zweckmäßiges Design einer Computersprache (B7).

KREOLL, G.A., USA: A Study of Instructional Research in Mathematics for Learning Disabled Students (B4).

KUDRYAVTSEV, L., USSR: On Mathematical Education at Higher Technical Schools (A4).

KUHN, U., FRG: COLORMULTIMAT - Konvex-konkave farbige Plastik-Gitter-Platten mit 6 ergänzenden farbigen Rechentafeln (A1).

LESKY, P., FRG: Ein unabhängiges Axiomensystem für Boolesche Algebren (A3).

LIND, D., FRG: Zur Prüfung 'theoretischer' Signifikanz bei dichotomer Klassifikation (B2).

LINDNER, H., FRG: Curriculumentwicklung und Programmiertes Lernen: Präsentation eines mathematikdidaktischen Projektes (B1).

LISTER, G., GB: Logic and Number (A1).

LIU, Fon-Che, Taiwan: Summary of a Report on Recent Activity in High School Mathematics Education in R.O.C. (B1).

LOMON, E.L., USA: Mathematics Education at Pre-school and Primary Level (A1).

LOMON, E.L., USA: Research Related to the Mathematical Learning Process (B4).

LOPATA, G., France: L'ordinateur pour tous et pour chacun (B5).

LOPATA, G., France: Agir ou laisser faire (B7).

LOWENTHAL, F., Belgium: On the Aquisition Process of the 'Formal Thought' (B4).

MALVERN, D., GB: The Use of Hints in Testing Applicable Mathematics - a New Examination Technique (B2).

MASON, J.H., GB: Tutor-Student Interaction in Universities: Comments Generated by Experience with the Open University (A5).

MATHEEV, A., Bulgaria: Une expérience a l'Université de Sofia - former des professeurs de mathématiques qualifiés (A6).

MATTHEWS, G., GB: Why Teach Mathematics to Most Children? (B3).

MBAEYI, P.N.O., Nigeria: Art vs. Technique in the Achievement of Goals and Objectives in Mathematics Education? (B3).

MEISSNER, H., FRG: Arbeitsgruppe Taschen-Rechner (B7).

MEYER, K., FRG: Stufung der Raumanschauungspflege (A2).

MOALEM, D., Australia: Secondary School Geometry Project for Years 7 to 12 (A2).

MOCCIOLA, M., USA: Probability and Statistics in the Elementary School: Survey and Analysis of Teacher and Curriculum Status (A1).

MÖLLER, H., FRG: Vereinfachte Analysis für den Schulunterricht (A3).

MORLEY, S., GB: The Initial Training of Primary Teachers (A6).

MRMAK, M., Yugoslavia: Notwendigkeit, Möglichkeiten und einige Erfahrungen mit der Anwendung des Films im Mathematikunterricht (A2).

MUNTER, C. de, Belgium: La pensée mathématique et les groupements de Piaget: Comment introduire les opérations sur les ensembles? (B4).

MÜNZINGER, W., FRG: Der Modellversuch KoRaG zur Förderung praxisnaher Curriculum-Arbeit und kollegialer Lehrerfortbildung u.a., dargestellt am Fachbereich Mathematik (A6).

NESHER, P., Israel: On the Notion of Mathematical Readiness (A1).

NESHER, P., Israel: Learning Isometric Transformations - an Operational Approach (B4).

NESTLE, F., FRG: Unterrichtstechnologie im Mathematikunterricht? - Hoffnung auf eine bessere Zukunft (B5).

NICHOLSON, A.R., GB: Mathematics and Language (B4).

NIKOLIČ, M., Yugoslavia: Neutral and Reserve Elements in School Teachings of Mathematics (A2).

O'BRIEN, T.C., USA: The Teachers Center Project (A6).

O'BRIEN, T.C., USA: New Directions in Mathematics Education (B3).

OLEJNICZAK, E., Poland: Die fundamentalen Komponenten der Befähigung in Mathematik (B4).

ORLOV, K., Yugoslavia: Mechanical Machine Working on Dialogue Principle in the Teaching of Mathematics (A2).

ORMELL, C., GB: On Piercean or Projective Applicability (A3).

OSHIO, S., Japan: On the Revision of Curriculum in Mathematics Education (B1).

PAASONEN, J., Finland: The Basic Material in the Finnish Mathematics Programme (A2).

PALMER, H., USA: Introducing Measurement with 'CHIIP' (B6).

PAQUETTE, G., Canada: Caracteristiques pédagogiques de la réalisation de projets de programmation (A5).

PAQUETTE, G., Canada: Un système co-géré de perfectionnement à distance (A5).

PAVELKA, E., USA: Moves, Strategies and Modes of a Concept Venture in Secondary School Mathematics (B4).

PELLEREY, M., Italy: The RICME-Project (A1).

RIEM, H., FRG: Soziale und mathematische Lernprozesse durch Gruppenunterricht (B4).

ROETHER, D., FRG: Unterstützung der Ingenieurausbildung durch Computer (A4).

ROMAN, T., Roumania: Action éducative de l'enseignement des mathématiques dans une école technique supérieure roumaine (B3).

ROUSE, W./WESSON, J.B., USA: Improved Evaluation of Children's Mathematical Performance (B2).

SADR, M., Iran: A Proposed Mathematics Curriculum for Iranian Universities (A4).

SANGIORGI, O., Brazil: TV as a Component in a Multi-media System for Teaching Mathematics (B5).

SCHÄFER, W., FRG: Anwendung einer höheren Programmiersprache zur Erstellung von Maschinenprogrammen (B7).

SCHRAMM, R., Israel: The Inverse Function Theorem for Functions of Several Variables (A4).

SCHUBERTH, E., FRG: Das Problem der mathematischen Unfähigkeiten (B4).

SCHWARTZ, J.L., USA: Infinity Factory - A Television Series on Mathematics and Problem-solving (A1).

SCHWARTZ, J.L., USA: A New Approach to the Assessment of Children's Mathematical Competence (B2).

SCHWARTZ, J.L., USA: The Development of the Semantic Aspects of Quantity (B4).

SHELLEY, N., Australia: Mathematical Illiteracy and some of its Implications (A5).

SHUARD, H.B., GB: The Training and Professional Life of British Mathematics Teachers, including those serving in Primary Schools (A6).

SIEMSEN, K.H., FRG: Über Behalten im genetisch-adaptiven rechnergestützten Kleingruppenunterricht (B4).

SLOAN, S.A., USA: A Proven Use of the Computer to Improve Mathematics Skills and Teaching (B5).

SMART, J.R., USA: 'BASIC' Education for Teachers (B7).

SMITH, J.C., South Africa: Carmel College Mathematics Project (A2).

SRINIVASAN, P.K., Nigeria: Teacher Resistance vs. Learner Resistance in Modern Primary School Mathematics Teaching (A1).

STEFFE, L.P., USA: A Teaching Experiment about Children's Learning of Addition and Subtraction (B4).

STEPHENS, W.M., Australia: Remediation in Mathematics Education (A2).

TARP, A., Denmark: On Different Perceptions of Mathematics (B3).

TAUBER, M.J., FRG: Das System CAVA - ein CMI-System mit computerunterstützter Aufgabenanalyse, - bewertung und -vergabe (A5).

TRAVERS, K.J., USA: An Ordering Principle for Curriculum Construction (B1).

TSUBOTA, E., Japan: Mathematics in Mixed Ability Groups (B4).

UETAKE, T., Japan: On the Ordering of Teaching Objects (B4).

USISKIN, Z., USA: Curriculum Development in the Applications of Mathematics (B6).

WAIN, G.T., GB: Mathematics Teacher Education Project (A6).

WALTER, W., FRG: Use of Programmable Pocket Calculators in High School (B7).

WEISSGLAS, J., USA: Mathematics for Elementary Teaching: A Small-group Laboratory Approach (A6).

WENZELBURGER, E., USA: Verbal Mediators in Mathematics for Transfer of Learning (B4).

WHEELER, D., Canada: Towards the Humanization of Mathematics Education (B3).

WINTHER, S., Norway: Some Results from an Investigation on Teaching of Mathematics in the 8th Grade of the Elementary School in Norway without Organizational Differentiation According to Ability Levels (A2).

WIRTH, D., Australia: Living Maths (A2).

YESHURUN, S., Israel: Some Researches in Psychomathematics and their Implications for Mathematical Education (B4).

Part 4

Further Congress Activities

4.1 Panel Discussion
What May in the Future Computers and Calculators Mean in Mathematical Education?

Chairman: H. Freudenthal, Netherlands

Participants: U. d'Ambrosio, Brazil; A. Engel, FRG; H. Meißner, FRG; J. Nievergelt, Switzerland; H.O. Pollak, USA

Secretary: H. Hielscher, FRG

The questions of the discussion, introduced and briefly explained by the chairman, *Prof. Freudenthal*, were divided into two groups, namely into three General Questions, formulated by the panel, and several Special Questions, formulated by the chairman himself.

The *General Questions* were:

- What to do with calculators today?

- How will calculators and algorithmic thinking influence teaching mathematics?

- What should a citizen know about computers?

Here the *Special Questions*:

(1) What are calculators, what are computers, and what is their use?

(2) Must we introduce handheld calculators in mathematical instruction, and if so what kind, and at which age?

(3) Is there a danger that numeracy will suffer if such calculators are introduced, or is this to be considered as a danger, or may numeracy be improved by using calculators?

(4) Will disadvantaged children be even more handicapped by including calculators into instruction, or will calculators be an advantage for them?

(5) Will the distance between education in developed and developing countries increase even more by the introduction of calculators or will it be an advantage for developing countries to jump over a bygone period of mental and written numeracy?

(6) Will teachers of science complain even more about the lack of mathematical abilities of their pupils, or will they be able to profit of the use of calculators in mathematical instruction?

(7) If we introduce computers in the classroom, then teachers must be prepared to work with these instruments. How to prepare teachers to work with these instruments?

(8) Should algorithmic thinking and programming be part of mathematical instruction or instruction whatsoever, and if so, for what kind of pupils, and at which age?

(9) Can teaching algorithmic thinking and programming improve learning and teaching mathematics?

(10) To which degree and in which way should programming be supported by the use of computers?

Before the panel members gave their statements to some of these questions, the question 'What are calculators, what are computers, and what is their use?' (Special Question No. 1) was dealt with separately by *Prof. Nievergelt* in order to standardize the terminology of the discussion. He pointed out that there is no sharp dividing line between calculators and computers but a continuous range of devices in between for which certain categories exist. Calculators for instance can be categorized into nonprogrammable and programmable ones, the latter into those with a lower or a higher level language. The primary application of calculators is numerical calculation. With computers, their periphery equipment is the main distinguishing feature, to which printers, card readers, plotters, terminals, second storage devices such as magnetic tapes and discs belong. Sizes and prices of the categories of devices may also vary largely but here again is no clear way to separate calculators from computers. As far as the use of computers is concerned, they have many applications where calculating in the conventional sense does not appear, such as writing reports, processing texts of another kind, making graphic designs.

In the first statement, *Prof. d'Ambrosio* emphasized the necessity to include the social context when attempting go give an educational rationale for handheld calculators. Calculators and computers play an important role everywhere in our life, just as other electronical devices do. If the general claim to make mathematics closer to real life, to combine school experience with outside school experience is to be put into practice, the use of calculators and computers in schools has to be considered a *must* and this not only for the developed countries, but for the developing countries, i.e. countries with a lower economic structure, too, because school education there has to make up for a lack of experience which is due to the economic situation of these countries. With respect to the influence of calculators on teaching mathematics, the importance of numeracy as a purely mechanical work has to be called in question. The chance to have more time available for a creative mathematical education in very elementary grades already must be taken. Experiments and investigations in this field, especially from the psychological point of view will be needed and must be urged. Not to teach pupils to perform the operations well will change the curriculum, but to supply the ability to use the calculator intelligently will mean a real change in the school curriculum.

The tremendous influence of calculators on the whole school curriculum was emphasized in *Prof. Engel's* statement, too. Starting from the definition that numeracy means being able to compute, he concluded that numeracy will not suffer if calculators are introduced but that it will be improved - much more than by hand computation. Calculators can also be of great help for disadvantaged children, if disadvantaged means not being able to compute. With a pocket calculator, these disadvantaged children will be able to compute, they will no longer feel frustrated but gain confidence. So, generally, the quality of mathematical education will probably go up by including calculators into instruction, because even though the calculator will help the fast learner more than the slow learner and will thus increase the gap between them, it will help the slow learner and improve his mathematical education. Correspondingly, calculators will help developed countries more than developing countries, because they can be used in countries with a sophisticated educational infra-structure in a better way. But to counteract this, it will be very important to give developing countries not only technical and financial, but above all more educational help.

In the next statement, *Prof. Meißner* first dealt with the question at which age calculators should be introduced into mathematical instruction. They must be introduced, because they are part of our real life, and they must be introduced early, because mathematical education reaches all pupils only up to the age of 15. The calculator must then be introduced as a teaching aid and pupils must be given not only the necessary knowledge but the methods to use the calculator properly. Mental numeracy, the abilities of estimating and approximating will be extremely important, otherwise mistakes will not be realized, results cannot be checked. The same applies to computers: school education should give pupils a basic knowledge and the working methods of computers and data processing, and since to programme is a way to learn, pupils should be taught to programme a computer. Besides, according to the theory of learning most of our thinking is algorithmic. So the use of calculators and programming may make our own programmed thinking more conscious and improve it - once research has found out what learning should be conscious. In addition, computers can offer a way to do team-work in school, which is a real life way to work. Considering the large technical disinterest however, it must be expected that computers will remain with specialists, but not the handheld calculator. Therefore one must teach all pupils to use it - and when teaching it, one has to be careful that the fear of mathematics does not become a fear of calculators or of computers.

Prof. Nievergelt agreed to the previous statements on the point that calculators are a great help in schools as far as arithmetics and the amount of time wasted by memorizing are concerned. But in his opinion, the effect of calculators on mathematics teaching is currently overestimated and has to be considered just a fashionable trend. The introduction of calculators will not change the school curriculum to the extent stated before. Differing from Prof. Engel's view that the design and analysis of algorithms are a major mathematical activity, algorithms in his opinion do not have a dominant role in mathematics, nor can most of our thinking and reacting be called algorithmic, as was pointed out in Prof. Meißner's statement, for many of our reactions are spontaneous. The main problem is how to programme computers, and this is a central subject of computer science. Mathematics are just the most convenient administrative organisation to bring knowledge about computers into the schools. To introduce computers into schools is also still a matter of finances. If computers become cheaper, they will be able to enter schools. Programming will then become essential, and computers will then have a real impact on the school system. Handheld calculators are just a first step in that direction.

A completely different approach to the questions of the discussion was exposed in *Prof. Pollak's* statement in which the pedagogical rationale for introducing calculators and computers into school education was given absolute priority. The question is not 'What can we do with calculators and computers?' but 'What are the pedagogical difficulties in teaching and how can calculators and computers help with these problems?' Teaching methods for solving linear and later non-linear equations, attacking the problem of inverse functions, simple data analysis can be done better with a calculator than with a slide-rule, and even better with a computer. With respect to the effect of calculators and computers on the school curriculum, the two orderings that are essential when designing a curriculum, the ordering of prerequisites and the ordering of importance, will change with the availability of calculators and computers. Referring to some other points of the discussion, an important effect of calculators was seen in the fact that they can remove the fear many pupils have of mathematics. Experience in the United States has shown that pupils who cannot do simple algorithms are with the help of calculators more willing to work and eventually learn to do algorithms. As the solving of problems comes up in many other fields, too, the calculator will make it possible to do more interesting and more realistic things in other school subjects; with larger machines in the schools, simulation for instance can become an important part of teaching. Another point is the different time constant of education and technology: the time constant in education is 5 years, the one of calculator companies 0.5 years. This causes the problem to make two systems with different time constants work together.

In a final comment, the chairman *Prof. Freudenthal* pointed out the high pedagogical, didactic and social value of the questions discussed. Many problems are not yet solved, in some fields there is little or no experience. To gain knowledge and experience about working with calculators and computers in schools and to publish this knowledge and experience has therefore to be considered an urgent need.

4.2 Exeter Working Groups (EWG)

Several Exeter Working Groups from the 2nd ICME met again in Karlsruhe and continued their work. In the following, short reports on the sessions of these working groups are given.

EWG 5
The Teaching of Probability and Statistics at School Level

Chairmen: L. Råde, Sweden; P.L. Hennequin, France

In the first session the chairman *L. Råde* reported on some activities in the field of teaching probability and statistics at school level. Special attention was drawn to the activities of the International Statistical Institute with headquarters in the Hague in the Netherlands. This institute, which is a learned society of probabilists and statisticians from all over the world, will through its educational committee strengthen its activities with regard to the teaching of statistics. This is planned to be done in cooperation with UNESCO. The institute has recently organised two round table conferences on the teaching of statistics at school level, one conference in Vienna in september 1973 and one conference in Warsawa in 1975. Proceedings of both conferences have been published.

The following papers were presented at the meeting:

G. Noether, USA: Report on activities of the Joint Committee (National Council of Teachers of Mathematics and American Statistical Association) on the Curriculum in Statistics and Probability.

P. Holmes, GB: Report on the Schools Council Project on Statistical Education.

I. Tajima/S. Fujimori, Japan: Report on the teaching of probability and statistics at the school level in Japan.

I. Tajima, Japan: How to introduce the normal distribution from the binomial distribution.

The second session was devoted to the presentation of the INRDP-IREM research group on the teaching of probability and statistics.

This group was set up in 1973 and is divided into three teams. The first one is concerned with the senior grades of secondary schools (14-18).

F. Decombe outlined the present situation, in which the syllabus provides for a rather dogmatic introduction of probability and the final examination (Baccalauréat) prescribes a specified type of examination leaving the student little initiative, especially for modelling a situation.

J. Badrikian mentioned how the team sought to change this situation by presenting students with random situations and studying their behaviour when confronted with these situations. A computer is required to present relatively complex situations involving a large number of selections. A graphical terminal (plotter) allows a more synthetic and more varied representation of the phenomena. Filming the development of the event allows its presentation to a large number of classes where computer terminals cannot be introduced. About 10 short films of 5 minutes each (mute films, size super 8) are now available. An experiment was conducted in 10 classes with the film 'Urne de Polya'.

P. Errecalde then presented the second team concerned with the analysis of tables of numerous and real data by children aged 12-16. This involves:

- searching subjects areas involving the treatment of data tables,
- placing children in open pedagogic situations and observing their behaviour,
- striking a critical balance of the experiments.

The children are expected to be able to:
- read, present and treat data (arithmetic or graphic),

- interpret the results of the treatment, criticize the treatment or its interpretation.

Seven teams are working on one subject each and prepared research documents containing a synthesis of the crude observations:

- 'Football' (Paris team)
- 'Sample surveys' (Paris team)
- 'Rate of flow of the river Loire' (Orléans team)
- 'The port of Rouen' (Rouen team)
- 'Car registration' (Rennes team)
- 'Study of the population of the Drôme on the basis of 1962-68-75 census' (Valence team)
- 'Height-weight' (Valence and Rennes teams)
- 'The Port of Bordeaux' (Bordeaux team)
- 'Analysis of sports results' (Grenoble team)
- 'Climatology' (Bordeaux team).

Finally, *G. Brousseau* presented the experiments conducted in elementary schools (age 10-12) aimed at studying in 35 sessions how the child behaves when confronted with the following problem: 'A sealed bottle, the neck of which only is transparent, contains 5 black and white balls. How can we find out how many white balls it contains without opening it?'

This problem is comprehensive enough for the progressing mathematization to be guided by the aim and the direction of the action. The roles of probability and statistics appear correctly in a child's answer to a complete epistemological problem.

EWG 6
Links between Mathematics and Other Subjects at Secondary School Level

Chairman: B. Dudley, GB

This working group was concerned with linking mathematics and other subjects, especially at secondary school level, and began its work in 1972 at the Second International Congress on Mathematical Education, at Exeter, England. At Karlsruhe the group explored precisely the meaning of 'links' and found its most satisfying expression to be that of teaching *equally* concerned with mathematics *and* with another subject, so that the teaching materials are equally attractive to the teacher of mathematics and the teacher of the other subject. In this respect 'links' offers to the teacher and to the pupil something in addition to that currently available as 'applied' or 'applicable' mathematics and as 'use of mathematics', where concern on the one hand is primarily for education in mathematics and on the other for education in the other subjects.

The working group was confident there are educational gains in making known existing linking materials and of developing further materials of this kind for the following reasons. First, the materials, being specific, would facilitate communication between subject specialists; secondly, it would help cause mathematics to be more acceptable to pupils not inclined towards mathematics while at the same time contributing to the teaching of another subject or other subjects; and thirdly, it would help bring cohesion to the school curriculum as a whole and mitigate against its continuing fragmentation by eroding not whole subjects but subject boundaries.

In so far as there is a concern as much for the other subject (or subjects) as for the mathematics the group found 'links' is concerned with a set of priorities for secondary school mathematics education that lie outside those adopted by the existing major national curriculum projects of any country. In consequence, this concern is not noticeably reflected in the curriculum materials of the major projects neither for mathematics nor for other subjects, though it became clear during the discussions that some work of this kind does seem to have been developed by individual teachers on a private basis in class, and also by some small groups of teachers working in isolation. There is a noticeable shortage of information about 'links' materials and experiences, even to those interested in this particular area. Little of what has been done has been published and so is not generally available.

As a result these instances are not generally known.

A number of problems associated with link work were identified. One is that the use of linking materials in class would not fit easily into the existing disjoint, subject-based, school curriculum and examination structure, because at their best, they would be difficult to classify under any single subject heading. A second difficulty arises from the fact that many teachers at secondary school level are single subject specialists and link work can be expected to generate in teachers anxieties and uncertainties. This in turn reflects upon the initial training and in-service training of teachers, and it is recognised that a capability to function adequately in the area of linking mathematics with other subjects on a reciprocal basis would need to be incorporated deliberately into initial and in-service courses. In that it is an area of mathematics education not covered by either applied mathematics or by applicable mathematics, information about existing materials suitable for use in class and in training need to be collected, and further materials need to be developed for this specific purpose.

Members of the working group, therefore, have agreed to gather together a collection of these materials and experiences, and if necessary to develop further materials which would enable the specific objective, namely, the mutual linking of mathematics and other subjects, to be achieved. To that end the activity of this working group continues with its immediate concern to collect examples, references, and descriptions of teaching materials that are known to exist, and attempts that are known to have been made relating to the kind of linking that the group had in mind - materials equally attractive to the teachers of mathematics and to the teachers of other subjects. The information should be sent to B. Dudley, Education Department, Keele University, Staffordshire, ST5 5BG, England, for distribution.

EWG 7
Application of Mathematics

Chairman: C.P. Ormell, GB
Reporter: M.E. Rayner, GB

The working group met once. *C.P. Ormell* of the School of Education, Reading University, England, started the meeting by giving a paper on 'New Applicability in the Classroom'. This carried forward the analysis begun in C.P. Ormell's paper to this working group at the 2nd ICME; both the present paper and the earlier paper (entitled 'New Ideas on the Applicability of Mathematics') can be obtained from the Sixth Form Mathematics Project at the School of Education, London Road, Reading RG1 5 AQ, England.

C.P.Ormell spoke of the idea of 'projective' or 'Piercean' applications as an idea about mathematics and about the characteristic relationship between mathematical modelling and reality. The idea can be very simply stated: that mathematics is the science of *ifs*. It is concerned with the possibilities, rather than with the actualities, of the world. Mathematics can provide us with definite conclusions about the implications of an hypothesis or plan. Mathematics used in this way is called *projective* modelling. C.P. Ormell gave examples to illustrate his talk.

Subsequent discussion was concerned with the process of teaching by the use of modelling:

(1) How artificial can applications be while still having usefulness?
(2) How can a teacher overcome the difficulties associated with a variety of background knowledge of pupils?
(3) Can we teach everything this way?
(4) Should the time sequence be applications-then-mathematics or mathematics-then-applications?
(5) What is the role of approximation?
(6) How can we assess the teaching of mathematics based on modelling?

EWG 11
History of Mathematics as a Critical Tool for Curriculum Design

Chairmen: P.S. Jones, USA; R.J.K. Stowasser, FRG

Papers presented to the group:

P.S. Jones, USA: Some Suggestions for Cooperative Activity in the Use of the History of Mathematics.

H.J.K. Bos, Netherlands: History of Mathematics in the Curriculum at the University of Utrecht.

R.J.K. Stowasser, FRG: The History of Mathematics as a Tool for the Construction of Problem Sequences.

B. Hughes, USA: Greek Geometric Invention: in History and the Classroom.

L.F. Rogers, GB: Implications of History of Mathematics for Teaching.

Mrs. J. Nicolson, GB: Interim Findings of a Project on the Teaching of the History of Mathematics in School.

G. Flegg, GB: The Philosophy and Materials related to the History of Mathematics Open University Course.

At both meetings, members - in Karlsruhe some 70 from 20 countries - stressed that only in the context of ICME was there a forum for such discussion, for meetings of mathematical or historical societies give little attention to the pedagogical uses of the History of Mathematics. Neglect such a large part of world culture seems quite shocking (in spite of the still leading formalist philosophy of mathematics for which there is no history of mathematics proper - LAKATOS).

In the connection with the contribution of *P.S. Jones* concerning the organization of a working base on the selection, preparation, and dissemination of historical material for teachers and students, the following three proposals were discussed by the members of the working group, and forwarded to the executive committee of ICMI for approval:

(1) There should be regularly scheduled sessions to consider the History of Mathematics at future meetings of ICMI.

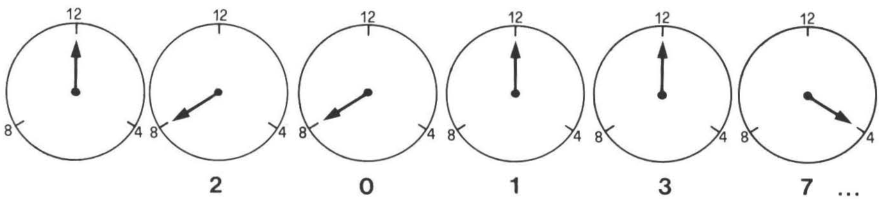
(2) A system should be set up to provide for continued consultation and planning with respect to the use of the History of Mathematics between the Congresses and in preparation for the next Congress.

(3) The chairman and secretary of the group of one Congress should hand on records and suggestions to the chairman and local secretary of the next Congress, and serve as members of the planning committee for that next Congress.

Proposal (2) concerns such between-congress activity, and the 'system' referred to might include:
a) maintaining mail contact and exchange of information with interested persons,
b) holding a small planning meeting in 1977 to discuss goals and possible international projects with reference to the production of materials,
c) holding a working conference in 1978 in connection with the International History of Science Meeting in Edinburgh or with the International Congress of Mathematicians in Helsinki,
d) continued planning for ICME 1980.

H.J.K. Bos reported that for about four years the History of Mathematics has had an established position in the mathematics curriculum at Utrecht University. Mathematics students (whose university study takes 5-6 years from matriculation to the final examination and is roughly equivalent to the Anglo-Saxon master degree or the German Diplom) can incorporate history of mathematics into their course of study in several forms:
(1) a main course;
(2) a seminar on aspects of 19^{th} and 20^{th} century mathematics;
(3) a final year-work on historical and modern aspects of one mathematical theme;
(4) a course on the social function of mathematics with an historical approach.

R.J.K. Stowasser presented the construction of two problem sequences tailored for the 11-year-olds with roots in the History of Mathematics. CLOCKS AND PASCAL'S ALGORITHMS using Pascal's essay 'Caractère de divisibilité des nombres,...' begins with a question such as, 'What time is it 2013754826085774459360075711O8 hours later than 12?'and asks for a very fast digital manipulation with the hour hand:



There are interesting transfer possibilities leading to all the divisibility rules normally treated - and many others never seen at school - which, with this method, come from a uniform point of view resulting in remarkable insight (for rediscovery of Pascal's general divisibility rule by 11-year-olds, see: Beiträge zum Mathematikunterricht 1972, Hannover 1973, pp. 125-32).

The other sequence EXTREMAL RECTANGLES related to Euclid 6; 27 can be found in: Der Mathematikunterricht 22,3, Stuttgart 1976. The reader should remember the important steps in solving the problem by coherent (funny) stories about the search of a spy and of the general (shortest diagonal), about the military examination and measuring the king (biggest stomach).

B. Hughes went back to the historical sources of proof making. Proclus refers several times, in his Commentary on the First Book of the Elements, to the analytic method, whereby Greek mathematicians discovered many theorems and their proofs. Pappus in his Treasury of Analysis wrote the classical description of analysis, which unfortunately was probably corrupted by a later scribe. The correct concept of analysis is a deductive argument that proceeds from the conclusion to something that was given; then the synthetic proof is found by reversing the analytic procedure. The educational use of the analytic method is most appropriate for introducing students to the making of proofs.

L. Rogers spoke about some implications of the History of Mathematics for mathematics teaching:

(1) The general attitude towards mathematics is poor, as a result of failure in traditional terms, and a whole array of cultural traditions which attributes 'mathematical ability' to only few students. The image of the mathematician is also poor, and the results can be seen in the low level of recruiting to mathematics and mathematics teaching.

(2)The general view held of mathematics in the culture is utilitarian and the traditional concerns of mathematics teaching are with tools for the improvement of the intellect and application to the real world.

(3) Traditional teaching of mathematics begins at points where the historical process ends and makes it increasingly difficult for pupils to come to grips with the field. The communication of a variety of different approaches is part of the nature of mathematics, and helps to establish a dialectic (reasoned argument) whereby the construction of mathematics may be examined. This is largely an historical-evolutionary study.

(4) By considering different styles of historical writing we see the relevance of the History of Mathematics for both teacher and student in the development of mathematics, the evolution of curricula, and the formation and communication of mathematical ideas.

(5) The aim is to help teachers to develop philosophy of mathematics teaching which includes the communication of the essential varieties of mathematics and provides a means of individual understanding of the dialectic of developing mathematics.

J. Nicolson reported about interim findings of a project on the teaching of the History of Mathematics, which is a research programme designed to evaluate the use of the film material associated with the Open University History of Mathematics Course to be taught in institutions of higher education (other than the Open University) and in sixth forms. The project was funded in the main by the Leverhulme Trust and has been carried out primarily by J. Nicolson. The programme (films, video-cassette copies, pamphlets) has, on the whole, been well received, as the evaluation of the responses of students and teachers to specially designed questionaires has shown.

The principal purpose of *G. Flegg*'s talk was to suggest questions for discussion. He maintained that the History of Mathematics has its role to play both as a motivating component of mathematics teaching and also as a discipline in its own right. Mathematics is an integral part of our culture and to neglect the teaching of its history is to have students with an inadequate concept of what mathematics is. He therefore posed the question of whether mathematics syllabuses should always include selected aspects of history? The History of Mathematics needs no more justification than do political history, history of science, art history, etc. We must however seriously consider who should be taught it and at what level. We must also consider the relationship with History of Science as well as with mathematics itself. In designing a History of Mathematics Course we have to take the advantages and disadvantages of the chronological and topic-based approaches into account. The topic-based approach enables us to control the conceptual difficulties, but how many topics must be covered so that the historical points can be made convincing? We also have to ensure a proper role for the study of primary source material. The History of Mathematics cannot be taught in isolation from cultural history. Mathematics teachers should have some training in the History of Mathematics and should be aware of what teaching materials exist in the various media, so that they can enrich their presentation of mathematics in the classroom. Finally, the History of Mathematics can humanize much of the present pedagogy of mathematics teaching, and we should seriously ask if it cannot be a bridge between the humanities and the sciences as well as between the 'man in the street' and the 'professional' mathematician.

EWG 12
The Psychologie of Learning Mathematics

Chairman: E. Fischbein, Israel

Three main topics were discussed during the working group sessions:

1. Image and Concept in Learning Mathematics

Are the pictorial, the figurative models always more direct, natural, 'intuitive' than conceptual solutions? In fact, in many cases, (as *Prof. Fischbein* pointed out in his paper) the conceptual network comes first, and the image used represents only a pictorial comment. Consequently, when teaching a certain new topic, we have to be very careful in choosing the place of the pictorial procedures, either as an introductory device or as a subsequent, iconic commentary. The decision depends on the age of the subjects, their previous experience concerning the given topic, and of course, the topic itself. The final decision must be determined by experimental research. Three other aspects were emphasized:

a) Every intuitive model used in mathematics education, such as games, structured materials, and graphs, possesses a symbolic function.

b) The same iconic model may communicate different meanings to the teacher and to the learner. For instance, the pupil may interpret a graph representing the functional relationship between time and space as the trajectory of a moving body.

c) Every model used in the teaching process must be *generative*. This means:
- It must be as general as possible.
- The isomorphism with the original should be as natural as possible.
- The 'code' relating the model to the original must be an explicit one.
- It must lead to solutions which are valid for the original.

R. Skemp, in his comments, has emphasized that figurative models may have different functions. There are models which are directly adherent to the original, and there are models which express, in a pictorial manner, abstract relations, such as graphs representing functions, Venn diagrams, etc. The kind of models we are able to use depends on our mental structures. Sometimes, 'parasitic' intuitive features may activate a wrong model.

A. Abele presented a paper and a film concerning the use by children of tree diagrams in solving a given problem. Two cars starting from A may use four ways to arrive at B. How many possible combinations exist concerning the ways, chosen by the cars? The paper exposed the steps followed by the pupils in the use of diagramatic models and the various interactions between conceptual and intuitive strategies.

During the discussions, it became evident that the classifications used by Bruner (the iconic, the enactive, and the symbolic models of representation) have to be understood in a new manner. Each representation is, in fact, at the same time iconic, enactive, and symbolic. The difference depends more on the kind of meaning expressed rather than on the nature of the model itself.

Brousseau thinks that there is a profound relation between intuitions and implicit models. He suggests that it is necessary to identify and to describe the implicit models used by pupils in order to correct or to reject them if they are wrong. Conflicting situations are the best way to improve or to correct wrong models.

2. Reflective and Intuitive Intelligence in Learning Mathematics

E. Fischbein suggested two categories of intuitions:

a) anticipatory intuitions,

b) affirmatory intuitions.

An *anticipatory intuition* is the global solution to a problem, at which we arrive before we reach an analytical (complete) argumentation. An affirmatory intuition is our direct interpretation (or conviction) of certain facts. For instance, it seems natural to believe that there are 'more' integers than even numbers. In fact, this is not true. This is an example of a wrong primary intuition.

The following aspects were emphasized during the discussions:

a) There are correct and false primary affirmatory intuitions. Before teaching a new body of knowledge, we must determine by way of research the intuitive background of the pupils concerning the domain we intend to teach.

b) The intuitive and the reflective levels of knowledge are both representations of the reality *(Vergnaud)*. By using symbolic procedures, the implicit (intuitive) becomes explicit (reflective). In every case, the pupils need a system of rules and skills enabling them to apply their knowledge to reality. A real mathematical concept is always an operational one *(Vergnaud)*.

c) The intuitive and the analytical ways of teaching are not contradictory. They must be used in a complementary manner. The analytical techniques help to correct the incorrect, primary intuitions and to transform them into coherent and productive interpretations *(Brookes)*.

3. Intellectual Structures and Mathematical Structures

A. Abele presented a paper and a film concerning an isomorphic problem. In the first stage, the children have to cover a 4x4 matrix table with numbers from 1 to 31, by using multibase blocks in base 4. In the second stage, they have to use Quadrimath discs (four different shapes with 1,2,3,4 holes) to express the same numbers. The children had to understand the isomorphism between the two types of materials. The following phases were described:
a) no planned trials,
b) partially systematic essays,
c) a general plan is adopted.

The main problem discussed was concerned with the utility and the aims of using such structured materials in teaching mathematics. *A. Abele* advocated that in solving such problems, children learn to support in a logical way their opinions and to behave in a creative manner. They learn to abstract a common conceptual structure from different categories of figurative materials.

The members of the working group decided to form a permanent international association affiliated with the ICMI, entitled 'The International Group for the Psychology of Mathematics Education' (IGPME).

EWG 23
Mathematics in Developing Countries: The Needs of the Average Student

Chairman: B.J. Wilson, GB

Many developing countries have rapidly increasing enrolments at secondary school level. The secondary school system was originally designed for a small minority of children, highly selected on academic criteria. Mathematics curricula, textbooks and examinations are all geared to the needs of this academic elite - and they are well suited to their needs. They are not, however, suited to the needs of the much wider intake now staying on into secondary school. Even in those countries where there is still a selection procedure for entry to secondary education, students of a very wide range of ability are to be found in the secondary schools, at least in the lower forms. The uniform course, taught at a uniform pace and leading to a common examination for all, is no longer suited to the needs of the majority of such students. There is an urgent need to diversify mathematics education in the lower secondary forms. It is a need that is widespread throughout the world, but it is particularly acute in the developing countries, who can least afford to mis-educate their secondary students, yet who have the most limited resources from which to attempt to solve the problem. What is being done?

This was the specific theme of the EWG 23 meetings. The group was essentially an opportunity for sharing information. Papers were received and circulated from the following countries and regions:

Definition of the problem	*B.J. Wilson*, The British Council
The Gambia	*M. Millard*, Gambia High School
The Caribbean	*D.R. Broomes*, University of the West Indies
India	*M.S. Arora*, National Council for Educational Research and Training
Nigeria	*G.A. Badmus*, Ahmadu Bells University
Botswana, Lesotho and Swaziland	*D.P. Ambrose*, National University of Lesotho and *E. Jacobsen*, UNESCO

Each author spoke briefly to his paper, emphasising the points that appeared to be of most widespread relevance to other developing countries. These short presentations were followed by lively questions and discussions, in which participants from many other developing countries contributed information and ideas from their own experience. The working group's main sessions were chaired by *R.P. Bambah* (India) and by *D.R. Broomes* (Barbados). *B.J. Wilson*, of the British Council, was the convener and overall chairman of the group. There was general agreement that the problem is in urgent need of solution in the developing countries. Several countries reported a two-fold or even three-fold increase in secondary enrolments during the past five years. Some major developing countries, such as Nigeria, are introducing universal primary education, and this will compound the difficulties at secondary level when the tidal wave of new students arrives at the lower secondary stage in a few years' time. Pass-rates in the public examination at the end of the main secondary course are already falling sharply. Where figures were available, it appeared that pass-rates in mathematics are declining slightly faster than overall pass-rates, and that this was independent of whether a traditional or a modern mathematics course was being followed. Nevertheless, widespread public anxiety means that scapegoats are sought, and the most usual scapegoat is 'modern mathematics'. There appeared to be no evidence to support this allegation, except insofar as teacher education and re-education lag behind the spread of modern syllabuses and teaching materials.

The shortage of competent and well-qualified mathematics teachers is still acute throughout the developing world, and few countries reported any signs of improvement. The causes are sociological and economic as much as educational, and as such lay outside the domain of the working-group; nevertheless the situation continues to give rise to great concern, and to frustrate all effort to improve the quality of mathematics education through the traditional processes of curriculum development. Accordingly, the group sought to identify other strategies for developing the confidence and the professional competence of those teachers already in the schools.

It was felt that the introduction of a new school course, accompanied by a programme of short in-service training courses for teachers, is an inadequate strategy in many developing countries, and only by bringing these two sets of activities together was significant progress likely to be made. Here, recent experience in the Caribbean, and in the Southern African countries of Botswana, Lesotho and Swaziland, was particularly relevant. In each of these regions, the dichotomy between curriculum development and teacher education - particularly in-service education - is being broken down by involving classroom teachers themselves in the processes of curriculum development, and by giving them the major responsibility for producing and refining new teaching materials. Work in these regions was illustrated by tape/filmstrip material initially produced for use in teacher education in the countries themselves, and by a film, 'A Way of Looking', of the Caribbean Mathematics Project. It was noted that the principle of giving ordinary classroom teachers a central role in the curriculum development processes of a country was one of the major themes that was being emphasised in the Congress as a whole; it is not a strategy that applies only to developing countries, though some developing countries are giving a lead to educators in countries of Europe and North America in this respect.

One of the effects of this integrated developmental strategy is that it automatically results in a measure of diversification in mathematics teaching. As the diagnostic skills of teachers increase, and they gain experience of selecting appropriate teaching methods and materials for their students, greater variety of classroom organisation and teaching-learning processes appear in the schools. Instead of focussing on the syllabus, the exam and the textbook, teachers gain the ability to focus on their students, their strengths and their weaknesses, their learning processes, their motivation, and their very varying needs. This new orientation may well result in the introduction of small-group work for some of the time, and even individual work where appropriate, the translation of diversification into the language of classroom management.

Of course external examinations still exert their baleful influence, and they cannot be ignored. There was a strong feeling in the group, however, that where countries or regions have their own independent Examinations Council, the examinations set can and must accord with the real needs of the majority of students in the schools, and not just continue to reflect a pattern of examination appropriate to bygone days. The possibility of multi-level examinations was discussed, on the analogy of the British GCE 'O' Level and the Certificate of Secondary Education, though it was recognised that there would be problems of public acceptability of a 'lower-level' alternative. Nevertheless, the situation where an ever-increasing proportion of secondary students finish their education branded as 'Failures' in mathematics cannot be allowed to continue. The use of a lower-secondary examination, such as in Malaysia, Botswana, Lesotho and Swaziland, was suggested as a possible alternative to two alternative examinations, one easier than the other, at the end of secondary school. In this way, a significant number of students could leave school after the lower secondary stage, having succeeded in their last public examination, rather than leaving as failures two years later.

Clearly there can be no uniform solutions. Each country must work out its own. The value of this group was in recognising a common problem, sharing experiences of what is being done to alleviate it, and exchaning ideas on strategies that might be considered in seeking to make mathematics education genuinely relevant to the needs of the great majority of students entering the secondary schools.

The group was strongly of the opinion that the special needs of developing countries should be given major attention at the 4th ICME, and so recommends to the organiser of the 1980 Congress. The group also requests the organisers to give urgent attention to the problem of identifying sources of funds to enable nationals of developing countries, particularly the smaller countries of Africa, the Caribbean and the Pacific regions, to attend the 4th ICME.

EWG 26
Initial Training of Primary (Elementary) Teachers

Chairmen: A. Morley, GB; F. Goffree, Netherlands

0. Introduction

The report of EWG 26 emphasised the need to develop courses which integrate the mathematical, pedagogical and school experience aspects of training. The Karlsruhe follow-up meetings and other project and poster sessions showed substantial progress in the construction of such courses in a number of countries. The interaction of the three components in the personal development of the student-teacher will be evident from the report which describes the discussions and materials on the *mathematical* and *pedagogic aspects* and the group's constructive efforts through simulation exercises on the *school experience part* of the programme. Provision is often made for students who wish to specialise in the teaching of mathematics in primary schools or act as consultants to colleagues in a school, but we are here concerned with the non-specialists who are, and likely to remain, the majority.

1. The Mathematics Component

1.1 *Content*. The degree of content re-enforcement needed by students depends on their knowledge on entry and the extent of the changes in the primary school curriculum in a given country. It is not wise to limit the content too closely to background to the existing primary school curriculum, since this will leave the student ill-equipped to judge new proposals or implement them. The level of treatment should not be such as to produce serious difficulties for students when a quite different level of treatment will be used in the primary school, though it should aim to give a deeper overall perspective. The mathematics component will also be the basis of reflective experience by the student on mathematics viewed as 'process' as well as 'product'. Whether the student is involved in classifying, generalising and proving are important criteria in judging the quality of the mathematics texts or worksheets used. A rich problem-solving background to the mathematics encountered is at least as important as its range, and an important factor in involving the student in the course and attempting to improve attitudes of confidence and enthusiasm for the subject. A superficial treatment of more advanced topics is positively harmful to the prospective teacher. The group reiterated the need for the mathematics to be taught using manipulative materials and apparatus which are used in primary schools, but with problems set at student-level. The production of good materials of this kind remains a continuing challenge.

1.2 *Method*. The wide range of ability and background and poor attitudes to mathematics which many of these students bring to the college course make a workshop-type approach with students working in small groups a necessary condition of their learning. Some useful self-evaluation tests have been produced which avoid the heightened anxiety which accompanies formal testing.

1.3 *Regressive Trends*. Some highly disturbing trends in the structural arrangements for courses came to light, often as the result of the reorganisation of teacher and higher education into larger institutions. These are either the elimination of special mathematics course components for elementary teachers or their replacement by a requirement to attend mathematics courses available to other students in the institution. These courses are usually of a quite unrealistically high level for the entry standard of these non-specialist students, accompanied by unsuitable content and taught in non-participatory large group lectures. Where these courses are compulsory students are failing them or where optional choosing not to do mathematics. In some an altered phasing in time of mathematics, pedagogic and school experience components makes integration more difficult. Staff with appropriate school experience but with degrees in mathematics who have been responsible for the training appear to be having to choose between being attached to the staff of either the Mathematics Department or the Education and Professional Departments of the larger institutions and not allowed to continue their valuable and necessary bridging role in being involved in all aspects of the students' training.

2. The Pedagogic Component

In this component the student begins to relate his mathematical knowledge to pedagogic problems. Only a brief time was spent in discussion of this aspect but some written material was available for review. Topics considered included concept and schema analysis, types of questioning, starting-points for lessons, constructive use of apparatus, decision-making using video-tapes, developmental levels (research findings), levels of treatment, critical analysis of texts and workcards, organising practical work. There is need for more detailed discussion of useful procedures in this part of the course.

3. School Experience

The importance of a progressive experience of observation, small-group teaching and longer periods of teaching practice involving teaching responsibility for a whole class as *the point of application* of the mathematics background component and the development of planning and teaching skills in the pedagogic component received considerable emphasis. Without it the college-based part of the course lacks relevance and meaning for these students. There is on the other hand a good deal of evidence of the freshness and enthusiasm for mathematics and its teaching which a well-planned programme of work with children brings. It is an essential basis of developing a professional attitude.

3.1 *Observation.* Observation of a group of children being taught mathematics was advocated as the first stage. This can be done through video-tape or film, or by observation of actual teaching. The merits of involving classroom teachers in this and/or the college lecturers doing it were mentioned. It was suggested that such a group activity or lesson should be based on a piece of apparatus such as the loop abacus or a problem involving manipulative materials - for example, building larger cubes from unit blocks and finding the number of units used, or ordering a series of parcels by weight. In suggesting observation guide-lines for students, members wished to concentrate on a few basic categories at this stage and recognised that in selecting these they are taking a point of view about what constitutes good teaching and effective learning.

3.2 *Suggested Guidelines.*

(1) How was the activity introduced? What was the problem or situation used and what is the mathematics in it?

(2) What questions did the teacher ask? (Was a spread of attention given to all the children?)

(3) What questions did the children ask?

(4) What were the children asked to *do*?

(5) What materials were available for the children to use?

(6) *Responses:* What did the children do? say? (language used?)
Were they interested in the problem? How did they solve it?
What difficulties did they have?
Which parts of the problem did the children discuss with each other?

Were there any non-verbal cues about the children's attitudes and evidence of their learning?
How did the teacher respond?

As well as recording their observations individually they will discuss with each other a shared experience and see other's differing perceptions.

3.3 *Small-Group Teaching.* The advantage of small-group teaching in the early stages is in limiting the problems of classroom management so that the student teacher can focus attention on the learning and teaching of *individual* children. Taking over an entire class and two students working with four children is a kind of organisation which has proved acceptable to schools in a number of countries because of the personal attention the children receive, even if the student teachers are inexperienced. A possible progression is first to provide the student teacher with an outline of a lesson, or a worksheet or workcard which they have then to prepare themselves to teach. Later they may receive outline suggestions of lessons including different and more demanding tasks - for example, teaching a number operation, organising a practical activity, introducing a concept, writing a worksheet. Finally they will be required to construct a scheme for a sequence of lessons.

The Karlsruhe group undertook a simulation exercise on a tiling (mosaics) worksheet with shapes by asking what things they would discuss with student teachers to help them prepare themselves to use the sheet with a group of children. A composite of the suggestions made by participants is given. There was some debate about the extent to which a student should try to structure the development of small-group activity. It was generally felt that it was most important to leave open the way children tackled the problem and used the materials.

3.4 *Student Preparation Guideline.*

(1) Work through the card yourself.
a) What are the fundamental mathematical ideas involved (ordering, area, rate, unit of measure, congruence, shapes)?

Identify any aspects of the task you do not understand.
b) Do it again concretely using cut-out pieces.

(2) Is the task appropriate to the background of the children? What previous knowledge is required? Concepts, facts, vocabulary?

(3) What do you think are the objectives of the worksheet?

(4) In what *ways* and at what *levels* can the problem be solved? (Check that you have enough materials for all the tasks to be solved concretely. Are any additional materials needed?)

(5) Can you anticipate some difficulties which children might have? Alternative interpretations of the questions?
Are some parts of the sheet more difficult than others?

(6) How will you start? What sort of introduction to help children to relate to the sheet? What motivation (connection with environment, recognition of shapes)?

(7) What sort of teaching mode do you envisage? How long will the activity last?

(8) What sort of recording, if any, will the children make? Verbal, written, diagrammatic?

(9) How will you check what, if anything, the children have learned at various stages?

(10) How will you deal with fast/slow workers?
What extensions or simpler alternative tasks could you provide?

3.5 *Evaluation.* A further simulation exercise was carried out with copies of students' evaluation of the small-group teaching using a three-part classification system of their reflections as given below.

The group were asked what points they would wish to discuss with the student teacher from their evaluation attempts - e.gs. mathematical misunderstandings, alternative materials, vagueness and generality of statements. It was suggested that in examples given to students to help them use the system, it would be necessary to focus on the kind of operational evidence of learning at various levels they would look for. Peer-group evaluation of a student's teaching, as well as individual evaluation, was used by some members to give a wider perception of the situation.

3.6 *Student Evaluation Guidelines.* In your 'Reflections' you should make an *Initial Statement* in which you:
(1) describe the pupils' reaction to your teaching strategies at various stages, and
(2) describe their final state of knowledge, skill or understanding.
This should be followed by an:

Explanation in which you
(1) explain the children's reaction to your teaching, and
(2) explain the possible reason for the success or failure in learning what you intended them to learn.

Finally, make *Recommendations* by answering the following questions:
(1) What follows from this evaluation for the next work you will do with the same children?
(2) In the light of this evaluation, if you were to begin afresh to teach the same sequence over again, would you tackle it differently?

4.3 Study Groups (SG)

SG A
Problems of Information and Documentation in Mathematical Education

Chairman: A.G. Howson, GB

The title given to this subgroup explicitly illuminates one of the major problems to arise in the group's discussions. There are those who, for their work, require a complete listing and classification of the hotels of Paris, there are also thousands who buy and use the Michelin Guide simply because it does not list all the hotels and restaurants but only those that attain certain standards. A similar situation arises within the field of mathematical education. There is a need to catalogue the vast number of papers, reports and dissertations being published, and yet another to sift and select from these so that writings of especial merit can be drawn to the attention of those educators for whom they are particularly relevant: one must both document and inform.

The working group, before beginning its discussions, heard short papers presented by a number of invited speakers. The first two concerned on-going work: that of *ERIC* in the USA and of the *Zentralblatt für Didaktik der Mathematik (ZDM)* in the Federal Republic of Germany.

Marilyn Suydam explained how ERIC (Educational Resources Information Center) began in 1966 as a system for providing ready access to educational literature in the English language. It consists of 16 'clearing houses', that for Science, Mathematics and Environmental Education (ERIC/SMEAC) being located at the Ohio State University in Columbus, Ohio. There documents on all phases of mathematical education are collected, analysed, evaluated for appropriateness, abstracted and assigned descriptors from a structured vocabulary of educational terms. The condensed information then appears either in abstract form in the monthly reference guide *Resources in Education* or in the *Current Index to Journals in Education*. Non-copyrighted documents in the system are made available to researchers, teachers, etc. through the ERIC Document Reproduction Service (EDRS). From time to time specialised documents - bibliographies on specific topics, compilations of research, critiques of research, etc. - are published. More than 100,000 entries are currently included in the document collection.

ZDM, described by *H. Wäsche*, has attempted since its establishment in 1968 to meet both the 'documentation' and 'information' needs so far as literature in the German language is concerned. The journal is published quarterly by Klett, Stuttgart, and consists of two parts. The 'documentation' section describes about 500 titles per issue, each item being characterised by certain key words. The descriptions are printed eight to a page - German on one side, English on the other - and are arranged so that the pages can be cut up to form record cards for an indexing system. Items are classified by means of a decimal system. The 'information' section is devoted to reviews, long essays, critiques and general information.

In recent years ZDM has extended its interests by including, for example, surveys of American literature and of writings in French. It is now hoped, as *G. König* explained, to produce an even more internationally-based ZDM. Firm plans have yet to be laid but it is likely that writings in English will soon be added to the documentation section.

Yet another new initiative was described by *D. Turner* of the USA who told how the National Council of Teachers of Mathematics is attempting to provide its members with increased information. Articles appearing in foreign journals on mathematical education were being reviewed by a volunteer panel and it is hoped presently to publish a selection of these reviews. If this first attempt proves successful then the service might well become established and even expanded.

The particular needs of the developing countries were described by *E. Jacobsen* of UNESCO who also told how some of the other science commissions were attempting to meet these needs through the medium of the commissions' newsletters.

It was clear from the general discussion that the need to provide some form of abstracting service was being recognised in several countries. It was obvious, however, that mathematics education presented different problems to those of mathematics itself. A theorem proved in Brazil has equal validity in Belgium, whereas a finding in mathematical education may have only local validity and interest. Again, although there is general agreement that much of what is published within mathematical education is ephemeral and/or second-rate, the problems of selection and classification would appear to be greater even than is the case in mathematics. The view was expressed, and found some support, that one should not worry about 'selection' in the sense of distinguishing between the bad and the mediocre, but rather concentrate on the 'promotion' of a limited number of exceptional, seminal works which might be identified with the help of an active, international panel.

The desirability of having not only national or regional documentation centres preparing locally relevant abstracts (as ZDM has successfully done), but also some means whereby the important papers of one region could be drawn to the attention of potential readers elsewhere was readily accepted. It is hoped that during the next year or so members of the working group will investigate possible ways in which this might be done.

The desirability of making available information not only of accounts of work completed, but of descriptions of work in progress was also stressed.

SG B
National and International Assessment of Mathematical Achievement

Chairmen: J. Kilpatrick, USA; J. Wilson, USA

Ongoing mathematics assessments at various places throughout the world made the topic of national and international assessment of mathematics achievement timely and appropriate for the 3rd ICME. Consequently a Study Group was organized with the following general purposes:

(1) to exchange information on some current activities in mathematics assessment;

(2) to discuss issues and problems of mathematics assessment;

(3) to establish a continuing international network of mathematics educators interested in mathematics assessment.

The study group met at the appointed times and began work toward these purposes through several brief presentations, discussions, and informal organization of a communications network.

1. Information on Some Current Assessment Activities

1.1 *IEA Second Mathematics Survey*. The International Association for the Evaluation of Educational Achievement (IEA) conducted its first mathematics survey in 12 countries in 1964. Preparation for the IEA Second Mathematics Survey is underway. The issues to be addressed are those of primary interest to mathematics educators and deal with (1) mathematics curriculum, (2) classroom processes, and (3) changes in mathematics education since 1964. *Kenneth Travers*, Chairman of the IEA International Mathematics Committee, presented a report. Approximately 20 countries are making preparations for this study. Current plans call for data collection during 1979-80. The two populations in the study will be (1) those students in the grade containing the majority of 13-year-olds, and (2) those students in their final pre-university year who are enrolled in mathematics courses. The IEA international center for the study is located with the Curriculum Development Unit, Department of Education, Box 12345, Wellington, New Zealand. Roy W. Phillipps is the Project Director.

1.2 *NAEP Mathematics Assessment.* The National Assessment of Educational Progress (NAEP) in the United States had its first mathematics assessment in 1972-73 and is preparing for the second mathematics assessment in 1977-78. *Dr. Jane Armstrong*, coordinator of mathematics exercise development for NAEP, reported on the preparations for the next mathematics assessment of 9-year-olds, 13-year-olds, and 17-year-olds. NAEP administers exercises to representative samples of these age groups. Each exercise represents a specific objective of mathematics performance and data is reported on each exercise in terms of the proportions of the sample making particular responses. The 1977-78 assessment will have a part of each age group using a hand calculator while responding to exercises. The NAEP address is Suite 700, Lincoln Towers, 1800 Lincoln Street, Denver, Colorado 80203, USA.

1.3 *Republic of Ireland Public Examinations Evaluation Project. Elizabeth Oldham* presented a report on the Public Examinations Evaluation Project (PEEP). PEEP is a project of the Government Department of Education. Its aims include operating the 15+ examinations, but also the education of teachers in the theory and techniques of assessment and general research in assessment theory, test administration of scoring, and data analysis. A major part of the presentation and discussion was given to PEEP use of Rasch analysis as a method of interest in the Rasch analysis techniques and claims for the relatively samplefree results one obtains. Some results from the Rasch analysis by PEEP were presented to illustrate its application.

1.4 *England: Tests of Attainment in Mathematics in Schools.* The Tests of Attainment in Mathematics in Schools (TAMS) Project in an exploratory project designed to produce assessment materials for use in a continuous survey of performance in school mathematics. The aim of the survey is to estimate trends by means of testing small samples of pupils at close intervals of time. Item banks have been produced for age 11 and age 15 pupils. Particular attention has been given to assessing practical mathematics. *Ray Summer* of the National Foundation for Educational Research in England and Wales provided materials and summaries for the ICME and *Alan Bell* responded to questions during the discussions.

1.5 *Project TORQUE*. A new approach to the assessment of children's mathematical competence is the goal of the Project for Tests of Reasonable Quantitative Understanding of the Environment (TORQUE). *Judah L. Schwartz* of the Education Development Center, 55 Chapel Street, Newton, Massachusetts, 02160, USA presented information on Project TORQUE and illustrated the innovative assessment procedures.

2. Discussion of Issues and Problems

Several issues and problems grew out of the presentation and discussion followed. The topics of Rasch analysis, calculator impact, and communication about our work drew special attention.

3. Continuing Networks

Informal contact is being continued by an exchange of addresses of the participants and a yearly newsletter to be coordinated by J.W. Wilson, 105 Aderhold Hall, University of Georgia, Athens, Georgia 30602, USA.

SG C
Problem Solving, Teaching Strategies and Conceptual Development

Chairman: L.P. Steffe, USA

Study Group C met at the four different scheduled times during the Third ICME. The purposes of the first meeting were to (1) acquaint the participants with the projects Problem Solving, Teaching Strategies, and Conceptual Development of Mathematics of the Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) and with the Project for Mathematical Development of Children (PMDC), and (2) plan for the remaining three meetings. After brief overviews of the GCSLTM and the PMDC, participants were separated by choice into three study groups - one on conceptual development of mathematics, one on problem solving, and one on teaching strategies. The work of the groups is outlined below.

1. Conceptual Development of Mathematics

Five investigators of the Conceptual Development of Mathematics Project of the GCSLTM discussed their work with the other participants. These five investigators were *M.L. Herman*, *T.E. Kieren*, *R.A. Lesh*, *L.P. Steffe*, and *A.I. Weinzweig*. Kieren and Herman discussed research on rational number learning they (and their associates) are conducting. Three questions being investigated are:

(1) What is meant by the conceptual framework of the rational numbers in childhood and adolescence?
(2) By what cognitive mechanisms does such a framework develop?
(3) What are instructional procedures which promote this development?

Various studies underway were abstracted by *Kieren* one of which was reported in detail by *M.L. Herman* on the development of the concept of the unit. Other studies underway include assessment of 14-year-olds knowledge of fractions, connections between measurement attributes and rational number learning, development of the operator concept, and identification of intellectual abilities which influence acquisition of rational numbers. The long term goal for the studies on rational number learning is an established theory of conceptual development for rational numbers and tested curriculum experiences for children of ages from 6 to 15 years.

L.P. Steffe discussed research on cardinal and ordinal number learning various investigators of the GCSLTM have underway. This research is organized around three problems. First, aspects of finite cardinal and ordinal number not studied by developmental psychologists are being identified and studied from a developmental point of view. Second, the problem of whether finite cardinal and ordinal number develop as unified constructs in children is being investigated. Third, models for learning and instruction of cardinal and ordinal number are being constructed and validated. Studies underway include the topics of counting, relations, operations, partitions, problem solving, numeration, and algorithms. An overview was given of a sequence of three ongoing studies in which children's counting is being studied developmentally and in conjunction with (1) acquisition of numeration concepts and (2) the ability to find sums and differences.

R.A. Lesh discussed research various investigators of the GCSLTM have completed in the area of space and geometry. These studies include:

(1) assessments of students understanding of selected transformation geometry concepts and frames of reference,

(2) an analysis of research needs in projective, affine, and similarity geometries,

(3) cognitive studies using Euclidean transformations,

(4) the role of motor activity in young children's understanding of spatial concepts,

(5) the influence of transformations on conservation of length, and

(6) mathematical models of children's geometry activity.

A.I. Weinzweig reported on his work in the formulation of mathematical characterizations of children's geometry activity.

2. Problem Solving

The participants of this study group discussed the Problem Solving Project of the GCSLTM as well as individual research efforts of some participants. *L.L. Hatfield* discussed the goals, history and progress, and future prospects of the Problem Solving Project of the GCSLTM. A summary of this discussion follows.

Goals. The goals reassert the significance of improving a learner's problem solving competence within the teaching and learning of mathematics. The fundamental aim of the Problem Solving Project is to produce practical, as well as theoretical information about the relationships of learning and instruction of mathematics and problem solving. Several distinct areas of study have been identified, including:

(1) Studies devoted to *identification of strategies and processes* used in solving various mathematical problems, including a *search for aptitudes* related to these strategies and processes;

(2) studies devoted to *development of clinical procedures* for observing and analyzing mathematical problem solving behaviors;

(3) studies devoted to *development of instructional prodecures* aimed at improving a student's problem solving capabilities;

(4) studies devoted to *development of teacher training procedures* to result in deliberate employment of instructional methods aimed at enhancing the growth of problem solving capabilities of students;

(5) studies of an *expository* nature, including analytical developments and interpretive reports.

History and progress. As a matter of strategy a theme of 'instruction in heuristical methods' was adopted for the Problem Solving Project to provide focus and direction. The adoption of this theme provided several general hypotheses or conjectures to be studied: heuristical methods of problem solving (a) can be learned, (b) can be taught, (c) if effectively used by the student, can improve mathematical problem solving performances, and (d) along with the pedagogical counterpart, heuristic teaching, can become a viable part of mathematical curricula. To study these hypotheses or conjectures five working groups were established.

The working group Task Variables started a 'problem bank', which is called the National Collection of Research Instruments for Mathematical Problem Solving (NCRIMPS). With the NCRIMPS now completely operational the Project Investigators, as well as all other interested persons, have at their disposal detailed documentation of all instruments in the bank. It is hoped that the problem bank will stimulate problem solving researchers to analyze more carefully and completely the problem tasks of their own and other investigations.

The working group Instruction in the Use of Key Organizers to Facilitate Problem Solving has been studying the kinds of abilities to be developed in students if they were to become better problem solvers. They have conceived of these abilities as organizers of information that facilitate sensible conjectures and provide direction and focus for the problem solver. The instructional approaches chosen by this group tend to introduce and practice each key organizer in relative isolation from the others. That is, unlike Polya's 'planning heuristic' which involves a comprehensive awareness and usage of possibly numerous heuristical ploys in each solution path, these problem solving episodes would be designed to feature one key organizer.

By contrast, the investigators of the working group Instruction Organized to Use Heuristics in Combinations are conducting small-scale exploratory investigations emphasizing the study of children's thinking during problem solving efforts. The approach with students to introducing and practicing the heuristical advice is holistic. That is, the genesis of a piece of advice for solving mathematical problems is prompted as naturally and spontaneously as possible from within a problem solving episode, rather than introduced as a predetermined, advanced organizer to be illustrated by a succession of carefully sequenced problems.

Future prospects. The Problem Solving Project will continue to build collaborative efforts from the more than 40 participating investigators. New participants are encouraged to join our research group. Future efforts will continue to focus on the study of processes and instruction in heuristical methods with special attention to analytical work related to basic constructs, to models of problem solving behaviors, and to teacher behaviors.

3. Teaching Strategies

The first of the three sessions consisted primarily of a report by *T.J. Cooney* on the research efforts within the Teaching Strategies Project of the GCSLTM. The report consisted of two parts. The first stressed the conceptual framework of the research including both theoretical bases and practical considerations for classroom use of teaching strategies. The second emphasized the current research activities of the three working groups within the Teaching Strategies Project.

The second session focused on identifying various teaching behaviors that deserve special consideration in teacher education programs. Among these identified were: asking questions, evaluation techniques, diagnosing student learning problems, flexibility, motivational techniques, classroom management techniques, and identifying mathematical situations in life-like situations. Techniques, for example microteaching, for prompting the development of these behaviors in preservice teachers were considered. There was some discussion on how to provide field experiences which focus on the totality of teaching and yet provide practice in developing specific teaching techniques.

The remaining session was devoted to descriptions of programs and problems in training teachers in various countries. *J.D. Austin*, *Th. Cooney*, and *C. Hollingsworth* elaborated on teacher education programs in the United States in general and on the programs at their respective institutions in particular. *H. Glaser* and *K. Heinkel* described programs at their institution in West Germany. *O. Zaslavsky* presented the teacher education program that she is a part of in Israel. *M. Mokgokong* discussed problems associated with development a teacher education program in Swaziland and also the problems of recruiting qualified young people into the profession. *P. Gazzola* elaborated on problems associated with teaching high school in Canada and West Germany. Although it is difficult to generalize across the various situations, it did appear that most programs are struggling with problems of recruitment, placement of teachers, and striking a balance between theory and practice in devising teacher education programs.

SG D
Minimal Competencies in Mathematics

Chairman: J.T. Fey, USA

One of the most urgent practical problems facing mathematics education today is the challenge to define minimum levels of competence in mathematics and to devise teaching strategies that ensure student achievement of that competence. The phrase 'minimum competence' has many meanings. For parents and employers of school graduates, dismayed by an apparent recent decline in student mathematical achievement, 'minimum competence' often implies the arithmetic skills essential for survival in daily life and occupation in business or skilled trades. Teachers, confused by urgings of curriculum innovators and the criticism of skeptics, seek the security of widely accepted standards for mathematical performance at various levels of schooling. College and university mathematics faculty also hope to improve the performance of entering students by specifying minimum levels of secondary school preparation.

These varied meanings of the phrase 'minimum competence in mathematics' attracted 40 people from 15 countries to the three meetings of Study Group D at the 3rd ICME. Short presentations by *U. d'Ambrosio* from Brazil (Specifying Minimal Competencies in a Developing Country), *H. Karcher* from the Federal Republic of Germany (Minimal Competence for University Entrance), and *W. Kilborn* from Sweden (Pedagogical Problems in Minimal Competence Programs) helped to identify dimensions of the problem and to focus subsequent small group discussions. While much of the work of these sub-groups involved sharing perspective and experiences from different countries, the discussions led to enumeration of major concerns that should be addressed by all who are involved in the 'minimal competence' problem area.

Survival Mathematics. In several countries - for example Brazil, Sweden, and the United States - public concern and governmental dictates have led to preparation of lists of specific mathematical skills to be acquired by all students before leaving school. Obviously the school leaving age and expectations differ from country to country. In some developing countries two or three years of schooling are the best that can be expected. But in nearly every situation where 'mathematics for survival' is being defined, mathematics teachers have expressed serious concern about the quality of such objectives and the impact of *minimal competence thinking* on the total mathematics curriculum. The common definitions of survival mathematics seem exceedingly modest goals for school instruction, yet there are indications that specified minimums become ceilings for the expectations of teachers. Is a student who has demonstrated such minimal competence adequately prepared to exercise intelligently his political rights and responsibilities? Is he prepared for lifelong learning that will open a full range of economic opportunities? Is the quest for certified minimal competence likely to produce higher or disappointingly low levels of general achievement? The verdict is still out.

Pedagogical Problems. When a school system sets its priority goal to bring all or most students to specified minimum levels of competence, several serious instructional problems arise. First, teachers at every school level are compelled to face, quite properly, the presence in their classes of many students who lack skills normally taught in earlier classes. The necessary remedial instruction proves extremely difficulty under best conditions; it is specially challenging when the students in need are members of the heterogeneous classes now dictated by social policy in many countries. The most natural approach to students of diverse mathematical background - some form of individualized instruction - has not been without problems. Several experiences reported to the Study Group suggest that the self-pacing aspect of individualization leaves many students far behind 'grade level expectation'. A report from Holland suggested that teacher-paced study on basic mathematical topics, supplemented by individualized enrichment, might be a promising approach to this difficulty.

The second major pedagogical problem area arising from minimal competence concern centers around evaluation practices. Many emerging programs make heavy use of testing to measure progress toward and attainment of minimum competence levels. In several countries satisfactory performance on a minimal competence test is necessary and/or sufficient for school leaving. However, Study Group D participants identified very serious issues in development of such testing programs. Who should be responsible for developing the evaluative criteria and instruments - teachers, administrators, or parents? What is acceptable performance for demonstrating mastery of minimal skills? Current practice in the United States is to set arbitrary 'percent correct' standards. The Study Group pointed out that schools which certify the competence of graduates in such capricious fashion face ethical and possibly legal accountability for their subsequent performance. Minimal competence testing is commonly in the form of collected short answer paper and pencil items. Is this a valid measure of functional literacy in mathematics? How can the information from testing be used effectively for improvement of instruction?

University Entrance Standards. In Germany, the 'Deutsche Mathematiker Vereinigung' (DMV) has recently proposed guidelines for the minimum mathematical knowledge and skills that should be possessed by all entering university students. The Mathematical Association of America (MAA) has a committee active on the same task, and Study Group participants reported similar efforts in Sweden and Austria. The Study Group identified three major aspects of the secondary/tertiary transition problem:

(1) Topics supposedly covered in the secondary curriculum are frequently not available when needed in later study. Consequently there is a widespread and growing need for university remedial courses.

(2) Many students entering university study are unable to see the relations between specific ideas or skills and other areas of mathematics, and they are equally unable to relate mathematics effectively to problems in the 'real world'. This problem is, in no small measure, a consequence of weak text material and inadequate knowledge on the connections between mathematical ideas and applications.

(3) Students leaving secondary school appear to suffer from a narrow and formal approach to mathematics. They have not had sufficient opportunity to experience genuine discovery in mathematics learning and thus lack valuable intuitive insight into the content and methods of the subject.

Specification of minimal mathematical preparation that will alleviate the problems identified above is an extremely challenging task. The Study Group urged active participation of university mathematicians in setting such expectations, but cautioned that all material and curricula that result from such consultation should be throughly classroom tested and revised where necessary before large scale implementation.

Conclusion. Despite the often vague and diverse meanings ascribed to the phrase 'minimal competence in mathematics', Study Group D participants found the exchange of experiences valuable. They look forward to continued communication with each other and anyone who has interest in an aspect of the problem is welcome to join this communication by contacting the reporter.

4.4 Workshops (W)

W 1
Continuous Workshop of Association of Teachers of Mathematics

Chairman: A. McIntosh, GB

The facts are these: ten members of the Association of Teachers of Mathematics (four teachers, six involved with pre- and in-service education of teachers), together with ten 13-15 year olds from two Midlands comprehensive schools, ran a continuous workshop from 9.00 a.m. until 6.00 p.m. throughout the Congress in a glass fronted room delightfully called 'The Green Grotto' on the Karlsruhe campus. We brought with us a large variety of carefully selected mathematical materials ranging from clocks to counters, from games to geoboards, together with various books, pamphlets and all sorts of paper.

During the course of the Congress we were visited by about one third of the delegates, many of their wives and children (some of whom stayed for several hours each day) and a trickle of teachers and children from local schools.

The Association had organised a workshop with children at the 1968 Lyons Congress, and had been associated with the British presentation at the 1972 Exeter Congress which took a similar form.

Our intentions in organising a continuous workshop at the 1976 Congress were diverse, but they included:

a) to provide a milieu in which doing mathematics and talking about mathematics were combined: and, knowing from past experience that much of the work of large congresses is of necessity formal and pre-planned, to provide an atmosphere in which individuals from different countries could meet and be stimulated to share experiences. In this respect the stimulus provided by the presence of children was often crucial in breaking down barriers between delegates;

b) to declare by the range of materials that we believe them to be essential in the mathematical education of children (as well as stimulating to adults);

c) to show delegates a group of children engaged in mathematical investigations directed by teachers: the fact that these investigations were carried on over four days allowed delegates to observe the progress (and the continued concentration) of children working in this way and to talk to them freely;

d) to indicate by the arrangement and style of the room (some easy chairs, tables informally grouped, areas for browsing, for talking informally or arguing heatedly, for working with materials, for reflecting, for relaxing) an atmosphere which we think conducive to mathematical studies (whether amongst adults or children);

e) to use positively the fact that many delegates bring husbands, wives, children to such a Congress: to provide a room where they could feel a welcomed part of the Congress; a room in which at the same time adults could talk, pupils work at mathematics, visitors observe, children make models or play games, and in which the presence of each group actually stimulated rather than interfered with the purposes of other groups;

f) to create an atmosphere which reflected some attitudes of the Association of Teachers of Mathematics and so introduce the Association and its work to others.

On reflection we believe that, given the inevitable limitations of a relatively small-scale venture arranged largely through correspondence in another country, we succeeded in making a unique and distinctive contribution to the Congress. We were lucky in occupying a room imaginatively chosen by the Congress organisers. We were also greatly impressed by the extraordinarily thoughtful and friendly co-operation we received both before and during the Congress from the local organisers at every level from Congress secretary to student helpers.

It is difficult to be objective about the impact made by the workshop on delegates who approached the workshop with widely differing expectations and from so vast a range of backgrounds and frames of reference. Some came, saw and retreated; some enjoyed a stimulating glimpse of a distinctive attitude to mathematical education. Many, we know, found it a welcome oasis in a vast and bemusing landscape of sometimes turgid and overformal sessions: a place where stranger could meet stranger and communicate on a personal and professional level.

Much valuable work took place because of the Congress; but it is important to consider how much of that work necessitated the presence of 1800 delegates during the week, and what delegates gained which they could not almost equally well have gained by reading the preliminary documents and the proceedings of the Congress. We believe that the workshop is one style of presentation which specifically uses the personal presence and participation of delegates, and that such a contribution is capable of playing a major role in future Congresses; but finances alone will make it difficult if not impossible for the Association on its own to offer even so limited a facility at the 1980 Congress in the United States. We hope earnestly that the enthusiasm and expertise of the Association will be called on by those responsible for the next Congress to help in organising a similar contribution on a larger scale and playing a wider role.

W 2
Spontaneous Modelling Exercise

Chairman: Christopher Ormell, GB

Two sessions were held. The two final sessions had to be cancelled, however, because of Ch. Ormell's departure.

The first session began with a brief introduction to the idea of modelling situations, both as a way of tackling existing problems and of tackling new problems arising from the exploration of specific innovative ideas. *Ch. Ormell* started by disclaiming any intention to wish to do advanced applicable mathematics badly in public. The object of the two sessions was not to get into really difficult mathematical areas, but to think aloud and to demonstrate the 'art of applicative simplification' on real problems. In many cases it was possible to find 'intellectually honest reductions' (to use Bruner's memorable phrase) of variants of real problems which could be understood by intelligent 16 and 17 year olds. This was the aim.

The starting point was always a real 'problem', 'awkwardness', or 'difficulty'. Mr. Ormell mentioned an example which had occurred only a few days earlier whilst on holiday with the family. His neice's 'flip-flop' (beach sandal) had broken after only about a week and a half's wear on the beach. This produced a real-life 'difficulty' or 'problem' in the sense that it was awkward and uncomfortable to walk with a broken flip-flop. It immediately prompted the question: why do not the manufacturers make their flip-flops stronger? The diagram shows the construction of the sandal in side view:

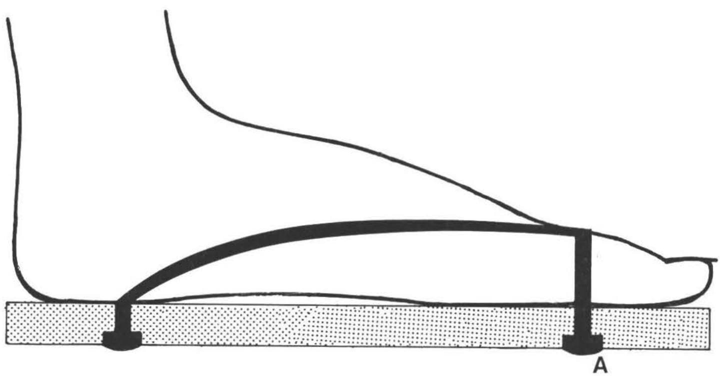


Figure 1

The bolt on the end of the post at A had broken so that there was nothing to attach the straps of the sandal to the front end of the sole.

The problem facing manufacturers was that there was a conflict of criteria. It would be possible to use a harder and stronger grade of plastic on the bolt but this meant that the straps of the sandal would be harder and stiffer. So improved reliability could be purchased at the cost of loss of comfort. A flip-flop could no doubt be made on which the probability of the bolt breaking was effectively zero. The reliability of such a sandal would be excellent: but this would hardly be relevant if the sandal were so uncomfortable that no child would wear it!

The 'problem' therefore lay in striking a balance between reliability and comfort.

To tackle it we needed two kinds of information. First we needed the functional model connecting the grade of plastic used, n, with the degree of subjective comfort c experienced by the wearer. Second we needed a functional model connecting the grade of plastic used, n, with the probability, p, of fracture of the bolts after a fortnight's average wear. Realizing that much of the problem could be reduced to this constituted 80% of the hard work.

The applicative simplification which would bring the problem within the reach of 16-17 year olds consisted in assuming that in certain circumstances *linear* models for the two connections would provide a reasonable approximation to the situation. One then obtained the following graphs:

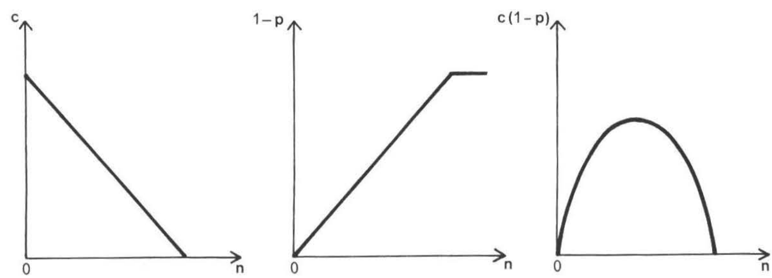


Figure 2

Having given this example, *Ch. Ormell* then called for problems from the audience. Four main problems emerged:

(1) What was the minimum gap within which a car could be parked against a kerb? How could one tell whether it was feasible to try

to park a car in a given gap?

(2) How to set course in a sailing boat to cover a given distance *AB* up-wind in the shortest time, the wind being from the North?

(3) How to tell from the information one could infer about the flow of people walking *away* from a restaurant, whether it was worth while continuing *towards* it?

(4) The problem of lack of tickets for the Congress Concert. How large an auditorium should the organisers have hired in order to satisfy as many people as possible, but with the minimum risk of over-provision?

The following conclusions were reached:

The car parking problem could be approached in two different ways, (a) in which it was desired to park the car using one movement only, and (b) in which unlimited numbers of reversals were allowed. The solution to (a) was found using classical geometry. The solution to (b) was suggested by *Dr. John Baker* of the Open University: a car of length l may park in a gap of length g if and only if $l < g$. After much argument and counter argument it was agreed that this was correct. *Owen Storer* estimated that, with a mini and a gap 10 cm longer than the car, at least sixty reversals would be needed. *Ch. Ormell* observed that there would be a bald patch on the two front tyres by the time that the car had been actually parked! Combining the two variations of the parking problem *Ch. Ormell* suggested a diesel-electric 'caster car' with its four wheels driven by independent electric motors, and each capable of turning through 360°. Various aspects of this idea were discussed, including the need for a 'turn inhibit device' to prevent inexperienced drivers from turning the car over in their haste to turn through acute angles!

The sailing problem was tackled by postulating that it was possible to sail the yacht at a speed $v(\theta)$ metres per second in a direction making θ degrees with the East. This gave a closed polar curve which was symmetric about the North-South line. An horizontal tangent to this was drawn to the curve as shown in Figure 3: this defined two courses θ_1 and θ_2 as most efficient for covering northerly distances.

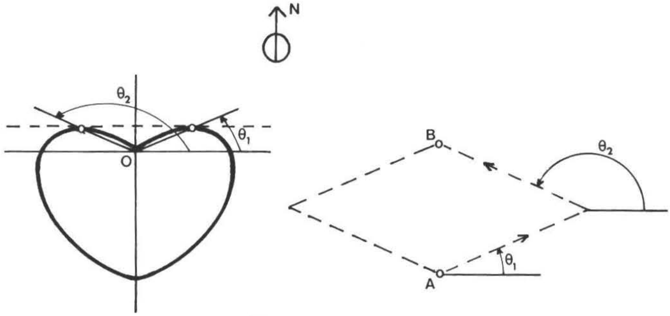


Figure 3

The best course was one of two legs: the first at θ_1 and the second at θ_2 (or vice versa).

The third problem was tackled after a long discussion had led to a particular set of simplifying assumptions being adopted. An algorithm involving graphs was then developed to decide whether or not the chance of getting lunch was greater or less than a pre-assigned value.

The fourth problem, too, depended on adopting suitable simplifying assumptions, but time ran out before a suitable set could be agreed.

To sum up: the two sessions produced a variety of interesting formulations, ideas and solutions. Of the problems for which simplified formulations were eventually adopted, all produced interesting discussions and all were successfully solved.

W 3
Introduction to Computer Graphics

Chairman: W. Zorn, FRG

1. *The purpose of 'Computer Graphics'* (CG) is to simplify the dialogue between man and machine. If we look at the following Table we can see where CG comes into data processing.

output / input	picture	alphanum. or function
picture	1	2
alphanum. or function	3	-

Field 1 and field 2 represent the areas of computer graphics called 'pattern recognition' or 'picture processing'. In both cases the input is a picture or a pattern.

Field 3 with a mathematical function, a user action, or some alphanumeric data as input, and a picture as output, represents the field of data processing normally known as 'Computer Graphics'.

The above diagram may give the impression of widespread use of CG, whereas actually, the share of CG in the total amount of data processing is still quite small.

Reasons for the limited application are:
a) in the past, graphic input and output devices were very expensive;
b) the lack of standards in CG;
c) the 'batch orientation' of earlier operating systems heavily restricted the efficient use of CG due to the lack of interactivity.

The dramatic reduction in hardware costs during the last few years and the fundamental research in CG are going to cause a gradual increase in the use of CG. Some applications of CG will be mentioned in the following.

2. *Graphic display of functions:* In many cases, the graphic display of functions or computed results is more illustrative than long columns of numbers. Two and three dimensional functions are good examples of this. The use of colour enables the display of a fourth dimension.

3. *Simulation:* Simulation of technical processes and their graphic

display is an area of increasing use of CG, favoured by considerable reduction in costs and complete certainty of experiments.

4. *Computer-aided design:* The speed of making a design, using the computer and CG in an interactive way, can be a multiple of that achieved with pen and paper only. Modern techniques use large data bases containing the description of parts, competent groups and complete constructions.

5. *Pattern recognition and picture processing:* Pattern recognition and picture processing belong to the field of artificial intelligence and use CG in nearly all steps of processing, e.g. the scanning of the graphic input data, the processing of pictures and producing graphic output as the result of the classification process.

There are still many unsolved problems in this field. Due to the fact that pattern recognition is of great interest for military observation, its research is strongly promoted.

6. *Computer art:* While computer art may not be approved of by the public, related areas such as the computer-aided design of fabric patterns, computer-aided illustration or computer-aided picture animation for advertising or motion pictures are valuable applications today.

7. *Computer-assisted instruction (CAI):* The high instructional value of the graphic display of information is evident. For this reason CG is favoured for CAI. However, up to now high terminal and computer costs prevented its widespread use.

8. The points mentioned above show that CG has a wide range of uses in data processing. Reduced hardware prices, software improvements and technological advances will bring about its large-scale use.

W 4
Teaching of Elementary Programming Concepts with a Flexible CAI-System

Chairman: A. Schmitt, FRG

In this Workshop one of the teaching programs of the Karlsruhe LEKTOR installations was presented and analysed.

LEKTOR is a flexible CAI-System which has been designed for writing complex programs controlling man-machine dialogs. The system was developed and implemented by the Computer Based Learning Research Group at the Faculty of Informatics, University of Karlsruhe.

The LEKTOR-System is in operation since the end of 1973. It is used to implement interactive programs of widely differing strategies in order to test new concepts of computer based learning as well as man-machine communication. Some of the unique features of the system are the graphic input and output allowing more natural communication with a computer than with alphanumeric terminals only.

One of the more frequently used teaching programs within the Karlsruhe installation of the LEKTOR-System is a simple interpreter for the definition and evaluation of possibly recursive function. This interpreter, called MATHCALC (German: 'Rekursivarithmetik') was developed to test some important didactic principles. The goal was to create a comfortable, simple to use 'desk-top-calculator' with a maximum of mathematical principles integrated.

The major didactic thesis of the MATHCALC design may be stated as follows: An interactive programming language as a tool in mathematics instruction has to reflect mathematical structures and mathematical manipulation of objects as far as possible. Therefore, the overall notational conventions in mathematics, guided by very few rules, have to be the basis of interactive programming in mathematics.

Some of these rules are:

(1) A new function definition may only contain names of functions and operations which have already been defined.

(2) More complex and complicated functions are formed by successive definition from simpler and more elementary ones.

(3) Functions may be altered or completely changed by redefinition. The possibly new definition substitutes the old one.

(4) The naming of free variable is of no semantic import. The definitions F(X):= X - 3; and F(A):= A - 3; define the same function.

(5) A once created function, similar to a verified mathematical theorem, can be used wherever it is of use.

The workshop was carried out in the following manner:

(1) Introductory lecture, lasting about 30 minutes, where the CAI-System LEKTOR was introduced and the teaching program MATHCALC ('Rekursivarithmetik') was analysed from a didactic point of view. The applications were demonstrated using several problems settings relevant to the classroom.

(2) Practice phase on terminals lasting about one hour. Participants used MATHCALC and other teaching programs of the LEKTOR-System in the terminal room assisted by tutors in case of operating difficulties. Each workshop participant received a worksheet with several problems of various levels of difficulties which were to be solved with the help of MATHCALC.

4.5 Projects

CAVA
Computerunterstützte Analyse und Vergabe von Aufgaben

The CAVA project (computer aided analysis und distribution of exercises) contributes towards the automatic/partially automatic correction and evaluation of exercises in a general sense. The collection of exercises developed for this purpose in mathematics by the Institut für Bildungsinformatik of the Forschungs- und Entwicklungszentrum für objektivierte Lehr- und Lernverfahren were specifically tested in the correspondence course 'Grundkurs der Mathematik' (Deutsches Institut für Fernstudien). The demonstrations especially emphasized the problems raised by multiple-choice exercises in mathematics. The multiple-choice exercises on differential and integral calculus developed by CAVA were presented. As it is impossible to cover all common mathematics exercises by the multiple-choice type, freely formulated exercises which, however, must be corrected according to a scheme that can be processed by the computer were also allowed.

The participants in the demonstration were very interested in both types of exercises as well as in the solution proposed here. The attention of the participants was also drawn to the computer aided organization of the correspondence course 'Grundkurs der Mathematik' and to the computer aided distribution of exercises. The incoming solutions of the exercises, the treatment of the exercises by the corrector and the computer, and the computer-made letters sent out to the students containing both the annotated solutions of the exercises and the new exercises were shown on the screen. The significance of the project for teacher in-service education in Nordrhein-Westfalen (FRG) was stressed.

The main aim of the demonstration was to make plausible that computer technology helps organize correspondence courses and gives tutors the possibility to provide the students with individual counselling, since part of the routine work is carried out by the computer.

Address: Forschungs- und Entwicklungszentrum für objektivierte Lehr- und Lernverfahren (FEOLL), Rathenaustraße 69-71, 4790 Paderborn 1, FRG.

CRDM
Centro Ricerche Didattiche Ugo Morin: Self-Administered Project for the In-Service Training of Mathematics Teachers

1. The Centre

The Ugo Morin Centre for Didactical Research was founded in 1968 as an autonomous and private association of teachers with different backgrounds representing various kinds of instruction to permit an exchange of views and mutual assistance for a permanent pedagogic and cultural retraining of mathematics teachers.

The centre has expanded very rapidly and has by now reached as many as one million Italian teachers, above all in the regions of Venezia, Lombardy and the Piedmont. It publishes a journal and organizes annual seminars and periodical retraining sessions for teachers.

From this point of view, the Centre has elaborated a theory linked with some postulates derived from Bruner's work and with a few principles inspired by Piaget. Proceeding from this theory, the idea was to contribute concretely to the retraining of teachers by setting up a service that could be administered by the teachers themselves and used by them either directly in classes or as an initiator of new pedagogic activities.

As a consequence, a few exhibition-laboratories were organized in collaboration with the University of Pisa (Mathematics Institute) and the University of Pavia (Mathematics Seminar).

Two examples are given here, one prepared under the leadership of Prof. Mario Ferrari from Pavia University.

2. The Exhibitions

Both exhibitions are designed to create situations in which visitors are invited to manipulate objects, to take part in group discussions, and to arrive at mathematical results, simultaneously assuming a correct behaviour towards reality and a permanent capacity for self-correction and self-retraining.

These situations are generally inspired by games and stories chosen from the best known entertaining mathematical literature, whereas the objects offered for manipulation can easily be realized in versions different from those presented here, and with the means at one's disposal, by using objects made by the pupils themselves.

2.1 *The Checcucci Exhibition.* It proceeds from the premise (inspired by Bruner and Piaget) that any programming of a teaching activity in class must progress through 3 levels, which may be called:
- the level of perception,
- the level of internalization,
- the level of the final acquisition of the concept.

It must be based on concrete and interesting situations provided by real life, with which the pupil must stay in continuous interaction, the final end being to help him develop his imagination, his creativity, and his social sense.

The mathematical content of this exhibition is virtually the conquest, or the mastery, of the 'number' understood as a powerful means for grasping reality. Hence: all topics which constitute the skeleton of a primary school curriculum are implicated: multibase calculation, positional systems, arithmetic operations, logical operations, geometry (e.g. the use of symmetries in combinatorial analysis refers to isometries), combinatorial analysis and probability. The situations lend themselves to extensions and more thorough investigations of important mathematical character; for this reason, the exhibition can be used at the level of junior secondary school and even of senior secondary school (expansion in series, advanced descriptive statistics, systematic use of calculators etc. are but a few examples).

2.2 *The Ferrari Exhibition.* It is inspired by the same principle as the one above but develops a single subject: plane isometries at the level of the lower grades of secondary school (age group 11-14).

The preparation of this exhibition is easier than in the preceding case in that it tries to avoid language, which, in the other one, is often intentionally ambiguous and provocative. It seeks to lead visitors step by step to the mastery of all concepts in a general vision of the theory of plane isometries. Accordingly, the content is strongly unitary: exhibition of plane isometries following the axiomatics proposed by Choquet, in which axial symmetry plays a fundamental role. The objects offered for manipulation in this exhibition are few in number and can be constructed with the means at disposal in a school workshop. The posters, on the other hand, are a unit that cannot be easily replaced or altered, and in this regard they differ substantially from those of the Checcucci Exhibition, which are more open to all kinds of variations and adaptations at different levels.

2.3 *Conclusions.* Both exhibitions have been subjected to different control tests in several Italian cities from the extreme North to the extreme South of the Peninsula. Everywhere visitors showed great interest: the best results were of course registered with those who were fully absorbed by manipulation, discussion, calculation, etc. The Checcucci Exhibition proved more difficult to approach, especially for older teachers, owing to its 'break' character, whereas it created more enthusiasm among pupils.

These exhibitions are now on permanent display at the Centre and are periodically visited by teachers and their pupils; they are also often placed at the disposal of teachers and schools.

Address: Centro Ricerche Didattiche Ugo Morin, Casella Postale 141, 36061 Bassano del Grappa, Italy.

DIFF
Deutsches Institut für Fernstudien: Distance Study Courses in Mathematics

DIFF tried to give the participants of the 3rd ICME an impression of its work. Further information about the courses could be received from the presentation of the printed materials and by discussion with members of the project group.

The *German Institute of Distance Studies (DIFF)*, affiliated with the University of Tübingen, was established as a foundation in 1967 in co-operation with the University of Tübingen. The main function of the DIFF is to investigate the possibilities for a scholarly education imparted by means of non-personal media.

The Mathematics project group of the DIFF is located at Rheinstraße 12, 7800 Freiburg, FRG.

1. Basic Mathematics Course (Grundkurs Mathematik)

During 1967 - 76 the distance study course 'Grundkurs Mathematik' was developed.

The 'Grundkurs Mathematik' gives the scientific theory needed as a background to school mathematics. It is used to train teachers of mathematics for the lower secondary level (grades 5-9/10), as a refresher course for teachers at both lower and upper secondary levels and at teacher training colleges to instruct student teachers. The distance study system enables teachers to take this further-training course without being released from teaching duties.

The course covers the following five subjects with altogether 19 study units:

I. *Sets, Mappings, Natural Numbers*
Sets / Mappings, Relations / Natural Numbers / Elements of the Theory of Numbers.

II. *Arithmetic, Algebra*
Algebraic Structures, Extension of Number Systems / Elements of Group Theory / Rings, Divisibility / Order, Real Numbers.

III. *Geometry, Linear Algebra*
Elementary Geometry, Part 1 / Elementary Geometry, Part 2 / Systems of Linear Equations and Inequations / Vector Spaces, Affine Spaces / Linear and Affine Mappings / Euclidean Geometry.

IV. *Calculus*
Real Functions, Limits, Continuity / Integral Calculus / Differential Calculus.

V. *Theory of Probability, Data-Processing*
Electronic Data-Processing / Theory of Probability.

2. Introduction to Integral Calculus (Einführung in die Integral-Rechnung)

In 1970 - 71 the DIFF and the Zweites Deutsches Fernsehen produced a multi-media distance-study block 'Introduction to Integral Calculus' intended for students of higher education. This block includes print-base materials and TV broadcasts. It was tried out at various universities using tutorials and tests.

3. Mathematics for Elementary School Teachers (Mathematik für Grundschullehrer)

The distance study course 'Mathematics for Elementary School Teachers' (Mathematik für Grundschullehrer) was planned in 1970/71 and developed between 1972 and 1976.

Addressees of the course are teachers of grades 1-6. The course includes 19 units, which are grouped in four large study packages:

Package 1
Sets and Their Representation / Operations with Sets / Cardinal Numbers, Addition and Subtraction / Multiplication and Division / Number Systems.

Package 2
Elementary Logic / Relations, Order and Equivalence Relations / Quantity Systems - 'Größenbereiche' - / Algorithms, Written Calculation Methods.

Package 3

Operator Games / Mappings and Their Composition / Algebraic Systems and Their Properties / Groups / Elementary Geometric Concepts.

Package 4

Theory of Divisibility / Prime Numbers, Factorization / Fractions (Positive Rational Numbers) / Integers, Rational Numbers / Elementary Functions, Applied Arithmetic.

Each unit is arranged in three parts: The *F-part* (mathematical theory) supplies the teacher with the background knowledge which is indispensable for modern mathematics instruction, independent of special methodical approaches. After this the *D-part* (methodology) points out the significance of the treated subjects for school lessons as well as offering motivational elements, illustrations and types of exercises. The exercises of the *A-part* (worksheets) are intended to encourage the teacher to work actively on the treated mathematical subjects and to make him thoroughly familiar with concept formation and argumentation in every case.

4. Mathematics for Teachers at Lower Secondary Level / General High Schools (Mathematik für Lehrer der Sekundarstufe I/Hauptschule)

The distance study course now being developed tries to treat General High School maths mainly as applied maths. The applicability of the maths taught should not only be evident within the discipline of mathematics, or of natural sciences, but must also include the students' everyday life and the world of their later careers. These characteristic features should include topical applications and a mathematical approach. Together with these comes the acquisition of techniques, including automatic processes and calculating skills, not neglecting modern aids.

Planning of the course started in 1976.

The new distance study course is a continuation of the course 'Mathematics for Elementary School Teachers', the entire 4th package of which is suitable for the General High School teacher too.

The projected contents of the new units are:

Applied Arithmetic (Percentages, Interest, Ratios) / Geometry (possibly in two parts, one of which will deal with technical, constructive, descriptive aspects) / Calculation of Probability, Statistics / Decimals (e.g. approximations, approach to real numbers) / Equations and Inequations / Language of Algebra (e.g. working with variables, formulae) / Special Functions (e.g. mappings of increase and decrease, trigonometric functions) / Simple Calculators (use of basic procedures, programming small calculators) / Finite Graphs (e.g. schedules, flow charts, tree diagrams, networks).

As in the correspondence course 'Mathematics for Elementary School Teachers', the units will be divided into three parts: mathematical theory, methodology, and worksheets.

5. Mathematics for Teachers at Upper Secondary Level (Mathematik für Lehrer der Sekundarstufe II)

This course (since 1976) is designed for teachers at the upper secondary level in 'Gymnasien', Technical 'Gymnasien', and Vocational High Schools, which lead to Abitur or entrance examinations to Polytechnics. For some topics from Probability/Statistics, various examples of different methodical approaches are presented and compared. Special attention is paid to applications. The alternatives presented for individual topics should be immediately applicable to the school situation.

Calculus

Approaches to Differential Calculus / Approaches to Integral Calculus / Different Levels of Stringency in Working with Calculus / The Mean-Value Theorem and its Applications / Exponential and Logarithm Functions, Increase Functions / Functions of Angles and Oscillation Processes.

Probability and Statistics
Approaches to the Concept of Probability / Calculation of Probabilities /Random Variables and Distributions / Introduction to Statistical Inference.

The DIFF is very interested in making contact with institutes in other countries which are working on similar projects in order to exchange ideas. For instance, an exchange of published materials could be envisaged.

Address: Deutsches Institut für Fernstudien, Rheinstraße 12, 7800 Freiburg, FRG.

DMP
Developing Mathematical Processes

Developing Mathematical Processes (DMP) is a research-based, activity oriented instructional management mathematics programme currently under development at the Wisconsin Research Center for Cognitive Learning. It is designed for use in a system of Individually Guided Education (IGE) in a multi-unit elementary school; however, it can be used effectively in any elementary school. The DMP programme will consist of approximately 90 curriculum packages, each of which will contain appropriate Teacher's Guides, Student Booklets, Textbooks (for intermediate levels), testing materials, and kits of manipulative devices and supplementary printed materials.

Address: DMP, Wisconsin Research and Development Center for Cognitive Learning, University of Wisconsin, 1025 West Johnson Street, Madison, Wisconsin 53706, USA.

INRP
Institut National de Recherche Pédagogique: Mathematics at the Elementary Level

Several research projects are being conducted at the moment by the research unit 'Mathematics at the Elementary Level'.

1. Research Project 71-02.2.03: Study of the role played by means of expression in mathematics learning

1.1 *Research topic*. The object of the research project is to study the role of means of expression and/or representation in mathematics learning.

These means of expression and/or representation are considered as means for communicating (for instance, in a message) mathematical information (relations, organization, and so on).

They are investigated from two points of view: The first one is to know to what extent children, on completion of a learning process, use means of expression and/or representation in situations which cannot be solved without using these procedures (mastery). The second is to know to what extent they use these procedures in situations in which they are not mandatory (availability).

They are treated from a pedagogical viewpoint, i.e. from the aspect of the communication of a message (as opposed to a mode of investigation that could be more psychological: means of expression envisaged as expression by the individual).

1.2 *Population*. The research project is organized around 20 teams distributed over different school districts. Each team includes a maths teacher, psychologists and teachers serving in experimental classes. The experimental population consists of about 50 classes (1200 pupils) at each level.

1.3 *Publications*. A first experiment at the preparatory level (6-7) was completed in 1974 and gave rise to the following publications:

a) Recherche Pédagogique No 2381 (June 1976) SEVPEN Mathématique au Cours Préparatoire
Compte-rendu d'une expérimentation: 'Représentations ensemblistes' (196 pages).

This document contains:
- the description of the chosen methodology,
- a description of the tests and of the corresponding evaluation systems,
- the outcome with the corresponding treatments and analyses,
- the pacing at the preparatory level (ages 6-7).

b) Revue Francaise de Pédagogie no 36 (July 1976) SEVPEN 'Expérimentation sur les représentations ensemblistes au Cours Préparatoire'.

This essay is particularly devoted to the description of the goals and methodology of the experiment.

Also published were a series of documents indicating a pacing and control test for each level of Primary School:

c) Mathématique au Cours Préparatoire
Document de Recherche no 1 (72/73), 102 pp; no 2 (73/74) 128 pp; 'Numération' (74/75), 70 pp.

d) Mathématique au Cours Elémentaire 1
Document de Recherche no 1 (73/74), 144 pp; no 2 (74/75), 194 pp; no 3 (75/76), 192 pp.

e) Mathématique au Cours Elémentaire 2
Document de Recherche no 1 (74/75), 190 pp; no 2 (76/76), 228 pp.

f) Mathématique au Cours Moyen 1
Document de Recherche no 1 (75/76), 198 pp.

g) Mathématique au Cours Moyen 2
Document de Recherche no 1 (76/77), 180 pp.

h) Géométrie au Cours Moyen 1 et Cours Moyen 2 (76/77), 127 pp.

2. Research Project 74-02.2.06: Evaluation of the mathematical behaviour of Primary School pupils

2.1 *Research topic*. This is a survey concerning the study of how pupils of the third year (CE 2) and of the fifth and last year (CM 2) of the Primary School behave in mathematics after 6 years of implementation of the new curricula. This survey has two complementary aims: on the one hand, an evaluation of teachers' attitudes towards mathematics instruction; on the other hand, an evaluation of pupils' attitudes toward mathematics instruction.

2.2 *Population*. A population of 4000 pupils for each level was determined at random from a representative sample of the population of the country.

2.3 *Timing*. This investigation will terminate in 1978. The first results concerning the CE 2 will come out in 1977.

3. Research Project 74-02.2.07: System for the permanent evaluation of maths instruction at the Primary School

This project will be started in September 1976. The purpose of this research is to extend the foregoing evaluation to all levels of primary schools and to study more thoroughly some important aspects which have emerged from the preceding project (74-02.2.06).

4. Research on D.G.R.S.T Programme: Study of problem solving procedures

4.1 *Topic*. The general goal is the analysis of the solving procedures of the pupil aged 8 to 11 faced with problems covering a very wide range of cognitive activities. Two types of cognitive activities will be particularly investigated: the information processing activity and the search for the information which is useful for solving a problem.

The problem will be presented in different forms and studied with diversified methods of observation.

4.2 *Timing*. Beginning: December 1976; end of project: December 1978.

Address: Institut National de Recherche Pédagogique, 29 rue d'Ulm, 75230 Paris, Cedex 05, France.

INTER-IREM
Instituts de Recherche pour l'Enseignement des Mathématiques

The 25 IREMs organized 6 information sessions (two in French, two in English, one in German, and one in Spanish) on their structure and fields of activities. Each session started with a film (in the language of the session) followed by a discussion during which questions raised by the audience were answered.

The following points were stressed:

(1) Teachers from all levels, from the primary school teacher up to the university professor co-operate in an IREM.

(2) All staff-members perform at most half of their work load at the IREM, and for the rest of time, go on teaching at the institution where they originally come from.

(3) The aims of the IREMs are research, pure and applied, related to the teaching of mathematics and in-service training of teachers. Far from being incompatible, these diverse aims complement one another. The work at the IREMs, which is conducted by teachers in service, essentially takes place in contact with daily practice and progresses through a continuous dialectic process: practice-theory-practice-theory etc.

(4) The 25 IREMs and the various teams working in each IREM are autonomous in the organization of their work. There are no 'IREM doctrines': diverging opinions can be voiced. The exchange of views, the related discussions and the preparation of research take place at workshops at the regional and national levels.

Parallel to these information sessions about the IREMs, an exhibition prepared jointly by the IREMs, the INRP (Institut National de Recherche Pédagogique) and APMEP (Association des Professeurs de Mathématiques de l'Enseignement public) presented at the congress some of the main directions followed recently by French research activities. The following topics of research should be cited among others: numerous investigations related to the Primary School; the study of the possible role of all kinds of calculators in mathematics education; the study of the problems raised by the teaching of geometry, of probability and statistics; the study of the liaison between the teaching of mathematics and of other disciplines (physics, technology, mother tongue, music, and so on); the study of learning processes and heuristic behaviours; the use of audio-visual aids; historical and epistemological research on the development of mathematics. A leaflet titled 'Nouvelles tendances de l'enseignement des mathématiques en France' was circulated in three languages (French, English and German) during this exhibition.

The IREMs wish to co-operate with institutions and teams which pursue similar aims in other countries.

Anyone who wishes to receive the above mentioned leaflet or wants to contact the IREMs should write to IREM (see below), which will then put him in touch with the teams whose work is of interest to him.

Address: Institut de Recherche pour l'Enseignement des Mathématiques, Université Paris VII, 2 place Jussieu, 75 Paris 5e, France.

IOWO

Instituut voor Ontwikkeling van het Wiskunde Onderwijs

In the first session IOWO presented an example of a mathematical theme 'Four-cube-houses' for primary schools.

By means of slides the audience was given an account of three lessons in the third grade of a primary school in Holland. The problems were presented to the pupils in the context of a story about dwarfs who wanted to build a new village in which each couple of dwarfs had equal houses, consisting of four cubes (a kitchen, a living room and two bed-rooms). The main problem presented was: how many essentially different houses of four cubes can be built for the village. In an open educational-learning situation, in which stress was laid on a discovery attitude, the pupils themselves constructed the 15 different possibilities. Systematization of the experimentally found solution led to the 'proof' of the exact number of 15 houses.

Other problems, presented in the same context, with aspects on computation, measurement, geometrical orientation, mathematical language and logical inference, concerned:

- groundplan (coordinates) of the village;
- different 'costs' of the 15 houses depending on the number of groundfields and cube-faces 'to be painted';
- constructing a central building in as many different shapes as possible (factorization of the number 60);
- constructing different houses, given their front- and side-views (and vice versa).

The theme was concluded with a number of working-cards in which the concepts, abilities and strategies, presented in these introduction lessons, were broadened and deepened.

In the second session IOWO presented Belvia - an example of curriculum development in practice on a geometrical subject. Participants received the Belvia textbook and the teacher guide.

In this geometry theme, pupils solve problems around the building of a bungalow - the name of the design is Belvia - they meet counting problems, scale problems, they estimate the areas of different plots, they make cut-outs of the bungalow. Groups work out the total amount that will be spent in building materials, design a new groundplan for the holiday bungalow, and so on.

In Belvia an attempt is made to build mathematical education on rich problems, fitting in the world of the 13-14-year-old child - problems with a large applicability for the pupil, inducing different levels of solution.

Viewing video pictures of pupils working with the Belvia materials the participants could get an impression of what is meant by words as rich problems, different levels, etc.

Belvia was designed in interaction with pupils, teachers, staff members. The principle of IOWO's activity is the simultaneous development of students' books, teachers' manual, teacher training materials as well as material suitable for refresher courses - a simultaneous development having the advantage of mutual influence. The strategy in the distribution stage was shown by means of the same theme.

After the presentation many questions were asked, discussions started. Contacts were made for further exchange of ideas and cooperation.

Address: Instituut voor Ontwikkeling van het Wiskunde Onderwijs, Tiberdreef 4, Utrecht/Overvecht, Netherlands.

MADIME
Integriertes Grundstudium von Mathematik und ihrer Didaktik im Medienverbund

The MADIME project is a university experiment conducted in two phases of 3 half-years each (winter semester 74/75, winter semester 75/76, winter semester 76/77) at the Koblenz Division of the Erziehungswissenschaftliche Hochschule Rheinland-Pfalz by the Mathematics Seminar and the Institut für Mediendidaktik. Following a recommendation by the Bund-Länder (joint Federal-State) Committee for Educational Planning, the project is funded jointly by the Federal Republic and the state of Rheinland-Pfalz.

1. *Aim of the project:* The aim of MADIME is the experimentation of a model for a 1 1/2 year basic education for mathematics teachers for the primary level and the junior secondary grades.

The following points are essential components of this model:

a) Instruction in pedagogy of mathematics begins in the first semester parallel with mathematical instruction.

Arrangement of time and content can be made such as to integrate the mathematical and subject-oriented pedagogical components into unified *basic studies*. This model pursues a reciprocal motivation between the mathematical studies and the studies in subject-oriented pedagogy as well as more practice-oriented teacher training.

b) The offered lectures and exercises are supplemented by the application of other media. All media used are coordinated into a *multi-media system*, which seeks to improve the offerings through a specific assignment of media and corresponding differentiations as well as through permanent availability of the non-personal offerings.

2. *Content:* Essential mathematical concepts for instruction in the first ten years of schooling are dealt with from the point of view of the discipline and of the pedagogy of mathematics studies. Sets, relations, functions, ordinal and cardinal numbers are treated in the first half year. Algebraic structures are treated in the second half year, the extension of the number system being especially stressed. The third half year is devoted to elementary geometry problems.

Following an introduction into problems related to the formation of concepts and the planning and performing of lessons, special questions of methodology are discussed and several teaching models are developed and compared with one another, parallel to the treatment of the mathematical content.

3. *Multi-media system:* Special textbooks give a relatively short presentation of the offered content; the other media are related to them. Lectures play the role of a guiding medium and supplement the presentations with other media. Practical courses give the students the opportunity to acquire a deeper knowledge by solving problems on their own. Transparencies assist lecturers and tutors in their lectures and tutorials. TV-transmissions (school TV, recordings of *teaching sequences*) are used for 'problematizing' teaching questions. Written teaching programmes provide individual learning opportunities for selected topics. Slides with synchronised sound accompanied by written working

material are offered at the audio-visual center for themes for which a visualization with a fixed picture seems especially appropriate. Tests are used for diagnosing performance and orienting learning behaviour. The lectures, exercises, and tests take place at definite periods of time. The use of the other media is largely free of time constraints.

4. *Accompanying Research:* Parallel to the implementation of the model, the attainment of its goals has to be assessed, and possible improvements must be suggested. Tests and questionnaires are the essential aids.

The outcomes of the first phase are the basis for revising the material and supplementing the offerings.

Address: MADIME, Erziehungswissenschaftliche Hochschule Rheinland-Pfalz, Abteilung Koblenz, Rheinau 3-4, 5400 Koblenz, FRG.

MM
Modular Mathematics

Modular Mathematics is a set of materials for use with classes of 11 to 14 year olds containing children of *all abilities*. The scheme was developed and tested under the auspices of the *Scottish Education Department*, and is designed to give children throughout the ability range a firm grounding in mathematical and numerical skills, and to do this in such a way that they are motivated to continue with the study of mathematics.

Within each topic, or Module, children work through a flow pattern of Workcards and Worksheets, and frequently using Apparatus. A strong feature of the scheme is the large amount of practical work involved. After completing a common core of work, children are then directed by their teacher into various optional patterns of work according to their performance on the core.

The material has been written with an individualised learning approach, so that children may work singly or in groups. This leaves the teacher free to sort out individual problems as they arise, while the rest of the class continue working on material which is within their own capabilities.

Modular Mathematics thus represents a new development in the teaching of mathematics to 11-14 year olds in the U.K., not only in terms of the presentation of the pupils' material and its style of organisation in the classroom, but also in the role of the teacher in the classroom.

Following a brief description of the design structure of the material, the discussion broadened to cover the problems which had been experienced by teachers from a number of countries in teaching mathematics to mixed ability classes. There was a feeling from most participants that this style of teaching does require the aid of material which has been specially designed to overcome the problems involved. The full range of published material was available for examination, and a number of teachers made arrangements to see the scheme in action at Scott Lidgett School.

Address: Modular Mathematics, Heinemann Educational Books, 48 Charles Street, London W1, GB.

MMP-MPSP
Mathematics-Methods Program, Mathematical Problem Solving Project

The Mathematics Education Development Center has been in existence at Indiana University since June 1971, and during this time has been primarily involved in two research and instructional development projects.

1. The Mathematics-Methods Program (MMP)

The first project, which has been supported by grants from the National Science Foundation since 1971, has culminated in the development of the Mathematics-Methods Program, a teacher education program in mathematics and mathematical pedagogy for general elementary teachers. (General elementary teachers are required to teach all subjects to children. Mathematics is just one of these subjects.) There are two components of this program: a university classroom component and a field experience component.

The university classroom component of the Mathematics-Methods Program integrates mathematical content and methodology in twelve units entitled:

Numeration, Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Analysis of Shapes, Measurement, Transformational Geometry, Experiences in Problem Solving, Graphs: The Picturing of Information, Number Theory, and Probability and Statistics.

Each unit focuses on a mathematical topic and on how that topic relates to the elementary school curriculum. The mathematics is developed in a small-group format, using a laboratory, problem-solving approach. Mathematical and pedagogical problems are posed to the learner who then has the responsibility to work toward a solution. The written units emphasize concrete embodiments of mathematical concepts, activities relating mathematics to its real-world applications, and the development of a repertoire of techniques for teaching each concept.

The field experience component of the Program has evolved as a *model* for early teaching experiences in mathematics. The purpose of these experiences is to allow prospective teachers to engage in a series of carefully sequenced, supervised activities in the elementary classroom designed to give them

a) confidence in working with children;
b) insight into child thinking and learning of mathematics;
c) an acquaintance with a wide variety of manipulative materials for enhancing the learning of particular mathematical concepts;
d) experience with developing and implementing a lesson.

An important goal of the Mathematics-Methods Program is to relate mathematics to the real world of the learner. Since the elementary teacher's real world includes the elementary school classroom, it is important to relate the mathematics training of a prospective elementary teacher to the classroom. The units attempt to do this in the university setting; the field experience component of the Mathematics-Methods Program carries the relationship much further.

At the present time more than fifty colleges and universities are using the Mathematics-Methods Program in the training of prospective elementary teachers.

2. The Mathematical Problem Solving Project (MPSP)

The second project of the Mathematics Education Development Center is the Mathematical Problem Solving Project, which was funded for two years by the National Science Foundation. The general goal of this project is to investigate ways to improve the problem-solving performance of children in grades 4, 5, and 6 (ages 9-12).

During the past two years, the efforts of the project have focused on classroom observation of children, on the development of materials, and on the trials of these materials in classrooms. An analytical approach and an experiential approach have been used in materials development.

In the analytical approach an effort has been made to identify the prerequisite skills for using a given problem-solving strategy, for example, trial and error or the 'guess-and-test' strategy. It was conjectured that children had to be willing to make a guess, able to see ways of testing their guess, and able to interpret the test in order to make a revised guess. To help children acquire these skills, modules consisting of lessons and problem decks were prepared. An inservice program with 24 teachers was conducted to provide feedback for module revision and to enlist the teachers' help in developing a module.

In the experiential approach to materials development, the materials consisted of a bulletin board display with four parts: the problem; hints to help the child get started; examples of how other children attacked the problem; and a set of questions which served to evaluate and/or extend the problem solution to new problems. Teachers were asked simply to give the children some time to work on the problems, on the assumption that the best way to improve problem-solving performance is to solve problems.

A significant component throughout the entire life of the project has been the evaluation efforts. These efforts include not only the formative evaluation of the classroom materials but also research activities related to the development of an instrument to measure the process of problem solving in children. Other evaluation efforts encompass the measuring of changes in teachers' willingness to teach problem solving and of attitudinal changes on the parts of both teachers and children.

Address: Mathematics Education Development Center, 814 E. 3rd St., Indiana University, Bloomington, Indiana 47401, USA.

OU
The Open University

The presentation of the Open University was centred on an exhibition that described the major features of the University, its general teaching methods and the particular methods developed by the Faculty of Mathematics. As well as looking at the graphic display, visitors were able to watch television programmes, listen to radio tapes, browse through correspondence texts, see interactive computer packages, and experiment with a small teaching computer.

The main sections of the display were as follows:

(1) *The Student and his degree* - how to become an Open University student, what it entails and the structure of an Open University degree.

(2) *Course Production* - each course involves many months of preparation by central staff and the B.B.C., the course team being the central feature of production.

(3) *The Mathematics Faculty* - a diagram showing the courses offered by the Faculty and how they are related.

(4) *Course Description* - details of some Mathematics courses including interfaculty courses.

(5) *Course Operation* - details on assessment procedures and how correspondence teaching operates in the regions.

(6) *Student Computing Service* - some aspects of how the computer is used in correspondence teaching.

(7) *Summer School* - describing the philosophy behind the week's residential course, with particular emphasis on problem solving.

(8) *T.V. and Radio* - illustrating the work required in the production of a television program from 3 months before recording up to the studio day; itemizing the different ingredients that make up a programme and the specialist people required to produce them.

Wherever possible, the exhibition was manned by members of the Open University who were attending the conference, and they were able to answer many questions from visitors who wanted more detailed information than the exhibition provided.

In addition to the exhibition, the OU/BBC presented colour films. The films shown included:

Real Numbers (PM 981/1)
Computing and Computers Case Studies (PM 951/1 and 10)
History of Mathematics (AM 289/2 and 5)
Finite State Machines (M 202/9).

Open University also presented a brief mathematical cabaret, entitled 'Chez Angelique', which was based on the Summer School programme for M 100.

Address: The Open University, Walton Hall, Milton Keynes, MK7 6AA, GB.

SMP
The School Mathematics Project

The School Mathematics Project presentation provided a visual exposition of the project, its current activities and its published materials. The static display explained the nature of the SMP as an independent educational body, whose published texts provide courses for pupils in the 11-16 and 16-18 age ranges, with appropriate examinations. Draft material and information on SMP 7-13 for younger pupils, due to be published in 1977, was also on display, as was draft material for the SMP/NEC (National Extension College) correspondence course. The full range of pupils' texts, with the supporting Teachers' Guides and Handbooks which the project has produced were available for inspection. The display emphasised that SMP works closely with practising teachers, both as authors, and in the in-service courses which it organises each year.

Considerable interest was shown in the many different language versions of the original texts that now exist, and other versions produced for use in countries outside the United Kingdom. Many participants were also especially interested in the project's materials for computing in schools, produced as enrichment to the main courses; a video-tape of lessons using this material was also available.

Another video-tape explained something of the history of the project and its present aims and activities, while a sequence of slides showed various aspects of the project's work and the texts in use in schools.

Besides members of the full time project staff, teachers and others involved in the work of the project were present throughout the conference to talk with visitors about the project, its work and materials.

Address: The School Mathematics Project, Westfield College, University of London, Kidderpore Avenue, London, NW3 7ST, GB.

SMTR
Scuola Media Tasso di Roma

Exhibition of 220 posters and a great variety of didactical material prepared by pupils (ages 11-18) from three Rome schools :

Scuola Media Tasso: E. Castelnuovo
Scuola Media Bartolomasi, Pomezia: S. Conte
Liceo Virgilio: L. Mancini

in order to illustrate a method used experimentally in many Italian secondary schools.

The pedagogic idea is that one must refrain from burdening pupils with a premature rigour, which taxes them with a formalization that they do not understand, but always give them the same feeling of constructing and developing often provided by reality. The following aims are pursued in particular:

(1) motivating pupils to study through an instruction which follows the natural learning process;
(2) stimulating the imagination, intuition, critical mind, and rational thought, thus promoting the searching attitude and a taste for rigour and abstract constructions;
(3) attaching great importance to experiments and to the most varied mathematizations with a view to connecting mathematics with the other disciplines, with society and, more generally, with the outside world.

Today, this world, this reality, raises ever more motivating problems: from social and political matters to economics, from physics to biology, from technology to architecture. One would like to gain insights into everything, to be able to answer all questions. It is precisely mathematics which, abstracting from nonessential data, is often in a position to give a simple interpretation of reality. But this mathematization can only be attained if one has previously been immersed in a natural way in the mathematical method. Proceeding from the observation of suggestive 'objects' and of concrete experiences, one wishes to provide the pupil with the instruments and methods of research as a function of their applications.

The exhibited materials show what can be done in school classes and in team work, in which pupils are free to participate and select themselves the topic they are most interested in. Such a laboratory provides the framework for organizing research, for elaborating the material, for historical investigations and for discussing the esthetic design of the posters, where the main lines of the subject are synthesized. The result of this work is presented by the pupils to the public towards the end of the school year. This oral exposition is a very valuable training in logico-synthetic organization, which helps children realize what difficulties impair the transmission of thought; it is an incentive for them to compare themselves with others and makes them more familiar with the world of culture.

Below the work presented by the three schools:

Scuola Media Tasso (ages 11-14)

The development of the various topics is notably different from traditional text books and also often from what happened in history, especially in that one always proceeds from observations and applications. The following items were treated: areas, perimeters, volumes, surfaces, affine transformations, conics and quadrics, the laws of growth, questions of probability, numeration systems, bary-center, bary-centric calculation and its applications to probability, to economic programming, and to colorimetry; the cycloid; cartographic projections.

Scuola Media Bartolomasi (ages 11-14)

It is a school for psychologically weak children. Hence the use of the concrete is essential. One topic very clearly showed the transition from the concrete to the abstract: the study of the form of a plane-tree leaf (the trees in the school yard). From the measuring of the essential elements of these leaves we proceeded to the abstract form by means of statistical methods (the cardioid). Thus, the children came to grasp the value of the mathematical object and, hence, of the symbol.

Liceo Virgilio (ages 14-18)

The poster showed how secondary school pupils can be led to master logical thought, proceeding from the concrete: from the logic of propositions to electric circuits, from probability to questions of genetics, from economics to linear programming, from special structures to abstract structures. Spurred on by these topics, the pupils are capable, after two years of instruction, to understand and appreciate an axiomatic presentation of probability. From then on a strongly rational development becomes apparent in the class, especially with respect to geometry: one proceeds 'by restriction' from the vector plane and affine analytical geometry to the metric plane and, 'by extension', to the projective plane. Finally deleting other axioms (e.g. continuity), one then arrives at the construction of affine and projective finite geometries. The aim of the last topic is obvious: to consolidate the value of the hypothetico-deductive process in 'simplified' situations.

Address: Scuola Media Tasso di Roma, Via S. Angela Merici 48, 00162 ROMA, Italy.

SWF
Südwestfunk Baden-Baden: A School Television Programme

Since 1969, the SWF ('Südwestfunk', Southwest Broadcasting Corporation, Baden-Baden, FRG) has been producing school television programmes for mathematics instruction in cooperation with the Ministries of Education of the states of Baden-Württemberg, Rheinland-Pfalz, and the Saarland.

A school television programme on mathematics for pupils of grade 5 was presented in Exeter in 1972 (FRG).

This time, the programmes for grades 7-8 were shown, which have been newly conceived on the basis of the trial-runs and extensive evaluative investigations of the fifth grade programme and which consequently represent new developments. The broadcasts, which are watched during the lessons with the teacher, are divided into introductory and instruction broadcasts. 'Introductory broadcasts' are shown at the beginning of a new topic, giving a general survey as well as motivating the pupil to see his environment in mathematical terms. The subsequent 'instruction broadcasts' then deal with the mathematical contents.

In addition to the broadcasts, the following materials also belong to the system of combined media packages: instruction by the teacher (teacher's handbook), accompanying written material in partially programmed form, and learning goal achievement tests with reference to the accompanying material.

The broadcasts as well as the accompanying materials are divided into two achievement levels.

The projects introduced here were accompanied by simultaneous evaluative investigations.

Address: Südwestfunk Baden-Baden, Hans-Bredow-Straße, Postfach 820, 757 Baden-Baden, FRG.

USMES
Unified Sciences and Mathematics for Elementary Schools

In addition to a continuous display of the USMES materials and presentations with slide illustrations, three special events were held: (1) a mini-workshop on 'Choosing the best beer and factor analysis'; (2) a panel presentation; and (3) a videotape presentation of 'Real Problem Solving at the United Nations' International School'. In addition, information about USMES use by primary students and evaluation of USMES was presented at poster sessions A1 and B4 respectively.

What Is USMES? USMES is an interdisciplinary program, developed under grants from the National Science Foundation, that challenges elementary school students to solve real problems from their school and community environment.

What Are Real Problems? Children working on USMES units tackle problems in getting to school; problems in the classroom, the lunchroom, and the playground; problems in consuming, producing, and advertising goods; problems in finding the best way to learn things, to make decisions, and to communicate certain kinds of information. They work on real problems like a dangerous crosswalk near the school or classroom furniture that doesn't fit the students in the class. The problems are 'real' in that (1) they have immediate, practical effects on students' lives, (2) they have no 'right' solutions, (3) they require students to use their own ideas about what the problem is and how to solve it, (4) they can be resolved by students, and (5) they are 'big' enough to require many phases of class activity for any effective solution.

The mini-workshop for the participants provided experience with the initial steps in the process of problem solving and indicated some of the mathematical topics that arise in the process. For adult participants at a German Congress, beer seemed to be an appropriate choice, although for children the challenge is 'Invent a new soft drink that will be popular and can be produced at a low cost', perhaps to be used at a class party or to be sold in the school cafeteria.

The adults, as usual, were not as thorough as the children in considering all the important factors that may be involved. (The children, however, usually take several weeks to finish their work!) Nevertheless, the participants tasted with enthusiasm. It is difficult to admit in this report that two Danish beers were preferred over all the German beers! Is anyone surprised that a diet beer and an 'alcohol-free' beer were least favored?

At the panel presentation, it was noted that USMES provided a context to develop proficiency in applied mathematics. The fact that much of the mathematics that *can* arise in real problems *does* arise in USMES classes is borne out by the research of S. Krairojananan of Michigan State University:

The following mathematical behaviors were observed to have permeated throughout all the four USMES units here: counting and measuring which lead to meaningful usage of whole numbers, decimals and fractions; place value; the four fundamental operations of arithmetic; percentage; ratio and proportions; estimation; collection and tabulation of raw data; graphs; proposing hypotheses and testing them; elementary logic; abstracting concrete situations into mathematical relations; sets and their elements;

ordinal and cardinal numbers; ordered pairs; simple geometrical shapes like triangles, rectangles, and circular cylinders.

The videotape presentation showed excerpts from third, fourth, and sixth grade classes at United Nations' International School working on USMES challenges.

More information about USMES development, use and evaluation may be obtained by writing to the Project Director.

Address: Unified Sciences and Mathematics for Elementary Schools, Education Development Center, 55 Chapel Street, Newton, Massachusetts 02160, USA.

4.6 Films and Exhibitions

1. Films

Films on topics of mathematical education were shown on several evenings during the Congress. There was no fixed programme, but rather the opportunity for Congress participants to have films shown and discussed, either to supplement and further explain their contributions to the Sections and Poster-Sessions or completely independent of these.

The contents, form and technical level of the various films differed widely. For example, extracts from whole teaching programmes were presented, as well as short classroom sequences. Many films were designed as aids for the teacher, whereas others attempted to replace the teacher through the medium of film.

Due to the informal organisation of the film evenings, it was not possible to set up a complete list of all the films shown.

2. Exhibitions of Scientific Institutions and Societies

The following institutions and societies presented their materials during the Congress:

The British Council, GB

The Mathematical Association of America, USA

National Council of Teachers of Mathematics, USA

Ontario Association for Mathematics Education, Canada

3. Exhibitions of Publishers and Firms

The following publishers and firms displayed their books, teaching aids and equipment on the occasion of the Karlsruhe Congress:

Addison-Wesley Publishing Company, Menlo Park, USA
Aristo-Werke Dennert & Pape KG, Hamburg, FRG
F.J. Arnold & Son Ltd, Leeds, GB
Bayerischer Schulbuch-Verlag, Munich, FRG
J. Beltz KG, Weinheim, FRG
Boxerbooks Inc, Zurich, Switzerland
Cambridge University Press, London, GB
Cedic, Paris, France
W.& R. Chambers Ltd, Edinburgh, GB
Cochranes of Oxford Ltd, Oxford, GB
Bokförlaget Corona AB, Lund, Sweden
Verlag M. Diesterweg, Frankfurt, FRG
F. Dümmlers Verlag, Bonn, FRG
Frankonius-Verlag, Dornburg-Frickhofen, FRG
F.W. Ganske KG, Karlsruhe, FRG
Gesellschaft für Regelungstechnik und Simulationstechnik GmbH, Darmstadt, FRG
Verlag W. Girardet, Essen, FRG
Heinemann Educational Books Ltd, London, GB
O. Heinevetter, Hamburg, FRG
Verlag Herder, Freiburg, FRG
Hueber-Holzmann Verlag, Ismaning, FRG
IBM Deutschland GmbH, Stuttgart, FRG
E. Klett Verlag, Stuttgart, FRG
Leybold-Heraeus GmbH & Co, Cologne, FRG
K. Mildenberger, Offenburg, FRG
Oldenbourg Verlag GmbH, Munich, FRG
Pan Atlantic Computer Systems GmbH, Darmstadt, FRG
Phywe AG, Göttingen, FRG
Rand McNally & Company, Chicago, USA
Verlag F. Schöningh, Paderborn, FRG
H. Schroedel Verlag KG, Hannover, FRG
Pädagogischer Verlag Schwann GmbH, Düsseldorf, FRG
Taylor & Francis Ltd, London, GB
B.G. Teubner Verlag, Stuttgart, FRG
Texas Instruments Deutschland GmbH, Freising, FRG
V-Dia-Verlag GmbH, Heidelberg, FRG
Schulverlag Vieweg, Düsseldorf, FRG
G. Westermann Verlag, Brunswick, FRG

Part 5

Appendices

5.1 ICMI and Congress Recommendations

Hans-Georg Steiner, Vice-President of ICMI

Ongoing ICMI-activities, recommendations and plans for future work were considered at the General Assembly (GA) of ICMI and at a meeting of the Executive Committee (EC).

1. New Member Countries

At the beginning of the 3rd ICME, the International Commission on Mathematical Instruction had delegated members from 42 countries. The president has received 6 new applications from: Bangladesh, Iran, Malaysia, New Zealand, The Philippines, Singapore. Membership of these countries was unanimously supported by GA and EC. Since only New Zealand and Singapore are members of the International Mathematical Union (IMU), final approval for the other four countries is to be asked from the EC of IMU and has already been received for Bangladesh and Malaysia.

2. Forthcoming Conferences

ICMI will assist in jointly organizing and co-sponsoring several forthcoming conferences:

(a) Pécs, Hungary, 23-27 August 1977
International Colloquium on:
The Problems of Training of Teachers of Mathematics
Organized by: Bolyai János, Mathematical Society,
H-1061 Budapest, Anker köz 1-3, Hungary

(b) Varna, Bulgaria, 13-18 September 1977
International Conference on:
Informatics and Mathematics in Secondary Schools - Impacts and Relationships
Organized by: The Third Technical Committee (TC-3) of the International Federation for Information Processing (IFIP).

TC-3 European Secretary: Dr. B. Penkov, P.O.Box 373, 1000 Sofia, Bulgaria

(c) Manila, Republic of the Philippines, April/May 1978
Southeast-Asian Regional Conference on:
Various Aspects of Mathematical Education at the Senior High School and College Level in the Southeast-Asian Region
Organized by: Mathematical Society of the Philippines, Southeast Asian Mathematical Society (SEAMS)

(d) Helsinki, Finland, August 1978
ICMI-Symposium on the occasion of the 1978 International Congress of Mathematicians on:
What Kind of Knowledge of Mathematics should a Teacher of Mathematics have?
Further information available from: Professor Dr. Hans-Georg Steiner, Institut für Didaktik der Mathematik (IDM), Bielefeld University, Universitätsstraße, D-4800 Bielefeld (FRG)

(e) Campinas S.P., Brazil, February 1979
5th Inter-American Conference on Mathematical Education
Further information available from: Professor U. d'Ambrosio, Instituto de Matemática, Estatistica, e Ciência da Computacao (IMECC), Universidade Estadual de Campinas (UNICAMP), Caixa Postal 1170, 13 100 Campinas S.P., Brazil

(f) Echternach, Luxembourg, (1977 ?)
4th Echternach Conference

(g) Conference on mathematics education for disadvantaged children.

3. Place of 4th ICME in 1980

The United States of America expressed an interest in hosting the 1980 ICME. Further details will be forthcoming in a formal written proposal at a later date.

4. ICMI-Journals and Bulletin

ICMI's official journal is *L'Enseignement Mathématique - Revue Internationale*, edited by the Institut de Mathématiques, Université de Genève, Case postale 124, CH-1211 Genève 24, Suisse. It regularly publishes reports from the EC of ICMI. During previous periods of time the journal contained a considerable number of papers devoted to mathematics education. It is intended to reactivate the educational component of the journal.

The Center for the Didactis of Mathematics at the University of Karlsruhe (FRG) and ICMI are jointly editing the *Zentralblatt für Didaktik der Mathematik (ZDM)*, a journal for documentation and information on mathematical education, published by Ernst Klett Verlag, Stuttgart. Office of ZDM: Hertzstraße 16, Westuniversität, D-7500 Karlsruhe, FRG.

Under Sir James Lighthill's presidency the secretariat of ICMI has begun to produce an *ICMI - Bulletin* which is now appearing biannually and sent to the members of ICMI including the representatives of the national subcommittees.

5. Recommendations

GA und EC are in support of the following recommendations:

(a) ICMI should continue and elaborate its cooperation with other associations devoted to furthering mathematical and scientific education, such as IACME, IFIP, ICSU, and the newly founded African Mathematical Union.

(b) National Subcommittees of ICMI should be reactivated. While the national subcommittees of a small number of countries are very active both with matters of national concern and in international cooperation and contacts, the committees of several other countries do not operate in a satisfactory way: Addresses are obsolete, information received from the Commission is not being distributed, letters and invitations are not being answered etc. Reactivation of the national subcommittees will be tried through the official channels and on individual bases. Reactivation should be a concern of all who could contribute to and profit from the national committees' work.

(c) Internationally composed committees, working groups and study groups which have been established in preparation for or during an ICME in order to study and further particular areas of research and development in mathematical education should be invited to continue their work and to establish appropriate working structures and information systems. Special proposals were presented in this direction by the Working Group (EWG 11) on the Relations between History and Pedagogy of Mathematics (Secretary: Dr. L.F. Rogers, 29, St. Winifred's Road, Teddington, Middlesex TW II 9JS, England) and the Working Group (EWG 12) on the Psychology of Learning Mathematics (Secretary: Prof. Dr. A. Abele, Schlittweg 33, D-6905 Schriesheim, FRG).

After the meetings of the GA and the EC had taken place the President received a recommendation from a Group of Congress participants which was then read by Professor Steiner in his final address and applauded by the audience. This recommendation is concerned with the theme 'Women and Mathematics' and the poor representation of women in official positions and activities during and in the planning of the Congress. It was suggested that for the 1980 Congress a group of women be included in the organizational committees, that a main speaker (preferably a woman) be invited to speak on some aspect of 'Women and Mathematics', and that this theme be a subject for a discussion group.

5.2 Conferences 1972 – 1976 Co-Sponsored by ICMI

Hans-Georg Steiner, Vice-President of ICMI

Between the 2nd and the 3rd ICME the following conferences co-sponsored by ICMI were held:

(1) Echternach, Luxembourg, 4-9 June 1973
3rd Echternach Conference:
New Aspects of Mathematical Applications at School Level
Proceedings available from: Director L. Kieffer, 1 Rue Jean Jaurès, Luxembourg, Luxembourg

(2) Eger, Hungary, 18-22 June 1973
International Colloquium on:
Theoretical Problems of Teaching Mathematics in the Primary Schools
Information available from: Bolyai János, Mathematical Society, H-1061 Budapest, Anker köz 1-3, Hungary

(3) Vancouver, Canada, 22-24 August 1974
ICMI-Symposium on the occasion of the 1974 International Congress of Mathematicians on:
Evaluation of Modern Mathematics Curricula
Information available from: Professor Sir James Lighthill, University of Cambridge, Dept. of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, England

(4) Nairobi, Kenya, 1-11 September 1974
UNESCO-CEDO-ICMI Symposium on:
Interactions between Linguistics and Mathematical Education
Proceedings available from: UNESCO, Division of Pre-University Science and Technology Education, 7 Place de Fontenoy, F-75700 Paris, France

(5) Bielefeld, FRG, 16-20 September 1974
ICMI-IDM Regional Conference on:
The Role of Geometry in Present-Day Mathematics Teaching
Proceedings available from: Institut für Didaktik der Mathematik (IDM), Universität Bielefeld, D-4800 Bielefeld 1, FRG

(6) Tokyo, Japan, 5-9 November 1974
ICMI-JSME Regional Conference on:
Curriculum and Teacher Training for Mathematical Education
Proceedings available from: Japan Society of Mathematical Education, c/o Tokyo University of Education, 29-1 Otsuka 3-chome, Bunkyo-ku, Tokyo 112, Japan

(7) Nyiregykaza, Hungary, 18-23 August 1975
International Colloquium on:
Evolving a Mathematical Attitude in the Secondary Education
Further information available from: Bolyai János, Mathematical Society, H-1061 Budapest, Anker köz 1-3, Hungary

(8) Warsaw, Poland, 25-28 August 1975
International Symposium on:
Combinatorics and Probability in Primary Schools
Proceedings available from: Professor Z. Semadeni, Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, 00-950 Warsaw, Poland

(9) Marseille, France, 1-5 September 1975
IFIP 2nd World Conference on:
Computers in Education
Proceedings published by North-Holland Publishing Company 1975

(10) Caracas, Venezuela, 1-6 December 1975
4th Inter-American Conference on Mathematical Education
Proceedings to be published by Oficina de Ciencias de la UNESCO para América Latina, Montevideo, Uruguay.

(11) Bharwari, India, 15-20 December 1975
Regional Conference on:
Development of Integrated Curriculum in Mathematics for Developing Countries in Asia
Further information available from: Professor P.L. Bhatnagar, Dholpur House, Shahjahan Road, New Delhi 11011, India

(12) Rabat, Marocco, 26-31 July 1976
1st Pan-African Congress of Mathematicians
Further information available from: Faculty of Science, Rabat University, Avenue Moulay Cherif, Rabat, Marocco

5.3 Meeting of Association Leaders

Chairman: Erwin Baurmann, Karlsruhe

In the following, *organisations participating* in the meeting are listed, inclusive of some preliminary information. Finally some *suggestions* made in the course of the meeting are reproduced.

Abbreviations

NAM Name of association
CON Contact address
MEM Number of members
BRA Branches
SER Services to members
ASS Associated organisations

1. NAM The Australian Association of Mathematics Teachers
   CON Hon. Secretary: R. McCreddin, Education Dept., Construction House, 37 Havelock Str., West Perth 6005, Western Australia, *Australia*

2. NAM Mathematical Association of Victoria (Australia)
   CON President: Denis C. Kennedy, Clunies Ross House, 191 Royal Parade, Parkville, Victoria 3052, *Australia*
   MEM 800 post-primary teachers, 300 primary teachers, a few teacher-educators, a few tertiary educators (mathematics), some student members
   BRA Several Victorian country branches, one metropolitan branch, Study Group for Mathematical Learning (State wide interest group)
   SER Journals: 'Vinculum', post-primary journal, 5/year
   'Set Two', primary journal, 4/year
   Newsletters: 'Link', post-primary, 12/year
   'Murmur', primary, 12/year
   Meetings: 5 lectures, 15 in-service conferences and other meetings from one day to a fortnight
   ASS Australian Association of Mathematics Teachers

3. NAM Société belge des professeurs de mathématique d'expression francaise

CON A. Festraets, rue Vandercammen 36, B-1160 Bruxelles, *Belgium*

MEM 500

BRA 1, mathematics

SER Journal: Mathématique et Pédagogie, 5/year
Meetings: 1 or 2/year

4. NAM Association Scientifique de Côte d'Ivoire

CON B.P.4322, Abidjan, *Côte d'Ivoire*

5. NAM The Mathematical Association

CON The Honorary Secretary, 259 London Road, Leicester LE 2 3 BE, *England*

MEM School: 4000
University: 1000

BRA 36, some regional, some institutional

SER Journals: The Mathematical Gazette, 4/year, for teachers in schools and universities
Mathematics in School, 5/year, for teachers of 7-16 year old pupils
Meetings: Annual conference at Easter, frequent regional meetings, curriculum development by committees

6. NAM Association of Teachers of Mathematics (ATM)

CON Market Street Chambers, Nelson, Lancashire BB9 7LN, *England*

MEM 6000 (made up of teachers from schools at all levels, teacher training establishments and universities)

BRA 20

SER Publications: Mathematics Teaching, 4/year
Recognitions, 4/year
Supplement, occasional
Occasional pamphlets about the teaching of mathematics
Meetings: Annual Easter conference
Occasional other conferences and weekend meetings
Local branch meetings and activities

7. NAM National Association of Teachers in Further and Higher Education - Mathematical Education Section

CON K.L. Gardner, Brighton College of Education, Falmor, Brighton BN 1 9 PH, *England*

MEM 400 teachers of mathematics and mathematical education in colleges of education

SER Journal: Mathematical Education for Teaching, 2/year - from G.C. Hume, Mathematics Dept., Homerton College, Cambridge CB 2 2 PH, England
Annual conference, second week in September

ASS Section of the National Association of Teachers in Further and Higher Education, Hamilton House, Mabledon Place, London WC 1, England

8. NAM Association des professeurs de mathématiques de l'enseignement public (APMEP)

CON 29 rue d'Ulm, F-75005 Paris, *France*
General Secretary: Paul-Louis Hennequin, 15 rue du Pavin, F-63000 Clermont, France

MEM 15000 (pre-school to university)

BRA 1, mathematics

SER Bulletin of the association, 5/year
Booklets: Informative documentation for teachers, summaries of examination topics
Annual national meetings, seminars
Regional sections (one per teacher training centre), 'sections départementales'

ASS Affiliated with other teacher associations within the 'Conférence des Présidents des Associations de Spécialistes' (19 associations: humanities, physics, biology etc.)

9. NAM Centre de Recherche de Mathématiques de l'Ouest (CEREMO)

CON Chairman: Bernard Heraut, CEREMO, B.P.858, F-49005 Angers-Cédex, *France*

SER Journals
Meetings: pedagogical sessions, 3/year, for primary teachers

in the west
In-service training of primary teachers at Anjou

ASS Institut de Mathématiques Appliquées (IMA)
Institut Supérieur de Promotion de l'enseignement catholique (ISPEC)
at Angers

10. NAM Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts

CON Chairman: Erwin Baurmann, Märchenring 11, D-7500 Karlsruhe 51, *FRG*

MEM 5050 at schools, 100 at universities, 50 others

BRA 4: mathematics, physics, chemistry, biology

SER Journal: Der mathematische und naturwissenschaftliche Unterricht (MNU), 8/year
Information leaflet to section heads, 8/year
Meetings: 1 annual meeting, about 8 regional/year

ASS Deutsche Physikalische Gesellschaft
Deutscher Philologenverband
Association for Science Education (U.K.)
Verein schweizerischer Naturwissenschaftslehrer

11. NAM Verband Bildung und Erziehung (VBE)

CON D-5000 Köln, *FRG*

MEM 8000, primary and secondary school

BRA Special working groups for special subjects, special levels and formed if necessary

SER Journals: 'forum E', 12/year, for all members in the FRG
'informationen für erzieher', 12/year, for members in the land of Baden-Württemberg

ASS Deutscher Lehrerverband
Deutscher Beamtenbund

12. NAM Bolyai János Matematikai Társulat

CON General Secretary: Ákos Császár, Anker köz 1-3, H-1061 Budapest, *Hungary*

MEM Research mathematicians and teachers of mathematics on all levels

BRA 3: pure mathematics, math. applications, math. instruction

SER Journals: Közepiskolai matematikai lapok (Math. Journal for Secondary Schools)
A matematika tanitása (Math. Teaching)
Matematikai lapok (Math. Journal)
Periodica mathematica hungarica (research papers in math., problem section)
4-day Meeting for mathematics teachers, 1/year
Colloquia and many other services

ASS Member of the Hungarian Association of Technical and Scientific Societies
Member of ICMI and IMU

13. NAM Association for Improvement of Mathematics Teaching

CON Jagadbandhu Institution, 25 Fern Road, Calcutta 700019, *India*
President: D.K. Sinha
General Secretary: B. Ray Chaudhuri

MEM school 450, university and college 50

BRA 5

SER Journal: Indian Journal of Mathematics Teaching (English), Ganit Parikrama (Bengali)
Newsletters
Seminars, meetings, symposia (10), lectures, conferences (1), workshops, training programmes (4), textbook writing (6), mathematical exhibitions (1), talent search test in mathematics (1), debate, quiz

14. NAM Japan Society of Mathematical Education (JSME)

CON President: Zenichi Kobayashi, Tokyo University of Education, 3-29-1 Otsuka, Bunkyo-ku, Tokyo 112, *Japan*

MEM 5000

BRA 50

SER Monthly journal
Quarterly journal
General annual meeting and 9 regional meetings/year

15. NAM Association des Professeurs de Mathématiques du Mali

CON General Secretary: T. Male, Professeur à l'Ecole Normale Supérieure, B.P. 241, Bamako, *Mali*

16. NAM Nederlandse Vereniging van Wiskunde Leraren

CON Chairman: Th.J. Korthagen, Torenlaan 12, Warnsveld, *Netherlands*

Secretary: J.W. Maassen, Traviatastr. 13, den Haag, Netherlands

MEM school 2400, university 100

BRA School Mathematics

SER Journal: 'Euclides'

Meetings: Annual meeting, some other meetings, short courses (4 days) on different levels and special aspects

17. NAM Mathematical Association of Nigeria (MAN)

CON President: P.M. Igboko, Public Service Commission, Owerri, *Nigeria*

Secretary: E. Ukeje, Dept. of Mathematics, University of Nigeria, Nsukka, Nigeria

MEM drawn from universities and secondary schools and teacher training colleges

SER Journal: 'Abacus'; Editor in Chief: O. Fajuyigbe, Dept. of Mathematics, University of Benin, Benin City, Nigeria

18. NAM Societatea de ztiinţe matematice diu R.S. România

CON Str. Academiei 14, Bucureşti, *Roumania*

President: N. Teodorescu, member of the Academy

General Secretary: Stelian Turbatu, maître de conférence

BRA 3 national sections: education, research, publications

40 regional sections

SER Bulletin mathématique (journal with contributions on research, in foreign languages, 3/year)

Gazeta matematicá (journal for students of grades 5-12, 10-18 of age, monthly, in Roumanian)

Matematikai lapok (Hungarian translation of the Gazeta matematicá)

Meetings (research or educational topics, at least 10/year)
Collaboration with summer courses organised in Roumania
Summer courses for teachers in public education

19. NAM Association des Professeurs Africains de Mathématiques au Sénégal (APAMS)
CON Bouna Gaye, Ecole Normale Supérieure, B.P. 5036, Dakar-Fann, *Sénégal*
MEM APAMS groups the African teachers of mathematics on the primary, secondary and university levels
SER Bulletin of APAMS, 3/year
ASS APAMS envisages its affiliation to UMA ('Union des mathématiciens africains') and to ICMI

20. NAM Mathematical Association of America
CON President: Henry L. Alder, Dept. of Mathematics, University of California, Davis, California 95616, *USA*
Executive Director: A.B. Willcox, Mathematical Association of America, 1225 Connecticut Avenue NW, Washington D.C. 20036, USA
MEM school 1000, university 15000, other 2500
BRA 29 geographical sections
SER Journals: The American Mathematical Monthly
The Two-Year Mathematics Journal
The Mathematics Magazine
Meetings: 2 national meetings/year (in January and August) and at least 1 meeting/year of each of the 29 sections
ASS My Alpha Theta (national high school and junior college mathematics honor club)

21. NAM National Council of Teachers of Mathematics (NCTM), *USA* and *Canada*
CON 1906 Association Drive, Reston, Virginia 22090, USA
President: John C. Egsgard
MEM 80000 (elementary school teachers, high school teachers,

teacher college professors, university professors, institutions)

BRA 185 affiliated groups (175 from the USA, 10 from Canada)

SER Journals: The Arithmetic Teacher, 8/year
The Mathematics Teacher, 9/year
The Journal for Research in Mathematics Education, 8/year
Newsletters, quarterly
Meetings: 10/year

Suggestions made

a) The above list should be copied and sent to all organisations mentioned here, as well as to other addresses available of this kind.

b) Further information is wanted as to facilitate bilateral contacts among organisations.

c) Its dissemination might be accomplished by inclusion in or by an annex to the already existing ICMI bulletin.

d) Eventually, the final edition of a booklet (loose sheet version like the 'Directory of Science Teachers Associations Worldwide' of the Association for Science Education on mathematics teachers organisations) would be appreciated.

5.4 Congress Committees

Hon. Chairman of the Congress

Prof.Dr.Dr.h.c.Dr.h.c. H. Behnke, University of Münster

Local Organising Committee

Prof.Dr. H. Kunle, University of Karlsruhe
Secretary: J. Mohrhardt, University of Karlsruhe

West German Sub-Committee of ICMI

H. Athen, Elmshorn b. Hamburg; M. Barner, Freiburg i.Br.; H. Bauersfeld, Bielefeld; E. Baurmann, Karlsruhe; H. Behnke, Münster; A. Bergmann, Düsseldorf; W. Böddeker, Castrop-Rauxel; H. Griesel, Kassel; K.P. Grotemeyer, Bielefeld; K. Kirchgäßner, Stuttgart; H. Kunle, Karlsruhe; G. Pickert, Gießen; E. Reiche, Ettlingen; H.G. Steiner, Bielefeld; H.J. Vollrath, Würzburg.

The International Programme Committee

H.G. Steiner (FRG, Chairman), U. d'Ambrosio (Brazil), A. Bergmann (FRG), P.L. Bhatnagar (India), B. Christiansen (Denmark), T.J. Fletcher (Great Britain), H. Freudenthal (Netherlands), C. Gaulin (Canada), M. Glaymann (France), S. Iyanaga (Japan), A.Z. Krygowska (Poland), J. Lighthill (Great Britain), G. Matthews (Great Britain), B.H. Neumann (Australia), G. Pickert (FRG), H. Pollak (USA), A. Revuz (France), S.L. Sobolev (USSR), J. Surányi (Hungary), C.O. Taiwo (Nigeria), Bakary Traore (Mali).

International Commission on Mathematical Instruction (ICMI)

Executive Committee

President: S. Iyanaga

Vice-Presidents: B. Christiansen, H.G. Steiner

Secretary: Y. Kawada

Members: E.G. Begle, L. Kudryavtsev, Sir James Lighthill

Ex-Officio Members

Sir James Lighthill (GB), D. Montgomery (USA), J.-L. Lions (France), H. Freudenthal (Netherlands).

Members-at-large

E.G. Begle (USA), P.L. Bhatnagar (India), E. Castelnuovo (Italy), B. Christiansen (Denmark), S. Iyanaga (Japan), L. Kudryavtsev (USSR), J. Lelong-Ferrand (France), B.H. Neumann (Australia), Z. Semadeni (Poland), J. Surânyi (Hungary).

National Representatives

L.A. Santaló (Argentina), M.F. Newman (Australia), E. Hlawka (Austria), G. Noel (Belgium), L. Nachbin (Brazil), B.L. Petkančin (Bulgaria), A.L. Dulmage (Canada), Shing-Meng Lee (China-Taiwan), J. Vyšín (Czechoslovakia), J. Hoffmann-Jørgensen (Denmark), H. Kunle, H.G. Steiner (Federal Republic of Germany), L. Kaila (Finland), J. Giraud (France), K. Härtig (German Democratic Republic), C. Papaipoannou (Greece), J. Szendrei (Hungary), P.L. Bhatnagar (India), C. Lanczos (Ireland), M. Maschler (Israel), E. Castelnuovo (Italy), Y. Kawada (Japan), L. Kieffer (Luxembourg), M. Cundy (Malawi), P.G.J. Vredenduin (Netherlands), S.A. Dada (Nigeria), A. Johansen (Norway), M. Raziuddin Siddiqi (Pakistan), Z. Semadeni (Poland), J. Sebastião e Silva (Portugal), G.C. Moisil (Roumania), S. Niang (Senegal), J.H. van der Merwe (South Africa), P. Abellanas (Spain), R.J. Waterston (Swaziland), H. Wallin (Sweden), A. Delessert (Switzerland), M. Bougila (Tunisia), J.V. Armitage (United Kingdom), P.J. Hilton (USA), I.M. Yaglom (USSR), D. Kurepa (Yugoslavia), S.M. Bayat (Zambia).

SUPPLEMENTS TO LIST OF CONGRESS PARTICIPANTS
(ADDRESS CHANGES ADDITIONS)

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