ICME-10 Proceedings

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Roskilde University, Denmark

Proceedings of the 10th International Congress on Mathematical Education, 4-11 July, 2004

Printed book + CD

2008

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Layout: Birgitte Clematide

Print: Kailow Graphic, Denmark

Published by: IMFUFA, Department of Science, Systems and Models, Roskilde University, Denmark

www.icme10.dk

ISBN: 978-87-7349-733-3

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Preface

The present Proceedings of the 10th International Congress on Mathematical Education (ICME-10), which took place at the Technical University of Denmark. Lyngby, just North of Copenhagen, Denmark, 4-11 July 2004, are composed of two parts. A book of 560 pages and a CD.

The book contains accounts of
- the Opening and Closing Ceremonies,
- the eight Plenary Activities, including the Plenary Lectures delivered at the Congress,
- the lectures based on the work of the five so-called Survey Teams,
- lectures by the first two recipients of the ICMI Awards
  – the Felix Klein Medal (Guy Brousseau)
  – the Hans Freudenthal Medal (Celia Hoyles),
- reports of the five themes of the so-called Thematic Afternoon,
- of the 29 TSG and
- of the 24 DG.

The CD contains all this, but moreover, and more importantly, the papers based on 64 of the 74 Regular Lectures given at ICME-10.

Despite the fact that the total of these Proceedings occupy well over a thousand pages it has not been possible to include reports on several other important Congress activities such as the five national presentations by Korea, Mexico, Romania, and Russia, and the Nordic host countries (Denmark, Finland, Iceland, Norway, and Sweden), the 46 Workshops, the 12 Sharing Experiences Groups, the more than 220 Posters, the five ICMI Affiliated Study Groups, and the several informal meetings on a large variety of issues and themes. Yet we do hope that what is contained in the Proceedings gives a fair and representative reflection of the work of and at ICME-10.

The contents of these Proceedings have been subjected to peer reviewing. A large number of anonymous reviewers have played a crucial part in the editorial process. Although they are far too numerous to be listed here, they all deserve our sincere thanks for their immense and highly valuable work. The authors, too, deserve our warmest thanks, first of all for having written their papers and reports but also for having responded in such a constructive manner to the reviewers' comments and requests.

It is most unfortunate that it has taken almost four years to complete these Proceedings. This is mainly due to two reasons: Long-lasting turbulence caused by fundamental changes at the editors' university and serious illness (now happily overcome) with a key staff officer.

It is my pleasant duty to thank a number of protagonists in the editing and production of this work. Above all *Elin Emborg*, whose sustained commitment and extremely hard work under difficult conditions has been absolutely crucial to the creation and completion of the Proceedings. *Morten Blomhøj* has provided first-class financial management of the work, while *Birgitte Clematide* has taken care of the lay-out in a highly professional and efficient manner, and has shown great patience with the delays and ruptures that occurred during the editorial process. *Eva Branner* and *Henrik Nielsen*, Congress Consultants, have continued to provide most valuable assistance with the logistics of the production several years after their formal obligations to ICME-10 have ceased to be in force.

It is often the case that conference proceedings are not really read after their publication (except, perhaps, by the individual authors). It has been our endeavour to make the Proceedings of ICME-10 worth reading. We hereby submit the outcome of the work by hundreds of people to the critical yet (hopefully) benevolent reception by the international mathematics education community at large and the participants of ICME-10 in particular.

Roskilde, June, 2008,
Mogens Niss, Editor

Around ICME-10

Morten Blomhøj, *Gerd Brandell* and *Mogens Niss*

In the beginning of July 2004 (4th-11th) the *10th International Congress on Mathematical Education* (ICME-10), took place on the campus of the Technical University of Denmark at the outskirts of Copenhagen. In this article we give a brief report on the statistics of the congress and a condensed outline of the scientific programme. ICME-10 was organised as a joint effort amongst the five Nordic countries: Denmark, Finland, Iceland, Norway, and Sweden. This is a novelty in the history of the ICME's. Therefore, in this article, we pay some attention to how the co-operation was organised and to some of its effects on the continuing Nordic co-operation in mathematics education research.

Statistics

More than 2.300 researchers in mathematics and mathematics education, teacher trainers and mathematics teachers representing all levels of the educational system from pre-school to university attended ICME-10. There were participants from nearly 100 different countries, (see figure 1 and 2; complete statistics can be found at www.icme10.dk). In total we estimate that close to a thousand of the participants contributed to the scientific programme of ICME-10. In addition 317 accompanying persons and some 100 professional exhibitors took part in the event. Assisted by the so-called Solidarity Programme, funded primarily by a 10% solidarity tax on the registration fees, it was possible to partially support about 175 individuals from less affluent countries with grants for participation.

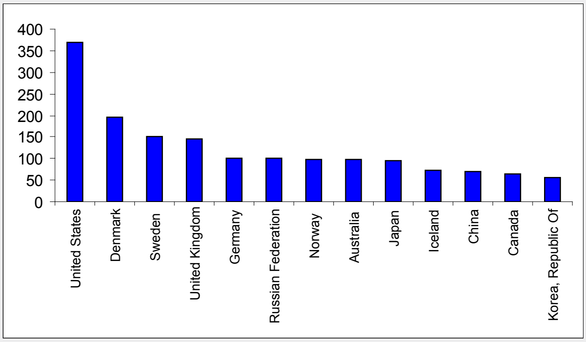

Figure 1: The number of participants from the top 13 countries.

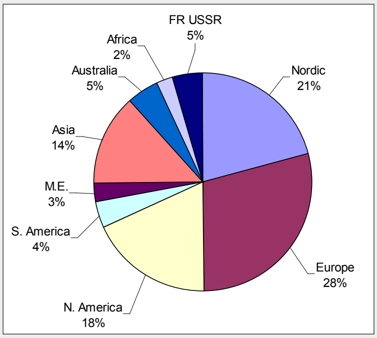

Figure 2: The distribution of participants on geographical regions.

The scientific programme

The scientific programme of ICME-10 was composed as a rich mixture of classical, renovated classical, and quite novel and innovative components.

The classical components comprised 6 Plenary Lectures, and one Plenary Panel Debate, 79 Regular Lectures, organised in parallel sessions, 29 Topic Study Groups in which particular topics were considered in presentations during four "mini conference" sessions, poster presentations, commercial and non-commercial exhibitions, presentations of recent ICMI studies, National Presentations and meetings of the five ICMI Affiliated Study Groups.

The National Presentations were given by Korea, Mexico, the Nordic countries, Romania and Russia. Each of them included very nice exhibitions and an entire afternoon programme with lectures and workshops allowing participants to get an insight into the mathematics education research and the practices of mathematics teaching in each country taking part in the presentations.

Renovated classical components included 24 Discussion Groups (in which there were no presentations, apart from an introduction, but structured discussions on pertinent challenges, issues, and dilemmas), and 45 Workshops established on the basis of submissions made by individuals or groups.

The novel components were pretty numerous: A Plenary Interview Session, in which four highly prominent mathematics education veterans were interviewed by another prominent scholar, proved to be very well received indeed. Five so-called Survey Teams worked for three years to survey recent developments in a particular sub-field of mathematics education. The outcomes of their work were presented in plenary or regular lectures. One afternoon was designed as a so-called Thematic Afternoon during which the entire Congress was divided into five parallel themes. Twelve Sharing Experiences Groups, established on the initiative of individuals, allowed participants to share and discuss mathematics education experiences in small groups. In addition to typical poster presentations a new scheme was adopted: Poster Round Tables in which 3-5 posters were grouped thematically and were discussed together under the direction of a moderator. More than 100 of the 217 posters were discussed at such Round Tables.

In addition to the scientific programme there were two other activities with an interesting mathematical content running through the entire congress. The Congress was pleased to be able to display an interesting travelling exhibition on mathematical objects and phenomena mounted by UNESCO and ICMI. Also a so-called mathematical circus, initiated by the Nordic Contact Committee (NCC, se below), was organised during ICME-10. Contributors mainly from the Nordic countries offered a number of entertaining and mathematically rich activities to Congress participants and their family members. The day before the congress the Circus also had some activities in the local community centre to inspire and entertain the general public.

Newcomers programme

As a newcomer it is not easy to navigate in the programme of an ICME congress. The programme structure is very complex and the programme offers a wealth of opportunities, and therefore also requires a lot of decisions to been taken by each individual participant. Many have indicated that their first ICME was quite a bewildering experience. Therefore, as a novelty a special newcomers programme was set up during the planning of ICME-10. The programme was initiated by the ICMI executive committee and launched by NCC. The aim of this programme was to help newcomers improve their outcome of ICME-10 scientifically as well as socially. All participants who considered themselves as newcomers were given the option to participate in the newcomers programme during ICME-10. Experienced ICME-participants among lecturers, members of the executive committee or the programme committee, people responsible for TSGs or DGs and so on were invited to act as mentors. The interest was large and about 500 newcomers and 50 mentors participated in the programme. The programme consisted of several components. One mentor and about 10 newcomers were grouped together and some groups had made contact by email already before ICME-10. The outcome varied a lot between the groups. The evaluation amongst participants also brought out a number of interesting suggestions for the future ICMEs – if the newcomers' programme is to appear again.

* * *

It will be up to the organisers of the next ICME, ICME-11 to be held in Mexico in 2008, to decide which of all these programme elements should be kept, perhaps in a modified form, and which should be dismissed.

Although we are not the right people to judge the success of the Congress, based on the feedback we have received from a large number of participants it is, we think, fair to say that the scientific programme of ICME-10 went rather well in most respects.

The Nordic co-operation

As ICME-10 was organised as a joint effort amongst the five Nordic countries, Denmark, Finland, Iceland, Norway, and Sweden, a couple of questions seem natural to pose. Why do "small" countries like the Nordic (the number of inhabitants range from 300 000 to 9 millions) offer to host a huge congress like ICME? Why do several countries volunteer to cooperate in organising such a congress? The answer to the first question is obvious; for the same reasons as large countries, the two main ones being to accelerate the development of the field within the country and to make mathematics education and research of the country more visible on the international scene. However, the demands on resources – both economic resources and resources in terms of scientific infrastructure – to manage the organisation of an ICME congress are greater than those at the disposal of most small countries. This is the obvious reason why earlier congresses have been hosted by large countries, and this is why the five Nordic countries made a joint offer to host ICME-10.

As arranging an ICME through co-operation between countries may be relevant in the future it is interesting to address questions like: What are the specific requirements on countries for such an endeavour? How was the organisation of ICME-10 made possible from this perspective? What positive side effects have appeared so far from the intensive and extensive co-operation during the preparation of ICME-10?

The first question may be answered in a tentative way only, based on the experiences from ICME-10. The success of the co-operation seems to require first of all an already well established co-operation between the communities of mathematics education in the countries, and working networks. Secondly, the structure of the organisation must be decided at an early stage and the actual co-operation should start as soon as the preparations of the congress are launched. A realistic picture towards what is possible to achieve is important, considering that projects will take longer to carry through when several countries are involved.

In the case of ICME-10 the organisation and working format of the co-operation was straightforward and built on existing contacts. A Nordic Contact Committee (NCC) was established in 1999 with representatives from the five countries - one from Iceland and two from each of the other countries. The commission of NCC was to ensure support of the organisation of ICME-10 from the four countries beside Denmark, and to encourage participation and contributions to ICME-10 from all Nordic countries. The committee was able to meet and work intensively for a couple of days twice a year during the whole period, alternating between the countries. Four members of NCC were members also of the local organising committee, thereby ensuring easy spread of information in both directions.

The co-operation between the Nordic countries concerned a number of projects related directly to the congress as described below. However, an even more significant outcome was the general boost of the Nordic collaboration within mathematics education during the four years of planning. Common projects aiming beyond the congress developed through the continuing contact.

The members of NCC were appointed by the national ICMI representatives and in Denmark and Sweden by the national ICMI committees. Through its members NCC had close links to all Nordic research environments in mathematics education, to national organisations for teachers, and to existing Nordic networks within mathematics education. New contacts over the national borders between persons involved in these networks were developed through the initiatives of NCC. National and Nordic networks within mathematics education were strengthened by the work of NCC and more people became involved.

Funding

The Nordic Council of Ministers, a Nordic governmental co-operation body, supported the Congress financially to some extent. However, it turned out to be impossible to get funding for the Congress organisation directly from any of the countries other than Denmark. This was a disappointment and left a heavy burden on the local Danish organisation. On the other hand, governments and non-governmental organisations in the other countries made substantial grants for the preparations of ICME-10 within each country. The costs directly related to NCC's work were covered by such means. The

Malmö symposium, *Educating for the Future*, described below, offered a possibility to cover travel expenses for one of the ICME-10 programme committee meetings by resources from Norway and Sweden.

Conferences

Two international conferences were arranged on the initiative of the NCC in May 2003, the year before the Congress: one was an international symposium - *Educating for the Future* - on the education of mathematics teachers and one was a *Nordic Pre-conference to ICME-10* with the aim of encouraging contributions from young Nordic researchers to ICME-10. Both conferences were documented through proceedings (Växjö University, 2003; Brandell, G., Grevholm, B. and Straesser, R., 2004).

The symposium *Educating for the Future* was arranged in May 2003 in connection with an ICME-10 programme committee meeting. About 40 researchers from all over the world were invited to the symposium, among them the members of the ICME-10 programme committee and a number of key persons within mathematics education with special interest in research on teacher education, such as the members of the editorial board of *Journal for Research on Mathematics Teacher Education*.

A number of internationally recognised researchers in mathematics education were invited to the *Pre-conference to ICME-10* in Växjö in May 2003. Invited lecturers addressed questions and raised issues of great importance for the development of mathematics education as a research field. Over 150 doctoral students, researchers, teachers and educators from the Nordic countries attended the conference and there were 80 presentations in parallel sessions. The majority of the presenters were young researchers who had not earlier presented their work outside their own country. Each presenter got feedback from two of the invited experienced researchers, who could give constructive critique on how to improve and develop a contribution. The aim was to encourage as many young Nordic researchers as possible to contribute to ICME-10. Almost all of the active presenters in the Pre-conference actually managed to contribute to ICME-10, in most cases to one of the Topic Study Groups of the congress or within the Nordic presentation.

The two conferences were planned to be close in time and space in order to make it possible for those invited or interested to conveniently attend both.

Satellite meetings of PME and HPM, two of the affiliated subgroups of ICMI, were organised in the Nordic countries directly after ICME-10. The yearly meeting of the International Group for the Psychology of Mathematics Education: *PME 27* was held in Bergen, Norway and the Summer University of the History and Pedagogy Group (HPM) was organised in Uppsala, Sweden.

NOMAD

The Nordic research journal on mathematics education – *Nordic Studies in Mathematics Education*, NOMAD - has played a crucial role for the Nordic research community in Mathematics Education since it first appeared in 1994. Starting in 2002 the journal had been suffering from administrative and financial difficulties. Due to this the journal did not appear as planned for some time. The NCC judged it to be of utmost importance that the journal would start to appear again before ICME-10 in order to have the necessary platform to publish research and reflect the development of the research field in the Nordic countries. Therefore the NCC took the initiative to finance and support the appearance of a special issue in connection with ICME-10, and suggested guest editors for this special issue (Bekken, O. B., Jaworski, B., and Kristjansdottir, A., 2004). The special issue contains nine articles by authors from all of the Nordic countries. The aim of the issue is to give an idea of some current research in mathematics education in the Nordic countries. This initiative and other efforts helped to reactivate NOMAD, and the journal has been published regularly since 2004. The special issue was funded by grants from Finland, Norway and Sweden. It was printed in a large number and was spread to a larger audience during ICME-10.

Nordic Presentation

NCC took on the responsibility of arranging a common presentation of mathematics education in the Nordic countries at ICME-10. Obviously, the NCC wanted to avoid a solution with five "small" presentations, one for each country. The fact that there were contributions from teachers and researchers from five countries enriched the programme. In a few cases there were even joined presentations given by researchers and teachers from different countries. On the other hand the joint Nordic presentation also created some difficulties since the educational systems, national curricula, teaching traditions, and learning outcomes varies between the countries. Hence it was not possible to create a common frame by describing one educational system and its outcomes. The solution was to have two invited "plenary" speakers to give their outline of mathematics education and research in mathematics education, with examples from several countries. The other presentations – about 60 lectures, workshops and exhibitions – each dealt with just one country. The NCC had defined a number of common themes, sent out together with the invitation, in order to create some structure. The themes grew out of discussions within the NCC – and with others – where the aim was to identify common characteristics within mathematics education in the countries. The programme will be documented in a separate proceedings.

A book was prepared by the NCC (Stedöy, I., 2004) to accompany the Nordic presentation. It was printed in a large number and distributed during ICME-10. The book gives a picture of different aspects of mathematics education in the Nordic countries. In an introductory article, an effort is made to identify and discuss a "Nordic dimension" in education.

The KappAbel-competition

In Denmark summer school holidays occur in July. Hence it was not possible to offer visits to schools in the Copenhagen area during ICME-10. However, the KappAbel competition presented an opportunity to meet Nordic mathematics students during the congress. The KappAbel competition for pupils in school year 9 (or 8) started in Norway around 1999. A number of special features make this competition interesting from a didactical point of view. Some of these features are the following: KappAbel is a mass competition, the first round is for large groups of students (school classes) , short tasks are complemented by class projects, the final is a competition between teams of two boys and two girls selected by the classes. The problems used in the competition are offered to teachers to use in follow-up activities in their ordinary teaching. The competition was spread to the other Nordic countries through the support and networks of the NCC. The first Nordic final was held in 2003. The second final was held during ICME-10, with five competing teams, one from each country. The members of the teams showed what joy and excitement mathematics can produce and especially the project presentations were highly appreciated by the 300 congress participants that experienced the final.

Although the planning of ICME-10 required an enormous effort we all feel that it has been worthwhile and we are convinced that the ICME-10 co-operation will have long lasting positive effects on mathematics education research in the Nordic countries.

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Opening Session

Keld Jørgensen, Royal Danish Brass

Your Excellencies. Good morning.

We have the pleasure of being the musical frame of this opening session and we started with a truly festival piece, a piece written by Henry VIII of England. It is called – and this could be the *bon mot* of this event – "Pastime with good company" and we continue with another royal piece, a Danish piece written by an Englishman, John Dowland, to the Danish king Christian the Fourth. The title is "The King of Denmark's Galliard".

Ingvill Stedøy, Mistress of ceremonies

God morgen, velkommen til København. Good morning and welcome to Copenhagen and the 10th International Congress of Mathematical Education.

It was the Royal Danish Brass that welcomed you with this great music. I myself am Ingvill Merete Stedøy, I am a member of the Nordic Contact Committee. Since this is a Nordic event we have been working for almost five years in the Nordic countries and now we are finally here. I am the scientific director of the Norwegian Center for Mathematics Education. I represent Norway in the NCC and this morning I am going to be your mistress of ceremonies. First I want to introduce to you Mogens Niss. He is the chair of the International Programme Committee and a professor at Roskilde University, Denmark.

Mogens Niss, Chair of the International Programme Committee

Dear Minister, dear ICMI Executive, dear Mayor, dear Dean, dear colleagues and friends.

Almost five years of planning have passed since the Nordic countries were first asked, in July 1999, whether we would consider to host ICME-10 in 2004. Today, this very morning, the bugles will sound to mark the opening of the 10th International Congress on Mathematical Education, here at the Technical University of Denmark, DTU.

In my capacity as the Chair of the International Programme Committee, it is my immense pleasure and honour to welcome you all to our Congress in Lyngby, Copenhagen, Denmark, the Nordic countries, as we call them at these latitudes, to enjoy – we hope - the fruits of all our efforts.

An ICME is not only a conference, focusing on themes or topics; it is a *congress*, addressing the entire community of all those in the world who perceive themselves as mathematics educators, whether as teachers, developers, curriculum authorities, administrators, researchers in mathematics education or in mathematics as such, or combinations of several of these materialisations of the profession of mathematics educator.

A congress is obliged to provide new experiences, new knowledge, insights, and food for thought, and above all opportunities for new contacts, to everyone present, regardless of her or his particular position and fields of interest. This calls for a rich, challenging, multi-faceted, and fascinating scientific programme.

The International Programme Committee has done its utmost and worked hard to establish a programme with these characteristics. We have deliberately attempted to compose the scientific programme as a mixture of 'classical' and 'novel' elements as is reflected in the overall time-table for the congress.

Classical elements include plenary and regular lectures and posters. Furthermore we have kept the dual structure of activity groups - the Topic Study Groups and the Discussion Groups - while emphasising in our briefs to the Organising Teams that the former are designed to be focused sub-conferences based on presentations, whereas the latter should really be what the title suggests, groups for focused discussion and not for presentations.

Novel elements include the five Survey Teams, each of which have worked for several years on the community's behalf to survey a field, a topic, or a *problématique*. A plenary interview session and a plenary debate on a truly controversial issue are new as well. The same is true for the Thematic Afternoon, in which the entire congress will concentrate on just five themes, albeit broad and overarching ones.

The Workshops and the Sharing Experiences Groups, too, are new inventions, at least if we keep the ICME-10 notion of a workshop in mind. The same is true with poster round tables where poster presenters are given an opportunity to have their posters discussed together with a few others in small groups, led and moderated by an experienced mathematics educator. And on top of all that, we have also organised a Mathematical Circus demonstrating to the general public, should that be needed, that mathematics offers lots of opportunities for entertainment and fun.

It goes without saying that we hope that these new initiatives, most of which are of a somewhat experimental nature, will be met with your approval. In fact we would very much like to have your opinion on the scientific programme, not because we in the Nordic countries are planning to host yet another ICME, but because we want to assist our successors in organising still better congresses. In your congress bag you will find a questionnaire on the programme. Would you please be so kind as to fill it in – it's anonymous - and return it to us at the end of the congress?

One consequence of today's globalised world, largely ruled by the market place, is that individuals tend to be reduced to being consumers rather than responsible and committed citizens. By its very nature, an ICME *is* a kind of market place, a supermarket or a department store, if you like. This suggests the presence of forces that turn delegates into consumers. Based on my personal acquaintance and friendship with hundreds of mathematics educators, it is my strong hope and expectation that the participants of ICME-10 will not accept to be just consumers shopping around, but will insist on the right to be true citizens of our congress, helping as best they can to ensure coherence and continuity of activities, in particular multi-session activities such as Topic Study Groups, Discussion Groups, Workshops and Sharing Experiences Groups.

It is time for me to conclude. I shall do this by extending, on behalf of the International Programme Committee, my sincerest and warmest thanks to all those hundreds of colleagues and friends who, in a pretty decentralised planning organisation, have worked for years, so altruistically – and indeed without the slightest financial remuneration - to organise the scientific programme of ICME-10. My thanks also go to all those of you who will contribute to the scientific programme as speakers, presenters, inventors of small group activities, special meetings and so forth. Last but not least, my thanks go to all the participants who have come here to listen and to learn and to make new friends from different places round the world. After all, mathematics education is a deeply humanistic activity, and our field will greatly benefit from strong international links and networks amongst those who profess it.

I wish everyone all the best for a stimulating, fruitful, and enjoyable ICME-10!

Ingvill Stedøy
Thank you to Mogens. The next person up here will be the chair of the Local Organising Committee, Morten Blomhøj. He is also a professor at Roskilde University in Denmark.

Morten Blomhøj, Chair of the Local Organising Committee
Dear friends and colleagues.

On behalf of the Local Organising Committee it is a great pleasur for me to welcome all of you to ICME-10. A special welcome to the Minister, the Mayor, the Dean and all the special honoured guests of the congress. After more than four years of preparation it is very exciting for us to see the fruits of our work. In close cooperation with the International Programme Committee and the Nordic Contact Committee we have worked to create the logistic frames for a congress enabling the realisation of the multifaceted and highly interactive programme planned by the IPC. It is a unique thing that ICME-10 has been organised in cooperation among five countries, the Nordic countries. This has been a great benefit for the congress but even more importantly, perhaps, the ICME-10 cooperation has fertilised the research *milieus* in the Nordic countries and strengthened the Nordic cooperation within the field of mathematics education. ICME-10 provides multiple opportunities for interaction between mathematics teachers, mathematics education researchers and mathematicians within the framework of the scientific programme, but also for more informal interaction within the framework of the social programme. It is our hope that all of you, contributors and participants, will take advantage of these particular opportunities. Hopefully, you will find, as we do, the campus of the Technical University of Denmark (DTU) an ideal venue for ICME-10 with its many well equipped lecture halls, group rooms for discussion, and nice, green surroundings. The only less ideal aspect of DTU is that this main hall is not large enough to hold all the 2300 ICME-10 participants. However, I am confident - looking down to the technicians -that our solution to this problem, i.e. video transmission, will make it possible for all participants sitting in building 116 and 303 to watch and hear what is going on now here in the opening session. So, a special welcome to you, too.

It is no secret that the congress programme and partly also the budget, I am afraid, was planned for some thousands of participants more than the 2300 that have actually registered for ICME-10. It is our hope that this fact can be turned into an advantage for the congress participants and for the scientific outcomes of the congress. A smaller number of participants in the many different parallel sessions will enable us to deepen the discussion and the reflections, hopefully raising the scientific outcomes of the congress. So as a contributor to the scientific programme, please use the extent and level of interaction as the main criterion for success rather than the number of participants attending your activity. And as a participant, please do take the opportunity to discuss with lecturers, presenters and group organisers. I ensure you that they are more than willing to discuss their work and ideas with you.

ICME-10 is a truly international congress with representatives and contributors from more than a hundred countries, and from all parts of the world. Thus it provides an excellent basis for the further development of multi-cultural approaches to mathematics education research. A large number, in fact nearly half of all the participants, are to contribute to the scientific programme in one way or the other. Please help and support contributors to do the best possible job. One way of doing so is to be supportive of contributors and others not having English as their mother tongue. The Local Organising Committee wishes you a fruitful and enjoyable congress.

I finish by paying my sincere respect to all parties who have helped and supported the congress and the organisation. First and foremost I thank all the members of the Local Organising Committee for their work and enthusiasm through the whole process, and special thanks to Elin Emborg, my dear friend and colleague who has worked incredibly hard, being the administrative secretary for both the IPC and the LOC. Also special thanks to our Congress Bureau, Congress Consultants, for their professional and loyal cooperation. Many thanks to the main sponsors, Casio and Texas Instruments, to the Technical University of Denmark for lending us their campus, and to the home institution of Mogens Niss, Elin Emborg and myself, IMFUFA, Roskilde University for supporting the planning process. And to other education and research institutions, foundations and organisations and in particular to the Danish Ministry of Education for a very early and very substantial support. Many thanks also to agencies and bodies from the other Nordic countries. The full list of sponsors can be found at the end of this proceedings. Without this support ICME-10 would not have been possible. Thank you all very much!

Ingvill Stedøy
As you have understood from what the previous speakers have said, this is a Nordic event between Denmark, Finland, Iceland, Norway and Sweden. The next speaker is Gerd Brandell, chair of the Nordic Contact Committee during the five years of the planning of the congress. She is a professor at the Technical University of Lund, Sweden.

Gerd Brandell, Chair of the Nordic Contact Committee

Dear Minister and invited guests, dear colleagues and friends.

On behalf of the Nordic Contact Committee I wish you very warmly welcome to the 10th ICME. I speak for all five countries involved: Denmark, Finland, Iceland, Norway and my own country, Sweden.

Our countries are neighbours, and we have a long history of close contacts and collaboration in many areas. We feel like brothers and sisters in a family - certainly all five are strong individuals, but we also have many things in common and understand each other well. This also goes for mathematics education; there are many different characteristics for each country. At the same time we have much in common in the teaching and learning of mathematics.

For the first time, ICME has been organised by several countries in cooperation. We hope you will find that this model proves to work well. The role of the Nordic Contact Committee corresponds to that of a national committee, securing necessary support on a regional level for this huge enterprise. As Morten Blomhøj pointed out the committee has been working in close cooperation with the Local Organising Committee. Our common vision for the congress is an efficient organisation and a friendly atmosphere. We feel strongly about the importance of gender balance and have had that in mind during the long process of preparation.

Great visions are fine but the only thing that really counts is the result. Most important for us is that you will all find good opportunities to gain interesting experiences and get new insights during the congress. Hopefully you will also find plenty of time to develop new contacts and get new friends in mathematics education.

A special program for newcomers is launched at this congress, organised by the Nordic Committee with support from the ICMI Executive Committee. We are happy about the overwhelming interest among newcomers to participate, and we sincerely hope it will help those who attend ICME for the first time to find the things of special interest for each one in the rich program, and to grasp the structure and aims of the congress.

As Mogens Niss said it is now five years since we developed the concrete plan to host ICME-10 in Copenhagen. I remember speaking to Gilah Leder – who was at that time a member of the Executive Committee of ICMI. We were talking together at a decisive moment during the process. She told me about the strong impact that the 1984 ICME in Adelaide had on the development of mathematics education as a research area in Australia. Not only the congress in itself but also the efforts to plan and organise the congress created inspiration and energy and offered fruitful experiences for all those involved on the national level.

I am convinced that ICME-10 will help to bring research in mathematics education a big step forward in our countries. Therefore I am happy that ICMI and the Executive Committee decided to let Denmark and the Nordic countries host ICME-10.

I am pleased to know that many participants from the Nordic countries have found their way to the congress, and that many of them will take active part and contribute in various ways to the program.

I am convinced that I speak for all members of the mathematics education community in the Nordic countries when I, once again, express my warmest welcome to all of you who attend this congress from outside our northern corner of the world. We are happy to see you all here.

I wish to express our special and warm thanks to the ministries of education in all the Nordic countries and to the Nordic Council of Ministers, for their generous financial support.

Thank you!

Ingvill Stedøy
As this congress is taking place in Denmark, it is of course a special thing for the host country and the Danish Ministry of Education. I am now calling upon the Danish Minister of Education, Ulla Tørnæs, to address the audience.

Ulla Tørnæs, Minister of Education, Denmark
It is a great pleasure for me – on behalf of the Danish government – to welcome you and to wish you success with all your activities and work the coming week.

In Denmark we are in a process of adjusting and reforming the entire educational system, from primary through upper secondary school to education at the universities.

In recent years quite a few countries have carried through similar reforms, or are planning do to so. In almost all primary and secondary education reforms, focus is placed on three specific subjects:

- mother tongue education, for our part Danish
- English, and
- mathematics.

In some sense you can say that there are the two international languages of our time: English and mathematics. At this congress both are in the game.

Reforming raises two main concerns: What to learn and how to learn?

But before trying to find an answer to those two questions, I believe it is just as important that we dare to ask: Why learn? Why is it, that it is so important to learn mathematics?

Putting this question to professionals, the answers you will get fall in three categories:

Some professionals claim that learning mathematics is important because it advances general analytical competences more than do other subjects. Please forgive me for challenging this statement: you may be right but can you prove it?

I believe that all subjects taught in school could and should sharpen all pupils' and students' abilities to reason, to infer logical implications and to disclose arguments that are valid. One of the greatest Danish mathematicians Harald Bohr once put it this way: "Mathematics may not enable us to learn how to think right, but rather make it clear for us how easy it is to think wrong."

Other professionals stress, that learning mathematics is important because a modern society needs mathematicians of all kinds.

I very much agree and in the process of reform, much attention is drawn to the problem of stimulating the interest in mathematics and natural sciences – from the first years of primary school to the last years of upper secondary school. This is a field where exchanging ideas and experiences of "best practise" are of utmost importance.

We do not expect, nor do we need, all young people to study or work in the field of mathematics. But why then have we decided that all pupils should learn mathematics?

This leads to the third category of answers: Mathematics for all is crucial for the democratic process in a modern society. It gives citizens a better understanding of public matters and debates and helps individuals to form their own opinions.

Therefore it is very important how we teach mathematics in our schools. What sort of mathematical teaching do we need in order to improve the abilities of every individual to become a democratic citizen?

The answer is not a simple one, and some people may argue, that you don't need to be an electronic engineer to operate a television set. However, that is not the issue, because: Mathematical skills at a basic level can furnish you with the self-confidence that it takes to dare to doubt and ask questions, relating to anything from economical policies of the government or propositions of real estate agents.

I believe there is a tendency to give priority to these aspects of teaching mathematics. Therefore we need more research in the didactical problems of teaching mathematics: How do we ensure that "mathematics for all" will really be for all in the end?

If we are to inspire more pupils to take mathematics to heart, and more students to apply for mathematics studies, we need more than politics. There is a Danish saying that you can force a horse to the trough, but you can't force it to drink. We politicians can put mathematics on the agenda and we can drag the pupils to class. But only the teacher can make them learn.

In the Danish Ministry of Education we are highly aware of the importance of congresses like this, and were happy to have made a considerable donation for the carrying out of the ICME-10.

Some weeks ago three leading professors of mathematics wrote a very interesting essay in one of the Danish national newspapers. The main point, translated into English, was that: "Beauty is a very important and strong incitement in mathematical research. All real mathematicians are chasing beautiful theorems and proofs."

Is it possible that the beauty of mathematics could also be a dynamo in the teaching of mathematics? I leave this for you to answer.

I wish you an inspiring congress and good luck in communicating the beauty of your science.

Ingvill Stedøy

The next person I will call upon is the Dean of Research at the Technical University of Denmark, the host institution of this congress. His name is Kristian Stubkjær.

Kristian Stubkjær, Dean of research, DTU

Ladies and gentlemen, your excellencies.

On behalf of the Technical University of Denmark, DTU, it is a pleasure for me to welcome you and the 10^{th} ICME to our university. We are a technical university and extensive skills in mathematics are a necessity for almost all of our activities. This is reflected by the fact that courses in mathematics account for 15-20% of our teaching load, thus emphasising the importance of mathematics. DTU is a modern university and as of September this year studies at DTU will be structured according to the Bologna declaration, which means a 3+2+3 stucture. The new students starting in just two months will meet this study structure which will enable them to move more freely between universities. Here they will be offered a number of specific specialisations at bachelor's level, including one in mathematics and technology, and afterwards, at master's level, we offer a specialisation in applied mathematics. It is an absolute priority for us to offer challenging and stimulating study environments for talented and enthusiastic students. Our teaching, including our teaching in mathematics, is research based. Our professors are continuously exploring new ways to communicate mathematics and to develop new teaching methods in the field. I am sure that the 10^{th} ICME will be important also for further improving the teaching of mathematics here and in other places. I wish you a very successful conference here at DTU. You are very much welcome!

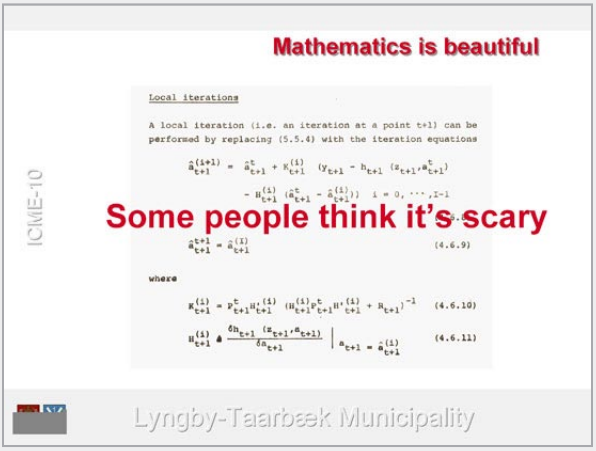
Ingvill Stedøy

And now the local host of this event, the mayor of Lyngby-Taarbæk municipality will welcome us. I call upon Rolf Aagaard Svendsen.

Rolf Aagaard Svendsen, Mayor of Lyngby-Taarbæk municipality

Thank you very much. Mrs. Minister, ladies and gentlemen, welcome to Lyngby-Taarbæk.

We are happy to host such an important conference here. I happen to be among those who think that mathematics is beautiful.



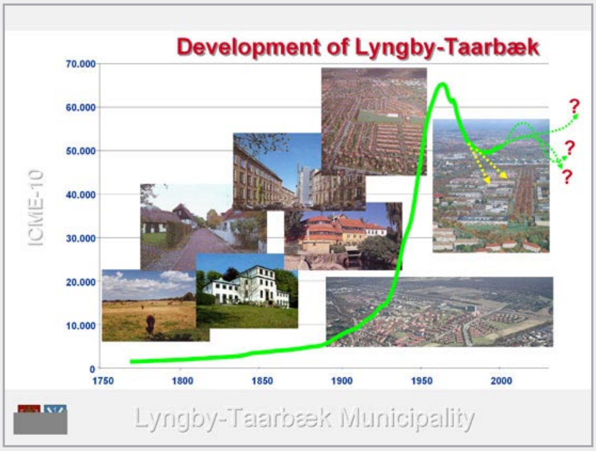
What is that? That is the Danish economy. Why is it so beautiful? It is because it is a model. The original is a catastrophe! These drawings I made for a chapter front page in my Ph.D. thesis made here at DTU in 1979. At that time it was true that the model was beautiful and that the Danish economy was a catastrophe. But somehow the economy has converged to the beauty of the model. So we certainly need you to teach future economists to make beautiful models. My Ph.D. thesis was called *Econometric methods and Kalman filtering*. I think it is sold out, but here is a page from the thesis:

Isn't that beautiful? Well, the problem is that most people will find it rather scary. If you haven't broken the code you are not able to see the beauty. So you have an important task to enable people to break the code and see the beauty. To be honest, after all these years it looks a little spooky to me too.

Because now I am the mayor of Lyngby-Taarbæk municipality, 12,5 km North of Copenhagen, occupying an area of 38,76 km^2 with 51.500 inhabitants.

You may say that I am the living proof that mathematics can lead to anything. Or for those of you who come from California, you don't need to be a movie star to be a mayor, or a governor. Try with math!

The landscape of the municipality was formed by the ice when it melted 15.000 years ago. Around year 1800 the number of inhabitants was approx. 2.000. They were mostly farmers. There were also three small castles, and some wealthy people had built mansions on the lake sides.



In the next century the population grew, and so did industry in connection to the nine mills along the Mill Stream. They are called the cradle of Danish industry. In the first half of the 20th century, the population growth really took off and farm land was converted to business and residential areas. Then it was decided to move DTU from Copenhagen to Lyngby-Taarbæk. That occupied almost all of the residual areas for housing so the population declined between 1965 and 1991 when it started to slightly grow again. But how will the development be? We need you to teach some people mathematics so they can make some better forecasts because we were misled by the forecasts of the past.

Lyngby-Taarbæk is a green city. More than 50% consists of parks, forests, open land, lakes and the Mill Stream. And most of this is preserved area. The 51.500 inhabitants live in 25.000 dwellings. We try to keep the residential areas, whether private houses or apartment buildings, green as well.

We have 32.000 workplaces, and many companies have chosen to place their headquarters here. The workplaces of course include the DTU but we have also some other educational institutions in the municipality. So we consider ourselves to be a university city.

Our shopping centre is a big mall area with a turnover that equals that of Copenhagen City. So you don't have to go to Copenhagen to buy things to bring home. Shuttle busses to and from the centre will be provided.

And while you are in Lyngby, you are most welcome to take a look at the different sights. We have a medieval church with some characteristic Danish frescos, and you can take a stroll in the well preserved village, *Bondebyen*, nearby. Moreover, you can visit the *Open Air Museum* with old farmhouses from different parts of Denmark.

You can visit the big forest park called the *Deer Garden*. There you will also find *Bakken*, the world's oldest amusement park. You can take a look at the three castles and the mansions. Unfortunately the castles are not open to the public, but you can take a look from the woods.

The mills along the Mill Stream are worth visiting and you can rent a canoe or take a boat trip on the lakes. But please don't forget the mathematics! You are always welcome back. Have a very nice stay!

Musical interlude by the Royal Danish Brass:
Jeremiah Clarke: Trumpet Voluntary
Keld Jørgensen: Lur Cha-cha
H.S. Paulli: "Retrait" from the ballet Napoli

Ingvill Stedøy
Thank you again to the Royal Danish Brass. Now we have come to the official opening of ICME-10 and I will call upon the President of the International Commission on Mathematical Instruction (ICMI), Hyman Bass, who is a professor at the University of Michigan, Ann Arbor, USA.

Hyman Bass, President of ICMI
Minister Tørnæs, Dean Stubkjær, Mayor Aagard Svendsen, Chairmen Niss and Blomhøj, Dr. Stedøy, guests and participants of the 10th International Congress on Mathematical Education.

As President of the International Commission on Mathematical Instruction, it is my honor and pleasure to welcome you all to this auspicious congress, and to express our collective appreciation of the hard work, imagination, and gracious hospitality of our Nordic hosts.

About the ICME and mathematics education
This Congress vividly reminds me of the complexity of mathematics education, and of how hard it is to globally comprehend.

This contrasts with mathematics, as a discipline. Mathematics has a universal character. Mathematicians throughout the world have a largely shared sense of the nature of their discipline, its central problems, its methods, and of the nature, genesis, and warrants of mathematical knowledge. Mathematicians know each other, and speak a common technical language.

Mathematics education, in contrast, is not simply a discipline, a body of knowledge, a field of scholarly research. It is *partly* that – things one *knows*. But, much more than that, and perhaps more importantly, it comprises things that people *do*, a field, or rather a constellation of fields, of *practice*. Who are the professionals that populate this enterprise? They are, first and foremost, teachers of mathematics, at all levels, from kindergarten through university levels. And they are teacher educators, a diverse community of which many mathematicians are (often unconsciously) members, as well as teacher leaders and professsional developers. They are mathematicians, curriculum developers, assessment specialists, school administrators, district and state level supervisors and policy makers. And overlain on all of this are educational researchers who study all aspects of this loosely organized system.

This character of mathematics education was not always so. For most of history, few of the professions I have mentioned, except for teaching, existed. The evolution toward this vast enterprise, that we now inhabit – and here assemble – was long and gradual.

It was marked by certain transforming events such as the invention of the printing press, the industrial revolution, the emergence of science as a foundation for security and commerce, the digital revolution, and the spread of democratization. These have had certain consistent and cumulative effects on education, and on mathematics education in particular:

- Higher leverage resources for the conservation and transmission of knowledge.
- The need for acquisition of more, and more sophisticated knowledge.
- The need to provide such knowledge to growing numbers of people.
- The challenges of diversities: Of resources and expertise needed for the educational enterprise; of cultural and social contexts; of institutional and curricular organization; of learners and learning styles; of appropriate pedagogical methods; and of the formation of education professionals.

The core of contemporary mathematics education remains what Deborah Ball and David Cohen have called the "instructional triangle," the interactive dance of the teacher, the students, and the mathematics, in a classroom setting. Scholarly work on mathematics instruction has progressed from an early focus on the mathematical ideas, and how best to render them in the school curriculum, then to a close cognitive study of learners and how they process and assimilate new mathematical ideas, and now increasingly to teaching, a complex and multidimensional phenomenon for which effective methods of research are only now being developed.

The size and diversity of this Congress mirrors that of the mathematics education enterprise itself. An added special feature, and benefit, of this Congress is its *international character*. Mathematics education is culturally situated, and takes different forms in different societies. Here you will be able to learn about, and from, those differences. Here, in one environment, you will meet and communicate with co-professionals with whom you rarely have occasion to interact, be they from another continent, or from your home institution. It is a unique event, perhaps at times bewildering, but I hope also edifying, and even inspiring.

A tribute to Miguel de Guzmán and Igor Sharygin

There are many dedicated individuals in the ICMI family who carry forward the work and organization that make these Congresses, and the other work of ICMI possible. We have sadly this year lost two members of that family.

Miguel de Guzmán, my predecessor as President of ICMI, passed away suddenly and prematurely on April 14, 2004. He was a distinguished harmonic analyst, and an intellectual and spiritual leader of the current blossoming of mathematics and mathematics education in Spain. Among Miguel's important contributions to ICMI is the creation of the Solidarity Project, whose aim is outreach to help improve mathematics education in developing countries. He was a man of broad culture, deep compassion, and an inspiring communicator and teacher. His passing away is a sad loss for our Spanish colleagues, and for the many communities of mathematics and mathematics education worldwide.

Igor Sharygin, a name perhaps less familiar to you, was a member of the last ICMI Executive Committee. I report with the sadness of all who had the good fortune to know him that Igor passed away on March 12, 2004. Igor was a high school teacher who exemplified the highest Russian traditions of problem-based mathematics education. His love and deep understanding of geometry is evident in his writings. And Igor was culturally a mathematician, who typically used the word "beautiful" in describing both mathematics and mathematicians. We fondly remember his personal warmth and generosity, and his passion for life and ideas.

The ICMI Awards

There are many important new developments in the ICMI world since the last Congress in Japan. To conclude these welcoming remarks, I wish to speak of one of them, the inauguration of two new ICMI awards – the *Felix Klein Medal* for lifetime achievement in mathematics education research, and the *Hans Freudenthal Medal*, for a major program of research on mathematics education during the past 10 years. Michèle Artigue will shortly chair the presentation of these awards.

When I arrived in the ICMI environment, the possible establishment of ICMI awards was one of the first issues I encountered. This question had already had a long and inconclusive history. The ICMI Executive Committee in 1999 appointed a committee of distinguished and respected leaders in the field, chaired by Jeremy Kilpatrick, to study the question and make a recommendation to the ICMI EC. The medals to be awarded today inaugurate a design that follows the essential principles recommended by the Kilpatrick Committee. Suffice it here to share some of the views, partly personal, that shaped this action.

Opposition to giving awards was based on concerns for things like elitism, potential or perceived bias, superabundance of qualified candidates and consequent disappointment of deserving individuals, lack of sufficiently objective and consensual criteria for selection, immunity of the selection process from undue external pressure, etc. All of these are serious concerns, to which substantial attention was given in the design of the award process.

Of the many kinds of important contributions to mathematics education worthy of recognition, we chose, for now, to focus on *mathematics education research*, since this is a domain where norms of evaluation are most developed, and now most demanded. Indeed, we felt that the awards themselves, and the quality controls on the selection process, could help contribute, through such public recognition of exemplary work, to the evolution and better articulation of broadly accepted norms in the field. The Awards Selection Committee consisted of an international group of six distinguished scholars in the field. Its membership remains confidential until expiration of term, except for its chair, Michèle Artigue.

These awards honor extraordinary work of individual scholars, and, in so doing they are meant to encourage the efforts of others in the field. But they have broader purposes as well. As I just indicated, they offer a process for developing, over time, a publicly sanctioned definition of quality in a field that has often struggled to find one. The absence of such awards was, in some minds, and in the outside world, a confession of the lack of possibility of such a consensual definition. To less generous critics of mathematics education, it signaled an absence of work worthy of high recognition. When we are now asked to cite exemplary mathematics education research, we should be able to point, with conviction and pride, to those recognized with the award of these medals. A further salutary effect of the awards, an effect already witnessed, is that they will help breach some of the provincial boundaries in mathematics education scholarship, wherein much important work is known only within national or regional boundaries. The works of today's and future medalists will more quickly gain the wide international audience that they deserve.

Opening of the Congress

It is now my privilege and joyful duty, on behalf of the International Commission on Mathematical Instruction, to declare officially open this 10th International Congress on Mathematical Education.

For my first act within the congress, it gives me great pleasure to introduce Professor Michèle Artigue, Chair of the Awards Selection Committee, for the ceremony of presentation of the Felix Klein and Hans Freudenthal medals.

Michéle Artigue, Chair of the Awards Selections Committee

As was explained by Hyman Bass, a moment ago, when the ICMI Awards Committee was built, he proposed to me the immense honour of chairing it. I accepted the task, conscious as were my five colleagues in the Awards Committee of the responsibility which was put on our shoulders, of the tremendous difficulty of the work given to us and of the decisions we would have to take.

On the one hand, the creation of these two awards was the official acknowledgement of the maturity acquired by the field of research in mathematics education, of the role that this research could play and should play for improving mathematics education at large; the ICMI gesture had thus a high symbolic value. On the other hand, the field was so diverse, so multicultural, as are educational cultures, that selecting two persons among those, so many, who for more than thirty years have worked for the development of this field of research and contributed to it, looked as a nearly impossible task.

Our first task was to reflect on the criteria we would use, and also of course to disseminate the information about the awards through different channels: ICMI national representatives and affiliated study groups, journals in mathematics education, national and international associations. I would like to thank all those who contributed, thanks to this process, to the diffusion of the information, and all those who then, through the nominations and documentation they sent me, helped us so much in our task.

Soon enough, we converged on some main criteria: *impact, sustainability, depth and novelty.*

These awards had to go to scholars who had played or where playing a central, decisive role in shaping the work and identity of the research communtity, but such a role can be played in many different ways and we had to be open to this diversity.

These awards had to go to scholars who had offered to the field deep and original contributions, whose research had been in some sense seminal for the field and had influenced its evolution.

And, in a research field which, too often, seems to change directions, following one ephemeral trend or another, we also wanted to award contributions that have proved to have lasting effects, to resist to the erosion of time.

Finally, we also thought that these awards had to go to scholars who were not just prominent researchers but tried to put their research advances and their research notoriety at the service of the improvement of mathematics education.

These criteria were, in our opinion, common to the two awards. What differentiated these was more the way the criteria had to be taken into account: thinking about the Klein award, we had to take into account a lifelong achievment in all its possible dimensions, and not limit ourselves to the current state of the field to judge the impact, sustainability, depth and novelty of the research work. Thinking about the Freudenthal award, we had to develop a different vision, focusing more on a specific area and a limited period of time.

For a while, our idea was to reserve the Freudenthal award for young scholars in the field, rising stars in some sense. But the selection between these quickly appeared as too random. The field of mathematics education is a field belonging to the human sciences, even if it deals with mathematics and requires of its scholars strong mathematical knowledge. Substantial advances don't result from flashes but from patient and long term work; novel approaches take time to reach a reasonable state of maturity; the possible influence of ideas and results on the field can rarely be appreciated correctly soon after they have been published and known. This was the reason why, after long discussion and case study examinations, we changed our mind and opened the Freudenthal award to mature researchers.

Defining criteria certainly was an essential step in our enterprise, but not necessarily the most difficult one. Much more difficult was to think about how these criteria could be made operational, when looking at specific cases. How could these be used in order to make selections, comparisons, and finally choices? How to appreciate for instance the exact influence of a researcher? There was no doubt to us that just counting her or his publications, how many times she or he is quoted, which tends to become the general trend in research evaluation, was too much of a superficial view. How to appreciate the deepness and novelty of a research contribution, without knowing intimately this work, and also all those which tend or have tended to address the same or similar issues in other contexts, relying on other theoretical frames?

More and more, we were seeing our task as an extremely demanding task, and the two years given to us which appeared as such a long time at the beginning, was soon seen to be too short.

I will not enter any further in the details of our work. I would only say that along the road, this work became more and more fascinating. We came from very different educational cultures; all of us had a lot of international connections, this was one of the reasons for us being apointed; but we were discovering the limits of our respective knowledge of the field, we were discovering the incredible richness of the field, we were discovering disconcerting proximities… We learnt a lot from each other in the friendly and scientifically challenging atmosphere of our exchanges. And, progressively, through an e-mail discussion with so many rebounds that, at some moments, we had the feeling that we were characters in a suspense novel, we came to a final agreement on two names: one for each award.

We perfectly know that the story could have had another end: there were several excellent candidates with so many different personalities and contributions that the final choice could not be something really objective, even if we used our criteria meticulously. But we are deeply convinced that the choice we finally made is a choice which responds to what ICMI has decided to value when creating these two awards. The two eminent scholars who will receive these awards in the next minutes perfectly exemplify what can be an outstanding scholarship in mathematics education and the multiplicity of dimensions that this scholarship takes. They exemplify also a cultural diversity which is an essential characteristic of the field of mathematics education, also an essential source of its richness and productivity, while at the same time making research in this field so challenging and communication between cultures so crucial.

Before coming to the next phase of this ceremony, as the chair of the Award Committee I would like to express my deep gratitude to my five colleagues in the Committee. Their scientific and human qualities, the sense they had of their responsibility allowed our group to work during these two years, free of any kind of ideological and political pressure, only for the benefits of science.

I am now proud to officially present the first awardees of the Klein and Freudenthal Medals.

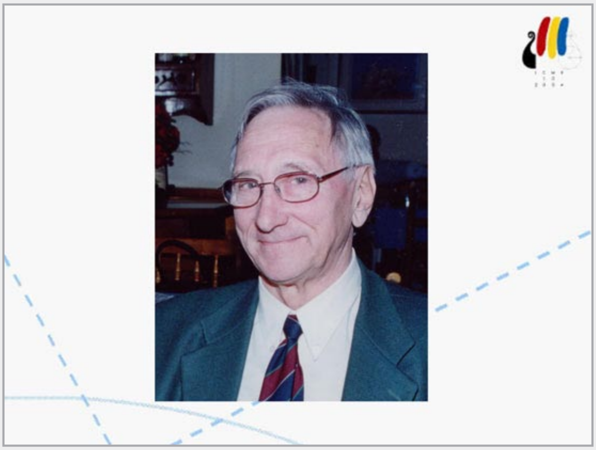
The Felix Klein Medal for 2003 is awarded to ***Guy Brousseau***, Professor Emeritus of the University Institute for Teacher Education of Aquitaine in Bordeaux, France

This distinction recognises the essential contribution Guy Brousseau has given to the development of mathematics education as a scientific field of research, through his theoretical and experimental work over four decades, and to the sustained effort he has made throughout his professional life to apply the fruits of his research to the mathematics education of both students and teachers.

The Hans Freudenthal Medal for 2003 is awarded to ***Celia Hoyles***, Professor at the Institute of Education of the University of London, United Kingdom

This distinction recognises the outstanding contribution that Celia Hoyles has made to research in the domain of technology and mathematics education, both in terms of theoretical advances and through the development and piloting of national and international projects in this field, aimed at improving through technology the mathematics education of the general population, from young children to adults in the workplace.

Guy Brousseau's work will be presented by Carmen Batanero who is a member of the current Executive Committee of ICMI.



Carmen Batanero, Member of the Executive Committee of ICMI
Dear authorities, dear organisers, dear friends.

It is for me an honour and a really great pleasure to introduce to you professor Guy Brousseau who has been awarded the first Felix Klein medal of the International Commission on Mathematical Instruction. I am sure you all agree that this distinction is well deserved because of all the work he has done throughout his professional life. Brousseau began his career as an elementary school teacher but his interest in continuous training led him to major in mathematics and also to do a Ph.D., *Doctorat d'Etat*, in mathematics education and to start doing research on his ideas. He entered the University of Bordeaux in 1996 where he became a full professor at the Institute of Teacher Education in 1998. He is now a professor emeritus at the University Institute for Teacher Education of Aquitaine, doctor honoris causa at the University of Montreal and doctor honoris causa at the University of Geneva.

One main achievement is his theory of didactical situations, which he began to create as part of his doctoral dissertation and continued to develop over the years, and which has inspired a large number of researchers around the world. At the time where the dominant vision of the field was psychological/cognitive, he was convinced of the need of implementing and introducing also social, mathematical, and epistemological dimensions in the study of mathematics education. Thus he helped clarify the relationships of mathematics education with other disciplines and to characterise the object of study, while at the same time developing concepts and models to interpret and analyse mathematical teaching and learning. This theory has been a constant source of inspiration and has given rise to many constructs such as adidactic and didactic situations, institutionalisation, action, devolution, and so on. Furthermore notions such as didactical contract, memory, milieu, informational graphs of types of obstacles, etc. have been made widely accessible through the translation of his work to many different languages and, in particular, by the publication of the Kluwer book in 1997 *Theory of Didactical Situations in Mathematics Education*. His research addresses all levels from primary school to university but has mainly concentrated on primary school, dealing with several different topics, from the learning of algorithms of multiplication and division, numeration, rational numbers and proportions, decimals, to the transition from arithmetic to algebra, geometry and probability. At the same time he has explored and used a variety of mathematical models, such as statistics, multivariate statistics, graphs and game theory to model and explain numerous didactical phenomena, and also to propose an original methodology of research, which we now call didactical engineering.

Brousseau is not just an original and inspired researcher in our field but he has also contributed to mathematics education in many other respects. For example, at the national level in France, he was involved in the creation of the association of mathematics teachers, the *IREM*s (the research institutions for mathematics education), the journal *Recherches en didactique des mathématiques*, the association for research in mathematics education, the summer school and the national seminars on *didactique*. At the international level he was involved in creating of the group *Psychology of Mathematics Education* (*PME*) at the ICME-3 in 1976 in Karlsruhe. He also played a major role in the *CIEAEM* (*Commission Internationale pour l'Etudes et l'Amélioration de l'Enseignement des Mathématiques*) for 30 years. He was its secretary from 1981 to 1984. He has been invited and continues to be invited to give talks, contribute papers and chapters in books, to participate in international conferences and so on and so forth. Moreover, he has helped initiate mathematics education as a research field in many different countries. For instance, this is the case of my university, University of Granada in Spain, where Brousseau coordinated a team of five researchers who came for four years to give courses and help supervise dissertations when only two people in the department held doctoral degrees, thus enabling us to start a doctoral programme. This was the first doctoral programme in Spain in mathematics education, the establishment of which would certainly not have been possible without his help. He has done the same in many other countries in Europe, Africa, Asia, Latin America where he has supervised more than 50 doctoral theses, many of these by doctoral students from different countries who, when returning to their home country, established research groups. In that way he has contributed to spread his ideas of mathematics education and research all over the world. Brousseau has taken part in many international committees and projects related to research, teaching and teacher training. This tremendous work has been reflected in an impressive number of publications in main journals. Throughout his scientific career his passion for and interest in mathematics education, combined with constant energy, untiring determination, great curiosity, extreme precision and his critical intellect, have led him to develop the most thorough and complete theory in the past 30 years. At the same time he was generous enough to spend his time and effort in the service of the national and international mathematics education community, in particular helping the training of young teachers and researchers.

It is to recognise all these different contributions to the advancement of our field that the Felix Klein medal for lifetime achievement in mathematics education has been awarded to Guy Brousseau. I am convinced that we are all happy that the first presentation of this prize has been given to a colleague, or better to a friend, who so well fits our model of the ideal mathematics educator from both a scientific and a human point of view. So now I invite professor Brousseau to come on stage to receive the Felix Klein Award.

Hyman Bass presents the Felix Klein Award to Guy Brousseau
accompanied by fanfare by Royal Danish Brass.

Michéle Artigue, Chair of the Awards Selections Committee
I am now very proud to announce that, as mentioned before, the first Hans Freudenthal medal has been awarded to Celia Hoyles, professor at the Institute of Education, University of London, United Kingdom.

Celia will be presented by Frederick Leung, who is also a member of the current executive committee of ICMI.

Frederick Leung, Member of the Executive Committee of ICMI

Ladies and gentlemen.

It gives me great pleasure to introduce Professor Celia Hoyles, the recipient of the first Hans Freudenthal Award of the International Commission on Mathematical Instruction. Celia Hoyles studied mathematics at the University of Manchester winning the Dalton Prize for the best first class degree in mathematics. She began her career as a secondary teacher and then became a lecturer at the Polytechnic of North London. After earning her Ph.D. she became professor of mathematics education at the Institute of Education, University of London, 1984. Her early research in the area of technology and mathematics education began by exploring the potential offered by LOGO and she soon became an international leader in this area. Later, in 1996, her book *Windows on Mathematical Meanings Learning Cultures and Computers*, co-authored with Richard Noss, has inspired major theoretical advances in the field and notions such as *webbing* and *situated abstraction* are now ideas that are well known to researchers irrespective of the specific technologies they are studying.

From the mid-nineties, Celia's research on technology integrated the new possibilities offered by information and communication technologies, as well as the new relationships children develop with technology. She has recently co-directed two projects funded by the European Union: the *Playground* project in which children from different countries designed, built and shared their own video games, and the *WebLabs* project, which aims at designing and evaluating virtual laboratories where children in different countries build and explore mathematical and scientific ideas collaboratively at a distance. As an international leader in the area of technology and mathematics education, she was recently appointed by the ICMI Executive Committee as co-chair of a new ICMI Study on this theme.

However, Celia Hoyles' contribution to research in mathematics education is considerably broader than this focus on technology. Since the mid-nineties, she has been involved in two further major areas of research. The first, a series of studies on children's understanding of proof, has pioneered some novel methodological strategies linking quantitative and qualitative approaches that include longitudinal analyses of development.

The second area has involved researching the mathematics used at work and she now co-directs a new project, *Techno-Mathematical Literacies in the Workplace*, which aims to develop this research by implementing and evaluating some theoretically-designed workplace training using a range of new media.

In recent years Celia Hoyles has become increasingly involved in working alongside mathematicians and teachers in policy-making. She was elected Chair of the *Joint Mathematical Council of the U.K.* in October 1999 and she is a member of the *Advisory Committee on Mathematics Education* (ACME) that speaks for the whole of the mathematics community to the Government on policy matters related to mathematics, from primary to higher education. In recognition of her contributions, Celia has recently been awarded the *Order of the British Empire* for "Services to Mathematics Education".

On a more personal note, I am fortunate enough to have some personal acquaintance with this great scholar. When I started my Ph.D. study in London, Celia was originally my supervisor. Her insightful advice had helped shape my Ph.D. study as well as the research agenda that I embarked on afterwards. Unfortunately, for some reasons she had to withdraw as my supervisor but she eventually became one of the examiners for my Ph.D. thesis. I can still remember, at the oral examination, the very sharp and critical questions she asked but also the very perceptive and constructive comments. The exam lasted two hours and she and the other examiners gave me a hard time. But eventually I passed my Ph.D. After my graduation from London we continued our contact through e-mails and occasional meetings in conferences. I am always struck by her dedication to scholarship and her amicable and caring character. In Celia there is no trace of arrogance that you sometimes find in some highly accomplished scholars.

Well, ladies and gentlemen, a personal friend, a former teacher of mine and a great scholar of the time, it is my great honour to present to you for the Freudenthal Award, Professor Celia Hoyles.

Hyman Bass presents the Hans Freudenthal Award to Celia Hoyles
Accompanied by fanfare by Royal Danish Brass.

Ingvill Stedøy

Congratulations to the two great prize winners! The next person to come to the podium is the Secretary General of ICMI, Bernard Hodgson who is a professor at the University of Laval, Québec in Canada.

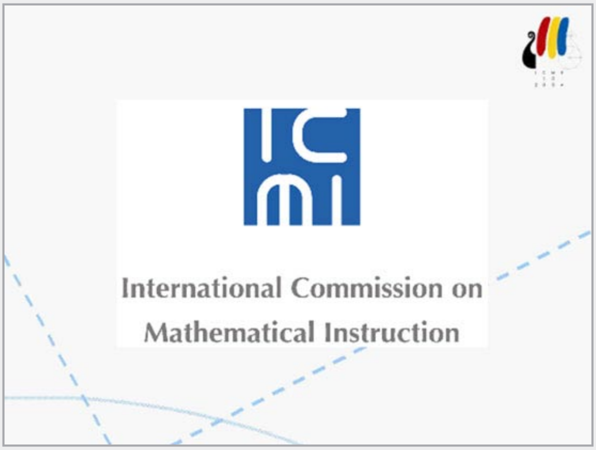
Bernard R. Hodgson, Secretary-General of ICMI, presentation of the ICMI Medals and Logo

Dear friends and colleagues.

The ICMI Awards in mathematics education research, as you have just witnessed, consist of a certificate and a medal, accompanied by a citation describing the contribution of each recipient, but unfortunately, I have to say, without a financial component. The establishment of these awards, announced four years ago at the Closing Session of ICME-9, induced a new challenge for the Executive Committee of ICMI: the design of the Felix Klein and of the Hans Freudenthal medals, serving as a tangible sign of recognition. This in turn reinforced a need often expressed in the past in other circumstances, namely the need for a visual identification of the International Commission on Mathematical Instruction in the form of a logo to appear on the reverse side of the medals. I would now like to present briefly the ICMI medals and the ICMI logo.

The ICMI medals were conceived and made by Thomas Soufflard, a current student of École Boulle in Paris, a renowned French school of art and design. Founded in 1886, École Boulle is named after a famous cabinet-maker of king Louis the 14th, André-Charles Boulle, after whom a well-known curved chest of drawers is named also. The medals were produced as a project in a course for students completing their degree at École Boulle. The technique used is that of modelled engraving, where a hollowed or relief motif is obtained by cast, strike, or ornamentation. The engraving was made by hand and the medals were stroke-pressed a few weeks ago at the Monnaie de Paris, using a special 600-ton press.

On one face of the medals are shown the past Presidents of ICMI whose names are attached to the awards. On the reverse side the logo of ICMI appears, surrounded by the name of the Commission, written in French and in the form of a circle, *Commission internationale de l'enseignement mathématique*. This is a testimony to the intensive use of French in the early years of ICMI, as is reflected, for instance, in the issues of that period of the journal *L'Enseignement Mathématique*, the official organ of ICMI since its inception.



The logo of ICMI, chosen a few months ago by the Executive Committee of the Commission, is also the result of students' work. It was designed by Anick Légaré and Priscilla Lavoie, two students from the Studio École of the School of visual arts of Université Laval, in Québec. More than 35 proposals of logos were received, in response to a call made already in 2000 at ICME-9 and repeated in the *ICMI Bulletin*. These proposals, representing a remarkable richness of visions and creativity, mainly originated from three groups located in Denmark, France and Canada. The concepts on which the various proposed logos rest are very different one from each other, and it was not an easy exercise for the members of the Executive Committee to come to a conclusion. Among the criteria used in the final decision were the simplicity and the efficiency of the design, as well as its flexibility.

The visual identification adopted for ICMI is described as follows by its designers: "The square is a simple geometrical object, one of the very first shapes met by a child. The square refers here to the world of education and its structure is intended to convey stability, solidity and support. This square has been opened up by other geometrical shapes representing the acronym of the Commission, I, C, M, I. These openings introduce rhythm and movement, and the network of lines they create evoke communication and transfer of information. The letters are built from simple shapes, straight lines and circles, and recall basic symbols used in mathematics while suggesting some kind of symmetry. The curved ends of the letters introduce suppleness as well as harmony to the whole. The acronym ICMI has been integrated into the logo so as to facilitate recognition and create a lasting image. The colour blue is traditionally associated with the world of education and suggests learning and knowledge. The colour white brings in some fresh-ness while the grey colour of the signature, more neutral, refers to communication and technology". The designers concluded that because of its simplicity, the logo will be easy to use in various settings.

Those wishing to know more about the ICMI medals or the ICMI logo are cordially invited to the ICMI Awards stand located in the registration area, where they will be hosted by a student of École Boulle who will show them some of the material used in the preparation of the medals and of the logo. I also invite those who wish to know more about the two past Presidents of ICMI, Felix Klein and Hans Freundenthal, after whom the ICMI Awards are named, to attend a lecture to be given tomorrow by Geoffrey Howson, past Secretary General of ICMI. The title of the lecture is remarkably clear, namely "Klein and Freudenthal". Thank you.

Ingvill Stedøy
Thank you to Bernard. And now we are approaching the end of this opening session, but before that Elin Emborg, the congress secretary, and Morten Blomhøj, both from Roskilde University Center will come with some house keeping remarks.

Elin Emborg, Congress Secretary,
Morten Blomhøj, Chair of the Local Organising Committee
Provided various comments concerning the practicalities of the congress, such as time table, programme changes and corrections, guidelines for power point presentations, use of computers, lunch arrangements, happy hours and social gatherings, including the new-comers programme, campus lay-out and so forth.

Musical postlude by the Royal Danish Brass:
Svend Asmussen: Oh, What a Day
Joseph Zawinul: Birdland

Ingvill Stedøy
Hereby the opening session has come to an end. I wish you all a most successful congress.

P 1: Mathematics, mathematicians, and mathematics education

Hyman Bass[1]
University of Michigan, Ann Arbor, USA

I am one of a growing number of research mathematicians who are substantially engaged with school mathematics education. Such outreach has a long and honorable tradition. In this lecture, I illustrate some of the ways that I think this can be helpful, and even essential, for the improvement of mathematics education.

Upon his retirement in 1990 as President of ICMI[2], Jean Pierre Kahane spoke perceptively of the intimate connection between mathematics and mathematics education, in the following terms:

- In no other living science is the part of presentation, of the transformation of disciplinary knowledge to knowledge as it is to be taught (*transformation didactique*) so important at a research level.
- In no other discipline, however, is the distance between the taught and the new so large.
- In no other science has teaching and learning such social importance.
- In no other science is there such an old tradition of scientists' commitment to educational questions.

It is this last point that frames my paper.

Let me begin with some background observations. While university teaching is a substantial part of the academic mathematician's professional life, recent years have seen many research mathematicians involved in school mathematics education as well. There has been much attention to the so called "math wars," an unfortunate term coined in the U.S. to describe the conflicts between mathematicians and educators over the content, goals, and pedagogy of the curriculum. Although these "wars" attracted a great deal of attention, the involvement of mathematicians has a much longer history in our profession. And most of that history is *not* primarily a history of conflict. In what follows, I will offer some snapshots from that history to provide a more robust picture of our tradition of concern for pre-college mathematics education. That tradition is both edifying and inspiring.

I choose specifically to focus on the involvement of *research mathematicians,* in part to dispel two common myths. First, it is a common *belief among mathematicians* that attention to education is a kind of pasturage for mathematicians in scientific decline. My examples include scholars of substantial stature in our profession, and in highly productive stages of their mathematical careers. Second, many *educators have questioned* the relevance of contributions made by research mathematicians, whose experience and knowledge is so remote from the concerns and realities of school mathematics educa-

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1 This paper is adapted from my plenary address at the International Congress on Mathematical Education in Copenhagen, 5 July, 2004. Variants on this were presented also as my retiring Presidential address to the American Mathematical Society in Phoenix, Arizona, January, 2004, and to the Center for Proficiency in the Teaching of Mathematics at the University of Michigan, December, 2004. I am greatly indebted to my colleague, Deborah Ball, for generous discussions and critical feedback in its construction.

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tion. I will argue that the knowledge, practices, and habits of mind, of research mathematicians are not only relevant to school mathematics education, but that this mathematical sensibility and perspective is essential for maintaining the mathematical balance and integrity of the educational process – in curriculum development, teacher education, assessment, etc.

Mathematics education is not mathematics. It is a domain of professional work that makes fundamental use of highly specialized kinds of mathematical knowledge, and in that sense it can, I suggest, be usefully viewed as a kind of applied mathematics. I will argue that, just as in other domains of "applied mathematics," the first task of the mathematician who wishes to contribute in this area is to understand sensitively the domain of application, the nature of its *mathematical problems*, and the forms of mathematical knowledge that are useful and usable in this domain.

The paper has three parts:

I. A brief look at the work of two major historical figures.
II. Some observations on the contemporary scene.
III. A sample immersion into some of the work in which I have personally been engaged.

I. A historical view (1870-1970): The tradition of involvement in mathematics education

For well over a century, a number of eminent research mathematicians have devoted substantial professional attention to mathematics education, even at the pre-college level. I have chosen two notable examples – Felix Klein and Hans Freudenthal – to illustrate the sorts of work that leading mathematicians have done in education. I chose them because of their stature in the field, the significance of what they did in mathematics education, and because their stories illuminate what mathematicians are able to contribute professionally. [3, 4]

Felix Klein

Felix Klein was born in Düsseldorf on 25 April, 1849. (He was fond of pointing out that his birthday ($5^2/2^2/43^2$) was formed of squares of prime numbers.)
He was the first president of ICMI, an international organization founded in Rome in 1908 by mathematicians in order to focus on educational issues of concern to mathematicians.

Klein's most famous mathematical legacy, inspired largely by conversations with Sophus Lie, is his Erlanger Programm, which re-conceptualizes geometries as the invariants of their symmetry groups. This was presented in a published paper on the occasion of his appointment, at age 23, to a professorship at Erlangen University.

But this was not, as is commonly believed, the subject of his inaugural lecture (*Antrittsrede*) there. He chose instead, as the theme of his lecture, "the pedagogical prin-

3 I have benefited here from the excellent account at ICME 10 by Geoffrey Howson (2004), of the careers of Klein and Freudenthal, and particularly of their relations of to ICMI.

4 Other mathematicians are similarly exemplary of this tradition – Henry Pollak or George Polya, for example – but I choose to focus here on Klein and Freudenthal for the interesting features that characterized their work in mathematics education.

ciples and goals for my future academic activity." In other words, he gave an inaugural lecture on mathematics education (Rowe, 1985).

In his inaugural lecture Klein emphasized the unity of knowledge, in particular emphasizing close ties between science and the humanities. In mathematics, he advocated focused attention on applied, as well as pure, mathematics, and on connections with the other sciences. While attending to rigor and logical skills, at the same time he strongly urged the cultivation of intuition and imagination. And, noteworthy for our theme here, he proclaimed the importance of giving serious attention to the mathematical preparation of school teachers.

Klein published over 30 articles and books dealing with educational matters. Notable among these is his book, "Elementary mathematics from an advanced standpoint." (Klein, 1924). This has been translated into several languages, and continues to be read with profit. His aim there is to provide for school teachers, and for their teachers as well, a robust mathematical perspective on the school mathematics curriculum. At the same time, he does not pretend that this fully prepares students for mathematics teaching, a task toward which he shows the greatest respect, sensitivity and even humility. For example, he writes (pp 7-8):

> "What high regard one must have for the performance of elementary school teachers. Imagine what methodological training is necessary to indoctrinate over and over again a hundred thousand … unprepared children with principles of arithmetic! Try it with your university training; you will not have great success!"

Mathematicians who have not turned serious attention to mathematics education often fail to appreciate the cognitive and epistemological subtleties of elementary mathematics instruction. Here is a sample passage that evokes Klein's sensitivity to these matters.

> "Let us realize once and emphatically how extraordinarily difficult in principle is the step, which is taken in school, when negative numbers are introduced.... Here, for the first time, we meet the transition from concrete to formal mathematics. The complete mastery of this transition requires a high order of ability in abstraction."

What can we say about Klein's contributions to mathematics education? Klein embodied abundant qualities rarely seen in such harmonious combination in a single individual. He was a mathematician of astounding precocity and cultural breadth, with a lofty and unified view of the whole of mathematics of his day. He respected rigor, but favored intuition and imagination, and the meaning that mathematics takes from the sciences and the experiential world. He shed the light of disciplinary mathematics on school mathematics in ways that were remarkably sensitive to young learners, and compassionate toward the challenges faced by their teachers. He was himself a gifted teacher, to mathematicians, and to future school teachers, whom he treated as professional partners, and whose calling he honored.

Hans Freudenthal

Freudenthal had remarkably broad mathematical and cultural interests. In mathematics he worked in topology, Lie groups, logic, and probability and statistics (for which he wrote a text book). He also wrote a book on popularization of mathematics, and he worked and published extensively in mathematics education. He was a gifted linguist, and even developed a proposed language for extraterrestrial communication. Howson (2004) reports that, during a heated argument with Dieudonné, Freudenthal protested – "Don't shout at me; for I can shout louder than you – and in more languages." Freudenthal was charming, mischievous, argumentative, autocratic, and an activist, who accomplished many things.

Born and educated in Germany, Freudenthal took his first position with L. E. J. Brouwer in Amsterdam, just ahead of Hitler's rise to power. But the German invasion of Holland in 1940 forced Freudenthal, a Jew, into hiding for the duration of the war. During this time he wrote novels, one of which won a competition that he entered using the name of a non-Jewish friend!

Freudenthal viewed mathematics not primarily as a body of knowledge, but as a human activity, and he urged that mathematics education should do likewise. It should, he argued, be based in reality, around phenomena that "beg to be organized" – a process he called "mathematization," a form of mathematical modeling of real problems, or of organizing and synthesizing mathematical ideas. He opposed deductive approaches, and favored instead development from the concrete to the general.

He was highly critical of most educational reform (New Math, or *Mathématiques Modernes*), and of the educational research of his day (both statistical and psychological). He believed in "mathematics for all," and favored small heterogeneous classes (no tracking/streaming).

Among Freudenthal's enduring cultural/institutional legacies in mathematics education are:

- Launching of the ICME's[5] with the inaugural congress held in Lyon in1969. This year's meeting in Copenhagen was the tenth such Congress.
- Founding of the international journal, Educational Studies in Mathematics
- Founding of what has come to be called the Freudenthal Institute, at Utrecht University, which has had a pervasive influence on mathematics education in Holland, and more broadly.

How can we characterize Freudenthal's contributions to mathematics education?

While Klein was a mathematical ambassador to mathematics education, Freudenthal became a full fledged, and even very prominent, citizen of the field. He became a knowledgeable and intellectually disciplined critic of the prevailing educational theories of his day. He brought strongly held beliefs and principles of his own, conveyed in his prolific writings. He was also a man of action. He enacted instructional experiments out of which he developed curricular and pedagogical ideas. And he built enduring institutions that continue to carry his legacy forward.

Klein and Freudenthal: Setting a standard

Although their engagements in the domain were quite different, Felix Klein and Hans Freudenthal exemplify the long history of mathematicians' interest in pre-college mathematics education. Each brought his aesthetic dispositions about mathematics to his view of the desirable nature of young learners' encounters with mathematics. Each considered in fine grain the special issues important to the mathematical integrity of the school

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curriculum. Still, the work that each contributed was distinctive. Klein examined and re-wrote a vision of the school level curriculum, offering a view of that curriculum that situated it in the larger mathematical landscape. He also sought to model both that mathematicians had important contributions to make, and that humility in those contributions was essential. The creation of the International Commission on Mathematics Instruction (ICMI) set the foundation for institutionalizing connections between the mathematics and education community.

Freudenthal also worked explicitly to build new structures for interactions between mathematicians and mathematics educators; in particular the institution of the international Congresses provided a regular context for exchange, across national and disciplinary boundaries. But, unlike Klein, Freudenthal also engaged directly in improving students' mathematics learning opportunities. He developed substantial ideas about how young people should engage in mathematization and how this could support the development of mathematical skill and knowledge. His ideas provided the foundation for significant research and curriculum development work that continues to this day.

Klein and Freudenthal, each in his own way, exemplified how articulation of mathematical sensibility and perspective could influence the mathematics education of young people. And each helped to establish the legitimacy and possible nature of mathematicians' involvement in mathematics education. It is this tradition that I seek to highlight in this lecture.

II. The contemporary scene (1970-2004): Current involvement of mathematicians in mathematics education

In this brief survey of the contemporary scene, I shall concentrate mainly on the situation in the United States, with which I am most familiar. Variants of this scenario seem to have transpired in many other countries. First, I offer some background.

The "New Math" reforms in the US (paralleled by the *Mathématiques Modernes* or *Modern Mathematics* in Europe) can be seen to have emerged from the convergence of several trends (See, for example, Dow, 1991):

1. Cold War competition, and the growing public appreciation of the importance of mathematics, science and technology for national security;
2. The triumphs in mathematics of axiomatic methods, enshrined for example in the writings of Bourbaki;
3. The recognition that the school curriculum gave no hint of these spectacular scientific developments; and
4. The generally impoverished quality of school mathematics instruction.

Largely guided by mathematicians' views of the subject matter, new curricula, prominently featuring axiomatic treatments of basic mathematical structures, were developed for the schools, and teachers were quickly (and inadequately) schooled in this "New Mathematics," with the presumption that this knowledge equipped them to teach these novel ideas and perspectives to young children (cf. Sarason, 1996).

In my view, the focus within the New Math on *mathematical structure* remains an appropriate theme for school mathematics, and its loss has weakened the curriculum. New Math's critical failure was to naively implement this structural approach via abrupt axiomatic formalism, rather than through a process of organic generalization from intuitive beginnings. This was fatal to its vision.

The shortcomings of the New Math precipitated in the U. S. a "Back to Basics" reactionary movement in the 1970's, one that left mathematics education in a somewhat rudimentary state, and whose outcomes came to again raise national alarm in the 1980's, signaled by the publication of "A Nation at Risk" (1983) from the U. S. Department of Education.

This time the response came not from mathematicians, but from educators, notably the National Council of Teachers of Mathematics, the professional organization of mathematics teachers. This led to the promulgation, for the first time in U. S. history, of new national standards for mathematics education (1989, 1991, 1995).[6] This was followed by the federally funded development of rather adventurous standards-based curricula, in whose construction research mathematicians had little voice. When these new curricula first entered classrooms and came to be known more concretely, some mathematicians (notably those with school age children) helped lead public protests whose effects we are still reconciling. They have expressed concerns over curriculum, standards, assessment, teacher preparation and professional development, and pedagogical practices.

In this environment of criticism and conflict, a number of eminent mathematicians have since rolled up their sleeves, taken their critiques to the ground level, and begun digging into important problems of school mathematics education. As mathematicians have by now gained stronger voice in educational policy environments, they have also begun to gain better capacity to hear, and listen, as they work on the ground with practicing teachers, and in more disciplined discourse with educational researchers.

Over time, the discourse has progressed from lengthy debate about standards, curriculum, and assessment to now a more deliberative attention to teachers and teaching, which is seen to be the critical, and most challenging, domain for potential improvement in mathematics education. Mathematicians are particularly concerned with teachers' knowledge and understanding of mathematics. Recognition and support of the role and responsibility of mathematicians in this area was highlighted by the publication by the CBMS (an umbrella organization for the U. S. professional societies in mathematics) of *The Mathematical Education of Teachers* (2001), a report that has stimulated widespread and continuing efforts to strengthen the mathematical preparation of teachers.

Mathematics departments are giving more concentrated and higher quality attention to the mathematics courses they offer to pre-service teachers, and this has become a lively area of professional inquiry and growth. Individual mathematicians have engaged in various forms of professional effort, such as participation in professional development programs for practising teachers, development of curricular materials for teacher education courses, and collaboration with education professionals in the design and implementation of instructional programs.[7]

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6 In fact, the NCTM Standards, having breached the sacred US political tradition of local control of education, precipitated a "standards movement" in the US, one that extended beyond mathematics across other disciplines, and that opened this arena to a somewhat politicized array of competing authors and critics of standards documents.

7 Among the many U. S. research mathematicians who have been active in these ways, I mention, by way of illustration: Dick Askey (Wisconsin), Sybilla Beckmann (Georgia), Herb Clemens (Ohio State), Roger Howe (Yale), Jim Lewis (Nebraska), Bill McCallum (Arizona), Jim Milgram (Stanford), Tom Parker (Michigan State), Robin Pemantle (Pennsylvania), Judy Roitman (Kansas), Paul Sally (Chicago), and Hung-Hsi Wu (Berkeley).

III Lessons learned by 2004:
Then need for immersion in the problems of school mathematics education

One observation, based on my brief narrative, is that the tradition exemplified by Felix Klein and Hans Freudenthal continues to thrive. The mathematicians to whom I alluded have devoted significant professional expertise and time to serious problems of school mathematics education.

But a second observation is that those contributions, the products of significant research mathematicians, are possible because they have developed a deep knowledge of the problems of pre-college mathematics. They have invested effort to learn about phenomena and environments, people and problems, very far from the everyday world mathematical research.

That knowledge has, in turn, allowed them to see ways in which their mathematical knowledge and sensibilities offered resources for these problems, ways that have not always been those they would have first assumed or expected. A mathematical colleague, after his recent immersion in an eight-day intensive summer institute for mathematics teacher educators, testified, "I found myself thoroughly immersed in serious mathematical conversations about division of fractions, a fact that I still find rather amazing."

To illustrate the intensive mathematical engagement possible in problems of education, I turn finally to an example from my own experience. After a period of serving on education committees and in policy environments, I became more seriously engaged in mathematics education as an intellectual and professional endeavor. Over the past decade, I have been working with Deborah Ball and her research groups at the University of Michigan. A central question of our work is to better understand the mathematical knowledge and resources that elementary teachers need to do the work of teaching mathematics, work that must simultaneously respect the integrity of the mathematical ideas and also attend closely to the mathematical development of their students. What do teachers need to know, in what ways, and for what purposes? And how can they gain such knowledge? Unlike earlier approaches, ours has treated these as empirical questions, starting not with the school curriculum and the mathematical topics covered, but rather with the practice of teaching itself. We are doing a kind of "job analysis," to understand what kinds of *mathematical problems* teachers have to solve in the course of their daily work, and what kinds of mathematical resources they deploy in solving those problems (Ball & Bass, 2003).

This has taken me into a closer study of the challenges and possibilities of elementary mathematics instruction than I first imagined. One point I want to emphasize is that my perspective and sensibility, *as a research mathematician*, gave me a lens in this kind of observation of instructional practice that made visible important things that would be missed by others with different training and expertise, just as their lenses have expanded my own vision.

This work has provided an emerging theory of what we have named, *mathematical knowledge for teaching* (MKT), together with the development of instruments to measure such knowledge and its growth (Ball, Hill, & Bass, 2005; Hill, Schilling, & Ball, 2003). I shall say more about this below. But first, in order to give you a more vivid sense of this kind of work, I invite you now to examine an episode from a real elementary mathematics lesson. What does it mean (and feel like) to *look analytically at real teaching practice with a mathematical eye?*[8]

Using a mathematical perspective to study teaching[9]
We now visit a class of 19 third graders (8 year-olds).[10] The class is culturally and linguistically diverse (many speaking English as a second language, and some only recently arrived in the U.S.). This is not meant to illustrate exemplary instruction or a particular pedagogical style, but rather to provide an example that makes vivid and visible some of the complex *mathematical work of teaching*. The philosophical orientation of the instruction was: (1) to work on *substantial mathematics* and treat the mathematics with integrity; (2) to take *students' thinking* seriously, and make it an integral part of the instruction; and (3) to treat the construction of mathematical knowledge as the work of an *intellectual collective*, with mathematical justification and critical evaluation of solutions and claims being a central demand of the student work.

The children were working on even and odd numbers. They came to third grade "knowing" which (small numbers) were even and which were odd, but without any formal definition of these notions. In this class, the topic was introduced through investigation of problems such as this one:

Mick has 30 cents in his pocket and he wants to spend it all (on gum and pretzels), and not have any change left in his pocket. What can he buy for 30 cents? What are his choices?

The solution to this problem pushed the children into encounters with notions of even and odd numbers, and eventually to make conjectures about their arithmetic properties (e.g., even + odd = odd, odd + odd = even, etc.).

On one particular day, the students were preparing to work on these conjectures, seeking to determine whether they were true for all numbers. Near the beginning of the lesson, one of the boys, Sean, reflecting on a discussion they had had the previous day, raises his hand and says,

> "I was just thinking about six, that it's a... I'm just thinking it can be an *odd* number, too, 'cause there could be two, four, six, and two, three twos, that'd make six... And two *threes*, that it could be an odd *and* an even number. Both! *Three* things to make it and there could be *two* things to make it. "

Hearing this, it is difficult to resist forming quick opinions about Sean's thinking or asserting what the teacher should do ("Why doesn't the teacher just set the students straight?") But, before even considering such judgments, we should address the more difficult question, "What is significant mathematically about what is actually going on

8 In my lecture, I showed the seven-minute video of the lesson; here, instead, we shall have to make do with a narrative based on its transcript.

9 These data were collected under a 1989 National Science Foundation grant to Ball and Magdalene Lampert, then at Michigan State University.

10 These data were collected under a 1989 National Science Foundation grant to Ball and Magdalene Lampert, then at Michigan State University.

in this episode? What can we see? What might be helpful for a teacher to see, and be sensitive to, mathematically?"

The teacher does not immediately challenge or correct Sean. She re-voices and tries to publicly clarify what he is saying, at which point she invites comments from the class. His classmates quickly express disagreement. Everyone already knew from second grade that *six is even*. We watch as this mathematical debate unfolds, attending to how the children are processing mathematical ideas and claims, and to the mathematical moves of the teacher to shepherd this discussion.

Cassandra, the first to object, points to the number line above the blackboard, saying,

> "Six can't be an odd number because this is *(she points to the number line, starting with zero)* even, odd, even, odd, even, odd, even,….. Because zero's not a odd *number*."

Sean persists,

> "… because there can be three of something to make six, and three of something is like *odd*…"

Then Kevin protests, "That doesn't necessarily mean that six is *odd*." Several students chime in, "Yeah." When the teacher asks Kevin "Why not?" he responds,

> "Just because two odd numbers add up to an even number doesn't mean it has to be odd."

At this point the teacher, thinking that Sean may be just confused about the meaning of "even," makes an important mathematical move, asking,

> "Sean. What's our working definition of an even number? Do you remember from the other day the working definition we're using?"

When Sean can't remember, she asks several other students, until Jillian offers,

> "It is um, if you have a number that you can split up evenly without having to make ______ to split one in half, then um, it's an even number."

When the teacher then asks Sean if he can do that with six, he agrees, so she says,

> "So then it would fit our working definition, then it would be even, okay?"

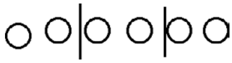
To which Sean comfortably responds,

> "And it could be odd. Three twos could make it."

Sean, contrary to the tacit understanding to the class, seems to allow that a number can be both even and odd. The teacher then realizes that, to mediate this discussion requires a definition of odd numbers as well as one for evens, something she had not before thought necessary. After some discussion, the class agreed that odd numbers were those you could not split up fairly into two groups. But Sean is tenacious, saying that you could split six fairly (two threes) and not fairly (three twos).

To clarify Sean's thinking, the teacher pursues a new line of questioning, and asks Sean if he thinks all numbers are odd then. When he says no, she asks him which numbers are not odd. He says that 2, 4 and 8 are not odd, but that 6 can be odd or even. Several students shout, "No!" And Tembe challenges him: "Show us." Sean only repeats, "There are three twos; one, two; three, four; and five, six." Unconvinced, Cassandra and

Tembe insist, "*Prove* it to us that it can be odd." The teacher then invites Sean to prove it to the class, and asks everyone to pay close attention. Sean goes to the board, where there is a drawing of six circles, which he then proceeds to separate into groups of two,



saying, "There's two, two, and two. And that would make six." To which Cassandra rejoins, "I know, which is *even*." And Tembe backs her up.

Then Mei raises her hand to say, "I think I know what he is saying." The teacher asks Sean to remain at the board while Mei explains,

> "… I think what he's saying is that you have three groups of two. And three is a odd number so six can be an odd number *and* a even number."

Notice here that the question is no longer whether Sean is right or wrong, but whether Mei has correctly interpreted Sean's idea and argument. The teacher first gets Sean's confirmation of this, and then she asks if others agree with Sean. After having clearly articulated Sean's argument, Mei herself then says,

> "I disagree with that because it's not according to like... here, can I show it on the board? "

At the board, facing Sean, Mei continues,

> "It's not according to like...... how many groups it is. Let's say that I have *(long pause while she thinks)* Let's see. If you call six an odd number, why don't *(pause)* let's see *(pause)* let's see -- *ten*. One, two,... *(draws circles on board)* and here are ten circles. And then you would split them, let's say I wanted to split, spit them, split them by twos... One, two, three, four, five,... *(she draws the dividing lines and counts the groups of two)*

Then why do you not call *ten* a, like -... a, an odd number and an even number, or why don't you call *other* numbers an odd number and an even number?"

What is Mei doing here? First she has understood and given a clear public expression of Sean's idea, one with which she in fact disagrees, and she has pinpointed the fault in Sean's argument. ("It's not according to how many groups.") But she goes well beyond the mere statement of that critique. She cleverly constructs an argument that she is persuaded will make Sean, in his own terms, see the error of his ways. She *generalizes* the principle of Sean's reasoning – that six is made of an odd number of groups of two – and so sees that this same criterion would usher in an unlimited supply of new odd-and-even numbers, to her a menacingly uncertain predicament that she fully expected Sean to back away from. Her reflective pauses were needed to search mentally, while the class waited quietly, for the next example – 10 – of an odd number of groups of two.

To Mei's surprise, and then dismay, Sean responds,

> "I disagree with myself… I didn't think of it that way. Thank you for bringing it up, *so* – I say it's – ten can be an odd and an even."

In this ironic exchange, Mei, intending to shock Sean with the extravagant implications of his reasoning, in fact succeeds instead in giving Sean an expanded understanding and appreciation of his own idea, which he embraces with thanks. Mei's argument is mathematically astute, well expressed, and well understood by Sean (and the class, as we later see). Mei and Sean differ in the significance that they each attach to it. Exasperated, Mei then proclaims,

> "Yeah, but what about *other* numbers?! Like, if you keep on going on like that and you say that other numbers are odd and even, maybe we'll end it up with *all* numbers are odd and even. Then it won't make sense that all numbers should be odd and even, because if all numbers were odd *and* even, we wouldn't be even having this *discussion!*"

Noteworthy here is Mei's mathematical sensibility about definitions, that they fail in their purpose if they lose the capacity to make significant distinctions, to give concepts appropriately sharp boundaries.

In these few moments of mathematics instruction, what can we observe about the mathematics going on? First of all, what mathematics are the children doing and learning? On one level they are exploring aspects of even and odd numbers. But, perhaps more significantly, they are engaged in substantial *mathematical discourse and reasoning*. The children are making mathematical claims and counterclaims, and critically examining each other's ideas. There is an imperative for justification of claims that the children take to heart, and to which they hold each other accountable. Such mathematical practices, as much as we rhetorically advocate them, are not learned if they are not taught, and practiced. That entails an instructional investment that we can see manifested in this episode.

To reconcile mathematical disagreement, the teacher recognizes the need for *definitions* of the mathematical terms in play. She asks the class to make explicit the "working definition." In fact three definitions of even (and odd) numbers are implicitly in use: *fair share* (a number is even it can be split into two equal groups), *pair* (a number is even if it is composed of groups of two), and *alternating* (the even and odd numbers alternate on the number line, with zero being even). These are not all explicitly stated or shown to be mathematically equivalent, but they are tacitly assumed to be so. Some students (not Sean) assume the "even" implies "not odd." Noticing these different definitions in the children's reasoning, realizing the need to reconciling them, and considering what is entailed in establishing their equivalence are all crucial for teachers to know. It is also important for teachers to know what are mathematically appropriate and usable definitions of even and odd numbers for third graders. Mathematical reasoning is not feasible without some careful attention to commonly understood mathematical definitions. For example, proving the conjectures (e.g., odd + odd = even) depends on the use of definitions.

Though Sean misuses the mathematical terms "even" and "odd," he nonetheless has a clear *mathematical idea* about six: he notices that it has 'an odd way of being even.' But, lacking vocabulary to name this feature, he misguidedly appropriates the name "odd-and-even" for it. Sean is thinking only about six. But Mei recognizes that Sean's argument about six is generalizable, and opens the door to far reaching possibilities that she assumes would cause Sean to retreat from his claim: No such luck.

A side comment to indicate some of the mathematics hovering around this lesson: Mathematical ideas typically admit multiple generalizations. Mei generalized Sean's idea to numbers that are an odd number of groups of two (twice an odd number). Another plausible candidate would be numbers that are an odd number of groups of any size, in other words, numbers with an odd factor (>1). In other words, numbers which are not powers of two. Would Sean have equally well embraced this generalization? Had the teacher, or a student, wanted to probe this possibility, an interesting mathematical question is then what number to ask Sean about. In this case, the first test case would be, not ten, but twelve (= 2 x 6 = 3 x 4).

What are these "Sean numbers," (as the teacher came later to call them) introduced by Mei? Odd multiples of two. Is this a topic worthy of instructional time? Even and odd are about mod 2 arithmetic. Sean has cracked the door open on mod 4 arithmetic, identifying numbers congruent to 2 mod 4. These turn out also to be exactly those natural numbers that are not a difference of two squares. So, the idea surfaced by Sean's natural curiosity about numbers in fact has some interesting mathematical significance, that he could not have anticipated, but that might figure in the teacher's evaluation of how much instructional play to give it. Indeed, once Mei had essentially defined these Sean numbers, the students eventually began an exploration of their properties – finding patterns (every fourth number, starting with two, is one); making and proving conjectures (a sum of Sean numbers is not one); etc.

But, more importantly, what the children are learning is, beyond the properties of Sean numbers, the skills of mathematical exploration and reasoning, generalization, use of mathematical definitions, etc. For people who wonder in frustration over our students' failure to gain proficiency with or appreciation of mathematical reasoning, you might consider that this provides one image of what it might look like for young children to begin to develop such skills.

Mathematical Knowledge for Teaching (MKT)

Let me turn now briefly to the area in which I have focused, working closely with Deborah Ball and others in our University of Michigan research group. I begin with a bit of context.

Many teachers have not had good mathematical preparation, and lack adequate mathematical knowledge for teaching. This is an enormous problem in the U. S.; and from what I have learned from reading and from my international colleagues, teachers' mathematical preparation is a problem in many other countries as well. I focus here on those who teach at the primary level, but strong evidence suggests that there are similar problems as well at the secondary level.

This is an important problem – for practice, policy, and theory – and many solutions are offered. Most solutions consist of increasing the requirements for teachers. But what these requirements should consist of is too often taken for granted, and left unspecified.

Our method is to turn this problem "upside down" and begin, not with the school curriculum, and the related disciplinary mathematics, but rather with teaching practice itself. The basic question is thus transformed into, "What is the *mathematical work of teaching*?" To answer this question, we, naturally enough, study actual teaching, including the work that teachers do inside and outside of classrooms to teach mathematics.

Our examination above of an episode of teaching gives one glimpse of what is entailed in such study of practice. From such observations we analyze the mathematical demands of that work, demands that often go unnoticed, and that we are learning are quite substantial. This in turn informs our evolving, practice-based answer to the question, "What is *mathematical knowledge for teaching (MKT)*?" Understanding this is important to improving teachers' mathematical preparation.

We use the term "mathematical knowledge for teaching" to represent the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching. And by the "work of teaching" we mean the daily tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom. For example: planning lessons, designing and modifying tasks, communicating with parents about their children's work and progress, introducing concepts, writing and assessing tests, etc. These comprise the specialized tasks in which teachers need to know *and use* mathematics in a variety of ways.

An important strand of our work, led by Heather Hill and Deborah Ball, is the development of *measures* of mathematical knowledge for teaching (Hill, Schilling, & Ball, 2004; Hill, Rowan, and Ball, 2005). In the course of this we have found it natural and useful to distinguish four different categories of MKT: (1) *Common mathematical knowledge* (expected to be known by any well educated adult (mathematical knowledge widely required for practices other than teaching); (2) *Specialized mathematical knowledge* (strictly mathematical knowledge that is particular to the work of teaching, yet not required, or known, in other mathematically intensive professions (including mathematical research)); (3) *Knowledge of mathematics and students*; and (4) *Knowledge of mathematics and teaching*. For example, in connection with multi-digit multiplication, MKT includes things like: (1) Knowing how to calculate; (2) Knowing how to analyze both correct (non-standard) and incorrect solutions; (3) Identifying the student thinking that might have produced an incorrect answer, or knowing likely student errors; and (4) Knowing what kinds of materials or representations would be best suited to explaining why and how some standard algorithm works.

A noteworthy finding is the identification and characterization of *specialized mathematical knowledge* for teaching (category (2)). It is to be expected that teachers need some knowledge – about students (3), and about pedagogy (4) – that other professionals would not need, or be expected to know. But specialized knowledge of mathematics is *strictly mathematical knowledge* (not about students or about pedagogy) that proficient teachers need and use, yet is not known by many other mathematically trained professionals, for example, research mathematicians. Thus, contrary to popular belief, *the purely mathematical part of MKT is* not a *diminutive* subset *of what mathematicians know*. It is something distinct, and, without dedicated attention, it is not something likely to be part of the instruction in content courses for teachers situated in mathematics departments.

To summarize some of what are we learning:

- A practice-based approach to asking about mathematical knowledge for teaching reveals that there is much mathematics *deep inside* the school curriculum as well as *beyond it*.
- Knowledge needed for teaching is different from that needed for other occupations or professions where mathematics is used.

- Knowledge needed for teaching must be usable for the specialized mathematical problem solving and reasoning that teachers have to do.

Conclusion

Let me conclude here by summing up my argument about productive interactions among mathematics, mathematicians, and mathematics education.

- The mathematics profession has a long and honorable tradition of involvement in mathematics education.
- Eminent mathematicians from around the world, and throughout history, have exemplified this tradition.
- Important contemporary mathematicians are continuing, and expanding, this tradition.
- This work can be productively pursued in the spirit of "applied mathematics," by first deeply understanding the domain of application.
- As practitioners of the discipline, research mathematicians can bring valuable mathematical knowledge, perspectives, and resources to the work of mathematics education.
- This is a tradition worthy of continued development and support.

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P 2: Mathematics education for whom and why? The balance between mathematics education for all and for high level mathematics performance

Plenary Panel Debate

Moderator: *Stephen Lerman*, London South Bank University, United Kingdom[1]
Panellists: *Richard Askey*, University of Wisconsin, USA
Susana Carreira, University of Algarve, Portugal
Yukihiko Namikawa, Nagoya University, Japan
Renuka Vithal, University of Kwazulu-Natal, South Africa

Education communities around the world, and mathematics education communities in particular, are facing strong pressures for change of one sort or another at a time when, in many countries, fewer people are studying mathematics at universities and, year on year, fewer people are coming into mathematics teaching. How can we balance the needs of all for mathematical knowledge, what has been called a critical mathematical literacy, with the specialist knowledge needed by those who could study mathematics beyond school? How can we make mathematics challenging and exciting for all given the range of needs and interests across the population? Can limited resources be stretched in so many directions? Should we make mathematics optional beyond a basic stage? What should the mathematics curriculum, or curricula if we differentiate, look like? How can we challenge all students in different ways? Who decides which students should receive what kind of mathematical education? These were some of the issues that were addressed at the Plenary Panel Interviews.

Richard Askey: Mathematical content in the context of this panel

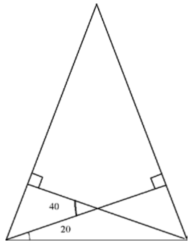
In dealing with the question of mathematics for all versus mathematics for high achievers, there is a more basic problem: getting mathematics right. To see if this is being done for high achievers, and not just the very top who do well in something like a national or international mathematics Olympiad, we can look at the results of the TIMSS study of students who are taking advanced mathematics in their last year of high school. The results are not good. Nor are they good for all students.

Consider two problems. The first is a simple three-step problem in geometry. A triangle is given with some data and students are asked to show it is an isosceles triangle (see below).

The angles which look like right angles are right angles.

The results were 35% correct internationally and 10% correct in the U.S. For the U.S. this was a sample from 14% of those in school in the last year of high school, so a very small fraction of the age cohort could do this problem. The problem is one which a vast majority of all students should be able to do. It is a much easier problem than ones included in eighth grade books in Singapore. The fact that such a small percent of the top U.S. students could do this problem illustrates some of the problems in mathematics education in the United States.

1 For personal reasons, Stephen Lerman was unable to attend the Congress. In his place, Abraham Arcavi most kindly accepted to moderate the session based on Stephen Lerman's notes.



The second problem is:
Which one of the following conics is represented by the equation (x-3y) (x+3y) = 36?

	U.S.	International averages
Circle	24.5	27.9
Ellipse	23.8	25.4
Parabola	25.6	12.8
Hyperbola	25.1	29.5

These numbers do not add to 100% because there were other options such as not answering the question.

These figures are shocking, and for a circle some countries had a score of over 40%. The most reasonable way to interpret the U.S. result is that the answers were random guesses, not even intelligent ones.

All students should know how to find the distance between two points in the plane, know that the distance formula comes from the Pythagorean theorem, and most should be able to prove this theorem. Only a small fraction of the calculus students I have taught recently have been able to prove the Pythagorean theorem, and some tried to prove it using the distance formula. In the original Standards published (NCTM 2000) by the National Council of Teachers of Mathematics (NCTM), there was a call for a decreased emphasis on conic sections. Since the only thing done on conics was a geometric definition and derivation of an equation, the only possible decreased emphasis was elimination, or just using the words as is done in many U.S. textbooks when $y = ax^2$ is called a parabola but no geometric description is given for a parabola. However, I doubt that NCTM wanted elimination of an equation of a circle as something students should know. The quoted results suggest that few U.S. students know enough about equations of circles for this to have influenced the answers to the question above.

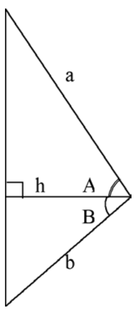
There were harder geometry questions in this TIMSS test and the results were poorer. How did we get to this point? Part of the answer probably goes back to the New Math period. To illustrate this consider the talk Dieudonne gave in 1959 at a seminar at Royaumont. Most of his talk dealt with the geometry students who will attend college should learn in high school, and a little on the geometry they should have learned before high school. Before age 15 he recommended experimental geometry. From age 15 on his geometry would be based on axioms, but not those used by Euclid or modified by Hilbert and others in the late 19th century. His axioms are those of a two-dimensional vector space with a scalar product. His own summary in one sentence was: "And if the whole program I have in mind had to be summarized in one slogan it would be: Euclid must go!"

What Dieudonne proposed as a substitute was more modern geometry, vectors and transformations. About more traditional topics, he claimed that triangles were an artificial notion with practically no applications outside of astronomy and geodesy. There was a call to bring algebra and geometry together. One result in the U.S. is programs which claim two lines are perpendicular when the product of their slopes is -1 without any geometric basis for this. It was talks like this one of Dieudonne which helped set the stage for the demise of good multistep problems in geometry, which had been one place in the curriculum where students had to think and do multistep problems.

Not everything Dieudonne proposed was recommended in the summary. However, such things as developing algebra informally first and then getting to an axiomatic version of the properties of positive and negative numbers or fractions as a culminating topic was recommended. It is an indication of the temper of the times that such things were suggested. In this regard, we are likely a bit more sensible now. Something else which was suggested in the summary was elimination of quadratic functions with parameters as coefficients. That has happened, and it has hurt. Now all except the very best students are mystified when asked to find the points of intersection of the circle $x^2 + y^2 = 1$ and the line $y = t\,(x+1)$ even after, with suggestions, they get to $x^2 + t^2(x+1)^2 = 1$. This example is mentioned because it deals with important topics in mathematics, Pythagorean triples and stereographic projection being two. What this does is give a rational parameterization of a circle.

One wants ideas to develop as students mature. Here is an example of how this can be done. In late primary school or early secondary school a formula for the area of a triangle is usually derived. This is done by starting with a right triangle and drawing a rectangle with the triangle as half of it. Then the general case is reduced to the special case of a right triangle, sometimes omitting the case when there is an obtuse angle on the base. Another result introduced about the same time is the sum of the angles of a triangle. This is usually done by having students tear off the corners and fitting them together to see that a straight angle seems to arise. This is a good starting exercise, but there is often a three or four year gap before the conclusion is shown to be correct. That is not necessary since the same argument which gave the area of a triangle works for the angle sum. Heath mentioned this in his comments on Proposition 32 in Book 1 of his translation of Euclid's Elements and cited earlier comments on this proof. This should be done within a year of deriving the formula for the area of a triangle by this method. The idea that the same set of pictures can be used for more than one important result is important.

Later, in high school, the same idea will be used to derive the law of sines and the law of cosines. It can also be used to derive the addition formula for sin(A+B) using the formula for the area of a triangle in trigonometric form.



Twice the area is ab sin(A+B) = ah sin(A) + bh sin(B) = ab[sin(A) cos(B) + cos(A) sin(B)].

This proof only works when A and B are acute angles, but the general case can be obtained from this special case either by adding angles of 90 degrees or by an argument which uses the rational parameterization of the circle mentioned earlier. Another way of extending this formula is to give the usual proof using the rotation of the unit circle. The addition formulas are vital so giving more than one derivation is a good idea. I learned this proof from the trigonometry book by Gelfand and Saul and was so struck by it that I looked for earlier instances of it. Tom Apostol has included it in one of the Cal Tech videos. It is essentially in Dickson's "Plane Geometry" published in 1922 with the law of sines used rather than the area. It is also in Davies' "Elements of geometry and trigonometry from the works of A.M. Legendre" published in 1857.

To give an illustration of the distortion of mathematics, consider the following from "Mathematics Teacher", a magazine for high school teachers.

"The trigonometry teacher can use the graphing calculator in teaching identities. These equations can be used:

$y_1 = \sin(2x)$
$y_2 = 2\sin(x)$
$y_3 = 2\sin(x)\cos(x)$

Have students graph y_1 and y_2. Two appear! Next have students graph y_1 and y_3. The students are now learning identities not by the rote method of pencil and paper but by experiencing and seeing an identity."

I suggested to two officers of NCTM, the organization that publishes "Mathematics Teacher", that one or both of them write an article saying this is not what their *Standards* (NCTM 2000) calls for in the way of reasoning. When nothing was done in a year and a half, I wrote a short note showing how to prove the double angle formula by the method above and suggested that students could extend the argument to derive the full addition formula. This was turned down both as a paper and as a letter.

When one gets to volumes, the problems become harder, but it is possible to motivate the factor of 1/3 in the volume of a pyramid by the same type of argument used to find the area of a right triangle. Consider a unit cube and decompose it into six congruent pyramids by drawing lines from the center of the cube to the vertices. The area of the base is 1, the height is 1/2, and the volume of each of the six congruent pyramids is 1/6 so there is a factor of 1/3 needed to get the volume. What we get in most of our textbooks is just the statement of the formula or a task appropriate for 12 year olds, poring sand into a pyramid and seeing how many times this must be pored into a prism with the same base and height to fill it. One text gives data and asks students to use the regression feature of a calculator to find a linear model for the data. The data was chosen so 1/3 occurs. What annoys me most about this treatment is a comment in the Teacher's Guide:

"This task attempts to help students see how volumes of pyramids and cones are related to the volumes of like-based prisms and cylinders. This is a nice use of data analysis to introduce a relationship that is not always easy for students to understand."

This is the only treatment of volume in this high school program. The geometrical argument sketched above should be for all students, and for the high achieves there should be a derivation of the formulas for the volumes of a general pyramid, a cone and a sphere. These derivations are missing in too many of our high school programs.

This example is an illustration of what I think should be the difference between mathematics for all and mathematics for high achievers. Both need serious mathematics, but not at the same depth. Neither group is getting an adequate education now. For those seriously interested in mathematics, the connection between volumes done geometrically and done via calculus needs to be done. An interesting observation is that the argument sketched in three dimensions works in n-dimensions and gives the integral of x^{n-1} without an explicit sum being used. I do not want this argument to be used in a calculus class, for explicit sums as approximations to integrals are very important, but it would make a nice math club talk or project for an interested student.

Further work should be done with students who want to study mathematics seriously. Contests are one thing, a mathematics club can be useful, and a good student magazine with articles which go beyond school mathematics is needed. This exists in some countries, but not all, and not in the United States. Further education of teachers is also vital, but that is a topic to be treated elsewhere.

Please don't ignore what has been done in the past, and think that mathematics education is a new field. Serious people have thought hard about different methods of teaching for a long time. One of my favorite quotes about teaching is in "New Plane and Solid Geometry" by Beman and Smith.

"It is sometimes asserted that we should break away from the formal proofs of Euclid and Legendre and lead the student to independent discovery, and so we find text-books that give no proofs, others that give hints of the demonstrations by a series of questions which, being capable of answer in only one way, merely conceal the Euclidean proof. But, after all, the experience of the world has been that the best results are secured by setting forth a minimum of formal proofs as models, and a maximum of unsolved propositions as exercises." [By "minimum" they do not mean none, as is seen by the proofs they give and the problems students should do.]

Susana Carreira: Dark and bright sides of mathematics teaching and learning: An inner perspective

In the very title of the panel – *Mathematics education for whom and why? The balance between mathematics education "for all" and "for high level mathematics performance"* – we may hint or take as implicit the fact that there may be two kinds of mathematics education serving different individuals and aiming at different purposes. While it can be sensibly argued that not everyone should have to take the same mathematics training, to be exposed to the same topics and to learn by the same books, I suppose that the kind of dichotomy displayed in the title can easily drag us into a misleading opposition. On one side we would find the "soft, accessible, low level, poor mathematics, available to all" and on the other side there would be the "hard, demanding, high level, rich mathematics, restrained to a few".

As I am choosing to step out of such a duality, I propose to concentrate on the implications of teaching and learning views in addressing the slightly rephrased question of "Mathematics Education for whom, why and *how*"? This generally means to acknowledge that one can not consider students' attainment without looking at classroom practice and implicit pedagogy, namely in the form of different teaching methods and teaching approaches (Gates & Vistro-Yu, 2003).

The inner perspective I am taking are elucidated by a few contrasting stories of mathematics classes. These are being referred to as bright and dark sides of mathematics teaching and learning and their purpose is to interrogate from the ground of practice and its ideological basis "what is happening in the name of mathematics education" (Davis, 2001, p. 22).

Frame 1

Apprehension and enjoyment: A class given in the school garden

My first story is about a group of prospective teachers in their last stage of academic studies. They were assigned the task of giving a mathematics class, centred on mathematical problem solving, in a school.

This group of five university students contacted and visited a pre-secondary school (grades 6-9), talked to the executive board and to the tutor-teacher of a 9th grade class to get permission to organise and teach in a class period of 90 minutes. They found out that the school had a nice garden with a pond, and started to prepare their intervention based on such features in the surroundings.

The following is a brief report of their initial worries, expectations, and intentions and also of the 9th grade students' performance and reactions.

We have decided to explore the theme of "Mathematics in the Garden". We wanted students to realise how mathematics can be related to the outside world. For that purpose we divided our lesson into two parts.

In the first part, our aim was to capture students' attention towards mathematics in nature by looking at a few examples presumably new to them. For the second part of the class, we prepared a set of mathematical problem solving activities to be carried out in the school garden. We separated the students into two groups that would be guided by different members of our team.

We did not really know the students, except from the previous conversations with their teacher. Mostly we were mentally prepared for a possibly hard time and we feared failing in keeping students involved, controlled and well behaved. As everyone knows, mathematics is not the favourite subject for the majority of students and we were afraid that they would reject us or manifest a special aversion to mathematical problem solving.

As students came in and sat down we noticed they were calm and nice and also curious about our presence.

We had brought some daisies and pines, a pineapple, and also a few pictures of sunflowers, butterflies and snails and we addressed ideas on number sequences, symmetries and geometrical shapes.

Then we invited them to go out in to the garden since we had a few problems for them to solve outside. We had questions on determining the height of trees, the diameter of the pond and the best shape of a rectangular flowerbed to be fenced with a certain length of string.

Students' performance was really better than we expected. Their engagement in the activities was always high and enthusiastic.

They showed creativity and original solutions. One student, instead of using the sticks we gave them to measure the shadows, decided to use his own shadow and his height. He also used the length of his foot as a unit measure and quickly arrived at an answer.

Both groups were able to deal with the problems proposed. Some students needed more assistance from us than did others. The whole class and their teacher were very collaborative *and fully involved. It was obvious that nobody noticed the time passing and some of the students even told us they would like to continue.*

For us it was a very gratifying experience and we sensed that we had accomplished something with those kids. We believe that our class was fruitful and stimulating and we find ourselves with a stronger conviction of what mathematics teaching is all about.

Frame 2

Ninety minutes of sacrifice …

A recent educational reform in the curricular organisation of schools sought to provide teachers with a more autonomous role in curriculum development while promoting new methods and innovative classroom activities. One resulting practical measure was to change the length to 90 minutes for each lesson period instead of the traditional 50 minutes. The purpose was to provide conditions for developing problem solving activities, practical and investigative work, the use of technology and other didactical resources.

The next story is a short description of a 90 minutes class period in a 12th grade class. The teacher claims that giving classes to 12th graders is highly demanding given the mathematical content and the pressure resulting from the final national examination. She argues that time is too scarce to try different and alternative approaches. She also feels that many students find the topics too hard, mostly because they lack the previous preparation needed to cope with many aspects of the curriculum, such as mathematical proofs.

The class is on probability theory and starts with the teacher writing a definition on the board. Students are quietly listening and recording in their notebooks what is put on the board.

The teacher continues the lesson by stating Theorem 1. She goes on explaining what the assumptions and the conclusions are in this theorem. Only one student says something about the givens in the assumption and the teacher carries on with her explanation. She finishes the proofs and waits for the students to write it down.

She proceeds to Theorem 2. Students listen and follow the teacher's talk: "Let's make a drawing here, let's consider the subset B as part of A". Occasionally, the teacher asks: "Where do we find the complement of B?" Two students give contradictory answers. The teacher makes a new drawing, this time using several colours. She then continues with the formal proof.

"Let's move to Theorem 3". One student dares to ask how many theorems there are to be proved. The teacher answers in a calm but almost guilty tone: "They are six altogether, but the next ones are really much easier..."

Theorems 4 and 5 and their proofs are presented. The class begins to show signs of impatience and boredom. Having completed another demonstration, the teacher tells the students they may take a little rest before the last theorem. A few moments later, she restarts and spells out the final proof.

Looking more at ease, she tries to be vigorous and exclaims – "Now it is time for you to work! Let's do some exercises. Open your book on page 89." Students seem to be willing to work on the exercises but most of them are not able to make any progress. The teacher tries to help with some clues written on the board. Nothing seems to be effective.

The class ends. Students leave looking discouraged but relieved. The teacher feels tired. She plans to encourage students to practice more and she expects them to overcome their initial but normal negative reactions. She finds the topic very dry and of little relevance except to a few who may possibly want to take a mathematics degree in the future.

Frame 3

Technology in the classroom – Good intentions and bad practice

The introduction of dynamic geometry software like the Geometer's Sketchpad is still not a regular practice in most of our mathematics classes. Different sorts of reasons are put forward to justify this fact, from the deficient equipment available in schools to the lack of teacher's preparation or the time deemed necessary to cover the syllabus.

Below is a description of a 9th grade mathematics class on the geometry of the circle with the use of Sketchpad. By using the software as a tool, the goal was to have students exploring geometrical properties of the circle.

Students were organised in small groups of three for each computer and were given a worksheet to guide their activity. It was intended that students discussed ideas, approaches and results and produced their own conclusions.

The worksheet started with the following sentence: "Investigate the definition of circle with Geometer's Sketchpad". A sequence of steps was then presented.

Step 1: Start by creating a circle.
Step 2: Create a point on the circle.
Step 3: Use Sketchpad to measure the distance from the centre to the point on the circle.

The last paragraph of the worksheet was highlighted: "Investigate! Drag a point along the circle and see what happens with its distance to the centre."

This group of three students engaged in a dialogue. One of the students reads the instructions: "Drag a point along the circle and see what happens..." Another student asks (shrugging her shoulders) – Is this to say that the points are at the same distance from the centre? Let's call the teacher.

The teacher comes and informs them that all conclusions must be recorded. The group assents. One of them reads the question again – "What happens with the distance …?" The distance is always the same … She ends up by asking the others in the group – Are there any other things you would like to say on this?

Up to this point the students have not attempted to create anything on the Sketchpad.

After a while, another student says – This is just to make us look at the radius! An exclamation came from one of the others – Ah! Is that what we were supposed to do?

Another activity followed, this time focusing on the ratio between the perimeter of the circle and its diameter. The same group of students struggled to discern the purposes of the task. Much of their time and efforts were spent on guessing what the teacher would like their conclusions to be.

The second task had a similar structure and was entitled: "Investigate the relation between the perimeter of the circle and its diameter".

Again, there was a set of steps formulated on the worksheet. The final paragraph read: "Investigate! Use the formula you already know, $\pi = P/d$. Calculate the ratio between P *and d on Sketchpad. Drag a point to change the diameter of the circle and see what happens to that ratio. What can you conclude from this mathematical investigation?"*

Frame 4

Students' good questions: Connecting mathematics to the real world

The last episode relates to a 10th grade class where the teacher had been working with her students on the quadratic function. They had studied transformations of the graph of a quadratic function by altering the coefficients in the polynomial.

She then decided to do a lesson devoted to mathematical modelling and problem solving in relation to the quadratic function. The purpose was to explore, with the help of a spreadsheet, the trajectory of water jets springing from little holes in a cylinder full of water. A small video was produced as it was considered unmanageable to perform the actual experiment in the class.

Here is a brief account of some of the ideas, questions and conclusions from the students.

Students were asked to guess the location of the hole for the water jet to have the longest reach. Next, they watched the video and a period of discussion followed. Students revealed their surprise with the unexpected result.

Afterwards the teacher split the class into small groups around the computers and handed out some information on the mathematical model describing the trajectory of the water jet. Their task was to model a sequence of water jets on a spreadsheet, by assigning a fixed value to the height of the container and taking different heights for the holes. After a while, the groups had produced tables and graphs of various parabolas, which represented several water jets.

Different groups of students used different values for the height of the container. This was used to explore the influence of that parameter on the graphs of the functions, thus connecting the problem to their previous learning about the transformations of graphs.

Later on, a student came up with a good question, which was shared with the whole class: – What if the container was not standing on the ground but rather on a higher plane above the ground, say on the edge of a chair or a table? All, including the teacher, found the idea interesting and started to imagine what the result would be. It led to the translation of the previous graphs along the vertical axis. This gave the opportunity to add a new parameter to the quadratic functions already represented on the spreadsheets. The new graphs were easily obtained. The maximum reach when touching the ground was no longer the same!

The class ended but students' discussion on the water jet phenomenon continued. They decided to do the actual experiment and record on video the effect of placing the container above the ground.

To conclude

Doing mathematics out in the school garden can be seen by many as a distraction from the serious practice of mathematics. Even students may perceive it as a detour from their usual experience at school. Nonetheless, all the signs of a meaningful learning process were displayed in the story above: students' involvement, discussion, original strategies, and a notion of achievement from both teachers and students. This means good teaching and learning for all. To work out the methods of deductive proof is one of the aims of including probability theory from an axiomatic point of view in the mathematics curriculum. G. Polya has proposed that mathematical proof is a fundamental part of mathematical reasoning. To initiate students to it requires a particular mediating action on the part of the teacher that can not be subsumed under exhibiting ordered sentences and symbols on the blackboard. Moreover, it is likely to be of little meaning for the majority of students if such an experience looks detached from and inconsequential in their ordinary school practice. It may equally well result in a waste of time for all. The integration of technology in mathematics classes hinges on particular aspects of teaching practice, such as beliefs about mathematics, the role of the teacher, the nature of the tasks for learning, learning processes and many others. Although technology tends to be seen as a good device to motivate students and to foster investigative work it often happens that technological tools are distorted to fit conventional practices and serve the teacher's implicit purposes of directing students to specific answers. As a consequence, we may find students and teachers highly absorbed in activities where mathematics is almost absurd and the immense potential benefits of using technology are washed away. Pretending becomes the prevailing spirit in the classroom and attainment is an illusion. Finally, mathematical modelling and applications deserve credit as a rich source for mathematics learning and as a powerful activity to get students involved in interpreting and understanding the world. Here again, the teaching and learning activities require the development of a culture of inquiry and liberty. Having students posing questions, having those being valued and sustained and collectively engaging in the search for answers is having all doing mathematics.

Asking, for each of the given stories, what was the mathematics learned demands to address the question of how it was learned. Above all it emphasises the problem of how to make mathematics education a realm of purposeful learning experiences. This does not imply that all mathematics is adequate for all students but rather that all students should have the chance to develop a "mathematical competence" and to be acquainted with those aspects that are inherent in the nature of mathematics. Undoubtedly, content is not a minor issue but contexts are equally decisive. Mathematics for all should not be seen as the same thing for every learner, neither in terms of content nor in what concerns the ways and trajectories of understanding (Freudenthal, 1991). As a final remark, let me iterate the words of our late colleague Paulo Abrantes on this point: "mathematical competence cannot be seen as independent from the educational experiences that all children should live in school" (2001, p. 135).

Yukihiko Namikawa

Introduction

In April 2003 the Subcommittee of Mathematical Education of the Liaison Committee of Mathematics of Science Council of Japan made public a document of 2 pages with the title: "Why is arithmetic/mathematics necessary for school education?"

Though the document is short, the significance of it is not small. It aims to give school mathematics teachers enough reason to teach mathematics at school. The content of the document is not so surprising for experienced mathematics educators. What is important is the seriousness of the situation that we *need* such a document since many mathematics teachers cannot answer with confidence when pupils or parents ask them: "Why do we need to learn such abstract mathematics which is of no use after finishing schools?"

However, this is not a phenomenon existing only in Japan. The clear proof is that we are organizing such a panel here at ICME-10. It is a worldwide problem we are confronting.

This is intimately related with another important problem of math education, namely *weakening of mathematical ability*. This does not mean only that children are less skilled in calculation but that it is difficult for them to learn mathematics in the following respects:

1. they are less motivated to learn mathematics;
2. there are more misunderstandings about mathematics;
3. they have a passive attitude to the study of mathematics;
4. they easily give up answering questions or problems without thinking or even trying.

Such tendencies are observed in Japanese universities (Nishimori, 2004), but we are afraid that they are actually widespread.

Therefore we need to analyze more systematically the significance of mathematics education. We have, traditionally, reasonable answers, but if we cannot persuade people to let children learn mathematics at school on the basis of these reasons, there will be mismatches in school mathematics education, which we must overcome.

Hence in what follows we try first to answer the question "WHY?" in the title of our panel, namely to clarify the significance of mathematics education. My approach is to consider characteristics of mathematics as an academic field. Among them we choose two characteristics which are related to the essence of mathematics education, mathematics as a universal language and the use of mathematical models.

This then gives a way to answer the problem of balance in the title, since we can see how far or how much each child is to learn mathematics according to his or her interest and needs from each point of view.

As a final remark, we mention an even more severe situation which appears when analyzing the role of mathematics education at school, namely the poverty of experience in early childhood. This influences even school education in general, and we cannot avoid facing it.

Mathematics as a universal language

Galileo Galilei said, "This great book, the cosmos, is written with the language of geometry [which meant "mathematics" at that time]." Being independent of natural philosophy, modern natural science has been established by employing mathematics as its own language. Nowadays almost all principles and laws in physics are written in terms of mathematics.

A modern use of mathematics is therefore as a *universal language* to describe theories in (natural or social) sciences. In this sense math education is closely related with language education.

For example, it is often said that learning mathematics is good for the training of logical or abstract thinking. This is true but in fact such logicalness and abstractness are characteristics of language. To write mathematical formulas is nothing but writing sentences.

This gives good reasons to learn mathematics.

Many people who are interested in or need to use natural or social science where mathematics is used as a language, should learn mathematics. In this sense we need more and higher mathematics as science is being developed. But here the difficulty appears. The higher mathematics they use, the more mathematical theories they need. We shall reconsider this problem again in the next section. Here I only want to say that the organizing principle for the curriculum is to prepare students to be ready to learn the mathematics they need when they need it,

On the other hand as human beings living in the modern era all people should acquire some knowledge of natural or social science, and hence a certain knowledge of mathematics. But what mathematics we should learn is a rather controversial question, and no "right" answer exists. For example, should the notions of derivative or integral be taught to everybody? (I am inclined to answer "yes".)

At least, however, we can say that mathematics is worth being taught to everyone at school as a tool of communication along with usual language.

In particular, we should notice that mathematical proof is nothing but a *debate* with the people who is reading the proof. Note that here the supposed reader is not a teacher, but a classmate who has the same level of knowledge of mathematics.

Mathematics as the science of mathematical models

For science, mathematics is useful not only because it gives a tool for description but also because it gives various powerful methods to solve problems, in that mathematical theory can be described as a study of mathematical models with abundant and useful results so as to be applicable.

For example, when an equation of motion is written explicitly, we can apply the theory of differential equations to deduce many properties of the motion considered and often we can even determine the motion explicitly.

As Bourbaki claimed half a century ago (Bourbaki, 1948), mathematics is the *study of structure*. Though structuralism seems to be outdated today, this characterization of mathematics still remains to be true. Pure mathematics is the study of mathematical models based on classical mathematical materials such as numbers and figures.

But the study of mathematical models has not been done in isolation. On the contrary, mathematics found rich mathematical models and important concepts from nature. We mention here two typical examples, trigonometry from astronomy, calculus from dynamics. Such interaction is stronger and wider nowadays and the term mathematical science is well suited to express such an "open" nature of mathematics.

What is important here is that some mathematical models are so widely used that they might be called universal mathematical structures. They have many different realizations in natural or social sciences. This is the reason why calculus, linear algebra and statistics are taught in basic courses at universities all over the world.

Thus again we have a good reason to learn mathematics for people who use mathematics as a tool. In organizing the curriculum we should prepare to teach important mathematical models systematically. Careful consideration and good analysis will lead to better solutions in organizing the curriculum in secondary education from this point of view. By this reason we need also to take the curriculum of other subjects such as science into account.

Another important point is to give more opportunities to use mathematics as mathematical models in sciences in real life so as not to lose learners' interest to study mathematics because of its abstractness. Here again we need cooperation with teachers of science.

Then how about the part "for all"?

The reason for all people to learn mathematics from this point of view is to train *systematic thinking* (not only logical or abstract thinking). Philosophy is to describe knowledge as a system but by using ordinary language. Some mathematical models e.g. from elementary geometry are much simpler and even children can handle them.

This reason applies to advanced level, too. Systematic thinking is necessary in everyday life. But it is even more necessary in academic study.
Hence for advanced children of advanced levels mathematics gives "toy" models to conduct training of academic study.

This consideration leads to a re-evaluation of classical Euclidean geometry as material for mathematics education. Many scientists (not only mathematicians) say that they became enthusiastic about elementary geometry when they were schoolboys/-girls. This is a proof that it provides a very nice "toy" model. There one can experience the depth of the theory, the importance of systematic thinking, the pleasure of finding solutions and sometimes even the rejoice at new discovery, all essential factors of usual academic study. As a mathematical theory, classical Euclidean geometry is not so useful any more after Descartes' invention of coordinate geometry which connects the world of figures with the world of numbers. But as an educational material we have still nothing more useful than classical geometry (the closest rival would be elementary number theory). It is more valuable for people of advanced levels.

Problem solving is usually considered to be a standard method for training of thinking. It is, however, difficult to develop systematic thinking, and the best way still is to learn classical rich theories.

Basic mathematical sense

In the last section we propose to consider a severe problem relating to "why mathematics for all".

The object for learning mathematics at elementary schools traditionally has been considered to be clear:

1) to acquire basic skill to *calculate with numbers* (arithmetic);
2) to acquire basic sense concerning *geometric figures*.

Some people say that we need not learn to calculate numbers any more since we have calculators, but this is too simplistic a thinking. We need also a fundamental sense of numbers, which can be acquired only by handling numbers in calculations. We might call this basic numeracy or basic mathematical sense.

Similarly in elementary school children learn to acquire the basic skill of human communication (including language) and a sense of life in human society and in nature together with hand work skills.

Children usually acquire these skills in daily life by playing together with friends, by playing outdoors, and by making handicrafts. Conversations with family or neighbors are also important.

The severe problem in Japan now is that children cannot develop enough such sense or skill because of the *poverty of experience* in social or natural life before entering school. This seems to have a serious influence on the skill of communication and the sense of life.

These senses and skills become very important for the learning of mathematics at higher levels, since the ability to learn mathematics as a universal language or as a study of systems is deeply rooted in them. Of course they have more influence on language education and science education, and in this respect there is a general problem in education.

I recognized this problem when, I asked the following three questions to students in the faculty of natural sciences in my university,:

Have you ever built model airplanes?;

Have you ever built radios or suchlike?;

Have you ever collected insects? (this was a common summer vacation homework when I was a child).

For all questions more than 70% students answered "no". Then I made the same inquiry with a group of mathematics teachers and more than 70% of them answered "yes" for at least one of the three questions.

This tendency seems to be stronger in universities of the higher level in Japan, hence this is a problem also "for children at a high level". And I am afraid that this phenomenon is not limited to Japan only.

There is a problem too big and too serious to be considered here in detail. I confine myself to mentioning that we need more systematic analysis of basic skills and sense of mathematics and the processes through which children acquire them. For that purpose the recent developments in brain science may give us very strong methods to study this problem.

***Renuka Vithal*: A battle for the soul of the mathematics curriculum**[2]

The long standing rather old and tired issue expressed in the title of our panel is at the core a mathematics curriculum question. It has endured over several years in mathematics education debates because it is a battle for the "soul" of the mathematics curriculum, and especially the mathematics school curriculum. I use the metaphor of "soul" quite deliberately to refer to that which we might believe exists but struggle to describe or analyse and cannot quite grasp. But then we immediately run into some difficulty with this metaphor because it might be used to assume that mathematics has some essential essence, some single universal meaning, when this is precisely what needs to be interrogated. Yet such a metaphor easily comes to mind when observing the fervour with which this debate unfolds in some countries.

In this contribution I attempt to develop three broad points. First I hope to argue that there is no "one size fits all" solution to the question being posed; that how the question is debated and addressed is shaped by a particular historical, socio-political moment of a country, and I use the South African mathematics curriculum experience to this end. Second I attempt to approach the question from the vantage points of the producers, users and consumers of mathematical knowledge and skills within society to question whether we are in fact talking of different kinds of mathematical competences; and raise it also at the margins of society for those who are usually ignored and

2 I express my appreciation and gratitude to John Volmink, Jill Adler and Cassius Lubisi for their reflections on the ideas expressed here.

invisible in this debate – the poor and those failed by mathematics education. Third, I conclude that we have perhaps a rather narrow, inadequate and limited understanding of the question because we choose to remain largely within an insider perspective of mathematics and mathematics education when the reasons and goals for learning and teaching mathematics are rapidly changing and may lie outside.

The South African case

South Africa offers a useful case for engaging in this question given the unique opportunity of its young democracy of ten years within which three waves of curriculum reforms have already occurred; and one in which mathematics educators, mathematicians, and a range of other "stakeholders" have had the possibility to participate and shape the mathematics curriculum. It may be asserted that there are two main imperatives driving and shaping curriculum debates in South Africa. The first is the post-apartheid challenge for greater equity and social justice to redress decades of deliberate inequalities and to entrench and deepen democratic life. The second is the global competition and development challenge to provide opportunities to learn and access knowledge and skills to participate effectively in the internationalised and globalised economy of the twenty-first century (Vithal and Volmink, 2005)

With regard to the first imperative there is no doubt that the South African curriculum takes as one of its key points of departure, the need to address the ravages of apartheid:

"When I have control over native education I will reform it so that the Natives will be taught from childhood to realise that equality with Europeans is not for them… People who believe in equality are not desirable teachers for Natives… What is the use of teaching the Bantu mathematics when he cannot use it in practice? The idea is quite *absurd."* (*House of Assembly Debates* Vol 78, August-September 1953, p. 3585)

These often cited words of the then Minister of Native Affairs and architect of apartheid, Dr H. F. Verwoerd, in a speech he delivered on the Second Reading of the Bantu Education Bill (Khuzwayo, 2005) allude to the ways in which those in political power, more than fifty years ago, used and understood the role and function of mathematics and mathematics curricula. It may be argued that scholars in the field have not adequately taken into account the curriculum policy implications of their positions, nor analysed or theorised on the influence of their views on mathematics teaching and learning. It has long been recognised that performance in mathematics serves as both "gateway" and "gatekeeper" depending on which side of performance a learner is located. Only now is the participation and power of politicians, policy-makers and the impact of policy in mathematics education research and practice being raised as a much stronger mainstream concern. (For example the *Second International Handbook of Mathematics Education* has a section on the "Policy dimensions of mathematics education" (Bishop et al., 2003)). The enormous neglect in mathematics education accrued through the implementation of apartheid curriculum policy and laws are well documented and continue to be felt in real terms. Suffice it to point to one piece of statistics for the purposes of this contribution: of the approximately half a million learners who take the high stakes national matric exams at the end of grade 12 presently, about 40% do not take or write mathematics; 5% pass on the higher grade which determines eligibility for the high status, well paid professions in science, technology, and economics related fields; and less than 1% black African students pass mathematics on the higher grade.

Addressing the first and last of these statistics has been taken up at the highest levels of government and continues to influence the development of curriculum policy (Kahn, 2001, 2004).

This point brings us to the second imperative driving curriculum reforms in countries like South Africa and that is the imperative to compete in internationalisation and globalisation agendas, not only of politics and economy but quite directly in mathematics education through studies like the Third International Mathematics and Science Study (TIMSS). The media impact of international studies like TIMSS on politicians, policy makers, public opinion, heads of higher education institutions, funders and other decision makers in shaping curricula decisions cannot be underestimated and is a somewhat neglected area of study in mathematics education. Access to mathematics is seen as having to address a *mathematical literacy* need argued as more than a practical or functional literacy but rather as integrating a democratic and critical competence for a citizenry that can participate meaningfully in a young democracy. But given the history of inadequate mathematical provision in South Africa, it is also argued that access to high level mathematical knowledge and skills, to the *mathematical sciences* to serve the economy and deal with unemployment and poverty within the tensions of an increasingly technological society and globalised world must be maintained and strengthened. The attempt to reconcile these different goals was aptly captured in the rather cumbersome naming of the mathematics learning area in the second wave of reforms for grades 1 to 9 as "Mathematics Literacy, Mathematics and Mathematical Sciences", but has now been dropped in favour of "Mathematics".

The new mathematics curricula reforms for grades 10-12 to be introduced in 2006, currently under intense debate, are instructive of precisely the difficulty of the question posed in this panel where it has been proposed that the present largely abstract mathematics curriculum offered as the higher grade and a watered down standard grade would be replaced by one mathematics curriculum and a new mathematical literacy curriculum for all those not taking mathematics. This is intended to ensure that unlike the present situation of students being allowed to opt out of mathematics at grade 10, in the future all students not taking mathematics will have a mathematical literacy on leaving school. The question of how these are distinguished is important because both intended curricula are organised around four similar learning outcomes. Both refer to "shape, space and measurement". "Data handling" is a relatively new addition and includes probability in the mathematics curriculum. "Number and number relationships" is referred to as "Number and operations in context" in the mathematics literacy curriculum; and "functions and algebra" is "functional relationships". Both curricula explicitly refer in the preamble to the new South African constitution in which healing the divisions of the past, improving the quality of life for all and so on serve as a socio-historical framing. Access to education and mathematics education in particular is seen as a basic right. Both curricula list as underpinning principles: "social transformation, high knowledge and high skills; integration and applied competence; progression, articulation and portability; human rights, inclusivity, environmental and social justice, valuing indigenous knowledge systems, credibility, quality and efficiency" and identify critical and development outcomes learners are expected to achieve. It is in the assessments standards spelt out for each outcome and grade level that one can analyse how these principles are to be given meaning. It is here that the curriculum struggles and the tensions become visible. Although the social, cultural, and political dimensions of these principals are present in the assessment standards they are unevenly expressed within and across each curriculum.

These inconsistencies may be explained in a number of ways. One is that curriculum conflict is resolved through a now well established South African tradition of reconciliation. Inclusivity and diversity of theoretical and ideological orientations are often achieved at the expense of coherence (Vithal & Volmink, in press). Second, the curriculum is attempting to balance both a vertical and a horizontal integration in mathematical knowledge and skills. In this respect a significant shift is toward understanding rigour in mathematics not only hierachically but contextually, requiring some kind of inter- or multi-disciplinarity. The move toward application brings in for the first time areas such as statistics and modelling seeks to develop a more contextualised mathematics curriculum. A third explanation relates to the curriculum development process itself. Two different committees were tasked with developing each curriculum in which the policy development process unfolded through different sets of debates and consultations. Fourth, a lack of consensus and clarity in how each – mathematics and mathematics literacy – are understood, as well as the relation between the two, is yet another reason. The difficulty of conceptualising a mathematical literacy different from what previously existed as "standard or practical grade" as a "lower order" mathematics but rather as a different, integrated contexualised competence that requires a different rigour, remains a significant challenge. If such an assertion is accepted then the question of articulation often posed about how those who take" mathematical literacy" may be disadvantaged, may then be turned around to ask how equally those who come through the mathematics curriculum will acquire mathematical literacy competences. Studies exploring the mathematical literacy competence of health science students (Prince, Frith & Jaftha 2004) who in South Africa are typically "high performers" in mathematics opens new challenges in this regard.

Are mathematics and mathematics literacy different kinds of competences?

The question of what exactly is meant by mathematics literacy and what is or could be its relation to mathematics is a long-standing one. Deriving from a broader "mathematics for all" movement, there is a lack of consensus on the name or term itself in the literature. It has been variously labelled within policy, theory, research and practice as: numeracy (in UK policy – Brown 2003); quantitative literacy (Steen, 2001); matheracy (in ethnomathematics, D'Ambrosio), mathemacy (in critical mathematics, Skovsmose, 1994). This is not only a debate in semantics but points to different ideological orientations, intentions and goals of mathematics teaching and learning across contexts, which extend from a concern to acquire basic numeracy to a sophisticated critical integrated mathematical competence. Mathematics education debates are different in character for "high level performance", typically raising questions of how advanced; how much abstraction; what application; and is dominated by the concern of the distance between school mathematics and higher education.

The question that must be taken up is: does mathematics and mathematical literacy embody different kinds of competences and what are their relation to society? One approach to this question may be to refer to the thesis of the formatting power of mathematics offered by Skovsmose (1994, p. 42): "mathematics produces new inventions in reality, not only in the sense that new insights may change interpretation, but also in the sense that mathematics colonises parts of reality and reorders it". It may then be argued that certainly a school mathematics curriculum has an obligation to produce both: a) those who come to participate in this formatting (as high performers), who are constructors or producers of mathematics and operators or users of mathematics; and b) those who must face and react to that formatting as "critical readers" who may be consumers of mathematics or as the marginalized of society (Skovs mose, 2003; Vithal, 2003). A key point here is that competence in the one cannot be assumed to produce competence in the other. What the above analyses points to is that access to mathematics serves different purposes. The questions of "for whom" and "why" are closely linked to the "what" question. Mathematical literacy is more than a functional or practical literacy. It is expected that one needs to integrate a mathematical, democratic and critical competence to participate meaningfully in a young democracy and growing economy. It needs to be responsive to a diversity of contexts, providing a mathematics that is inclusive of the majority who do not enter further education – the labourers, the poor, the unemployed, etc. Mathematics for the "formatters" or "high performers" needs equally to recognise the diversity of contexts and goals of mathematics, since transfer cannot be assumed, and nor does "high level" or abstract mathematics necessarily produce an integrated mathematical literacy competence.

Going outside mathematics and mathematics education to understand it from inside w*hat counts as mathematics* has shifted and opened. Drawing on a broad range of other disciplines scholarship in areas such as ethnomathematics and critical mathematics education have forced a recognition of a much broader set of practices, knowledge and skills as mathematics. By holding on to a narrow definition of mathematics not only do many get excluded, pursuing a limited meaning fails to prepare the diversity of learners for life in an increasingly technological but unequal and unjust local and global world. Arguably far more attention has been paid to the disadvantaged, those who fail to learn, and far less to the quality of the mathematics education of those who succeed and participate in the formatting of society through mathematics – "the high performers" – who do not acquire the tools for challenging the use that the products of their labour is put to – for example in warfare or in systems that do not serve the poor and marginalized of society. This is, in part, because a critique of the use and application of mathematics cannot be produced with reference to the expertise and language of mathematics itself. The need for the development of ethical, social and political responsibility of the so-called high performers who become producers and operators gets masked in the dichotomy of mathematics for all versus mathematics for some and does not serve learners well in seeing and working with the complexity of the relation between mathematics and society.

Bringing a "mathematical gaze" through research to the broad range of social, political, cultural, economic and other everyday practices and artefacts, both in the present and historically, has increased the recognition that all peoples produce and use mathematics. This has raised the question of *whose mathematics counts* and is drawn on for inclusion in the mathematics curriculum. Whose mathematics is excluded is increasingly contested through multiple perspectives such as gender, race/culture and class.

Who decides what counts as mathematics. The question being posed in this panel cannot be dealt with by referring only and narrowly to mathematics education as dealing primarily with the mathematics. Many more analyses are needed that connect the micro with macro patterns and systems of differential economic, political, social and cultural power to recognise how mathematics curricula can and do participate in challenging or entrenching inequalities, and to deepen and broaden our understanding of how the very posing of a separation of mathematics for all and for high level performance becomes taken for granted as the lens through which to view the question of mathematics for whom and why. Who gains access to what kind of mathematics through any mathematics curriculum must in the global world of today take account of the open trade and migration of mathematical labour, mathematical workers (in the broadest sense) and their products. Mathematical curricula questions cannot be addressed without recognition of students' perspectives that shape how they participate in mathematics as they are making choices about their futures, depending on their location within particular race/culture, gender, class, urban-rural, poverty, conflict settings. Mathematical knowledge and skills are in a complex relation with science and technology skills and knowledge, and these in turn are linked to rapidly changing work opportunities within and across countries. What students regard as interesting and important to learn in mathematics, and the reasons for investing in this learning, may be quite different from what mathematicians or educators may hold. Decreasing student numbers pursuing subjects like mathematics and physics for their own sake are testimony to these shifts. Those who shape political decisions related to mathematics curricula are sensitive to and will respond to these shifts to make decisions, unless mathematics educators begin to take many more risks to participate and develop their own capacities in more frequently crossing disciplinary and other boundaries to widen and sharpen how they understand and act in their own field of study and practice.

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P 3: What could be more practical than good research? On mutual relations between research and practice of mathematics education

Plenary Lecture based on the work of Survey Team 1

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1. Why survey?

Let me open with an anecdote borrowed from Etienne Wenger (1998). A person strolling through the streets of a city comes across two stonecutters toiling over identical pieces of marble. "What are you doing?" she asks. "I am trying to turn this stone into a perfect cube", responds one of the workers. "I am building a cathedral", says the other. This story is the perfect parable with which to introduce and justify the kind of task we are going to implement collectively in the next hour. Throughout our professional lives as mathematics educators we are building cathedrals even as we are shaping individual stones. And yet, it is not easy to keep the big picture in mind while going through everyday activities. Faced with thousands of concrete tasks that constitute our professional lives as researchers, developers, teachers, and policy makers, we prefer to do rather than to ponder on how the little stones of our daily chores fit into the huge, collectively constructed edifice. ICME-10 organizers' decision to launch the survey on what has happened to the relations between mathematics education research and practice over the last few years is an invitation to pause for a moment and try to see the cathedral in what usually appears to us as but a heap of individual bricks.

The wish to engage in this kind of reflection at this particular moment is not surprising. These are rather special times for mathematics education. While the public interest in the topic has never been greater, the press has not always been friendly. Claims were repeatedly being made that mathematics education research is "not very influential [or] useful" (Burkhardt & Schoenfeld, 2003). In the case of mathematics education, voices could also be heard accusing the research and its products of being more harmful than helpful. In the United States, the criticism has been followed by appeals for teaching "grounded in scientifically based research" and for instructional methods that draw on "reliable evidence that the program or practice works." In particular, the authors of the Mathematics and Science Initiative, launched on February 6, 2003, speak about "the need for better mathematics and science education for every child", and declare that perhaps the most important means to this end is "a research base" with which one can "improve our knowledge of what boosts student learning in mathematics and science."[1] Add to this the fact that, in the four years that passed since ICME 9, the world changed almost beyond recognition – enough to mention September 11, 2001; the

1 See www.ed.gov/rschstat/research/progs/mathscience

unprecedented attempts to unify the globe and, at the same time, stronger than ever, the tendency for tearing this globe apart; and the saturation of our lives with wireless communication that irrevocably transforms our conceptions of space, time and human relations – and you cannot but agree that we need to reflect on our past deeds in order to decide what needs to be changed in the future.

2. How to survey?

With such periodic stocktaking in mind, the ICME-10 Programme Committee created *Survey Team 1* whose members are *Yoshihiko Hashimoto* from Japan, *Gelsa Knijnik* from Brazil, *Aline Robert* from France, *Ole Skovsmose* from Denmark, and the author of this talk who is well acquainted with both Israeli and North American scenes. The five of us embarked on the project, convinced that answering the question about contributions of research to the practice of teaching and learning mathematics is a matter of our community's professional accountability.

While always useful, such critical self-reflection becomes a necessity at present times, when the quality of the collective cathedral building is being publicly questioned. We thus interpreted our task as guided by the following questions: How well have we been doing as researchers? What do we have to change in order to do better in the future? It did not take long before we became aware of the extreme complexity of the task. After intensive deliberations, we decided that rather than play the role of observers and attempt to tell an "impartial" story of the research community, we would try to help in constructing this community's own account. We turned to our colleagues asking them to tell us their stories. In the fall of 2002, we issued the call to mathematics educators in academia, likely to be involved in research, to answer three questions that are presented here in a slightly abbreviated form:

Research: How would you describe your work in mathematics education over the last five years or so?

Practice: During this period, to what extent was your work influenced by the current state of mathematics education?

Impact: Do you think that your work had, or is going to have, an actual impact on the practice of mathematics education?

The questionnaire had been posted on the ICME-10 website. In addition, to ensure a uniform distribution of responses between continents and countries, we had sent a number of individual requests to as many colleagues as our group could reach. Over the next 18 months we were able to collect 74 responses of varying length – from answers in the form of a single paragraph to many-pages long essays. Some of the survey participants joined the community quite recently; some others were "veterans" who have been around for many years and are well known to the rest of us. Through energetic recruiting, not to say nagging, we arrived at a reasonable, if not entirely balanced coverage of the globe (see Table 1). Although the sample cannot count as truly 'representative', we are proud of our bulky data base that spreads over six continents and 250 pages.

Continent	N
Europe	28
South America	15
Asia	14
North America	9
Africa	5
Australia and New Zealand	3

Table 1: Distribution of responses across continents

In launching the survey, our overall aim was to combine the individual responses into a collective narrative. What follows is an executive summary of our study. In this brief talk I will present the highlights of the findings regarding our three central themes: the current research, practice, and the relation between them. For each highlight, the actor's own story will be followed by another one, told in the participant-observer's voice. This second account will be more of a commentary than a separate tale. In creating the observer's version I will draw on materials such as other team members' contributions, newspapers, policy documents, research publications, and last but not least, the team's own speculations. I will complete my discussion by taking a critical look back at the past and a hopeful one into the future.

And one last remark. The picture to be presented here is, inevitably, *my* version, *my* revoicing of the community's own story. I cannot even say that it is our team's narrative because, aware of the immensity and the controversial nature of the task, my colleagues decided in advance to have a number of individual contributions rather than one collective article. Their work can be found on the web (www.icme-organisers.dk/st1/).

In the analyses of the data I was helped by Jagdish Madnani, whom I wish to thank. Throughout the rest of this report please keep in mind that although the picture I am painting is a result of the team's work, I am the only person to blame for all of its shortcomings.

3. Research

To identify recent trends in research in mathematics education we scrutinized the survey participants' responses to the first question, *How would you describe your work in mathematics education over the last 5 years or so*? In our analysis we concentrated on four topics: (a) the prevalent *focus* of research, (b) the dominant *research paradigm*, (c) the *quality* of research, and (d) the academic *identity* of the mathematics educator. The categorizations and the statistical assessments to be reported are crude. There is no space here for subtle distinctions.

3.1 Research focus

3.1.1 Actor's voice

The first salient feature of the research, as described by the survey participants, is its prevalent focus on the teacher and teacher practice. The initial indication for the teacher's centrality was found in a simple word count: In the responses to our questions, the word *teacher* appeared 832 times, which is nearly three times as many as the 317 appearances of the words *student*, *learner* and *pupil* (some of which, by the way, might refer to pre-service teachers!). We then examined the issue in a more direct manner and found out that teacher-centeredness in research could be identified with those two thirds of the respondents who claimed to be engaged in research. This is a striking finding, especially when contrasted with the mere one-quarter of the researchers whose investigations focus on the school student.

3.1.2 Observer's voice

This finding is significant, as it seems to be showing a considerable change with respect to what was true about mathematics education research in the not-so-distant past. Twelve years ago, in her plenary PME-16[2] address in New Hampshire, Celia Hoyles deplored the scarcity of teacher-focused research which, at that time, was particularly salient in comparison with researchers' preoccupation with student's cognition. She said:

"Of the 45 papers included in the published proceedings of the third PME conference in 1979, all but three focused on student understanding of mathematical concepts.… If the teacher was mentioned at all, s/he was discussed purely as a facilitator…. In 1980, the majority of papers again concentrated on [the] student." (Hoyles, 1992)

According to Steve Lerman and Anna Tsatsaroni (2003), students' learning did not lose its place of honor in research of the 1990s. In their insightful study on the development of theories in mathematics education, based on detailed analysis of leading mathematics education journals and PME proceedings in the period 1990-2001, the authors conclude that although there has been a certain growth in publications on teachers and teacher practice, there was no real turnaround.

The decisive shift in research might have occurred in the last four or five years, a period not covered by Lerman & Tsatsaroni's data. We also need to remember that those latter data did not include the specialized teacher-oriented journals, notably the relatively new *Journal of Mathematics Teacher Education*, or special publications such as the 1997 volume of *Recherches en Didactique des Mathématiques* (see, in particular, Margolinas & Perrin-Glorian, 1997), to which most research on teachers might have been channeled. The very fact that such publications were created may serve as evidence of the growing centrality of the subject. Similar confirmation comes from the proliferation of books on teacher-focused research, many of which became widely popular and some of which stirred public debates (see, e.g., Ma, 1999; Stigler & Hiebert, 1999; and Lampert, 2001).

3.2 Research paradigm

3.2.1 Actor's voice

At least three features are mentioned frequently enough to be regarded as fairly general characteristics of the survey participants' research. First, the basic type of empirical data is a carefully recorded classroom interaction, as opposed to the past attempts to document the learning of the individual student while concentrating on the result rather than on the process of teaching and learning. Second, this research emphasizes the broadly understood social context of learning. The wish of one of our respondents to "systematically analyze and report... the messy real-life classroom development" seems typical. Third, the majority of the research is qualitative and does not make any reference to the quantitative argument. As many as 74% of the responses mentioned at least one of these characteristics.

2 PME: The annual conference organized by the international study group for the Psychology of Mathematics Education.

3.2.2 Observer's voice

All this shows that the dominant type of research in our sample is one that can be called *participationist*, since it conceptualizes learning as a change in one's participation in a certain type of activity rather than as an ongoing attempt to acquire, or just enrich, a system of individual's internal representations of the world. This latter, more traditional vision of learning will, for obvious reasons, be referred to as *acquisitionist*.

Our respondents' preference for participationist, qualitative research is a phenomenon well known to the incumbent editors of mathematics education journals. Ed Silver, until recently the editor of the *Journal for Research in Mathematics Education*, marvels in one of his editorials that, "These days it seems that mathematics educators are a bunch of quantitatively competent individuals who are inclined to conduct qualitatively oriented studies". With the help of a deftly chosen metaphor, he implies that for some authors, "qualitative" does not mean much more than "number-free".

	Acquisitionist focus on the *product* of		**Participationist** focus on *process* of	
			Learning	Teaching
Interventional (teaching experiment)	Process-product (controlled experiments) 19%		Design experiments 19%	
Non-interventional (no intended teaching intervention on the part of the researcher)	Student's (mis)conceptions 7%	Large-scale achievement comparisons (TIMSS, PISA) <2%	Ethnographical studies on learning (classroom norms, development of discourse) 55%	Ethnographical studies on teaching (teacher practices; e.g. TIMSS video studies)

Table 2: Distribution of types of research in mathematics education in the survey sample

This qualitative preference of our respondents is counterbalanced by the increasingly popular international comparative studies, such as TIMSS and PISA, that focus mainly on students' measurable achievements. Only one of the researchers in our sample seems to have been engaged in any of those large-scale projects. Even so, it is quite telling that in our data, we find no more than 3 references to these studies. Our respondents do not help themselves to TIMSS or PISA findings even when responding to our questions about the state of mathematics education in their countries. The gulf that separates the qualitatively and quantitatively inclined mathematics education researchers appears difficult to bridge, and this is true in spite of our frequent declarations about the need for a balanced mixture of methods.

Table 2 combines the acquisitionist/participationist distinction with the classification based on the question whether a given study involves an intentional teaching intervention. The numbers present the distribution of the different types of research among the survey participants.

3.3 Quality of research

3.3.1 Actor's voice

Since research can be defined as an exploratory discourse that aims to interpret and enhance the practice of teaching and learning, the question about the quality of research becomes almost tantamount to the question about the researchers' ability to communicate effectively among themselves and with others. On this all important point, the survey participants sound rather skeptical. With striking repetitiveness, they complain about "fragmented mathematics [education] community", talk about the lack of theoretical infrastructure and about their efforts to provide what is missing by constructing theories of their own, but above all, they wonder about the "reason why it is necessary for authors to coin their own vocabulary." As observed by a number of survey participants, lack of communication entails the impossibility of cumulating and the habit of "reinventing the wheel." The putative communication deficiency is rather puzzling in the view of two other findings: In research, there is a tendency for team work and for mutual inspiration – 40% of our respondents report to be working with others and half of them explicitly link their research to the work of others.

3.3.2 Observer's voice

With quite a lot of similarities between the individual images of research drawn by our respondents, one might expect the complaint about imperfect communication and insufficient accumulation to be somehow exaggerated. Indeed, there is much convergence in the research focus; there is the general preference for qualitative methods, and there is a wide agreement that research should be socially minded. And yet, evidence gathered in a number of independent reviews over the last few years confirms our survey participants' grievances. Thus, for example, Lerman & Tsatsaroni (2003) summarize:

> "[I]t is not uncommon to find a substantial and informed review of literature in an article, in which the range of theoretical resources drawn on by others are noted, but then for the authors not to use any theory themselves, at least explicitly." (Lerman & Tsatsaroni, 2003, p. 19)

More often than not, words central to the research discourse – from the most basic, such as *learning*, *understanding*, *meaning*, or *mathematical object*, to more specific, such as *belief*, *identity*, *improvement*, or *disability* are used without being operationally defined, their communicative power taken for granted. And yet, without an operational definition, the reader who is told, "The student did not understand functions" or, "The class built a shared meaning of functions" has no means to unpack the reported findings into what the students actually did or said, and can have no reasonable expectations about these students' future sayings and doings. Moreover, it is not uncommon for researchers to use the same words in different ways. Unaware of this fact, they are only too likely to fall prey to confusions and misunderstandings. It is plausible that many of the controversies that split the mathematics education community, including the arguably most famous one on the individual/social dichotomy, would simply disappear if the differing uses of the relevant terms were made explicit. This kind of research cannot be very effective in informing the practice. As such, it does not live up to its principal commitment, and some would go so far as to say that it does not justify its existence.

3.4 The identity of the mathematics education academic

3.4.1 Actor's voice

Based on the survey, our professional activities are strikingly numerous and multifarious: 76% of our respondents do research, 56% work as teachers' teachers, 33% are engaged in curricular development and 15% are busy with policy making. No additional statistics are necessary to understand that the mathematics education researcher is often engaged in as many as 3 or 4 additional types of professional activity. The following remark by one respondent echoes a concern expressed in one way or another by almost everybody else in the sample:

> "Being overwhelmed, like many of my colleagues, by teaching and other responsibilities..., I find it difficult to develop my own research and to keep contact with worldwide research in mathematics education."

Over-commitment and chronic time pressure are evidently a universal affliction in our community. This is why many well-meaning colleagues failed to answer our questionnaire even though they intended to do so, while others apologized for long delays followed by terse answers, saying that they would love reacting more at length if they only could allow more time for this kind of activity. Our business may be the price we pay for the fact that the position of mathematics education as academic discipline solidified in these last few years, [3] and led to new responsibilities within universities. The sense of becoming full-fledged citizens in the world of academia has been aptly captured in the following declaration by one of the survey participants: "I have noticed that the whole Department has been increasingly respectful regarding the field of Mathematics Education."

It is interesting to see how the researchers position themselves with respect to other actors in the educational drama. Our respondents' remarks about politicians and funding agencies are markedly negative in tone, which contrasts strongly with the caring, warm timbre of their references to teachers. While the politicians and funding agencies are presented as constraining, if not downright oppressing factors, the teacher is portrayed as an ally, a kindred spirit, a partner, a colleague. This egalitarian self-positioning toward the teacher is a rather dramatic change in the research discourse which, only a few decades ago, was imbued with patronizing undertones. Today, the researchers stress that their studies are done *with* the teacher rather than *about* her, that they go to classrooms to listen to the teacher and to think with her rather than to tell her what to do, and that they "support teachers and learners to develop their own powers... rather than trying to make changes for them."

3.4.2 Observer's voice

The alliance with teachers constitutes the very heart of the mathematics education academic's self-definition and provides his or her professional raison d'être. The tendency toward the dialogical relation with the practitioners may be a result of the growth in the number of researchers who began their careers as teachers. Whatever the reason, there is a remarkable blurring of the boundaries between the communities of researchers and of practitioners.

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3 Only a decade ago, the academic status of mathematics education was widely questioned. This was certainly true at the time when ICMI launched the study that resulted in the two volumes edited by Anna Sierpinska and Jeremy Kilpartrick (1997), titled *Mathematics education as a research domain: A search for identity*. There is much independent evidence that, indeed, if not everywhere in the world then at least in many places, research in mathematics education is more alive and well than ever. One of the most convincing signs is the proliferation of graduate programs in mathematics education. Although exact statistical data are unavailable, I can testify on the basis of informal evidence that in many universities, mathematics education attracts many more graduate students than pure, or even applied mathematics. The recent Carnegie Doctoral Initiative enthusiastically embraced this trend, whereas many North American universities strive to increase their mathematics education faculty – and fail to do so because of the shortage of eligible candidates.

Interestingly, we seem to be witnessing yet another, apparently less likely, border crossing. Although the external policy makers and funding agencies embody values that the research community tends to oppose, they do seem to have a distinct, and not necessarily desirable, cultural impact on the culture of academia. While under the growing pressure for engaging in large funded projects, mathematics education researchers are sometimes acting more like corporate employees than scholars: They think in "PowerPoint bullets" rather than full paragraphs, write "documents", "memos" and "proposals" instead of articles and books, and replace deep solitary reflection with collective "brainstorming" and "instant" creativity. They even start speaking in the corporate language – with my own use of the term "executive summary" being a case in point.

4. Practice

For the sake of this report, practice of mathematics education has been defined as any kind of activity that belongs to, or results from, the actual learning and teaching of mathematics. While it was risky enough, but still justifiable, to generalize about research, the story of school mathematical practice involves too many people and societies to try to tell this story in general terms, bracketing national or cultural idiosyncrasies. Not to mention the fact that there are places in the world where school mathematics practice is simply absent along with the extensive regions that our research has left uncharted. These "other" places, according to statistics quoted by Ole Skovsmose (2004), may be the great majority of the world. After all, says Ole, the dominant, prototypical site of our research is a "well-equipped classroom from countries ranking high on the world's welfare scale". Sadly, UNESCO (2000) statistics let it be understood that a great many children in the world may not have access to such classrooms – suffice it to mention the 16% of the children of the world who do not attend any school at all. On top of that, whatever I may be able to say about learning and teaching mathematics in those parts of the world where children are born into incontrollable hostilities, would probably be misleading. It would not reflect the fact that in the face of pervasive life loss, when the universe itself appears fragile, the abstract mathematical certainty may have little appeal and there may be no wish to invest in its learning for the sake of future rewards. But let me do the little that can reasonably be done.

4.1 Actor's voice

It seems to be generally agreed upon that research in mathematics education is not an end in itself. In their responses to the second survey question, "To what extent was your work influenced by the current state of mathematics education?", more than half of the participants confirm that it is the situation in the practice of mathematics education in their country that motivates their work. Close to one third of our sample present a little wider perspective, saying that they are driven by the awareness of social and political *wrongdoing*, and that for them, mathematics education is a pathway to the much needed socio-political change in the increasingly globalized world.

Exactly half of our respondents express varying degrees of distress with the present state of mathematics education in their country. The other half simply does not offer any evaluation. In general, the complaints vary widely in tone and pitch, depending, mainly, on the nationality of the respondent. While many are merely disheartened, approximately one third of the complainers speak about their being "deeply disturbed" by the situation, which is subsequently described with words as strong as *dire*, *bleak*, *reactionary* or *retrograde*. This uneven emotional charge notwithstanding, the grievances seem to converge in their content: They are mainly about classroom practices that refuse to change and, in particular, about the fact that the lessons learned by pre-service teachers do not seem to "transfer" to the actual school classrooms.[4] If there is a reform, say the complainers, it is distorted. Sometimes it seems as if the pendulum of educational change were on its way back to where it was decades ago, especially if its movement is fueled by the *back-to-basics* slogan.

And what is it that puts the backlash to reform in motion? One aspect of the respondents' vision is common to all: Almost nobody blames the teacher. Rather, the teacher is pictured as a victim of external forces that run against his or her attempts to revamp the classroom discourse. Among factors that hamper the change our respondents mention, with an emphasis depending on their nationality, governmental interventions, economic shortage, and the insufficiency of teacher education programs. What makes the situation even worse, the teacher educators themselves may be constrained by external impositions, notably by governmental regulations. "Whereas my research is primarily theoretical and critical," says one of them, " it is increasingly difficult to incorporate such perspectives in any serious way into work with pre-service teachers."

Another frequent complaint is about a veritable explosion in testing and assessment, evidently driven by the view that "accountability" means liability to measurement. This measuring and labeling tendency is, naturally, not without its consequences, one of the most disturbing of which is the industry of private tutoring, flourishing in those parts of the world where the parents are sufficiently well off. This, needless to say, makes the distribution of opportunities for learning even less equitable than ever.

4.2 Observer's voice

Research done by Susanne Wilson, who, in her recent book (Wilson, 2003) tells the history of the reform in Californian schools, confirms the picture drawn by our respondents: Although there is a certain visible change, the American mathematics classroom is rarely a reasonable fulfillment of the reformers' dreams. Wilson describes what she saw in an elementary mathematics classroom:

> "... there was change. Most teachers... added some new practices and problems to their teaching. For some teachers it felt revolutionary. But what seemed radical to them appeared more incremental to us.… Other teachers more actively resisted the reforms." (Wilson, 2003, p. 207)

But the voice of outside observers is not just the voice of another researcher. In this last decade, the public debate on mathematics education has been probably more common and much louder than ever. One can name a number of events that occasioned this unprecedented exposure. To begin with, the world seems to have been swept with reform movement. The launching of *NCTM Principles and Standards for School Mathematics* in April 2000 was a momentous event, the importance of which transcended the borders of North America. The *Principles* are the revised version of *NCTM Standards* (1989) and

4 It is interesting to note that while the concerns seem evenly spread around most of the globe, only one complaint came from East Asia. Further, while the prevalent reason from the complainers' dissatisfaction are the traditional, classroom practices that are difficult to change, the single East Asian complaint was not about the lack of reform, but on the contrary, about the reform as a result of which "pupils are active, talk, exchange ideas, ... but they are not thinking mathematically".

subsequent, related publications and they follow the lessons learned from research on learning and teaching mathematics in general, and on the first decade of the *Standards*' implementation, in particular. Against this development one has to mention again the broadly publicized, often disappointing, results of TIMSS and PISA. Confronting the unsatisfactory outcomes of educational efforts, mathematicians, parents, mathematics educators and politicians let themselves be drawn into heated debates on the reform and its impact on students' learning and achievement. The vociferous participants of what came to be known as "math wars" are not any less concerned about the state of mathematics education in their countries than those who are "insiders" to the educational project. And yet, the focus of the outsiders' concern is quite different. While the mathematics education researchers deplore the conservatism of the mathematics classroom, parents and politicians are disturbed by children's low achievement, and the mathematicians worry about the nature of the mathematics learned by the student. While the insider deplores the destructive impact of external forces that counteract implementation of the reform, the others often view the reform as the main culprit. While the mathematics education academics feel for the teacher, who is seen as constrained by the system and unable to act to the best of her understanding, the others do not hesitate to put the responsibility on the teachers' shoulders.

It is notable that while the battles are being fought over the question of who is responsible for the pervasive failure in mathematics, nobody seems to consider the possibility that the present cultural climate may play one of the leading roles. Mathematics, once a highly prestigious type of activity, seems to have lost most of its luster and appeal. In the unprecedented flow of books[5], films[6], and plays[7] about mathematicians, the protagonist is portrayed as a curiosity, sometimes admirable but always too detached from reality to serve as an example to follow. School mathematics is often ridiculed by the media as a contrived activity that plays no real role in one's life and is practiced only by "uncool," socially ill-adjusted individuals.

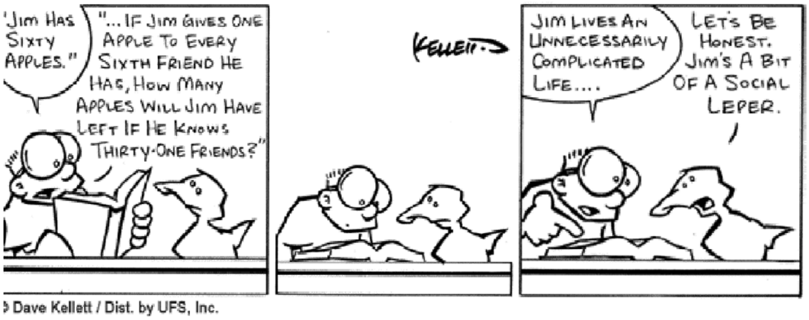

Figure 1: School mathematics in the eyes of a cartoonist (www.comics.com/comics/sheldon)

The comic strip (Figure 1), chosen at random from an infinite supply, is a representative example. Its hero, a 10-year old billionaire, made his fortune in the world of high technology but is still unable to make sense of school mathematics. In the first picture the

5 See, for example, Silvia Nasar's (2001) *The Beautiful Mind*, Paul Hoffman's (1999) *The Man Who Loved Only Numbers*, Apostolos Doxiades' *Uncle Petros and Goldbach Conjecture* or Simon Singh's (1998) *Fermat's Enigma*.

6 *The Beautiful Mind, Good Will Hunting, Pi*.

7 See, for example, *The proof* by David Auburn.

boy reads a word problem that tells the story of a person by the name of Jim who "gives an apple to every sixth of his friends". After a thoughtful pause the boy concludes: "Jim lives an unnecessarily complicated life", and his friend adds, "Let's be honest, Jim's a bit of a social leper"[8].

All this leads us to the last question to be dealt with in this report: What is it that shapes the educational practice and its results, and in particular, what is the role of research in making it the way it is?

5. Impact

In the third item of our survey the respondents were asked to assess the impact of their research on the practice of mathematics education. Let me report the findings by answering the following three questions: (a) What kind of impact are we hoping for? (b) Do we have an impact? And, last but not least, (c) Can the latter question be answered at all? As before, each query will now be answered by the actor-observer duet, which does not always sing in unison.

5.1 What kind of impact are we hoping for?

5.1.1 Actor's voice

In the light of what was said about the centrality of the teacher to the mathematics education researcher's work and identity, it is not surprising that 55% of those who responded to this question hoped to influence teacher practice. The other fields of intended impact, in the order of the frequency of reference, are: society at large (25%), curriculum and educational policy (17%), and other researchers (3%).

5.1.2 Observer's voice

The dominant wish to make a difference in teacher practice implies that we came a long way since the time, just a few decades ago, when it was believed that one improves students' learning simply by "fixing" the curricula. In that period, all we expected from research was to show whether this or that instructional idea worked. Our disillusionment with process-product studies is what brought about the participationist-qualitative turn (cf. Silver, 2004). The question that must now be asked is, "Why do we have more confidence in this new type of research, the one that focuses on teacher practices?"

As remarked before, research can be conceptualized as a form of discourse that, if properly constructed, can lead to a reorganization of teacher practice so as to make it more effective. To illustrate this point, let us consider the following episode, in which 7th grade students are discussing the expression 15000-300w for calculating somebody's dwindling savings as a function of the number of weeks (w) during which the money was regularly spent[9]:

[95] Teacher: Would anyone do anything differently? Martha?
[96] Martha: I'd do 15 000 minus brackets, 300 and number of weeks …
[writes: 15 000-(300w)].
[100] Teacher:.. All right. Do we need brackets around this? [points to 300w]

8 www.comics.com/comics/sheldon

9 The episode is taken from a study by Carolyn Kieran and is described in (Sfard, 2000). For more details about the study see (Kieran, 1994) and (Sfard & Kieran, 2001).

[104] Simon: Yes, you do, because you have to know that there's an operation. A person, now, he'll probably think 300 weeks, not 300 times weeks.

[105] Teacher: OK, anyone who now knows algebra will know there is an operation.

The researchers who analyzed this scene concluded that algebraic expressions may have been initially read by the children as abbreviated colloquial sentences, in which letters, such as w, were a shorthand for nouns, such as weeks, rather than placeholders for numbers. The teacher was clearly unaware of the children's interpretation. In all likelihood, once she gets acquainted with the researchers' analysis, her teaching of introductory algebra will change.

The example shows how discursive habits become an obstacle to communication and how research could come to our rescue. The teacher could hardly be blamed for being a captive of her own discursive ways. While in the midst of intensive interaction with a group of children she could not allow herself the luxury of multiple interpretations. To set herself free from the discursive entrapment, the teacher needs a much more detached and relaxed glance at classroom communication – which is exactly what research is all about.

But the emancipatory power of research goes further than that. The established ways of communication also set well-defined limits to one's ability to interpret his or her own experience. The discursive exclusivity of the traditional classroom may be oppressive. Indeed, educational discourses tend to become dangerous if left unchallenged by additional ways of communicating and alternative narratives about the world. Their ostensible innocence, their reputation of being "just words", endows discourses with a great power to hurt students' feelings. Moreover, unquestioned ways of communicating may turn each one of us into oppressor even as we are acting with the best of intentions. Think, for example, about the way in which the teacher whom I just quoted divided the world into "those who know algebra" and those who don't, signaling the privileged position of the algebra knowers and de-legitimizing the children's query. With this casual, seemingly self-evident utterance, the teacher contributed to the vision of mathematics as a universal yardstick with which to measure, gauge, and compare people. This kind of use turns mathematics into a safeguard of the social order that, in its inner workings, rests heavily on a variety of splits and divides. This order would be in danger without the possibility of distinguishing the "mathematically knowledgeable" from the "mathematically deprived". Once again, the power of educational research lies in its being the art of multiple interpretation. By making clear that there are many narratives to be told about any given instance of educational practice, this research loosens the oppressive grip of old discursive habits and sets us free to consider new options. The next question to ask is how close we have come to attaining this worthy goal.

5.2 Do we have an impact?

5.2.1 Actor's voice

On the basis of the responses to our last question, I can say that although there is a measure of optimism about research that makes a difference – only 8% said they do not believe their work had any impact at all – there is also little confidence in the possibility of a decisive, far reaching influence. Even the most upbeat tones are cautious. Those who declare that their work did have an impact (45%) use qualifiers such as *some,* *certain, little, limited*. Others say that while five years is not enough to let an educational innovation take root, they are optimistic, if also a bit leery, about the future.

Not surprisingly, nearly 2/3 of the reported impact is in the domain of teacher practice. Approximately 1/4 of those who claim to have had an influence speak about changes in curriculum and policy. A few respondents mention their contribution to research, and only two people conjecture that their work had a certain impact on the issues of equity and social justice. Whenever impact is mentioned, it is understood that the change is in a desirable direction and nobody seems to consider the possibility of unintended harm.

5.2.2 Observer's voice

Lately, there has been a sharp increase in studies that feature the word "impact" or "relationship" in their title – and the present survey is a representative example. Probably, in response to the often unsatisfactory results of international achievement assessments and to the subsequent criticism toward all those who are held responsible, there is the easily understandable wish to exhibit some solid, uncontestable evidence for a positive causal relation between the investment and what can count as its outcome.

Although widely spread, this wish may also seem somehow unrealistic. The complexity of the educational machinery precludes the possibility of identifying clear-cut cause-effect relationships. The difficulty with telling the impact does not imply, however, its non-existence. As stated by a group of social scientists reflecting on their own work, "It would be quite irresponsible to deny the real effects of research in our disciplines", (Cameron et al., 1992/1997, p. 142) and especially those that were neither intended nor envisioned by the researcher. While anything we do is bound to have some effect, the real question is whether this effect is for better or for worse. Yet another question is, "Who is to tell?" This leads me to our last query about impact which, I wish to argue, though not the same, may have a similar answer.

5.3 Can we tell or foretell the impact of research?

5.3.1 Actor's voice

There is a consensus among the survey participants that the answer is closer to NO than to YES. They all stress the difficulty stemming from the fact that the influence of research is never direct, whereas some deny the very possibility of telling the impact.

5.3.2 Observer's voice

The first thing to stress is that the current rapprochement between the researcher and the teacher means, among other things, that the impact is mutual rather than one-way: that is, there are cycles of research that observes practice, practice that feeds back and inspires new research and, eventually, research that returns to practice as a modifying agent. Due to the nature of our survey, however, let me focus on the research-to-practice direction. As an observer, but also a participant, I share the position of the more extreme among our respondents and claim that while the existence of our impact is unquestionable, evaluating this impact or controlling it, for that matter, is almost as difficult as trying to predict or to tame the effect of the Hawaiian butterfly on the weather in Boston. Let me list some possible reasons for this situation.

First, the researcher's message must travel through a long chain of mediating factors before it reaches its ultimate end, the student. Even the teacher rarely receives the message directly from the researcher. For one thing, say both Aline Robert (2004) and Susanne Wilson (2003), teachers do not read research reports: They are too busy with everyday chores, and even if they weren't, they would probably be put off by the specialized language, not to say jargon, in which research reports are usually written. Teacher education programs, which could bring teachers and researchers together, are few and far between.

The researcher's message usually comes to the teacher in the form of a policy document, a textbook or an external examination. All these rarely present the rationale for what is suggested and, more often than not, do not reflect the overall spirit of the researcher's advice. In the "broken telephone" exchange of successive re-interpretations the original message is often lost and the practical implications may have little to do with what the researcher had in mind. A good example is our current exaggerated reliance on children's own mathematical inventions – the instructional idea inspired by the Piagetian claim that "children build their own knowledge". The interpreters overlooked the fact that, according to Piaget, learning is one's own construction whatever the teaching method.

The most consequential distortion in the researcher's message is inflicted by mediating factors that are not mere passive transmitters, but active agents who have their own vested interests. Thus, when a government overtakes the role of educational policy-maker, even the direct encounter between the researcher and the teacher may become subject to regulation. One of the survey participants reminds us, in this context, that politicians tend to "devalue research that does not have immediate, obvious classroom implications". Textbooks written with an eye to financial gain are another factor likely to counteract the researchers' message. Assessors and testers, whose voices these days sound stronger than ever, impose their own curricula. Faced with the assessment frenzy, one begins to suspect that rather than measuring what we believe important, we consider as important what is being measured. Finally, students' own agenda may sometimes override researchers' proposals, forcing the teacher into a discourse quite different from the one she had in mind while entering the classroom. Among the main issues at stake in this context are certain widely accepted norms and values that do not necessarily agree with what the researcher considers necessary for successful learning.

To counter-balance this long message about the bumpy road from research to practice, let me now observe that, imperceptibly, the researcher's message is also traveling on its own. "Any utterance... reveals to us... words of others," says Bakhtin (1986/1999, p. 131), meaning that discourses penetrate other discourses whether we want them to or not. Through the process of communicational osmosis, the researcher's words are likely to make their way into other discourses. Perhaps this is what one of our respondents had in mind when he said, "changes in education occur by 'stealth'". This means that research, like revolutions, may change the world even when officially silenced. But this also means that our responsibility as researchers may be greater than we think.

6. Looking back critically and ahead with hope

This is the time to try to answer our initial questions. So far, I have played the ventriloquist for actors and observers. In concluding this report, I wish to become myself again and will thus switch, more fully, to the first person singular. In this way, I will be able to share with you the personal lesson that I, as a researcher, have learned from our survey. It will be up to you to decide whether this has been your lesson too.

The first thing I wish to say is that I am pleased to find out that the last few years have been *the era of the teacher* as the almost uncontested focus of researchers' attention. This is quite a change with respect to the last two decades of the 20th century which were almost exclusively *the era of the learner*. And we have certainly come a long way since *the era of the curriculum*, roughly corresponding to the 1960s and 1970s when the main players in the educational game were the developer and the textbook. I consider the re-conceptualization of the relationship between the teacher and the researcher a big leap toward research that plays a genuine role in shaping and improving practice.

Secondly, I was not surprised by the finding that, as researchers, we are not communicating well either among ourselves or with other communities, notably those of practitioners and policy-makers. In my professional life, this familiar phenomenon is a source of much frustration. The principal culprit, I suspect, is a certain abuse of the important principle of tolerance toward discursive diversity. Although I have argued for the plurality of outlooks myself, I am also aware that this principle may sometimes be misinterpreted as a license for doing one's own thing without regard for the work of others. This may well be the main reason why educational research does not count as highly potent. Indeed, no cathedral can be built by people who do not understand one another. Let me immediately add that the concern about the effectiveness of communication does not imply the request for a full discursive uniformity. Personally, I interpret it as the need for "conceptual accountability" – the need for being explicit about the ways in which I use words and about how these uses relate to those of others. And if the words are to serve me in research rather than in poetry writing, it would be better if they were defined operationally, so as to make sure that those to whom I speak know how to identify the phenomena I refer to. For this advice to be workable, I feel I need to oppose the trend of 'corporatization' of academia – of remolding it in the image of profit-oriented business organization, and above all, to bar the corporate interpretation of the term "time-on-task" – the idea that any task may be implemented in no time, provided many people give this task their passing attention.

Thirdly, my work, like that of the majority of the survey participants, is participationist and qualitative, and this means that rather than trying to arrive at a mechanistic view of "what works in the classroom", I focus on how things work and try to make myself aware of alternative possibilities. I am also wary of the other kind of research, the one that aspires to tell what works in the classroom and relies too heavily on the power of numbers. Only too often does this type of research seem to honor the principle, "Take care of measurement and the question of what is being measured will take care of itself". In the eyes of a politician, measurement is full of an irresistible appeal: When research results come disguised as numbers, decision-making becomes simple and the decisions themselves appear externally imposed rather than man-made. And yet, the putative scientific reliability of the purely quantitative research is a dangerous illusion: Numerical results, with their reputation of "objective truths," gloss over individual differences, leading to potentially harmful interpretations. Indeed, interpreting quantitative research unassisted by a qualitative outlook is a highly implausible mission – a fact that no politician seems to care about.

Finally, while claiming the impossibility to control or measure the impact of our research, I also claimed that this impact may be greater than we think simply because research discourses have the tendency to infiltrate all the others. This means that our work is consequential not just to the mathematics classroom, but also to society at large.

I conclude that if I am not alert and open-minded enough to oppose some time-honored, never-questioned norms, I may inadvertently spoil more than I improve. Thus, for example, my research may be helping in perpetuating the widespread practice of using mathematics as a gatekeeper and a tool for exclusion. To bar this abuse, I try to combine a continued struggle against mathematical failure with an ongoing protest against measuring people's "quality" according to their achievements in mathematics.

As researchers, we are producing just words. And yet, words are more than sounds. People do things with words, and sometimes what is being done is wrong. When the latter happens, it does not help to say that we had little influence on what was done with our words or that we were unaware of these words' possible misuses. The responsibility for our words and for what is done with them, I believe, is always ours.

Acknowledgment
My thanks go to those who are, in practice, co-authors of this talk (but who are free from responsibility for its shortcomings), Yoshihiko Hashimoto, Gelsa Knijnik, Aline Robert, and Ole Skovsmose, the members of ICME-10 Survey Team 1; to the colleagues who answered our questionnaire; to Jagdish Madnani, who helped in analyzing the survey responses; and to the first readers of the manuscript, who provided much helpful feedback: Leon Sfard, Nathalie Sinclair, Joan Ferrini-Mundy, Ole Skovsmose and Kenneth Ruthven.

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P 4: Mathematics education and learning sciences

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Introduction

The aim of this chapter is to analyse how some general findings of learning research and educational sciences can contribute to the development of mathematics education. Teaching in different domains is a complex (cultural) activity and it requires many kinds of background knowledge. It is obvious that successful teaching in any domain is necessarily based on sufficient expertise and content knowledge in this domain. We also know that there is a rich tradition of teaching conventions and pedagogical practices which are transferred to new generations of teachers through discussions with older colleagues and through participation in pedagogical communities. It is not always clear, particularly for practitioners, what the role of more general research-based knowledge of learning and teaching processes could be.

The question of what kind of knowledge is adequate for making decisions about educational systems or designing learning environments has become more important over the last few years. Current discussion about the impact of research is related to attempts to develop evidence-based practices and policies. This discussion started in medicine but is gradually extending to other fields of society including education. In some countries the main pressure towards an evidence-based approach comes from policy makers and administrators. A well-known example of this kind of initiative is the "No child left behind Act" by the US government. This document has aroused a lot of criticism among educational researchers because it defines the criteria of useful knowledge in a very narrow way. Similar policy-driven discussion is taking place in many European countries as well.

Generally speaking, the evidence-based approach is a very attractive idea because it emphasises the use of the best possible information in developing important practical activities and in decisions about the systems which surround these practices. The problem is that these political programmes emphasising evidence-based approaches have very narrow and insufficiently elaborated ideas of what counts for evidence. On the other hand, it is difficult to believe that complex cultural activities such as education can ever be completely organised according to evidence-based policy and practice. Thus the wording "evidence-informed" activity or policy might be more realistic. There is a need for more elaborate analysis of the types of evidence and knowledge which teaching and educational systems can be based on. (e.g. Shavelson & Towne, 2002; Burkhardt & Schoenfeld, 2003)

What kind of knowledge do we need in evaluating and developing mathematics education? The main source of knowledge used in mathematics education originates from the scientific tradition of mathematics, which has been considered the fundamental basis for decisions on the content of teaching. But even this foundation for mathematics education is neither simple nor self-evident. There are many branches of mathematics and it has been interesting to see that among the mathematicians there is a continuing controversy over the content which should be included in the curriculum on different levels of education.

A second form of knowledge base which is very influential is the informal tradition of mathematics education; a kind of craft knowledge of teaching mathematics. There is a great deal of informal and tacit knowledge about good teaching and learning that teachers share with each other and also partly with their students. This informal and tacit knowledge is important for everyday practices but it can also be problematic if we aim to develop evidence-informed practice. It is obvious that teachers need informal craft knowledge about conventions and local practices when coping with the complex and varying problems they face in the classroom. On the other hand, we know that this informal knowledge is sometimes based on incorrect beliefs and tends to lead to erroneous or superficial teaching and learning processes. Alan Schoenfeld (1985) has reported many fancy beliefs students have about mathematics and mathematics learning. It is obvious that at least a part of students' harmful beliefs are emerging from these informal teaching conventions (Staub & Stern, 2002).

In spite of these limitations in craft knowledge about teaching it is important to ask to what extent can teaching be based on the scientific knowledge base and how much do we have to rely on teaching conventions and craft knowledge. In any practice, even in evidence based medicine, conventional craft knowledge plays an important role. However, it is important to have a better understanding of the fundamental beliefs dominating informal and tacit craft knowledge and how these are related to and can be integrated with the scientific knowledge of teaching and learning processes. The aim of this chapter is to deal with the role of research-based knowledge in developing mathematics education.

Scholarly content knowledge and general learning theory

The mathematics curriculum and the sequencing of teaching and student assignments are typically organised according to the knowledge base of the established scholarly mathematics tradition. In some cases, university mathematicians are directly involved in curriculum development and text book writing. Often the scholarly mathematics tradition is mediated to the schools through teachers who have studied mathematics but do not, strictly speaking, belong to the academic mathematics community. In both cases, however, some fundamental beliefs of the scholarly community about mathematics teaching and learning are mediated to schools (Lehtinen, Merenluoto & Kasanen, 1997; Nathan & Koedinger, 2000). One of the self-evident beliefs is that teachers' good content knowledge in mathematics leads to good teaching. Another example is the belief that mathematics learning depends on more or less stable mathematical abilities and learning difficulties are caused by the lack of capabilities. In addition, the community tends to believe that the hierarchical mathematical knowledge structure defined by the scholarly mathematics community provides school students with an optimal learning trajectory without any discontinuities (Greer, 2004).

These beliefs are not formally documented but can be seen in the ways in which the curriculum and teaching materials are organised. They are also visible in teaching practices and widely spread attitudes about mathematics teaching and learning. It means that the structure of mathematical knowledge as defined in the scholarly mathematical tradition is taken as an inherent and self-evident basis for organising teaching and the curriculum. In other words, the expert view predominates and students' perspectives are seldom taken into account in developing the content and order of the curriculum and text books (Lehtinen, 1984; Ritter, Nerb, Lehtinen & O'Shea, 2007).

The other research-based knowledge base for pedagogical practices in the mathematics classroom consists of the findings of general domain-independent research on learning, motivation and social processes. Many theoretical ideas applied in mathematics education have not been originally developed in this field but have been imported from other fields or from seemingly "content-independent" learning research.

Pedagogical practices based on empirical theories of learning and behaviour modification (association psychology, behaviourism) have been highly influential in mathematics education during the entire twentieth century and they still dominate many of the informal and tacit practices. During the last few decades, cognitive, constructivist and situated approaches have been widely accepted as desirable bases for developing school teaching and learning environments (e.g. Bransford, Brown & Cocking, 2000; Greeno & Goldman, 1998).

In many cases, the different approaches to learning have been mainly applied in mathematics classrooms as domain-general principles. In the spirit of these theories, general pedagogical practices such as problem solving, inquiry learning, authentic tasks and collaborative learning have been emphasised. The common idea in all these ways to apply general theories of learning and behaviour modification is that there are general design principles of learning environments and teaching practices which can be applied for teaching in any content.

It has been typical of traditional mathematics education to see content knowledge and knowledge on teaching and learning as different aspects which can be used separately in planning teaching practices. There are differences between the different levels of mathematics education in emphasising the source of the knowledge as a basis for teaching. Pre- and primary level educators often emphasise the general learning theories and pedagogical approaches and see the mathematical content as a set of given facts and operations without any domain-specific characteristics which would have an influence on learning processes. In secondary education, the scholarly mathematical knowledge base often predominates and general instructional design principles based on learning research are only used if they seem to fit with the content. In traditional mathematics instruction in higher education, only domain-knowledge matters and principles of teaching and learning are seldom consciously taken into account. For example, university level students' learning difficulties are almost entirely attributed to their lack of effort or ability.

Problems of the overemphasis on general theories of learning and instruction

As argued above, it is typical for primary level education to overemphasise general theories of learning and neglect the learning requirements of the domain-specific features of mathematical knowledge. My favourite example of this kind of problem originates from our early studies on learning arithmetic algorithms in primary schools (Lehtinen, 1986). These findings are based on our classroom studies in the late seventies and early eighties.

We followed classroom teaching and learning with very experienced and successful primary school teachers. These teachers were very skilful in applying behaviourism-based training practices in teaching arithmetic procedures such as paper and pencil addition of multi-digit numbers:

$$\begin{array}{r} {}^{1} \\ 347 \\ \underline{+234} \\ 581 \end{array}$$

When we observed the students carrying out addition tasks we realised that they learned very quickly to perform the standard procedures including the use of carrying and made practically no errors.

Accidentally, we happened to give them a slightly different task with three numbers:

$$\begin{array}{r} 5\ 7\ 8 \\ 1\ 8\ 7 \\ \underline{+\ 3\ 5\ 6} \end{array}$$

When solving this type of tasks some students applied an interesting procedure:

$$\begin{array}{r} {}^{1\ 1} \\ 5\ 7\ 8 \\ 1\ 8\ 7 \\ \underline{+\ 3\ 5\ 6} \\ 1\ 0\ 2\ 2 \end{array}$$

They calculated 8 plus 7 plus 6 is 21, wrote 2 under the line and carried 1. Then they continued 1 plus 7 is 8, 8 plus 8 is 16 and 16 plus 5 is 21 and again wrote 2 under the line and carried 1.

When we asked the students to repeat the tasks and interviewed them simultaneously, they told that they did what had been taught in school. They told that the rule they followed was that if the sum of a column is ten or greater, the digit one is always carried. Interestingly, students used explanations which resembled so-called production rules in cognitive architecture (Anderson & Lebiere, 1998). Most of the training tasks the teachers had used consisted only of two numbers and for all these tasks the procedure that students had learned was correct, though not adequately justified.

Afterwards, we interviewed several students when they were applying addition and multiplication algorithms and found several rules of thumb for carrying which were based on the *ad hoc* generalisations they had made during the drill and practice experiences but not on the integration of calculation algorithms with the decimal system. The message here is not that it is exactly these kinds of procedures young students typically learn. The main point is that in various drill and practice models without deeper mathematical reflection, the seemingly good learning results can overshadow serious incorrect learning. We have called these kinds of learning outcomes "computation without mathematics" (Lehtinen, 1986).

The above described problems are not typical only for mechanistic procedural skill training based on behaviourist ideas. Many learning experiments based on constructivist principles have resulted in similar problems. For example, learning models making use of authentic problems and discovery type processes might be motivating for students but the learning outcomes easily remain on a superficial level if there is no adequate domain-specific expert guidance (Mayer, 2004).

Knowledge about theories of learning and different pedagogical models is important for mathematics educators. However, it seems that general learning principles and teaching approaches are not enough for high level mathematics education if they are used independently from adequate content knowledge. (Kunter, et al., 2007).

Why mere content expertise of teachers is not the solution?

Traditionally we tend to assume that high content knowledge in mathematics by itself results in better abilities to guide students' learning processes. However, current research on so-called experts' "blind spots" refers to possible problems with this view (Nathan, 2003; Nathan & Koedinger, 2000).

Normally we think that expertise always facilitates performance, and this is true for typical activities that experts are to perform. However, many studies on expertise indicate that high level expertise also creates certain limitations. Think-aloud reports from experts and novices show that experts are less likely to have access to memory traces of their cognitive processes when engaged in tasks within their domain of expertise.

This also means that they have more limited opportunities to understand the problems which less advanced persons have when trying to learn new skills or solve novel problems in the area (Nathan, 2003). There might be several reasons for this. Studies in cognitive psychology have shown that automation of cognitive processes makes it difficult to be aware of all the sub-processes of the activity. It is also obvious in many fields that experts have experienced irreversible learning which may have changed the whole interpretation framework they use when dealing with fundamental issues in their field. This means that they have mentally located these issues into different ontological categories than novices. For example, the scientific concept of electricity is ontologically very different from the everyday concepts of material (e.g. flow of water) (Chi, Slotta, & de Leeuw, 1994).

Expertise can not be defined in individual terms only, and professionals' thinking and activity are highly dependent on the larger expert culture they belong to (Gruber, Palonen, Rehrl, & Lehtinen, 2007). As members of expert cultures they share certain beliefs and values which mean that certain things are self-evident and certain questions, typical for novices, are no longer asked.

Empirical studies and historical examples show that well-developed subject-matter knowledge in mathematics can lead curriculum developers and teachers to inaccurately predict students' learning difficulties. The most powerful examples of this problem are the failed curriculum reforms such as the so-called *New Math* movement in the last half of twentieth century. Based on different expert statements we can conclude that the New Math curriculum failed because the mathematicians who designed the new curriculum did not know enough about children's learning and socialising in their cultural context and about teachers' abilities to organise and support the desired learning processes in their classrooms. The high but narrow expertise in mathematics had made them blind to the struggles experienced by teachers and students (e.g. Nathan, 2003).

Nathan and his colleagues carried out a series of empirical studies in which they investigated how well teachers with different levels of mathematical content knowledge are able to anticipate students' difficulties with different tasks. According to their results, teachers with more post-secondary mathematics education (high school teachers) were less able to predict students' learning difficulties than teachers with less mathematics education (middle school teachers). For example: "High school teachers responded that symbolic problems (from a set of given problems) would be easiest for students because they were written in 'pure math'."

On the basis of their results Nathan and Koedinger (2000) concluded that teachers with advanced subject-matter knowledge of scholarly mathematics tend to use the powerful organising principles, formalisms, and methods of analysis that serve as the foundation of the discipline as guiding principles for their students' conceptual development and instruction, rather than being guided by knowledge of the learning needs and developmental profiles of novices.

Based on the studies described above, and on somewhat similar findings elsewhere, we can conclude that experts' (curriculum developers, teachers with high domain expertise) and students' views about mathematics curriculum and teaching are fundamentally different in terms of interpretation frameworks, the sequences of content units and aims of the different assignments (Figure 1).

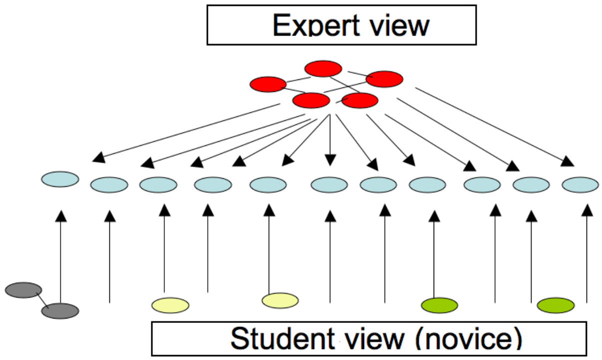

Figure 1. Curriculum content from the expert's and novice's perspectives

Experts who have planned the curriculum or the teaching processes according to the curriculum

- interpret the different concepts in a larger conceptual framework which makes them meaningful and logical,
- see the curriculum as a set of interconnecting elements,
- see the meaning of each assignment for long term learning aims, and
- interpret the curriculum as a gradually and continuously deepening and extending body of knowledge.

From a student's point of view the situation appears very different. For them there is

- a totally absent or only weakly developed conceptual framework for making sense of the new elements and concepts,
- a set of more or less isolated facts and activities with very little coherence,
- too little information about the long term aims of different assignments,
- a lacking continuity of different knowledge elements, and
- knowledge which violates his or her beliefs and is in conflict with the prior knowledge.

It is obvious that the fundamental differences between teachers' (expert) and students' (novices) perspectives in mathematics education are not adequately dealt with in the conventional teaching practices, and are still insufficiently taken into account in research on mathematics teaching and learning. My claim is that these different views can be treated neither with the help of pure content knowledge and domain specific craft knowledge, nor with general learning theories or didactic principles. High level domain expertise is important, but in addition to that teachers need "pedagogical content knowledge" which helps them understand the "novice perspective" of students and the different trajectories from initial ideas to more advanced knowledge and skills. This means that there is a need for learning sciences which take the domain specificity of learning processes seriously.

Conceptual change approach as an example of a domain-sensitive theory

In many fields of learning research we can find examples of studies which try to deal with the domain-specific learning processes. For example, studies on students' beliefs related to mathematics education, mathematics specific motivation and discourse processes in mathematics learning have opened up a much more detailed view into the personal and social factors affecting mathematics education than the results based on domain-independent studies. It is not possible to present findings of all these research areas within this paper. I focus only on one emerging domain-specific research area, conceptual change in learning mathematics, which has been very important in our own research during the last few years (Lehtinen, Merenluoto & Kasanen, 1997; Merenluoto & Lehtinen, 2002: 2004, 2006; Merenluoto & Palonen, 2007)

During the last decades, cognitive, developmental and educational research has produced a rich variety of models of the development of students' conceptual understanding and conceptual learning (Carey, 1985; Chi, Slotta & DeLeeuw, 1994; Duit, 1999; Hatano & Inagaki, 1998; Vosniadou, 1994; 1999, 2007). The notion of "conceptual change" refers to a situation where learners' prior knowledge is incompatible with the new knowledge and learners are prone to construct systematic errors or misconceptions. It can be interpreted that prior knowledge interferes with the acquisition of the new concept.

It has proved useful to make a distinction between two different qualities of learning: a continuous growth and discontinuous change. (e.g. Schnotz, Vosniadou & Carretero, 1999; Vosniadou, 1994). The easier level of learning based on continuous growth is often called *enrichment of* existing knowledge structures. More demanding is a situation which is characterized by discontinuity of learning; prior knowledge is incompatible with the new information and *radical revision* of existing knowledge structures is needed.

The results of several empirical studies show that in cases where radical revision is needed the prior knowledge of students is often resistant to teaching attempts. In a typical teaching situation, neither teachers nor students are aware of the nature of students' previously acquired knowledge and how it contradicts with the scientific knowledge delivered by the teacher. If teachers and students do not see or understand any reasons for radical change of previous knowledge and beliefs, they tend to focus on enriching prior representations rather than revising them (Duit, Roth, Komorek & Wilbers, 2001; Guzzetti, Snyder, Glass & Gamas, 1993; Vosniadou, 1999).

The conceptual change approach to learning has its roots in the philosophy and history of science and it has been initially applied to predict and explain students' misconceptions mainly in science learning. It is an important but still somewhat controversial question whether this framework can be applied fruitfully to mathematics learning. Many mathematicians argued that changes in mathematics can not be described in the same terms as in science. In her article about the implications of conceptual change analysis for mathematics curriculum, Jere Confrey (1981) argues that the selection and presentation of curriculum units depends on the (implicit) theories of knowledge. She describes three different approaches to knowledge which have influenced science and mathematics education: absolutism, progressive absolutism, and conceptual change.

Confrey refers to the criticism of Lakatos (1976) and Tulmin (1972) and claims that most people conceive mathematics as absolutist. According to that view, "concepts in mathematics do not develop, they are discovered, and the impression is given that their structure is unchanging." The absolutist view considers mathematical knowledge as a linear and hierarchical accumulation of truths; new inventions or discoveries are not treated as changes but as enlargements in which the previous ones are included as substructures. (Boyer, 1994/1949; Confrey, 1981; Merenluoto & Palonen, 2007). The absolutist idea of mathematical knowledge differentiates it fundamentally from knowledge in science. A logical consequence of the absolutist view is that the learning of mathematics follows this hierarchical accumulation and possible learning problems are the results of a lack of abilities or complexity of the concepts, rather than discontinuities in the conceptual system.

The very nature of mathematics is the attempt to develop a coherent knowledge base which is free from conflicts. In scholarly mathematics this is considered a fundamental guideline for research, and researchers are aware of the still existing conflicts and that the long and difficult history of mathematics includes revolutionary changes (Greer, 2004). Toulmin (1972) pointed out that the history of mathematics is not a hierarchical accumulation of new truths, but rather that "such fundamental concepts as 'validity' and 'rigour', 'elegance' and 'proof', and 'mathematical necessity' undergo the same sea-changes as their scientific counterparts 'soundness,' cogency,' and 'simplicity,' 'relevance,' and 'physical necessity.' Even the basic standards of 'mathematical proof' have themselves been reappraised more than once since Euclid's time."

The mathematical knowledge currently taught in schools has taken millennia to develop, and the development has been far from linear or smooth; on the contrary, there has been a background of discussions and controversies before certain novel constructs were accepted (Lehtinen, Merenluoto, & Kasanen, 1997; Merenluoto & Palonen 2007; Tirosh & Tsamir, 2006). However, in educational contexts, mathematical content is often considered to form a hierarchical structure where all the new concepts logically follow prior ones. A typical assumption in the mathematics curriculum is that this hierarchical nature of the content allows students to enrich their knowledge step by step.

On the basis of the studies carried out in science learning it is plausible to assume that in mathematics, as well, many concepts which are unproblematic parts of the coherent knowledge base from the teacher's (and expert's) point of view tend to appear as counter-intuitive from the learner's perspective. For example, many studies have shown that students who are skilled in multiplication with natural numbers have difficulties multiplying by rational numbers and particularly by numbers which are smaller than one. This violates their prior knowledge about repeated additions and beliefs that after multiplication the result should be larger than the multiplicand. (De Corte & Verschaffel, 1996).

In their experimental study Asmuth and Rips (2006) studied the learning of non-Euclidean geometry. They assumed that because students expect mathematical knowledge to be unchanging, it can be especially difficult for them to encounter advanced mathematical topics that force them to reconceive existing knowledge. From the prior knowledge point of view hyperbolic geometry, a form of non-Euclidean geometry, is an interesting topic because of its conceptual similarities and dissimilarities to Euclidean geometry. While many geometric theorems are true in both Euclidean and hyperbolic geometries, others change dramatically. They found that more complex training material applying closed figures instead of mere lines helped participants deal with the partly counter-intuitive tasks and carry out conceptual change.

There are several studies on students' conceptual change problems when they are learning extensions of the number concept (e.g. Merenluoto & Lehtinen, 2002; 2006; Prediger, in press; Vamvakoussi & Vosniadou, 2007). The beliefs related to natural numbers strongly dominates students' prior knowledge when they start learning rational numbers and, later, real numbers. However, many features of the new number system (e.g. rational numbers) seriously violate these very fundamental beliefs of the nature of numbers (Figure 2).

Numerical Value	Natural Number	Rational Number
– Order	– Supported by the natural numbers' sequence – Existence of a successor or a preceding number – No number between two successive numbers in the numberline	– Not directly supported by the natural numbers' sequence – There is no unique successor or a unique preceding number – Infinity numbers between
Relation to Concrete Ojects	– Numbers can be directly related to the objects in a given set	– No direct relation between the number and the objects in a set

Figure 2. Some differences between natural and rational numbers which cause conceptual change problems in students

Even more radical change is needed when students "clash" with the notion of real numbers in their first calculus courses (Lehtinen, Merenluoto & Kasanen, 1997; Merenluoto & Palonen, 2007).

There are many reasons why the intuitive idea of natural numbers so strongly predominates our thinking. Researchers of early childhood development widely agree that our ability to deal with numerosity is based on innate predispositions which make it possible to recognise and distinguish small quantities from very early on (Feigenson, Dehaene,. & Spelke, 2004; Lehtinen & Hannula, 2006). In addition, the large amount of practical experience with the number of objects and the formal learning of early school years strengthen the idea of positive whole numbers as a prototype for numbers (see Ginsburg, Balfanz, & Greenes, 2000; Ginsburg, Inoue, & Seo, 1999; Hannula, Mattinen, & Lehtinen, 2005).

Our own studies indicate (Merenluoto & Lehtinen, 2004) that the conceptual change process needed in extending the number concept is not a mere cognitive process but also a complex emotional, motivational and metacognitive challenge. In the process of conceptual change, students are forced to tolerate the ambiguity that comes from newly learned operations and characteristics of objects while they do not yet fully understand the concepts. The empirical results indicate that this kind of sensitivity and tolerance is significantly related to students' achievement level in mathematics.

Educational consequences

It has been demonstrated above that correct teaching from the point of view of scholarly mathematics or generally powerful learning environment designs are not always sufficient for supporting higher order learning. Teachers might have "blind spots" due to the lack of content expertise or insufficient understanding of students' perspectives and learning processes. How to avoid teachers' blind spots in teaching mathematics?

Without a deep understanding of the domain and simultaneous understanding of the functioning of cognitive processes, it is impossible to recognise these topics which are systematically resistant to teaching. It is important that mathematics teachers are informed about the mathematics-specific findings of learning research. Much more detailed information is needed about the specific topics in mathematics which might be particularly difficult due to the fact that they conflict with the knowledge and beliefs students typically have.

Some attempts have been made to develop instructional designs and learning environments which would optimally support student mathematics learning when more radical conceptual change in needed (e.g. Merenluoto, 2006; Prediger, in press). In the science education tradition, cognitive conflict has been emphasised as a powerful tool to support conceptual learning. Many studies, however, show that conceptual conflict experience might be necessary but not sufficient for adequate learning in situations which require radical restructuring of prior knowledge and beliefs. More elaborated metaconceptual awareness about the conflicting nature of prior knowledge and new concepts is often needed. For example, in the case of extension of the number concept, at least some knowledge of the history of mathematics could be helpful. This could help students understand why more abstract number concepts are needed, and what kind of problems the mathematical community has experienced when developing these concepts.

It is also obvious that cognitive support for conceptual change is not enough. Learning environments should also encourage students' sensitivity to novelty and conflicting information and give support for the emotional coping with ambiguity.

Conceptual change research is only one example of the emerging domain-specific research approaches which combine the content knowledge and general approaches of learning sciences from the very beginning of the research process. The approach I have described here highlights the cognitive and individual aspects of learning. The knowledge base we need for improving mathematics education, however, also requires other levels of analysis dealing with the social and cultural aspects of learning mathematics.

I believe that the integration of content knowledge and learning science will result in better accumulation of theoretical and empirical knowledge and help us create the knowledge base which is needed for evidence-informed practice and policy in mathematics education.

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P 5: The Plenary Interview Session

Moderator: ***Michéle Artigue***, University of Paris 7, France
Interviewees: ***Ubiratan D'Ambrosio***, Pontific Catholic University of São Paolo, Brazil
Gila Hanna, University of Toronto, Canada
Jeremy Kilpatrick, University of Georgia, USA
Gérard Vergnaud, University of Paris 8, France

The aims and structure of the session

The interview session was a new format introduced at ICME-10. It aimed at reflecting on the development of the field of mathematics education by asking people who had long been among the main contributors to this development to share their personal experiences and thoughts. It was expected that this session, intertwining in an informal way personal experiences and reflections, would offer to the congress participants, and especially to the youngest ones, a more intimate, rich, and insightful vision of the field than the reading of academic texts, irrespective of their value, could possibly provide.

Four distinguished scholars in the field had been invited by the International Programme Committee to take part in this session. They had some common characteristics: Their commitment to mathematics education had been the essential dimension of their professional life, and they had made outstanding contributions to the field. But they were also very different: They came from different countries and different cultures, and they had approached the field with different foci and interests. Thus they have had rich but different educational experiences and were able to reflect the intrinsic diversity of the field. These scholars were interviewed by Michèle Artigue, professor in the mathematics department of the University of Paris 7 and one of the ICMI vice-presidents at the time of ICME-10.

After a brief presentation of the four interviewees, the interview session was divided into three main parts. In the first part, the interviewees were asked to tell about their respective entrances into the field of mathematics education and to give the audience a flavour of the field in its early stages. The second part of the session was devoted to a reflective analysis on the development of the field, the difficulties and obstacles that had to be overcome, the major advances that were reached and how that occurred, and the initial ambitions that could not be fulfilled. In the third and final part, the four interviewees were asked to give their personal visions of the future of the field. In this report, we respect this global structure of the session while giving the opportunity to the four interviewees to add some retrospective comments at the end.

Introducing the four interviewees

In alphabetical order, these were Ubiratan D'Ambrosio from Brazil, Gila Hanna from Canada, Jeremy Kilpatrick from the USA, and Gérard Vergnaud from France. We briefly present them below as was done at the beginning of the session.

Ubiratan D'Ambrosio is from Brazil, where he is professor emeritus at the State University of Campinas (UNICAMP). He currently teaches at the PUC of São Paulo. First a research mathematician, he became progressively involved in the field of history of mathematics and mathematics education through his involvement in the Pugwash movement for nuclear disarmament. He was president of the Interamerican Committee of Mathematics Education from 1979 to 1987, and vice-president of ICMI from 1979 to 1983. His name is especially attached to the development of ethnomathematics and the sociocultural bases for mathematics education. In 2001, he was awarded the Kenneth O. May Medal by the International Commission on the History of Mathematics.

Gila Hanna is from Canada. She is professor emeritus at the Ontario Institute for Studies in Education of the University of Toronto. She holds a master's degree in mathematics and a doctorate in education. She has been convener of the International Organization of Women and Mathematics Education from 1988 to 1992, which is today an affiliated ICMI Study Group, vice-chair of the Canadian Mathematics Education Study Group from 1986 to 1990, co-editor of the journal *Educational Studies in Mathematics* from 1989 to 2000, co-founder and co-editor of the *Canadian Journal of Science, Mathematics and Technology Education* since 1999. She is particularly known for her contribution to research on the role of proof in mathematics education and on gender issues, and is a fellow of the Fields Institute for Research in Mathematical Sciences.

Jeremy Kilpatrick is from the USA and currently Regents Professor of Mathematics Education at the University of Georgia. He entered mathematics education in the 1950s, when he was still an undergraduate in mathematics at the University of California, preparing to become a mathematics teacher, and worked with Ed Begle and George Polya when a doctoral student at Stanford. He has been involved in ICMI activities for a long time, reporting on new trends in evaluation at ICME-3 in Karlsruhe in 1976, giving a plenary address at ICME-5 in Adelaide in 1984, in charge with A. Sierpinska of the ICMI Study on Research in Mathematics Education from 1992 to 1998, and also being vice-president of ICMI from 1991 to 1994. His research interests include teachers' proficiency, curriculum change and its history, assessment and the history of research in mathematics education.

Gérard Vergnaud first studied commerce and management before orientating towards psychology and preparing for a doctorate under Jean Piaget's supervision. He entered the field of mathematics education when the "new math" movement started and played a fundamental role in the building of bridges between the well-established field of psychology and the emerging field of mathematics education. In 1976, he was one of the founders of the International Group on the Psychology of Mathematics Education (PME), which is today an affiliated ICMI study group. He is especially known for the theory of conceptual fields he has progressively developed. During the last 12 years, he has also made fundamental contributions to the emergence of "professional didactics" and to the study of adults' competences at work.

The first phase of the interview: Entering the field of mathematics education

During this first phase, the four interviewees were interviewed in the following order: Jeremy Kilpatrick, Gérard Vergnaud, Gila Hanna, and Ubiratan d'Ambrosio. We summarize below the main questions posed to each of them by Michèle Artigue and their respective answers.

***Jeremy**, could you tell us when and why you entered the field of mathematics education?*

Fifty years ago last month, I was graduating from a two-year college in California. I had already decided to be a teacher but wasn't entirely sure what subject I would teach. I was transferring to the University of California at Berkeley and thought about majoring in psychology, but that was not a school subject. I had not had good mathematics teachers in secondary school, but I had had an excellent mathematics teacher in the two-year college and had taken all of the courses that he offered. When I went to Berkeley, I thought about majoring in English – which turned out to be my second subject (my minor) – but ended up majoring in mathematics. In other words, I decided on teaching first and then on mathematics. I went into teaching and taught mathematics for several years at a junior high school in Berkeley, getting a master's degree at the University of California and then going to Stanford to get another master's degree and then a doctorate.

What was your doctorate about?
My dissertation research concerned the problem-solving heuristics used by eighth-grade students. For my master's degree at Berkeley, I had studied what made various kinds of problems difficult, and I was interested in how students were solving the problems. So I interviewed students and asked them to think out loud as they solved problems. Then I tried to analyse their solutions using a framework based on the work of Polya. I had gone to Stanford to study with Polya, so that was a natural topic for me to choose.

What was your first ICMI experience, and what was the first ICME Congress you attended like?
My first ICME was the third Congress in Karlsruhe in 1976. At that time, UNESCO was supporting the development of a series, *New Trends in Mathematics Teaching*. I had been to a conference in Royaumont to produce materials for Volume IV, published in 1979. Alan Bishop and I had written the chapters on research and on evaluation. At Karlsruhe, I presented a "survey-of-trends" report on evaluation.

How many participants were there in this ICME?
It was not at all the size of ICME-10, but I don't recall the number.

Do you remember any of the plenary lecturers?
I remember Freudenthal's talk, and I remember a talk on false dichotomies by Peter Hilton.

And what were the main issues discussed?
I think there was a lot of attention to problem solving. Most of the research that was discussed was dealing with the learning of mathematics. As Anna Sfard said yesterday, the early research was looking primarily at students' learning, so there were topics dealing with the different subjects that students were learning. And computers were just appearing on the horizon, so there was some discussion of technology in mathematics education.

Gérard, *I read in your CV that you prepared a thesis under the supervision of Piaget, how did it happen and what was your doctoral research about?*
I started my scientific life as a psychologist and prepared a thesis, under the supervision of Piaget, on the problem solving activity of children from 4 to 10 years of age. The main tasks I used were of two different sorts: the unblocking of a system of bars blocked in one another, and the actions necessary to move from a permutation of four physical objects to another permutation of the same four objects.

The first task implies anti-symmetry of order relationships, and transitivity of the rules of action necessary to generate an adequate sequence of actions. Therefore my results could illustrate the development of implicit inferences and spontaneous algorithms, depending on the grasping of different level properties of the relationships involved. The situation with permutations allowed me to show, against behaviourism, that the concept of representation of goals and subgoals was necessary for giving an account of the organization of activity.

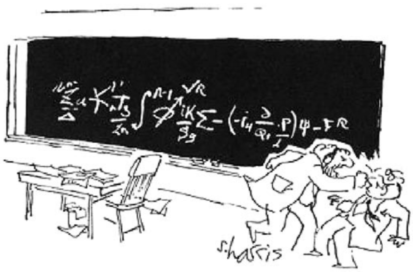
What made you move towards the field of mathematics education?
In 1967, a social and political circumstance changed my life, as I was asked to be a counsellor for teachers in a primary school when the school director wanted to introduce modern mathematics. I then attended many classes and helped teachers in choosing situations. I can add that the theory of conceptual fields was born at that time, even if it was only several years later that I could formalize it as a triplet of a set of situations, a set of operational invariants, and a set of symbolic and linguistic representations.

Together with Guy Brousseau, you are considered the father of the French school of didactics. When did you meet him and have the first opportunity to work with him?
I joined the small community of didacticians in France after a few years of research. Guy Brousseau and I were the oldest fellows of that community. There were also interesting and inventive researchers like Yves Chevallard, you Michèle, Colette Laborde, Claude Comiti, Nicolas Balacheff, André Rouchier, and many others. I was most impressed by the reflection and the experience of Guy Brousseau concerning didactic situations, and also by the creative work of Régine Douady. Then I could match the concept of 'scheme', which I had borrowed from Piaget, with the concept of 'class of situations'. I have already mentioned that the concept of situation had impressed me up to the point to consider that the theoretical couple of scheme and situation was the cornerstone of both psychology and didactics. If knowledge is adaptation (see Piaget for that principle), then it is important to recognize that it is schemes that adapt, and they adapt to situations. One of the professional competences of educators and researchers consists in reflecting on and inventing new situations to be offered to students in order to destabilize their former knowledge; we also need to identify at the same time the different classes of situations students are comfortable with. Stabilization and destabilization are essential in education.

And you ***Gila****, why did you enter this field, and how did it happen?*
I had been a teacher of mathematics and physics at the upper secondary level before deciding to pursue graduate studies in mathematics. After obtaining a master's degree in mathematics, majoring in set theory and logic, I did my doctoral studies in the field of education, and it was only natural that I would focus on mathematics education. I believe it was thanks to David Wheeler, though, that I became active in the community of mathematics educators. David Wheeler, who was then the president of the Canadian Mathematics Education Study Group (CMESG), invited me to join the group after reading my manuscript on proof. I am very grateful for that because at the time I was doing research on the role of proof in teaching mathematics, but I had no local colleagues who were working on that or on any other aspect of mathematics education. Thus the members of CMESG served as my main source of information on the state of mathema-

tics education at both the national and the international levels, and in the course of time many of them became very close colleagues. I must acknowledge that I found the CMESG to consist of people who were not only dedicated to mathematics education but were most collegial, most stimulating, most professional, and at the same time compassionate and fun to work with.

What was your first theme of research in that area and why?
It was the role of proof in mathematics education. As far as I could see, mathematical proof had not been given the attention it rightly deserved in mathematics education. I thought that research into the role and function of proof would help remedy this situation. In addition, I was very interested in the role of proof in mathematical practice, as well as in many other aspects of proof, relevant to education, such as philosophical, epistemological, heuristic, cognitive, discovery, and communication, including this one (see Sidney Harris cartoon).



"YOU WANT PROOF? I'LL GIVE YOU PROOF!"

Do you remember your first publication in mathematics education?
If I discount the publications on measurement and evaluation in education, I would say that my first publication in mathematics education was a book on the role of proof in mathematics education.

*And you **Ubiratan**, I have the impression that your personal experience was quite different. How did you become involved in this field, and what was your first decisive involvement?*
Indeed. I was born in 1932, and in 1949, guided by my father, who was a mathematics teacher, I was already giving private tutoring classes for adults preparing to enter public service (mainly tutoring financial mathematics).

In 1954, I graduated from the University of São Paulo with a major in mathematics. Even before graduation, I taught in elementary and high schools. I enjoyed it, and I believe I was doing well considering the reaction of my former students when, incidentally, I encounter some of them.

In 1958, I was hired as a full-time instructor and graduate student at the University of São Paulo and received my doctorate in 1963, with a thesis on calculus of variations and measure theory (very pure!). My thesis, in the context of geometric measure theory, was loaded with the historical background of this theory. Just after receiving my doctorate, in January 1964, I went to the USA as a research associate at Brown University on a one-year leave. But a couple of months later, the military coup in Brazil prompted my decision to stay in the USA, where I later became a tenured associate professor at the State University of New York at Buffalo. My first Ph.D. student wrote his thesis on stability of differential equations. My stay in Buffalo was very rich. Besides being a faculty member in a strong mathematics department, I benefited from having as colleagues – indeed mentors – pioneers of emerging fields from other departments. There come to my mind James F. Danielli [molecular biology], Charles Waddington and Sir John Eccles [sciences of the mind], David G. Hays [linguistics], Robert Rosen [biomathematics], Lejaren Hiller [computational music]. This is how my trans-disciplinary posture developed. During this time, my interest in education was occasional and minimal. But I became aware of the potential of the new technologies, particularly informatics, for education at all levels.

In late 1972, I returned to Brazil and became the Director of the Institute of Mathematics, Statistics and Computer Science of the State University of Campinas (UNICAMP), which grew as a major research institution. My first Brazilian doctoral student at UNICAMP wrote a thesis on measure theory and minimal surfaces.

Soon after engaging in academic life in Brazil, I recognized the cultural and social barriers in the schools that were responsible for the high rate of failure and dropping-out by children coming from marginalized communities, normally the first generation in their families with an opportunity for schooling. They could not compete with children coming from families with some schooling. I thus realized that mathematics education was a priority for Brazil. Thus motivated, I decided to enter the field. I was fortunate to have excellent funding provided by UNICAMP, and we were able to receive a host of visitors from all over the world. I eagerly learned from them the new directions in the field. I was curious about the historical evolution of these new directions, and this revived my interest in history of mathematics. I also realized that we had to move into the new technologies, particularly calculators and videos. In the early 1970s, this was very new in Brazil, particularly in education. The visitors were a strong support in my late involvement with mathematics education. I will not give the impressive list of visitors, but I must recognize the influence I received from Hassler Whitney, who visited us every year, for many years, for 1- to 2-month periods.

My definitive engagement came from my participation in the Second International Mathematics Study, and my first great challenge as a mathematics educator was to be in charge of Section B-3 on "Overall Goals and Objectives for Mathematical Education", at ICME-3 in Karlsruhe in 1976. I read many of the classics relevant to mathematics education in preparing the discussion paper.

The second phase of the interview: Reflecting on the development of the field

In this second phase, each of the interviewees was asked to reflect about her or his contribution to the field, and the vision she or he had of its major advances. At the end of this phase they were also given the possibility to react to what had been said. Once more, we summarize questions and answers.

***Gérard**, you are known as one of the persons who have especially contributed to the development of productive relationships between psychology as an established field and mathematics education. How did this happen?*

The creation of PME was crucial for the development of communication between psychologists and researchers in mathematics education. It took place after the Karlsruhe Congress in 1976. At the Congress, Fischbein had shown how psychologists could produce interesting results. After his lecture, we decided to meet the following year in Utrecht

(1977), in order to reflect on the possibility to intensify and improve the international communication between researchers. We met in Osnabrück (1978), then in Warwick (1979). I can even tell you a significant anecdote. In Warwick, Alan Bell had written a draft of a constitution, but we had not been able to reach an agreement. I invited Alan Bell and Hartwig Meissner to my home in the suburbs of Paris. We had good food and good wines. We reached an agreement rather easily. The text of the first constitution of PME was adopted in Berkeley in 1980 without any discussion.

How would you describe the evolution of research in that area of psychology and mathematics education?
Since that period, the evolution of research has been important. In the beginning, PME received a majority of contributions on the primary school level. Then there was a significant evolution towards the secondary level, and even the university level. There was also more and more research on the activity of the teacher, and on the use of computer software, in geometry and algebra mainly – up to the point that less than 10% of the contributions now concern the primary level. Fortunately enough, there has always been a good climate for discussion in the PME conferences. The problem is that there are too many sessions in parallel, and less and less psychology.

Ubiratan*, you are considered as one of the fathers of ethnomathematics. When and why did you enter into these cultural considerations?*
After my return to Brazil, I became more and more immersed in mathematics education. The experiences below were the seeds responsible for my interest in the relations between mathematics, culture and society, the backbone of the program in ethnomathematics. I feel very uncomfortable when I am called "the father of ethnomathematics". The ideas which are in the genesis of this concept are old. Surely, I am a critical trans-disciplinarian and trans-cultural reader of classics in many fields, particularly history, philosophy, anthropology, art, psychology and education, and I try to identify mathematical ideas in these writings. The development of my trans-disciplarian posture was explained before. How did the trans-cultural view come about?

It is important to mention that while at SUNY Buffalo, I was the director of graduate studies in mathematics, and in 1968, as a result of the affirmative action measures, our program had to admit, among the new students in the Ph.D. program, 25% blacks. This means 15 black Ph.D. students. But there were practically no applicants. So I visited what were labelled "black colleges" in the Southern states on a true recruiting mission. All this, including accompanying the students after they were admitted to the program, was my exposure to the academic facet of multicultural reality in the USA. This was the origin of my concerns about mathematics and society.

Another important experience was my work in Mali, since 1970, in the innovative Project Mali-1, a "Doctorat sur place" in the Centre Pédagogique Supérieur de Bamako, sponsored by UNESCO. Basically, while living and working in the USA, and also after my return to Brazil, I would be a frequent visitor to Bamako, for a 2- to 3-week period four times a year. This is how I got the nickname "Ubiratour"! In this program, I was responsible for about ten doctoral students, who did excellent academic work. My visits were very important for my understanding of the extant scientific and technological practices and the supporting theories and philosophy, obviously permeating current knowledge and behaviour of the Malian population. This was my introduction to a trans-cultural history and philosophy of science and mathematics, with particular attention to non-European civilizations. This is another basic pillar of my views on ethnomathematics.

In ICME-3, I presented a mélange of ideas, drawn from unusual sources, to answer the proposed question "Why teach mathematics?" The essence of ethnomathematics, as a program on the history and philosophy of mathematics and its pedagogical implications was there. But there was no name attached to these proposals. In 1977, during the Annual Meeting of the American Association for the Advancement of Science, I gave a paper in a section on "Native American Science", and I used the word *ethnomathematics* to designate the mathematics of the native cultures, similarly to what other participants were doing with their disciplines. The paper was never published and had a restricted circulation. In ICME-4, in Berkeley, a new opportunity came. I spoke on the theoretical background, from cognition theory, to support the methodological bases of ethnomathematics. It was mainly a paper on cognition. But I was not courageous enough to use the word *ethnomathematics*! In 1984, I gave a plenary talk in ICME-5, in Adelaide, and there I was explicit about ethnomathematics. It gave great visibility to the idea, and obviously much rejection. It was regarded as an intrusion of practices of primitive populations into the edifice of mathematics. Next year, during the NCTM Annual Meeting, the International Study Group on Ethnomathematics/ISGEm was founded. Since then, the evolution of the field is well documented [see www.rpi.edu/~eglash/isgem.htm].

Do you think that ethnomathematics has had a real influence on mathematics education in a country like yours?

The results of the creation of the ISGEm are remarkable, and not only in Brazil. The multicultural educational scenario, present everywhere in modern societies, presents a real challenge with multiple implications such as boredom, dropouts, recurrent failures, and various societal maladjustments.

Ethnomathematics clearly contributes to the reduction of all these deficiencies. It draws on motivation, which is related to daily occurrences, and to the students' cultural backgrounds and their expectations from schooling. It gives voice to the students in pursuing their mathematical growth. Of course, this may imply leaving aside some topics from the curriculum. But many will agree that a number of items in the syllabi are justified only by the fact that they are in the syllabi. I cite a phrase of David Hilbert's, in his famous talk of 1900 on the 23 problems: "History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones. … The close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future."

We are closing an era, no one will deny that. The world order and the main agents of the 20th century will soon be replaced by agents who have been born and raised in symbiosis with an entirely new technology, sometimes having been born thanks to technology, and with a new world scenario, with new everyday priorities, amenities, and dangers. It is a different world.

I believe all this is a very pertinent reflection when we discuss mathematics education.

What can be expected more globally from research in this domain, in your opinion?
Ethnomathematics is a very broad research area. It relies on many supporting sciences that define the strands of ethnomathematical research, which is obviously an interdisciplinary area. Its main strands – ethnographic, historical, epistemological, cognitive, sociological, political – must be regarded as interdependent. The most visible strand is the ethnographic one. The ethnographic strand is absolutely needed, but without a good reflection on its broad theoretical bases, the ethnomathematics project may be distorted.

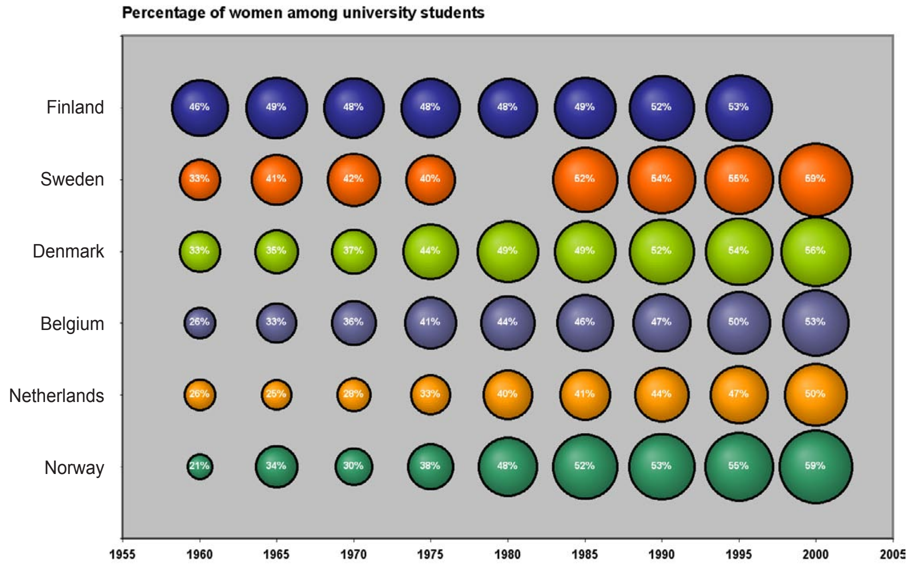
As an educational practice, ethnomathematics is growing everywhere in the world. Regrettably, many teachers and researchers who decide to enter the field immediately engage in the ethnographic strand, causing the frequent confusion of ethnomathematics with ethnic mathematics. Sometimes, the practitioners stress only amusing folkloristic mathematics. But these distortions are normal in every emerging field and are highly compensated for by the successes.

As a research area, ethnomathematics is growing as well. A number of books and papers have been published, and master's and doctoral dissertations have been accepted in very good universities, all over the world. The vitality of the field can be seen by visiting the web site indicated above.

Probably the most relevant aspect of current ethnomathematical research is the vision of the world and of education it offers. Ethnomathematics in teaching is necessarily critical in its approach.

Gila, *you certainly have foreseen that I would ask you some questions about gender issues. What made you especially sensitive to these issues?*
The fact that women were extremely underrepresented in mathematics at all levels of education, and that there were so few women in mathematics-related fields and in mathematics departments in universities, ought to have made everyone sensitive to gender issues. In the 1970s, when affirmative action started to call for minorities and women to be given special consideration in employment, education, and other decisions, both industry and governments set out to support educational programs aimed at increasing the participation of women in science- and mathematics-based fields. At about the same time, several associations were founded, among them three with the explicit purpose of encouraging women to study and to pursue active careers in the mathematical sciences. These are: the Association for Women and Mathematics (AWM), founded in 1971; the International Organisation of Women and Mathematics Education (IOWME), founded in 1980 and affiliated with ICMI; and European Women in Mathematics (EWM), founded in 1986.

What has been achieved in your opinion in that area?
The chronicle of women and mathematics in the last forty years is a success story. Women have made enormous strides in all areas of higher education, including in the mathematical sciences. Whereas women's share of enrolments in universities was well below 30% in most countries in the 1960s and in the 1970s, it is now well above the 50% level in most countries (see slide of data for Norway, the Netherlands, Belgium, Denmark, Sweden and Finland). In some countries the percentage of women among university students is now well above 60%.



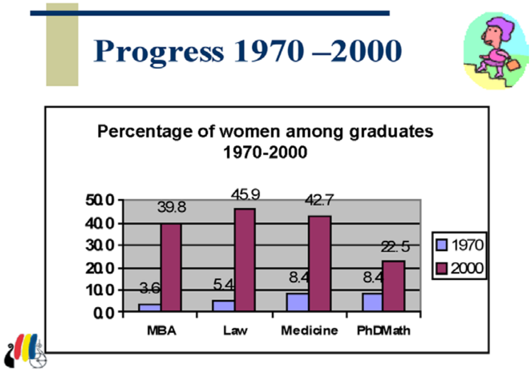
Reference: UNESCO: *Statistical Yearbook, 1972, 1988, 1998*, 7 Place de Fontenoy, Paris.

In an effort to look at the factors contributing to gender inequities, a flurry of work by researchers resulted in a voluminous body of research on gender differences. Researchers looked at a large number of variables, such as attitudes, beliefs, learning styles, teaching methods, and single-sex vs. mixed schools, as well as at controversial feminist and other models, such as the deficit model, the difference model, gender stereotyping, and more. Yes, there has been a great deal of progress at the school level.

Perhaps we should even look at this progress with some apprehension, because in some countries men are already a minority in higher education. This situation is worrisome, in my opinion, and should not be allowed to continue. Boys should not become an endangered species in higher education. I would suggest that the present generation of researchers take up the challenge of finding ways to increase the number of boys who attend university. Women have a great deal of experience in this area, and I am sure they would be willing to help.

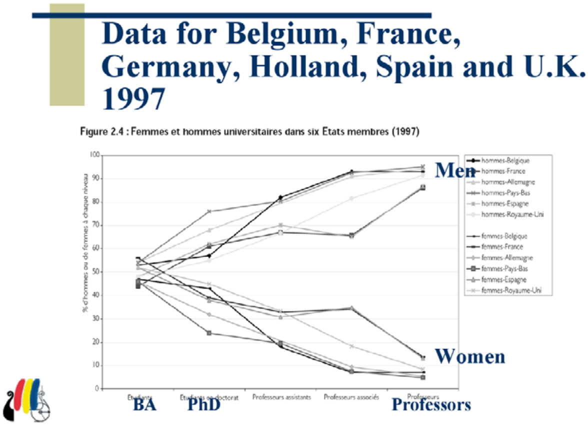
Do you regret anything about what you did or what has been done and not done in that area?

Though women have increased their share of university degrees dramatically in the last forty years, progress in mathematics has been slow. According to Canadian data, women have made more progress in business administration, law, and medicine, increasing their share of higher degrees to a level very close to the desired 50% by the year 2000 (see slide). In contrast, among recipients of doctoral degrees in mathematics, fewer than 25% are women. Although there was some progress in mathematics from 1970 to 2000, it didn't match that achieved in business administration, law and medicine.



Statistics Canada (1996), *Annual demographic statistics*. Catalogue no. 91-213 and Statistics Canada (2002). *Annual demographic statistics*. Catalogue no. 91-213-XIB20020000.

Another area where women still have a lot of catching up to do is their share of faculty appointments in universities. European data shows that in 1997, women and men had a more or less equal share of the B.A. degrees. But there were more men proceeding to a Ph.D. degree, and consequently women were severely underrepresented at the full professor level, holding fewer than 20% of the positions (see below).



Reference: www.education.gouv.fr./syst/egalite/etan.htm.

Efforts should be made to bring the share of women faculty positions closer to that of men, so that these lines become parallel. This is a challenge that the younger generation should perhaps take up. So it is clear that there is still a lot of work to do in the area of gender equity.

Jeremy*, together with Anna Sierpinska, you have been the co-chair of the ICMI Study on research, and I know that the history of research in mathematics education is one of your specialities. How would you describe in a few words the global evolution of the field during the last 20 years, first perhaps in terms of problématique?*

If I may, I'd like to go back further than 20 years because, as I look back at the history of research in our field, I see it going back more than 100 years, to the people who were moving from teachers colleges into universities. They were responsible for the preparation of mathematics teachers and in many cases, then, as university faculty, were expected to do some kind of research. The two great streams that I contend contributed to our field initially were mathematics and psychology, and they were the source of much research that went on during the first half of the twentieth century. When we get a little further along, we see research blossoming in other directions. I was very much taken with Anna Sfard's distinction yesterday among the eras in research, and I would certainly agree with her that the last 20 years have seen a considerable shift to research on teaching and teachers. But I would disagree with her a little bit, because I think the research on learners goes back much further than she claims – back into the 1920s and 1930s at least. So there has been a change in the *problématique* from a focus on learners and curriculum to a focus on teachers, but I wouldn't separate it exactly the way she would.

These are global evolutions in the themes of research, but what about the evolution in a given theme, for instance as regards the approach of the learner?
Twenty years ago, the learner was looked at as an individual learner. In the kind of work that I did even 40 years ago, we looked at the learner or the thinker as someone who did mathematics in isolation and not necessarily in the classroom with other children under the direction of a teacher. So there was a lot of attention to learning, thinking, and development following the models of Piaget and others. What we've seen is a huge change to looking at learning in context – in particular, in the context of the classroom.

Was there also an evolution in the research methodologies?
Oh, yes. There has been a tremendous change. When I started out, I did my own research, as I mentioned earlier, by interviewing children and trying to understand their thinking. At that time, most research was entirely quantitative, and my study was unusual in being at least partly qualitative. The change since then has been drastic, I would say, to the point where almost all the research that I look at these days is qualitative in its orientation. It is very much concerned with understanding what children are doing and what teachers are doing rather than trying to categorize and measure and correlate and so forth.

Do you consider this shift from the quantitative towards the qualitative as a good thing or that it can sometimes become an obstacle to development?
I think both. I think it has been a good thing for us to adopt newer forms of research because what it has meant is that we have looked outside of traditional psychology to try to borrow some techniques from other subject fields, and that has been healthy for our field. A lot of that – particularly anthropology and sociology – has been helpful to us in doing qualitative research. On the other hand – and this is one of my concerns about what has not been so good in our field – is that as a relatively young field, we tend to go to extremes rather than seeking a more balanced way. What has happened is that we have gone so far in the direction of qualitative research that now quantitative research seems to be endangered – from the perspective of the field and not necessarily that of the individual researcher. Quantitative research is in need of some rehabilitation.

Before going to the next phase of the interview, Gila, Ubiratan, Jeremy and Gérard, do you want to react to what has been said?

Jeremy Kilpatrick: I'd like to answer a question you didn't ask me. You asked about changes in the kind of research, but I'm more impressed by the way the field has developed professionally. It's not just the size of this gathering here, for example; it's where these people are coming from. There are places where research is being done now where it was not being done 20 or 50 years ago. One of the most rewarding things in my professional lifetime has been to see the development of mathematics education as a research subject in different countries. I think that has been probably the most important change over the past 20 years.

**The third phase of the interview session:
Some visions about the future of the field**

In this phase, as was planned, the four interviewees were asked to express their personal vision about the future of the field and the main challenges it has to face. In fact, they were asked, in an analogy with the famous 23 problems set up by Hilbert at the International Congress of Mathematicians in Paris, in 1900, to articulate the five questions that they considered as the most crucial problems to be addressed and solved by educational research today. Here is a summary of their answers.

Gérard Vergnaud: Michèle, yesterday you said four ideas and not five. I have three of them, plus one. Is it OK with you?

When one thinks of the future, one often confuses one's anticipations and one's wishes. As for myself, I cannot really separate them.

I can see three main social functions of mathematics education that should be reflected on a regular basis:

- the transmission of the wonderful heritage of knowledge that has been produced by mathematicians throughout history
- the formation of competences for a variety of uses of mathematics in different professions, not only in science or engineering, but also in farming, accountancy, industrial jobs, and so on
- the development of the mind, of "intelligence" as we usually say in psychology. In what sense does mathematics contribute to shape the mind differently from other disciplines?

I think that more research and more reflection are needed to identify what should be the most essential parts of mathematics for secondary school students, and also how they could be related to situations from outside mathematics. Therefore we should devote more research work to the analysis of situations at work that involve mathematics and which mathematics? There is more mathematics used at work, and more profound mathematics than we would have thought 20 years ago. This is true not only for engineers and technicians but also for nurses and farmers. Many situations borrowed from the analysis of activity at work can be transposed into the classroom, under certain conditions. Not only vocational schools are concerned by this idea, but also general schools. It is a challenge in a sense, but it would contribute to a more functional image of mathematics. I foresee important developments in that direction within the next 20 years because it is obviously a social need.

Finally, concerning the development of the mind, one recognizes easily today that mathematics offers students some experience of what a proof is, or an argument, or even a contradiction. This is certainly an important characteristic of mathematics, even though there are also proving processes in other disciplines, for instance in history. But mathematics has other interesting specificities beside proof. It is extensively used to model situations and processes in many fields outside mathematics, or even to theorize about them. Mathematical theorizing is often a crucial and intrinsic part of theory in physics, the life sciences, and the humanities.

If I take the example of geometry to illustrate these ideas, I can say, in the first place, that geometry is the core of the endeavour and achievement of mathematicians, and therefore must stand in a central position in the transmission process of the heritage. But it has also an immense metaphorical power for other disciplines and activities. Last but not least, young children develop very efficient representations of space that could certainly be better analysed so as to make us able to imagine interesting situations at the primary level (where there is little work on geometry) and at the secondary level (where one finds mainly geometry of figures, not so much geometry of positions and transformations).

Jeremy Kilpatrick: The first challenge I see arises in connection with the topic of the panel discussion on Monday, which put in opposition mathematics for all and high-level professional mathematics. There's another way to cast that. Before I begin this, however, I should say that I think problems in mathematics education never really get solved. It's not the same as mathematics itself, where theorems are proved and, unless somebody comes up with a more elegant proof or finds a flaw, the proof stands, and that's it. In mathematics education, we never quite get the stake through the heart, so the vampire rises again in the next generation to haunt us. This is characteristic of our field: we don't solve the problems; we just get them under some kind of control for a while. So it is a persistent issue: how do we reconcile the fact that, on the one hand, students respond very favourably – as do teachers – to a contextualized mathematics, a mathematics that's given in the form of things that they can see and touch and work with, that's full of context. I see the power of that. But on the other hand, there is an abstract, formal side to mathematics to which some if not all students should eventually go. Working on the balance between those – How do we get from one place to another? How do we help children abstract so they can de- and recontextualize their knowledge? – is a persistent problem for all teachers of mathematics. It will stay with us, but we need to keep working on it.

A second challenge connects with this first curriculum issue, and that is the curriculum as problematic in itself. We come to a meeting like this from different countries, speaking different languages, and we talk about geometry, and we talk about algebra, and we talk about number. Since this is mathematics, we think we're talking about the same thing. But in the context of our own local curriculum at home, it's not clear that these are always understood in the same way. So often, at a meeting like this, we have the illusion that we are speaking on a common topic, and we are not. Clearly, one of the problems for the international group of mathematics educators is to figure out how to talk about something like curriculum at an international level. People have often claimed that there is a canonical curriculum if you look around the world because there are all these labels that are common. And a study like TIMSS or PISA is built on that assumption. But as Freudenthal pointed out, that assumption may not be correct and needs to be considered and thought about.

Third, I see a lot of promising work coming from various countries that are taking us in the direction of more concern with equity – not just gender equity but equity of all kinds – and social justice issues. I just had the pleasure of editing the section on research in the *Second International Handbook of Mathematics Education*, for which Alan Bishop was the main editor. One of the most pleasant things about that was to see the good work that was being done by our colleagues in South Africa to put these issues on the table for us to think about. One of the advances of the last 20 years is that we now have these issues of social justice and equity on the table, but one of the problems for us how is how we work on these issues in a collective way.

My fourth challenge, I guess, concerns qualitative and quantitative methodology. I noted before that there has been too extreme a shift toward qualitative methodology. In the Topic Study Group 28, where we're talking about new trends in mathematics education as a discipline, it's interesting to note that one of the subthemes there has been this notion of putting qualitative and quantitative methods together in a better way in our research. I would like to see that happen. Several speakers in that group made the point yesterday that if you want to do studies that combine methodologies, studies that are quite labour intensive, then you have to have more cooperative research, and it needs to be much better funded than it is at the moment.

And finally, a challenge that I think connects with all of these and goes back to something I said earlier about the faddism that we are subject to and the extreme views that we often find ourselves taking on curriculum, methodology, and other topics, is something that I've tried to live my whole professional life. And my students will tell you that I can do this. It is to be critical, to criticize what's out there in what you hope is a constructive way. I had the benefit of getting some very good criticism myself when I was a doctoral student. It hurt at the time, but it made me a better person. I've always tried to help other people be better people by criticizing what they've done.

Ubiratan D'Ambrosio: In my view, the major objective is to offer an education that aims at the elimination of inequity, arrogance, and bigotry. This discussion cannot proceed without my critical view on current education, as regards both research and practice, particularly mathematics education. When I compare much of the ideas for the improvement of mathematics education in the last decades with what was priority in the late 19th century and most of the 20th century, I see a remarkable effort to improve sameness[1]. Proposals have been variants of theories and practices coming from the past. This is supported by sophisticated research instruments, mostly quantitative, developed in the course of the 20th century, which, in most cases, only confirm what is perceived by any critical observer. Sadly, philosophical arguments have lost favour.

It seems that the main goal is to have generations giving continuity to what we have been doing for hundreds of years. And this means to give continuity to what is going on. The goal seems to be to improve sameness. What do we want to offer the new generations that we are fathering? What do we expect for their new world? Do we want them to repeat all the mistakes of the past, which were based on beliefs, principles, and

1 By sameness is meant "doing the same" and improve sameness means "ameliorating the way of doing the same things".

values, that were mostly unsustainable? The scenario of the 20th century, which we are offering them, is ugly, deformed by inequity, arrogance, bigotry, and many other shameful behaviours. Regrettably, although outspokenly against these behaviours, many teachers, and even researchers, naively struggle for minor adjustments in the previous model without questioning the supporting bases of the model.

A big threat facing education all over the world is its subordination to unfit testing systems and to intellectually banal practices. Much of what we demand of our students is obsolete, trivial, and uninteresting. And ethically false. At the same time, we ignore their perception of the ugliness of the world, which they witness in streets, at home, and mainly in the media, accessible to practically all. They enter the school in all levels, and they are presented with false scenarios of their future. The subliminal message is that they are being prepared to kill or to be killed, to exploit or to be exploited, to defend themselves. They are taught to be better aggressors in order to survive because every other human being is regarded as a potential aggressor. They are trained to compete in order to survive. This is the philosophy of deterrence, which was so much practiced during the Cold War, and which is now being brought to even more insane levels. The insanity is evidenced by the futility of the deterrence devices. In antiquity and the Middle Ages, feudal rulers built majestic walls, incorporating and promoting advanced engineering technology, surrounding the cities as protection against aggressors. Aggressors developed powerful engineering technology, and the walls did not resist. Nowadays cities, nations, and blocks are repeating the pattern of building walls with very sophisticated technology. They are obviously futile, and the moral damage, auto-inflicted by their builders, is irreparable. I might give also as an example the deformed and cruel economic system. People live in fear and distress! Is this a sane world?

I am sure many will be asking: but what does this have to do with mathematics and mathematics education? I feel uncomfortable when a reading of history tells us that both the builders of the walls and the developers of the devices to destroy them relied on the support of mathematicians, and are indeed funding further development of mathematics. Deterrence philosophy has always been a great provider for mathematics and mathematicians. Is there a flaw in the ethical support of mathematics? At this point, I offer a metaphor for reflection. It is an agreement among philosophers and historians – indeed, this is a perception of all sectors of society – that the beautiful and perfect construct which is called Western (or academic) mathematics is the dorsal spine of the body we call modern civilization. It is well known that in any organism, the function of the dorsal spine is to support and to provide for the body. Paradoxically, the dorsal spine [mathematics] is recognized as beautiful and perfect, but few will disagree that the body it supports is ugly, deformed by inequity, arrogance, bigotry, and many other shameful behaviours. If we really want to correct the body, shouldn't we look into the dorsal spine? Maybe a malformation in its evolution is causing the problems with the body. It is possible that this malformation relates to a misconception of values, which are subordinated to a partisan ethics. A comparative reading of the history and philosophy of mathematics, with much attention given to non-Western environments, may reveal to us flaws in the development of mathematics that cause the distortions and malformation.

Summing-up, as educators, our main objective is a more dignified future for mankind. But what we see is a degradation in the quality of life, all over the world. What went wrong with this model of civilization? Since mathematics is inherent to this model, we should feel less comfortable with the progress of our field. Let us think anew about it.

After such strong sentences, it was difficult to say more. Gila Hanna said that she had nothing to add to her eloquent colleagues' statements, and Gérard Vergnaud took this opportunity to ask Michèle Artigue to leave her comfortable position of interviewer and give her personal vision of the field.

Michèle Artigue: Looking at the current state of the field, I see reasons for being optimistic. As was attested by this interview session and beyond this session by the whole Congress, major advances have been reached: We have built efficient conceptual and methodological tools for approaching the complex reality we aim at understanding and for solving the difficult problems we have to face; we better understand learning processes and teaching practices as well as the more global functioning of educational systems; and, even if the relationship between theory and practice remains problematic, we can rely today on many successful experiences. Our field is diverse, but our communication has improved, and today we can consider the richness resulting from this diversity. Nevertheless, as a researcher, I have often the uncomfortable feeling of facing a never-ending task. In this field, even when you think you have answered a question, understood a problem, found a solution, you soon realize that your success was only a local one. Some years later, the context has changed, the solutions are no longer solutions, and you can have the impression that you are recurrently facing the same questions. A rather strange and uncomfortable impression for somebody coming from the field of mathematics!

There are recurrent issues but there are also new issues. During this Congress, during this interview session, social issues, equity issues have received particular attention. This is perhaps the result of some personal blindness, but I have the feeling that this was not the case when I entered the field thirty years ago, nor even twenty years ago, at least for researchers living, as is the case for me, in developed and rather rich countries. Today nobody can escape these issues. Social issues are no longer the "privilege" of developing countries, and the Fourth World is present in every country, nearly in every town. This is for our community an essential challenge to face, and we have certainly much to learn from those who have tried for years and years to make mathematics education a vehicle for more social justice and equity. I hope that, if in 20 years from now a new interview session will be organized in an ICME Congress, our successors will be able to say: Yes, we have tried hard to solve this crucial problem for mathematics education, and today we have partly succeeded.

Retrospective comments

When an interview session comes to its end, in the hours and days that follow, many facts and ideas come to mind that, retrospectively, each of the interviewees would have liked to share with the audience. This is the reason we have added to this report a kind of post-interview where each of the interviewees was given the opportunity to express the most important things that she or he neglected to express during the session. Jeremy Kilpatrick and Michèle Artigue used this opportunity.

Jeremy Kilpatrick: For the record, in 1976 there were 1831 participants at ICME-3 in Karlsruhe, not so many fewer than the 2161 at ICME-10 as I thought. Freudenthal did not give one of the main lectures; what I was remembering was the panel discussion on computers and calculators that he chaired (and during which he memorably started translating one questioner's remarks from French into French). In addition to Peter Hilton, the main lecturers were James Lighthill, Michael Atiyah, Arnold Kirsch, and Georges Guilbaud. (Not surprisingly, strong complaints about the lack of gender equity were voiced during the closing session.)

Although plenary sessions are often the best prepared, delivered, and reported parts of an ICME, they are not necessarily what one remembers best. I have cherished memories of groups that met repeatedly during a Congress to work productively on common problems of mathematics instruction, of national presentations that opened up new vistas of research and practice, of collaborations begun at an ICME that led to coauthored articles and even books. I always appreciate the exhibitions of project work, talking to students who are presenting their research in poster sessions, and hearing about ICMI studies. Most of all at ICMEs, I've enjoyed meeting old friends and making new ones.

Large gatherings like ICMEs are bound to continue if only because technology will never be able to replicate the experience of face-to-face encounters. A summary report from a topic study group, for example, cannot express the give and take of the vigorous discussion that arose after the group had heard a challenging set of presentations. Chance meetings and subsequent engaging conversations in the registration line, at the happy hour, on an excursion, or on the train into town are now lost to history. Neither these proceedings nor a DVD recording of the plenary interview session can capture the surge of delight and appreciation that swept through the hall as the session ended. In Copenhagen, as at other ICMEs, you had to be there.

Michèle Artigue: Reflecting back on the content of this interview session, I was struck by the fact that technology was hardly mentioned. How to explain such an absence? Was it due to the fact that technology was not a major dimension of research for the four interviewees? Was it due to the fact that some decades ago a lot of hope was invested in technology and we are now in a less enthusiastic phase, and even for some of us fearing a world where technology would reinforce the cultural and social divide? I don't know, but being personally involved in research in that area, I retrospectively regret that discussion about the ways technology can help us face the difficult challenges we have to face has not entered the scene of the interview session. I certainly have a large part of the responsibility for that.

As was mentioned at the beginning of this report, the interview session was a new format at ICME-10, and it was for me an honour to be asked to be the interviewer of the four distinguished scholars chosen by the Programme Committee. I accepted the honour and tried to prepare myself to play a role that I had never played in my professional life. When arriving at Copenhagen, I was a bit anxious, but that anxiety disappeared when I discovered that my distinguished colleagues were even more anxious than I was. We tried to do our best, but none of us could imagine what would be the result of our endeavours. The standing ovation at the end of the session was a real surprise for me, a moment that I will remember forever with emotion.

P 6: Mirror images of an emerging field: Researching mathematics teacher education

Plenary Presentation based on the work of Survey Team 3

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Team Members: ***Deborah Ball***, University of Michigan, Ann Arbor, USA
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Abstract

Survey Team 3 examined research in mathematics teacher education from 1999-2003. We focused our survey on published research in international mathematics education journals, international handbooks of mathematics education and international mathematics education conference proceedings. Some regional sources from various parts of the world were also included. We investigated who was writing, from and in what settings, with what theoretical frameworks, and with what sorts of study designs for what core questions. We also examined the range of findings and conclusions produced in these studies. Our analysis presented here focuses on four themes that stood out from our investigation of almost 300 published papers, and offers a reflection on the current state of the field of mathematics teacher education research. Our purpose was both to provide a mirror image of the field, and to stimulate discussion that can support its development.

Introduction

Mathematics teacher education as a field of study is relatively young. It is also thriving, with substantial progress in the past decade. It was thus possible, desirable and indeed timely, to take account of our[1] progress at the time of ICME-10: This paper reports our international survey of published research in mathematics teacher education in the past five years[2]. We present some of the mirror images reflected back to us in the survey.

We begin with a discussion of the current field of mathematics teacher education, the emergence of related research, and the value of critical reflection on progress at this juncture. We then discuss why we focused on *research*, and the methods we adopted – where and how we looked in order to construct the survey we did. This process brought to the fore a number of themes, in particular, the research methods in use, issues of authorship and voice, and consequences for the substance of research being done. We observed a field currently dominated by small scale studies in English-speaking countries. The studies we surveyed focused on teachers' learning in the context of a reform agenda, and researchers, typically, were studying aspects of reform programs offered by or in their own institutions. We offer these observed themes as mirror reflections on ourselves

1 We use "our" and "we" in relation to the mathematics education research community and teacher education community as each of us, though in different ways, is involved in mathematics teacher education, and mathematics education research.

2 The research was done by a team (Survey Team 3) of mathematics education researchers – the five authors listed above – and presented at ICME-10 in Copenhagen in July 2004. A copy of the presentation can be viewed on the following websites: www.wits.ac.za/jadler/presentations.html. or www-personal.umich.edu/~dball/BallSelectedPresentations.html.

and our work as a community. We present these as claims, each of which is followed by a range of commentaries. We conclude the paper with some suggestions for the field of mathematics teacher education research, as well as reflections on our work as an invited international survey team.

Mathematics teacher education in 2004

We are currently witnessing what can be called the "massification" of mathematics as a school subject. In many countries today there is an extensive move to make mathematics accessible for all. Mathematics is viewed as a necessary competency for critical citizenship. In her opening address to the Congress, the Minister of Education in Denmark pointed out that in Denmark, competence in both mathematics and English language are viewed as priorities in a globalised world. An obvious consequence of the increasing demand for mathematics proficiency for all is an increase in the need for quality teaching[3]. That this need is evident at all levels of schooling is unprecedented. Although the demand for quality teaching is high at the secondary and tertiary levels of schooling, where mathematics is a specialisation subject, quality teaching is even more important at levels where mathematics is a general requirement. More teachers and better mathematics teaching are needed if mathematical proficiency is indeed to become a widely held competence. Of course, quality instruction depends on teachers, and so their preparation and continuing professional development is crucial.

To make the magnitude of this demand more vivid, we offer a brief glimpse of who the children are that our world's teachers must teach. The snapshots (Pictures 1-5) on the next page are from mathematics classrooms in different countries. At first glance, it is clear that all are classrooms, and they are differently organised. But if we focus more closely, what else can we see? These visual images convey different class sizes and material resource bases, with implications for what it might mean to enable quality teaching in different contexts. In some countries (e.g. South Africa), many mathematics teachers are teaching in large (over 40 learners) classrooms often severely lacking even basic resources. For example, one South African classroom shows one group of learners sharing a single concrete tangram as they explore conservation of area. Class size also varies within countries (e.g. in the US, there is a relatively high pupil-teacher ratio in urban schools while more affluent suburban schools may often enjoy lower class sizes). In many contexts, mathematics classrooms also include a greater range of learners who live in and bring with them diverse cultural practices and languages, as well as linguistic and mathematical competences. This diversity adds to the challenge of providing quality teaching. Globalization is increasing the dominance of English as a language of instruction around the world. More and more learners are having to learn mathematics in English, a language that is not their main spoken language. This phenomenon is no longer specific to (British) post colonial countries. There are similar pressures for English language competence in Scandinavian[4] as well as some European countries (e.g. the Czech Republic). This quick look inside a few different classrooms brings to life that a

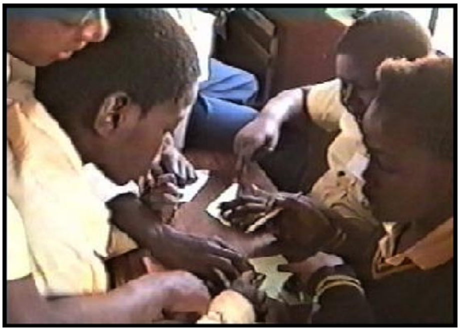
3 The scale of provision of mathematics teachers across countries varies, with enormous shortages of quality teachers in some countries (e.g. the USA) to over-supply in others (e.g. Taiwan). Across countries, however, is the demand for quality teaching at all levels and so a scale of quality provision like never before.

4 This point was made rather forcefully by the Danish Minister of Education in her opening address at ICME-10.

significant part of preparing mathematics teachers for quality instruction, includes preparing teachers to engage and mediate the increasing diversity of their learners.

1. Classroom in the USA

2. Classroom in Czech Republik



3. Classroom in South Africa

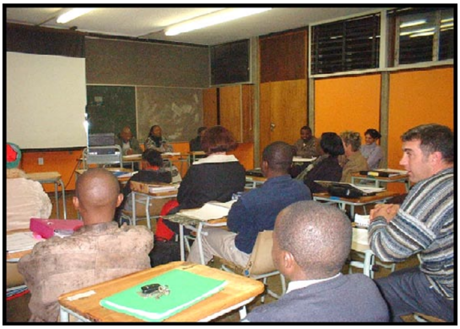
4. Classroom in Austria

5. Classroom in Taiwan

But what is it that mathematics teachers need to know and know how to do to enact quality instruction across these diverse conditions? How is teacher education research and practice dealing with these current challenges?

It is as instructive to look across a range of prospective as well as in-service teacher education classes. As we zoom in on the few snapshots (Pictures 6-10) on the next pages we see similar diversity across those learning to teach. There are some smaller and some larger (over 80) groups of teacher learners. There are also culturally homogenous as well as culturally diverse groups. We can detect diverse socio-economic conditions, with differences in the materials and resources being used across teacher education settings. Less visible, but a significant additional note about who is (re)learning to teach mathematics, is that differences are increasing between teacher educators and their ‘learners’ – i.e. prospective and practicing teachers. Teacher learners bring increasingly diverse mathematical histories. In many countries prospective elementary teachers have learned limited mathematics in school. In countries where there are great shortages, even prospective secondary teachers are entering training with relatively poor mathematical experiences and performance at school. This reveals that we are dealing with different kinds and levels of under-preparedness, a phenomenon that extends into in-service teacher education. Many practising teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required. In particular, curriculum reform processes in mathematics across different countries have resulted in many teachers now having to teach a curriculum that is quite different from the one they were educated for, and from one with which they had become experienced – and often also successful.

Teachers need support if the goal of mathematical proficiency for all is to be reached. The demands this makes on teacher educators and the enterprise of teacher education are great. These, in turn, shape the context in which research on mathematics education is developing.



6: In-service teacher educators South Africa

7: Pre-service teachers Taiwan

8: In-service teacher educators in Czech Republic

9: Pre-service teachers in South Africa

10: In-service teachers in Austria

The timeliness of a survey of research on mathematics teacher education

The timeliness of the survey reported in this paper is not only a function of the current demands on mathematics teaching and teacher education. While still relatively young, mathematics teacher education (MTE), as an area of research and development, has mushroomed in the past five years in particular with multiple approaches and initiatives evident. For example, there were over 60 contributions on mathematics teacher education across various parts of the ICME-10 program (relevant Topic Study Groups, Discussion Groups and the Thematic Afternoon) from a wide range of countries and regions as listed on the congress website[5]. It is also interesting to note that only ten years ago there was very little research on processes of mathematics teacher education, in contrast to research on teachers' beliefs, knowledge, practice, biographies, expert-novice comparisons. Now, in 2004, we have with the *Journal of Mathematics Teacher Education* a journal dedicated to researching teacher education. And we have focus strands in major conferences, particularly the PMEs, as well as increased attention to mathematics teacher education in recently published international handbooks in the field[6]. The importance of teacher education for our community is further signalled by the invitation to develop and present this Survey at ICME-10, and in setting up of ICMI Study 15, focused on teacher education, which is currently in the process of its work.

The Survey Team saw as its responsibility to describe "where are we", globally, in the field and within ICME, and so complement work of ICMI Study 15 upcoming in May 2005. We intended to both survey and report and also contribute to the growth of this relatively new, but critically important, research field. We believed that it was a good moment to hold a mirror up to ourselves and see what it is we are doing. Survey Team 1 (reporting on research and practice in mathematics education)[7] noted the shifts over time in the field of mathematics education research, starting with studies focused on curriculum, then shifting to a focus on learners, then teachers. We would add that the last five years in particular, has seen the emergence of teacher education research. And this emergence is signified in the presence now of journals with specific focus on mathematics teacher education, as well as of dedicated strands in mathematics teacher education in key conferences in the field.

5 See www.icme10.dk

6 See for example Bishop, A., Clements, M.A., Keitel, C., Kilpartick, J. and Leung, F. (2003) (Eds.) *Second International Handbook of Mathematics Education*. Kluwer. Dordrecht.

7 The paper by Survey Team 1 is also in these proceedings. See Sfard, A. and others.

Central questions

For the Survey Team and its work, this meant that a massive amount of material was available to be examined. We decided that, in order to do a useful survey of the field, a clear focus would be helpful. Both because of our own interests, and the demands for research knowledge about teacher development, we posed the following question:

What is research in the field contributing to the improvement of the education of teachers of mathematics?

More specifically, given that the task of mathematics teacher education is to work with large groups of teachers, in diverse contexts, so that they are able to teach mathematics well in diverse settings and conditions, then

- What stands out about research that focuses on mathematics teacher education over the past five years?
- What research is being produced that can contribute to the massive need for supporting teachers' learning and development? We were interested in inquiries of two basic types:
  - *Understanding* how teachers learn, and from what opportunities, and under what conditions
  - *Improving* teachers' opportunities to learn

Delineation of mathematics teacher education research

Mathematics teacher education is a very broad field, and so a key task, as with any survey, was to agree on the meaning of central notions. First, we agreed that by "teachers," we would include student teachers, classroom teachers, and teacher educators. For us that also – importantly – entailed delineating and agreeing on what we would count as *teacher education research*. The *Journal of Mathematics Teacher Education* initiated in 1998 became a useful marker for us, as the research reported there was clearly mathematics teacher education research. We needed, for example, to be able to identify those papers in PME, for example, or in journals not dedicated to teacher education, that would "count" in our survey. There is much work to do to define the broad field we encountered, and this will be developed in a more detailed paper on the first claims to be discussed below. Our perusal of JMTE revealed many studies that occurred in the context of teacher education and focused on teachers' learning and change over time. This is captured in the inner circle in the figure below. In addition to research on teachers' learning, there were numerous papers on teachers' beliefs and knowledge. Some of these were not focused on teachers' learning or changes in their beliefs, and some were not situated in the context of teacher education. We included in the survey presented here, those studies on teachers' knowledge and beliefs where the teachers being studied were those participating in teacher education programmes, but not studies that investigated teachers' knowledge and beliefs independent of questions of learning or change (see Figure.1). The boundary, therefore, around what does and does not count as teacher education research in relation to areas like teachers' knowledge and beliefs is somewhat blurred, and its delineation will require further work. In addition to the papers depicted in the diagram, we included a third set of papers in our survey: theoretical papers focused on mathematics teacher education and papers that provided some meta-analysis of the field.

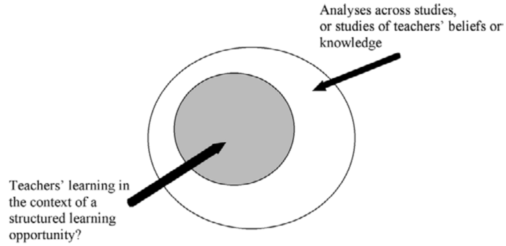


Figure 1. Delineation of areas of mathematics teacher education research examined in this survey

As we move on now to describe and explain what we did and what we found, we need to add that we see each of ourselves as deeply invested in what we are looking at. We are all researchers in mathematics education research, mathematics teacher education practitioners: hence the notion of the mirror. Our different experiences shaped our work, our interpretations, and the nature of our analyses. The differences among us were a resource for the quality of our work; our different perspectives also presented us with challenges. In addition, unlike other collective research endeavours where researchers come to work together over time, and usually in near locations, we were distanced, geographically, culturally, and in the work we do. We begin the next section with some brief comment on the processes we engaged to do this work as a team.

The method we used

Making the survey team work

We play intentionally with words here, capturing the critical dual dimension of our task. We are a diverse group from very different and distanced countries and cultures. We needed to find ways of making the team 'work'. It was clear that undertaking the survey was going to be hard work for each of us, and then together. So we needed to establish work patterns and deadlines. We also needed to find ways, set up processes that would enable us to accomplish team-work – to make this a joint, collaborative task.

We worked hard at both these dimensions of making the survey team work, developing a process that could transcend boundaries of geography, language, orientation and experience and that included: two meetings in person prior to ICME-10; sharing the extensive number of articles that needed to be read; developing a shared framework for this reading and then sharing the data that developed; deciding together on the claims we could make and then constructing common and different interpretations of our claims.

Our work began over e-mail in 2002, where we were able to agree on our focus on research (notwithstanding the massive development work in the field), and allocate parts of the survey. The two meetings, both in 2003 (and each facilitated by a conference that we could all attend), were pivotal in that at the first meeting (May 2003) we were able to discuss and agree on the scope of the work, and what we would and would not include. By the time of the second team meeting in July 2003, we had completed a substantial part of the reading and so were able to focus then on the themes that were emerging from the data, and begin a plan for the presentation at ICME-10.

What we looked at (included and excluded)

All the domains of mathematics teacher education were taken into account: pre-service and in-service, as well as primary and secondary teacher education. By this relatively broad definition of professional development, we hoped to gain insight into issues that are topical in particular contexts, and into the kinds of problems that appear to be common, or substantively different, across levels and contexts.

We selected from multiple outlets for this work, including peer reviewed journals, international handbooks and key conferences proceedings. We looked across international journals as well as a handful of journals in Asia, Europe, i.e. published in languages other than English where it was possible to access these. In general, however, we did not have the time and resources to investigate thoroughly journals written in e.g. French, German, Russian or Spanish. We capitalised on the advantages we brought as a diverse team from diverse and distanced countries. At the same time, we restricted the survey to published research between 1999 and 2003, that is, since the previous ICME Congress.

The full range of what we looked at is listed in Table 1 below. The focus of our report is nevertheless on the highlighted publications that constitute a careful selection of those journals and proceedings widely considered as either leading publications in our field, or central to the work of the survey[8].

Journals (126 papers)	Journal of Mathematics Teacher Education JMTE 1998 – 2003, No. 3	65
	Journal for Research in Mathematics Education JRME; Journal of Mathematical Thinking &Learning JMT&L; Journal of Teacher Education JTE; all 1999-2003	13
	Educational Studies in Mathematics ESM 1999-2002	2
	Mathematics Teacher Education and Development MTED 1999-2003	34
	Pacific Journal of Teacher Education Chinese Journal of Science Education	11
	Pedagogika	1
Conference proceedings (154 papers)	Proceedings of Psychology of Mathematics Education Conferences 1999 – 2003	88
	Papers from discussion group on teacher education in proceedings ICME9 2000 (a selection of these appears as a special issue of MTED in 2001)	15
	Cerme Conferences of the European Society for Research in Mathematics Education CERME	4
	Symposium on Elementary Maths Teaching SEMT 01 and SEMT 03 MedConf 2000 and 2003 Second and Third Mediterranean Conference on Mathematical Education	21
	NSC and TE conf Taiwan Fou Lai – also here	24
Handbooks	2nd International Handbook of Mathematics Education. Eds. A.J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick and F.K.S. Leung. 2003 Dordrecht: Kluwer Academic Publishers	4
TOTAL		282

Table 1: Journals and proceedings included in our survey

8 Our survey selection is similar to that of Lerman, Xu and Tsatsaroni in their study of the field of mathematics education research as a whole i.e. key journals and PME proceedings. See, for example, Lerman, S., Xu, R. and Tsatsaroni, A. (2003) "The Production of Theories of Teaching and Learning: The Case of Mathematics". Unpublished paper presented at the AERA Conference, 21 – 25 April, Chicago, USA

How we looked

As mentioned above, we are a diverse team, and one that was constituted by the International Programme Committee of ICME-10. In addition to our geographic spread, we brought different orientations to research in the field. This was a strength in that it broadens the scope of what is 'seen'. At the same time, we faced a considerable challenge in establishing a shared framework that was necessary if we were to carry out a consistent survey. To launch our work, we developed a framework for looking across ranging publications, reproduced in Table 2 below.

Source	Title	Authors + country		
Pre- or in-service Primary or Secondary?	Mathematical topic or process in focus?	Is topic object of study or means to studying something else?	Author field position	Comment on article (summary points, what stands out / what is missing/ problematic)
Central problem	Theoretical orientation	Assumptions	Findings	"Argument"
Methodology	Research design and methods used	Analytic framework	Rigor	

Table 2: Framework for analysis of papers

A great deal of information is contained in the summaries we produced of the 282 papers read and captured in through this framework. A glance across and down the rows and columns of the table reveals that we captured the *who* (who was writing/doing the research, and from where), the *how* (what methods were used) and the *what* (what was being studied, theoretical orientations, assumptions and outcomes).

The value of working this way was that it enabled us to look across and discuss the wide range of papers we had read. It also enabled the job to be done within a reasonable time frame. In addition, this kind of capturing of the data enabled us to examine trends that we might otherwise not have seen. And as with any framework or structure, there were also limitations to the way we went about this work. In particular, when a research team undertakes a survey, they typically do so with a more focused question and theoretical orientation and so are more directed in theoretical underpinnings of the survey. This kind of orientation is thus absent in our survey, by design.

There are interesting things to report about all that we noticed as we read. We focused here, however, on those things that struck us as we began to look across all that we had captured. We formulated four main claims about these major findings. Our claims focus on: (a) where the research in this domain is being done; (b) how it is being done, (c) by whom, and (d) the consequences of these trends[9].

The claims presented below are not necessarily surprising. They reflect the progress we see in some areas. At the same time, we discuss some trends that we believe are troubling. Each of the claims presented is followed by three different comments – each a particular interpretation of the claim by one of us (authors). These multiple commentaries reflect our collective, and sometimes differing, views on the implications of what we saw for the field.

9 Additional aspects of the study will be reported in papers that expand on each of the claims presented below.

Emerging themes

Claim 1: Small-scale qualitative research predominates

By "small scale qualitative research," we mean studies that focus on a single teacher or on small groups of teachers (n<20) within individual programmes or courses. For example, 69% of articles we surveyed in the 1999-2003 PME proceedings were studies of this type.

Table 3 below shows a detailed analysis of 65 papers in JMTE. The first line indicates ten studies dealing with one teacher or teacher educator's learning. Take the following example from a Danish researcher, Jeppe Skott, who investigates very carefully how Christopher, a novice teacher, copes with the complexities of his mathematics classroom[10]. Studies involving two to nine teachers were those that focused on, for example, a study of a group of teachers within one school site or program. The third row in the table refers to papers reporting on investigations with, for example, an entire faculty; the fourth an institute or larger group; and the fifth refers to survey research, and so far larger samples of teachers in the study. The table also indicates what we referred to earlier as meta-studies, those that are theoretical or conceptual with no explicitly stated empirical base. Summing this up, there are 38 papers where there were fewer than 20 teachers in the study. Hence, we observed that a significant percentage of papers are small case studies.

Number of teachers	Number of articles (N=65)
1	10
2-9	18
10-9	10
20-99	14
100-553	5
No data or not claiming to be empirical	8

Table 3: Numbers of teachers studied in each JMTE article

Commentary 1 (Konrad)

The distribution of cases along the five categories including the dominance of small scale research is a mirror of the complexity of the field. For example, study groups at schools and even most entire mathematics faculties at schools have fewer than 20 teachers. A large number of pre-service teacher education classes or summer schools have these numbers of participants.

Only recently, given the results of international comparative studies like TIMSS and PISA, and the growing demands on a better teacher education and more knowledge about its effects, educational policy has begun to realize the importance of research in teacher education. This might give rise to bigger projects where large scale studies are done. In addition, it makes sense that in a new emerging field researchers first refer to a small number of cases, and even to studies of one single teacher, in order to better understand these particular cases and to further develop theoretical frameworks, methodologies and instruments. On that basis it is then easier to build on hypotheses that

10 Skott, J. (2001). "The emerging practices of a novice teacher: the roles of his school mathematics images". *Journal of Mathematics Teacher Education*, **4**, 1, 3-28.

can also be examined with regard to larger studies. From that point of view, it seems natural that the interest in particularisation precedes generalisation. Also, investigating teachers always means to put into consideration their interests, to share the goal of the research with them and to negotiate their role and part in the study. This is a difference, for example, to research on students where such questions of participation, communication, validation of results is not necessary. In addition, research in teacher education is often more complex since it deals not only with the beliefs and knowledge of teachers but also with students' beliefs and knowledge, as well as with the interaction between teachers and students, and the interaction between teacher educators and teachers. Thus, having teachers as the focus of research leads to high complexity. This increases the tendency to keep the sample small in order to reduce complexity. Teacher education needs both – the particular, and the general. However, there is also some general in the particular, and there is always the particular hidden in the general.

Small case studies have an advantage for the theory-practice relationship since it is easier to integrate teachers into research. Also, research results from such studies can be written in the form of "stories" which give an authentic view of practice and give principals, administrators, policy makers, etc. an insight into the complexity of change in the teaching profession. They are a good contrast to percentages which by non-experts often generate the view as if teacher education and teachers' growth is as easy as counting numbers and calculating a means. In addition, such stories are also a good starting point for working with teachers, in particular because they compare their situation with those of the case.

Finally, it is also interesting to reflect on the need expressed by policy makers for large scale studies. We need to engage policy makers and show them a single teacher, so revealing how complex teachers' learning is – and so avoiding falling into the trap of having some narrow conceptions of "best practice" that they, the policy makers, believe can be disseminated.

Commentary 2 (Fou-Lai)

Indeed, it is a natural state that particularization comes before generalization for an emerging field. Developing a theory of teacher learning is a key issue for research on mathematics teacher education: conceptualizing, modeling and theorizing are considered as three stages of development. Small-scale qualitative studies make great contributions for conceptualizing the complexity of teacher education and modeling individual teachers' learning process. Some of the reviewed case studies have developed models of individual teachers' learning[11]. Studies based on different perspectives naturally produce different results. The results of those in-depth small scale qualitative studies could be used as fundamental data for secondary analyses that seek to contribute to theory across studies. When theorizing, large-scale studies are needed for testing the hypothesis.

Commentary 3 (Deborah)

I agree with Konrad and Fou Lai, and want to elaborate the last point made by Fou Lai. Three types of studies are missing in the survey. There is a notable absence of large scale studies, and these are needed to understand the larger landscape of teachers' opportuni-

11 See for example, Chen, I.R. & Lin, F.L. (2004). "A beginning mathematics teacher becoming professional through action research". *Chinese Journal of Science Education*, **12**(1), 83-105

ties for learning around the world and within countries and to contribute to theories of learning to teach. For example we know astonishingly little about the range of ways teachers acquire – or don't acquire – the mathematical knowledge needed for teaching. Small scale studies don't help us sufficiently to understand at larger scale what these learning opportunities look like on large scale. Also notably missing are cross-case studies. There are strong beliefs about methods that help teachers to develop particular kinds of mathematical knowledge for teaching – and I use this only as an example. Without cross-case analyses, we lack opportunities to test those beliefs, to treat them as hypotheses and so to learn about how different approaches, programs and settings affect the content knowledge teachers need to learn how to teach. Finally, we also lack longitudinal studies. Many of the studies we looked at were short term. By way of example, teachers' knowledge develops across many years as they participate in professional development activity, use new curriculum materials, and meet new students. Without studies that follow teachers over time, our understanding of how teachers learn and under what conditions is lacking.

Claim 2: Most teacher education research is conducted by teacher educators studying the teachers with whom they are working

A focus here on JMTE and PME proceedings in the last 5 years bears out this claim most forcefully. Of articles representing research that focus on teacher education, 90% of JMTE articles and 82% of PME articles were of this type. Across all the articles in our survey, we see that articles of this sort amount to approximately 70%.

Commentary 1 (Jarmila)

This is the case because of the very nature of the teacher education profession. Mathematics teacher educators' professional responsibilities include both research and teaching. Research is one aspect of teacher educators' professional development. This kind of research is also an important part of teacher educators' learning to improve their practice. Finally, institutions of education differ from other kinds of institutions in that they provide direct access to teacher education practice and to school. There is thus ready accessibility for teacher educators' pursuit of important research interests.

Commentary 2 (Konrad)

Research done in the context of teacher education is a special kind of research that intersects practice. Teacher educators have the double role of intervening and investigating, or in other words, of improving and understanding. In addition, both aspects are strongly interrelated. This contributes to the complexity of this field.

We do need more external research, in particular, large scale studies. However, this will entail more specifically funded projects.

It also seems to be very important to engage teachers in research activities, for example by integrating them into research projects led by academic researchers or by supporting them to critically and systematically reflect their own practice within collaborative action research projects. Teachers tend not to read research papers within the context of their work, but being involved in such projects mentioned above, bridges might be built. It is important that teachers learn to balance nearness and distance, and that they gain interest in their particular challenges but also in the general problems.

Commentary 3 (Jill)
While agreeing with much that has been commented on above, I would like to add to the issue of nearness and distance. It is difficult, when you have an investment in who you are teaching, to take a sceptical stance towards that work. Important questions that need to be asked might be missed. So, a critical question is what we need to do to help ourselves do this. One way is to invite "external eyes" to gaze in with us on what we are doing. Another way is to develop strong and effective theoretical languages that enable us to create a distance between ourselves and what we are looking at.

Claim 3: Research in countries where English is the national language dominates the literature
For example, in JMTE between 1999-2003, 80% of the articles were from such countries. It is less stark, but nevertheless prevalent, in PME, where the percentage is 43%.

	JMTE (n=65)	PME (n=88)
North America	68% (65% U.S.)	30% (24% U.S.)
Oceana	8%	9%
Europe	15% (5% U.K.)	25% (6% U.K.)
Africa	3% (all South Africa)	8% (6% South Africa)
Asia	5%	9% (7% Taiwan)
South and Central America	0	3% (all Brazil)
Inter-continental	0	7%
Middle East	2% (all Israel)	14% (all Israel)

Table 4: Where is research being done? Two major examples

The detail in the Table 4 above helps us to focus in further. Presenting the information across regions at the same time hides some interesting phenomena inside regions. For example, in the Middle East, all of the papers we read were from Israel. Similarly, in Africa, all the papers were from South Africa, and in Asia, all from Taiwan. In North America, the vast majority are from the US, and, indeed, there is a remarkable predominance of US authored papers in JMTE overall.

Commentary 1 (Jill)
These disparities are not surprising. The prevalence and increasing hegemony of English was referred to in the opening ceremony of the Congress. But the disparities are deeply troubling. For some people in our community, their "local" becomes global. Their particulars become the basis of the general. In others, their local *remains* local – indeed does not even get heard. What problems, and whose problems then come to constitute the field? This is a critical question for us, particularly if we reflect back for a moment on the pictures of diverse learners across selected classrooms earlier in this article.

Commentary 2 (Fou-Lai)
Mathematics education, as a field of study, can be traced back 30 – 40 years, with strong roots in the United States, Europe and Australia. The presentation of Survey Team 1, ICME-10, showed the shift of research foci in mathematics education starting from a focus on curricula in the 1970s, then shifting to a focus on learners in the 1980s~90s, and more recently there has been a shift to a focus on teachers. These developmental

shifts seem natural since information resulting from research on curricula and learners very often are necessary as foundation for research on teachers. Those that start first, then can base their accumulated knowledge on curricula and learners to move on studying teachers. Thus, "first start-first" achieved is rather a natural development. The dominance of research from English speaking countries we witnessed is thus understandable.

However, there are other factors that exacerbate this dominance. Many students from other countries take mathematics education programs in the US, UK, Australia, Canada. When these students return to their homelands, and undertake research, they often base these on the perspectives they have learned from abroad. For example, the following topics are pursued: Changes of beliefs, growth of pedagogical content knowledge, and different degrees of awareness of the complexity of teaching. Studies that are based on the same research perspectives are often merely seen as replication, and thus rejected for publication. This stands in interesting contrast to the natural sciences where replicated experimental studies have their value. Replication studies in mathematics education are not favoured by journal reviewers. Comments from reviewers are that the research is not innovative and so not contributing to the field.

Recently (2003) a new international journal in mathematics and science education[12] has been launched with a support system for authors whose mother tongue is not English. In this Journal, the editorials encourage researchers to take societal and cultural practices into account. Hopefully, the publication of this journal will gradually change the phenomena of dominance of authors with English as their first language in the field of mathematics education.

Commentary 3 (Deborah)

As a person who comes from one of these English-speaking countries, I share the sense of how disturbing this is – of what we don't learn about and how we become persuaded that what we know from local settings is somehow more general in our field. And what this caused me to reflect on is what this might mean for the induction of new researchers where English is the main language. For instance, it is important in the education of new researchers to include the development of a disposition and set of skills to actively seek broader literature from more countries, to hold a more sceptical stance about beliefs and generalisations developed in one's own context or country. It is important to develop a stance that avoids confusion between the local and the global. And so it is important to be able to work (read and speak) in more than one language.

Moving on to our fourth claim: The first three claims combine to shape this emerging field in mathematics teacher edudation and we ask the question: *What* are the consequences for a field that is characterized in these ways: by a predominance of small scale qualitative studies *(how)*; teacher educators studying their own contexts *(who)*; and a predominance of publications from countries where English is a national language *(where)*. In other words, *the how, the who* and *the where* have important consequences for the *what* we are learning, and that takes us to Claim 4.

12 *International Journal of Science and Mathematics Education*, Kluwer, **1**, 2003; **2** 2004.

Claim 4: Some questions have been studied, not exhaustively, but extensively, while other important questions remain unexamined.

What has *been studied extensively?* We noted many articles that involve efforts to establish that particular programs of teacher education 'work'. Interestingly, you can understand how this particular trend follows from our second claim: that the research is often conducted by people studying their own program. One designs a program and one wants to show that it works. It is not so surprising that efforts to show that things work predominate.

We also found a large number of papers dealing with reform processes, particularly in the US. These include studies of teachers learning or relearning mathematics, teachers learning about students' thinking, their language, their orientations and pedagogical practices – and you can understand this as an instance of the local becoming the global. In the last case, efforts about math reform dominate with US researchers who are themselves involved in the program they are studying. And then those in the US get to publish more – we find our ways into the journals and this becomes a dominant theme in the literature.

We saw a large number of teacher studies in professional communities and in other institutional settings and we see this, in part, growing out of our first claim, and the emphasis we saw on small-scale qualitative work in the context where it takes place.

What has been studied less? We list here some important examples that we think are notably missing. Clearly you could make a different list as many things have not been studied, or studied less. We chose as a group to identify a small set of things of what is notably missing that has consequences for what we understand and can do in the practice of teacher education and in policy surrounding it.

We have studied less:

- Teacher learning outside of "reform" contexts – many teachers are struggling to develop their teaching skills in environments where reform is not the dominant issue; but assisting a wide range of learners at learning mathematics is. How does the dominant thrust of research on and in reform contexts help to understand this?
- Teachers' learning from experience – we know much less than we should *what* teachers learn from experience, *whether* teachers learn from experience, and what supports learning from experience. Teachers spend most of their time doing teaching. We understand far too little about what helps some teachers to develop from their own teaching while others do not.
- Teachers' learning to directly address inequality and diversity in their teaching of mathematics – we know far too little about teachers' learning to directly address inequality and diversity within their teaching of mathematics and here we include culture, gender, language, socio economic status and mathematical background.
- Comparisons of different opportunities to learn – we lack comparisons in the field that compare different opportunities to learn – how does one approach to helping teachers to learn mathematics compare with another? – we have studied these sorts of comparisons much less.

- "Scaling up": i.e. What happens when programmes spread to multiple sites – we have also done less of studying what it means to scale up or what it means to extend a program that has worked in one setting to another setting – what works, what goes wrong, what do designers need to know and think about?

Reflections on our Survey Team work

As with any research endeavour, it is important to reflect on one's own process of production. Before concluding with what each of the team believes is important for the advancement of the field, we offer some reflections on our work, reflection both on what we have and have not done.

What we have done

What we have presented, and how we have done so, are a function of our interpretation and enactment of our task and how we carried it out. We set out to survey the field over the past five years – since the last ICME; to take advantage of the diversity and expertise of our group; to develop ways to share and develop and communicate shared and contested interpretations of what we found; and to identify accomplishments of the field, as members of it, as well as ways in which the field can grow.

What we have not done

While accomplishing much of what we set out to do, we did not conduct a complete survey of literature around the world. Nor did we move on to systematically evaluate the *quality* of research in mathematics teacher education. In particular we have not commented on: the use and development of theory; the use of appropriate methods; the quality of analysis and how well claims are supported by evidence.

These are important tasks that remain to be accomplished. So we conclude now with brief comments from each of the team members as to what we see as directions for the field.

So: What now? Comments and directions for the field

Jarmila: I am speaking from the position of someone outside of main teacher education theories, but someone who has access and/or is trying to have access to them. The field needs to find ways to transcend cultural and language boundaries to profit more from multiple traditions and schools of thought. A good practice in this direction is international summer schools where colleagues from various places can meet and discuss and work together.

Jill: The field also needs to focus on is what it means to teach both *mathematics* and *teaching* in the same program. We do not understand well enough how mathematics and teaching, as inter-related objects, come to produce and constitute each other in teacher education practice. We don't know well enough what and how this happens inside a teacher education program, and then across ranging or contrasting programs, contexts and conditions. The field needs to understand better how mathematics and teaching combine in teachers' development and identities.

Fou Lai: The field needs better "local" (geographic, topic-specific, etc.) theories of teacher learning before trying to accomplish general theories about how teachers learn.

Konrad: More creative forms for presenting research results are needed, in order to represent the complexity of the field. The field has such variety and this could also be mirrored in the presentation of our research. For example, we need authentic and interesting stories, both practice-grounded *and* theory-driven, and combinations of "reflective papers" by teachers with cross-analyses by teacher educators. In order to overcome the gap between theory and practice – to support teachers to come nearer to our field – more action research is needed, combining first-order and second-order action research: Teachers investigate their practice, and teacher educators investigate their support processes.

Deborah: Teacher education research has been dominated by – and has profited from – small-scale studies, and from teacher educators studying their own contexts. For the field to grow to contribute to policy and practice, and to teachers' learning, however, we need to build capacity for smart, probing, comparative and large scale studies.

Acknowledgements

The survey team members would like to thank the ICME-10 Programme Committee for the opportunity provided in doing this survey.

In addition we all thank our various agencies for support to attend the Congress in Copenhagen. In particular, we acknowledge the contribution by Jarmila Novotna who was supported by the Research project MSM 114100004 "Cultivation of mathematical thinking and education in European culture".

P 7: Structure formation in nature as a topic of mathematics

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Introduction

Structure formation in nature is clearly one of *the* fundamental topics in the sciences. Humankind wants to understand all sorts of structure formation processes, concerning galaxies and planetary systems, continents, oceans, mountain ranges, and river systems, molecules and crystals, weather patterns and ocean currents, individuals (plants developing from seeds or animals from fertilized egg cells), genera, orders and species, and so on, all of which come about by natural structure formation processes. Understanding such structure formation processes is not just one of the basic topics of the sciences. It is also an important topic of mathematics. The basic task of the natural sciences in this context is of course to unravel the laws that govern structure formation processes. But what, then, is the task of mathematics in this context? Mathematics has two tasks.

Deducing consequences of model hypotheses

The first task is to develop tools for rigorously deducing the *consequences of any model hypothesis* regarding the laws at issue. When Newton fostered his idea of gravitational theory, he tried to identify and fix the parameters involved and then went on to make computations on that background. This went completely wrong, as his results deviated from reality by one order of magnitude. So he dropped that theory, until ten years later when he went back to look at his computations and realized that at some point he had a mistake of one order of magnitude. Revising his computations he came to the conclusion that his gravitational theory was actually more or less correct after all.

There is one big tool box, one "grand scheme" of mathematics, for studying consequences of model hypotheses, namely dynamical systems theory. This theory includes, in particular, the theory of ordinary and partial differential equations, discrete versions of those, difference equations, cellular automata, etc. They provide ways to actually compute solutions, also when we cannot find an analytic solution. The theory of dynamical systems relies on fundamental branches of mathematics such as analysis, differential geometry, topology, etc. Although all this is indeed one of the grand accomplishments in mathematics of great relevance to the topic of my lecture, – perhaps more than anything I am going to say – it is not, however, the topic I shall deal with here, as I am not an expert in the field.

Book keeping devices for encoding and comprehending features of structures

The second task left to mathematic is to develop tools for *encoding and comprehending the characteristic features of the structures* resulting from structure formation processes, for example Fullerene type molecules. There is no "grand scheme" of mathematics for doing this, comparable to that of dynamical systems theory. Yet many important techniques for sophisticated *"book keeping"* were developed over the centuries. Book keeping is basically what I do in my research, and that is what this lecture is about.

1 Now at Department of Combinatorics and Geometry, Shanghai Institutes for Biological Sciences, the Chinese Academy of Sciences

What are these book keeping devices? There are many more than it is possible to consider here. As I see it, Fourier theory is such a device. If we have a periodic signal and look at the frequencies of the various harmonics contained in it, we forget about the harmonics with small amplitudes and just take note of those with large amplitudes. That is probably what the signal is all about. More generally, wavelet theory, nowadays a very popular and very important tool of mathematics, is also a book keeping device. The fact that today we can listen to Caruso in a much better manner than fifty years ago is due to wavelet theory, which is the best tool to get rid of the noise in the old shellac records. So, wavelets are an interesting topic and aspects of it may perhaps even be addressed at some point in upper secondary school

Both of these theories aim at identifying "the important message" in a perhaps noisy time-series signal, which is actually an issue of book keeping. One tries to find out what the message is, or one tries to encode it with a very few short code-words. Yet, the crucial thing is that these devices do not depend on any model hypothesis or any theory, such as gravitational theory or quantum-mechanical wave-functions. They can be applied whenever we just want to obtain some results.

Another well known book keeping device is linear algebra and its close kin analytic geometry. One of the first messages we encounter in linear algebra is that in a vector space all bases are equal. It does not matter which basis you take, since you can express all the the vectors in the vector space as linear combinations of the basis vectors. However, a little later we learn that some bases are "more equal than others", which is an equally important message. For example, let us think of the theory of eigenvectors. Generically, if you have a nicely behaved square matrix, its eigenvectors form a basis, and it is often much more useful to use this basis rather than the "natural" basis of the standard unit vectors. This is for instance the case in principal component analysis, a tool which is applied in so many distinct areas, from protein analysis to the behaviour of electorates. Principal component analysis tries to identify the principal components that form the best coordinate system to represent a given set of data, which is exactly to say that certain bases are more equal than others, relative to given data. There are many more important instances of this phenomenon, but we have to leave them aside here.

In summary, the goal is to represent data in the most appropriate coordinate system. Of course, Fourier analysis and wavelet theory are just special cases of this more general approach, provided that you allow infinite-dimensional vector spaces to begin with.

Group theory and combinatorics

Group theory, too, is a book keeping device. It serves to classify various modes of symmetry, it offers qualitative and quantitative analyses of symmetry-related phenomena, achieving magnificent degrees of "reduction of complexity". If we take, say, crystallographic tables, they form a total of 5-6 volumes, now all stored electronically. Without these tables crystallography would not be the basic tool in materials science and many other disciplines that it actually is today. The tables help to organize all we know about crystals by reducing the phenomenological complexity of crystal structures.

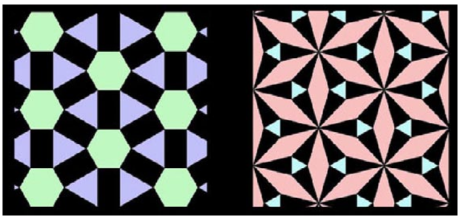
Combinatorics is another interesting book keeping device. It offers conceptual tools for constructing large varieties of highly complex discrete structures or objects, such as Fullerenes. Combinatorics deals with – and supports – the systematic enumeration of such structures or objects. You may want to know – or construct – just one

Fullerene; or you may want to know how many Fullerene type structures (assuming that we know what that is supposed to mean) we can have. You may have the idea of packing just hexagons or pentagons together on a sphere in such a way that at each vertex three of them meet. If you want to have, say, 60 vertices, how many distinct Fullerene structures does there exist with this specification? The answer is 1812.

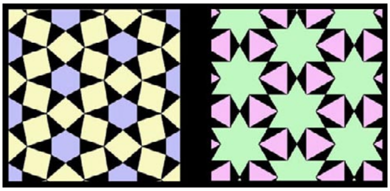
I shall restrict myself to looking here at just two connected topics lending themselves to book keeping devices. Firstly, we shall consider the task of classifying and designing *tilings* as models of molecular and crystal architecture by means of tiling *symbols*. This will introduce you to a piece of mathematics, which is not really new from the point of view of today but was new about twenty years ago. Secondly, we shall look at the task of reconstructing *phylogenetic trees* by means of *similarity analysis*. These trees try to capture the bifurcation of species. For example, we are related to monkeys, we now believe, to chimpanzees, to the great apes – the gorillas – and to orangoutangs. It is likely that everybody would agree just by looking at the animals that chimpanzees are closer relatives to us than to orangoutangs, although orangoutangs look only a little bit different from us. It is an intuitive similarity analysis that leads us to this idea. Now, if you look at human beings, gorillas and chimpanzees, the classical point of view was that man is further away from the other two. Chimpanzees and gorillas are the closest cousins of human beings, but man broke off a little earlier from the evolution that led to chimpanzees and gorillas. Now by means of molecular analysis it actually turns out that this is not the case. In fact *we* are the closest relatives of the chimpanzees, not the gorillas. The task of phylogeny is to reconstruct all such evolutions. Researchers now think that they should and could reconstruct one large phylogenetic tree for all two million known living species of plants and animals. (We are also aware that there are many more bacteria than the ones we do know, e.g. in the soil, billions probably. They are not included in the project.) In later sections I shall say something about the fundamentals of the activity leading to such a tree.

Tilings, molecules and crystals

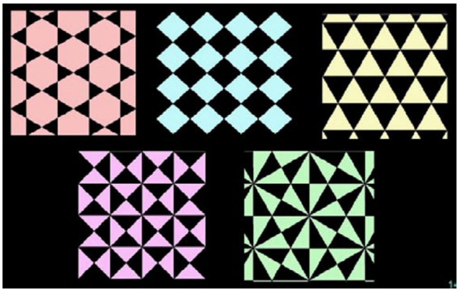
What is a tiling? In the real world, everybody knows, of course. For instance there are magnificent specimens in the Alhambra in Spain. Below we have two particular tilings. They contain different sorts of *tiles*, i.e. building blocks. In this particular representation they come in different colours (shades), and tiles of the same colour are symmetry-equivalent.



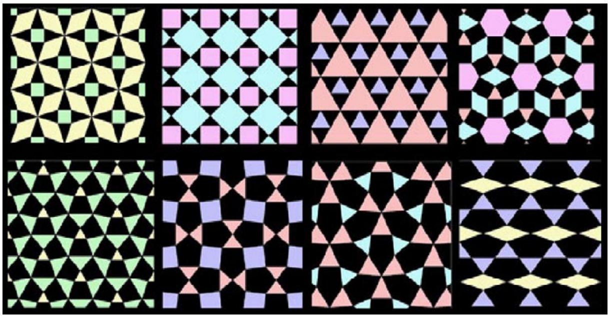
There is always a symmetry that maps the tiling into itself while mapping each tile of a given colour into another tile of the same colour. Here are two other examples.



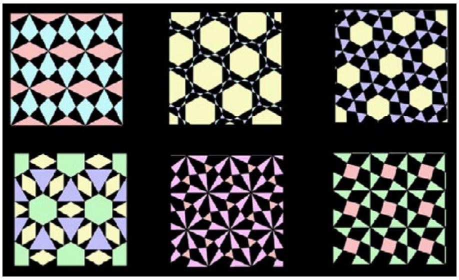
Note that in all four examples, the coloured (shaded) tiles share an edge only with black ones, not with tiles of another colour, and that black tiles share an edge only with coloured ones. So, these two pairs of tilings are coloured in such a way that we do not need drawn edges, colours are sufficient. Of the black tiles there is only one type, whereas there may be more than one type of non-black tiles. But there are more such tilings, like the following five:



They are very simple, composed of one type of black and one type of another colour. The following eight tilings each make use of one type of black and at least two types of coloured tiles:.

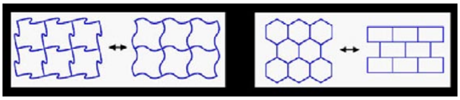


All tiles of the same type, black and coloured tiles, are symmetry equivalent. It is still the case that a black tile only has a joint edge with coloured tiles, and a coloured tile only with a black one. This property holds for the last six tilings as well:



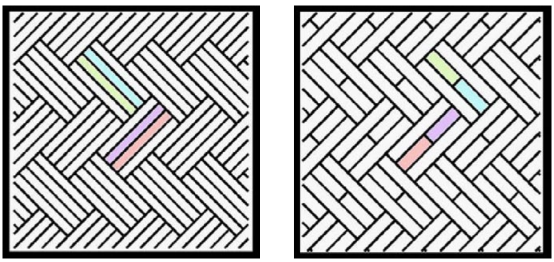
We have seen a total of 2+2+5+8+6 = 23 different types of planar tilings. The question then arises: Are there more such tilings? The answer is "no!" There is a theorem that states that there are exactly 23 such planar tilings, and above we have seen a representative of each type. This result resembles the famous classical theorem – known by the ancient Greeks – that there are exactly five different kinds of platonic bodies, i.e. regular polyhedra – three dimensional object with certain symmetry properties. The 23 types of planar tilings have been constructed and drawn by the compter programme "R.E.P. Tiles " (R.E.P. for "repetitive") written by Daniel Huson and Olaf Delgado (www.mathematik.uni-bielefeld.de/~huson/approach.html) for studying and designing 2D and 3D tilings. With this computer programme at hand, you do not have to write a paper proving that there are only 23 planar tilings. It is sufficient to ask the computer how many tilings there are of a given type.

Now, what exactly does "exactly" mean? Of course, by scaling up and down it is possible to get an infinitude of tilings of any sort. So, the finiteness of the different sorts of tilings has to lie somewhere else. Below we have two pairs of tilings that we consider to be intrinsically – i.e. topologically – identical, so we will not distinguish between them. In what sense are the two pairs topologially identical?



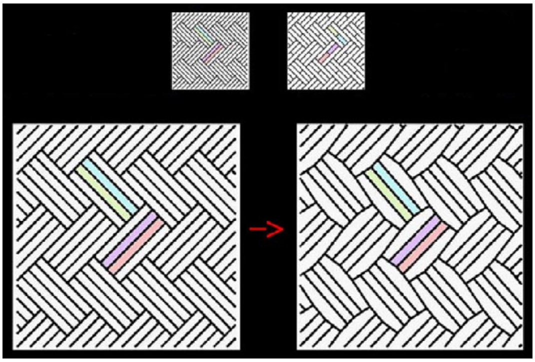
By a well-behaved transformation we can move one tiling in the left pair into the other without destroying the topology and without destroying the symmetry of the tiling, so the two tilings in that pair are apparently the same. If we insisted on distinguishing these two tilings we would never, of course, get a finite number of tilings. But that is not so different from what we do in the case of the platonic bodies. Whether we have a small cube or a large cube we just say that we have a cube. Now, let us look at the hexagonal and the brick tiling(s) in the pair to the right. Topologically they are the same – we can transform one into the other – but their symmetries are not the same. The hexagonal tiling has a higher symmetry. Every symmetry of the brick tiling is also a symmetry of the hexagonal tiling, but the converse is not true. For instance rotation by 120^0 is a symmetry of the hexagonal pattern but not of the brick pattern. But the topology is the same. Sometimes we will not take changes of the symmetry groups for the tilings into account, sometimes we will.

The most basis feature is the topology. That even this is not a trivial issue can be demonstrated by two tilings that were first studied by Lothar Collatz.



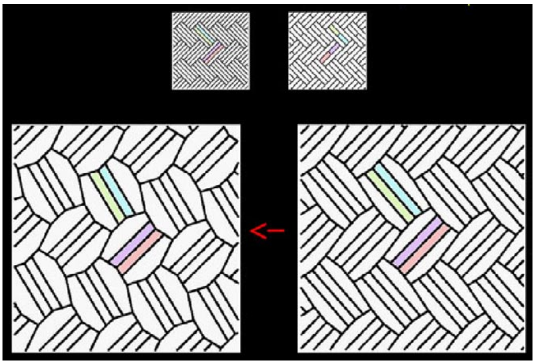
Two (crucial) tiles in each tiling have been coloured (shaded). Certainly the two tilings look very different. However, are they intrinsically distinct or are they just two representations of the "same" tiling? In other words does there exist a homeomorphism of the plane that transforms one tiling into the other? That was Collatz' question. Apparently the coloured tiles in the two patterns are the ones that give rise to this question.

Let us first note that counting arguments do not help to establish that the two tilings are intrinsically distinct. From a topological point of view the coloured tiles in both tilings are all quadrangles, each of which has four other tiles as neighbours. The other tiles are, topologically speaking, octagons. They do not look like octagons, but if we count the number of neighbours – eight – for each of them we see that they are indeed octagons. In both tilings vertices have degree three, i.e. at every vertex three tiles come together. So, these counting arguments do not suffice to prove that the two tilings are not the same. Nevertheless, Lothar Collatz believed them to be intrinsically identical, without being able to settle the issue. So, the question is whether this is correct, and of course I would not ask this question if Collatz was not right. They are in fact topologically identical. If so there must be a way to transform one into the other in a "legal" manner. And indeed, Olaf Delgado discovered a way to transform one into the other as we shall now see illustrated by a sequence of eleven transformations.



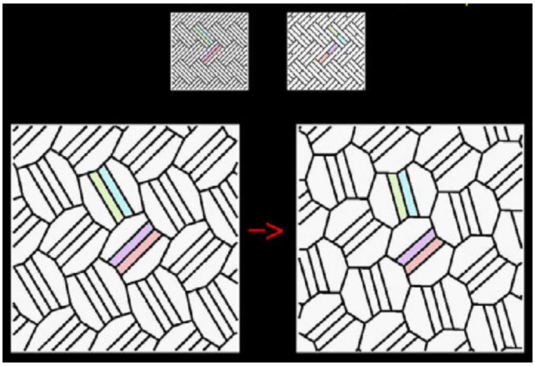
Transformation #1

The first transformation of the original pattern into a second one is depicted in the above figure. It is rather straightforward. Now the second pattern is transformed into a third (from right to left) one as seen below.

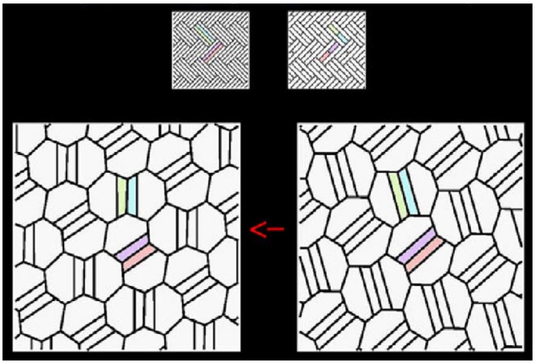


Transformation #2

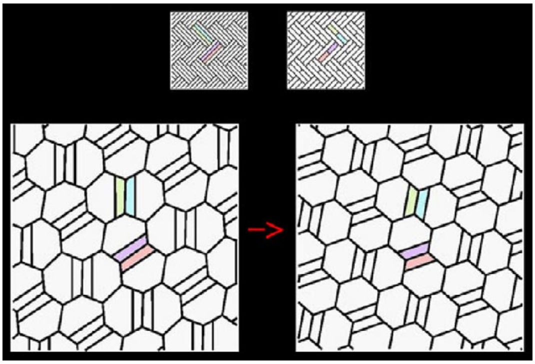
In each of the following transformations the middle part is contracted / compressed a little bit. Note that the odd-numbered transformations go from the left to the right pattern, the even-numbered from the right to the left.



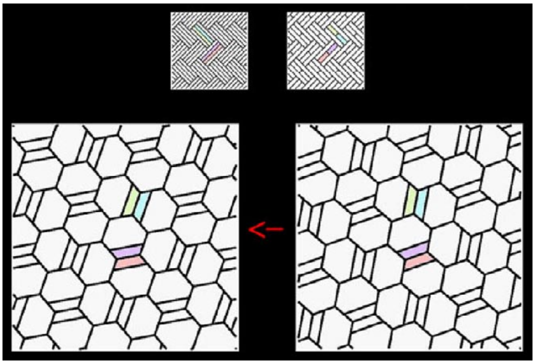
Transformation #3



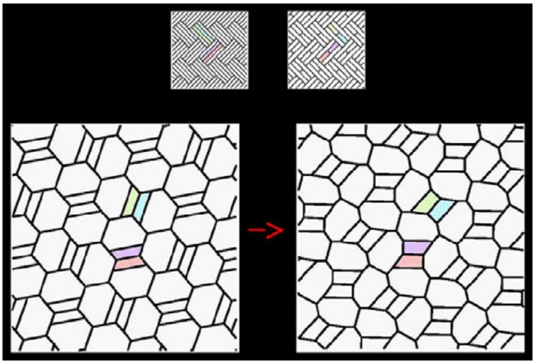
Transformation #4



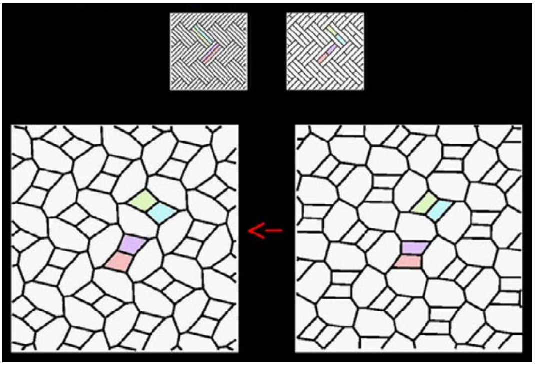
Transformation #5



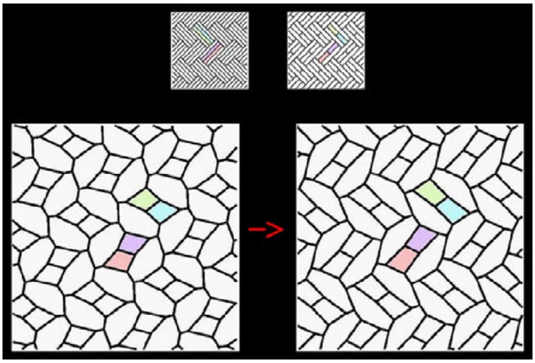
Transformation #6



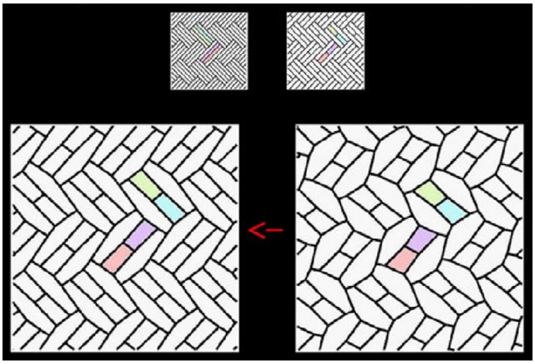
Transformation #7



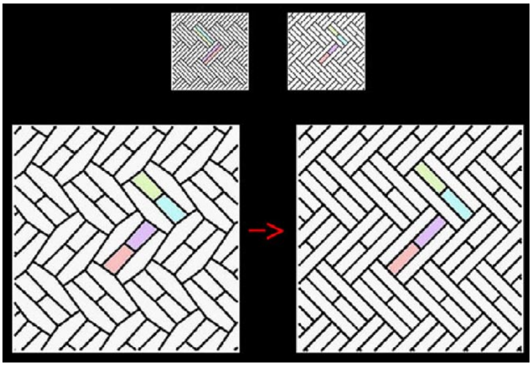
Transformation #8



Transformation #9



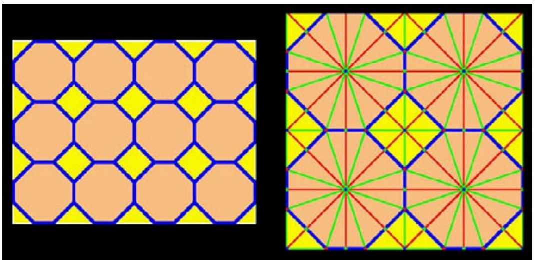
Transformation #10



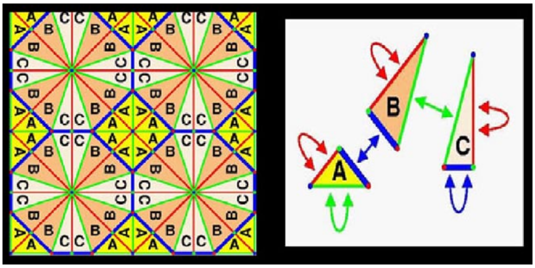
Transformation #11

What helped Olaf Delgado make this construction was that he saw that there must be a third tiling that is the same as each of the two original ones, That tiling is the result of transformation #5, the pattern on the right hand side. He constructed the resulting transformation by transforming this tiling both into the first and into the second Collatz tiling. Lothar Collatz was quite amazed to see this when he visited our group in Bielefeld. Sadly, he died a few months later.

Now the important question is, how do we establish such facts as mathematical facts? The transformation is visually impressive, but what is the mathematics behind it? To address this question let us consider just any tiling such as this one:

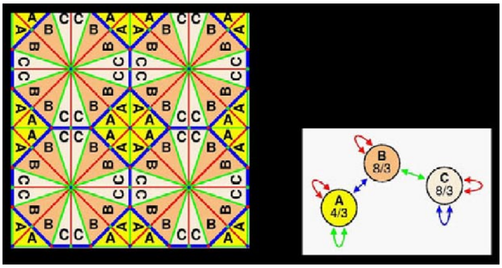


We now want to construct the so-called barycentric sub-division of this tiling. That is the pattern on the right hand side of the figure above. In addition to the vertices of the octagons, marked with light dots, we put another point (dot) in the centre of each tile and one in the mid-point of each edge. These points are used as new vertices in a construction where each centre vertex is connected consecutively to all the vertices on the boundary of the tile. This results in a set of triangles, which is already quite a simplification. Rather than having a complicated tiling – here consisting of octagons and quadrangles, but it could consist of many other tiles and be much more complicated – we now, after barycentric subdivision, have a tiling that consists of triangles only. Moreover, the resulting triangles all have one (blue) edge, stemming from the original edge, plus one light (green) edge and one dark (red) edge, The light (green) edges are those that connect the original vertices with the centre of the face, while the dark (red) edges connect the mid-point of an edge with the centre of the face. So, we not only get triangles, we get edge-coded triangles.



Now assume that we wanted someone, say a child, to reconstruct the whole pattern by means of the triangular building blocks only, without providing any information about the octagons or the quadrangles, or what have you. Then the first rule the child would have to learn, gluing the blocks back together, is that red a red edge should be glued to a red edge, a blue to a blue and a green to a green. Moreover, the symmetry group still acts on these triangles, because they were canonically constructed from the original structure. There are three classes of such triangles. All triangles in the original quadrangles are symmetry-equivalent. They constitue type A. All the triangles opposite an A-triangle are symmetry-equivalent. They give rise to type B. Finally there is type C, consisting of all the triangles.that are adjacent to a B-triangle. They, too, are symmetry-equivalent.

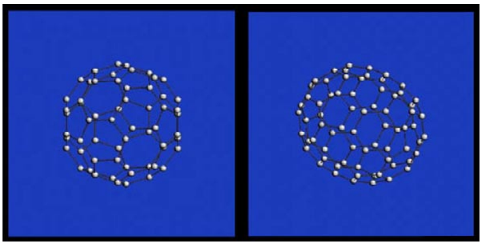
Imagine that the child has access to an infinite supply of triangles of each type A, B, and C, with colour-coded edges. How are these triangles to be glued together? The second rule is the following: On the green edge of an A-triangle, you put another A - triangle. On the red edge of an A-triangle another A-triangle has to be put. On the blue edge of an A-triangle you put a B-triangle. On the green edge of B-triangle a C-triangle has to be put. These rules are incapsulated in the right hand figure above, which serves as a kind of "blue-print" for the gluing process. However, these rules are not quite sufficient to complete the construction. For, if the child takes just any A-triangle and keep gluing another A-triangle on its red edge, and on its green edge yet another A, and so on and so forth, how does the child know when to stop? Of course, if we were dealing with claasical geometry, the triangles would all be rigid ones, each with well-defined angles. The gluing process would then stop exactly when these angles add up to 360^0. But if we have topological triangles – and we are trying to understand things up to topology – they are not rigid and do not have rigid angles. So the stopping rule has to be one that tells you how often you are allowed to glue the blocks together. The best way to encode that is by saying first how many pairs of triangles sharing a green edge are contained in a face. So within each octagonal face there are 8 pairs of triangles sharing a green edge between them and red edges with the adjacent pairs. The same question is interesting for going around the vertices. Around the vertices of the face we always have three pairs being put together along blue edges, and within the faces there are eight B-C pairs and four A-A pairs. This information is captured in the diagram to the right in the next figure. This diagram is called the *symbol* of the tiling, again a blue-print for its construction:



The above explanation may warrant a comment especially addressed to mathematicians. These symbols present *orbifolds*, as defined by William Thurston, in the way that simplicial complexes "present" topological spaces. In the 1920's, much topology was done in terms of simplicial complexes, a key notion in combinatorial topology, whereas orbifolds were defined much later. The tiling symbols *present* orbifolds, i.e. they are not orbifolds but they present them the way a simplicial complex present topological spaces. The problem is this: For each topological space there are many presentations and we want to understand when two presentations actually define the same topological space. In the 1920's and 1930's simplicial complexes gave rise to combinatorial topology. An analogous situation arises here, as the tiling symbols are presently giving rise to the notion of combinatorial orbifold theory. I shall state a few important results of that theory here. All the results are claims up to topological equivalence.

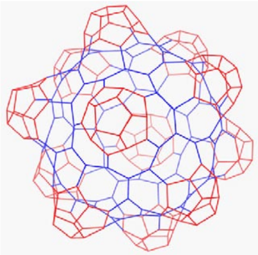
Any tiling of any simply connected manifold – like the plane, the sphere, three-dimensional space – is uniquely determined by its symbol. That is the fundamenal theorem in this context. Moreover, Olaf Delgado and Daniel Huson showed that up to and including dimension 3, all symbols of Euclidean tilings with a bounded number of building blocks can be determined algorithmically, as can the associated tilings. In fact that is what their programme "R.E.P Tiles" does, at least in dimension 2.

As a striking example of tilings you can construct on the basis of these results let us take the Fullerenes. The figure below shows two of them. Gunnar Brinkmann who was also working in my group wrote a specific programme dedicated to dealing with Fullerenes (www.mathematik.uni-bielefeld.de/~CaGe/).



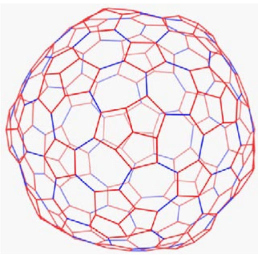
The webpage in Bielefeld is called CaGe because all the objects have some form of a spherical architecture, a cage-like structure. Nowadays these strcutures are much used by chemists who want to understand such cage-structures in molecules.

But there are more interesting objects of these kinds. Fowler -a chemist from the University of Exeter in the UK – said, well Fullerenes are built from hexagons and pentagons, and the Euler formula tells us that there must be exactly 12 pentagons and any number – except just 1 – of hexagons. But what if we tried to work with heptagons and pentagons instead? If we insist on icosahedral symmetry, the smallest possible object would have sixty heptagons and seventytwo pentagons. Do such Fullerenes – according to Fowler, Fantasmagorical Fullerenes, or Fulleroids – exist? With pentagons, three of them put together create positive curvature. With heptagons, if we put three of them together we get a kind of saddle, which creates saddle-type negative curvature. The next figure depicts such a structure.



It is interesting to see that here we have twelve heaps of six pentagons put together; there is high positive curvature in these places, There is negative curvature in the valleys. From a chemical point of view the structure is probably very fantasmagorical. One would guess that it cannot exists. At least it would be very unstable.

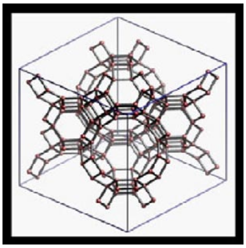
The next structure is more homogeneous. We have only three pentagons coming together at some points and just single pentagons surrounded by heptagons in other parts of the structure. This structure is more spherical. By using our theorem it can be proven that these are the only two examples of structures built from heptagons and pentagons.

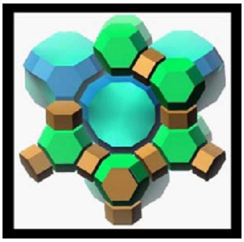


Applications to crystallography

Let us look at the relevance of our findings to crystallography. This takes us from two to three dimensions. Fullerenes have a 3-dimensional structure in 3-space but as they are really just tilings of the sphere they are intrinsically 2-dimensional. But crystals really live in 3-space. As an example we can take Faujasit, which is a zeolite.

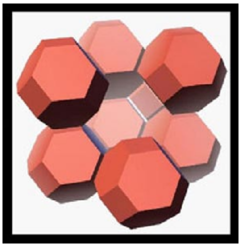
By using X-ray crystallography we can the find its atomic positions. And by connecting any two atoms that are relatively close to each other, we obtain a network. We cannot claim that the links between two atoms represent a chemical bond, because in inorganic chemistry bond is not such a clearly defined concept as it is in organic chemistry, and crystals are a matter of inorganic chemistry. Rather, the links represent geometrical close-ness. Now the network can be filled it with tiles, which results in the so-called Deloné (or Delauney) tiling associated with the network.





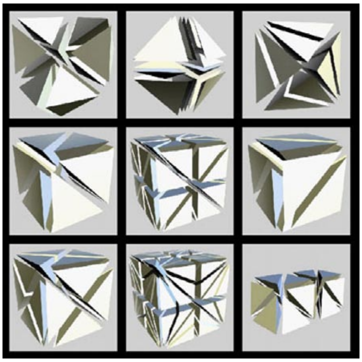
So, in this manner we arrive at a tiling of space. Sometimes it is better to modify it a little bit so that we do not get too many inequivalent tiles. The figure above is such a modified tiling. Now if we take out the big tiles we get a net with cages, which are of particular interest to crystallographers and chemists, because chemically interesting things can happen within them. For instance they can be used to purify petrol.

Crystal design is an important issue. Can we, for instance, find all distinct simple crystal tilings? By a simple crystal tiling we mean a tiling of 3-space, in which all vertices are symmetry-equivalent – as in the figure below – and have tetrahedral vertex figures. This means that just four edges or four faces come together at a vertex. We can think of them as forming a kind of dual of a tetrahedron.



Olaf Delgado, with the help of Daniel Huson, managed to find all simple crystal tilings, i.e. all geometrically possible simple crystal tilings. They showed that there are exactly nine simple crystal tilings. The proof was based on a proof of the dual theorem, that there are exactly nine distinct tilings of 3-space by symmetry-equivalent tetrahedra.

Aristotle (382-322 AD) claimed (in De Caelo III, 306b) that the regular tetrahedron can be used to tile 3-space. I do not know when people found out that that this is actually wrong, but it is easy to see just by computing the angles, thus realizing that it cannot work. Probably Euclid could have told him this. But here are the nine tetrahedral tilings of space by means of *non-regular* tetrahedra:

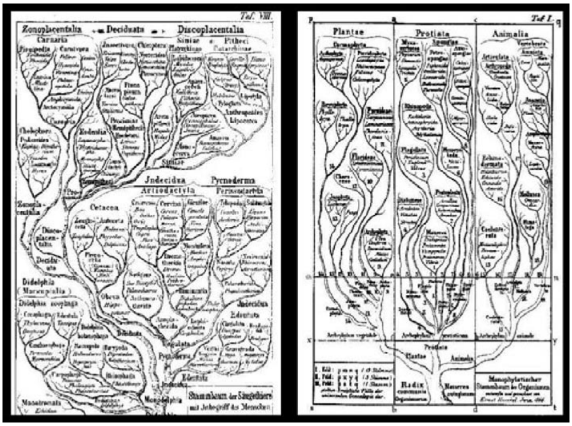


Actually seven of them can be realized by zeolites – e.g. diamonds – whereas the other two are probably chemically improbable.

The conclusion of this section is that while Cartesian coordinates allow us to describe arbitrary complex, rigid geometrical objects in purely algebraic terms, and thus to study such objects by using simple algebraic manipulations, tiling symbols allow us to describe rather complex topological objects in purely combinatorial terms, and thus to study such objects by using simple combinatorial manipulations. In both cases the manipulations can, in general, be performed by computers. So, you can do geometry on the computer thanks to Decsartes, and you can do tiling theory on the computer thanks to Olaf Delgado and Daniel Huson.

Reconstruction of phylogenetic trees

Even if it is difficult to see the details of the pictures below, they do give an impression of what we are going to deal with.

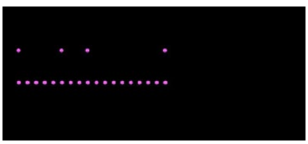


These are plates from *Monophyletischer Stammbaum der Organismen* (the monophylogenetic tree of the organisms) by Ernst Haeckel (1834-1919), published in 1866. After Darwin had published his *On the Origin of Species* in 1859, Haeckel, only seven years later, published a big book in two volumes on the systematics of living organisms with many plates. What we have above, to the right, is the phylogentic tree of all living beings, or organisms, plants and the animals in the picture on the right hans side, and mammals, especially, on the left hand side. Nowadays *the* monophyletic tree would of course look very different, as the eucharyotes and the bacteria were not known at Haeckel's time. On the left we see *Stammbaum der Säugetiere* (the genetic tree of the mammals), leading up to us, *homo sapiens*, at the top end of the tree. In an earlier version, homo sapiens was also at the top, while a little below the top we found *homo stupidus*. By that Haeckel presumably meant the Neanderthal, *homo neanderthalensis*. Now we are politically correct and refer to this species as *homo sapiens neanderthalensis*, while we have become *homo sapiens sapiens*, although from time to time one has some doubts about the justification of this label.

What is behind the idea of phylogenetic trees? Let us take a look at some of the heroes of taxonomy. The authors of so-called "herbal books" of the 16th and 17th centuries list all that is., as they saw it. They included dragons and elephants, even though they had not seen either creature, but you better record what other people have been recording. In their lists they listed plants separately from animals, but when it came to animals they distinguished between flying animals, like insects and birds, non-flyting animals, and swimming animals. Their clsassifcation systems are not accepted today, but it sort of worked for their time. But there were other authors, e.g. John Ray (1628-1705) an interesting character in England. He wrote his treatise *Theologia Naturalis* with the purpose of understanding the order in nature. Ray was the first to propose the concept of the invariance of species. We should study nature to learn about the divine and unvarying order of creation. Later on many English parsons went out into nature to study plants, rocks, fossils, and animals, a study which was justified by the good service it did to God to try to understand his creation. Another hero of taxonomy was Maria Sybilla Merian (1647-1717), who recognized, presumably as one of the first, that there is a 1-1 correeespondence between certain types of caterpillars and certain types of butterflies, and close a correspondence between these and certain types of plants that caterpillars eat.There are also other people, now forgotten, August Quirinius Rivinus (1652-1723) from Germany, and Joseph Pitton de Tournefort (1656-1708) from France, who all worked on the basis of the idea that there is order in life, and tried to classify and to understand it.

And the most famous of them all, not forgotten at all, were Carl von Linné (1707-1778), who devised his *Systema Naturae*, and the above-mentioned Ernst Heackel. How could Haeckel publish his phylogentic tree so soon after the publication of Darwin's work? Did he not first have to study all these living beings to understand how they are derived from one another? No, he could base his classification on Linné's binary classification into plants and animals, which in turn were sub-divided into invertebrates and vertebrates, and vertebrates again into mammals and other vertebrates,, and so on So, what Haeckel did was just to re-write this into a dynamic tree, i.e. a dynamic evolutionary interpretation of the static order found before, by Linné and others.

In a phylogenetic tree, similar species should appear in close proximity. However, this is not always possible, because of what we could call the notorious intransitivity of similarity. The fact that *a* is similar to *b*, and *b* is similar to *c* does not imply that a is similar to c. So, now we can see why mathematics and mathematicians are involved in the construction of phylogenetic trees. The key tool is hierarchical cluster analysis. We can get an idea of what is gong on by looking at the figure below. First, there are four points. If we were to cluster them, we would probably begin by clustering the two closest ones together, and then take the three leftmost ones, and finally all four points. This gives us three different clusters. Now if a few more points were added between the leftmost ones, our original cluster system would change, i.e. transitivity is missing. Trying to make similarity transitive would lead us to cluster all points together, not exactly a result that contains much information.



Hierarchical cluster analysis comprises a large body of methods for inferring collections Γ of hopefully significant and, if possible, non-overlapping subsets C (clusters) in a large set X of species – proteins, languages, manuscripts. Perhaps we want to identify the indo-germanic languages among all languages, the germanic or the celtic languages among all the indo-germanic languages. Or we want to find within the class of all proteins the sub-class of proteins with a specific function. Or we want to distinguish the mammals among all species from the non-mammals and so on and so forth.

What does "significant" mean in this context? This notion has to be formalized if we want computers to assist us in our cluster analysis. Let us agree to call a subset C of objects in X *significant* if all objects in C are more closely related to each other than to the objects in X\C, e.g. the subset M of mammals within the set X of metazoa are more closely related to each other than to the animals in the complement. So any two mammals should be thought of as more closely related to each other than any mammal is related to any non-mammal.

Next, what does non-overlapping mean? Here is a very formal definition: A collection Γ of subsets of X is *non-overlapping* (or *linnean*) if any two subsets in Γ are either disjoint or one contains the other. For example, if X are the metazoa, M the mammals, B the birds, and V the vertebrates, then {M,B,V} is a linnean set system, because every vertebrate, every mammal, every bird is metazoa, and because every mammal is a vertebrate, every bird is a vertebrate, but the set of mammals and the set of birds are disjoint – except in Greek mythology. Linnean systems are what we need for classification. For instance when the duckbill – partly a bird because it has a beak, and partly a mammal because it has fur – was discovered in Australia and specimens were sent to England, people first thought it was a hoax, just made up to make fun of the authorities of the Royal Society, and it took quite some time before the Royal Society realized that these beasts really exist. Then the animal had to be placed somewhere in the taxonomy. Eventually it was placed in the group of prototheria, with the mammals rather than with the birds.

Linnean set systems are sparse. None contains more than at most #X – 2 non-trivial subsets (i.e. of cardinality distinct from 0, 1, and #X). If you have, say, a thousand species to classify you cannot have more than 998 subsets with the property that any two subsets are either disjoint or one is contained in the other. This is an example that is typical of a mathematical result with which the biologists have to live. It is not a hard result to obtain but it serves to show that simple combinatorics is of relevance in this context. So, an arbitrary collection Γ of subsets of a given set X can be represented by a phylogenetic tree if and only if it is linnean. In other words, this property of being either disjoint or one contained in the other is equivalent to being representable by a phylogenetic tree. That is why Haeckel so quickly could construct his famous phylogenetic trees of the living species, which actually display a linnean structure. Haeckel believed in creation and thought that we recognize God's own order of creation when we recognize species and relationships among species. By the way, he tried to do the same for minerals. He knew a lot about minerals and studied them in great detail. He tried to classify them in a hierarchical fashion in a linnean system. Of course that classification did not survive. It does not make sense for minerals because minirals are not subject to Darwinian evolution as we understand it today.

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P

Plenary Session 7

One major problem in hierarchical cluster analysis is that we often find either too few clusters, which do not allow us to distinguish to a satisfactory extent, e.g. mammals and birds or certain proteins from other proteins of course that is not fortunate.

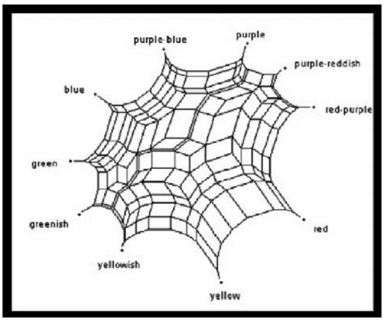
Another major problem is that we find pairs of clusters that appear to be significant, yet do overlap. This is the case for instance with the classes of warm-blooded animals and flying animals, respectively. Both concepts seem to be very reasonable but apparently they overlap – as there are warm-blooded animals that do fly – without one being contained in the other. What can we do to deal with this problem? We can either modify or drop one or both clusters, if they overlap, or we can accept both clusters and work with *phylogenetic networks* instead of phylogenetic trees, although the latter are, of course, the ultimate goal of phylogenetic analysis.

A natural approach to obtain a clustering would be to identify for all three objects *a*,*b*,*c*, in a given set X, that pair – *a*,*b*, or *b*,*c* or *c*,*a* – that clusters *best* realtive to the third objects, and then go on to find all non-trivial subsets A in X such that for all *a*,*b* in A and for all *c* in X\A *a*,*c* cluster best realitve to *c*. These subsets would indeed form a linnean cluster system, but often there are not enough clusters.

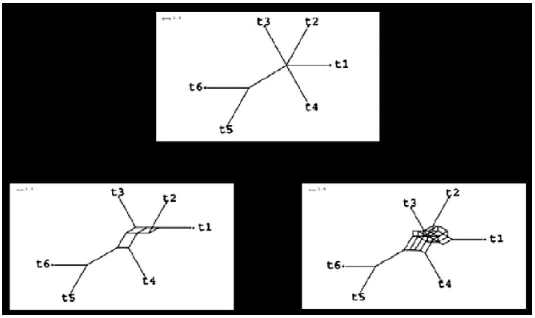
Let us take an example. We have X = {nightingale (1), man (2), alligator (3), dragon fly (4). We might begin by clustering {1,2} versus {3} because both nightingales and human beings are warm-blooded, whereas alligators are not. We could also cluster {1,3} versus {4} because both nightingales and alligators are vertebrates, but dragon flies are not. Or we could cluster {1,4} versus {2}, because both nightingales and dragon flies fly, in contradistinction to human beings. So, there really is no good system of clusters that would capture all these separations. Assume that C is a cluster with at least 2 distinct elements. Now, if 2,3 or 3,4 or 4,2 are in C then 1 has to be in C as well. If 1 and 3 are in C then 2 must be in C as well. If 1 and 4, are in C, then so is 3, and if 1 and 2 are in C so is 4. This implies that C = X, which is not a very informative clustering.

If we wanted to find more clusters we might, for any three objects *a*,*b*, and *c* in the given set X, identify that pair – *a*,*b* or *b*,*c* or *c*,*a* – that clusters *worst* relative to the third object, and then go on the find all non-trivial subsets A in X such that for all *a*, *b* in A and for all *c* in X\A *a*, *b* do not cluster worst relative to *c*. The resulting cluster system would not necessarily be linnean. Yet it is not too bad as it does not contain more than $O((\#X)^2)$ clusters.

In order to get an impression of what it would be like to work with phylogenetic networks rather than trees, let us take a look of one such network of perceived colour similarity



If the network is constructed with the use of automatic methods based on hierarchical cluster analysis, we get a colour circle rather than a colour tree, but that is actually all right.. The colours can be depicted in the following networks which we can think of as generalized trees. In the first tree we see that 5 and 6 are different from the rest. In the lower left tree 1, 2 and 4 are separated from the rest, while 1, 2, 3 are separated from the rest in the lower right tree. Although this does not give rise to disjoint clusters anymore, the network representations are indeed quite satisfactory.



The network of the human mitochondria DNA is really a network much more than a tree:

However it does have some tree-like features. One interesting feature which is captured in this network is that there are Africans everywhere in this human mtDNA. That probably means that the people who claimed many years ago that homo sapiens came out of Africa and spread to the rest of the world were right. So, 160,000 years ago there was a group of maybe 2,000 people to whom something happened, a small genetic change, probably not so much in the genome itself but in the way the genes were activated. And from that change homo sapiens somehow emerged. First they spread all over Africa, and them some of them went out to Europe, Asia, to Australia and to America. This can actually be deduced from the universal presence of African human mtDNA components in the network. So, with mathematical techniques that allow you to construct these structures automatically, we can see in just one glance why biologists nowadays claim that homo sapiens came out of Africa.

P

Plenary Session 8

P 8: Mathematical landscapes and their inhabitants: Perceptions, languages, theories

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Introduction

One of the main problems in teaching mathematics consists in the dramatic gap between the *worldly truth* in which our students make their concrete experiences and the *logical truth*, which represents the rigorous official side of mathematics (what in French is called *le savoir savant*).

The main goal of this lecture consists in analysing this gap. For reasons of clarity, the arguments will be exemplified by discussing some concrete cases, mainly concerning the concept of function (in a wide sense of the word)[1].

Each example shows a different perspective for analysing the genesis of the mathematical concepts in the classroom and a possible way to fill up the gap. I shall show that the two aspects above must be considered within a more complex landscape, where they do not appear so dramatically contrasted and where mathematical objects do really live.

The lecture is divided in four parts. In section 1 a concrete example will introduce the subject and its main components: a theoretical model will be defined to frame the complex processes that take place in the classroom when a mathematical concept is built up. In section 2 these will be investigated from different points of view and with different tools, all important processes in the economy of mathematics teaching. In section3 an example will illustrate the model. In section 4 some final conclusions will be drawn.

1. The meaning of the mathematical objects in the classroom

In order to introduce some important theoretical issues I shall sketch an example of a young girl, Eleanor (9 y.o.), who is introduced to the concept of function as a representation of motion. The example is taken from a case study; it has been video recorded and analysed by the research team of R. Nemirovsky (Nemirovsky et al., 1998). Eleanor makes experiences with a motion sensor that measures her distance from the tower on a table (see figure 1, left). Distances vs. time are recorded in a Cartesian graph which appears in real time on the screen of a computer (figure 1, right). Eleanor can see the screen and move the device as she likes. Doing that, she enters into some of the many different meanings of a function. Of course only the video can show completely what is happening, since not only Eleanor's words but also her gestures and body motions are important for understanding the multiform way in which she is able to build some of the meanings of a function. I shall try to give an idea of this discussing four excerpts from the video.

1 The examples are taken mainly from a joint Italian National Research Project, funded by MIUR and by the Universities of Genova, Modena, Pisa, Roma, Torino (COFIN3 n. 20030110729). The project involves researchers from different Italian regions and recent joint studies with R. Nemirovsky (TERC, Boston, USA) and L. Radford (Univ. Laurentienne, Ontario, Canada). I thank all of them for having allowed me to quote their findings. I wish to thank particularly: M. Bartolini Bussi, L. Bazzini, P. Boero, F. Ferrara, L. Giacardi, M.A. Mariotti, M. Menghini, O. Robutti, C. Sabena and, last but not least, all the teachers who have contributed to our research activities, especially C. Dané and D. Paola.

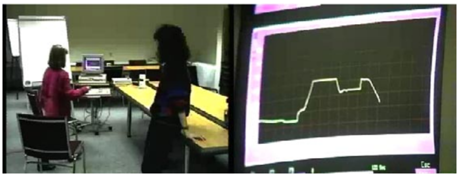

Figure 1

EPISODE 1. ***Eleanor and the device: first approaches***

1. Teacher:*"And this is how it works. I push F1 to start, and you can move that. ... And it's going to respond to the tower. So ..."*
   [Eleanor moves the device with her arm up, down, right, left and observes what happens on the screen]
2. E: *"Let me move farther away"* [E goes back slowly looking always at the screen to see what happens during her motion]
3. [At a certain point the graph stops its increasing and has a horizontal line segment (figure 2a)]
   E: *"If I move it … Maybe this is the farthest it can go"*

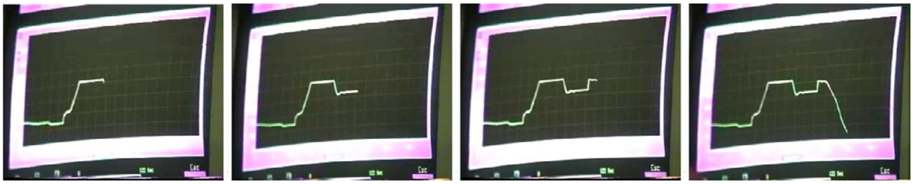

Figure 2 a b c d

   E: *"What if I move it higher?"*
   [E goes a bit forward then raises the device remaining immobile (the line segment remains horizontal: figure 2b); goes a bit back and raises again the device remaining immobile (figure 2c); then goes towards the tower (figure 2d)]
   "The closer to the tower it gets, the lower, I think"
   [she goes slowly back again looking at the screen]
4. E: *"OK, I'm going to try and make a pattern".*
   [E goes back and forth regularly and produces the graph of figure 3a]
   E: *"Actually this is not exactly the same pattern".*

EPISODE 2. ***Looking at what has happened before***

5. T: *"Let's stop it and look at it for a minute".*
   [E follows carefully the graph on the screen with her finger, figure 3b]
6. T: *"What was happening?"*

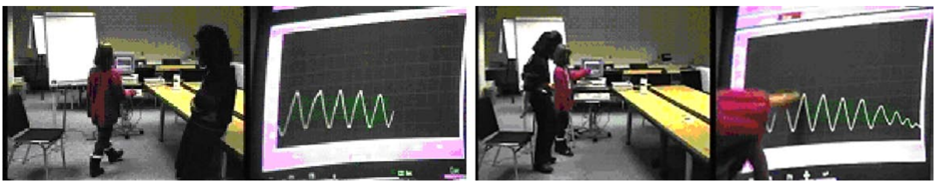

Figure 3 a b

7. E: "Well, I was going far – I was going like far, and a little bit closer but still far away then". [she repeats with her arm back and forth the movement she had done with her body]
   E: *"I was really going like this but kind of changing a little".*
8. T: *"So the line up was when you were walking …?"* [T points the line on the screen]
9. E: *"When I was walking backwards, and the line forwards was that way".*
10. T: *"And then all the … the whole thing has a sort of a shape too, doesn't it?"*
11. E: *"Yeah, it's all, …* [E inclines her head to look again at the different parts of the graph; she follows again the graph with her finger] *… like zig-zags through that side, but I mean they're all … they look like, kind like mountains or something".*

EPISODE 3. ***Interpreting the vertical line: distance over time***

12. E: *"Let's see … I wonder if you could get it to go straight up?"*
    [she follows the graph very fast with her finger]
13. E: *"Not like diagonal. Probably you couldn't because if it would go straight up it would have to just be the same time, because it's moving along* [she makes a horizontal movement on the screen with her hand across the graph], *no matter what you do"*
14. T: *"Right, it's…moving along in time?"*
15. E: *"Yeah. So you'd have to kind of stop the time and go like that.*
    [with her arm taut, E points the finger to the screen and produces the form of the graph on the screen]
    And go like this. [E moves back and hints the movement she had done previously]
    Because, because it's moving along that way or this way the same time".
16. E: *"It's going that way. So it kind of goes like, instead of just going like this...*
    [she makes a vertical movement on the screen with her forefinger]
    ... it kind of goes like that probably this".
    [she makes a slow oblique movement on the screen with her forefinger]

EPISODE 4. ***Slope and velocity***

17. T: *"Do you think you can make a steeper line than this? Maybe you can't make it go straight up but maybe you can make it a little bit ..."*
18. E: *"Maybe, maybe if you do it faster".*
19. T: *"OK, shall we try that?"*
20. E: *"No, I'm not going to worry about like ..."* [first E runs twice back and forth, then she stops and continues moving only the arm back and forth twice ...]
    "... and if you just go slowly" [then she runs again but very slowly ... figure 4]

Figure 4

The four excerpts illustrate how Eleanor can experience the concept of function as a model of her motion: interacting with the device through her motion and discussing with her teacher she realizes the relationships between the geometric properties of the graph and the properties of her motion. For example: the position of the prompt on the screen with respect to her distance from the tower (lines 3, 4, 7, and 9); the inclination of the line as an index of her speed (lines13, 18, and 20). The discussion shows that she is able to enter into concepts in a deep way. See for example the discussion in Episode 3, where Eleanor essentially develops the idea of slope as speed and as distance over time; she can do this because she is pushed to interpret the vertical lines in the diagram by the questions asked by the teacher. See also the Episode 4, where she tests her conjectures about the relationship between slope and speed, disregarding the inessential variable of the form of the graph and concentrating only on its slope.

All this appears very far from the concept of function that we find in the "holy" books of mathematics. For example, in Bourbaki's *Éléments de Mathématique* (1970, §3, n°4, p. EII.13) we find:

> Definition 9… a correspondence f = (F,A,B) is a function if for every x belonging to the domain A of f the relation (x,y)∊ F is functional in y ...; the unique object which corresponds to x under f is called the value of f at the element x of A, and is denoted by f(x)…[2]

Of course, there is a big gap between the *logical rigour* of Bourbaki and the *worldly experience* of Eleanor. But it is exactly this experience that allows Eleanor to enter into (some of) the meanings of the concept of function. In fact in teaching mathematics the first issue cannot be that of rigour. This has been pointed out by René Thom, who in his plenary lecture at the second ICME wrote: *"The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects"* (Thom, 1972; p. 202).

The word *meaning* in the teaching of mathematics entails at least the following components (taken from Vergnaud's definition of concept, Vergnaud 1990):

- *the reference situations* ("le sense", e.g. the environment in which Eleanor operates);
- *the operating invariants* ("les signifiés", e.g. the action she is doing first – Episode 1 – to produce regular graphs on the screen, and then – Episodes 3, 4 – to produce graphs which are more or less steep);
- *the external representations* ("les signifiants", e.g. the graphs on the screen, her gestures).

Our example puts forward a big didactical issue, namely the necessity of analysing the processes that happen concretely in the classroom in order to provide reason to the ways in which students can grasp the meaning of the mathematical concepts. To condense it into a slogan, our task is: *How to bridge the gap between the logical truth of Bourbaki and the worldly truth of Eleanor?* More specifically, the *real problem* consists in focusing on the genesis of the mathematical objects in the classroom.

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2 *… une correspondance f = (F,A,B) est une fonction si, pour tout x appartenant à l'ensemble de départ A de f, la relation (x,y)∊ F est fonctionelle en y…; l'objet unique correspondant à x par f s'appelle la valeur de f pour l'élément x de A, et se désigne par f(x)…*

Now, mathematical objects live in the classroom as clusters of different partially overlapping meanings due to the different perspectives according to which they are introduced. It is as if a person were introduced to a few drawings of a building made from different points of view and was then asked to rebuild the whole image in her/his mind.

In fact the teacher possibly introduces the different meanings (through examples, more or less rigorous definitions, analogies, and so on) but students are asked to get the whole out of it.

My claim is that the different perspectives can be combined in a shared environment for cognition. I call it *the cognitive space of action, production and communication* (APC space). An example of APC space is given by episodes 3 and 4 in Eleanor's protocols. I shall give a further example in section3, but to do that in a suitable manner I need to introduce its main ingredients and the tools necessary for their analysis.

The APC space is built up, developed and shared in the classroom. Its main components are:

- the body;
- the physical world;
- the cultural environment.

When students learn mathematics all these components (and possibly others, e.g. emotional ones) are active and interacting.

The APC space is built up in the classroom as a single dynamic system, where the different components are integrated each other into a whole unity. The integration is a product of the interactions among pupils, the mediation of the teacher and possibly the interactions with artefacts.

The three letters A, P, C illustrate the dynamic features of this system, namely the fact that three main components characterise the learning of mathematics: students' actions and interactions (in the situation at issue, with their mates, with the teacher, with themselves, and with tools), their productions (e.g. answering a question, posing other questions, and so on) and communication aspects (e.g. when the discovered solution is communicated to a mate or to the teacher, using suitable representations).

Here is a definition of the notion of APC space: an APC space is the unitary system of the three main components listed above, amalgamated into a dynamically evolving unity within a concrete learning situation in the classroom, though the action and mediation of the teacher, who orchestrates their integration in a suitable manner.

The system is not immediately active, neither does it exist only because the components are present in the class. As such it is a product of the amalgamating action of the teacher, who coaches the integration of the components. Sometimes the integration is successful, sometimes it is not: in the process the teacher is like a cook making mayonnaise. In fact an APC space is a typical complex system, which cannot be described in a linear manner as resulting from a simple superposition of its components. It is beyond the aims of this paper to analyse the reasons why the integration process may possibly fail, even if this is an important question.

The APC space model allows us to properly study the so-called *perceptuo-motor* features in the processes of knowing (Antinucci, 2001; Nemirovsky, 2003). More specifically it allows us to consider how action and perception determine the processes of learning and to describe them so that doing, touching, moving and seeing appear as their important ingredients. In fact, they are essential not only in the first phase of the cognitive development of children but also in the learning processes of older students.

This shifting towards the perceptuo-motor side in the approach to knowledge has been pointed out by many studies in the field of neuroscience in the past few years (see Lawson, 2003, Gallese, 2003). It is illustrated properly by the following quotation:

> "a) Mathematical abstractions grow to a large extent out of bodily activities having the potential to refer to things and events as well as to be self-referential.
> b) While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based on how we have learned and used the subject itself. [Moreover,] that of which we think emerges from and in these activities themselves.
> c) [As a consequence,] the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context."
> Nemirovsky (2003, p. 1-108).

Eleanor's episodes illustrate this very well: the modelling of velocity through the slope of the distance/time diagram is grasped by the pupil through a typical perceptuo-motor approach (see Nemirovsky & Borba, 2003).

This approach challenges the traditional one mainly based on the transmission of content through formal language, which is called *symbolic-reconstructive* by Antinucci (2001). In fact mathematics is often conceived as a pure formal language necessary for treating abstract concepts (the logical truth of Bourbaki!).

The three components of the APC space allow us to consider both the symbolic-reconstructive and the perceptuo-motor ways of learning within a unifying environment, where all such processes can develop.

Of course, a learning approach based on perceptuo-motor activities requires suitable modalities of teaching, in which the students are actively involved in the construction of mathematical concepts, as in the example of Eleanor. In this perspective, the artefacts that are introduced in the didactical practice can support and mediate, in an essential way, the construction of the experiential base that is necessary for learning: this is particularly easy today by means of suitable technological devices.

In analysing the APC space one must enter into its components and scrutinize them: so one obtains a list of its many ingredients. These include pupils' sensory-motor experiences, the embodied templates that they activate, the languages, signs, representations they use to interact with the environment (mates, teacher, artefacts and so on): again think of Eleanor. As pointed out before, the APC space does not result from the simple juxtaposition of its components; on the contrary, its ingredients must be amalgamated in a systemic and organic way. To obtain this amalgamation the work of the teacher is essential. In fact all the ingredients are always present in the class and one major task for the teacher consists exactly in producing a positive interaction among them. Generally speaking, a good teacher can do this almost in an unconscious manner. What I shall try to do here is to found this unconscious work on a more scientific and

systematic basis. To do that one needs suitable tools to investigate the main ingredients of the APC-space, e.g.:
– *history* to analyse the cultural environment;
– *ergonomy* to analyse cultural artefacts;
– *semiotics* to analyse signs and languages;
– *neurology* and *psychology* to analyse perceptions, actions and gestures.

Of course the same tool can be used to analyse different ingredients (e.g. gestures can be studied also through semiotics; languages can be studied also with psychological and neurological instruments, and so on). The list above gives only an idea of the different complementary tools which are necessary to carry out the multi-faceted analysis of the APC space.

2. The cognitive roots of mathematical objects

In this section I shall discuss the different tools that are necessary to investigate properly the APC space ingredients, listed above. I shall do that continuing my exemplification through the function concept.

2.1 Historical analysis

The function concept, as any mathematical concept, can be analysed in its genesis, developments and changes in the course of history. One can pursue its epistemological and historical roots (for examples of such an analysis see: Grattan Guinness, 1970; Monna, 1972; Youschkevitch, 1976; Ponte, 1990).

Some of these roots also have cognitive and educational interest, as widely discussed in the literature (e.g. see: Vinner & Tall, 1981; Freudenthal, 1983; Kleiner, 1989; Gravemeijer & Doorman, 1999; Lakoff & Núñez, 2000). Among others, the following historical roots have an important cognitive counterpart:

Motion (Oresme, Galilei, Newton, …)
Graphs (Leibniz, Euler, …, Klein, …)
Equations (Leibniz, Euler, D'Alembert, ...).

More specifically, such roots put forward different *concept images* ([3]) of functions, which illustrate their *multi-faceted features* that are historically, epistemologically, cognitively and educationally important. Only a multidimensional analysis of mathematical objects can help us in focusing on their specific components within the APC space. Let me spend a few words about motion and graphs as *cognitive roots* ([4]) of the function concepts.

Some of the roots of the function concept were developed as relationships between concrete, dynamic and continuous variables, to express the idea of change and phenomena of motion. This is a very old story. In fact our ancient colleagues lacked a mathematical description of motion. They saw distance and time as measurable quantities, but

3 A concept image in the brain is 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes,' (Tall & Vinner, 1981, p.152).

4 Tall, McGowen and DeMarois (2000) proposed the following definition of cognitive root. It is a concept that: (i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence; (ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction; (iii) contains the possibility of long-term meaning in later developments; (iv) is robust enough to remain useful as more sophisticated understanding develops.

not ‘velocity’. In fact the notion of change, according to Aristotle is philosophy, was only of a qualitative nature and had a very wide meaning (Generation and corruption, Alteration, Augmentation and diminution, Local motion). Ideas changed in the Middle Ages and it was in the XIV century that new revolutionary conceptions ripened in Oxford, at Merton College, and in Paris, with Nicole Oresme. Middle Age philosophers had realized that qualities have an *intensity*. This is a concept of degree. For example, ‘hotness’ may have different degrees of temperature. Qualities have also an *extension* or *quantity*. This depends on how widely distributed the quality is. Example: the concept of the ‘quantity of heat’ in a body. The same is true of motion, because motion can be viewed as a kind of quality of a body. Now velocity can be seen as an ‘Intensity’ of motion. Nicole Oresme discussed this point in his book published in 1353, *Tractatus de configurationibus qualitatum et motuum* (Treatise on the configurations of qualities and motions). There he developed techniques for graphing qualities: a horizontal line represents the subject and a vertical line represents the intensity of the quality. Shape of the graph shows distribution of the quality through the subject (figure 5).

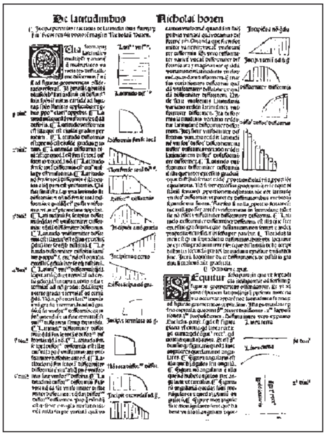

Figure 5. From Oresme’s *Tractatus de configurationibus qualitatum et motuum*

Examples (figure 5) are given by a body moving as a unit, but non-uniformly. The subject line becomes time, rather than spatial extension. So, quantity of motion is the area of the graph: which is the distance traversed. This makes it possible to apply geometry to claims about motion. For example the ‘Merton Rule’ or ‘mean speed theorem’ says that if a body accelerates uniformly from v_1 to v_2, then it covers the same distance as a body travelling at $(v_1+v_2)/2$ over the same time. Thus it was possible to measure changes and to represent them in some way (figure 6).

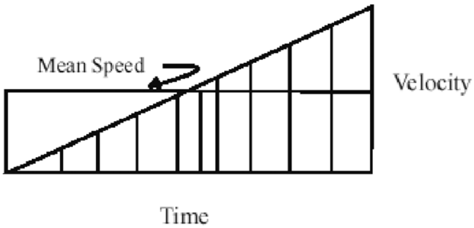

Figure 6

That motion is the right way to look at mathematical quantities and at their relationships is very transparent in Newton: for instance, he used the term *fluent* to indicate independent variables. His dynamic conception appears in the following quotations:

> "I don't here consider Mathematical Quantities as composed of Parts extremly small but as generated by continual motion …
>
> These Geneses are founded upon Nature and are every Day seen in the motion of Bodies." (Newton, 1964, QC, p.141)
>
> "The Conception is very easy and natural: We see by continual Experience that all Kinds of Figures are actually described by the Motion of Bodies." (Newton, 1964, F, p.33)

These historical data point out that the necessity to understand motion generated a concept-image, which would have generated a fresh concept-definition of a mathematical object, the function, whose name was to be given some time later (it seems that the word function was introduced only in 1673 by Leibniz, to indicate how some geometrical quantities, e.g. subtangents, depended on the shape of a curve: see Ponte, 1981).

Another aspect illustrated by the examples is that the visual and geometrical aspects have been crucial to support the developing of the function concept from the beginning. This was explicitly stated by Euler. He stressed the importance of the graph because it is a geometrical object and its geometrical features could suitably incorporate the properties of functions:

> "… let's look for a method equally convenient to represent any function of x geometrically.
>
> … Thus any function of x, geometrically interpreted in this manner, will correspond to a well defined line, straight or curve, the nature of which will depend on the nature of the function." (Euler, 1743, p. 4 -5; translation by the author)

The link between the geometrical and the kinematic aspects was represented by time, which was the ingredient through which a motion could be analysed and described in a geometrical fashion. Newton explains clearly how to achieve this:

> "But since we do not consider time here, any farther than as it is expounded and measured by an equable local motion; … therefore … I shall have no regard to time formally consider'd, but shall suppose some one of the quantities proposed, being of the same kind, to be increas'd by an equable Fluxion, to which the rest may be refer'd, as it were to time; and therefore by way of analogy it may not improperly receive the name of Time …
>
> I [did not] mean Time in its formal acceptation, but only that other quantity, by the equable increase or fluxion whereof, Time is expounded and measured." (Newton, 1964, F, p. 49)

However since the XIX century, e.g. with Bolzano and Dirichlet, and later with the so-called 'rigour paradigm' in the foundations of analysis (i.e. with Weierstrass and his school), time and motion have been abandoned as possible sources of misunderstandings (but see the critical comments in Lakoff and Núñez, 2000, on what they call Weierstrass' masterpiece). For example, Peano and people working in set theory or in functional analysis started to use a more abstract de-contextualised concept-definition. Eventually, the modern Bourbakist definition of function definitely abandoned the idea of motion and change and eliminated any reference to time. We are left with a *formal* definition, to be contrasted with the *natural* ones involving time and motion. This distinction is used here in the sense of Pinto (1998) which Tall describes as follows:

> "A *natural* approach builds on concept imagery to give a personal meaning to the formal definition. This means constructing examples of the definition that are sufficiently generative to be used as a basis for thought experiments to imagine possible theorems and possible strategies for proving them.
>
> A *formal* approach, on the other hand, focuses essentially on the definitions, using formal deductions to build theorems in a manner that attempts to avoid any appeal to intuition." (Tall, 2001, p. 204; emphasis in the original)

Students must be nurtured with natural examples that later can generate the modern definition. The natural approach in our discussion represents both a cognitive and a cultural basis upon which concepts must be built. The example of Eleanor underlines this approach (figure 7):

T: *So what happened? Do you remember what you were doing here?*

E: I was going more slowly

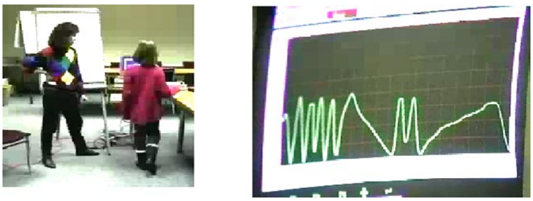

Figure 7

Phenomena of change and motion can produce a positive cognitive resonance in pupils and support their learning. Moreover, today appropriate technological artefacts offer the opportunity of showing functions as objects which represent how things change. A first example (e.g. see Laborde & Mariotti, 2001) is given by DGS (Dynamic Geometry Software, e.g. Cabri géomètre or Geometer's Sketchpad with their dragging, trace and animation functions). A second one (e.g., see Nemirovsky & Borba, 2003) is given by probe devices used in collecting data of moving objects and people in real time, and in representing them, e.g. on the screen of a calculator.

These tools represent a perceptuo-motor approach as discussed above and are natural in the sense of Pinto and Tall. An important feature is that this approach entails both the cognitive and the cultural roots of the function concept. I shall discuss the relevance of this two-sided approach in section 4.

2.2 Cultural artefacts

As we have pointed out, cultural artefacts can be used as mediating tools supporting a perceptuo-motor approach to mathematics. In fact, the artefact, under the guidance of the teacher, may produce, in a 'natural' way, important cognitive and didactical effects. To describe the dynamics of such processes, the approach elaborated by Rabardel (Rabardel, 1995; Verillon & Rabardel, 1995) is useful. It allows us to describe the role of artefacts within an APC space. I shall quote at length from Arzarello & Robutti (2004) to describe this point:

> "In this approach, the technical devices are considered with two interpretations. From the one side, an object has been constructed according to a specific knowledge, that assures the accomplishment of specific goals; on the other side, a user interact with this object, using it in different ways. So, the object in itself is called an *artefact*, that is, the particular object with its features, realised for specific goals. And it becomes an *instrument*, that is, the artefact with the various modalities of use, as elaborated by an individual, who is using it. The notion of artefact refers to the object, with its characteristics, while the notion of instrument is referred to the subject who uses the artefact, with particular modalities, related to a specific task. So, the instrument is conceived as the artefact together with the actions made by the subject, organised in collections of operations, classes of invariants, and utilisations schemes. The artefact, together with the actions, constitutes a particular instrument: so, different subjects can have different instruments using the same artefact, or the same subject can use the same artefact as different instruments.
>
> To make an example, the compass is an artefact, which can be used by a student to trace a circle, as the locus of points in a plane at the same distance from a fixed point. The transformation of the artefact in an instrument is made through the action of pointing it at a point, and tracing a curve with a fixed ray. But the same artefact can be used by a sailor on a sea-map, to control and measure distances between points on the map. The transformation of the artefact in an instrument is made through the action of pointing it at a point, and opening it up to another point. So, this instrument is different from the previous one, for the actions involved by the different subject, who has a different purpose.
>
> As different and coordinated utilisation schemes are elaborated successively (by the subject, with her/his actions), the relationship between the artefact and the subject can evolve, causing the so-called process of *instrumental genesis*, revealed by the schemes of use (the set of organised actions to perform a task) activated by the subject. In principle, it is not assured that this evolution is consistent with the original purpose for which the object has been designed. While the artefact is an object that can be considered static, in the sense that it doesn't change its features in time, the instrument can be thought dynamic, in the sense that it can change its features, according to the schemes of use activated by the user."

In our example, Eleanor is the subject who uses the device in different ways and with various actions in order to achieve different goals. The process of instrumental genesis can be integrated with the construction of knowledge, because Eleanor, solving a task with the device, does not only press or move it and herself in order to obtain a result (whatever it may be), but she must also control it, interpret it correctly and use it in her conceptual path through the task. As Lagrange wrote, the artefact "acts as a mediator for the action of students.... meeting new potentialities and constraints, the students have to elaborate utilisation schemes, potentially rich in mathematics meanings" (Lagrange, 1999, p.200). For example, the conceptualisation of the velocity concept through the slope of the curve in Eleanor's Episode 4 is a result of the instrumental genesis produced through her elaboration of the utilisation schemes of the device she is using.

2.3 Perceptions and actions

One of the main components of an APC space is the body. In fact, in learning processes we can observe a complex interaction and intertwining of many ingredients with/of our body:

– gestures, – glances, – speech, – signs

I shall elaborate a bit on gestures and signs in the next two sections.

2.3.1 Gesture: Old and new

As Vygotsky pointed out in his pioneering work, the genesis of gestures is a product of social interaction. Discussing the meaning of pointing gestures with children he wrote:

> "The grasping movement changes to the act of pointing.It becomes a true gesture only after it objectively shows all the functions of pointing for others and is understood by others as such a gesture" (Vygotsky, 1978, p. 56).

Recently the analysis of gestures and their role in the construction of meanings has become relevant not only in psychology, but also in mathematics education. Gestures are considered in relation with speech, and with the whole environment where mathematical meanings grow: context, artefacts, social interaction, discussion, etc. As such they are an essential ingredient to consider in the perceptuo-motor approach to knowledge.

Gesture analysis reveals an important tool for focusing the cognitive processes of students when they communicate and reason on a mathematical activity. Such an analysis requires the integrated contribution of different perspectives, e.g. mathematics education, psychology, neuroscience and semiotics. It is beyond the aims of this paper to enter into details. I shall confine myself to sketch some of the main items, namely:

- from psychology, the *Information Packaging Hypothesis* and the notion of *match* vs. *mismatch*;
- from neurology, the notion of *peripersonal space*;
- from semiotics, the idea of *semiotic means of objectification* (Radford, 2003).

The first two items will be discussed in this section, while the last one will be developed in the next one (§ 2.3.2).

In psychological research, Alibali, Kita and Young (2000) considered the Information Packaging Hypothesis (IPH) to describe the way gesture may be involved in the conceptual planning of messages. According to the IPH, gesture helps speakers to "package" spatial information into verbalisable units, allowing for alternative ways of encoding and organising spatial and perceptual information. Within the same perspective – that gestures play an active role not only in speaking, but also in thinking – *gesture-speech matches* and *mismatches* are defined (Goldin-Meadow, 2003). A match occurs when all the information conveyed by a gesture is also expressed in the uttered speech; a mismatch happens in all other cases. Mismatches are the most interesting since they appear to be a stepping-stone on the way toward mastery of a task.

Gestures are interesting also from a neurological point of view. On the one hand, there are deep connections between gesture and language, and hence between gesture and thought: "gesture is actually part of language and must be considered along with it" (McNeill, 2004, Ch. 7, p.1). McNeill illustrates how these connections live physically in the Broca area of our brain:

> "Broca's area is more than a 'speech center'. It is the action-orchestrating area of the brain under some significance – that is, the area of the brain that assembles sequences of movements and/or complexes of moving parts into performance units with the property of internal coherence, and with meaning. It compiles movement packages unified by goals, meanings, and adaptability ...
> ... performance units are also unified by imagery – a primary organizational factor – and Broca's area may be where actions (articulatory, manual) are organized around gesture images (visuospatialactional)." (McNeill, 2004, Ch. 7, p.1).

On the other hand, gestures are interesting since they contribute to realise the so called peripersonal space of a subject, which is important when people interact with each other or with tools. The peripersonal space is a physical and a cognitive space of action, which one can reach with one's body (shadow included) and possibly with artefacts (say a stick, but also a laser-pointer):

> 'The primate brain constructs various body-part-centred representations of space, based on the integration of visual, tactile and proprioceptive information. These representations can plastically change following active tool-use that extends reachable space and also modifies the representation of peripersonal space.' (A. Maravita et al., p. R531).

An example will show how gestures constitute an important ingredient of learning, hence of the APC space. The problem below illustrates a task given to 10th grade students in a teaching experiment of ours. In this case the students work in pairs, with or without the aid of a computer: they are used to work in groups; they know what a function is and also they have the habit of using finite differences for discussing the properties of function graphs; however they do not yet know the fact that if the second differences ($\Delta 2$) are constant one gets a parabola.

> Problem. Table 1 illustrates some data of a function B = f(A): $\Delta 1$ and $\Delta 2$ are respectively the first and second differences
> – Draw a sketch of the graph of function B = f(A);
> – Explain the strategies that you have used.

The videos of the discussions in some pairs of students illustrate manifestly how gestures enter into their thinking processes. The results we have collected from analysing the videos confirm the general conjectures sketched above (e.g. the role of the IPH) and show interesting properties, which seem peculiar to 'mathematical gesturing'.

A	B	Δ1	Δ2
-0,80	-1,15		
-0,75	-0,99	0,16	
-0,70	-0,84	0,15	
-0,65	-0,70	0,14	-0,01
-0,60	-0,57	0,13	-0,01
-0,55	-0,45	0,12	-0,01
-0,50	-0,34	0,11	-0,01
-0,45	-0,24	0,10	-0,01
-0,40	-0,15	0,09	-0,01

Table 1. The problem

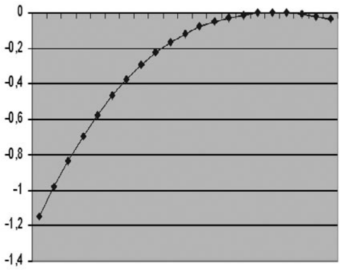

Figure 8. A solution

They can be summarised as follows:

> Gestures may: (i) be a thinking tool; (ii) be partially alternative to artefacts, as a prosthesis to carry out real experiments with virtual objects; (iii) have explorative, anticipative and organising functions; (iv) have social features: they belong to the peripersonal space of people who are interacting and may contribute to the dialectic of the social construction of knowledge, provided the teacher encourages gestures in the class.

I shall comment shortly on some of these items while using some photos from the video. Of course only seeing the video can illustrate completely the sense of my sentences.

(i) *Gestures as thinking tools*. Figure 9 illustrates how gestures can become thinking tools, insofar they support reasoning when the subject does not yet have words to express what he is imaging.

> Teacher: “What are you expecting from the differences, if the function goes up more and more?”
> Student G: “… the differences become bigger and bigger”.

Figure 9. Gestures as thinking tools

Here the gesture made with the two hands integrates the utterance and condenses two related properties into one single act: the function growing up more and more (utterance of the teacher, left hand of G); the increasing difference (utterance of G, right middle finger of G). In words you cannot condense all information in this way; but a graph can: the non-redundant gesture helps G to imagine and anticipate the graph, conveying an information, which is not contained in speech.

(ii) *Gesture partially alternative to artefacts*. Students' gestures at times may replace or integrate the role of instruments in the conceptualisation process. In fact they can produce virtual objects in their peripersonal space, which they can manipulate and with which they can carry out mental experiments. Moreover, students' gestures embody exploratory activities and may become a thinking tool which supports them in their speech.

Figure 10. Gestures partially alternative to artefacts

Figure 10 illustrates an interaction between two mates. The student on the left (G) says: "but if they go up more and more the graph slants more and more ... however if they decrease and decrease less and less …Yes … it must make a curve …". In saying so his hand describes very very slowly a curve in the air: his glances are looking carefully at his moving hand; the words he pronounces describe some properties of the object he is imagining and creating in that moment. Let us observe that G might have made a drawing but he *does* not do that: he prefers using gestures to 'write in the air', to quote Vygotsky (see § 2.3.2).

(iii) *Gesture with explorative functions*.

Figure 11. Gestures for exploring

In figure 11 the student on the left is exploring something: look at his gaze, which is looking in the air. While starting the gesture (figure 11, left) he says: "it starts"; the gesture is slow and he terminates it (figure 11, right) saying: "it ends". He is thinking for himself, possibly imagining the graph of the function. Gestures are accompanying his exploration and there is no communicative goal: compare them with the glances of his mate to the right.

(iv) *Social features of gestures*. Gesture can have interactive functions. Figures 12 and 13 illustrate these functions. In figure 12 the peripersonal space of two mates is shared: the hand on the left is drawing a graph of the function; that on the right stops the motion and pinches it; the pinch is used by the second mate to refer to the differences and to change the way the first mate is drawing the graph. Figure 13 illustrates another example of interaction and of sharing of the peripersonal space. This time, gestures are shared between the teacher and a student. It is important to observe that gestures may contribute to the dialectic of the social construction of knowledge, provided the teacher encourages gestures in the class.

The items (i)-(iv) about gestures can be summarized as follows: gestures produce and represent an overlap between perception and imagination and are an important root of abstraction (see also: Decety et al., 1991; Nemirovsky, 2003). Thus gestures are an important ingredient of APC spaces and illustrate widely the perceptuo-motor approach to knowledge.

Figure 12

Figure 13

2.3.2 The role of signs

Gestures are also significant from the point of view of semiotics if seen as *signs*. Already Vygotsky (1997, p.133) pointed out that "a gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture".

However semiotics is useful to analyse gestures only if one does not forget their cultural, physical and embodied aspects. Such a direction has been followed in mathematics education by Radford (2003) with the introduction of the so-called *cultural semiotic system*, which "makes available varied sources for meaning-making through specific social signifying practises" (L. Radford, 2003, p. 60). The relationship between the sign and its signified is not one of mere substitution: the cultural semiotic system provides the practical activity of the individual with meaning. A cultural semiotic system has a twofold nature:

(i) it is in interaction with the territory of the sign;
(ii) it is in interaction with activity

The crucial point in a cultural semiotic system is what Radford calls a semiotic node: it occurs when gestures and words achieve a coordination of time, space, and movement leading to the social objectification of abstract mathematical spatial-temporal relationships. Epistemologically speaking, the semiotic node has a sense-making constructive dimension.

Figure 14

An example of a semiotic node is given by Eleanor's episode 3: the meaning of the slope is obtained by the young girl's coordination of her gestures (figure 14) and words (the metaphor of stopping the time, which gives sense to the slope of the curve through a mental experiment).

2.4 The role of the teacher

In the previous sections 2.1, 2.2, 2.3 we have sketched some of the main ingredients of the APC space, namely the cultural, physical and bodily roots of mathematical conceptualisation. But all these ingredients must be suitably amalgamated in the classroom by the teacher, who is in charge of guiding the interaction between the perceptual, body level of students' proto-mathematics and the socially shared level of mathematics as a cultural, historically situated heritage.

More specifically, the teacher's role is fulfilled through different, integrated means of mediation: e.g. he/she can make direct or indirect mediation. In the first case she/he can intervene directly with students when the personal meanings they are attributing to a didactical situation appear to diverge from the culturally shared one. An example is when the teacher realises that the students do not read the table of differences according to a dynamic[5] of variation-covariation.

Figure 15

In the second case the teacher orchestrates the interaction or the discussion with the students in order to accompany them towards the shared meanings. An example is given by the interventions of the teacher in Eleanor's episodes: sometimes she takes the interaction for granted, posing indirect questions (episode 3), sometimes she poses more direct questions (episode 4).

3. The space of action, production and communication: An example

Now we have all the ingredients necessary to describe what an APC space is. I shall do it by exemplifying through a concrete example, which comes from another teaching

5 Namely they do not pass from the independent variable values to the dependent variable ones and conversely (covariational way; see Slavit, 1997) in an wholistic way; nor they examine entirely each column from top to bottom and vice-versa (that is in a variational way).

experiment of ours that was designed by G. Pezzi and his team in Faenza (see Arzarello et al., 2004). Some information is necessary to make the context in which the experiment has been developed understandable. In many Italian schools mathematics and physics are taught by the same teacher. The idea of the experiment, which is called "A classroom without walls", is that students learn physics and mathematics visiting an amusement park, Mirabilandia, near Ravenna. It is similar to the Tivoli Gardens in Copenhagen. Students experience some attractions and by doing so they also gain experiences of physics. They use sensors to collect data of some physical quantities (speed, acceleration, pressure) while going on a switchback or some similar machine, and then use graphical and numerical representations to discuss the model thus obtained. More specifically, the goal is that pupils engage more and more deeply with the physical concepts experienced while going on the machines, using the mediation of the mathematical model represented on the screen of the computer. The experiment requires some mathematical knowledge. Actually pupils are in the last years of secondary school (from 10 th to 12th grade). Figure 16 (left) shows the sensor-kit organised to measure the physical quantities (courtesy of Texas Instruments).The kit is assembled in a bag, which can be fastened to the experimenter's body or directly to the machine. Figure 16 (centre) illustrates one of the amusements (the Eurowheel, a big rotating wheel, like that of the Tivoli Gardens). Students go on the Eurowheel and can make online measurements of atmospheric pressure with a barometric probe connected to the CBL2 and the TI83 (figure 16 left and right).

Figure 16

We have orchestrated the same activity with prospective and in-service teachers: the photos below show them while discussing with their instructor the reasons why the graph pressure vs. time obtained while making one turn on the Eurowheel has the shape in figure 17. The discussion takes place immediately after they have experienced the Eurowheel. The episode, which I shall briefly comment on here, is very short (about 30 seconds) but also very dense. It represents an example of how the different ingredients discussed in section 2 are condensed in what I have called the APC space. Observe that all of them are present on the scene: from culture (the function pressure vs. time, the amusement park) and artefacts (calculators, probe devices) to the body (through which the subjects have experienced the motion on the Eurowheel) and the signs (the graph on the screen of the calculator), gestures (as we shall see), the physical world, the teacher (namely the instructor).

I shall sketch the main phases through which the APC space is realized. There are four steps, that I shall comment upon using gesture analysis to describe what is happening. To do that, I use the classification of gestures proposed by McNeill (1992), with some modifications introduced by Edwards (2003) and Arzarello and Robutti, 2004. McNeill classifies gestures in iconic, metaphoric, deictic and beat gestures. Iconic gestures represent “body movements or movements of objects or people in space and shapes of objects or people” (Goldin-Meadow, 2003, p. 7). Metaphoric gestures are “like iconic gestures in that they are pictorial, but the pictorial content presents an abstract idea rather than a concrete object or event. The gesture presents an image of the invisible – an image of an abstraction” (McNeill, 1992, p.14). Deictic gestures are pointing gestures: “pointing has the obvious function of indicating objects and events in the concrete world, but it also plays a part even where there is nothing objectively present to point at” (McNeill, 1992, p.18).

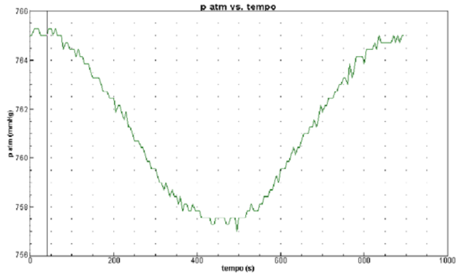

Figure 17

Edwards (2003) introduces a distinction between different kinds of iconic gestures: she calls *iconic-physical* those gestures that correspond to McNeill’s category, “in which the referent of the gesture is something concrete or physical”. And she calls *iconic-symbolic* those gestures that refer to “written symbolic or graphical inscriptions, and/or to the procedures associated with these inscriptions” (Edwards, 2003). However, according to Arzarello and Robutti (2004), the level of symbolisation can be divided into two further sub-levels, according to the type of representation. Typically we can have representations of a phenomenon, according to a graphical environment (as in our example) or according to an algebraic environment (for instance through a formula). When a gesture refers to graphs we can speak of an *iconic-representational gesture*, while when it refers to formulae, we speak of an *iconic-symbolic gesture* (like in Edward’s studies).

Figure 18. Iconic-physical gesture

In the following steps we find the three main components of the APC space (the body, the physical world, the cultural environment).

Step 1. ***The description of the phenomenon.*** The girl in the red circle of figure 18 (let us call her Sara) describes the physical phenomenon (i.e. the rotation of the wheel) with words and gestures. Gestures are iconic-physical: they mimic exactly the movement of the wheel. The utterances of Sara refer to the same physical situation: there is a match between her words and gestures.

Step 2. ***A new entry: pressure***. To explain the graph of figure 17 pressure must be considered. Sara's (iconic-representational) gesture simulates how this quantity enters into the graphical representation, namely through a projection on the vertical diameter; this is accompanied by matching words (figure 19, left). Then her (iconic-representational) gesture represents (figure 19, centre and right) how the pressure, which changes with height, combines with the motion of the wheel. Making this gesture, Sara says: "A projection of pressure values on a diameter, combined with the wheel motion, gives a cosine function on the calculator as a result".

Figure 19. Iconic representational gesture and mismatch

Figure 20. Communication: sharing gestures

Here utterances and gestures illustrate two complementary aspects of the mathematical model. It is a case of mismatch: the pieces of information conveyed in gesture and speech do not conflict but are different. The mismatch helps Sara to encode the complex notions that are necessary to grasp the graph of figure 17. Her gestures are also thinking tools.

Step 3. ***Communication: sharing gesture***. The ideas conveyed in Sara's gestures and utterances become shared among the people around the table. The circles in figure 20 show how Sara's gestures are mimicked and shared by people around the table.

Step 4. ***The instrument enters the shared peripersonal space.*** The instructor (on the right in figure 21) makes the instrument enter the scene: he explains how its data have been collected. He does so with many deictic gestures (figure 21, top) that only describe the phenomenology of what the probe has done. Glances and postures of people in figure 21 show that the instrument is thus entering their peripersonal space. Immediately after the instructor explains how the probe has produced the graph of figure 17. He does so with utterances that refer to the physical phenomenon, while with gestures he refers to the data and acts upon them (figure 17, bottom) to explain the way the differences of pressure can be measured and represented on the graph (the graph of figure 17 is on the screen of the device on the table throughout the discussion).

Figure 21. The instrument enters the shared peripersonal space.

In the end all ingredients of the APC space are active and integrated. There is action, communication and production of meaning. The genesis of the APC space is summarised in table 2.

4. Conclusion

In this paper I have underlined two main aspects in the processes according to which mathematical objects and their meaning are built in the classroom. In fact, in the construction of knowledge both a biological and a cultural component are present and deeply intertwined.

The biological aspects push us to consider everything that concerns our body: gestures, glances, speech and so on and to consider learning according to what I have called the perceptuo-motor approach. It is a naturalistic approach to knowledge, whose relevance has been remarked pointed out in much research in psychology and neuroscience in the last years. This quotation from a neurologist is a comment that features this very well:

> "Representational content, and thus – a fortiori – conceptual content, cannot be fully explained without considering it as the result of the ongoing modelling process of an organism as currently integrated with the object to be represented, by intending it ... [the] integration process between the representing organism and the represented object is articulated in a multiple fashion, for example, by intending to explore it by moving the eyes, intending to hold it in the focus of attention, by intending to grasp it, and ultimately by thinking about it." (Gallese, 2003)

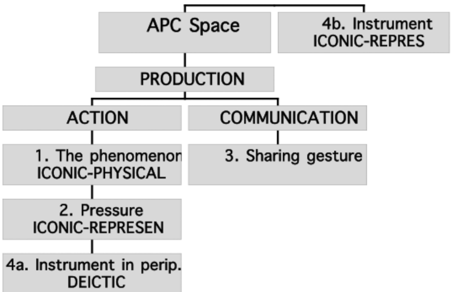

Table 2

On the other hand, biological entities do not exist independently of the cultural environment in which they are embedded:

> "From a neurobiological perspective … all cognitive functions are 'embodied' in brain functions and the latter can be accounted for only if one considers the dynamic interplay between the biological agent (who possesses the brain) and the environment." (Rizzolatti & Gallese, 1997)

Hence mathematics can be learnt only as a complex cultural phenomenon. Culture embraces not only its history but also all the ways according to which mathematics is embedded in our modern life, technology included. This approach is not a sophisticated one reserved for a minority. It is a general requirement of the teaching of mathematics requires, as an education for the rational thought.

The two components – the biological and the cultural – have been analysed through different lenses (history, ergonomy, psychology, neurology, semiotics, etc.): each has illuminated one or the other aspect. But the main point is that a genuine understanding of mathematical ideas can be achieved only if all these aspect are condensed and live in what I have called the APC space.

The role of the teacher is crucial in building situations which are favourable to the building of this APC space. Specifically, her/his task is to promote the integration of the cultural and the biological roots of the mathematical ideas in the teaching situations. This effect can be achieved by nurturing their cognitive resonance in students; for that to tbe possible a careful analysis of the cultural and cognitive roots of mathematical ideas must be carried out. An example has been given above for (some aspects of) the concept of function. This can produce what I call learning in a 'natural' setting (see the quotation from Tall in section 2.1). Its motto could be: "teaching a mathematics with a human face in the classroom" and a representation poster for it could be "La scuola di Atene" by Raffaello in Vatican's Stanza della Segnatura.

This naturalistic approach, which integrates the two components, is the route that can bridge the gap between the logical truth of Bourbaki and the worldly truth of Eleanor.

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SP

Sub-Plenary Lecture

SP: Reasoning, proof and proving in mathematics education

Regular Lecture based on the work of Survey Team 2

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"A Mathematician is a machine for turning coffee into theorems"
(Paul Erdös, 1913-1996)

Introduction

For a long time, mathematical proof has been at the core of an active debate in the community of mathematics educators: often blamed as responsible for pupils' difficulties, but also recognised as a crucial aspect of mathematics activity.

In the recent past the role and the place that proof occupies in the mathematics curriculum have often changed. For instance, in the United States, after a period of 'banishment' proof has got a central position in the Principles and Standards (Knuth, 2000).

> "Reasoning and Proof as fundamental aspects of mathematics.
> Reasoning and Proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied."
> (Principles and standards for school mathematics, NCTM, p. 342)

Nevertheless, certainly the idea of "proof for all" is not one that most teachers endorse, and even where there is a longstanding tradition of including proof in the curriculum (for instance in my own country, Italy, but also in France, and in Japan), the considerable difficulties encountered have lead many teachers to abandon this practice.

Thus the debate is certainly still open. In our opinion there are at least three main questions to be addressed:

- Is proof so crucial in the mathematics culture that it is worthwhile to include it in school curricula?
- What are the meanings of proof and proving in school mathematics and how are these meanings introduced into curricula in different countries? Important aspects include students' conceptions on proof, students' achievements, and teachers' conceptions on proof.
- How has research in mathematics education approached the issue of proof. In particular, is it possible to overcome the difficulties in introducing pupils to proof so often described by teachers?

Starting with a brief discussion on the status of proof in the mathematics culture we will attempt to provide a quick overview of proof in the reality of schools. A few snapshots from recent research studies will be presented together with possible directions for the future.

Proof in the mathematics culture

A historical and epistemological analysis serves to highlight the role of proof in the evolution and systematisation of mathematics knowledge throughout the centuries. Mathematics cannot be reduced to theoretical systems, but certainly its theoretical nature constitutes a fundamental component of it, as clearly expressed by Hilbert and Cohn Vosssen in the introduction to their book *Geometry and the imagination*.

> "In mathematics … we find two tendencies present.
> On the one hand, the tendency towards abstraction seek to crystallize the logical relations in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner.
> On the other hand, the tendency towards intuitive understanding foster a more immediate grasp of the objects, a live rapport with them, so to speak, which stress the correct meaning of their relations."
> (Hilbert & Cohn Vossen, 1999)

A dual nature characterises mathematics: on the one hand intuitive understanding and on the other hand systematic order within logical relations.

Actually theoretical perspectives in mathematics have old roots. This led us to the classic book Euclid's Elements and its particular way, the deductive way, of presenting the 'corpus' of knowledge which has characterised mathematics exposition since then. Heath, in his edition of the Euclid's Elements, reports the following passage from Proclus.

> "Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. (…) In all these ways Euclid's system of elements will be found to be superior to the rest."
> (Heath, 1956, vol. I p. 115-116)

The crucial point seems to be the appropriate order in which a set of known properties should be expressed and communicated. The problem of the transmission of knowledge was solved by Euclid in a very peculiar way. The elements were transmitted according to"logical arguments". This method soon became the style of rationality, not only in mathematics, but also more generally for discourses in any 'science' (Vegetti, 1983), This is not the case of other cultures, for instance in China, as we shall see in the following.

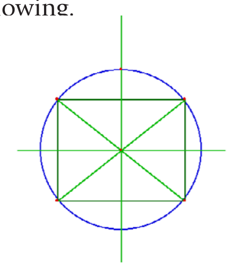


Figure 1.
The figure itself *was* the whole theory.

Let us briefly sketch the story, giving some examples from the history.

1. A historical reconstruction of geometrical argumentations in early Greek mathematics of the pre-Euclidean period (Becker 1975, 24/5) hypothesises that the theorems which Thales (ca. 600 B.C.) knew – such as "A circle is bisected by any of its diameters", "If two straight lines intersect the opposite angles are equal", "The angles at the base of any isosceles triangle are equal", "The diagonals of a rectangle are equal and bisect each other, that is, an angle inscribed in a semicircle is a right angle." – were summarized in a single figure. Apparently, the theorems weren't proved in a Euclidean sense, starting from axioms and definitions. Rather, the figure itself provided the conditions of their validity and, by symmetry, the mode of reasoning. That is, the figure itself *was* the whole theory. One could call this an early and pronounced example of Peirce's idea of 'diagrammatic reasoning' (Dörfler, 2005).

2. With Euclid's *Elements* (ca. 300 B.C.) Greek geometry and number theory was transformed into a deductive system and the very notion of proof in a modern sense came into being. Theorems have to be derived by purely logical conclusions from axioms and other theorems which have been proven before. Nevertheless, Euclid's notion of proof cannot be separated from his practice of working with figures. This is especially true of relations of incidence and position, as pointed out in the 19th century (cfr. the work of Pasch), but it is also true of other aspects of Euclid's theory (Netz, 2000).

3. In Chinese mathematics, as found in the compilation *Nine chapters on mathematical procedures* and its commentary by Liu Hui (ca. 300 A.D.), algorithmic procedures and theoretical arguments are inseparably linked. As Chemla (1996) has shown analysing of the procedures for determining the area of a circle, 'proof' cannot be separated from calculation as in Greek mathematics: in Chinese mathematics, processes of argumentation are dependent on the specific mathematical practice.

4. With the emergence and development of symbolic algebra in the renaissance and early modern times the established notion of proof was again substantially modified. Isaac Newton, for instance, saw no problem in 'proving' the rule that the integral of x^n is equal to $(n+1)^{-1} x^{n+1}$ simply by working through a numerical example. Nor did he hesitate to state the rule without specifically acknowledging the exception $n = -1$, since it could easily be seen that the formula does not apply in this case (Newton, 1667, 206 ff). Also, he did not prove the right implication. Rather, he showed that the derivative of $(n+1)^{-1} x^{n+1}$ is x^n. It became accepted practice in analysis and algebra in the 17th and 18th centuries that theorems might "suffer exceptions" which, as a rule, one does not need to point out. Mathematical thinking was dominated by manipulations of *indeterminates* and the accepted notion of proof reflected this practice. A valid proof was nothing else than a correct manipulation of algebraic symbols. It was not before the 19th century that – under the influence of Cauchy and Weierstrass – the domain of validity of a theorem was exactly specified, and the modern notion of proof in analysis and algebra emerged step by step. Again, we see that the notion of proof is dependent of the way mathematics is practised. It is plausible to think that Peirce's concept of diagrammatic reasoning can be applied to this case, too.

5. Nowadays, proof plays a different role in different sub-disciplines of mathematics. Of course, there are broad areas of mathematics in which the role and meaning of proof is unquestioned and adequately described by the Euclidean scheme (with Hilbertian refinements). However, with the growing role of computers we have witnessed new developments with new types of proof in certain areas. We mention only the computer proof of the four colour theorem and "zero-knowledge proofs". Above all, there is a growing amount of purely experimental work in mathematics with publications containing results which are suggested by computer experiments, but in many cases not proven (cf. the controversy between Jaffe & Quinn (1993) and Thurston (1994)). Today, numerical analysis contains a large amount of numerical experiments and a lot of algorithms whose optimality and limits of validity are not proven. Surely, numerical analysts try to prove as many results as possible, but the requirements of numerical practice are so extensive that a restriction to proved algorithms is not possible.

6. In applied disciplines such as, say, theoretical physics, the meaning of a proof might be different from its usual meaning in pure mathematics. Consider again the example of Isaac Newton. When he derived Kepler's laws of planetary motion from his supposed law of mass attraction he based empirically well-established laws upon an uncertain hypothesis. At his time, Newton's gravitational law was far from being generally accepted or just plausible. Thus, Newton's proof did not transfer truth from the assumption (the law of gravitation) to the conclusion (Kepler's laws) as is the notion in established fields of mathematics. On the contrary, the gravitational law was justified by Newton's proof because one could deduce Kepler's laws from it. Thus, what is proved may serve to legitimise the assumptions from which it is derived. On the other hand, Newton's proof put Kepler's laws in a broader theoretical context and, by this, made it possible to draw new conclusions, explain additional empirical facts, and formulate new predictions. The proof was a medium of *generalisation*. This encompasses also an effect of a more qualitative nature in that it opens up a *new perspective* on Kepler's laws. Their status was changed from a purely kinematical description to a dynamic view of nature.

Thus, epistemological and historical analyses show a rich variety of meanings and uses of mathematical proof, to which corresponds a complexity in the educational field, where epistemological distinctions, articulating different *functions of proof* (Bell, 1976; Hanna, 2000; de Villiers, 1990), have proved useful and have found a shared consensus across different studies.

There is a dialectical relationship between proof in the scientific practice of mathematics and proof in the educational realm. By division of labour most of the epistemological problems of proof are settled when a mathematician attempts to prove a theorem. Mathematicians seem to be mainly concerned with the *mathematical complexity* of the theorem in question. Of course, they have to evaluate proofs and theorems, but coming to terms with the mathematical complexity is the foremost problem. In the educational realm, however, it is the *epistemological complexity* which matters. For the students proof is above all a problem of meaning, and educators have to devise teaching contexts which make proof meaningful to them. We will come back to this point in the following.

Proof in the curriculum. A comprehensive account of proof at school

It is difficult to have a complete overview of the situation in the different countries; in most cases, no specific studies are available; what we can provide are snapshots coming from the most complete large-scale study carried out in the recent past (Hoyles, 1997; Hoyles & Küchemann 2003).

Proof cannot be considered as any other mathematics topic, such as trigonometry or functions. Thus very rarely one can find "proof" explicitly listed among other topics, in the official programs of national Curricula. The position and the status of proof in education is a complex issue and ideas about it can emerge only from an accurate analysis across different topics and the specific guidelines accompanying the list of the mathematics contents related to them. However, even more can be drawn from direct observation of classroom activities. As discussed above, different aspects (functions) of proof may emerge in different mathematical contexts and in relation to different social interactions among the basic elements of the didactic system. Actually, the presence of an explicit reference to proof is a crucial point. On the other hand the lack of such an explicit reference may not mean that proof is not required, or it is not much valued. Nevertheless, the absence of mentioning proofs might be a hint of the fact that the didactic issue of proof is not in focus. It is impossible here to sketch the story of proof in curricula, although this could be of great interest. Let me share with you some general observations and remarks.

In any country, and in any moment of history, the mathematical curriculum reflects the cultural changes of society, and of the mathematical community in particular (Chevallard, 1995). Of course, different positions can be found in different countries, and changes are sensible only over a long period. In fact, the effects of research studies begin to appear in the attempt at shaping national curricula according to experts' suggestions.

Consider, for instance, the impact of the Bourbaki inspired movement in the sixties: it is possible to notice how the education systems were affected in different countries. For instance, in some quarters in Russia (the Soviet Union at that time) a stormy enthusiasm of over-indulgence into rigorous exposition of material in school mathematics arose at the end of the 1960's, and a radical grandiose reform of the school mathematical education was realized. Old teachers were driven away, new textbooks were written in the manner of Bourbaki and remained in use for more then 20 years. All didactic literature was directed to popularization of the formal logical study of mathematics in school, even with very young pupils. A typical example at the elementary level sounded in the following way: "Let us consider a statement: "the river x flows into the Caspian Sea". To find out if this statement is true if a) x = Volga, b) x = Don".

Actually, some mathematicians found themselves involved in a widespread movement aimed to innovate and improve the teaching of mathematics, although not always so strongly influenced by the Bourbakian perspective.

Hans Freudenthal was one of them and the foundation of the Institute – now the Freudenthal Institute – shows the deep impact that the innovation had in certain countries.

It is important to stress that, at that time, mathematics education research was centred on issues related to curriculum design and innovation: changing the mathematical content was seen as the crucial contribution to solving didactic problems, and following the new trends of the discipline appeared to be the required solution.

The effects of the New Math revolution are well known, and this is not the place to come back to that discussion. As far as proof and theoretical thinking is concerned, the experience of the didactic transposition of Bourbaki inspired approaches, characterized by the aim of presenting mathematics from a structural perspective, is of interest mainly to understand the appearance in the curriculum of some elements of logic, intended to promote pupils' introduction to theoretical thinking, and generally speaking to 'correct' mathematical reasoning. The long term effect of such a cultural 'innovation' can be seen in the persistence of "truth tables" and "Boolean algebra" at the very beginning of some textbooks (Italian textbooks, for example).

International studies on proof

The rapid evolution of research in the field of mathematics education led not only to overcoming the negative influence of the structural approach, but more generally to criticizing didactic practices which were judged to be too formal and therefore to constitute an obstacle to understanding. From this perspective we can interpret the strong criticism of the so-called "two columns proof[1]", and the guidelines contained in the NCTM Standards published in 1989.

While, in the nineteen sixties and early seventies there was a worldwide normative attempt to align mathematics teaching in the school with a formal conception of proof, later on – especially under the influence of I. Lakatos' *Proofs and Refutations* (1976) – there has been a significant reorientation towards communication and understanding in classroom practice (Balacheff, 1987, 1991). The heart of this reorientation which reflected trends in mathematics, in the philosophy of science and in mathematics teaching could be called a *shift towards a pragmatic view of proof* (Hanna & Jahnke 1993, 422). The very term 'pragmatic view' is not meant to indicate a liberal attitude of 'anything goes', but to designate in a rigorous way the insight that the role and norms of mathematical proof are dependent of the specific mathematical practice in which proof is embedded. Proof is subject to historical and cultural change and its role and meaning are different in the different areas of mathematics. Thus we can say that, at least in some countries, the story of the status of proof and proving in the curriculum shows an alternate favour, depending on both cultural factors and on the research studies on the impact of changes of curriculum on students. In the U.K., we find a clear example of the impact that research studies, based on large scale (nation-wide) surveys describing students' view of a subject, may have on the reform of the curriculum. After the massive change following the imposition of the National Curriculum in the 1980's, proving and proof was one area of the mathematics curriculum that was radically altered in the face of persistent student difficulties and lack of motivation, as it emerged from research studies.

The main response to evidence of children's poor grasp of formal proof in the 60's and 70's was the development of a process-oriented approach to proof. Many argued, (Bell, 1976, Cockcroft, 1982), that students should have opportunities to test and refine their own conjectures, thus gaining personal conviction of their truth along with gaining experience of presenting generalisations and evidence of their validity. (Hoyles, 1997)

A survey on students' views on proof

Clearly, there are potentially considerable advantages in a process-oriented approach in terms of motivation and the active involvement of students in problem solving and proving. Indeed many researchers (see, for example, de Villiers, 1990), argue for such a shift in emphasis, suggesting that students develop an inner compulsion to understand *why* a conjecture is true if they have first been engaged in experimental activity where they have '*seen*' it to be true. The implications of these changes are discussed in a paper by Hoyles (1997). The author presents some of the findings of a nation-wide survey into student conceptions which point to a strong curriculum effect. Most notable was the evidence that student responses were strongly connected, in terms of format and language, to the investigations part of the curriculum they were now studying. Students

1 An account and an interesting discussion of the emergence of the two columns proving custom can be found in Herbst, 2002.

appeared to be imposing a 'type' of proof on every question regardless of whether it is appropriate. Moreover, students seemed to have shifted their notion of proving from one ritual to another – from a *formal* ritual to a *social* ritual – something added on at the end of an investigation. As the author says, it was salutary to trace the extent of the intended or unintended curriculum influences which indicates that student responses cannot simply be 'blamed' on the student. Their types of behaviour cannot be ascribed solely to properties of age, ability or even individual interactions with mathematics. The meanings students have appropriated about proof have been shaped and modified by the way the curriculum has been organised. For example, data concerning responses in geometry – much worse, from a mathematical perspective, from those in algebra, and in contrast to results from other countries – can be explained by the almost complete disappearance of geometry in the curriculum. Nonetheless, this finding casts doubt on how far proof can be considered as a unitary mathematical 'object' with its own hierarchy separated from any domain of application.

A recent study

A new survey study has been recently carried out in the UK, focusing on the influence of the curriculum and more generally on students' progress in recognising and constructing mathematical arguments (Hoyles and Küchemann, 2002).

For this project, survey instruments and questionnaires were developed and used to gather large-scale longitudinal data on a nation-wide large sample (n=1512) of high-attaining students in England, from age 13 1/2 years to age 15 1/2 years, as well as information about their teachers and schools. The longitudinal data were analysed descriptively to produce a rich picture of students' explanations and proofs in algebra and geometry and of how these changed over time. The data were also analysed using multilevel modelling to isolate student, teacher and school factors that seemed to promote mathematical reasoning. Using a mixture of quantitative and qualitative methods, the study reported clear progress in reasoning in response to standard items in algebra and geometry. However, on less standard items difficulties persisted, for instance, in basing explanations on perception in geometry or on numerical evidence in algebra. Besides, robust gender differences were identified along with subtle influences of curriculum and school organisation.

Overall, the study reports only limited progress and some regression in students' articulation of explanations based on mathematical structures; appreciation of circularity in arguments; use of analysis and deduction; and in distinguishing logical implication and its converse. Although there is clear progress in calculations involving several deductions and some progress in distinguishing perceptual from logical reasoning, students still have problems with the organisation of written argument and the understanding of 'reasons' in mathematics. The influence of the curriculum on students' responses is sometimes apparent, for example when new geometrical properties are introduced, students often discount their earlier correct heuristics.

Learning to reason is not automatically transferable across items and domains, is not linear (this is probably not surprising), and is not necessarily retained.

The longitudinal analysis allows to describe individual trajectories and show how single snapshots of student understanding can be misleading. The project also illustrated the power and utility of mixed research methods that incorporate longer and more detailed case studies to interpret classroom and teacher influences, alongside statistical analyses of different types.

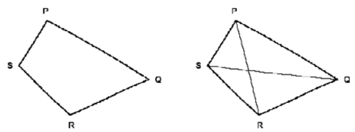
Progress on core items

In the study a key role was played by the so called *core items*, i.e. items that were set each year for 3 years. As a matter of fact, progress was analysed by considering students' scores on the core items. With progress defined as an increase in score (of any amount), 73% of students (N = 1512) progressed on the core items from Yr 8 to Yr 10, 5% stayed the same and 22% regressed, giving a 'net progress' of 51%. Similarly, the net progress from Yr 8 to Yr 9 and Yr 9 to Yr 10 was 23% and 34% respectively. From the case studies, this low net progress was interpreted as partly due to the pressures of the test situation along with influences of the curriculum.

The longitudinal analysis on the core items indicated progress from Yr 8 to Yr 10, along with a sex difference when the baseline mathematics test scores were included in the analysis: girls started from a lower base on the core proof items than boys in Yr 8, caught up in Yr 9 but were overtaken again in Yr 10, with overall progress from Yr 8 to Yr 10 not being significantly different for girls and boys.

Progress on individual items – illustrative results

Overall, the net progress on individual core items was small, of the order of 20%, with the highest at 32% and the lowest at just 2%. Consider the following item.



G1 Tim sketches a quadrilateral. He draws the diagonals of the quadrilateral.
Tim notices that one of the diagonals has cut the area of the quadrilateral in half. He says:
"Whatever quadrilateral I draw, at least one of the diagonals will always cut the area of the quadrilateral in half".
Is Tim right? Explain your answer

G1 is a non-standard geometry item and was designed to test whether or not students would succumb to a perceptual proof (i.e. argue that a false statement was true on the basis of a misleading diagram). In Yr 8, 39% of students claimed the statement was true, reducing to 26% in Yr 10, and with a net progress in score of 20%. Students' counter examples in Yr 10 tended to be more compelling than in Yr 8, but there was little evidence of a shift to a more analytical approach (e.g. starting with and just focussing on the relevant properties), with interviews suggesting the inappropriate use of recently taught geometry facts.

Overall, the study reports an improvement in the use of algebra, in spite of a big gulf between a numerical and the equivalent algebraic task and the persisting strong attraction of "pattern spotting". Data show students' reluctance to abandon an empirical way of thinking, while interviews suggest that students *needed* to calculate and were insecure about using number relationships even when they apparently understood them.

As far as geometry is concerned, students appeared competent at multi-step calculations, at least those based on fairly basic geometric knowledge and they improved markedly over the years. But their explanations tended to be vague and circular; many students were unsure about what is meant by a mathematical reason, and prone to perceptual reasoning, even in Yr 10.

Comparisons with Taiwanese students

Unfortunately, "comparable" data, concerning other countries are not available. Nevertheless some of the parts of the Longitudinal Proof project presented the previous section, were replicated in Taiwan. Though the samples were not exactly similar – the Taiwanese sample was not randomly selected as was the case with the UK sample – some interesting similarities and differences are notable. Similarities and differences can be found, which can be interpreted referring to and comparing the curricula of the two countries. But is interesting to note that differences appear that could not easily be foreseen, which illustrates that curricula cannot capture all the factors in play. Take the following example.

In answer to G1 in Year 8, many more Taiwanese students agreed with the false conjecture (45% as compared to 39%), and fewer produced an explicit counter example (e.g. by making a drawing). This comparison may simply be the result of sample differences, but colleagues in Taiwan suggest the following interpretation: Taiwanese students are not familiar with *refuting* an incorrect property.

From the experience described in this local study crucial variable related to the complex of relationships among the different elements of the didactic system arises. In particular, the impact of the didactic contract (Brousseau, 1997) set up in the classroom in relation to proof and proving is such a variable.

Proof in the classroom: The key role of teachers

Proof is not the same in the classroom as in the mathematical community. As Dreyfus (2000) and Yackel and Cobb (1996) remind us, what is interesting is to shift the focus to a key element of the didactic system: the teacher.

> "Teachers, as representative of the mathematical community in class, have the key role in establishing the various socio-mathematical norms in general and those related to justifications, argumentation and proofs in particular."
> (Yackel and Cobb, 1996)

Very few pilot studies have been carried out concerning teachers and proof (Barkai, Tsamir, Tirosh & Dreyfus, 2002; Dreyfus, 2000). Results show a great variability in teachers' evaluation of potential arguments.

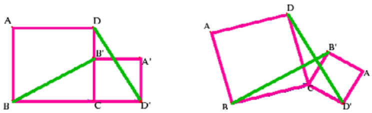
> "Different beliefs are likely to produce widely differing interpretations of guidelines, but also widely differing consequences teachers may draw for their classroom practice."
> (Dreyfus, 2000)

Interesting results come from the comparison between two nation-wide surveys. The first one investigates teachers' attitudes towards mathematics education (Nagasaki, 2001), and in particular the relevance attributed to proof, the second one investigates students' achievements on proof. Although not at the highest rate, proof seems to be perceived by the teachers as an important part of mathematics education and there is a shared agreement to introduce proof in compulsory education. In contrast, only about 20-40%

of students in lower secondary school achieves expected objectives on proof. It seems that there is some tacit support to include proof, even if it is difficult to understand. Because of the importance of proof, in spite of its difficulty, and although students cannot perform satisfactorily, the shared opinion is that they should have an idea of proof, even if it remains a vague image.

The report contains examples of the items used in the survey (for 12-13 year olds). It is interesting to compare these examples with those used in the UK, for instance for the 2nd grade of the lower secondary school (13 year olds). Consider the following item.

Square ABCD and square A′B′C′D′ have a common vertex C. Provide an answer to the next questions.



1. Three points B, C, D’ are on a straight line like the figure on the left, holding the relation BB’ = DD’. In order to prove this, we will show that triangle BCB’ is congruent to triangle DCD’. Write the condition for congruence of triangles that will be used in the proof.
2. Even if the three points are not on a straight line like in the figure on the right, they hold the relation BB’ = DD’. Write your reason to this.

Results: 1. rate for correct answer 56.8%, 2. rate for reasonable answers 38.1%.

The data available do not allow a reasonable comparison with other countries. However, even this single example shows the intrinsic complexity of a comparison. The difference between this item and item G1, presented above, is evident: it reflects different expectations of the researcher who designed the questionnaire.

Actually, it would be highly valuable to set up a large scale nation-wide survey across different countries and allowing for a comparison. Certainly the design of a common questionnaire, that could be used to test students of different countries, presents great complexity, but I believe that the effort needed to coordinate the work of different researchers, referring to different school systems, but also belonging to different cultures themselves, would be of great interest, and likely to highlight unexpected issues. In fact, culture has a deep impact on how tasks are selected and answers are conceived of by the researchers. Thus new perspectives coming from cross cultural interaction could reveal new interesting questions, but would mainly make some implicit aspects become apparent. It is unbelievable how much one can learn from diversity.

> “Cross cultural comparison also leads researchers and educators to a more explicit understanding of their own implicit theories about how children learn mathematics. Without comparison, we tend not to question our own traditional teaching practices and we may not even be aware of the choices we have made in constructing the educational process.”
> (Stigler and Perry 1988, p. 199).

A cultural perspective to explain differences

Different perspectives related to different cultures can be a powerful tool of interpretation. A comparative study (Knipping, 2002), analysing French and German contexts in relation to arguments and proof, will be used to illustrate how cultural difference may affect school practice and curricula. Consider the following quotation taken from an interview with a student of a German-French school (Knipping, 2003). Pupils had experienced both French and German mathematics teachers (in two different academic years).

> **Sophie**, classe de seconde "Oui, pour ma part je trouve que les mathématiques en allemand sont beaucoup plus concrètes que les mathématiques enseignées par les professeurs français, euh les professeurs français s'attachent plus à démontrer des théorèmes, ou à faire beaucoup de démonstrations alors que les professeurs allemands vont plus... directement au principal, et ils ne s'attachent pas à donner des choses, qui en fait sont superflues …"[2]

Knipping analyses the didactic context into which proofs are inserted. Data and results concern the introduction, the development and the justification of Pythagoras' theorem, together with its application in the solution of exercises. Differences among a French and a German didactic style are highlighted and described both in terms of proof processes and in terms of the functions of proof, as they are lived by the participants in the classroom activity.

Comparing the role of proof in German and French teaching contexts has uncovered different teaching patterns and different functions of proofs. As Knipping clearly discusses, in the observed German teaching the function of proof is to "understand why", and generally speaking to get an insight that makes students grasp, at the same time, a property and its reasons. In contrast, in French teaching it is important to "defend why" a statement is true. There is a general habit to divide the arguments into "sound bites" in a chain of reasoning, which reaches a public status in the class, reinforced by writing them down on the blackboard. Proving in French teaching is seen as an activity which characterises the whole teaching: even when students are not explicitly asked to prove something, they are implicitly asked to state the conditions of validity of a statement or a solution.

We may presume that this characterises distinct relations to knowledge and rationality as ingrained in culture. According to Knipping (2001), the German attitude towards proof that the teacher shows can be interpreted as an outcome of a hermeneutic tradition that has influenced education, and in particular mathematics education since Humboldt's reforms in 1810 in Prussia, which turned the ideas of enlightenment, defending Cartesian principles in reasoning, into another approach (Jahnke 1990).

2 "In my opinion, I find that mathematics in German is much more concrete than mathematics taught by the French teachers. Yeah, French teachers care more about proving theorems, to do a lot of proofs. In contrast, German teachers go more ...directly to the core, and they do not care to give things that in fact are useless ..." (translated by the author)

Research studies on students conceptions of proof

Two main streams can be considered, one related to the analysis of problems related to proof as they emerge from school practice, and another one related to proposals for introducing pupils to proof, generally speaking projects for innovation, within which the role of proof is recognized and addressed. There are at least two possible, opposite, perspectives, which often seem unable to communicate to each other, and which might be related to cultural differences (Balacheff, 1999). They differ with respect to the aspects they have in focus.

On the one hand, starting form the analysis of students productions in solving problems, different ways of thinking, related to the observable behaviours, are described and classified. On the other hand, starting from an epistemological perspective, difficulties and problems encountered by students are related to the specific nature of proof.

Analysis and classification of students' behaviour: Proof schemes

An example of the first type analysis is provided by the research study carried out by Harel and Sowder. The authors describe the solutions given to problems (mainly in linear algebra), classifying the different arguments provided by students. The large scale investigation came out with a taxonomy of what the authors call *proof schemes*, obtained by a highly refined classification, fully described in Harel and Sowder (1998). Any kind of argument is considered a proof, and convincing and the key elements in play are persuading. A teaching project (PUPA) elaborating this model according to specific pedagogical assumptions was set up, a description can be found in (Harel, 2001).

A historio-epistemological perspective: Argumentation versus proof

"Argumenter, démontrer, expliquer: continuité ou rupture?" is the title of a seminal work by R. Duval (1992). Starting from an epistemological perspective, Duval analyses the nature and the role of argumentation as well as those of proof in mathematics. Duval holds a very radical position; he focuses on one crucial point: the difference between the semantic level, where the epistemic value of a statement is fundamental, and the theoretical level, where only the validity of a statement is concerned, i.e. only the logical dependence of a statement on the axioms and the theorems of the theory, independent from the epistemic value that may attribute to the propositions in play.

As a consequence of this analysis, Duval stresses the *cognitive distance* between argumentation and proof and, consequently, the relevance of this issue from an educational point of view. A similar analysis, although not so radical, can be found in Balacheff (1987).

In spite of the strength of Duval's arguments, they are still debated. For instance, the contrast with a position like that of Harel and Sowder, just presented, is evident. Crucial questions arise:

Is it possible to overcome the rupture between argumentation and proof? Or, is there any real rupture between the two?

Some results, coming from a research project aimed at introducing pupils to proof, open a new perspective. Data were collected from a long term teaching experiment, centred on open-ended problems: pupils were asked to produce a conjecture and then to prove it. The phase of producing a conjecture showed the appearance of a number of arguments aimed to support or reject a statement. The analysis of the subsequent proof showed an essential continuity with these arguments. This is what the authors called *Cognitive Unity*.

> "During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain."
> (Boero et al., 1996)

Although the notion of Cognitive Unity within avery peculiar teaching experiment, and as part of a very peculiar teaching project, it showed its potential, both in explaining traditional difficulties and in suggesting new directions of investigation. In particular, this seminal work has provided a new tool of analysis, which in my opinion seems to be very promising.

Overcoming the dichotomy: The notion of Cognitive Unity

The notion of Cognitive Unity, developed with the aim of describing and interpreting students' approaches to theorems, can be used as an analytical tool in investigating the relationship between argumentation and mathematical proof, taking as an underlying assumption the parallel proposed by Balacheff:

> "Je résumerai en une formule la place que je crois possible pour l'argumentation en mathématique, allant dans le sens du concept d'unité cognitive des théorème, forgé par nos collègue italiens:
> L'argumentation est à la conjecture ce que la démonstration et au théorème[3]"
> (Balacheff, 1999)

In this way, the analysis proposed by Duval is not refused but further articulated with the aim of identifying the key elements of a comparison between argumentation and proof.

The very first analysis was limited to what may be called the referential field. A more refined analysis has been carried out by Pedemonte (2002), showing the complex relationship between the structure of an argumentation and the structure of the related proof: Toulmin's model (Toulmin, 1958) provides a powerful framework for this analysis.

An example of the structural distance between argumentation and proof

I would like to discuss the case of induction because it provides good opportunity to come back to the notion of proof scheme, and thus to relate to different research studies. Inductive types of argumentation are quite common, but the development of recursion has been difficult, raising a strong debate, so that only recently mathematicians have obtained an agreement about its acceptability.

3 I would summarize in a formula the place that I find possible for argumentation in mathematics, according to the notion of Cognitive Unity as it was introduced by our Italian colleagues: argumenttation relates to conjecture, like proof does to a theorem" (translated by the author)

According to Harel's analysis (Harel, 2001), two different types of arguments can be recognized both leading to a general statement. The first one is a case of an *empirical* proof scheme, whilst the second one is a *transformational* proof scheme.

Generalization may be achieved by recognizing a general pattern in the result itself (*result pattern generalization*), for instance, observing the regularity of the result of a calculation.

Generalization may be derived from the process that leads to the results *(process pattern generalization)*, for instance, observing a chain of steps relating the results to each other.

Although in both cases the arguments supporting the general statement are obtained from an inductive process, i.e. from the verification of a limited number of particular cases, in the former case the examples function as generic elements on which the arguments can be applied. In the latter case, examples are provided, but the passage from one step to the following is in focus. A more refined analysis reveals a further distinction between what Harel calls Quasi Induction and Mathematical Induction (MI): While both are instantiations of the transformational proof scheme, the latter is an abstraction of the former. In MI, on the other hand, the student views $P(n) \rightarrow P(n+1)$ as a variable inference form, a placeholder for the entire sequence of inferences. In spite of its brevity, this analysis clearly shows the variety of argumentations that may be produced and which have different relationships with the mathematical proof by induction. In particular, both the continuity and the cognitive gap between quasi induction and MI should be carefully investigated.

The distance is related to the fact that in the case of proof, the standards of acceptability for an argument are pretty strict and mainly related to a well defined and clearly stated set of paradigms of arguments. Among these paradigms the deductive reasoning is perhaps the most accredited one. From this perspective, a main characteristic of proof is its social dimension, i.e. proof makes sense with respect to a community which shares (more or less implicitly) the criteria of acceptability.

At school, this social dimension related to sharing the standards of the community of mathematicians must be articulated within the social dimension of the classroom community: the crucial role of the teacher comes to the fore, at the same time representing the mathematics community and the classroom community.

From a different perspective, the possible discrepancy between argumentation and proof, was recently analysed by Raman (2002), in terms of private versus social perspectives. Here the author expresses the two poles of the dialectics as *private* and *public* aspects, and identifies, in what she calls the *key idea*, the possible link between the two poles. The results, reported in Raman's work, open a new direction of investigation, i.e. the comparison between experts and novices. The analysis is consistent with the perspectives of cognitive unity research; a fine grained analysis of data, if carried out using Toulmin's model, in analogy with what was done by Pedemonte, could provide further insight into the potential continuity between the private and the public aspect of proof, as well the potential gap between them, both in the case of experts and that of novices.

Proposals for introducing pupils to proof

Results coming from survey studies as well as from research work focused on students' conceptions of proof provide a motivation and a base for projects whose aims are more or less directly related to the aim of introducing pupils to proof.

Different research studies clearly suggest an early start of a practice of proving. Accordingly, a number of research projects, at the primary and the lower secondary school levels, are widely based on thoughtfully selected open-ended problems investigated by children and collectively discussed by the whole class. The aim is very often that of establishing a "mathematical community in the classroom" (Arsac, 1992; Davis & Maher, 1993; Yackel & Cobb, 1996, Bartolini Bussi, 1991). Different approaches share as the common assumption that reasoning and arguments contribute to knowledge construction (Boero et al, 1995; Dueck, 1999). This widely accepted perspective expresses the need of coordinating psychological and sociological perspectives, i.e. developing a model where education is interpreted as entering and participating in a culture rather than as being subject to transmission of knowledge. New knowledge emerges from pupils' activities. However in the collective activity of the classroom it is systematized into a mathematical framework, and social norms determine what is considered acceptable and in particular mathematically acceptable.

At the primary school level the nature of "mathematical" is hardly questionable: "What makes the "objects" of Trevonda less mathematical than those of Jameel?" do the authors (Yackel & Cobb, 1996), analysing the transcripts of a classroom discussion, ask themselves. As a consequence, beyond the social norms, controlling what students are expected to do, socio-mathematical norms are established in the classroom, and as part of them the criteria for acceptability are negotiated.

> "The understanding that students are expected to *explain* their solutions is a *social norm*, whereas the understanding of what counts as an acceptable mathematical explanation is a *socio-mathematical norm.*"
> (Yackel, 2001)

Similarly, the development of young children's understanding of mathematical argumenttation constituted the key objective of another teaching experiment, designed to create classroom environments within which the sense making is a cultural norm, and a particular outcome of this culture is expected to be the emergence of argumentation, justification and proving in children's discourse.

The teacher's actions accomplish several goals: among others to calling attention to the argumentative support for conclusions thus contributing to the class' understanding of what is taken to be argumentative support. (Maher, 1996, 1998)

Consistent with this perspective, but more explicitly oriented towards framing mathematical arguments within a theoretical system, are the experiments carried out in Italy by the research group directed by Bartolini Bussi. The detachment from the conception of empirical verification as the only tool suitable to resolve conflict situations is carefully managed by the teacher by means of the collective construction of *germ-theories* (a germ-theory is an embryo of theory that has an expansive power and the potential for developing into a fully-fledged one). Within a selected field of experience (Boero et al., 1995), the solution of a rich collection of problems provides a basis on which a *germ theory* is established.

An example of a germ theory

Consider the field of experience of gears (Bartolini Bussi et al., 1999). From a variety of experiences a basic principle arises; this principle is not explicitly named a postulate, but its status is stated as a principle.

In the protocols, various argumentations are produced to justify individual statements, based on the stated principle. The following example aims to show the kind of argument that can be produced and how it can be related to proof in a germ theory. Consider the following problem, proposed to 5th grade class.

> "We have often met planar wheels in pairs. What if there were three wheels? How could they be positioned? You can build possible situations by drawing or by cutting. Remember you must always give the necessary explanations and write down your observations."
>
> Elisabetta's protocol

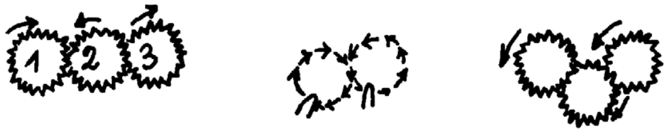


Figure 1: A sequence of three drawings produced by Elisabetta

1) *Wheel n. 1 turns, but we do not know in which direction; let us say that it turns clockwise, then Wheel n. 2 turns anticlockwise, this is sure, and n. 3, how do you think this one turns?*
   I know how: it turns like n.1. Do you know why? Because they have to be in gear in the opposite direction. We could do this with fingers too, remember. I've drawn two wheels with arrows in opposite directions.
   Yet, if we think hard, n. 1 could turn clockwise and n. 2 clockwise too, couldn't they? They could not, they would not be in gear.
2) *Let us try to draw the wheels in another way.*
   Yes, they are on the same plane; but if I try to turn one we would see that two turn in the same direction and the other turns in the opposite direction. They cannot be in gear, because the first turns clockwise, the second anticlockwise and the third clockwise too, but the first and the third touch each other and so they are not in gear.

> „The first two wheels turn in opposite directions; and this is OK, but there is a third wheel that is in gear with both; it is a kind of block as the teeth should break. Actually the two wheels go in opposite directions and a tooth would push a tooth of the wheel one way but there is the tooth of the other wheel that pushes this tooth the other way. Conclusion: if the wheels are put in this way they can't turn."
>
> Davide's protocol

In both the graphical and the mental experiments a property drawn from physical experience is used: *'two wheels in gear turn opposite ways'.* For the pupils, this principle does have the status of 'postulate' of a germ theory. Within this framework, only a 'small' step is needed, to shift the arguments to the status of mathematical proofs, thus explicitly building the reference theory. Within this theory, pupils' argumentations about the impossible motion assume the status of proofs by *reductio ad absurdum*. The mental experiments show all their power, as they allow the dynamic exploration of gears that do not actually work and permit production of statements and argumentation for any number of wheels. Analogous examples can be found in (Boero et al., 1996), where the field of experience of shadows functions to foster a rich context where the need of explanation leads to modelling and conceptualising, according to the main assumption on the teaching-learning process, which is modelled by a systemic interaction between the production of conjectures and mathematical systematisation.

The contribution of technologies

In recent years a number of different contributions on the theme of proof shared the choice of a Dynamic Geometry Environment (DGE) as their context. Some years ago a Special Issue of *Educational Studies in Mathematics Education* (vol. 44, 2000) was devoted to this theme. Certainly, the availability of graphing capabilities "has given a new impetus to mathematical exploration, and has brought a welcome new interest in the teaching of geometry" In fact, "Dynamic software has the potential to encourage both exploration and proof, because it makes so easy to pose and test conjectures" (Hanna, 2000, pag. 13).

But if it seems clear that dynamic figures may contribute to setting up conjectures, providing the students with a strong evidence that a property is true, their contribution to finding a proof, i.e. validating that conjecture, seems less clear. The possible contribution to introducing students to a theoretical perspective, i.e. to construct a meaning of proof, appears to be even more critical. It could be natural and reasonable if the student jump to the conclusion that exploration via dragging is sufficient to guarantee the truth of what can be observed (Mason, 1991; Healy & Hoyles (2001). Thus the critical point concerning the relationship between empirical evidence and theoretical reasons arises in this new context. As, using the notion of *milieu* (Brousseau, 1997), Laborde points out:

> "[…] a DGE itself without an adequately organized milieu would not prompt the need of proof. And it becomes evident the need of establishing a rich milieu with which the student is interacting during the solving process and the elaboration of a proof."
> (Laborde, 2000, p. 154)

The papers included in the ESM special issue were intended to discuss this question and to contribute to clarifying potentials and limits of DGE.

Although it is impossible here to give a full account of the discussion, I would like to focus on a specific point, which in my view is a crucial one: the relationship between the *dragging tool* and theoretical control within geometry.

Dragging tool and logical control

DGEs, as opposed to paper and pencil environments, contain within them the seeds for a geometry of relations: entering a DGE offers the opportunity of experiencing the break between these two worlds and to experience this break at the level of actions (Laborde, 2000). But I would like to go further by stressing the fact that actions are mediated by tools which, according to a Vygotskian perspective, can become "semiotic tools", exploited by the teacher according to her didactic objective related to making students develop mathematical meanings.

Consider the dragging tool, as it is used in a DGE, like for instance *Cabri*.

The dragging tool can be activated by the user through the mouse and can determine the motion of different objects on the screen. Two main kinds of motions are possible, as a consequence of the dragging mode: direct and indirect motion.

The "indirect motion" of an element occurs when a construction has been accomplished; in this case, dragging the basic points from which the construction originates will determine the motion of the new elements obtained through the construction. This motion will be consistent with the properties stated by the tools used in the construction. In other words, the use of dragging allows one to directly experience *motion dependency* which can be interpreted in terms of *logical dependency* within the geometrical theory.

Such a semiotic analysis highlights the link between the dragging tool and the meaning of theoretical control that is the complex of meanings related to the notion of theorem. Thus a *statement* that can be *proved* within a specific *theory* (Mariotti et al., 1997). In spite of the centrality of this interpretation for an effective use of dragging in exploration, both for posing and proving conjectures, its difficulty is well documented (Hoelz, 1996; Hazzan & Goldenberg, 1998; Chazan & Yerushalmy, 1998). As a consequence, it becomes crucial to face the didactic problem consisting in relating phenomena, visually and kinetically perceived on the screen, and logical dependency between geometrical properties.

According to a semiotic process, triggered by the teacher's actions, meanings should evolve from personal meanings, concerning the idea of dependent movement as it emerges from pupils' own experience in a DGE, to mathematical meanings, concerning the mathematical idea of logical dependence between hypothesis and thesis, as expressed in a theorem.

Evidence from different studies (see for instance Jones, 2000) indicates that using dynamic geometry software does provide students with access to the world of geometry, including definitions and explanations based on the logical relationships between properties.

Similarly, taking geometrical constructions in a DGE as the field of experience, a long term teaching experiment has been carried out, aiming at introducing students to theoretical thinking. Results from different classes provide evidence of both the complexity and the feasibility of this project. The validation test based on the dragging mode has been used as an instrument of semiotic mediation to introduce the meaning of theoretical control. More generally, different tools offered in the Cabri environment were used as instruments of semiotic mediation to make the meaning of "theorem" evolve (Mariotti, 2000, 2001, 2002).

It is interesting to remark that in a recent research project the general theoretical framework based on the notion of semiotic mediation has been used to promote a theoretical perspective in a completely different mathematical field, namely that of symbolic manipulation of algebraic expressions. Similarly to the case of geometry, a computational environment was designed to offers specific tools that may function as instruments of semiotic mediation to foster students' evolution of the theoretical meaning of symbolic manipulation (Mariotti & Cerulli, 2002).

Conclusions

I want to start these concluding remarks by going back to the epistemological distinction among different *functions of proof*, on which I think there is a large consensus (see Bell (1976) and de Villiers (1990)):

- *verification* (concerned with the truth of a statement)
- *explanation* (providing insight why it is true)
- *systematisation* (the organisation of various results into a deductive system of axioms, major concepts and theorems)
- *discovery* (the discovery or invention of new results)
- *communication* (the transmission of mathematical knowledge)
- *construction of an empirical theory*
- *exploration* of the meaning of a definition or the consequences of an assumption
- *incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective.

This long list clearly exhibits the complexity of the educational task concerning the introduction of pupils to proof, a complexity that cannot be transposed into educational practice without difficulties.

Nevertheless, nowadays (and maybe differently to ten years ago) there is a general consensus about the fact that the development of a sense of proof concerns an important objective of mathematical formation: this objective is strictly intertwined with other objectives (for instance the development of linguistic abilities and competence within different mathematics fields), which require long term strategies of intervention within an encompassing curricular perspective.

The design of curricula, at least in some countries, has been determined by pressures from the world of educational research, and there seems to be a general trend to include proof in the curriculum as highlighted by the change in the NCTM-2000 standards with respect to the 1989 standards, but also, to some extent, by the reform in the UK.

But, in spite of this consensus on the importance and value of proof, the complexity of the idea of proof and the difficulties that must be faced ask for a great caution.

Including proof in the curriculum is only the first step. It is also important to ensure that the goals for doing so and how these goals are operationalised, are clarified and taken into account.

Clearly proof has the purpose of verification – confirming the truth of an assertion by checking the correctness of the logic behind a mathematical argument. But at the same time, if proof simply follows after the conviction of truth rather than contributing to its construction, and is only experienced as a demonstration of something already known to be true, it is likely to remain meaningless and purposeless in the eyes of students (de Villiers, 1990; Hanna & Jahnke, 1993). For a long time an alternative approach has been claimed, characterized by proofs that are acceptable from a mathematical point of view but whose focus is on understanding (what Hanna calls explanatory proofs – Hanna, 1990, p12), rather than on meaningless formal deductive methods. One crucial point in operationalising this approach is that of encouraging student engagement and ownership of the proving activity; that means to add a social dimension to explanatory proving. A culture of validation has to be established in the class, leading students to explain their arguments to peers and to the teacher, as well as to convince themselves of the truth of their arguments (Hoyles, 1997).

In this same vein, interesting suggestions come form the recent research projects, mainly from what is called research for innovation. But the difference between experimenttal classes and reality must not be underestimated, and the problem of disseminating the results, mainly in teachers' training courses, must be taken seriously.

The teacher must be adequately prepared. In particular, I would like to stress the delicate role that the teacher has to play at the primary school level, where students' first beliefs are settled, and most of the basic meanings sprout but remain implicit.

Possible research directions for the future:

In face of the richness and the variety of issues concerning reasoning, proof, and proving, a number of different research directions are possible.

As far as studies on students' conceptions are concerned, it seems useful to enlarge the number of large scale surveys on students' conceptions, but instead of multiplying unrelated studies, we must profit from comparisons between different cultural backgrounds, which can provide deeper insight, highlighting unexpected points of view, e. g. a comparative analysis of different "cultures of proof", as proposed in the schools of different countries, with relation to the specific cultural features of curricula and, more generally, to the cultural values characteristic of each country. Cross-cultural studies may be of great value, and the seminal work of Knipping is a good example highlighting the potentials of such a perspective. In particular, students' conception on proof is tightly related to their beliefs about mathematics.

Moving into the field of beliefs, makes the role of the teacher come to the fore and reminds us not to forget that specific investigations should be devoted to describing teachers' views on proof. Both epistemological and pedagogical perspectives have to be taken into account: what is proof to a teacher? And also, what is a student's proof to a teacher?

Research studies concerning the analysis of argumentation processes and their comparison with the production of mathematical proof appear to be very promising. The construct of Cognitive Unity can be fruitfully applied to describe and to compare cognitive processes related to proof. We need to enlarge the number of case studies; in particular comparisons between experts and novices deserve great attention. This area of research has a natural dimension of investigation concerning the relationships between proof and knowledge construction, in relation with the study of the discursive constitution of both mathematical concepts and procedures constituting an important trend in the current educational research.

Several studies show that competencies of students in devising a proof as well as their understanding of proofs vary across the mathematical subject areas. A common message behind such studies is that not only the competence of interpreting a given and devising a new proof is bound to special areas and situations, but the same is true of the general understanding of what a proof is. Therefore, successful teaching of proof requires the construction of specific contexts. Different fields of experience have been used in specific long term teaching experiments, as discussed above, but also, interdisciplinary working contexts, for instance between mathematics and physics (Hanna & Jahnke, 2002). And a different organization of classroom activities, from dialogue between students to collective discussions orchestrated by the teacher. Certainly, further investigation is required, assuming that such investigation cannot be entirely detached from the classroom, in which the whole didactic system operates, and where beliefs are settled. Classroom investigations are greatly valuable, although methodological difficulties must be taken into account in a serious manner.

Since ancient Greece, Western thought has considered proof to be an essential characteristic of mathematics, and as such proof should be a key component in mathematics education. However, translating this statement into classroom practice is not a simple matter. There has been and there remains differing and constantly developing views on the nature and role of proof and on the norms to which it should adhere. Besides, mathematics education has to take general goals into account, for instance the promotion of mathematical understanding and the furthering of insights into the contribution of mathematics to human understanding of the world around us. All these goals must be tuned with the introduction of theoretical perspectives and in particular with practices of proof.

How to overcome these difficulties is the challenging issue that the future presents to us. The richness of the recent contributions and vivacity of the debate are very promising.

Acknowledgments

I want to thank the colleagues of the team Celia Hoyles, Hans Niels Jahnke, Alexander S. Mishchenko, Ren Zichao for the support and the contribution that they gave me in preparing this lecture and in writing this lecture.

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SP: The shaping of mathematics education through testing

Regular Lecture based on the work of Survey Team 4:

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Introduction

Modern testing, either as standardized mental or achievement testing, celebrates its almost one-hundredth birthday in 2005, a good opportunity for a look back on its history. So the organizers of ICME-10 might have been inspired by this coincidence when calling an "excavation team" to recall and revisit "contributions to the field" of the theme "The shaping of mathematics education through testing".

Curiously enough, and despite the fact that the role and impact of testing in current mathematics education is tremendous, its history has sunk into oblivion. It will not come as a surprise, therefore, that exploring the past may bring about some intriguing insights. The intrusion of testing into education takes place in a period of forceful child-centered pedagogical reform activities[2], and the antagonism of inside-school educational approaches and outside-school governing over education – by means of a politically, economically and administratively oriented testing structure – is in place from the very beginning. And history teaches us – as does present day reality – which of the two is the more powerful side overruling the other one. There may be eras in which society tends to focus more on student-centred or on stronger subject-matter oriented educational initiatives – as was the case in the U.S. in the 1960s to 1970s and again in the 1990s[3] – but testing as the predominant pillar of a firmly rooted organizational structure of the educational system tends to rule out all such endeavors. Our review of the history of testing leads to the recognition that testing has witnessed continuous refinement and enormous technical advancement, but testing itself has not undergone any substantial development or change with respect to its origins, its implicit assumptions and premises, its possibilities and constraints, and the functional purposes it has served and still continues to serve. Testing today is largely what it has always been.

If we agree on this view, we may be curious to know more about the premises, ideas, and aims of those who originally developed and propagated testing. The founders of

1 Testing as assessment and as evaluation of educational systems in comparative studies is comprehensively surveyed by David Clarke recently in two international handbooks (Clarke 1996, in particular 357-362, and Clarke 2003). These substantial reviews are well known and considered as part of our survey that is not repeated here in detail because of the limited space.

2 To just mention some of the child-centred movements we list, e.g., Montessori's educational principles in Italy, Reform Pedagogy in Germany, Dewey's Progressive Education in the USA and Kilpatrick's Project Method, cf. in Howson et al. 1981.

3 See the survey analysis in Howson et al. 1981, Keitel 1980; the recent reform initiatives in NCTM 1989, 1995, 2000, NRC 1989, 2001, and the fourth section of this survey.

testing exhibited their convictions and intentions quite frankly in their writings. What do contemporary apologists of testing think about the views and objectives of their fore-fathers, views and objectives which were never explicitly discarded and are still implicitly underlying constituents of today's actual practices? To put it cautiously, a lot of those ideas, which promoted the success story of testing are not quite compatible with what we today regard as socially and politically correct views and attitudes. Do today's testers know in whose footsteps they walk? Or would they rather not wish to be associated with their forerunners, whose theories remain buried in their books?

Early history of testing

"Tests" as assessment procedures with a simple question-answer format and based on memory and training were known early in Asian countries. Chinese colleagues (Leung, 2004) report the precedence during the Sui Dynasty in 600AD of such "tests" for rigid and strictly political purposes to select appropriate bureaucrats for government employment. Similar forms of "tests" were introduced in Europe with the establishing of public education to prepare for school admission and leaving certificates, and were used in the USA since the 1840s. In 1905 Alfred Binet and Théodore Simon published their "Méthodes nouvelles pour le diagnostique du niveau intellectuel des anormaux". These methods represented the first prototype of a mental test. The Binet-Simon-Test-design differed substantially from its predecessors insofar as the researchers discarded memory and sensory mental testing and emphasized the independence of their instrument of formation or training:

> "It seems to us that in intelligence there is a fundamental faculty, the alteration or the lack of which is of the utmost importance for practical life. This faculty is judgment, otherwise called good sense, practical sense, initiative, the faculty of adapting one's self to circumstances. To judge well, to comprehend well, to reason well, these are the essential activities of intelligence … Indeed the rest of the intellectual faculties seem to be of little importance in contrast with judgment." (Binet and Simon, 1948, 147)

Binet and Simon, who were psychiatrists, designed their test in order to better identify mental deficiencies in children, and thus to become able to develop more appropriate individual treatment. Binet's and Simon's findings received great attention and were an immediate success in the USA. In particular with Edward Thorndike, who had adopted a similar research approach concerning animal behavior and associative learning, like Pavlov's discussed in American journals in 1909, and considered his learning theory published in his "Educational Psychology" in 1913-14 as the most important basis for a science of education. As the founder of American behaviorism, Thorndike understood that mental testing could provide a perfectly fitting tool of measuring behavior – subsuming mental performance as behavior. He became one of the most important psychologists to elaborate and propagate mental testing in the USA. Since 1908 he operated with the term of intelligence test as did Terman, the author of the Stanford-Binet-Test, and Goddard, one of the leading psychologists and test-designer, by whom Binet-Simon's mental test has been extended and generalized to standardized intelligence tests (Terman 1916, Goddard, 1917, Thorndike et al., 1927).

But there was another socially extremely important movement, to which testing fitted as a constitutive contribution: The development of the scientific management principles of Frederick Taylor, by which he inaugurated what Callahan named "The cult of efficiency" (Callahan, 1962). The social efficiency movement is based on the belief that the problems of industrialization and urbanization can be and should be solved by means of science and appropriate scientific methods. Taylor's objective was to optimize industrial production and economic processes in general, and to search for the most effective working procedures by cutting a production process into minute fragments and defining the most effective performance, measurable in terms of time. These fragments are then to be connected as "assembling line production" and "scientific planning and controlling of production" (Callahan, 1962, 14-41). At the beginning of the 20th century, the blatant contrast of a flourishing industry and a quite low performance in the management of the educational system brought public and also school administrators to demand that efficiency criteria and methods be applied to the system. The most radical educator to develop overall efficiency claims and the means by which to achieve an efficient management for schools was Franklin Bobbitt. The development and subsequent adaptation and transfer of Taylor's scientific management principles into the educational system intended to efficiently reorganize schooling and to establish a body of efficient and specialized school administrators and institutions accordingly. Behaviorist psychology and learning theory (Thorndike, 1913, 1922, 1923), which claimed not only to explain learning (of mathematics), but also to develop efficient learning and teaching methods and ensure their outcome by means of precise standards of test-retest-procedures, represented what was seen as the appropriate educational philosophy. The idea of a teacher-proof curriculum with prescribed teaching methods followed by teachers accurately and carefully, and controlled by tests as the necessary controlling instrument, offered a rationale and scientific base. An additional claim for efficiency called for "*scientific measures … needed to predict one's future role in life and determine who was best suited for each endeavor*" (Shepard, 2000, 4).

Standardized intelligence, ability and (scholastic) aptitude tests promised to best do this job, and achievement tests could complement them in measuring effects of schooling in general and in controling teaching efficiently. In order to "eliminate waste in education", according to Bobbitt the most important goal of the social efficiency movement, it was necessary to design curricula by analysis and determination of socially valuable activities and related competencies. Efficiency in schools meant to educate students according to their inherited capabilities, and this demanded highly differentiated curricula with a strict utilitarian focus. Teaching procedures had to be based on effective learning theories and scientific measurement principles (Bobbitt, 1912). Thorndike's "associationism", which conceived of learning as the accumulation of stimulus-response associations and promoted contingent reinforcement by rote learning, has been further developed and refined by behaviorist studies of followers such as. Skinner and Gagné, influential in the reform movement of the 1960s. Shepard lists the key assumptions of the behaviorist model and their consequences for teaching and testing:

1. "Learning occurs by accumulating atomized bits of knowledge";
2. "Learning is tightly sequenced and hierarchical";
3. "Transfer is limited, so each objective must be explicitly taught";
4. "Tests should be used frequently to ensure mastery before proceeding to the next objective";
5. "Tests are isomorphic with learning (test =learning);
6. "Motivation is external and based on positive reinforcement of many small steps."

(Shepard, 2000, 5)

In an educational system as strongly governed by concerns outside pedagogy as was the American at that time, the political climate and actual political requirements have full effect on the system and determine trends and shifts. Thus WW I, and later again WW II, concentrated efforts on best serving the demands of military and economic requirements. Equally, at the beginning of the 20th century, mass immigration into the USA demanded clear, easy to handle, and scientifically justified tools for the selection of immigrants to reliably identify and sort the best fitted and reject those who are susceptible to become a burden of society. Hence these periods were particularly favorable to strong positions of a behaviorist orientation and the related testing apparatus got particular support (Kliebard, 1995).

Social dimensions of tests

Only in recent years we observe a kind of "globalization of testing", aspects of which will be discussed in section 5. For the greater part of the 20th century, the belief in the benefits and an unlimited application of testing was peculiar to the USA. From the beginning, testing found the biggest acclaim in the USA, and there grew to its invincible, inevitable, indispensable position. This is not to say that testing was not and is not being criticized in the USA. American educators and researchers are among the sharpest and clearest critics of testing (e.g. Lippmann, 1922, Hoffman, 1962, Houts, 1977). However, their voices were and are little heard or appreciated by those responsible for public education.

The acceptance, or appraisal, of intelligence, aptitude and achievement tests was very different in other countries. Many European countries were much reluctant in accepting testing in education. The reason was that many countries had, since a long time, well established educational systems, in function in their societies and, by and large, they corresponded to their aims and values. In contrast, in the USA, liberty of education, one of the most important democratic principles, had resulted in allotting responsibilities in educational concerns to local communities that are different in competence and resources. What may have been appropriate in the pioneering days, did no longer function when the nation rapidly integrated in cultural, geographical and economical terms: a unification or the establishing of standards was needed. Since there was no question to touch the democratic right of liberty in education, a solution could not be easily found within the traditional system. Testing filled the gap.

In the USA, education has always been regarded as a bastion of human liberties and democratic rights. This gives rise to a very strong and recurring concern for equity, social fairness and objectivity in every respect pertaining to the place ultimately assigned to an individual in society: Social "nobility" should not be defined by birth, but by talent and innate mental capacities. If it is possible to identify intelligent people easily and as early as possible, it would be best for society, and fair, to support them by good education and to offer them leading societal roles. Barriers between social classes could then be diminished on fair and reasonable grounds. And an overwhelming majority holding the belief that testing serves this purpose best. The further development and eventual dominance of the "objective" test is considered the "single most striking feature of achievement testing in the USA" (Shepard, 2000, 5). Even nowadays many teachers and educators who know the deficiencies of testing with respect to subject matter and educational effects quite well do not see any way to do without it.

It is true that school systems of countries that do not totally rely on testing, have deficiencies in equity and objectivity. When the assessment of students is left to teachers, or when different standards are applied in different schools or different parts of the country, unjust and unfair judgments do occur. But it is a prevalent view in those countries that such deficiencies can be tolerated for the sake of the possible benefits of the system with respect to subject matter formation and fair treatment of students. The idea is that a teacher would rather convey a notion of what a discipline is as a whole if this discipline is not put through "the mill of testing", and that it is also harmful to equity if capturing those parts of formation not contained in testing and test-driven curricula, is not within the reach of every student. Also, a teacher is more likely to come to a fair judgment about a student if she knows her well and she has a holistic view of her personality. There are many such considerations to take into account. This is crucial to secure organizational conditions such as high standards of teacher formation, harmonization of curricula and the like. The underlying insight of systems characterized by the views just outlined, may be phrased as follows: Education is so complex, contradictory and contains so many parts that only a wise balance in distributing emphasis and constraints can secure satisfactory outcomes. The more radically emphasis focuses on one demand only, the greater the risk will be of seeing shortcomings in all other respects.

"Objectivity" and "social fairness" have become catchwords immediately put forward when testing is discussed. Catchwords often assume an aura of evidence in the course of time, and there is little interest in questioning this aura. The success of testing in the USA was not only due to complying with an urgent demand for standardization as well as with democratic claims. It also was a simple diagnostic method, based on good or common sense, which could become a handy mechanical device for sorting people according to ability and become extended to a generalized scientific measurement instrument. The mixture of scientific methods with popular assumptions and social prejudices about traditional theories created the fertile ground for a wide acceptance and unlimited range of applicability of testing. The most convincing aspect was, and is, the ease by which the modes and ways of tests are designed, constructed, administered, scored, summarized, and reported, in particular after the marriage between testing and computers.

The usefulness of tests did not foster inquiry into the assumptions and theories on which testing originally had relied. Sheldon White and Lorrie Shepard are amongst those who try to identify the premises on which the huge realm of testing is built. White emphasizes the politically conservative characteristic of tests:

"Our notion of intelligence has transcended questions of definition and proof. … Who sanctified intelligence and made it prior to proof? …Before Binet, or Thorndike, people had made up their minds about the centrality of intelligence as the epitome of human merit. When the tests came along, they were not required to prove their way. The tests could not then – and they cannot now – prove their way. They were exemplifications, definitions, and manifestations of an entity whose scientific and social sanctity was given. …The argument then is that the intelligence test exploded into public acceptability and public use not because of its merits, but because it could be seized on as part of a more fair and more just system of social contracts. Tests could be used as part of the system of allocating social opportunity. Needless to say, the tests could not have been so accepted if the people in power… saw the tests as potentially destroying their children's power. But the IQ tests of that time had the rather happy property of being a conservative social innovation. They could be perceived as justifying the richness of the rich and the poverty of the poor, they legitimized the existing social order."
(White, 1977, 36-38)

Curiously, assumptions and theories which in part were conservative and outdated already when educators first constructed and justified tests, have, since then, undergone another century of growing older. Shepard reports about teachers and principals, partners in a collaborative research project about new and alternative assessment designs, who, however, strongly demanded "objective" assessment modes and believed that fairness is only ensured when assessment methods are uniformly administered. They were, therefore, reluctant to using the additionally proposed individualized assessment models intensively. (Shepard, 2000, 5) She also studied beliefs and perceptions held by psychometricians, educational politicians and administrators in the state of Colorado about implicit learning theories, noticing that they hold very similar convictions to those of their colleagues during the efficiency movement at the beginning of the century. (Shepard, 1991). She concludes that the findings show that *"dominant theories of the past continue to operate as the default framework affecting and driving current practices and perspectives. Belief systems of teachers, parents, and policymakers today still derive from these old theories."* (Shepard, 2000, 4)

The perception that everything is measurable and definable in terms of numbers is a belief dating back to Pythagoras' times, but it is repeated and reinforced as a general belief of scientists and psychologists of 19th century:

"Whatever exists at all exists in some amount. To know it thoroughly involves knowing its quantity as well as it quality. Education is concerned with changes in human beings; a change is a difference between two conditions; each of these conditions is known to us only by the products produced by it – things made, words spoken, acts performed, and the like. To measure any of these products means to define its amount in some way so that competent persons will know how large it is, better than they would without measurement. (…) This is the general Credo of those who in the last decade have been busy trying to extend and improve measurements of educational products."
(Thorndike, 1918, quoted in Cremin, 1964, 185)

Because socially valued mental abilities are non-visible, existing inside individuals as a conglomerate of "general intelligence", they have to be detected to serve as a valid justification for assessment or selection. To make them "visible", physical phenomena like behavior or actions can be used. This requires a process of translating phenomena from the invisible to the visible world. In claiming that this can reliably be accomplished, early test constructors adopted a view of the concept of intelligence, that seemed to be very helpful for their purposes, and still is for their successors. It was crucial for tests, which concern decisions of long-term or life-long impact on the candidates, that a reliable prognosis can be derived from an instantaneous picture, possibly taken at a low age. A necessary condition for this is a notion of intelligence as an innate, static equipment of human beings that is not easily changed by external influence like socialization or school education. With respect to intelligence a person will always remain the same. This reflects Darwin's theory of evolution, in which evolution does not occur in one individual but in the genus over a long period of time. An important assumption and presupposition for any test construction is the belief that the distribution of mental abilities follows the Gauss-model of normal distribution, as do most physical phenomena, e.g. measurements of the length and the weight of objects. In fact, tests were the first "scientific tools for measuring", in which the normal distribution of data was a precondition of construction and not only a result of the measuring process.

On IQ-testing White concludes:

> "If one reviews the situation persisting from Binet through Thorndike to the present, we find that we have in some astonishing way managed to continuously upgrade a technology for directing an uncertain measurement paradigm toward an undefined entity."(White, 1977, 34) (see also a strong and critical analysis in Liungman, 1970)

In Laughland and Corbett, two school principals, there is a ring of distress. They start their look at standardized tests as follows:

> "No people on the face of the earth have been bitten quite as hard by the testing bug as the American people. We take quizzes on every aspect of our lives and derive self-satisfaction or self-mortification from the results we achieve. We have even been known to cheat a little in order to get that warmth and security we feel when the little computation at the end of the quiz yield a result indicating that we are 'above average' or even 'superior' to others… We have an unfortunate belief that every skill, every talent, every body of knowledge can be broken down into finite parts, that we can draw a sample of those parts, test ourselves with that sample, and thus prove our ability in any particular area. We employ this technique in business and industry to determine the right people for the right job, which label to assign to people and sort out those who should be refused admission … The blind faith of testing is matched by no other belief. Testing can mark innocent children with stigmata for the rest of their lives, tarnish schools' reputation and ruin teaching careers." (Laughland and Corbet, 1977, 331-332)

In these reflections one aspect pertaining to "blind faith" is less considered than it deserved to be: Devices and tools created for interference in social life are never created without specific intentions. The tool has to accomplish what it is construed for. These tools mostly do not openly show the underlying intentions, for they would not be necessary if these intentions were easily accepted. If we want to understand the effect of the tool we have to detect the purpose behind it. Mathematics plays a prominent role in the tool, because it provides credits the tool with scientific correctness and hence legitimisation.

Testing is a perfect example of a basic constructive approach and its practical use, as exhibited in the promoters' writings. This was possible because these writings were hardly within the reach of those concerned, and the scope of the new tool could hardly be anticipated. Moreover, at the time the ideology conveyed in the papers would not generally be seen – as it must be today, in the age of social, political, and moral correctness – as transgressing absolute limits of acceptability. The best quality a tool of this kind can provide is pervasiveness.

The incorporation of mental and achievement tests and testing into larger parts of the American society created long lasting facts: fast growing commercial test companies and private industries turned the testing machine into a perfect and complex technology continually reinforcing its importance in education. By developing additional standardized tests, also for situations outside the educational domain, these products contributed to a general social acceptance of tests. New administrative institutions for educational testing purposes, on local, regional and state levels, were established accordingly outside schools, with NAEP (National Assessment of Educational Progress) as one of the most influential agencies providing national reports at regular intervals. Despite educational fashions and wild debates in favor of or against tests[4] during the last century, the established social structure of testing and the respective certifying institutions made the belief in testing and the underlying assumptions be taken for granted along with the actual acceptance of the testing enterprise.

Is this sketch overdrawn? Judge from the following quotations. Terman, one of the main authors of the Stanford-Binet-Test, and Goddard, one of the leading psychologists and test-designer, praise the social gains of testing results after the first success with group tests for military purposes, in particular for coping with problems of immigration:

> "… in the near future intelligence tests will bring tens of thousands of these high-grade defectives under the surveillance and protection of society. This will ultimately result in curtailing the reproduction of feeble-mindedness and in the elimination of an enormous amount of crime, pauperism, and industrial inefficiency." (Terman, 1916, 6-7)

> "… the number of aliens deported because of feeble-mindedness … increased approximately 350 percent in 1913 and 570 percent in 1914… this was due to the untiring efforts of the physicians who were inspired by the belief that mental tests could be used for the detection of feeble-minded aliens."
> (Goddard, 1917, 271, in Kamin, 1977, 65)[5]

4 Cf e.g. the surveys and debates in: American Federation of Teachers 1977, Clarke 1996, Hoffmann 1962, Haney 2002, Houts 1977, Kamin 1974, Kober 2002, Linn 2000, 2002, Popham 1999, Wood 1991

5 After testing immigrants from various parts of Europe, differences among those from Northern Europe to those from South-Eastern Europe were explained as innate superiority or inferiority and justified to define different numbers for admission.

Kliebard, in looking back on the early history of testing mania, outlines some social consequences and political misuses:

> "Educate the individual according to his capabilities has an innocent and plausible ring; but what was meant in practice was that dubious judgment about the innate capacities of children became the basis for differentiating … along the lines of probable destination for the child. Dominated by the criterion of social utility, these judgments became self-full-filling prophecies in the sense that they predetermined which slots in the social order would be filled by which 'class of individuals'." (Kliebard, 1975, 56).

The upsurge and success of the testing movement furnished both the ideological basis and the instrumental basis for school practice by sorting students rather than educating them.

Testing and mathematics

Since the very beginning of test construction, mathematics has played an important role in both intelligence or aptitude and achievement testing, in two different respects: as subject matter of test items, and as the fundamental methodological device in test construction itself. Mathematics imposed itself on testing for various reasons:

– Test construction developed in close relation to the efficiency fever and the leading ideology of utilitarianism of those days. Hence "socially necessary" skills and useful knowledge were viewed as central tasks of public schooling. Among these arithmetical skills and logical analysis – mostly on a common-sensical level- occupied prominent places.

– Logical reasoning was viewed as the medium of intelligence, more or less a synonym of intelligence. Mathematical test items therefore seemed particularly appropriate for intelligence and aptitude testing. Mathematical structures like relationships, patterns and conceptual connections could be viewed as facts like any other factual relationship in reality. Ability, knowledge and skills in solving mathematical tasks therefore could be generalized as a capacity to acquire more general abilities, more factual knowledge and skills required in all domains of practical life. Choosing mathematical content as representative test items in IQ or aptitude tests granted an advantage: Mathematical tasks were the least problematic, without any ambiguity the easiest to be measured.

– Traditional mathematics instruction in a way preconfigured the organization of subject matter in tests by its fragmentation of subject matter in tasks, which served as exercise as well as achievement control. The school mathematic curriculum was much more structured in clear facts and skills than was the case in any other subject domain, the dimensioning and distribution of content matter were established and accepted, thus best fitting Bobbitt's claims for orientation towards social needs. And the complete lack of ambiguity in mathematical facts, rules and tasks in schools allowed to simply use them in test construction as items for which it can be clearly decided whether an answer is right or wrong.

– The visible presence of mathematics in testing: in test construction, as testing subject matter, and in the rating of outcomes, lent scientific seriousness to the whole enterprise. The popular perception of mathematics as dealing with and producing objective truth was readily extended to testing and enabled it to become a universal and accepted, though often incisive, social tool. Ironically, the mathematical interference in test making is the way by which subtle steering, in fact manipulation, is most effectively creeping in and, in the guise of mathematics, passing unnoticed.

The call for changes and reform grew to bigger dimensions in the 1950s when advances in the sciences, mathematics and learning theory had continually widened the gaps between school practice and new scientific standards. Eventually a new period of reforms came into being, initiated by a political event frightening the Western countries during the cold war: the Sputnik-Shock after the launching of the Sputnik satellite by the Soviet Union in 1957. One reaction to this event was a massive criticism of the state of mathematics and science in school. Well funded by federal and state agencies, an over 20 years long epoch of exceptionally intensive and rich research in mathematics and science education allowed educators to experiment with designing and applying new curricula. This epoch has been surveyed and evaluated in detail (e.g. Howson et al., 1981). It is sufficient here to outline what this development meant for testing in mathematics education.

– Behaviorist theory and practice in mathematics education, was in an ideal position for investigating improvement of the testing approach. Above all, the effectiveness of teaching methods had to get a new base. This was expected to come from introducing "programmed instruction" as a "teacher-proof teaching method" in connection with computer technology: a major part of teaching could be transformed into computer-aided-instruction. Programmed instruction could be designed in analogy to test design, as pre-testing-learning-post-testing chains made testing an integral part of the learning program. Teaching and testing converged.

– Projects initiated by university mathematicians, cognitive psychologists and educationalists claimed quite a different scientific foundation. Mathematicians complained about the low level of mathematical competence and higher order thinking of university freshmen. School mathematics should be urgently adapted to significant modern developments of the science of mathematics: fundamental principles of modern mathematics focusing on comprehensive and abstract thinking, on a holistic view of mathematics, on the use of a formal and unified language, on reasoning and proof, were seen as the cornerstones in teaching and learning modern mathematics. There was no special focus on developing new teaching methods nor any opposition to testing in principle; it was supposed that new test programs could easily adopt the new vocabulary and content. But it was left unnoticed that principles of compartmentalization of subject matter in test items were opposed to a comprehensive and holistic understanding of mathematics; the new curricula failed in old tests.

– 'Structuralist' or 'formative' projects (see Howson et al., 1981) with a stronger focus on connecting new fundamental mathematical ideas with new constructive learning methods, influenced by research by cognitive psychologists like Piaget and Bruner, mainly addressed early learning in primary schools. However, processes and outcomes of child-centered teaching and "discovery learning" allowing children to construct their own perceptions of mathematics and enrich and revise them gradually, could not be captured by traditional standardized tests. This also applied to projects focusing on applications of mathematics in contexts, or integrating several disciplines like mathematics, science, social sciences in collaborative project work with a focus on problem solving strategies and social competencies. Traditional tests were no appropriate assessment means to capture other achievement than skills or knowledge of facts.

The reform in the USA had met an invincible antagonist: the established behaviorist orientation of public education with a narrow view of efficiency, united with the lobby of test producers, both of which carefully sustained the myth, held until today, (Kober 2002) of equity, fairness, and objectivity granted by standardized testing. Test production had grown to a big, profitable industry with excellent connections to educational administrators and stakeholders in education:

> "For the past several decades, standardized testing has been a growth industry in this country, and if we look to the future, the forecast for the industry is, as Wall Street people like to say, bullish." (Houts, 1977, 13)

Reform projects on the other hand, were limited enterprises with respect to staff, time and funding. Reform projects were doomed to failure in as much as they were not compatible with standardized tests.

Although the success of the reform projects in this period is considered as rather limited in general and, in particular in the USA, the big efforts gave a push to similar activities in European countries and elsewhere. The best outcome and long lasting merit of these activities ware the creation of a growing community of qualified mathematics education researchers, at universities as well as at research centers, who continued and extended mathematics education research. They developed a variety of new and fundamental research problems in mathematics education, e.g. by studying the social and cultural dimensions of classroom teaching and learning with newly developed research designs, which were to overcome the antagonism of quantitative versus qualitative methods; and incorporated research into teacher education programs by collaborating with teachers in action research in various countries. Researchers were encouraged to follow their own ideas and to pick up suitable project designs from others as well, which eventually led to international co-operation, reinforced by international organizations in mathematics education like e.g. ICMI, PME, CIEAEM et al. (Jacobsen, 1996).

In the 1980s, new theories of learning, such as constructivism, social constructivism and cognition in practice entered educational debates, but also fostered new research methodologies for classroom studies. The underlying perception of learning claims that mathematical concepts and knowledge are developed by construction and negotiation within social groups, that the classroom as the place where students can learn with and from others allows for another quality of insightful and satisfactory learning than can be found with individual learning. Restricting assessment methods to those taking the shape of standardized tests was a threat to teaching methods that concentrated on students' discussions and collaboration. Therefore the invention of new and more appropriate assessment modes seemed to be crucial and urgent (see e.g. Barnes et al., 2000, Clarke, 1992, Leder, 2004, Linn, 2000, Madaus, 1992, Popham, 1999, Schoenfeld, 1999, 2002, Wood, 1991).

Persisting and increasing criticism of the constraints of testing and similar endeavors eventually led the American National Council of Teachers of Mathematics to commission activities for defining national standards of curriculum and of assessment (NCTM 1989, NCTM 1995, NCTM 2000), based not only on wide discussions among researchers and teachers, but also on research studies. These publications were the first in the USA that offered a broad unified new vision of mathematics education practice and were rather well accepted within the field. The Mathematics and Science Education Board of the National Academy of Sciences supported this work by publishing related examples of alternative, non-schematic assessment tasks closer to students' activities allowing several ways of solving problems and different sophisticated solutions, incorporating port-folios and other formats that contrast and complement the usual test-formats for teachers use.

The government act "No Child Left Behind" (2002) is the latest though most consequential change in the US school system. It aims at raising test scores in reading and mathematics by 100% until 2013/14. The way to achieve this aim comprises two additional, compulsory federal tests ("reading first", "mathematics first"), recording of all test results, and supervision of compliances with schedules broken down to minutes. Success is rewarded by financial benefits, failure punished by financial and other disadvantages for teachers, principals, and schools, up to firing those who are seen as accountable.

> "The federal act, signed by Bush in January 2002, expands testing programs and imposes sanctions on schools whose students do not meet state standards in reading and mathematics. The purpose of the law is to close the achievement gap that finds some groups of children – such as minorities, special education students and children from low-income families – lagging behind others." (Hartford Courant, Wednesday, September 8, 2004)

Criticism is fierce. Long lists of grotesque side-effects and faulty constructions are reported. The concern that the law destroys achievement of public education gained through decades of engagement, growing professionalism, and hard work, is uttered too frequently to be singular over-reactions.

All this also regards testing. In mathematics education, teaching time is not sufficient to prepare actually for all the tests, let alone to allow for sense-making in teaching and learning. Schools are forced to cut down on the time-slots allotted to "soft" disciplines such as arts and music. The impossibility to comply with the goals and subsequent penalties threatening the existence of teachers and principals compels them to adopt tricks and cheating, and if it was but for that, tests are invalidated. Stress and anxiety become prominent features of schooling.

> "The test publishing industry gears up to produce new exams on an industrial scale, the result of a federal law that requires the greatest expansion of standardized testing in American history. Many states now test students in only a couple of elementary grades, but the law known as No Child Left Behind requires states to test every public school student in third through eighth grades and one high school grade every year. Educators have nicknamed the law, 'No Child Left Untested'." (Dillon, 2003)

At first sight one might take the law as an expression of a somewhat bizarre nostalgic turning back to efficiency mania. There are indications, however, that it springs from strategic planning rather than from lack of professionalism: In fact, the Act stipulates that private schools being exempt from fulfilling the regulations, do not have to apply the new tests, and hence risk no penalties. This has induced observers to surmise "a plot to discredit public education to the point where privatization and choice are seen as the only answers." (Lewis, 2002), or a "war against America's public schools" (Bracey, 2002).

Tests in international comparative studies: Testing mathematical literacy

International comparisons in mathematics education on a large scale focusing on performance assessment tasks were first carried out and published in 1968 (FIMS), then – more prominently – SIMS was carried out in 1988 and TIMSS since 1995[6]. In the first study, a rather small group of mainly wealthy countries compared their successes and failures in certain areas of mathematics and science instruction and used some results for different national or local reform activities. Discussions raising problematic features and weaknesses of the first study led to a refinement of the measures and to efforts of improving comparative analyses by adding other, including qualitative, data and forms of interpretation, such as video-based lesson studies and curriculum analyses, which however never reached the same public attention as the ranking lists of average student achievement. Few criticisms of the studies addressed questions not only concerning the research designs or data analyses, but also the needs and interests served by those studies. Clarke (2003) raised the issue of cultural authorship of international studies and argued in favour of more explicit collaborative processes through which educational, philosophical and cultural positions are given voice in the interpretation of data and in the reports. The effects of such global projects for participants, as well as the high demands on resources in terms of budget and expertise in international comparative testing, seemed not to be fully justifiable for some countries, in particular as it could not be shown that there is a substantial gain of new insights from quantitative results in comparison to other, e.g. local, research activities. The acceleration in international comparative test making leaves those concerned with mathematics education breathless: The masses of data produced by TIMSS had not been analyzed when yet another TIMSS-R (TIMSS Repeat) was already being conducted; TIMSS-R was immediately followed by 2000-PISA I, and 2003-PISA II[7],. The problematic nature of the globalization of achievement tests has come out much more in these years.

Undeniably, the most prominent feature of such testing projects, the ranking of systems or countries, pushed back everything else. The ranking has encountered popular interest; it has become another discipline of international sports competitions. Public perception is constrained to the ranking, media push politicians to action, decisions affecting public education in a serious way are taken very hastily. The repercussions of sports like testing may have an uncontrolled irrational impact on the development of

6 IEA's First and Second International Mathematics Study and the Third International Mathematics and Science Study, today Trends in International Mathematics and Science Study, are referred to here. An early very strong criticism of FIMS is given by Freudenthal 1975; a richer debate exists for SIMS, e.g. Travers et al.1988 give a survey and point to some deficiencies, and Atkin & Black 1996 discuss perils and challenges of international comparisons; in Kaiser et al. 1999, a broad and controversial debate on TIMSS among mathematics educators is presented; Clarke 2003 provides a most substantial and critical overview of fundamental problems and opportunities of international comparisons in mathematics education.

7 PISA is the Programme for International Student Assessment, conducted by OECD, the most recent and most ambitious comparative study so far.

the public education in individual countries, in particular if cultural self-perception is being affected. The testing results of PISA I and the activities that followed world-wide were discussed in an APEC Summit meeting early in 2004. Representatives of Asian countries complained that, although their performance is considered as rather good, many countries try to make up what is missing in their education systems by replacing their traditional practices with new ones adopted from other countries, and called for keeping in mind the problems resulting from reform, particularly as regards strategies arrived at through international comparisons. The danger might be that in an attempt to learn from others, own strengths might be too easily abandoned. The focus on the successful parts of other's achievements only, might neglect what did not work for them. The tendency to disregard the conditions that might have enabled the achievement of the others, i.e. borrowing techniques while ignoring the cultural, social, and systemic contexts that help making them effective, tends to look at practices in isolation, ignoring the possibility that they interact with one another to produce the desired effects. Mathematics educators in particular warn against taking too early actions in adopting other cultural habits or policies. Park and Leung (2004) argue that strange reactions have followed some debates, e.g. that in order to lower the gap between the East and the West, Western countries adopt methods, techniques or materials from successful Eastern countries without being able to, or wanting to, adopt the attitudes or habits of students and teachers in these countries or the learning climate usually cultivated. But the two authors are more concerned that:

> "Eastern high achieving, but low attitude countries are urged to adopt approaches that are considered as student-centered and context-related. A uniform curriculum might not only be not transferable or ineffective by respecting only superficial aspects of the other and not the substance of underlying principles and conditions, but might destroy traditions and characteristics of countries. On the one side: countries might abandon their abstract and mathematically rigorous features, but do still judge them as the most important; on the other: effective methods are not independent of content and context, as well as many other factors that contribute to professionalisation of teachers and students." (Park and Leung, 2004)

The tremendous costs in terms of researchers and resources were rather a burden for some of the participating smaller countries at a lower socio-economic level. However, the competition seemed to force politicians to deny these problems and to participate, as international studies of this type now tend to be regarded as superior to research work on a (geographically) smaller scale, although such work may be of more immediate interest in these places. Many of those working in large international test projects have withdrawn from other studies, and after the release of the test results, they are often absorbed by studying and evaluating them. Researchers may be urged to more consider test results or to better accommodate their research themes or aims to the trends set by the tests.

International cooperation and intensive collaboration and exchange among academics or research institutions from various, even distant, areas of the world have been celebrated as a great success supporting the scientific orientation and development of mathematics education research. The focus on looking for differences, while celebrating diversity, integrating views from various cultural and social traditions in studying similar questions of mathematics education, and developing a variety of methodologies particularly appropriate to local and national conditions, have been affected and rendered less important by international comparative test projects. These projects have challenged and weakened international collaboration not only because they are attracting much attention, but also because trustful co-operative relationships may, in one way or the other, be infected by the competition. All of these considerations make it doubtful whether prospective gains of international comparisons by testing measure up with the many uncontrolled effects, if not disadvantages, that they entail. The constructors of the tests may not look much for them, for the "consumers" it is quite different.

For assessing the usefulness of large scale comparisons, two aspects seem to need a clearer understanding: Firstly, to what extent can 'positive effects' of such evaluation of national performance be expected. 'Positive effects' here mean that countries act upon the suggestions given in the study's evaluation for every participating country, e.g. by national representatives in PISA. Secondly, in giving such advice, the project assumes a model role for the participating countries basically relying on the syllabus material indirectly established by the material included in the test item. It ought to be examined whether such test items as e.g. the PISA ones are sufficiently substantiated – theoretically, empirically, and politically – so as to serve as a model for the syllabus in many countries.

Accompanied by complementing school, teacher and student questionnaires, testing outcomes in e.g. PISA can be interpreted in wider socio-economic contexts, and could thus provide valuable statements, as shown in the following quotation:

> "In Belgium, the Czech Republic, Germany, Hungary, the Slovak Republic and the United States, and the partner country Uruguay, the between-school variance in student performance that is attributable to students' socio-economic background accounts for more than 12% of the OECD average between-student variance … and for Belgium, Germany and Hungary this figure rises to over 40% if the additional effect of the whole school's socio-economic composition on each student's performance is taken into account as well."(OECD, 2004, 187)[8]

Due to the well-established cooperation and exchange in the scientific community, few researchers and educators will be unaware of deficiencies in their own school systems. If no ameliorations take place it is because massive political obstacles prevent it. The prestige of PISA can raise a media storm of public debates but does not provide substantiated and realizable suggestions for political changes.

Suggestions addressed by PISA II to the participating countries in individual evaluations, may read as follows, quoted from the "Brief summary of key findings for Ireland" (2004,4)

> "Students in Ireland were expected to be very familiar or familiar with the mathematical concepts underlying between 50% and 70% of PISA items (depending on the syllabus level taken). This suggests that any future review

8 Regarding Germany, to researchers and educators the reported phenomenon is all but new. For many years, the horizontally tripartite school system has been criticized, but only half-hearted ameliorations have been attempted. The problem is a political one: The actual school system is among the most fiercely defended properties of conservative and liberal politicians, a relic of a clear social class structure of the educational system. When a PISA representative urged Germany to take measures against the problem, he earned a storm of indignation from some politicians and media.

> of mathematics at post-primary level should consider if important mathematical content is being omitted. Again, any debate around the differences between PISA and the (Irish) Junior Certificate mathematics would need to recognize that there are important elements of the Junior Certificate mathematics syllabus that are not assessed by PISA (e.g. sets, geometry and trigonometry)".

The explicit suggestion here is merely that if Ireland wants to score better, it has to better accommodate its syllabus to the content of the PISA material. Implicitly it is imputed that the selection of test material in PISA is in any respect better, more competent, more authoritative than a national selection would be. It may be asked what this self-confident evaluation of the study is derived from.

In fact, despite of all extensively given information about the way PISA has been constructed, the precise way in which the test items have come into being remains blurred. We are informed about the contributions of national experts and different consortia, about pretests and meticulous care about measurability and validity of test items, but we do not learn anything about the scientific approach to "assessment areas" about "definition and distinctive features", "content dimensions" and how they have been specified, about the "process dimensions" and "situation dimensions". This is even more badly missing as PISA has embarked on a quite new testing format: In earlier international comparisons with achievement tests, the body of content matter to be tested for test items was established by compiling the national syllabuses of participating countries and there finding their "greatest common divisor". This is to say that in the tests of former international comparisons there is a direct step from syllabus content to test items, the selected items being credited with the legitimation of an assumed universal(ized) mathematic curriculum. In contrast, PISA established a new overall goal, namely to measure "Mathematical Literacy". Test items are constructed explicitly for this purpose. It is a particular feature of the test that all test items are given in terms of reality: PISA-items consist of very short narratives that describe a situation or a scene, in which it is supposed that some mathematics can be applied to solve a given problem, or has been applied and the application should be evaluated.[9] The concept of Mathematical Literacy is defined in slightly different words in PISA 2000 and PISA 2003:

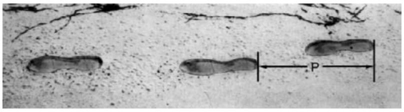
> "Mathematical Literacy is the capacity to identify, to understand, and to engage in mathematics and make well-founded judgments about *the role that mathematics plays, as needed for an individual's current and future private life, occupational life, social life with peers and relatives*, and life as a constructive, concerned and reflective citizen." (OECD, 2000, 10)

> "Mathematical Literacy is the capacity to identify and understand the *role that mathematics plays in the world*, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen." (OECD, 2003, 26)

9 Testing of skills, which are not limited to mathematical concepts and procedures, has been also a component of TIMSS 1995: "Reasoning and social utility were emphasized in several items. A general criterion in selecting the items was that they should involve the types of mathematics questions that could arise in real-life situations and that they be contextualized accordingly" (IEA 1997, iv).

Neither definitions refers to a theoretical elaboration or grounding[10]. It is implied that the meaning of Mathematical Literacy is embodied in the new item format, and it is stated that the operationalisation of the rather ambitious definitions involved *pragmatic* decisions on the selection of test items in order to guarantee inclusion of a variety of mathematical ideas and contexts. (OECD, 2000, 10). For the validity of the test items, it is crucial not only that all aspects of mathematics expected to be mastered by 15-year-olds be appropriately represented, but also that reality be represented correctly, in particular as embedded in a comprehensive view of subject matter implications and the context-related aims and interests involved. Moreover, it is essential that the "role that mathematics plays in the world" is adequately represented, not just with respect to the variety of applications but regarding the major functions mathematics serves.[11] It goes without saying that it seems difficult to comply with such demands in the very limited dimension of short test items, and some doubt may be proper about how PISA fulfills them. A short look at a few test items may provide certainty.

UNIT: WALKING (2000)
Context: Personal



The picture shows the footprints of a man walking. The pace length P is the distance between the rear of two consecutive footprints. For men, the formula, $n/P = 140$, gives an approximate relationship between n and P where
n = number of steps per minute and P = pace length in metres.

WALKING QUESTION 3 (Item code: M124Q03)
Domain: Change and relationships. Item type: Open constructed response:
Bernard knows his pace length is 0.80 metres. The formula applies to Bernard's walking. Calculate Bernard's walking speed in metres per minute and in kilometres per hour. Show your working out.

Empirical verification of the postulated formula given, $n/P=140$, establishing a constant relationship between pace length P and number of steps n for men can not be confirmed in general. Besides, it is theoretically questionable. Although the same formula may not

10 For an analysis of conceptions of "mathematical literacy" with respect to the social practices implicitly or explicitly promoted see Jablonka, 2003; see also Gellert & Jablonka, 2002a, 2002b, Jablonka & Gellert, 2002, Jablonka, 2000

11 PISA aims at ensuring that the tasks are based on 'authentic' contexts, which means that the situations in which the tasks are embedded are 'real-world' settings. The problem situations are classified as 'personal', 'occupational', 'scientific', 'public', or as combinations of these categories. Other ways of classifying mathematics in context would have been, for example, in terms of different associated aims (e.g. description, prescription, explanation), of different technological aids (such as calculators, tables, visualisations) or of different perceptions of the relation of the mathematical concepts to "reality" (e.g. empirical models, theoretical models, simulations).

be applicable for little children, who keep pace with adults, it is obvious that also adults with smaller pace length can modify the frequency of paces per minute, e.g. in order to keep pace with a companion, without running.

MathematicS UNIT: PATIO (OECD, 2000, 73)
Nick wants to pave the rectangular patio of his new house. The patio is 5.25 meters long and 3.00 meters wide. He needs 81 bricks per square meter.
Domain: Shape and space,
Sample Question 13 (Open-Constructed Response)
Calculate how many bricks Nick needs for the whole patio.
Scoring scheme: 'correct' (score 2) for answers like: ″which indicate 1275 or 1276, e.g.5.25 x 3 = 15.75 x 81 = 1276.″ (units not required), 'partially correct' (score 1) and 'wrong' (score 0)

It must be supposed that, at best, the hobby paver is accidentally successful with his calculation of the number of bricks he needs. On the one hand he has forgotten that there are joints between the bricks (about 5 to 8 mm wide) that save him bricks. On the other hand, there is indication that bricks will have to be cut (81 bricks per square meter). This not only means losses because of the rests left over, but also because of breakage. Count 15 to 20%, for that is an empirical value. Furthermore it makes a difference for the number of bricks required whether they are of square or oblong shape, and in the latter case whether they are placed in brick length parallel lines, or form right angles. It also is reasonable to keep some bricks in reserve for later repair. And bricks are normally sold in lots… Not considering aesthetic problems… Already these examples, which, with respect to the reality context involved, are of very modest complexity, reveal that reality is condensed to a relatively high degree of stylization[12]. In the first example, a mathematical relationship is claimed, which – if it exists at all – is so inconsistent that it would not make any recognizable sense in reality. The second example shows that disregarding even minuscule aspects of reality transforms the real context into abstraction. In reality, the hobby paver would probably know what to consider, the student who refers to his everyday knowledge about paving would maybe deliver a more differentiated answer than expected and hence fail the test, whereas the 'clever' student knows that in a school context there are only just word problems and that he is expected to give a formal answer to a formal problem. This student is realistic, the test constructor is not. The fact that side-aspects of the reality context, though of sensible importance, are not even noticed shows that there is little interest of the test-makers to reality.[13]
Finally two test items may show most of the important aspects of mathematics in reality, which are touched upon in the item, however – as far as materials are accessible – without being explicitly addressed anywhere:

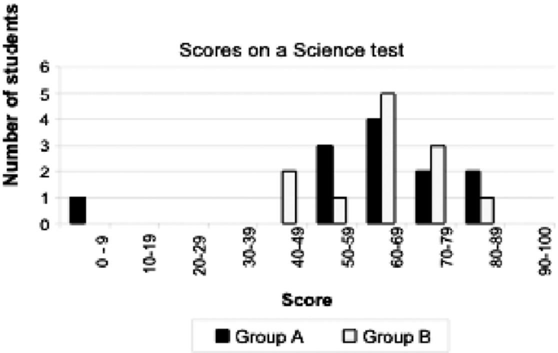
12 Authentic texts containing mathematics (e.g. from newspapers, manuals, reports of scientific studies) are not used in the mathematics test, but are indeed part of the reading literacy test in PISA.

13 Authenticity in the mathematics test turns out to be a contradiction in itself, if "authentic" means conforming to an original situation in a way that reproduces essential features of the original. But these essential features are predetermined to be the mathematically interesting ones. In this respect the PISA items are extremely heterogeneous.

UNIT: TEST SCORES (2003)

Context: *Educational.*

The diagram below shows the results on a Science test for two groups, labelled as Group A and Group B.



The mean score for Group A is 62.0 and the mean for Group B is 64.5. Students pass this test when their score is 50 or above.

TEST SCORES QUESTION 1 (Item code: M513Q01)

Domain: *Uncertainty.* **Item type**: *Open constructed.*

Looking at the diagram, the teacher claims that Group B did better than Group A in this test. The students in Group A don't agree with their teacher. They try to convince the teacher that Group B may not necessarily have done better.
Give one mathematical argument, using the graph that the students in Group A could use.

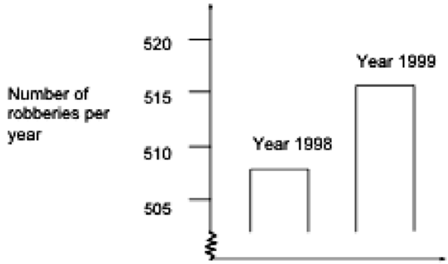
Invoking different criteria to evaluate a diagram apparently causes antithetical outcomes: Yet, both of the criteria are equally valid. So the choice of the criterion on which of the two student groups is declared better than the other, is arbitrary. Knowing the groups, as a teacher does, knowing that group B is more homogenously centred around the average, whereas group A has a stronger distribution towards the extremes, the teacher could, according to the criterion chosen, favour one group in a correctly, though in fact arbitrary manner. That is to say: While giving an impression of objectivity, mathematical assessment lends itself to manipulation. Is it a self-referential test item?

A similar effect of manipulation is shown on the next page, in this case produced by way of graphical representation. Optically, or by a graphical representation deliberately chosen, a tremendous increase of robberies of around 50% is being suggested, whereas in reality it is about 1,57%. Who might be interested in such manipulation? Maybe a TV-Chain sympathizing with the political opposition could show this diagram, with very small figures on the left, and a short moment of keeping the diagram on the screen. This perfectly discredits the achievements of the government, though in fact it may be dealing with safety problems quite well. To be able to discover such strategies and tactics and question the intentions behind them would, in fact, be a sign of what one could imagine as being mathematically literate.

UNIT: ROBBERIES (2000)

Context: *Public.*

A TV reporter showed this graph to the viewers and said:
"The graph shows that there is a huge increase in the number of robberies from 1998 to 1999."



ROBBERIES QUESTION 1 (Item code: M179Q01)

Domain: *Uncertainty.* **Item type**: *Open constructed response.*

Do you consider the reporter's statement to be a reasonable interpretation of the graph? Give an explanation to support your answer.

The latter examples refer to an assessment area (according to fig.1.2, OECD, 2003, 26), which could be labeled "interpretative dimensions" and, in the sense it is understood here, it is not represented in PISA. It refers to the enormous impact of mathematics on daily life. It is sufficient to take a look at political or economic or even sports news in a newspaper to get an idea of the ubiquity of numbers as arguments in the information conveyed. And in general it is not for illustration only that numbers, calculations, statistics, and such like are established and promoted: They serve to substantiate, corroborate, and justify an argument with an interest. At the same time mathematics lends its beneficial appearance of objectivity, correctness, and truth to the argument and thus conceals the interest driving its author. Interests become constraints of the matter. This functional role of mathematics in social life actually is a property of areas of the greatest concern to the individual, whether it regards political debate, taxes, health care or social conflicts. It is certainly becoming more and more difficult, even partly impossible, to see through the mathematics being implicit in an argument, and modern technology is multiplying and complicating mathematical interference still further, in fact to invisibility. But becoming aware of these problems is not only crucial for survival, it can start at an early age. Relatively simple tasks like the test item "Robberies" could be used to arouse awareness of and interest in this additional level of understanding. A more comprehensive understanding of the problem requires a rather small amount of mastery of mathematics. What it demands mostly is skepticism supported by mathematical insight, perspicacity, and common sense, attitudes rather than abilities, which can be trained from early on.

Or imagine the test item on sizes of pizzas (OECD, 2000, 56, discussed in Gellert & Jablonka, 2002a,116). What a wonderful classroom discussion could be orchestrated around this subject: Confront the question, which is in publicity language, relating the association "better money value" – better for me – to either the problem of sharing or the problem of obesity actually recognized as a major health problem among youths. If tests determine more and more what counts in the classroom in the near future, this kind of discussion no longer has any value, in particular as in this case the discussion could contradict the 'correct test answer'. It may be true, though, that context problems promising for a lively and critical classroom discussion are not suitable for testing, in particular if the test item – which in the above case only asks for "conceptual understanding of growth rate of area" and "space and shape" – disregards its own contexts. This turns our attention back to the limitations of tests that shape mathematics education.

The definition given for Mathematical Literacy is comprehensive (even more comprehensive if we put the PISA 2000 and 2003 definitions together), although in such general terms that it is easily acceptable for everybody. However, the test items do not attain the ambitious goal set by the PISA authors: to measure what they have defined as Mathematical Literacy. One cannot seriously claim to measure how students "understand the role that mathematics plays in the world" if those aspects of the "role of mathematics in the world", that are, by far, the most meaningful for an individual who is not actually going to be a mathematician (or even for mathematicians?) are simply left aside.

We may presume that the level of understanding sketched above was omitted because it cannot really be grasped by the format of a standardized test that makes reality run through the testers' fingers like water. This leaves us with an abstract structure of traditional 'word problems', where the richness of reality is lost, while multifaceted connotations and links to a wide range of domains are cut down to the very narrow context of the task and to the loss of authenticity. Thus the Mathematical Literacy Test seems to come close to a kind of (Mathematical) Intelligence Test based on common sense assumptions, with the difference that IQ-testing meanwhile has undergone considerable processes of refinement, whereas the testing of Mathematical Literacy is rather in its infancy.

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ICME 10 2004

SP

Sub-Plenary Lecture

SP: Information and communication technology in mathematics education

Regular Lecture based on the work of Survey Team 5:

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Introduction

The incorporation of information and communication technology (ICT) into mathematics education constitutes one of the most important themes in contemporary mathematics education. For example, one of the six principles of school mathematics as espoused by the National Council of Teachers of Mathematics (NCTM, 2000) is that "Technology is *essential* in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (emphasis added).

The purpose of this paper is to survey and review the state-of-the-art of the most important developments and contributions in the area of ICT in mathematics education in the past ten years or so, and of current trends and tendencies, with particular regard to identifying and characterizing important recent developments, new perspectives, and emergent issues. This task of summarizing the developments in the field is not an easy one, as "in the past ten years or so, there has been ample research into the impact of technology on mathematics education, resulting in a proliferation of research literature in the area. Moreover, because of the rapid advancement of technology, the relevance of the publications has a very short "half-life", resulting in an ever-moving body of literature of the field. And it is this moving body of literature that makes this task of trying to document the state of the field … so daunting" (Leung, 2003: 233).

This paper attempts to tackle four aspects of this daunting task. The first part of the paper provides a categorization of the current ICT tools available as far as incorporation into mathematics education is concerned. This will define what is meant by ICT in the paper, and give an idea of what technology is available for mathematics teaching and learning. Part two summarizes research findings on the effectiveness of the use of the various ICT tools identified in part one. This will delineate the potential of ICT for mathematics teaching and learning. Part three reviews surveys on the actual use of ICT tools in the real classroom. This will provide a picture of the extent of the actualization of the potential of the tools. The discrepancy between the potentials and the actualization of the use of ICT in mathematics education will also become clear. The last part of the paper will discuss various issues concerning the incorporation of ICT into mathematics education. These include the reasons for the discrepancy between the potentials and the actualization of the use of ICT in mathematics education, the essential features of ICT, and the impact of ICT on our understanding of the nature of mathematics and the implications for mathematics teaching and learning.

Categorization of current ICT tools available

ICT tools may be categorized in different ways. The classical categorization is suggested by Taylor (1980), who classified IT or ICT tools according to whether they are playing the role of a Tutor, a Tool, or a Tutee. Alternatively, we may categorize ICT tools according to the subject areas (e.g., Arithmetic, Algebra, Geometry, Statistics, etc.) in which the tools are applied. A further way of categorizing ICT tools is to classify them according to the stage of the teaching and learning process in which the tools are applied. So there are tools for the stages of lesson preparation, lesson presentation, classroom interaction, and evaluation of learning, respectively.

A comprehensive categorization is given by the Mathematical Association in England (Mathematical Association, 2002), where ICT tools are classified according to the following categories:

Whole class displays
Hand-held technology
"Small programs"
Programming languages
General purpose software
Mathematics teaching software

It should be pointed out that categorization of ICT tools is not a trivial task. Categorization of the tools in a certain manner presupposes a certain view of ICT, of mathematics, and of the relationship between the two. For example, classifying ICT tools according to their applications in the subject areas (of Arithmetic, Algebra, Geometry, Statistics etc.) presupposes the existing conception of mathematics, in particular the division of the discipline of mathematics into Arithmetic, Algebra etc.

In this paper, an information processing approach is taken of the application of ICT in mathematics teaching and learning. Mathematics learning is conceived as information processing, and ICT tools are classified according to the different ways they contribute to students' processing of information. In accordance with this approach, ICT tools are first classified into hardware and software, as they constitute two different kinds of tools.

Two types of hardware will be considered in this paper. They are the processor and the input/output devices. As for the software to be considered, they include general purpose software, programming software, mathematics specific software, and other software. The hardware and software to be considered in this paper are summarized in Table 1 below:

Hardware	Software
• Processors ◦ Computers ◦ Calculators ◦ Other hand-held devices • Input/Output Devices ◦ Input ▪ Data loggers ▪ Internet ▪ Digital cameras etc. ◦ Output (whole-class display) ▪ Interactive white boards ▪ OHP panels etc.	• General purpose ◦ Spreadsheet • Programming ◦ LOGO ◦ Calculator (BASIC) • Mathematics ◦ Computer Algebra Systems (CAS) ◦ Dynamic Geometry Software (DGS) ◦ Graph Plotting Software (GPS) (which run on computers or graphing calculators) ◦ Data Handling Software (DHS) • Other specific programs

Table 1: A categorization of ICT tools

Using this typology of ICT tools, we will summarize below research findings on the potential effectiveness of some ICT tools of different categories in mathematics teaching and learning.

Summary of research findings on potential effectiveness of ICT tools

In this section, each of five selected ICT tools will first be briefly described, and then research findings on their potential effectiveness will be reviewed. These five tools are, for hardware:

1. Interactive Whiteboard
2. Graphing Calculators;

and for software:

3. Spreadsheet,
4. Computer Algebra System,
5. Dynamic Geometry Software.

Interactive Whiteboard

The interactive whiteboard is both an output and an input device. On the one hand, it allows outputs from a computer to be displayed on an electronic board. On the other hand, the board responds to "electronic pens" or touch (e.g. with a finger), and the information will be sent back to the computer for processing. In contrast to the standard input device of a computer, an interactive whiteboard allows the teacher or a student to click at the point of action on the display screen rather than via a mouse or a keyboard. There is also a "flipchart" mode which allows handwriting on the board to be recognized by the computer. In the flipchart mode, screens may be created, saved and/or printed.

Claimed advantages

Some of the claimed advantages of the interactive whiteboard are (Ball, 2003):

- The one shared image in the classroom encourages discussion and other whole-class activities.
- Numbers, diagrams and graphs can be changed quickly, so the pace of lesson can be increased.
- Dynamic images are readily available and can be amended using a pen or finger.
- It is relatively easy to switch between different modes of use and different programs.

Graphing Calculators
A Graphing Calculator is basically a scientific calculator having the following additional features:

- RAM for programs, values etc.
- ROM to store application software, archive data etc.
- LCD display screen.
- Built-in graph-plotting software for a variety of algebraic and trigonometric graphs.
- Some even have symbolic manipulations facilities.

The graphing calculator can play different roles in the teaching and learning process. It can be used merely as a computational tool (when used this way, the graphing feature is not being capitalized upon, and the device is used just as an ordinary scientific calculator). But it can also be used as a transformational tool, a data collection and analysis tool, a visualizing tool and/or a checking tool. Doerr and Zangor (2000) summarized the different roles the graphing calculator plays, and the corresponding student actions, as follows:

Role of the Graphing Calculator	Description of Student Actions
Computational Tool	evaluating numerical expressions, estimating and rounding
Transformational Tool	changing the nature of the task
Data Collection and Analysis Tool	gathering data, controlling phenomena, finding patterns
Visualizing Tool	finding symbolic functions, displaying data, interpreting data, solving equations
Checking Tool	confirming conjectures, understanding multiple symbolic forms

Table 2: Patterns and modes of use (Doerr and Zangor, 2000)

Contribution of the graphing calculator to mathematics teaching and learning
In the research literature, it has been reported that the graphing calculator contributes positively to mathematics teaching and learning. For example, Embse found that "the graphing capability of the graphing calculator helps students make connections among the numerical, symbolic, and graphical representations of a mathematical relationship" (Embse, 1992: 78), and Ruthven (1990) reported that students using graphing calculators made a significantly stronger linkage between the algebraic form and the graphic form of functions than students not using it. Shoaf-Grubbs (1995) also reported that in using the graphic calculator to teach algebra, students' performance has been improved, especially on spatial visualization and mathematical understanding. And Penglase and Arnold (1996) found the graphing calculator to be a powerful tool for teaching and learning the concept of function.

Constraints and limitations of the graphing calculator
Like all tools for teaching and learning, the graphing calculator has its constraints and limitations. According to Doerr and Zangor (2000), there are at least two constraints on or limitations of the use of the graphing calculator in mathematics teaching and learning. First, some students attempted uses of the device as a 'black box' without attending to meaningful interpretations of the problem situation they are dealing with. Secondly, the personal or private use of the tool served to breakdown group communications.

Spreadsheets
Spreadsheets are software for handling data. They were originally developed for and are still most commonly used as accountancy tools. In a spreadsheet, data are organised in "cells". Some cells can be defined in terms of others such that changes to the latter will automatically cause changes to the former. So, in essence, spreadsheets are rule-using tools that require the users to identify relationships and patterns among the data they want to represent in the spreadsheet. When used in mathematics teaching and learning, the relationships are modeled mathematically using rules to describe the relationships in the model. Spreadsheets can also be used as a hypothesis tester (playing "what if" games). According to Pea (1985), spreadsheets are a typical example of a cognitive technology that amplifies and reorganizes mental functioning,

Teaching and learning algebra using spreadsheets
Spreadsheet first found their use in mathematics teaching and learning in the area of algebra. Pioneering work was done in the 1980s by researchers such as Healy and Sutherland, who found that spreadsheets helped students develop powerful mathematical ideas such as generalisation, symbolisation and explorations of functional relationships.

Studies in the 1990s (e.g. Filloy, Rojano and Rubio, 2000; Kieran, 1992; Rojano, 1996; and Sutherland and Balacheff, 1999) analyzed spreadsheets as an intermediate expression means between natural or numeric language and algebra. It was claimed that spreadsheets would enable students to cope with the transition from a numeric or verbal representation to a symbolic representation, from the specific to the general, from the known to the unknown, and from intuition to abstraction. Studies of students working with spreadsheets on arithmetic or beginning algebra problems found students' communicative power enhanced, and interesting and powerful thinking strategies evolved from students' use of spreadsheets as a problem-solving tool (Ainley, 1996; Sutherland and Rojano, 1993). Spreadsheets were also found to contribute positively to students' mathematization in the domain of beginning algebra, and spreadsheets were successfully used as a technological tool for other areas in mathematics as well as in science (Hershkowitz et al, 2002).

Heid (1995) argued that spreadsheets demand new visions of school algebra that shift the emphasis away from symbolic manipulation towards conceptual understanding, symbol sense, and mathematical modeling. No longer can the main purpose of algebra be the fine-tuning of techniques for by-hand symbolic manipulation or the acquisition of a predefined set of procedures for solving a fixed set of problems. Students will spend far less time on many of these techniques, will execute a majority of them with computing technology, and will completely forgo the study of others. The concepts of variable and function in a technological world are much richer than those found in current school text-books or in the minds of today's students. In a technological world, variables actually vary and functions describe real-world phenomena. Variables represent quantities that change, and algebra is the study of relationships among these changing quantities

In sum, the very nature of algebra may be changed because of the impact of ICT. In an ICT environment, we are not sure whether it is "symbolic arithmetic" or "algebra" that students are learning. The distinction between the two is becoming more and more blurred.

Advantages of spreadsheets

From the discussion above, it can be seen that spreadsheets are in essence computer programs for making multiple calculations that do not require the use of a programming language. Spreadsheets explicitly demonstrate values and relationships in any problem or content domain in numerical form. Identifying values and developing formulas to interrelate them enhance learners' understanding of the algorithms used to compare them, and of the mathematical models used to describe content domains. Students' willingness to monitor solution methods and their results have been increased considerably by being released from computations and algebraic manipulations, and by being able to relate to the meanings attached to the problem situations.

Computer Algebra System (CAS)

A computer algebra system (CAS) is software that works with strings of symbols. Popular examples include *Mathematica*, *Maple*, *Reduce*, *Derive*, and *LiveMath*. CAS is available on some graphing calculators as well. In addition to handling routine manipulation, the software executes rules for symbol manipulation in algebra and calculus (e.g., differentiation of x^n; factorization; solving equations). As such, it is applicable in the field of calculus, complex numbers, and matrices etc.

Contributions of CAS to mathematics teaching and learning

Artigue (2002) argued that CAS provides a means of facilitating and extending experimentation with mathematical systems, including generalization. Instrumenting graphic and symbolic reasoning through using CAS influences the range and form of the tasks and techniques experienced by students, and hence also the resources available for more explicit codification and theorization of such reasoning.

Keller and Russell (1997) found that students who used CAS technology were more successful at having correct solutions and at producing correct answers given that they had a correct solution method, and Shaw et al (1997) argued that students using CAS technology were better able to develop "their mathematical skills by freeing themselves to focus on understanding the problems and doing the mathematics". Keller and Russell (1997) also found that students were more able to concentrate on developing their conceptual understanding of calculus and their "metacognitive behaviours which support problem solving".

Uses and limitations of CAS

One of the problems of CAS is that in algebra problems, there may be many equivalent forms of output, and the best that the software can do is to "guess" which form the user expects.

For users of mathematics (e.g. engineers), CAS is an efficient mathematics tool. But for mathematics educators and learners, CAS functions as a cognitive, motivational and social tool. It leads to beneficial results when used as a "cultural reorganiser" (Pea, 1987), but if used primarily to increase efficiency and speed in implementing standard approaches to solving problems, the outcomes are less positive (Mayes, 1997).

Dynamic Geometry Software (DGS)

Dynamic geometry software (DGS), as the name suggests, is software that deals with geometry in a dynamic manner. A distinctive feature of DGS is the dynamic aspect of the geometrical diagram – sometimes referred to as the fourth dimension of the diagram. Laborde (2004) described the essence of DGS as follows. In a DGS environment, a diagram is the result of sequences of primitives expressed in geometrical terms chosen by the user. When an element of such a diagram is dragged with a mouse, the diagram is modified while all the geometric relations used in its construction are preserved. So the diagram is "quasi-independent" of the user: when the diagram is dragged, it is being modified according to the geometry of its construction rather than the wishes of the user.

A typical example of DGS is the *Geometer's Sketchpad*. According to Healy and Hoyles (2001), the Geometer's Sketchpad allows students to use a mouse to interact directly with the tools provided by the system so that they can build, manipulate and explore figures. Through using the Geometer's Sketchpad, students are able to make conjectures that can be tested. It offers fast and non-judgemental feedback to students, and opens up their minds to accept lots of possibilities. Many of the problem solving strategies students use and the experimental stance they adopt to challenging problems are unimaginable before the advent of DGS.

The dual nature of geometry

Osta (1998) pointed out the dual nature of geometry: as the study of space and spatial relationships, and as an axiomatic system and a context for deductive reasoning. Osta argued that DGS can provide a valuable means for visualizing geometrical situations. The animation capabilities of DGS provide ways for constructing, moving and rotating configurations, for observing them under various angles, and for modifying some of their features.

Laborde (1998) maintained that DGS provides visual evidence in solving geometry problems. She argued that visual evidence plays an important role in the problem solving process. Visual evidence is interpreted in geometrical terms and generates questions which are solved by means of geometry. Geometrical analysis triggers new questions which may be empirically explored, giving rise to experiments with the software. Questions generated from the visual evidence are probably due to the solvers' mathematical backgrounds which enable them to interpret in geometrical terms what they observe.

Visual phenomena and the construction of geometrical knowledge in a Cabri environment

Another popular example of DGS is *Cabri*. Cabri provides a 'real' model of the theoretical field of Euclidean geometry in which it is possible to handle, in a physical sense, the theoretical objects which appear as diagrams on the screen. The behaviour of Cabri is based on geometrical knowledge in two ways:

(1) diagrams can be drawn, based on geometrical primitives which take into account relevant geometrical objects and relationships, and
(2) it offers feedback which can distinguish diagrams drawn in an empirical way from diagrams resulting from the use of geometrical primitives.

DGS and geometric proof

DGS has given rise to questions about the place of proof (Hershkowitz et al, 2002). Since conviction can be obtained quickly and relatively easily with DGS, and the dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence convince themselves of its truth, does DGS then conceal the need for proof?

Hanna (1998) differentiated two kinds of proofs: proofs that show only that the theorem is true, by providing evidential reasons (these are sometimes referred to as an informal approach to proofs), and proofs that explain why the theorem is true, by providing a set of reasons that derive from the phenomenon itself (referred to as deductive proofs). It can be argued that DGS only enhances the first kind of proof, and does not provide the second kind of proof. However, DGS, in affording students greater access to exploration, heuristics and visualization, actually increased their understanding of the limitations of informal approaches and thus of the need for deductive proof.

Mariotti (2000) conducted a long-term teaching experiment for pupils in the 9th and 10th grades aimed at analyzing the process of semiotic mediation related to the emergence of the meaning of proof, and also the specific role played by DGS. She found that when a construction problem is presented in a dynamic geometry environment, the justification of the correctness of a solution figure requires description of the procedures used. The intrinsic logic of a dynamic geometry figure, expressed by its reaction to the dragging test, induces pupils to shift the focus onto the procedure, and in doing so it opens up to a theoretical perspective.

So experience of geometrical constructions in the dynamic geometry environment could effectively facilitate 'semiotic mediation', which was mainly achieved by using the dragging function as a tool to check the correctness of the constructions. This 'semiotic mediation' helps students make sense of the process of exploring, conjecturing, and arguing as a way of arriving at a valid proof, which has a significant contribution to their understanding of 'theoretical' geometry. Thus, it seems that DGS has the potential of breaking down the traditional separation between action (as manipulation associated to observation and description) and deduction (as an intellectual activity detached from specific objects).

Strässer (2001) summarized the roles of DGS in the following terms: teaching and learning geometry through DGS is a human activity integrating the use of modern instrument; DGS strengthens students' problem solving capacity; and it deeply changes geometry and its teaching and learning.

Survey of the actual use of ICT tools

In this section, we will review the actual use of ICT tools in mathematics teaching and learning from two perspectives: the official policies of ICT use in different countries, and surveys of the actual use of ICT tools in the classroom. For the former, we will recount the official positions in various countries as reported in the TIMSS 1999 study, and will then look at the official curriculum documents in two places: Hong Kong and Sweden. For the latter, we will report on the results of the teacher and student questionnaires of the TIMSS 1999 Study (Mullis et al, 2000).

National Policies on Calculator Use

Official documents of 23 of the TIMSS 1999 countries included an explicit policy on the use of calculators. Wide variation was reported across countries in their official policies on calculator use, ranging from encouraging unrestricted use, use with restrictions, to banning calculator use entirely. Out of the 23 countries, seven allowed unrestricted use of calculators, 14 permitted some restricted use, and two countries had policies which varied across different regions in the country. In several countries, calculators were not permitted in lower grades of primary school. In others, use of calculators in these grades was limited so that students could master basic computational skills, both mentally and using pencil and paper (Mullis et al, 2000: 213).

Policies of ICT use as stipulated in official curriculum documents

Example 1: Hong Kong

In Hong Kong, use of IT for interactive learning is one of the four key means to develop students' independent learning capabilities so that they would acquire the necessary generic skills for life-long learning (Curriculum Development Council, 2001). In mathematics, students should use IT tools such as graphing software for various exploratory activities so that they could learn at their own pace and develop the habits of self-learning (Curriculum Development Council, 2002).

Example 2: Sweden

The official Swedish documents stipulate that the teaching of mathematics should strive for the goal that students with familiarity and sound judgment can see and use the potentials of calculators and computers (Utbildningsdepartementet, 2000). Students should develop their knowledge about how mathematics is used within information technology, as well as how technology can be used in problem solving, in visualizing, and when investigating mathematical models (Skolverket, 2000).

The reality in the Swedish classroom, however, may fall short of what is expected in the official documents. Access to technology is not the problem. It is not the accessibility of technology that limits its use in the teaching and learning of mathematics, instead it is the views and opinions with the user (Lingefjärd et al, 2004). Samuelsson concluded that "the computer supported teaching of mathematics (in Sweden) is still infantile. The possibilities and conditions for teaching that the technology brings does not always meet appreciation from teachers, who probably are far too busy with all issues involved in their everyday teaching" (Samuelsson, 2003: 223). This description of the situation in Sweden may well apply to Hong Kong as well.

Actual use in the classroom

In TIMSS 1999, teachers and students were surveyed with regard to the use of handheld calculators, computers and others ICT tools such as the internet. Some of the results are presented below.

How are calculators used in the classroom?

Results of TIMSS 1999 show that in 14 countries, teachers reported that nearly all students (more than 90 percent) had access to calculators in class. For students in classes with access to calculators, most teachers reported some type of restricted use (for about two-thirds of the students on average, internationally). The different ways calculators were used in mathematics classrooms of TIMSS countries and their reported frequencies of use are displayed in Figure 1 (reported by teachers) and Figure 2 (reported by students) below.

Exhibit R3.14 Ways in Which Calculators Are Used*

	Percentage of Students					
	Never or Hardly Ever Use Calculators	Ways in Which Students Use Calculators At Least Once or Twice a Week				
		Checking Answers	Tests and Exams	Routine Computations	Solving Complex Problems	Exploring Number Concepts
International Avg.	28 (0.5)	44 (0.6)	20 (0.4)	43 (0.6)	43 (0.6)	26 (0.5)

Figure 1: Teachers' Report on Ways of Calculator Use (Mullis *et al*, 2000: 300)

Exhibit R3.12 Students' Reports on Frequency of Calculator Use in Mathematics Class*

	Almost Always		Pretty Often		Once in a While		Never	
	Percent of Students	Average Achievement	Percent of Students	Average Achievement	Percent of Students	Average Achievement	Percent of Students	Average Achievement
International Avg.	19 (0.2)	469 (2.0)	20 (0.2)	482 (1.1)	29 (0.2)	488 (1.2)	32 (0.3)	476 (2.2)

Figure 2: Students' reports on frequency of calculator use[1] (Mullis et al. 2000: 298)

It can be seen from Figures 1 and 2 that the calculator is not heavily used in the mathematics classroom internationally. Teachers of 28% of the students worldwide reported never or hardly ever using the calculator in the mathematics classroom (Figure 1). This is consistent with the figure as reported by the students (32%, Figure 2). Also, only less than 20% of the students reported that the calculator was "almost always" used in their mathematics lessons, and another 20% reported using the calculator "pretty often".

For those countries which participated in both the 1995 and the 1999 TIMSS studies (there are 24 of them), the "trends" for the frequency of calculator use were also computed, and the results are reproduced in Figure 3 below. It can be seen from Figure 3 that there is even a slight but statistically significant decrease in the percentages of students who "almost always" use calculators in the mathematics classroom (and there is also a slight but significant increase of the percentages of students in the "once in while" category). This shows that the use of calculators in the mathematics classroom is actually on the decrease for these TIMSS countries between 1995 and 1999.

1 In Exhibit R3.12 of Figure 2, the percentages of students who almost always, pretty often, once in a while or never use the calculator in mathematics class are shown, together with the corresponding average mathematics achievement of each category of students.

Exhibit R3.13 Trends in Students' Reports on Frequency of Calculator Use in Mathematics Class*

TIMSS 1999 8th Grade Mathematics

	Almost Always		Pretty Often		Once in a While		Never	
	Percent of Students 1999	1995-1999 Difference	Percent of Students 1999	1995-1999 Difference	Percent of Students 1999	1995-1999 Difference	Percent of Students 1999	1995-1999 Difference
International Avg.[§]	23 (0.3)	-3 (0.5) ▼	25 (0.3)	-1 (0.4) ●	27 (0.3)	4 (0.5) ▲	25 (0.4)	0 (0.5) ●

▲ 1999 significantly higher than 1995

● No significant difference between 1995 and 1999

▼ 1999 significantly lower than 1995

Significance tests adjusted for multiple comparisons

Figure 3: Trends in calculator use (Mullis et al. 2000: 299)

Calculator use and students' achievement in mathematics

TIMSS 1999 developed an index of emphasis on calculators in mathematics classes (ECMC), and computed the relationship between ECMC and student achievement. It was found that within a country, positive association was found between calculator use and mathematics achievement. Across countries, there was enormous variation in the emphasis on calculator use as measured by ECMC. For example, the Netherlands, Singapore, and Australia had more than 4/5 of their students (from 84 percent to 95 percent) in the high ECMC category, but Chinese Taipei, Iran, Korea, Japan, Malaysia, Romania, Thailand, and Turkey had half or more of their students in the low ECMC category.

How are computers and the internet used in the classroom?

Results of TIMSS 1999 show that across the TIMSS countries, the vast majority of students (80% on average internationally) reported never using computers in their mathematics classes. When compared with the data in 1995, the trend data show a small but statistically significant shift from the "never" to "the once in a while" category, showing that although the use of computer in the mathematics classroom is very scarce in countries all over the world, there is sign of a slight increase in the use between 1995 and 1999.

On access to the internet, TIMSS 1999 found a great variation across countries. About a quarter of the students across all the countries had access to the internet at school. But the international average for using the internet to access information for mathematics teaching and learning in the classroom on even a monthly basis was only 10 percent (less than half of those reporting having access to the internet).

In summary, we can see that although there are ample research findings testifying to the great potentials of the effectiveness of using ICT tools in teaching and learning mathematics, the actual use of ICT tools in mathematics classrooms across the world is still very limited. Some of the reasons for this discrepancy between the potentials and the realization of application of ICT tools in mathematics teaching and learning will be discussed in the next section.

Issues concerning the use of ICT in mathematics education

Discrepancy between the potentials and actual use of ICT

From the discussion in the previous section, it can be seen that in general, *access* to ICT tools is not a problem for most countries. The problem is in their *actual use* in mathematics teaching and learning. As Hoyles and Noss (2003, 324) observed, "changes in the computational domain open up only the potential for change, not actual change in the didactical field". On the particular tool of the microcomputer, Bottino and Chiappini (2002) lamented that the high expectations regarding its potential to drive change and innovation in schools appear to have remained largely unfulfilled. And in reviewing the practice across countries, Pelgrum and Plomp (1993) concluded that "computer use has had a limited impact on schooling throughout the world".

Why is there such a great discrepancy between the potentials and the actual use of ICT tools in mathematics teaching and learning? Can this be explained simply by the customary conservatism of teachers? To tackle this question, one needs to ask the underlying question of what, in essence, ICT tools are. Are they merely computational and presentational devices?

Essential features of ICT

From the literature summarized in part two of the paper above, it can be seen that there are three essential features of ICT tools:

1. *Efficiency in mathematics manipulation and communication.*
2. *Multiple representations of mathematics, especially the efficient coupling of visual representation with other forms of representation.*
3. *Interactivity between the learner and mathematics.*

So the advent of ICT does not merely represent an increase in the repertoire of tools available for mathematic teaching and learning. It actually brings in a fundamental cultural change, and some even compare it with the invention of the written language in human history. As Kaput (1999) put it, "the computer heralded a new kind of culture – a virtual culture – which differs crucially from preceding cultural forms. Not only is there a new representational infrastructure but also the externalisation (from the human mind) of general algorithmic processing". An ICT tool is not just an artifact, it is an "instrument" in the sense of an artifact plus a cognitive scheme. As Artigue (2002) pointed out, an ICT tool becomes an instrument through instrumental genesis.

Recognizing this essential feature of ICT tools, the implications for the teaching and learning of mathematics is momentous. In the introduction of ICT tools to the mathematics classroom, what is involved is not simply the addition of one more computational or presentational tool, but the actualization of a paradigm shift in teaching and learning mathematics. We will return to this point later in this paper.

To fully capitalize on the use ICT tools to achieve the paradigm shift, teachers need to know more about *ICT and the potentials* that it offers for the teaching and learning of mathematics. They also need to know *how students learn*, and most research reviewed here is based on learning theories related to constructivism. But maybe more importantly, teachers need to know more about ***what mathematics is*** before they can decide on whether and how ICT is able to assist in students' learning of mathematics.

What is mathematics?

Sherin (2001) suggested that programming (in *Boxer*) could shift the ontological foundations of school physics and mathematics, and that "the nature of the understanding associated with programming-physics might be fundamentally different than the understanding associated with algebra-physics". The argument should hold true for mathematics as well. Guin and Trouche (1999, 198) pointed out that "there is an unavoidable gap between 'real' mathematics and the image reflected by calculators". The question then is: are there two "mathematics", one as understood through "traditional" methods (we may call it traditional-mathematics), and one arrived at through the use of ICT (ICT-mathematics)? If there are two mathematics, which is the 'real' one (Guin and Trouche seemed to suggest that traditional-mathematics is the "real" mathematics)?

For example, Weigand and Weller (2001) reported that their "investigations did not show a better understanding (of the function concept) for students working with a computer, but they got *different* understanding compared to students working with pencil and paper". Are there then two different concepts of "function", one as understood through working with pencil and paper, and the other as understood through the use of ICT? Or are the two merely different representations of the same concept of function? If there are actually two different concepts of "function", which is the "real" one?

Take the learning of algebra through ICT as another example. Is "CAS algebra" (i.e., the algebra as learned through CAS) algebra in the traditional sense? Is "spreadsheet algebra" (algebra as learned through spreadsheet) the same as the algebra we learned in school before the advent of ICT tools or is it something else? As mentioned in part two above, spreadsheet algebra may be considered as "symbolic arithmetic" rather than traditional algebra. If that is the case, how do we draw the line between the two?

Similarly, is dynamic geometry the same kind of geometry as Euclidean geometry? Goldenberg (1995; 2001) opined that dynamic geometry is not merely a new interface to Euclidean construction. It is a new style of reasoning and it generates new heuristics. Dynamic geometry mediates the nature of explanation, verification and proof[2].

These questions touch on the philosophical issue of the nature of mathematics. If a Platonic view of mathematics is taken where mathematics is considered as having real existence, and mathematical knowledge is regarded as absolute, infallible truth (Ernest, 1991), then there is *ipso facto* only one mathematics. Different ways of approaching the same mathematics (e.g., through using different learning tools) may give rise to different representations of mathematics. But the tools won't change the mathematics under study, and the different approaches will only add to the richness of our understanding of the mathematical truths. Taking this approach, ICT provides an alternative way for us to understand mathematics and enriches our understanding of the traditional, absolute mathematics.

But there is an alternative to this understanding of the nature of mathematics based on a fallibilist view of mathematics. According to this view, mathematical knowledge does not possess absolute truth, but is merely a fallible social construction perpetually open to revision. As a result, mathematical knowledge and concepts develop and change with time. One particular school of thought under this fallible view is *Conventionalism*,

2 Even before DGS exerts its impact on our understanding of the nature of explanation and proof, the impact is already felt from another area of ICT. The famous proof of the Four Colour Theorem by Appel and Haken in 1976 (Appel and Haken, 1978), which utilized 1200 hours of computer time to complete the proof, challenges our very concept of the nature of mathematical proof.

where mathematical knowledge and truth are considered as being based on linguistic conventions. According to the conventionalist Wittgenstein (1978), "mathematics is a collection of 'language games', and the notions of (mathematics) … depend upon our accepting the conventional linguistic rules of these games". Human language, rules and agreement play a key role in establishing and justifying the truth of mathematics.

Understood this way, mathematics is defined by the language and the tools used to study it. As Artigue (2001) remarked, "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computation". With a change of tools as drastic as the introduction ICT tools, the nature of the mathematics that we study will inevitably be changed.

So the "mathematics" studied through ICT is no longer the mathematics in the pre-ICT era, just as the mathematics after the invention of written language is fundamentally different from pre-written-language mathematics. The very use of ICT in teaching and learning mathematics changes the nature of the subject matter that we teach and learn, and challenges us as to what mathematics is. This has fundamental implications for what mathematics students should learn, how mathematics should be learned, and how mathematics should be taught. What it amounts to is a paradigm shift in the teaching and learning of mathematics.

Concluding remarks

Given the understanding of ICT and the nature of mathematics discussed in the last section, we teachers and learners of mathematics need to constantly adapt ourselves to the potentials of what the available tools unfold for us – not for the sake of using the tools per se, but for the capitalization of the tools to the fullest extent in order to solve the ever changing problems that we encounter in the modern world, and in order to fulfill the aims of education in general and those of mathematics education in particular.

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ICME 10 2004

SP: Research in mathematics education

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SP
Sub-Plenary Lecture

I imagine that this morning some of you are here with a number of questions – for instance:

What is this "*Didactique*"? What is the difference between it and pedagogy?
How does it relate to psychology?
Do we really need this term to talk about our work?
What is the Theory of Didactical Situations?
Do we have to have yet another theory in order to approach questions of mathematics education scientifically?
What is the value of a scientific approach to teaching?

I cannot give a proper academic response to all of those questions in the time available, and I will not try.

Many things have led to my being here in front of you today. Some were opportunities offered by circumstances of timing. Others resulted from the support of many people who were convinced by my arguments to try out some ideas, and then convinced of the value of these ideas by the results we obtained. I am grateful to all of them.

Without these favorable events, none of our reflections could have led to experimental results. But the central question was how to interpret such a multitude of facts, observed and validated but apparently having such a variety of possible causes, in terms of scientific facts?

You said *Didactique?*

In the classical sense, pedagogy is the art of *educating* children; didactics is the art of *teaching* a science, an art, a language, to somebody.

Thus pedagogy takes into account an educational and moral *intention* not shared by didactics: teaching is only educational in terms of the virtues specific to the thing being taught. For Comenius (17th century) "didactics" did not depend on the nature of what was being taught.

Common usage tends to mix the two terms. Further, since art and science are habitually taken as opposites, pedagogy and didactics seemed condemned to elude scientific procedures. But since the end of the 19th century, considerable effort has been made in this domain, to complement the *art* with *scientific knowledge* and techniques imported from numerous domains.

To integrate these contributions and monitor their compatibility and adequacy, a science appropriate to the field was needed.

We have shown that the nature and practice of the knowledge being taught plays a far larger role in the organization of teaching than Comenius thought. So *Didactique* of mathematics is not *a priori* a specification of general didactics.

We have chosen the term *Didactique*[1] to designate this field, with the following rather broad definition:

Didactique as a science studies the diffusion of the knowledge that is useful to people living in society. It deals with the production, diffusion and learning of knowledge as well as with the institutions and activities which facilitate it."

Thus, *Didactique* as a social or professional activity consists of everything that is directed toward the teaching of a piece of knowledge, of a science, of an art or of a language.

This is the activity that is the object of study of *Didactique,* the science. When we speak of *Didactique*, we are speaking of the relations among a learner, something that needs to be learned (by someone's decision) and an environment that produces learning.

We have shown that cultural knowledge cannot be learned without the presence in the environment of a teaching system. (In *Didactique* the thing being learned is generally a mathematical concept.)

I restricted myself to the use of the classical methods of experimental science, principally *observation, modeling* and *statistical methods*.

But in this new field of *Didactique* we have sometimes had to adapt these methods or to break away from common practice. I will speak a little later of how we did this as regards observation. At times we have even had to devise new instruments such as *the statistical analysis of implicative dependence* or such as *the Theory of Situations.*

Many people today are inquiring into the relationships among theories, research methods, experiments, results and the practices of teachers. Perhaps my account might be of assistance to them. For example, our observations consisted of watching ordinary classes. But beyond that, the observation school made it possible to modify the teaching conditions and observe the result. We learned more about mathematic education from what we had to do in order to observe classes than we did from the observation itself.

Another example: in our experimentation, we did not compare the results of the students to determine whether one method was better than another. Instead we restricted ourselves to having the results be equally good, *despite the modifications* we made, and compared the *efforts* required by the students and teachers in each case. I am therefore going to describe how the observation and modeling of knowledge and processes led us to clear the way for this new scientific field.

Like most of you, I learned the teaching of mathematics as a profession, and I have practiced that profession throughout my career. The rest came from my taste for three things: mathematics, the pleasure of school children in doing and discovering mathematics, and the observation of teachers who love prompting these activities in their students.

1 At the time when this lecture was given we were consistently using "*Didactique*" untranslated to refer to the field. Since that time we have begun to favor the use of "Didactics" (as parallel to Economics or Linguistics). As of 2007, both versions are very much current.

The observation instrument: The COREM (Center for Observation and Research in Mathematical Education)
I devised the COREM back in 1964, and it opened at the École Jules Michelet in Talence in 1973 after a number of attempts. I led it until 1988 with the help of at least a hundred people, and I continued to work there until the end of its activities in 1999.

The functioning of this institution is of interest for more than one reason. In the first place, it presents a model of the respective positions and relations among teachers, researchers and an academic support system – a model which is both very concrete and very different from the standard models.

The relationship between theory and practice was an essential part of the functioning of the center. But above all, it provides a perfect illustration of the foundations of the Theory of Situations. It does so on two levels, that of the ideas being studied and that of the sources of those ideas.

- In addition to being an "ordinary" school, this school had a special building equipped for recording observations. Its staff constituted an excellent team, since they were selected purely on the basis of their ability to cooperate and their willingness to be observed. All of them were there on a voluntary basis and had contracts which had to be renewed every 3 years.
- In addition, some members of the mathematics faculty of the *Ecole Normale* (College of Education) of the University of Bordeaux were regularly available for the observation school. They helped the teachers to carry out the regular program and to coordinate and moderate the research propositions.
- The researchers belonged to what we call in France a University Laboratory. Teachers were considered *for part of their service* to be "technicians" under the responsibility of the University
- The technical activities (seminars and training, recording and maintaining videotape supplies, didactical materials, the computer system, statistics, etc.) were maintained by members of the group. All members had training in this sort of observation and in the specific methods and interactions employed within the COREM

The school absolutely did not function as a place of pedagogical innovation or research, nor as a demonstration school, nor as a training ground for teachers. It was an ordinary school that followed the official programs. The teachers employed did not adhere to any particular school of pedagogy. Only a very small portion of the lessons were observed, but the progress of every student was carefully monitored. Some of the observations were done in an experimental set-up agreed upon by the researchers and the teachers. The rest were prepared by the teachers only.

There were many rules and they were very rigorous but they were established in collaboration and constantly re-explained. They were defined in terms of particular functions, not of personnel. This was necessary, since, for instance, a teacher might also act as a researcher at some level other than the one at which he or she was teaching. The purpose of the rules was to permit the teaching and the research to function normally, and independently and insofar as possible to prevent illegitimate contamination. The most important principle was that the powers should at all times stay balanced among the different categories of participants.

The lessons or sets of lessons being observed could in exceptional cases be rather long, but in general they were fairly short and did not happen terribly frequently.

For the first ten years or so our observations had to do with watching students solve problems, or with lessons where the teacher's interventions were minimal and could be defined in advance. The objective was a direct observation of the mathematical activity of the students. Constructing the necessary set-ups led us to establish the foundations of *didactical engineering* (I will give an example in the next section) and of a theory of mathematical situations (about which I will talk later.)

However, we also used and studied modifications in the relations between teachers. For example, to try to find out what concepts teachers really use in making their decisions we set up the following arrangement: two teachers prepared together a pair of successive lessons to be taught to the same class. One of them carried out the first lesson without the other being present. We then looked at the questions:
– What information did the second teacher request of the first?
– What answers could the first one give?
– Was that information enough to manage the second lesson? etc.

We were able to use this to bring to light many phenomena, including certain roles of what we may call the "didactical memory"

Modeling: Enlarging a puzzle

Let us take up a well known example of the type of situation we constructed for eleven year old students. The issue is to model mathematical situations where proportionality comes into play.

Almost all of the functions introduced in elementary school are proportionalities, and this property is either presented as obvious or taught without justification. What should we do so that the students choose proportionality from amongst several possibilities, and that they do so for mathematical reasons, and not only empirical ones?

The teacher holds up a square puzzle (Tangram) 11 centimeter on a side (figure 1) which can be used to make different shapes (figure 2),

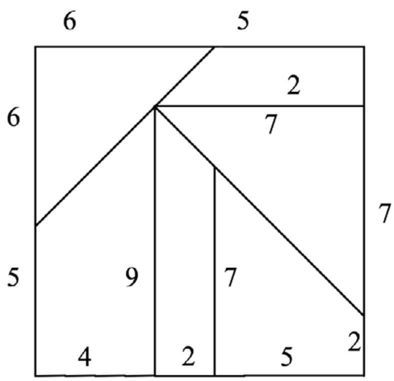


Figure 1

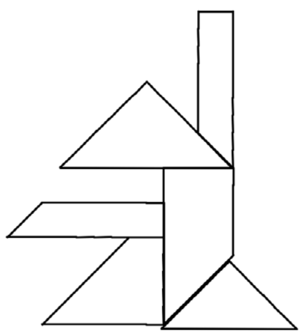


Figure 2

He tells the class: "You have to cut out a puzzle for the kindergarten class that looks like this one but you need to make it bigger. The side of this piece that is 4 centimeters long needs to be 7 centimeters long on the new puzzle. But they have to be able to make the same shapes with the big pieces that we have been making with the little ones.

To make this big puzzle you are going to get into groups. Each group will make one piece and then we will see how well they fit."

Almost all of the groups start off thinking they need to add 3 centimeters to every measurement. Disaster! The pieces don't fit together! The first hypothesis is always that somebody messed up the cutting (figure 3).

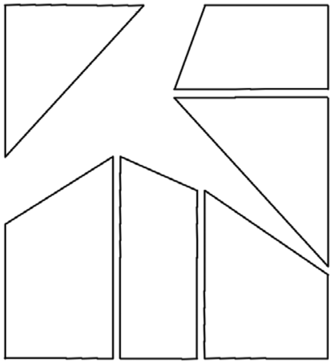


Figure 3

After eliminating various hypotheses, the students eventually assume proportionality. "The side of length 2 is going to have to be half of the side of length 4." Unfortunately, even if this observation is accepted by the other students, it still doesn't give a method of calculation that turns the side of length 4 into one of length 7.

Then some of the students suggest doubling the side length and subtracting one. The method is "almost acceptable".

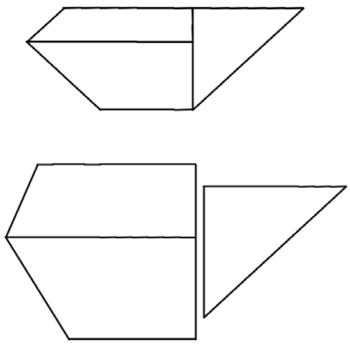
$4 \rightarrow 2 \times 4 - 1 = 7$

$6 \rightarrow 2 \times 6 - 1 = 11$

$2 \rightarrow 2 \times 2 - 1 = 3$

Others have managed by this time to make a solution by trial and error that is "acceptable" because they can play with it as with the original.

Finally the students have to notice that the *image of the sum of two segments must be the sum of the images of those segments* (figure 4), which is not true in the other "solutions".



$2 \rightarrow 2 + 3 = 5$

$4 \rightarrow 4 + 3 = 7$

$6 \rightarrow 6 + 3 = 9$

Then

$2 + 4 = 6$

But

$5 + 7 > 9$

Figure 4

So they verify that the ratios are indeed conserved. Thanks to which they are able to use the definition of measure-fractions either directly: 7 is 7 fourths of 4, or by calculating the image of 1.

I chose this example because it makes it possible to understand our method of work. To characterize a Situation, we establish a set of desirable conditions such as:

- The mathematical knowledge aimed at should be the only good method of solving the problem.
- The assignment should not make reference to any of the knowledge that one wishes to have appear. It determines the decisions permitted, the initial state, and the gain or loss represented by the final states.
- Students can start to work with inadequate "basic knowledge".
- They can tell for themselves whether their attempt succeeded or failed.
- Without determining the solution, these verifications are suggestive (they favor some hypotheses, bring in some appropriate information, neither too open nor too closed.)
- Students can make a rapid series of "trial and error" attempts, but anticipation should be favored.
- Amongst the empirically acceptable solutions only one takes care of all objections.
- The solution can be found and tested by some of the students in a reasonable amount of time in an ordinary class, and swiftly shared and verified by the others.
- The situation can be re-used, and will then provide some questions that relaunch the whole process (for example, is this the way to do all enlargements?)

These are the conditions that ensure the maximum amount of autonomy of the student.

Our goal is for each piece of knowledge (here it was proportionality) to find Situations satisfying as many of these conditions as possible. The Situation of the Puzzle is one that satisfies this list of conditions. Moreover it gives a good illustration of the catalogue of technical means used.

There exist *a priori* different types of knowledge: knowledge required to produce decisions, messages and proofs, and knowledge as repertories of schemes, languages and theories.

We associated to these types of knowledge typical organizations of mathematical Situations: Situations of action; Situations of formulation; and Situations of validation. We observed that different modes of learning are associated with each of them. Competencies (schemes, languages and theories) are acquired in Situations where performance (decisions, messages and proofs) must adapt itself to difficulties. The types of Situations are illustrated in the successive phases in the progression of the lesson on the puzzle.

We therefore verified the possibility of realizing these Situations with students and observed the results (length of work or probability of success) as well as the effects of a number of variables.

Most of these Situations are not models to be reproduced in the classroom. In fact, it is not necessary for *every* lesson to satisfy all of the conditions I just listed. It would be a considerable waste of time and energy to attempt that.

Questions, exercises and classical problems can be described, in a natural manner, as special cases and combinations of these types of Situations. From this perspective, simplifications, restrictions, conventions and economies introduced by practice become apparent. Modeling the situations consequently makes it possible to analyze the effects of these systematic "economies".

The lesson outlined above also demonstrates the degree to which it is difficult to describe the acquisitions obtained and the progress made in the course of this lesson in terms of the habitual system. To do so requires the development of a different epistemological repertoire, adapted to real *Didactique*. In *Didactique*, the value of a lesson is in what it permits future lessons to achieve, the things that would not work if this lesson had not taken place. This value shows up in terms of possibilities for the students (opportunities for learning) and possibilities given to the teacher. Thinking just in terms of what they know and can immediately do leads to ignoring many events and conditions that are indispensable for learning but hidden.

Here the rejection of the additive model and then that of doubling and subtracting 1, and moving on to use the proportional model is more than a simple action. It is a piece of knowledge which is local but solves a problem. It cannot easily be formulated exactly or evaluated: when we presented a similar problem a few days later the students needed to go through the same type of attempts and reproduce the same type of reasoning. But then they recognized the model and after that they were able to learn to use it, state it and put it to the test. The lesson of the puzzle is just one step (the 37^{th} lesson out of 65) in the study of rational and decimal numbers.

Between 1964 and 1990, we constructed situations of this type for most of the important knowledge that must be taught to school children.

A *Theory of Mathematical Situations* – Why?

I made the assumption that

- to every piece of mathematical knowledge there corresponds a collection of Situations which can be resolved using this knowledge,

and reciprocally that

- in any real environment of a student it is possible to choose elements of one or more Situations that make it possible to identify the knowledge being brought into action by the student's actions.

The Situation determines what is worthwhile for the student to do, either because he already knows it or because he discovers it in the course of adapting to the Situation. This is the type of reflection that governs anyone designing problems, exercises or textbooks.

Modeling the Situation makes it possible to study the consistency of the choices made and their consequences by looking at the repertory of knowledge that the students put into action.

Modeling also lets us reconsider the *Didactics* and epistemology that we use here and there. The goal of the Theory of Situations is to verify the consistency of the different models themselves. For example, this theory makes it possible to inquire whether knowledge in action, formulation of knowledge, and its logical validation can be used and developed in the same type of Situations by the same kind of process.

Modeling also makes it possible to study a phenomenon that we call "didactical transposition": Teachers have to adapt the meaning of a piece of mathematics to their classrooms ("number" presented to a first grader cannot include irrationals!). Then students taking in the knowledge automatically adapt it in the course of building their understanding.

It follows that subsequent lessons require returning to the previous learning and modifying it, not simply adding something on top of it.

We predicted and then observed how certain necessary pieces of learning can turn into *obstacles* to later learning.

Following up on the number example, children have to learn first about natural numbers. Every natural number has a successor, but a decimal number doesn't. So any implicit reasoning that depends on enumeration, like addition, has to be rethought. Students use the fact that multiplication gives an answer larger than either number being multiplied to help tell multiplication from division. With decimals that is not true any longer. So the intuitive understanding of 0.3 x 0.2 is not easy, and it takes a while to verify it. The similarity between decimals and natural numbers which makes it easy to learn estimation and some of the arithmetic algorithms, also encourages some misunderstanding and errors.

The Theory of Situations offers a good means for a coherent approach and for experimental verification based on observation and centered on the teacher's means and methods of working. Actually, the only true and legitimate means a teacher has for influencing her students are situations – those which she either invents or reproduces. The direct and authority-based implantation of ready-made ideas cannot always be avoided, but it always presents dangers, and in the end is less effective.

Returning to the organization of the COREM, we can now see that that system was a S*ituation* organized on the same principles in order to allow the production of relevant knowledge about *Didactique* while respecting the obligations of teaching itself. I conceived and maintained the relations among the teachers, the students, the researchers, the observers, the technical support staff, the administration, the civil authorities, the teachers' unions, the parents of the students, and the media with the same care and with the same types of methods as for the Situations in the classroom. The system was intended to help us produce the knowledge about *Didactique* needed for the management of the teaching. But we had no intention whatsoever of teaching anything at all to this system as a system.

The *Theory of Didactical Situation in Mathematics*

At one time I thought that the study of mathematical Situations to be used didactically, like the Puzzle Situation we just looked at, would be enough to account for all of the activities of the teacher. But then observation revealed that the teacher had to step in to maintain a certain equilibrium between what is known, what can be expressed, what has been shown and what is agreed on as being known.

For example, time after time the teachers did not want to keep on going even though they had been through an elegant sequence of situations that theoretically took care of everything: action, then formulation and then validation. We wondered what was going on.

Then various paradoxes became apparent. A student can develop a piece of knowledge similar to what has already been established in society but without knowing its place, its importance, its future, etc. So, eventually we had to admit that another kind of Situation was indispensable: the *Institutionalization* of the knowledge acquired.

More concretely, the teacher has to recognize and interpret some of the students' actions, forget others, and organize it all into a coherent history. From such history the students will know what they ought to learn or know and what they have to do for that to happen. The teacher has to maintain the necessary equilibrium by specific interventions of a type that cannot be represented in the theory I just presented – that of mathematical Situations.

We began to suspect the limitations of our initial point of view in 1975. After that, the observation of some struggling students brought to light the nature of the difficulties that the teacher has in trying to get a student to take "ownership" of a problem the teacher has proposed. We call this transfer of ownership *devolution*. Asking a question, transmitting an assignment or a problem statement to the student, or getting the student to enter into a given situation, etc., posed quite different types of problems. From the learner's perspective, a student is supposed to produce what she does or says personally as if she were the author, and not by citing anyone or reciting anything, using knowledge that she does not yet have and that nobody has taught her. No professional would accept a contract like that!

The teacher and the student thus enter into what, at the time, we saw as the negotiation of a type of "*didactical contract*". Neither party was able to make the contract explicit, or even to maintain it. It was always being broken and renewed, and it was through that contract that the student's knowledge was created.

In the end we showed that the teacher needs, on the one hand, to organize an activity adapted to the students and, on the other, to "re-read" it, re-shaping it to make it as close as possible to current mathematics.

Moreover, the memory of the system cannot be reduced to the student's memory. It is shared with the teacher's memory and with the one representing the state of knowledge.

The process of teaching consists of an alternation of devolutions of autonomous situations and institutionalizations. We established the necessity of this alternation by theoretical study and, in reality, by observation and experimentation. It should be noted that at all levels of this research we have made use of a wide variety of mathematical theories and techniques, which there is no point listing here. It is noteworthy that in this particular domain, mathematician-didacticians avoid using mathematics to calculate the didactical properties of the situations they are studying. Perhaps they are too fascinated by the research within their domain, or perhaps it is because they fear they will be misunderstood.

Conclusions

What is the place of these works in the world of Mathematics Education research? It is clear that it does not replace any one of the approaches that we assemble under this title. Every source, every subject has the possibility of producing interesting results:

- pragmatic or technical studies
- construction of didactical materials
- research on the teaching or learning of mathematics using any discipline or methodology like those listed below:
  psychology, linguistics, sociology, pedagogy, epistemology, history of mathematics, economics, medicine, psychoanalysis, anthropology, logic, artificial intelligence, semiology, neurophysiology, and of course mathematics itself.

The scientific and social functions of *Didactique* is rather to assign to this knowledge *that comes from outside sources* a status and a mode of intervening in didactical decisions. What matters is to prepare and permit genuine progress and prevent the irremediable destructions caused by the uncontrollable and incoercible unfurling of reforms proposed without much relationship to their declared objective.

The way to realize this project is to know and understand *Didactique* as what is specific to the transmission of a piece of knowledge from one generation to another. The capacity of each generation of humans to communicate the fruits of its experience to the next generation is as old as humanity, and perhaps its principal characteristic. Because of this capacity, each human being has a personal experience of learning and of teaching, and as a result, these activities appear to present no mystery. The prevailing impression is that the only question is the quality of execution. As is so often the case, trouble arises not from not knowing something, but from knowing as a certainty something that is false!

The Theory of Situations is just one attempt in this direction. Here I have only presented micro-*Didactique*: the study of the interactions specific to the diffusion of mathematical knowledge between two or three systems. Many phenomena remain to be studied in this field, as can be seen from numerous recent works. But we are seeing difficulties of a different nature arising from macro-*Didactique*, the study of the relationships between social and cultural systems and different sectors of mathematics.

All forms of research on mathematics education are welcome as long as their goal is knowledge. I am less enthusiastic about research aimed at radical modifications of teaching itself without concern for predictable results. What seems important to me is the improvement of the teaching of mathematics. Not tumultuous or surreptitious reforms hazarded at the pleasure of practices and modes, but ones that are supported by a deeper and surer knowledge of teaching and act with respect for humanist ethics.

Furthermore, I focus most at the level of the mathematics required in school. We demand of our children that they accomplish a veritable civic service: learn a lot of knowledge of which we know that a large part may be of no personal benefit to them. They learn this knowledge because in due course society is going to need doctors, engineers, bakers and mathematicians… and because all human beings need to make themselves understood by each other in order to take part in the decisions that are relevant and interesting to them.

So, we need to pay the price of that demand and do our best to make their civic service easier. Our work should make it possible for teachers to better understand and help others better understand the necessities of their profession so as to restore a dialogue and sharing of responsibilities with society.

In the history of humanity, mathematics and the teaching of mathematics first appeared as a set of practices. Reflection on mathematics emerged later, and only long after that, reflection on the teaching of mathematics. It was only at the beginning of the 20th century, during the international congress of mathematics held in Rome in 1908, that ICMI was created as an international institution serving this purpose, with F. Klein as its first president. After the convulsions of both world wars, research in learning and teaching of mathematics took its inspiration from specific scientific approaches and was included among the study groups within ICMI, such as PME (The International Study Group for the Psychology of Mathematics Education) created in 1976 at ICME 4 in Karlsruhe.

A century of these activities produced a considerable number and variety of initiatives, reforms, new practices, knowledge and projects. The harvest is considerable but difficult to synthesize. This tends to discourage the mathematicians, future teachers, new researchers, or simply people who, out of curiosity, would like to be acquainted with it. Syntheses and a better organization of the field would probably be very useful. The Klein Award of the ICMI could favor the identification of these developments while crystallizing them around certain symbolic figures.

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SP: Reflections and transformations: A mathematical autobiography

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Introduction: Facing challenges

My talk at the ICME Congress in 2004, and this paper, serve as a modest attempt to offer heartfelt thanks to the mathematical and mathematics education communities for choosing me as the first recipient of the Hans Freudenthal Medal, an honour that belongs to my colleagues and collaborators as much as to myself, and which represents a step forward for our mathematical community perhaps even more than for me personally.

In this paper, I map out some of the challenges facing mathematics education that I have tried to address during my professional life to date, and that I intend to continue to work on in subsequent years. How to exploit the potential of new technologies for rigorous rather than superficial engagement with mathematics, either face-to-face or at a distance? How better to design and operationalise teacher development programmes that foster a professional culture that is subject-specific, relevant to teaching and learning and is sustainable over time? How to help employees, in workplaces that have been transformed by the presence of computers, come to terms with the growing obsolescence of craft skills and the emergence of new mathematically-based skills relevant to their new situation? How to find ways to engage in genuine collaborations that respect and learn from the diversity of different cultures, curricula and research paradigms? How further to develop knowledge in mathematics education that derives from theoretical effort, yet which is sufficiently robust to have policy and practical implications? And finally – perhaps most crucially – how to ensure that students of all ages come to recognise that mathematics is about structures and relationships with an internal coherence, an elegance and an aesthetic, and that an exploration of these structures and the development of mathematical habits of mind (Goldenberg et al, 1998) need be neither boring nor irrelevant. As an aside, and before addressing any of these challenges in any depth, I should add that I am committed to rigorous and systematic research methods in the field of mathematics education and have made efforts over many years to become familiar with a range of research methods (qualitative, quantitative and mixed) to be used as appropriate to the goals of the study in hand.

In this paper, I sketch my personal response to some of these challenges and draw together some overarching themes that have underpinned my work. I want to state from the outset that any success achieved in my endeavours, has been the result of working in an environment that is both challenging and supportive, that prioritises research and scholarship while acknowledging the importance of practice, and – even more crucially – of enjoying collaborations with brilliant and diverse groups of people, many of whom are co-authors of papers arising from our joint work[1]. In particular, I must acknowledge from the start the huge contribution to most if not all of my work of Professor Richard Noss, with whom I have collaborated for many years.

I have approached this chapter as a selective summary of my research efforts. I present brief descriptions of different research strands and some projects within each, and give references to relevant papers so readers can follow up the work if they wish. I rarely refer to work of researchers who have not been part of my research teams, in order to achieve a coherent and relatively succinct overview. I hope this will not be interpreted as ignoring the work of others: appropriate references are made in the papers to which I refer.

Fostering engagement and respecting rigour through design experiments

A major objective in my teaching and research has been to find ways to motivate students to engage in mathematical thinking. I recall in my early teaching that the worst moments were when students complained: 'it's boring'. In my PhD, I chose to investigate students' affective responses to mathematics, being convinced that we must find ways for all students to benefit from mathematics learning (Hoyles, 1982). Although not continuing this theme as an explicit research focus, it was undoubtedly its motivational potential alongside the possibility of enhancing accessibility that led me to work with *Logo* in the early 80's. I still believe that the potential of using software to motivate rigorous engagement with mathematics should not be underestimated – although much still needs to be done to realise this potential in practice, in schools, colleges and universities. Some of these early ideas concerning students' work with Logo were explored with my colleague, Ros Sutherland, and described in for example, Hoyles and Sutherland (1989) and Hoyles (1985). In that corpus of early work, we noted the importance of students being able to build their own models through Logo programming, which served as a medium to construct and express their evolving mathematical ideas in collaborative, long-term projects.

Thus my early Logo work was focused on ways to motivate and engage students in mathematics, and the role of the teacher in promoting this engagement. Gradually my research took on a more conceptual focus, in collaboration with Ros Sutherland and Richard Noss, through investigating new representations for mathematical objects, such as variables, ratio and functions (see, for example, Hoyles & Noss, 1986; Sutherland & Hoyles, 1986). This strand of work, shared among a group of researchers who became known as the *Logo Mathematics* community, culminated in an edited volume (Hoyles

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1 I note here the organisations with which I am – or have been – involved since 1970:

- Research projects based at the Institute of Education, University of London, UK that have included teams of researchers and teachers.
- The British Mathematics Education Community, including specifically the British Society for Research into Learning Mathematics (BSRLM) and the Joint Mathematical Council of the U.K. (JMC).
- International Group for the Psychology of Mathematics Education (PME), International Congress on Mathematics Education (ICME) and ICMI Study groups.
- Logo Mathematics Education Community, a group of researchers in mathematics education who carried out research, during the 1980s and 1990s, on how Logo could be used as a tool to explore mathematics.
- Basic Components of Mathematics Education for Teachers (BACOMET), a small group of scholars who worked together from 1980 to 2004 and produced four edited volumes.
- The Advisory Committee on Mathematics Education (ACME), established in 2002 in England as a single voice for the mathematics community. I was a founder member.

& Noss, 1992a). Increasingly, I recognised that the public character of the screen on which the Logo programs were written could serve as a window on a range of issues that shaped mathematics learning: for example, on the way students were construing their mathematics, on how mathematics was discussed, on how teachers intervened in the process of learning, on gendered ways of working and on teachers' beliefs about mathematics teaching and learning (Hoyles, 1988; Hoyles et al, 1991; Hoyles, 1992).

During this time, I became increasingly involved in what would now be called design research; in particular in designing microworlds around particular mathematical topics (see for example, Hoyles et al, 1989). The research evolved, in collaboration with Richard Noss and Lulu Healy, into designing and building programming tools (in Logo and later in other software) through which students could investigate deep and challenging mathematical ideas in a playful way, by building, expressing and debugging their ideas and by making sense of, and possibly arguing over, the computer feedback. As part of this research, we designed activities in which the tools could be used to maximise the potential for learning mathematics; for example, where students would invariably be faced with surprising feedback that would challenge them to reflect on any 'incorrect' mathematical assumptions they might have made, or would foster collaboration and the sharing of perspectives in order to support alternative approaches (Hoyles & Noss, 1992b). We iteratively tested the designs through extensive qualitative and sometimes quantitative evaluations. One of our most successful microworlds was called *Mathsticks* (Noss et al, 1997), designed to help students construct meanings for algebraic generalisations by forging links between the rhythms of their actions (the enactive mode), the graphical output (iconic mode) and the corresponding symbolic (Logo) representation, which served as a language to describe the patterns generated. The results touched on a key insight that has been at the core of my subsequent work: namely that at the heart of mathematics learning is first, the medium of expression, and second, the ability to coordinate different mathematisations of any situation, possibly using different media and different metaphorical or real connections and networks. I extended my design research by moving from a focus on tasks to the creation of activity sequences, embedding computer use, whose goal was, for example, to foster collaborative work (Hoyles et al, 1992; Healy & Hoyles, 1999), or to provoke students to explain and prove their conjectures in the domains of number/algebra and geometry (Hoyles, 1997; Hoyles, 2001; Healy & Hoyles 2001).

Following these studies, my research moved outside conventional classroom environments and in doing so I had to face another design challenge: how to exploit the potential of informal learning for mathematics. In the *Playground Project* (1998-2001), Richard Noss and I set out to design systems and activity structures where mathematics learning was the outcome of a synergy of building, sharing and playing, face-to-face and at a distance. We started this work with two premises. First, when playing games, children necessarily follow rules passed down from above by the game designers. And second, there was a range of strategies that we knew could promote reflection in computer-based environments on underlying structures, relationships and rules, namely, strategies stimulating discussion, prediction and explanation in and across multiple forms of expression, planning for cognitive and socio-cognitive conflict, and orchestrating their resolution through debugging and the exploitation of feedback. What we did not know, of course, was how far these strategies could be incorporated into game-building and game-playing without destroying the game, and how all this could be achieved in a distributed community of young students collaborating over the web.

The Playground Project aimed to place children in the role of producers as well as consumers of games, so our overriding design criterion was to build a computational environment in which children were able to program their games in ways that were inspectable, modifiable and shareable over the web. The project achieved some success in terms of long term engagement in the development of a joint product (the game) during which time students engaged with issues of inference and the meaning of rules (see for example, Noss et al, 2002a). We also were able to identify and describe the practical challenges of this work, which provoked us to revisit our theoretical ideas on microworlds and tool use (Hoyles et al, 2002a).

In a later project, WebLabs (www.weblabs.eu.com/), we *extended* our design work still further and *shifted* its focus to iteratively building tools and activity sequences in which students, in different sites across Europe, program models of their mathematical and scientific knowledge and then share, discuss and modify the models through a web-based system, WebReports. We try to ensure that the potential of collaboration is exploited in all its forms, by including asynchronous discussion and exchange around WebReports as part of any activity sequence, alongside synchronous interchange, both face-to-face and at a distance. We also aim for a more explicit promotion of learning mathematics through the processes of modelling and sharing, collective reflection and participation in a joint enterprise. We do not underestimate the challenge in this approach, but are heartened by the productive interchanges we have recorded and analysed that have demonstrated conceptual change and the overcoming of known mathematical obstacles (see, for example, Simpson et al, 2005). Inevitably, much is still to be done and the work has, once again, stimulated more theoretical reflection in two directions. First, design issues raised by this new kind of microworld involving collaboration over the web alongside the constructive use of tools. And second, the roles teachers need to play in building the socio-cultural mathematical norms of a distributed community, where students are expected to post their products on the web, and expected to comment on, modify and discuss the contributions of others.

Before ending this section, I wish to suggest that design research may have an important role in mathematics education, beyond the experiment and its evaluation. Design research should perhaps play a more explicit role in a teacher's everyday practice? Teachers not only shape the culture in the classroom through interactions at a meta-discursive level, but also (possibly in collaborative teams with researchers) can play an active role in shaping the tasks and activities at an object level, evaluating students' tool-mediated responses and generating the norms of language and communication. Design research could also play a more central part in programmes of teacher development, as, again, it necessarily focuses attention on the mathematical knowledge that teachers need to operationalise in their classroomas. We need of course to work together to build a robust theoretical basis for research in all its forms, and it is to theory development that I now briefly turn.

Working on theory alongside design

I first became involved in theoretical work in the early '80s, partly stimulated by the need for more clarity about the meaning of a microworld, partly provoked by the complexity of the task of integrating computers in teaching and learning mathematics and the need to better understand the trajectories of students' mathematics learning, and partly through my early involvement in BACOMET (Hoyles, 1991; Hoyles, 1993; Hoyles, 2005). With the benefit of hindsight, I now recognise that in my early Logo research, I may not have given sufficient attention to the complexities underlying the introduction of microworlds into institutionalised mathematics teaching, even microworlds that had been carefully designed in terms of computer tools, sequenced activities and the teacher's role. These complexities include a recognition of the ways the 'computer' shapes mathematical knowledge and the interactions between learners and between learners and teachers, and crucially, how computers frame the language in which mathematics is expressed and the meanings of 'doing maths' and communicating mathematically. Moreover, I have come to recognise that if these ways of doing and communicating mathematics are not legitimised, computer use will inevitably be marginalised in mathematics classrooms or defined as 'learning about computers' rather than learning mathematics.

Reflecting on my early aspirations, I am struck how similar they are to much of the rhetoric around computer use that we hear in 2004, and that despite our energetic, enthusiastic and disciplined research with teachers and students, rather little has changed in mathematics classrooms that are not supported by research or development teams. Certainly the technology has changed and there is now a plethora of excellent software available with which to explore and to build mathematics. There is also a growing corpus of fine research in the field of computers and mathematical learning: I will mention just one source with which I have been associated, the *International Journal of Computers for Mathematical Learning;* there are, of course, many more. But rather little has altered in the way mathematics is conceived, the way it is taught, the hierarchies in place and who is and is not judged as mathematically competent. In the '80's, I was filled with huge optimism that computers (alongside teachers) would change mathematics teaching, making it more exciting and more inclusive. I did not recognise adequately the complexity of using computers for mathematics learning; complexity in terms of curriculum and pedagogy and access to hardware and software, but more profoundly, complexity around epistemology, conceptual understanding and how to 'transfer', to relearn in new contexts, the mathematical knowledge developed through interaction with computers.

In our early Logo research, Richard Noss and I noted how students' work with Logo served as a window on their evolving mathematical knowledge, knowledge which not only shaped the Logo language and programs they chose to use, but also was shaped by Logo, the medium through students largely expressed their mathematics. We described this process as one of 'situated abstraction' (see Hoyles & Noss, 1992a), and elaborated this notion, along with that of webbing – the connections made during this conceptual process – in our book (see Noss & Hoyles, 1996a). Underpinning both notions is the recognition of the crucial role that symbolic tools play in mediating mathematical expression and mathematical communication.

As I argued in Hoyles (2002), discussion of tool mediation as a unit of analysis is all too often missing in mathematics education research. The reference for this remark was the mediation by computer tools, but is not necessarily limited to computer interaction. Written production or verbal remarks are as much shaped by the medium as by computers; it is simply a matter of our familiarity with the outputs that renders the medium invisible. I discern the beginnings of a convergence of views in recognising the centrality of tool mediation in mathematics education (see the overview of uses of digital technology in Hoyles & Noss, 2003, the discussion in Hoyles et al, 2004, along with the recent edition of *Educational Studies in Mathematics*, edited by Nemirovsky & Borba, 2004). It may be that establishing and elaborating a tool-mediation focus to mathematics education research could help to build bridges between the individual and the social, as it places the spotlight on the expression of mathematics and on the communication of mathematical ideas within and between communities. Additionally this focus, while keeping mathematics at the centre of inquiry, foregrounds the role of the design of activities, the design or choice of the tools or sign systems that introduced to foster mathematics learning and, of course, requires investigation of the transformative potential of these tools (see Hoyles, 1995, and more recently, Kaput et al, 2002; Hoyles & Noss, 2003). Further work in this direction is planned in an international setting in the forthcoming ICMI Study that I am co-chairing with Jean-Baptiste Lagrange. As part of it remits around digital technologies and mathematics teaching and learning, this Study will consider how cultural factors impinge on computer use, in different phases of education, both within and outside educational institutions by particularly incorporating analyses of the situation in developing countries.

Clearly there will be a continual need to revisit this agenda as technology changes. But at the core must be the theoretical resources, necessary to optimise the chances of sustained, systemic strategic change for the benefit of mathematics teaching and learning. We know now that the use of computers to motivate investigation of mathematics – even if undertaken after careful design experiments as we did in the early Logo research – may not result in any long-term change and improvement. We have long recognised that computers could not achieve anything on their own and that ultimately, despite the best tools and resources, "It's down to the teachers". But we need nonetheless, to elaborate and refine what this might mean in different circumstances and with different mathematical goals, and so be better placed to anticipate the challenges that inevitably must be faced. Design research has a growing theoretical base alongside the development of tool-rich activity sequences (see, for example, diSessa & Cobb, 2004). But due research attention must also be paid to how to 'transfer ownership' of the design innovation beyond its initial sites, and to how to do this without trivialising the innovation and losing sight of its essential mathematical goals[2] (see, for example, my work with Teresa Smart around the professional development of mathematics teachers in London, Smart & Hoyles, in preparation).

Researching students' conceptions of proof and reasoning

As well as researching the use of computers in mathematics education, another major strand of my research, that links with the agenda of investigating 'engagement with rigour' has been around the investigation of students' conceptions of proof. My motivation once again grew out of my teaching. Why was it that students did not feel the need to prove, the need to struggle to ensure their arguments were water-tight – something I had always so enjoyed myself. At the time of my first proof project with Lulu Healy, the mathematics curriculum in England had undergone considerable change and we set out to investigate the effects of this change on mathematical reasoning. Given that the English

2 Computer-catalysed innovations all too often are trivialised: take for example what happened so often with Logo, which simply became a tool to draw a square!

National Curriculum was statutory, there was consistency in the intended curriculum across the country, so it was reasonable to adopt a methodology to investigate students' responses to proof tasks that comprised a large national paper-and-pencil survey, followed by classroom observations and interviews with teachers and students. This project entailed adopting (for me) new research techniques, quantitative statistical analyses alongside qualitative studies: for example, we used multilevel modelling of data to identify predictors of success in proof and to identify schools and students with exceptional success, and case study to elaborate and contextualise possible reasons for this success.

The multilevel modelling analyses showed consistently that students' successes in constructing proofs, and their choices of arguments that best exemplified their approaches, were strongly influenced by mathematical attainment as measured by standardised national tests, but were never determined by this factor alone: students' views and evaluations of proofs, their gender, and their experiences of the curriculum, all exerted significant influences on responses. Our findings also suggested that classroom climate might be influential, in that we consistently found that students in classes in which a larger percentage of students were to be entered for the most challenging assessment at age 16 years, produced better responses from their students than equivalent students (that is, students matched in terms of all other predictors), in classes with a smaller percentage. This finding is particularly interesting, not least as it has been replicated in my later research. We found that students revealed many of the problems identified in previous research on proof, but we also found, through this large-scale research, new factors that seemed to frame their responses. For example, students simultaneously held two different conceptions of proof; those about arguments they considered would receive the best mark and those about arguments they would adopt for themselves. We also noted that students tended to confer status on proofs because of factors quite apart from their generality or logical nature, such as the presence of named geometrical facts or relationships, or the inclusion of algebra (see for example, Healy & Hoyles, 2000).

Of course, longitudinal studies were needed to draw out any causal links between the factors identified and the outputs measured. I therefore was fortunate enough to be able to follow up this first proof project with a longitudinal study, the *Longitudinal Proof Project* (1999-2003), this time in collaboration with Dietmar Küchemann. The project analysed students' mathematical reasoning over time, focusing on students aged from 13 to 15 years old. Again mixed methods of analysis were used (see Hoyles et al, in press), for a discussion on the strengths and challenges of mixed methods). We carried out an annual written survey of high-attaining students from randomly selected schools within nine geographically diverse English regions. In the first year (June 2000) 3000 students, aged 13, from 63 schools were tested in number/algebra and geometry. The same students were tested again in 2001. Some of the questions were from the previous test, others were new or slightly modified questions. The same students were tested in a similar way in June 2002. Case studies and interviews of selected students and teachers were also carried out every year. Findings from this longitudinal study confirmed many of the findings of the earlier cross-sectional study, but also produced new insights, for example concerning the sustainability of mathematics learning over time (or lack of it), and how the introduction of new curriculum content can have unanticipated effects on students' proof responses. The challenge for teachers and researchers is therefore to develop sustained programmes of activities, in which new ideas are introduced into the curriculum, that build connected and layered mathematical knowledge and ways of explaining and proving, rather than simply replace old content or modes of expression with new ones (Küchemann & Hoyles, in preparation).

I will end this section with a remark that could not be presented as a research finding but captures one general lasting impression I have of my work on proof. It is that I have been continually amazed at the originality of so many of the students' proofs and explanations, proofs that would perhaps not be judged as adequate mathematically (they might not display a fluency with mathematical language and procedures for example), but nontheless proofs that somehow displayed mathematical integrity and mathematical creativity. How can we ensure we do not suppress this creativity, this student voice, in our quest for rigour?

Windows from the workplace

I now move on to the last strand of my research, which is how mathematics is used in the workplace. In this corpus of work, we continued to refine our notions of situated abstraction and webbing, as we were constantly forced to address issues of 'transfer' (or lack of transfer) and tool mediation: why was it that workers so often could not use the mathematics they were supposed to know, or paradoxically, why could they clearly display mathematical competence yet deny it or fail to answer correctly any traditional mathematics questions? To investigate these questions I began to research how mathematics was used in an investment bank (Noss & Hoyles, 1996b). This work triggered so many new challenges and insights, that we continued with longer studies of nurses and pilots (in collaboration with Stefano Pozzi; for overviews of this work, see Hoyles et al, 1998; Noss et al, 2000), and then of workers without professional qualifications in a range of sectors (see Hoyles et al, 2002b).

Studies of the workplace raise methodological challenges: what to observe, for example, as mathematics is so often invisible to practitioners. Our findings consistently showed that what we termed the 'visible' mathematics of a practice was almost invariably associated with routine activities, often involving measurement and recording, the use of algorithms to find unknowns from one or more known quantities, or the communication of results so as to inform action or decision. However, we were also aware of what we would describe as mathematical activity, which tended to be invisible to practitioners and managers alike. For example, we noted how practitioners often used a range of apparently idiosyncratic strategies finely-tuned by the tools available for solving particular problems in specific circumstances. Yet despite their specificity, we could discern sound mathematical models underpinning practitioners' strategies, many of which only came to light at times of disruption to routine or the need to communicate to others. For example, in the nurses' study, we identified largely unarticulated ways that expert nurses undertook the calculation of drug dosages on the ward and noted how they actually used a range of correct proportional reasoning strategies based on the invariant of drug concentration to calculate dosage on the ward, rather than the single taught method they described outside of the practice. These strategies were tied to individual drugs, specific quantities and volumes of drugs, the way drugs are packaged and the organization of clinical work (see Hoyles et al, 2001). We used data from the nursing study to elaborate the notion of situated abstraction as an analytical tool to understand nurses' conceptions of the intensive quantity of drug concentration, which we argued, was webbed to the mathematics of ratio and proportion as well as to the contextual artefacts and procedures of the practice. We also noted the fragility of the nurses' knowledge when they were no longer able to coordinate – to web – their mathematical knowledge with their professional expertise (see Noss et al, 2002b).

In our later work, in collaboration with Phillip Kent and Arthur Bakker, Richard Noss and I are specifically investigating the shifts of perspective in practice, the 'transfer' of mathematical ideas across boundaries in technology-rich workplaces seeking to improve the efficiency of their production processes, to increase the quality of their products, or the return on their financial products (see www.ioe.ac.uk/tlrp/technomaths/). In this drive to improve, abstract computer-based models are developed of the work process, and employees at all levels have to develop new skills that we are seeking to describe: for example, systematic and precise measuring and data entry techniques, monitoring systems against targets, and interpreting and communicating progress in relation to targets. Increasingly therefore workers are involved in interpreting, manipulating and communicating numerical, symbolic and graphical information.

Situated abstraction and webbing have again proved to be useful analytical tools, enabling us to appreciate the challenge of meaningful mathematical communication at work: the former offers a tool for valuing and making sense of what is mathematically understood by people in work, and the latter helps us understand how the pragmatic, specific and concrete connect with the theoretical, general and abstract. At the same time, we are developing design experiments that aim to foster this communication, using constructionist computers tools and activities: thus bringing several strands of my research together.

Conclusions and future challenges

I now briefly summarise what I see as some of the key challenges for the mathematics education community today, both old and new. They include to foster engagement with mathematics and promote legitimate diversity without forfeiting rigour; to achieve a more robust understanding of the complexity of introducing computers, particularly in collaborative endeavours; to develop short, medium and long-term innovations and implementation strategies; to devise systematic methodologies to design and evaluate innovations over time; to investigate (theoretically and practically) 'crossing boundaries', that is the processes by which knowledge has to be re-learned in new contexts and in order to address new audiences (e.g. employers, employees and policy makers), and to seek to exploit learning in informal as well as formal settings. Outside the constraints of 'school', we might be able to foster 'real' engagement with mathematics, without the need to 'please the teacher', while at the same time face new challenges about what is actually being learned, and how can this be aligned with institutional modes of mathematical expression.

I have sketched my personal response to some of these challenges in a form that is loosely autobiographical. I have always loved mathematics and taken pleasure in the struggle to solve problems and to explain solutions – at first to myself, but later to others when I became involved in education. I have enjoyed the challenge of moving into an 'unknown territory', whether it be the design of activities with new technologies in teaching and learning mathematics, the critical adoption of different research paradigms to investigate new research questions, or the identification and fostering of mathematical skills required in a range of workplaces. I have welcomed the testing experience of addressing new audiences or making new connections – both can be catalysts to develop novel modes of communication and to challenge hidden assumptions. I have tried to address the general public in the popularisation of mathematics (Hoyles, 1990); employers, in attempts to communicate research results and to re-negotiate with them the skills they require at work; and researchers in countries with different traditions in mathematics and schooling, so that we can better learn from each other. Most recently, I am facing a new challenge, a new boundary to negotiate, in working with Government in the U.K. and seeking to convince politicians of the strategic importance of mathematics; politicians who may be driven by a different agenda from mine but one that would benefit from evidence derived from research. In this endeavour, I am fortunate to be able to draw on the rich and diverse results of mathematics education researchers from all over the world, whom I thank again for the award of the Hans Freudenthal medal.

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TA A: Teachers of mathematics: Recruitment and retention, professional development and identity

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Introduction

The focus of this Thematic Afternoon – mathematics teachers – reflected the renewed research interest in mathematics teachers and teaching noted by several of the plenary speakers at ICME-10: The Survey Team, chaired by Jill Adler, reported increased research on teacher development, learning and associated curriculum reform. Anna Sfard also noted, in her talk, the change of research focus from learners to teachers. The acknowledgement of the central role of teachers in students' learning of mathematics has encouraged research to consider more closely the nature of the teaching demands, the ways in which teachers manage these demands in the realities of their classrooms (Stein, 2001, Strässer et al, 2004). With respect to mathematics reform in particular, studies include: investigating teachers as learners, a more critical examination of the pre- and in-service development provisions and the associated formation of teacher identity. Research presented in the Thematic Afternoon reflected this closer examination of the professional formation of teachers. The papers[2], offering a variety of theoretical frameworks and models, highlighted the collaborative nature of emerging research methodologies.

Recruitment, supply and retention of mathematics teachers

The issue of recruitment, supply and retention of mathematics teachers was addressed by contributions from England and Sweden. The small number of contributions offered for this strand is possibly indicative of the relative scarcity of related research. Collective concerns were the decrease in the number of students studying mathematics courses, the quality of mathematics teachers' qualifications, and teacher attrition related to work conditions and aging teacher populations. Contributors argued that all of these issues impacted on the quality of teaching within schools. An additional concern raised by *Johnston-Wilder* (TA, 2004) related to difficulties of engaging teachers in 'out of school' curriculum development projects. Schools, faced with difficulties finding relief teachers and fears of teachers not wanting to return to school after project involvement, were becoming increasingly reluctant to release quality teachers for curriculum development projects.

Solutions offered within the English context to address recruitment included changes in schools to address workload issues, the adoption of an entitlement of continuing professional development (*Zhang*, TA, 2004), and diversification of routes to qualified teacher status, including flexible training options. *Angier* (TA, 2004), reporting

1 Now Agder University.
2 Papers referenced by (TA, 2004) refer to the Thematic Afternoon presentations and are available on the ICME-10 website (www.icme10.dk) programme page.

on the experiences of students completing a flexible Post Graduate Certificate in Education, claimed that such courses may make little impact on recruitment numbers. However, on a positive note she argued that the impact of flexible pedagogies may better equip teachers to "enjoy the complexities and challenges of teaching" and thus improve teacher retention. Using a similar strategy involving changes in initial teacher education programmes, *Thunberg* (TA, 2004) reported a Swedish initiative to combine engineering and teacher training culminating in a double diploma qualification involving practicum experiences in both schools and science centres.

Pre- and in-service education of mathematics teachers

The need to understand and support with appropriate professional development all stages of the teacher professional learning continuum – pre-service, induction, early career, and experienced – was advanced in several contributions within this strand. *Van Zoest* (TA, 2004) posited that the third stage, approximately years 4-7 in a teacher's career, may well be a time of experimentation and consolidation that shapes the future teacher.

In light of current reforms the need for effective teacher education and increased knowledge about what and how teachers might learn was a central issue addressed by several papers. While some of the papers provided examples of ways in which progress is being made, others also highlighted challenges still to be addressed. *Sztajn*, *White*, *Hackenberg*, and *Alexsaht-Snider* (TA, 2004) emphasised the need to develop trust within professional development programmes: trust between the facilitators and the participants. Van Zoest warned that the quest for the ideal model of professional development needed to be clearly linked with outcomes, arguing that we need to more clearly understand and articulate the nature of transformation in teachers' knowledge, understandings, skills and commitments. In this respect, *Morony* (TA, 2004) considered the potential of recently developed professional teaching standards (AAMT, 2002) as a tool for professional development and *Baber* (TA, 2004) noted the role of professional teacher associations in developing "networks of learning".

Contributions also highlighted the various models of teacher education across the international spectrum. A study by *Peterson* (TA, 2004) compared expectations of pre-service practicum in both Japan and US. Cultural differences at a discipline level were also highlighted within *Groves*' (TA, 2004) discussion of integrated curriculum studies. Initially introduced as a response to a crowded curriculum, the integrated curriculum studies course compounded growing concerns about the adequacy of time available to support mathematics education within initial teacher education. Continued reports such as Groves' are needed to monitor this trend and are clearly linked to the wider issues of teacher knowledge expressed in the parallel strand.

Professional development using distance learning and associated technologies was explored by *da Ponte* (TA, 2004). Within the virtual community, the strong presence of collaboration and reflective writing led da Ponte to question the impact on teachers' professional identity: the fundamental roles, norms and values of the mathematics teacher.

Missing from this strand were studies that focused on the early years of teaching. Given the concerns expressed about retention of mathematics teachers there appears to be much scope for studies that examine the nature and effectiveness of support for beginning teachers.

Mathematics teachers' identity

Contributions in this strand interpreted the issues related to mathematics teachers' identity in many ways, and from a range of theoretical perspectives. Some papers raised issues that mirrored discussions on teacher recruitment, particularly in relation to the potential disjunction of identities and related images of the mathematics teacher:

Alignment with the mathematics community – in the sense of doing well in your degree and taking on the characteristics of a mathematical person – may well be at odds with alignment to school teaching. (Rodd et al., 2003, cited in *Winbourne*, TA, 2004).

Thornton (TA, 2004), provided an official version derived from teacher input of teachers' identities (AAMT, 2002). Integrating the standards document into assessment and portfolio tasks, Thornton argued that the signposts and guidelines enable student teachers to effectively map their developing teacher identity against a vision of what it means to be an excellent teacher. Likewise, *Wilson* (TA, 2004) provided observations on teacher excellence in relation to a sense of self in terms of motivation, commitment and feelings about teaching. However, both *Proulx* (TA, 2004) and *Parker* (TA, 2004) challenged the use of pre-designed official identities. Proulx suggested that student teachers appropriate teacher education programs in unique ways – their identities continuously unfold as new opportunities and possibilities are realised. Based on a series of interviews, Proulx provided a range of characterisations of pre-service teacher as 'Technician', 'Mimic', 'Self-assured', 'Reflective practitioner' and 'Natural teacher'. Applying Bernstein's theory Parker argued that local teacher identities emerge within specific pedagogic contexts as a 'form of consciousness' embedded in the social practices of a community. Within the context of South Africa Parker discussed the duality of identity formation experienced by novice teachers: that of a mathematics teacher and a mathematics learner. Also mindful of the multiplicity of identities, Winbourne applied Wenger's (1998) theory of participation with the notion of 'figured worlds' (Holland et al., 2001) to develop a theory of identity formation within a community of practice.

The question of how the emerging work on teacher identities might be usefully used within teacher education was a recurring focus. Reflection on the characterisations offered in the papers was seen as a positive way of increasing student teachers awareness of the development of identity, not only enabling teachers to become the teacher they want to be, but also being able to articulate and justify this.

The mathematical competency of teachers

Today, in a climate of reform, many teachers are being asked to teach in ways that are very different from how they learned, and the expectations of teacher knowledge often outstrips that which teachers, especially those in generalist roles, can confidently realise. While acknowledging the many factors involved in effective teaching, the papers in this strand addressed the central role of teacher knowledge, both in terms of classroom practices and issues of competency related to expectations of professional standards. Case studies (e.g., *Christiansen,*TA, 2004; *Kaldrimidou, Sakonidis, & Tzekaki*, TA, 2004) focusing on the complexity of the teaching process highlighted the importance of effective teacher scaffolding, interactions and the creation of space and time for student learning. Explorations centred on teachers' ability to 'notice' – to have a sense of when something happens that can carry the learning forward – and the nature of interventions in relation to student difficulties and errors. Kaldrimidou et al. noted the need to focus on the subject-matter structure within lessons, claiming an interplay between the epistemological organisation of the mathematical content and the organisation of the mathematics classroom.

While the majority of papers focused on mathematical knowledge and pedagogy, *Forgasz* (TA, 2004) presented research from the Australian context indicating the need to address teachers' beliefs. Reviewing studies from a range of school sectors Forgasz noted that despite changes in contemporary students' beliefs about the gendering of mathematics (Leder & Forgasz, 2002), gender-stereotype expectations remain prevalent among teachers, especially in relation to the interaction of technology and mathematics.

Assessment of teachers' competency is increasingly becoming a focus of government agency within a range of countries. *Fraser* and *Morony* (TA, 2004) discussed the AAMT *Teaching Standards Assessment Evaluation Project* aimed at the development of a process for acknowledging outstanding teachers. Assessed through a portfolio and interview, knowledge of students, knowledge of mathematics and knowledge of students' learning of mathematics all contributed to the professional knowledge domain. Concerns about pre-service teachers' mathematical knowledge base were also addressed in several papers (e.g., *Oh*, TA, 2004; *Arvidson*, TA, 2004). *Amoto* (TA, 2004) reported an action research project involving pre-service teachers' exploration of a series of children's activities. Increases in mathematical understanding were attributed to the unlearning and re-learning process that facilitated student teachers ability to work backwards from their symbolic ways of representing mathematics to more informal representations.

Conclusions

The papers in the thematic afternoon provided a snapshot of the issues and directions that we as a community are concerned with. This focus on mathematics teachers, their knowledge, their identity and their learning will play a critical role in ensuring quality teaching and effective learning of mathematics. However, the papers also indicate gaps and questions still to be addressed. Despite advances in our research capability and increased focus on reform teaching practices, there remains the interminable challenge to provide equitable mathematical access to all children irrespective of culture, ethnicity, gender, economic and social positions.

The panel debate triggered important questions from participants, such as "We talk about 'beneficial, efficient, excellent, improve, change, develop' without making clear what we mean by these words. Teachers are not good but need to become good. Do we know what we are aiming at?" Future research needs to listen to such questions and try to include them and address them in the work.

This challenge makes issues of recruitment, teacher education and retention of quality teachers all the more pressing. Using a metaphor of teachers "cleaning the path on which they walk" van Zoest (TA, 2004) reminded us that the journey to reform is difficult and exhausting. For example, within current reforms in South Africa, Parker (TA, 2004) argued that the focus on mathematical practices (e.g., investigating, making conjectures, justifying, generalising etc.) and on making meaning, rather than simply skills and product, has created new demands on mathematical competencies to teachers. Within this context, teachers need to develop new images of 'good practice' for mathematics teaching and new pedagogic identities. Although our research efforts must clearly be directed to making the pathway less hazardous, it is evident that we must be patient in our efforts to reach the destination. The interest expressed and generated in this thematic strand bodes well for the forthcoming ICMI Study: *The Professional Education and Development of Teachers of Mathematics.*

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TA B: Mathematics education in society and culture

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Introduction

In this theme the focus was on the complex relationships between mathematics education, society and culture, and it had four sub-themes:

1. Multilingual and multicultural classrooms: Increasing diversity
2. Mathematics education within and across different cultures and traditions
3. Social and political contexts for mathematics education
4. Equity in mathematics education: Culture, gender, and social class.

Within the entire programme structure for this ICME, there were several other programme items which dealt with themes and issues close to those of Theme B, for example, TSG 25 Language and communication in mathematics education, and TSG 26 Gender and mathematics education, among others. However, in the perception of the IPC, it was the task of Theme B to depict the entire *problématique* in relation to mathematics education in science and culture, whereas the other programme elements will address special aspects of that *problématique*.

The four sub-themes which the IPC for ICME-10 determined for this Theme represent significant growth areas, and so, as each of the sub-themes involves both research and development, the aims of the Theme B afternoon were:

- To highlight current foci of research and development in each sub-theme.
- To demonstrate and contrast the various approaches that researchers and developers are currently using.
- To allow practitioners at all levels to engage with, and critique the latest developments in these sub-themes.

The afternoon's program was organised around the four themes, and there were three papers selected for each sub-theme. Their summaries follow:

Sub-theme 1:
Multilingual and multicultural classrooms: Increasing diversity

In the first paper, *Leo Rogers*, from Roehampton University, Surrey, UK, gave a talk entitled "Multicultural classrooms in 4 European countries", in which he described a Comenius Project on teaching and learning mathematics during the transition from primary to secondary school (pupils aged 9 to 14). This project involved UK, Italy, Cyprus and the Czech Republic; and he noted that these countries are experiencing the effects that ethnic 'minorities' are having on their school population. As school classes become less culturally homogeneous, methods of teaching mathematics have to be re-examined. From their experience, the control of the curriculum, and any modifications that teachers may see necessary, have to confront the political ideologies of the governments concerned.

The second paper in this sub-theme was by *Yun* and *Tina Zhang*, and was called "The influence of culture and parental guidance: a case study of a group of Chinese students in England". International comparisons of the mathematical competence of secondary school students have shown that Chinese pupils achieve a higher level than their English counterparts. Some research findings have suggested reasons behind this, in areas including pedagogy, social and cultural factors, and teachers' knowledge of mathematics. So the Zhangs considered in their talk what happens when Chinese students attend the same schools as English students. They found that the students' group was a rather selective one, where the parents were willing to spend a great deal of time discussing their children's school education and helping them with their homework, even to the extent of being able to teach their children Mathematics up to Advanced-level. This was truly unusual in the UK.

The third paper in this sub-theme was by *Alan Bishop* and was called "Immigrant students in transition: dilemmas and decisions". Being an immigrant school student in a new country is a difficult matter. Language problems predominate, compounded by not knowing which other students to trust in the school, not knowing the school rules (except that you know there are likely to be many school rules), and not knowing the teachers. As well as their own self-imposed pressures to survive in the new environment, there are pressures from their parents who may be ultimately dependent on their ability to earn money for the family. These social pressures are exacerbated by the cultural conflicts experienced by every immigrant person, but particularly by immigrant students. In the talk Bishop explained the particular problems revealed by secondary school mathematics students because of the predominantly cultural nature of mathematics education.

Sub-theme 2:
Mathematics education within and across different cultures and traditions

In Sub-theme 2, the first paper was by *Jerry Lipka*, *Barbara Adams*, and *Ferdinand Sharp* of the University of Alaska, Fairbanks, USA and *Nancy Sharp*, from the Southwest Region School District, Alaska, USA, and was called "Connecting out of school learning to school mathematics: qualitative and quantitative data from Alaska." The talk explored the implementation of a culturally-based mathematics module in one Yup'ik Eskimo teacher's classroom. The development of this module connected Yup'ik cultural activity with school based geometry, and it showed how Nancy Sharp, the Yup'ik teacher, developed a classroom space that connected her home culture to the culture of schooling in some unique ways. On the project's pre- and post-tests this class performed better than average. She effectively used modelling and joint activity as a means of teaching geometrical relationships as students learn to fold and cut geometrical patterns out of paper.

The second paper was by *Charoula Stathopoulou*, from the University of the Aegean, Greece, entitled "Mathematics education as an acculturation process: the case of a Romany student group in Greece". She supported the argument that whenever we refer to students from minority and marginal groups, we can only talk about mathematics education as an acculturation process. More specifically she examined the phenomenon of school failure by a Romany group of students in a Greek school in Athens, in relation to their cultural particularities and the cultural conflicts that occur within the school as well as in the classroom. She also expanded the argument about how these cultural conflicts are connected with equivalent cognitive conflicts and how they influence the learning of mathematics more generally. For the purposes of this project she relied on ethnographic material, some of which she reported at the conference.

The third paper in this sub-theme was by *Victor Zinger* from University of Alaska Southeast at Ketchikan, USA, and was called "Key issues of teaching mathematics to Alaska Native students". Victor shared his experience in using the state-wide exit examination (High School Graduation Qualifying Exam-HSGQE) as a valuable and flexible tool to increase the effectiveness of learning and understanding mathematics by native students. He argued that the implementation and further development of the teaching program he described, with classroom practices based on the teacher's cultural awareness, wide usage of culturally sound mathematics with high expectations, understanding, and community involvement would help eliminate the performance gap of the native students on the HSGQE, and increase their overall level of understanding.

Sub-theme 3:
Social and political contexts for mathematics education

In Sub-theme 3, the first paper was by *Frank Davis*, Lesley University, Cambridge, MA, USA and was entitled "The Algebra Project – social movement and educational intervention". This talk was about the work of the Algebra Project, Inc., founded by Robert Moses, a noted civil rights activist and mathematics educator in the USA. The talk described the project's work as both facilitating a social 'movement' and mounting an educational intervention. However, these two faces of the project raised different types of evaluation and research questions that are difficult to link. Davis analyzed this difficulty through the idea of 'communities of practice', and suggested that a distinction should be made between practices aimed at engineering a solution to an educational problem or creating a new design, and practices aimed at intervening within a current set of practices, or what can be characterized as finding the "what works" solutions.

The second paper in this sub-theme was "International and global contexts in mathematics education: friends or foes?" by *Bill Atweh*, of the Queensland University of Technology, Australia. This paper presented firstly various arguments about the pros and cons of the international and globalised contexts of mathematics education. He also summarised some findings arising from a research study with mathematics educators in many countries on internationalisation and globalisation of mathematics education. Finally Atweh proposed a model of social justice as a useful tool to study international collaborations in mathematics education in global and international contexts. This model involved consideration of the four constructs of Aid, Development, Multiculturalism, and Critical Collaboration.

The third paper in this sub-theme was by *Lena Licon Khisty* from the University of Illinois at Chicago, USA, and was called "Language diversity and language practices: Why should mathematics educators care?" In this paper she discussed the nature of academic discourse and its connection to academic socialization processes and competence in mathematics particularly for linguistically diverse students. She argued that to understand development is to understand the relationship of how language is used in classrooms, which cultural language is used, and how students participate within the language structures. Two studies were reviewed to highlight these ideas. It was suggested that these concepts are crucially linked to effective instruction of mathematics with linguistically diverse students, if they are to be full participants in their respective societies.

Sub-theme 4:
Equity in mathematics education: Culture, gender, and social class

For Sub-theme 4, the first paper was by *Marta Civil* of the University of Arizona, USA and was called "Lessons learned from research on the intersection of culture, social class, and mathematics education: implications for equity." This paper drew on research aimed at connecting school mathematics with everyday experiences in low-income, Latino / Hispanic communities in the Southwest of the USA. The author discussed the challenges in developing school learning experiences in mathematics that acknowledge and build on the resources and experiences from the community. Some of these challenges have to do with the different values and beliefs associated with different forms of knowledge and how these differences influence the implementation of certain forms of mathematics in school. There are two groups of people who played a key role in the research approach used: the teachers (i.e., what support mechanisms are needed to help teachers implement these culturally-based teaching innovations?) and the parents (i.e., what do we mean by viewing parents as intellectual resources?).

The second paper was by *Maitree Inprasitha* from Khon Kaen University, Thailand, and was called "Reforming the learning processes in school mathematics in Thailand with an emphasis on mathematical processes." This talk centred around three themes:

1) to investigate learning processes in school mathematics of elementary and junior high school students using open-ended problems,
2) to construct a model for developing students' learning processes by implementing open-ended problems and meta-cognitive strategy, and
3) to disseminate the developed model to mathematics teachers in the Khon Kaen provincial areas.

The third paper in this sub-theme was by *Marcelo Borba*, of the State University of São Paulo at Rio Claro, Brazil, and was called "Social dimensions of internet based distance mathematics education in Brazil". In the education community in Brazil at large, positions have emerged that oppose the haste and superficiality of the distance courses compared with the face-to-face courses. In this presentation the author showed that distance education is important for a country that has 75% of the GNP in just one part of the country. He also showed data about how Internet-based continuing mathematics teacher education is already taking place in Brazil, and he discussed the problems and possibilities of this modality of education as means of mitigating social inequality.

Conclusion

The brief descriptions above give little indication of the depth of the papers, and of the interesting discussions which followed in the small group sessions which were organised especially to enable the participants to interact with the speakers. It was exciting and revealing to see the range of social and cultural contexts in which the current research is being carried out, as well as the different foci of the studies. Each paper, as well as each sub-theme, indicated promising agendas for further research. They amply demonstrated the potential and significance of this research area for enabling greater numbers of learners to benefit from a relevant mathematics education instead of suffering and failing under a socially irrelevant and culturally exclusive one.

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TA C: Mathematics and mathematics education

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Introduction

The guidelines as outlined in the announcement of the congress could be described as follows.

- Trends in the mathematical sciences and their influence on mathematics education
- The role of research mathematicians in mathematics education
- New and old mathematical topics, and the balances between them, in mathematics curricula
- The mathematics educator: Mathematician or pedagogue?

In the opening remarks of the afternoon *Jean-Pierre Bourguignon* gave his vision of the many issues to be discussed. It is a fact that mathematics is part of the school curricula at schools in almost every country in the world at various stages of the education process. The issue to concentrate on is whether and, if so, how the mathematical training in schools should be influenced by the evolution of mathematics as a science and in its relation to society. It is widely accepted that exposure to mathematical ideas at school is part of the education to systematic thinking, and basic mathematical objects such as numbers and geometrical figures are used for that purpose. The following questions arise. What about giving a glimpse of both the achievements of present day mathematics and of its multiple uses in society? Should one make the fact perceptible that mathematics as a science is thriving and is presently developing at an unprecedented pace?
First, an overview of the present situation would be useful. It is certainly desirable to form a global idea of the content of the pre- and in-service training of teachers. Of particular importance is the impact of this training on the personal relation teachers entertain with mathematics.

How can teachers maintain contact with present-day mathematics and the new involvement of mathematical facts, products and ideas in many areas of the society? Which documents are available for that purpose? By whom and how can their requests for contacts and explanations be answered? What kinds of contacts with research mathematicians are institutionally organized:

- Conferences and workshops?
- Cooperative projects in schools?
- Internships in research labs?
- Other modes of exchange?

It would be most important to attempt to analyze difficulties or insufficiencies that can be identified:

- At the level of training
- In the contacts with present-day mathematics and mathematicians.

Looking towards the future some avenues have to be explored:
– Are there web resources that can contribute? If yes, in what format?
– What kind of events, or structures, can help meet the needs?
– Who should be responsible for establishing and maintaining them, teachers or research mathematicians?
– What kind of agencies should take the lead in such matters?
– How can one get users of mathematics to testify about their uses?

After the introduction the following presentations based on previously distributed papers were given. We shall give a brief outline of them in the sequel.

Presentations

Lucia Grugnetti, ***Carlo Marchini***, ***Angela Rizza***, Local Research Unit in Mathematics Education at the University of Parma, Italy, lucia.grugnetti@unipr.it:
The long way (from primary school to the end of secondary school) for constructing the concept of limit
The concepts of limit, continuity, derivative and integral of real functions are generally introduced in the last or two last years of secondary school into Italian high schools in a fairly formal way, enriched by technical details and the demonstration of theorems. Results of research on the question "what kind of intuitive ideas are present in the students' minds and how can teaching support or obstruct their development?" were given. The speaker pointed out the presence of propitious intuitions about approximation which often are neglected in didactical practice. A study on epistemological problems in the concept of limit was presented. The interviewee sample encompassed a total of 600 people including students (ranging from 14 to 19 years of age from different types of schools) and adults without specialised mathematical knowledge. The interviewees were asked to describe their ideas on the terms 'limit' and 'infinite'. The natural language register (especially in Italian) does not give a hint to the mathematical meaning of 'limit'. The word limit denotes something which is associated with concepts like 'barrier', 'rule', 'restriction' and other words with an idea of 'finiteness'. Infinity is something that has no limits. On the basis of these findings it appears that the central point is to identify teaching strategies and constructive activities capable of enriching the learning experience and stimulating an evolution of intuitive understanding. An important point is to use approximation as a teaching resource. Students should learn the proper place of empirical methods leading to legitimate approximation schemes which can favour the gradual early development of the concept of limit. It is important to use rich and unusual contexts. The need of approximation can be introduced for example through the question of measuring an area with a curved boundary (a lake, say).

Vagn Lundsgaard Hansen, Department of Mathematics, Technical University of Denmark, V.L.Hansen@mat.dtu.dk:
Education in mathematics – Mathematics in education
The impact of mathematics has been absolutely fundamental to society in the past as well as for the shaping of modern society. Nowadays, not only many sciences, but also society in general rely to an increasing extent on applications of mathematical models. Even though mathematics plays a decisive role in many of the technological advancements in present day society, it is invisible to the general public and difficult to com-

municate. This may provide a platform for tackling the negligence of mathematics by the general public and may lead to actions in order to prevent reductions of the mathematics curriculum in schools. Mathematics education as a scientific subject has emerged in a meeting between two cultures: Theoretically interested educators at the universities and practice-oriented teachers in the school system. There is a great need to relate to the pedagogical and the didactical methods applied in the teaching of mathematics in the many diverse types of educations supported by mathematics. It is important that researchers in mathematics as well as researchers with mathematics education as their speciality participate in developing suitable teaching materials. The involvement of mathematicians is important to ensure the quality of the mathematics taught and specialists in mathematics education are needed to monitor the learning process. Large-scale mathematical experiments can now be performed on the computer. An important task for mathematicians and mathematics educators will be to maintain that mathematics is more than experiments; for a true mathematical education, you need proofs of mathematical results. Quality education of mathematics teachers in primary and secondary schools is the fundamental key to changes and improvements not only in the teaching of mathematics in schools but also, in a longer perspective, for increasing the general level of mathematical knowledge in the population of a country. New media, such as CD-technology and the Internet, will provide many possibilities for valuable mathematical activities, but can never be a substitute for engaged mathematics teachers. And it should never be forgotten that the place where one can really make a difference in shaping the public's attitude to mathematics is by delivering first class teaching of mathematics in schools. Teacher training in mathematics should be given a new impetus!

R. Cantoral and R. Farfán, Cinvestav – IPN, Mexico, rcantor@mail.cinvestav.mx:
Mathematics and mathematics education: A vision of its evolution
Mathematics education is a discipline of knowledge the origin of which dates back to the second half of the 20th century. In general terms it could be described as the study of educational problems linked with mathematical knowledge. During the last decades we have seen university academic activities appear at the heart of the community as mathematics teachers, learners of mathematics and educational mathematicians (corresponding with the term 'Matemática Educativa'). The following questions are the starting point of Cantoral's view: How do new developments in mathematics influence the teaching? How are teachers trained in mathematics? How can mathematicians and educators collaborate? It would be important to see plans and models for this interaction. Three domains must interact: mathematics education as a scientific discipline, mathematics as a scientific domain, and mathematical teaching as a field of practice. To achieve this goal specially designed courses for mathematics teachers are recommended. There are some examples in history for such a fruitful exchange, when Felix Klein and others at the beginning of 20th century had great influence on the changes in curricula. On the other hand the problems in understanding analysis may be related with the development of new models for the 'infinitesimals'.

Urs Kirchgraber, Department of Mathematics, ETH Zürich, kirchgra@math.ethz.ch:
Popularization: The case of ill-posed inverse problems
The starting point of this paper is E.C. Wittmann's view of mathematics education as a design science, and in particular what he calls the core tasks of the field. They include:

- Exploration of possible contents that focus on making them accessible to learners;
- Critical examination and justification of contents in view of the general goals of mathematics teaching;
- Development and evaluation of substantial teaching units, classes of teaching units and curricula.

One out of many questions that follow from Wittmann's list is how to find examples of mathematical results which are both 'beautiful' and 'important' and yet can be popularized. Pythagoras' theorem and Euclid's proof of the infinitude of primes are classical examples. There are a few lucky cases from more recent research like public key cryptography, in particular the RSA method. In this example the prerequisites are minimal but the result of a combination of a few elementary though nontrivial and highly powerful mathematical ideas is intellectually amazing, and, as it turns out, of overwhelming practical use. At times the use of metaphors may open a gateway. Here there exists a broad scale of possibilities ranging from metaphors that are quite close to the objects they mimic to more and more remote ones. If used to mimic the system of partial differential equations used for weather predictions, Birkhoff billiards are but a metaphor from a technical point of view. Yet they are reasonably suitable to explain some phenomena of an important nature from a general educational point of view. The extent to which metaphors can help transfer technical mathematical language into semantically available information certainly needs further study. A third approach relies on a process that can be called elementarization. It tries to (re)discover and expose key features of a more advanced topic in a setting that is more easily accessible, i.e. with fewer prerequisites.

As an illustration of the last mentioned approach Kirchgraber proposed to look at so-called 'ill-posed inverse problems.' An inverse problem amounts to reconstructing a cause from its effects. An example is provided by computerized tomography. The ideas behind the solution of ill-posed problems, in particular the concept of 'regularization' (due to A.N. Tikhonov) usually described in a functional analytic setting, can well be illustrated with tools from elementary linear algebra. Due to measurement errors inverse problems suffer from imperfect data. The goal of 'regularization' is to reduce the precision requirements on the data. As a concrete and rather spectacular example the reconstruction of a simple mass distribution from measurements of its gravitational field is presented.

Giorgio T. Bagni, Department of Mathematics, University of Roma "La Sapienza", Italy, gtbagni@tin.it:
Similar problems in different contexts: An example from model theory to elementary algebra
Looking for analogies and differences in different situations is educationally and interesting from a scientific point of view. Abstraction itself is based upon the interpretation of analogies and upon the consideration of similar problems in different contexts. The different propensities for self-correction should be considered when we compare researchers and young students. Topics must be considered with respect to their roots and to present-day context. A mathematics educator should be a mathematician, an educator but to some extent also an historian. He or she should be aware of analogies as well

as of differences. This can be illustrated with an example from model theory. Robinson Arithmetic is weaker than Peano Arithmetic. This can be shown by giving an explicit model, namely by taking $Z^*[x]$, the set whose elements are 0 and all polynomials with integral coefficients whose leading coefficients are positive. This model is not isomorphic to the set N of natural numbers (as the standard model for Peano Arithmetic). Clearly, the order in $Z^*[x]$ must be defined in a suitable way. The order is defined in $Z^*[x]$ as follows:

> $f(x) \leq g(x)$ if and only if $g(x) - f(x)$ belongs to $Z^*[x]$ and clearly $f(x)<g(x)$ if $g(x) - f(x) \neq 0$.

It is interesting to confront both models with famous problems in number theory: Fermat's Last Theorem, Catalan's Conjecture (solved recently by to Mihailescu and therefore not anymore a conjecture), and Goldbach's Problem. Interestingly Goldbach's Problem has been solved for non-constant polynomials.

These considerations are connected to fields whose roles in traditional mathematical curricula, referring to primary and secondary schools are weak: mathematical logic and number theory. One must add that even the concept of proof is not known to most pupils.

The presentations were followed by lively, and at some times rather controversial discussions. One central point was the question on whether proposals from research mathematicians could be suitable for teaching in school. However, it should be emphasized that the collaboration of educators and mathematicians is the central message of Theme C 'Mathematics and mathematics education', message that was considered unrealistic by some of the participants in the audience, a view which was, however, refused energetically by some other participants.

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TA D: Technology in mathematics education

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Team Members: *Jean-Baptiste Lagrange*, IUFM de Reims, France
Shoichiro Machida, Saitama University, Japan
Richardo Nemirovsky, TERC Inc., Cambridge, USA

Introduction: Overview of the theme

The thematic afternoon "Technology in mathematics education" provided an opportunity for participants to find out about current and future technologies, to focus on how ICT can be and is being used for teaching and to engage with current research perspectives from around the world. Novice users of technology in education were able to use the afternoon as an opportunity to see the potential for ICT to enrich mathematics teaching while experts could further explore current and emerging issues. There were sessions of relevance to all levels of education: elementary, secondary and tertiary.

Through a range of lectures, panels and hands-on sessions, the afternoon addressed four major topics for elementary, secondary, tertiary and teacher education:

- New developments in information and communication technology for mathematics education
- Advantages and pitfalls concerning technology in mathematics education
- The Internet and mathematics education: Accessibility, use and misuse
- Technology in distance teaching and learning.

In keeping with the wide-ranging nature of the theme, there were a total of 17 sessions and 52 speakers, addressing highly diverse topics from many different perspectives. Six hands-on workshops for groups of about 30 participants enabled participants to obtain direct experience of new software products and new uses of established technologies for all levels of schooling, including teacher education. Simultaneously, there were 11 lecture hall sessions, mostly lasting 105 minutes, where speakers demonstrated new possibilities and discussed some of the many pedagogical questions that arise when technology with mathematical power is put into the hands of students. Both information and communications aspects of ICT were featured, since both these areas have seen significant development since the last ICME. Recent developments include a widening of who has access to ICT (although this remains a source of great inequity between and within countries), a greater sophistication and range of mathematical tools and imaginative teaching devices and the growth of internet resources and communication. The descriptions below indicate in broad terms the major concerns and interests of the sessions.

Introducing ICT: Experiences and issues

This session, chaired by *Kaye Stacey*, drew together experiences from around the world on productive ways to introduce technology in school systems. From practical and theoretical viewpoints, it addressed issues such as equity and teacher training and the nature of systemic support needed to make the introduction of ICT positive for school systems. Sharing experiences from many countries, and appreciating the differences, was a major outcome of the session. There are common problems but they are experienced in different ways and to differing extents. *Wong Khoon Yoong* (Singapore) described Singapore as an example of a technologically rich learning environment. Schools are well equipped, official support to use ICT is strong and there is adequate in-service training. Wong attributed the lower-than-expected uptake of technology use in classrooms to the pressure to prepare pupils for public examinations.

Some countries have addressed this pressure by exploring how the public examination system may be changed so that the assessment itself promotes technology use. *Peter Flynn* (Australia) provided an example of this. From 2002, student use of CAS calculators has been permitted in some high-stakes assessment. Assessment cannot be left unchanged when the learning environment is different, but the changes require mathematical, pedagogical and ethical considerations.

There were three presentations from countries where the economic circumstances severely restrict access to technology. *Luckson Kaino* (Botswana) reported on ICT availability and utilization in Botswana primary and secondary schools, with insights for other developing countries. *Yuriko Yamamoto* (Brazil) discussed introducing hand-held technology in mathematics classrooms of basic schools in Brazil, a developing country with many economic and social problems. The main challenges are to guide teachers to discover the didactical potential of technology and to link previous content knowledge with technology-appropriate activities. *Louise De Las Peñas* (Philippines) explained how, since the late 1990s, technology has slowly been introduced to mathematics teaching in universities and schools and nationwide teacher training and outreach programs have been conducted. She identified the main challenges confronting teachers in the Philippines as equipment, learning environment and curriculum.

Whilst the above presentations served to highlight inequities between countries, *Penelope Dunham* (USA) reminded us that inequities are also evident within systems. Inequities arising from differential access to and use of educational technology for groups characterized by gender, race/ethnicity, or social/economic class can limit the impact of ICT. She suggested public policies and pedagogies that may remove the boundaries between technology "haves" and "have-nots".

Showcase surveys

Three lecture hall sessions and most of the workshops showcased exciting new possibilities for teaching and learning with ICT. *Robyn Pierce* (Australia) chaired the session entitled "Advances in undergraduate education with ICT". *Neil Challis* (UK) gave examples to illustrate that we must encompass symbolic, graphic and numeric thinking, and he emphasised that "doing mathematics" includes the whole problem solving process from its initial source to solution and appropriate communication of conclusions. Robyn Pierce gave examples from her teaching of both mathematics and statistics to demonstrate how technology can support and enhance the learning of diverse student cohorts, by promoting better understanding of concepts and providing access to real world problems. *Jack Bookman* (USA) demonstrated how students negotiate roles and meaning as they learn in technologically rich environments and compared active learning in technologically-rich and pencil and paper environments. *Paul Igodt* (Belgium) demonstrated a web-platform and database architecture for multiple choice problems and course-specific tests which allows sharing of exercises between teachers and courses. By using so-called `learning objects meta-data'', the high effort requested from authors of exercises gets a beneficial return in ease of sharing and ease of reusing over time.

The session "Teaching primary and junior secondary mathematics with ICT: Changing pedagogy and learning", chaired by *Kaye Stacey* (Australia) also showcased mathematics and statistics examples. Kaye Stacey presented a theoretical overview of how technology in classrooms can operate to increase engagement and achievement of students, in particular by bringing real world problems to life in the classroom, and she gave examples of doing this with digital and video images. *Douglas Butler* (UK) showed how a teacher can use one computer in a classroom to enliven demonstrations and problem solving. Use of dynamic images can play a crucial role in inspiring pupils to want to be more successful, and therefore to want to take the subject on at a higher level. *Tim Erikson* (USA) used statistics software to capture and analyze data from the internet. By choosing the data and the context carefully, we help students make use of the mathematics they are learning already, to accomplish something of genuine interest to them. *Jenni Way* (Australia) demonstrated a new suite of digital learning objects for the first years of school, to be delivered to every school in Australia and New Zealand on demand.

Ricardo Nemirovsky (USA) chaired a session that showcased learning mathematics with physical phenomena and involving kinaesthetic, bodily experiences. *Ornella Robutti* (Italy) reported on teaching experiments logging body motion with sensors and calculators from kindergarten to secondary school. *Karen Marrongelle* (USA) reported on investigations of the interface between students' understandings of the integral and subsequent performance on physics problems. The use of the integral in physics is not simply the application of a mathematical technique to numerically solve a problem but needs to be reinterpreted for each problem situation. *Michal Yerushalmy* and *Beba Shternberg* (Israel) showed software to develop the concept of function from physical experiences with technology. *Apolinario Barros* (USA) reported how kinaesthetic activities with a 2D motion detector can assist students to understand the relationship between sine and cosine.

Two workshops showcased classroom activities using dynamic geometry packages. *Sophie Soury-Lavergne* (France) explored new types of tasks made possible by dynamic geometry, emphasising the new ways in which mathematical properties appear to the learner when the new tool is used. *Nicholas Jackiew* (USA) extended the use of dynamic geometry from investigations of shape to number patterns and elementary number theory, fractions and early algebraic reasoning.

The algebra theme

In recent years, algebra teaching has been significantly influenced by technology, so four lecture hall sessions and three workshops specifically addressed this. The influence is due to the way in which software gives students access to symbolic, numerical and graphical representations of algebraic ideas. It is also due to the fact that teaching and curriculum has to adjust to a technological environment where many of the calculation aspects of algebra can be taken over by technology, although this needs to be handled carefully to get the best for learning.

Jean-Baptiste Lagrange (France) chaired a double session, which reviewed a range of technologies that can enhance understanding of algebraic ideas and track students' progress. *Carolyn Kieran* (Canada) described research on algebra learning and teaching that has been carried out in various technological environments, with either multiple representations, dynamic control, or structured symbolic calculation. The duality of algebra with its multi-representational functional approaches on the one hand, and symbol-based manipulation perspectives on the other, provided a framework. *John Olive* (USA) showed how to connect motion, geometry and algebra using dynamic geometry and simulation software. *Jean-Baptiste Lagrange* (France) considered the place of classroom situations involving experimental approaches and algebraic treatment of functions with the help of technology. Although the use of CAS seemed promising, difficulties were experienced and curricula now tend to privilege approaches to functions with non-symbolic software. *Brigitte Grugeon* and *Elisabeth Delozanne* (France) presented prototypes of software for diagnosing students' competencies and building cognitive profiles for algebra, which is based on an artificial intelligence approach. *Rosamund Sutherland* (UK) focussed on using spreadsheets for enhancing the learning of algebra. She identified a gap between students' idiosyncratic approaches to solving problems and a more socially accepted school algebra approach, and explained why the use of ICT could exacerbate this problem. *Alain Bronner* (France) reviewed various uses of the "Aplusix" software for learning to solve equations or systems of equations. He analysed the role of the various interactions with the technological environment in the evolution of the concept of equation and strategic knowledge of the students to solve equations.

"The teacher and the tool", chaired by *Paul Drijvers* (The Netherlands), presented findings from three research studies which have examined how teachers adapt to using CAS in their classrooms. *Lynda Ball* (Australia) asked what "assessment" looks like in a CAS classroom. She explored the ability of students to communicate solutions when intermediate steps were assigned to CAS and how to help students communicate well. *Bärbel Barzel* (Germany) noted that the integration of CAS in teaching mathematics leads quite often to a change from classical instruction to a more constructivist approach. Both of these changes, integrating CAS and opening up learning, can be challenging for teachers. *Rose Mary Zbiek* (USA) offered several insights into the complexity and promise of teaching with CAS. Examples of classroom-teaching events lead quickly to factors that impinge on teaching with the technology. She described how the multi-faceted relationship between and beyond the teacher and tool emerges from mere acquaintance to effective partnership.

In a second session, entitled "Instrumentation and CAS" two views were presented on the process by which teachers and students come to be skilled users of computer algebra technology. *Paul Drijvers* (The Netherlands) began by distinguishing an artifact from an instrument and describing the process of instrumental genesis by which the transition is made. This instrumental genesis includes the development of mental schemes for using the artifact for the target activity. In such schemes, technical and conceptual aspects interact. Then *Luc Trouche* (France) provided evidence to show that the more complex the environment, the more diverse the students' work methods, and, consequently, the more necessary the 'orchestration' of the teacher in order to assist instrumental genesis.

Two workshops gave participants first-hand experience of integrated mathematical environments with substantial mathematical power. *Renée Gossez* (Belgium) gave an introduction to the power of CAS in teaching for school years 9 and above. She examined use of capacities such as the automatic updating of documents, the use of sliders to change the values of parameters, the capacity to pass worksheets between the students and teacher, and the substantial mathematical calculation capacities. *Steen Grode* (Denmark) demonstrated teaching mathematics with "Mathcad" and "Smartsketch".

Together these make an integrated environment for performing and communicating math-related work, which has been trialled in Danish schools.

Internet resources for teaching mathematics

Although use of the internet featured in many sessions, *Shoichiro Machida* (Japan) chaired a session which specifically presented internet resources from around the world for students at all levels of education and their teachers. *David A. Thomas* (USA) observed that modelling technologies are empowering students to formulate, test, and support their own mathematical conjectures. Integrated mathematical modelling and web-based communication technologies can help achieve better mathematical dialogue. *Akimichi Tanaka* (Japan) demonstrated a tool called "linkWorks," which helps learners search for information related to given subjects on the Internet and collaborate with each other. Students using the tool learned actively and collaboration among them raised their learning motivation. *Vincent Jonker* and *Frans van Galen* (The Netherlands) demonstrated the "RekenWeb", a website providing mathematics internet games for primary education and printable activity sheets for teachers, which provides many opportunities to support teachers for their daily lessons. *David A. Smith* (USA) described the Journal of Online Mathematics and its Applications (JOMA). JOMA contains articles, modular learning materials, reviews, "mathlets", and a Developers' Area for assistance in creating online materials.

One workshop showcased small stand-alone software programs called applets, to use across the internet. Drawing their examples from teaching introductory algebra, *Peter Boon* and *Martin van Reeuwijk* (The Netherlands) illustrated model applets for concept development and practice applets to reinforce skills.

Distance learning

Shoichiro Machida (Japan) also chaired a session on developments in distance learning; an area which has undergone rapid change in recent years. Machida reported on a digital learning environment for supporting teachers to encourage students' self-directed learning in the mathematical classroom. Hypermedia mathematical textbooks, called e-subtextbooks, included a section that is regenerated by teachers themselves every lesson. Teachers' group collaboration was supported through a mailing list. *Shuhua An* (USA) reported on teaching mathematics methods for pre-service teachers by an integrated hybrid course, combining the best of traditional and on-line teaching. It was particularly appropriate for independent, focused, and goal-oriented students. *Lyn Leventhall* (UK) reviewed collaborative teaching resources on the web and "web ready" software using the underlying technology "MathML" for displaying mathematical equations. *Masami Isoda* (Japan) reported on a different style of distance collaboration, which involved students from different countries collaborating to solve mathematical problems via regular e-mail exchange. The projects illustrated that the major significance of communication between countries is cultural awareness in mathematics. Mathematics is a communication tool and developing communication ability is an important aim in mathematics education.

Video-based technologies in teacher education

In recent years, there have been significant changes in the ease of creating, storing and accessing video-based information. Video is now a highly practical, as well as extremely rich, data collection tool. This has had a major impact on teacher professional development, pre-service education and also on research. *Ricardo Nemirovsky* (USA) designed and chaired the session. *David Clarke* (Australia) discussed the use of video material for mathematics teacher education in a climate of standards-based reform. Professional standards have the potential to communicate the findings of research and the wisdom of practice in the most practical manner, but also have the potential to become prescriptive and constraining. *Chronis Kynigos* (Greece) illustrated how teachers' epistemology and perceptions of teaching and learning mathematics were challenged through their interactions with exploratory software during a professional development course. He believes that it is important to perceive technology as a medium for the empowerment of teachers, rather than just a powerful tool for students. *Robert Tinker* (USA) reviewed the provision of online teacher professional development, noting that it has great potential but has been marred by poorly designed and executed courses. He described an on-line course for algebra teachers using video case studies and software tools delivered during the school year, and presented evaluation results.

The workshop with a focus on teacher education, offered by *Federica Olivero* (UK) and *Dan Cogan-Drew* (USA) described how self-study projects using the "VideoPaper Builder" software can be used to teach mathematics pre-service teachers to reflect on their practice. This is directed use of video for self-study, not for use by others. The workshop discussed progress on transforming videopaper creation into a new teacher education model.

Conclusion

Overall, the field is characterised by rapid change, as new products and possibilities become accessible to more people. Educational responses are strong, both to mould new opportunities to improve learning at all levels, including teacher learning, and to refine pedagogical practices to strengthen the value of technology in teaching.

This report has been written by Kaye Stacey and Paul Drijvers. They will be happy to be contacted at the University of Melbourne k.stacey@unimelb.edu.au and The Freudenthal Institute p.drijvers@fi.uu.nl for further information on the work of this Thematic Afternoon.

TA E: Perspectives on research in mathematics education from other disciplines

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Introduction

The thematic afternoon on perspectives on research in mathematics education from other disciplines provided an opportunity to focus on the contributions of psychology, cognitive science, philosophy, sociology, anthropology and general education to research in mathematics education. Such contributions include theories, issues, problems, concepts, methodologies, studies, and results that are of significance to the international mathematics education research community. An overview of such contributions was complemented by accounts of specific research projects incorporating such elements from outside disciplines. These external references balance the traditional attention to 'homegrown' (versus 'imported') theories (and by extension, concepts, etc) in mathematics education research. Here we were explicitly focussing on the role of such intellectual imports and appropriations, and illustrating them with exemplary mathematics education research projects, as well considering the overall role of external disciplines in our work.

In addition to the Team Chairs *Brent Davis* and *Paul Ernest* there were three further plenary speakers: *Tommy Dreyfus*, Tel Aviv University, Israel; *Christine Keitel-Kreidt*, Free University of Berlin, Germany; *Robyn Zevenbergen*, Griffith University, Australia.

Summary of Strand 1: The perspectives of psychology and cognitive science in research in mathematics education

"Psychology and cognitive science" sweep across neuroscience, cellular biology, developmental psychology, linguistics, and cultural anthropology – to name only the few disciplines that were explicitly invoked in Strand 1 presentations during the Thematic Afternoon.

The fact that these topics should be included among so many others during an afternoon devoted to the exploration of other domains highlights how things have changed within mathematics education research over recent decades. There was a time that this field looked almost like a subset of psychological research. In fact, the learning of mathematics is still a favourite phenomenon of study among cognitive psychologists. But psychology is no longer such a favourite domain of inquiry among mathematics education researchers.

Each of the contributors to Strand 1 looked at a different and quite distinct phenomenon, as one might expect given the very different discourses that frame their work. That said, however, despite the clear differences in objects of interest, there were some striking and provocative similarities in the manners of description offered. By way of a conceptual organizer, and as became very apparent through the course of the presentations, the phenomena of interest in Strand 1 seem to be nested in one another, beginning with subpersonal phenomena, and moving through personal, interpersonal, and transpersonal. So framed, and despite at least one major tension that arose, it was clear that there are deep complementarities among the discourses invoked. They need not be treated as competing, or even disparate fields, but as overlapping and intertwining areas of inquiry that might better be considered in terms of their collective contribution to mathematics education research than in terms of their particular foci.

For example, the first presenter, *Daniel Ansari* of Dartmouth College (United States), started the strand presentations with a review of current research in cognitive neuroscience into children's development of numerical and mathematical skills. Daniel argued that early developing approximate number skills contribute to the gradual development of exact number representations and that these systems are represented differentially in the adult brain. Among the consequences of this research, Daniel argued that the findings urge educators to place greater emphasis on early education of basic numerical skills and how they point to the importance of basic quantity understanding.

Daniel's topic, while focused on the subpersonal, clearly pressed into the space of personal understanding, which is where *Willy Mwakapenda* of the University of Witwatersrand (South Africa) located his presentation. Willy focused on concept mapping, which he offered as a methodology for researching student understanding. Through a series of examples, Willy argued that students' understanding of concepts is highly related to the contexts and experiences in which they learnt mathematics – a conclusion that pressed the discussion into the space of the interpersonal.

That was the principal site of the third presentation, from *Joyce Mgombelo* of Brock University (Canada). Joyce argued for the significance of Lacanian psychoanalysis for mathematics education research, contending that Lacan's (1977) distinction of objective knowledge and knowledge-as-enjoyment presents a significant reframing of questions of knowing, knowledge, and experience. Focusing her interpretations with the 19th-century writings of Mary Boole, Joyce looked at the relationship of the mathematics teacher and her or his students.

Thomas E. Kieren, University of Alberta (Canada), moved us from the space of the interpersonal to the transpersonal. Specifically, Tom's contribution was concerned about "conversations" among fields of inquiry. He argued that the influence of one domain on another can never by unidirectional – that is, that the mathematics educator not only takes on, but necessarily transforms ideas developed elsewhere, which in turn presents the potential for the changing of the ideas from the home field as well.

Part of the character of such interdisciplinary conversations was powerfully illustrated in the final presentation of Strand 1. *Terezinha Nuñes* of Oxford Brookes University (United Kingdom) drew on developmental psychology to critique the use of any single discourse, in particular cognitive neuroscience, to make sense of human competencies that arise in and unfold through the interweaving of biological processes and the invisible symbolic web of culture. Such competencies, she argued forcefully, must be studied in their wholeness, neither as reducible to subpersonal processes nor as by products of more global processes.

Summary of Strand 2:
The perspective of philosophy in research in mathematics education

What might the perspective of philosophy in research in mathematics education mean? Philosophy involves a critical examination of fundamental problems and assumptions, and systematic analysis, reasoning, judgement, resulting in conclusions, knowledge and beliefs. However, these are what we expect of good research in mathematics education too, although philosophical enquiry may be more thorough, or may pay more attention to the process of conceptual clarification itself. The unique contribution of philosophy is the use of the substantive concepts, theories and results of past philosophical enquiry.

Research problems in mathematics education are typically multi-faceted and require an awareness of the complexity of the teaching and learning of mathematics and the surrounding social context. So philosophy cannot usually be applied directly to solve such problems. *Anna Sierpinska*'s (Concordia University, Canada) contribution "The philosophical perspective in mathematics education" pointed out the danger that philosophy may end up leading to generalities, and understanding nothing in particular rather than understanding specific mathematical concepts.

Where most successful in applying philosophy, researchers in mathematics education draw upon philosophical theories and concepts as resources to help clarify research problems and their conceptual frameworks. Typically they start with a problem in mathematics education research, then search for resources drawing on relevant philosophical concepts and theories before importing and adapting them in constructing and clarifying a conceptual framework for their research. The following are brief sketches of sample uses of philosophy in maths education

Philosophy of mathematics. It is well known that there are different (and contrasting) philosophies of mathematics. 'Postmodern' developments in the philosophy of mathematics have been concerned with mathematical practice and what mathematicians do to create (and justify) new mathematical knowledge (e.g., Lakatos, Davis & Hersh, Tymoczko)

These developments have been a useful resource for mathematics education researchers wanting

1. To give a dynamic and humanistic account of mathematics
2. To find philosophy of mathematics compatible with problem solving in the classroom
3. To research the processes of doing maths. *Leone Burton*'s (University of Birmingham, United Kingdom) contribution in this area was "Mind the gap" and was about her work in exploring knowledge and knowing, epistemology and pedagogy, in mathematicians' practices.
4. To find a philosophy compatible with multiculturalism and ethnomathematics. *Bill Barton*'s (University of Auckland, New Zealand) contribution in this area was "Culture and mathematics" and explored philosophical perspectives concerning anthropology and ethnomathematics and mathematics education.

Personal knowledge and knowing. Researchers have been concerned with what it means to know mathematics and different forms of knowledge. One important distinction due to Gilbert Ryle is between 'knowing that' and 'knowing how'. A number of philosophers and researchers in mathematics education have made the distinction between explicit vs. tacit forms of knowledge, including: Polanyi, Kuhn, Wittgenstein, Skemp, Mellin-Olsen, Hiebert *et al.*, Kitcher, and Ernest. Several of these, especially the last two, have proposed multi-dimensional models of mathematical knowledge encompassing such distinctions. *Cristina Frade* (Universidade Federal de Minas Gerais, Brazil) drew upon several of these latter authors in reporting her investigations of the tacit-explicit dimension of the learning of mathematics.

Research methodology and paradigms. The philosophy of science, especially the work of Popper and Kuhn, has been influential in educational research, especially in the scientific research paradigm. *Gerald Goldin* (Rutgers University, USA) made his contribution entitled "Toward reproducibility and generalisability" in this area, where he offered perspectives on mathematics education research from the philosophy of science.

In addition to this important area of philosophical influence, thinkers including Weber, Schutz and Habermas have contributed much to the philosophy of the social sciences underpinning the interpretative research paradigm, sometimes called the qualitative paradigm. Habermas and the Frankfurt school have also led to the foundation of the Critical-Theoretic research paradigm in education research. Thus philosophy has been especially important in the area of educational research methodologies.

Although these areas included all the individual contributions to the philosophical strand *Paul Ernest* also indicated a number of other important areas of influence and controversy. These included the following.

Theories of learning. There are different philosophical traditions underlying theories of learning. Empiricist theories stem from Locke, Hume and Mill. Constructivist theories can be traced back to Kant and Piaget. Social theories are more recent, and can be found in Mead, Wittgenstein, and Vygotsky. Heated controversies over theories of learning mathematics still abound, with empiricism, cognitivism, radical constructivism, enactivism and embodied cognition, social constructivism and socio-cultural theories of learning still slugging it out. As Ernst von Glasersfeld (1983) said, to introduce epistemological considerations into a discussion of education has always been dynamite.

Ethics, values, feminist theory. Research on gender and mathematics education has been strongly influenced by philosophical theories of moral and epistemological development. The strongest inputs have been from Gilligan (1982), and Belenky *et al.* (1986) distinguishing between separated and connected values, separated and connected knowing. However, not all of the potential for growth in mathematics education research based on feminist theories has been realized yet, in *Paul Ernest*'s opinion.

Several other areas of philosophy have the potential to further influence and contribute to research in mathematics education. For example:

1. Philosophy of biology – this is important for enactivism and embodied learning.
2. Postmodernism, post-structuralism and political theory – these can contribute much on researching power and the social impact / context of mathematics education.

3. Philosophy of language – the interpretation of discourse is an increasingly important dimension of mathematics education research.
4. Hermeneutics – has likewise much to offer on textual interpretation.
5. Semiotics – can further contribute to a deeper understanding of the sign systems of mathematics and mathematics education.

Summary of Strand 3:
The perspective of the social in research in mathematics education (incorporating sociology, anthropology and general education)

The contributions introduced by *Robyn Zevenbergen* were the following.

1. *Tine Wedege* (Malmö University, Sweden): "Import and reconstruction of concepts: the social dimension of mathematical knowledge"
2. *David Wagner* (University of Alberta, Canada): "New directions for analyzing mathematics classroom discourse"
3. *Paula Ensor* (University of Cape Town, Republic of South Africa): "Sociological perspectives on research and practice in mathematics education"
4. *Derek Woodrow* and *Janis Jarvis* (Manchester Metropolitan University, UK): "Learning preferences of mathematics students compared to students of other subjects"
5. *Ubiratan D'Ambrosio* (Unicamp, São Paulo, Brazil): "Is integrating science and mathematics a promising option?"

In the final hour the three strands were brought together in a shared plenary session chaired by *Christine Keitel-Kreidt*. In addition to brief comments from the three strands, a presentation was made by *Tommy Dreyfus* addressing "The power of homegrown theories in the discipline of mathematics education." His presentation serves as an important cautionary tale that encourages us to look elsewhere for ideas when researching in mathematics education, but reminds us not to be overly committed to or swayed by any singular domain, or to lose sight of the complexity of the phenomenon at hand.

It was a rewarding if densely packed afternoon. Naturally there are problems associated with an afternoon conference-in-a-conference that is about every other field of inquiry except the one that serves as the focus for the rest of the conference. In addition there are many other fields that might – and, in fact, do – inform work in mathematics education research, and that were, by necessity, ignored here. Every field that we did manage to address has a terrain that is as varied and as contested as that of mathematics education research.

One of the things that came through pretty powerfully was that we need to be mindful of the discourses that we draw on, and in particular, the subpersonal, personal, interpersonal, and transpersonal consequences of what we bring together.

One final concluding thought concerns where discourses are focused. Educational research has to be educational – it has pragmatic concerns that other domains of inquiry do not have. For that reason, drawing from domains in which discussions can be described to be mainly descriptive in character, rather than pragmatic, means that we are borrowing ideas that are not educational – that is, they are not framed, by the educators' pragmatic concerns. And if we are not careful with that, history shows that problems will arise.

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ICME 10 2004

TSG

Topic Study Group 1

TSG 1: New development and trends in mathematics education at pre-school and primary level

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Team Member: ***Marit Johnsen-Høines***, Bergen University College, Landaas, Norway

Aims and focus

TSG 1 had a broad focus, calling for papers that examined contemporary developments in mathematics education at the preschool and primary level (approximately ages 0 through 12 years). This provided an exciting opportunity to explore issues across the early childhood and school sectors and the team quickly decided to integrate early childhood and school presenters, with sessions organized around common themes, rather than by sector. The range of nationalities of the presenters and the participants in this group also allowed the ideas to be considered as they applied in a number of different contexts. This report highlights the themes that were considered, describes the papers that were presented, and summarises some of the key ideas and issues that arose in the group's discussions.

Key themes

In their chapters on preschool and primary mathematics education in the *Handbook of International Research in Mathematics Education* (English, 2002), Perry and Dockett and Jones, Langrall, Thornton, and Nisbet raise issues about access to powerful mathematical ideas and new mathematical ideas for this age range. These authors are calling for policy makers, curriculum developers and educators to ensure that *all* children in preschool and primary school learn the kind of mathematics that will begin to nurture their lifelong mathematical thinking. Moreover, they see these powerful mathematical ideas extending what has been traditionally thought of as "elementary mathematics" and incorporating new strands such as geometrical reasoning, algebraic thinking, data and chance.

This strong thrust in mathematics education research flowed over into the presentations and discussions in TSG 1 where powerful and new mathematical ideas, along with problem solving, were again key themes. These ideas, and related curriculum considerations, gave rise to a fourth theme on teacher education and development.

Paper presentations and discussions

The TSG began with an introduction by one of the Chairs, *Graham Jones*, followed by a keynote presentation by *Carole Greenes*, from Boston University USA. Greenes outlined the details of a mathematics curriculum for preschool and kindergarten called *Big math for little kids*. The programme capitalizes on a number of key elements: children's knowledge and interests, highlighting the mathematics in routine classroom activities, organizing instruction in which activities are sequenced, incorporating complex mathematical ideas, emphasising mathematical language development and promoting "thinking like a mathematician" (Greenes, 2004, p. 5). Assessment processes were still being developed, but observations of the programme led Greenes to conclude that there had been benefits for children and teachers during its implementation with children showing "some remarkable student competencies" (p. 12).

Papers during the second and third sessions were organized into the four main themes. The issues that arose within each theme are identified here as a series of questions. In the first theme ***powerful mathematical ideas***, *Yukio Sugawara* explained how mathematics lessons and curriculum standards have been developed to foster children's mathematical thinking in a Japanese study. *Jill Waters* and *Lyn English* described mathematical patterning in two early childhood settings in Australia. They discussed the importance of mathematical patterning, and the lack of current research in this field. *Margaret Curry*, *Michael Mitchelmore* and *Lynne Outhred* examined Australian Grade 1-4 children's understanding of length, area and volume and the relationship between them. Their intention was to explore the relationship between the learning of measurement in the three domains, and to gain an understanding of how curriculum in this area could be sequenced. *Christina Misailidou* and *Julian Williams* looked at improving English students' performance on 'ratio' tasks. For students who use an additive approach, several strategies such as working on shared context tasks and using a pictorial representation of the problem, assisted in moving students towards multiplicative thinking.

Issues

- *Patterning* (mathematical reasoning where children recognise or build an arrangement of shapes or numbers that repeat or change in a predictable way, for example, 3, 9, 27) was not only a powerful mathematical idea, it was seen to be fundamental to children's mathematical development. Why is there a scarcity of research on children's development of patterning skills? How do we develop teachers' knowledge of mathematical patterning and their understanding of children's knowledge of patterning?
- Understanding of *length*, *area*, and *volume*, and the relationship between these mathematical ideas was of key importance for young children. How do we develop concepts like unit, unit iteration, and the relation between measure and unit size? How do we use research to assist teachers to foster the notion of unit structure (the pattern formed when the units fill the object to be measured)? How do we enable students to deal with the increasing complexity of the unit structure as we move from length to area to volume?
- The power of *proportional reasoning* is critical to children's mathematical thinking. How do we assist children to move from additive strategies to multiplicative thinking? How do factors like the following facilitate this movement from additive to multiplicative thinking: a sharing context task? pictorial models? grouping strategies?
- In facilitating the learning of all these powerful mathematical ideas there was recognition of the importance of *knowledge-creating* type lessons. How do teachers create or locate tasks to foster this knowledge creation? How can extensions to these tasks be developed? How do we get children to express and discuss their ideas during knowledge creation?

The second theme, ***New mathematical ideas for the early years*** included two presenters from the USA who had been exploring children's algebraic thinking, and when this might be introduced into the curriculum. *Sue Brown* focused on children's algebraic thinking in kindergarten through to Grade 2, and described a number of activities that allowed children to work with arithmetic sequences and equations. *Zhonge Wu* used teaching experiments with older (5th grade) students "to encourage students to participate in algebraic reasoning and justification for patterning problem situations" (p. 3). This theme also included a paper by *Chrisanthi Skoumpourdi* on probability as a new trend in Greek primary education, and *František Kuřina* from the Czech Republic argued for the importance of geometry in primary school.

Issues

- *Algebraic* thinking (including recognising and building geometric and number patterns, identifying and applying relationships to make predictions, and making and explaining generalizations) is a new strand in elementary mathematics but we are only beginning to understand its potential for young children. What does research tell us about the value of algebraic thinking for young children? How do we encourage children to discover, describe, and develop algebraic patterns? How do teachers organize activities in algebraic thinking that build on children's existing knowledge? How do we enable students to pose their own problems/patterns in algebraic thinking?
- Learning *probability*, which explores and measures the likelihood of random events occurring, is a new experience for young children and their teachers. Why is the study of probability appropriate for the preschool and primary school curriculum? How do we convince teachers that probability is an important learning area for young children? What contexts provide useful learning experiences in probability? (Although the group focused on probability, many of the same issues arise in relation to data analysis).
- *Geometry* is an effective source for young children's mathematical investigations. What is the nature of challenging and open-ended geometrical investigations for young children? What kind of learning environments work best for geometrical investigations with young children? What kinds of professional development for teachers would facilitate challenging investigations in geometry for young children?

Problem solving was the focus of *Tom Lowrie*'s and *Noor Azlan Ahmad Zanzali*'s work. Tom Lowrie examined the influence of cultural artifacts (brochures, menus and bus timetables from a theme park) on Grade 5 Australian children's problematising of problem scenarios (e.g. formulating a budget or constructing a timetable for a family at a theme park). Working with children of a similar age, Noor Azlan Ahmad Zanzali from Malaysia examined Year 5 children's problem-posing abilities based on three different stimuli. He recommended that children should engage in both the posing and solving of problems.

Issues

Problem solving can be a method for creating mathematical engagement and for developing mathematical meaning in young children. How do we use problem solving to achieve an appropriate balance between conceptual knowledge and procedural knowledge? What is the role and value of contexts/ authentic artefacts in problem solving? How does problem solving enable us to make connections among mathematical ideas? How do we use children's voices/experiences in problem solving? What strategies can teachers use to help students gain ownership of problem solving tasks?

The final theme brought together presenters who were exploring ***Trends in teacher education*** at the early childhood and primary school levels. *Shiree Babbington* and *Gregor Lomas* had developed a video, *The magic of mathematics in the early years,* for use with early childhood education students at Auckland College of Education (New Zealand). The video highlights the mathematics in a range of examples of young children's play. *Kwok-cheung Cheung* reported on his work with teachers attending an in-service programme at Macao University. Examples were given of how Gardner's multiple intelligences could be used as a base for planning mathematics teaching in kindergartens. Finally, *Saulius Zybartas* and *Allan Tarp* from Denmark described and illustrated a post-modern approach to elementary mathematics that regards mathematical concepts as culturally constructed names for social practices.

Issues

Teacher education is of prime importance in the development and implementation of curriculum programs in mathematics for early childhood and elementary children. Effective change in curriculum and instruction in mathematics is dependent on the nature of partnerships between policy makers, educators, teachers, parents, and children. Why is it important for numeracy to be linked to notions of context and a sense of holistic learning within cultures? How can teacher education and teacher development programs address the diversity of theoretical perspectives faced by schools and teachers?

One additional paper by *Jenny Young-Loveridge* and *Sally Peters* was presented by distribution on the TSG 1 website. This considered mathematics teaching and learning in early childhood and early school and provided a chronology of events in this area in New Zealand since the early 1990s.

TSG 1 concluded with a plenary session. This comprised two presentations, reports from the paper presentations over the previous two days, and a general discussion of issues arising from the work of the Topic Study Group. The first plenary presentation was by *Mike Askew* from King's College, University of London. His paper shared findings from two large UK studies. The first demonstrated the importance of relationships in the teaching–learning process and explored the connections among child, teacher and the learning of mathematics. The children whose teachers were able to connect with mathematics, and to the children's knowledge, made the greatest gains. The second study confirmed this finding and showed that the peer group was also an important factor in these connections. For example, some children were driven by the dynamics of working with the group.

The final presentation was by Sally Peters from the University of Waikato. Her paper considered the New Zealand situation, where early childhood and school have different histories, curricula and pedagogy. An increased focus on literacy and numeracy in the early school years, with assessment against specific frameworks and levels, contrasted with the integrated and holistic approach of the early childhood curriculum. These differences had led to considerable debate about how connections can be made between children's mathematics learning in early childhood and at school. The paper explored these issues and discussed how teacher awareness of opportunities for mathematical learning in everyday activities, contextualised narrative assessments, and views of progression that included the range of contexts in which mathematics was used, learning dispositions, and mathematical complexity, could all assist in 'crossing the border' between the sectors.

Conclusion

In the opening address and throughout the ICME-10 conference, there were many references to the beauty and power of mathematics and the need for students at all levels to experience these characteristics of mathematics. TSG 1 focused on students in the early childhood and primary years and there was strong support amongst presenters and participants for rich mathematical experiences, which introduced young children to a broad range of powerful mathematical ideas. Knowledge, skills and attitudes were all considered to be important. In the discussions many insightful questions and responses were raised about appropriate pedagogy for fostering young children's mathematical development. Notwithstanding the many fruitful examples of activities and ideas that were considered, the *Issues* noted above indicate possible directions for further research and development to ensure that all children in the early childhood and primary years gain access to the beauty and the power of mathematics.

Reference

English, L. (Ed.) (2002). *Handbook of international research in mathematics education*. Mahwah, NJ: Lawrence Erlbaum.

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ICME 10 2004

TSG

Topic Study Group 2

TSG 2: New developments and trends in mathematics education at secondary level

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Athanasios Gagatsis, University of Cyrpus, Nicosia, Cyprus
Elaine Simmt, University of Alberta, Edmonton, Canada

Introduction

TSG 2 aimed at discussing and sharing opinions, experiences and research results within the ICME Community related to this broad theme.

Several movements characterized secondary mathematics education during the past decades. Most of them are deeply related to changing societies and technological worlds and at the same time, they are often inspired by the results of leading research in mathematics education. There is much diversity in mathematics education research depending on communities and academic societies in the world, but the common aim of mathematics education research has been to improve curricula, teachers' practices, students' learning, assessment, and teachers' education.

There are several trends and projects in the world that represent the reform of mathematics education at the secondary level. These include policy, curriculum or textbook development research; developing teaching practices based on classroom research such as lesson studies and the development of teaching-learning environments for mathematics using new technologies; and the results and the impact of international comparative studies.

TSG 2 focused on future movements in mathematics education at secondary level and exemplarily illustrated these movements by presentations on:

1. Research projects for curriculum development having the potential to influence mathematics education in the next decades;
2. Policies of secondary schools' reforms having the potential to generate new trends in secondary mathematics education;
3. Developmental studies of teaching new contents in mathematics;
4. Developmental studies of new ways of teaching mathematics;
5. Influential research results in mathematics education for the secondary school level.

In the first session of TSG 2, internationally known specialists presented their ideas on three main issues related to the central theme of TSG 2, namely the impact of new technologies, curricular developments, and the role and results of international comparative studies. In the second and third session, papers were grouped around two themes. The first, *Curricular Developments and New Contents* provided our community with a description of current events in different parts of the world, and the second, *Learning from Research and Classroom Practice*, provided us with some of the means to critically reflect on the new trends. In the fourth and final part, the chairs and members of the Organising Team presented their personal opinions and reflections on *New Development and Trends in Secondary Education* and on the different papers that were presented in the first three sessions.

Keynote presentations

Paul Drijvers (Freudenthal Institute, the Netherlands) in his presentation addressed the integration of technology in mathematics education at secondary level. From a retrospective point of view, he argued that initial pedagogical approaches sometimes turned out to be too optimistic and too straightforward. Furthermore, important questions concerning the influence on the curriculum, the relation between paper-and-pencil work and technology use, the need for traditional practice and the role of the teacher did not receive satisfactory answers. This brought the presenter to the "state-of-the-art" concerning the integration of technology in mathematics education. He observed positive trends on the research and development front (technology is no longer the concern of a group of "enthusiasts" with little communication with research in mathematics education in general), but also with respect to the practical conditions, at least in the Netherlands. The original optimism concerning the influence of technology on the learning of mathematics has changed so as to become more nuanced ways. By now, it is fairly widely accepted that skills and understanding cannot be separated, and that machine techniques and mental concepts are related. Looking into the future, infrastructural arrangements, adequate research, curriculum development and teacher training were identified and discussed as critical factors affecting a productive integration of ICT in mathematics education.

In a second keynote presentation, *Florence Glanfield* (University of Saskatchewan, Canada) discussed curricular developments from a Canadian perspective. She first described the current trends and the backgrounds of these trends. Across Canada, mathematics curricula now include a focus on problem solving, applications, the development of mathematical concepts from a concrete approach, the design of programs intended for those students who are university-bound but not going into science and mathematics, and the integration of computer technology. According to the presenter, this last aspect is strongly related to the idea of "humanizing" mathematics: rather than mathematical work being accessible only to those patient and diligent enough to develop the many procedures for calculation needed for completing the foregoing tasks, technology may widen access to all pupils. She believed that the trend to humanize mathematics for all students and teachers will continue to prevail in Canada.

Ross Turner (Australian Council for Educational Research) focused on the impact of international comparative studies on mathematics education, more specifically on the role of PISA (OECD's Program for International Student Assessment). The presenter outlined some of the objectives and features of PISA, comparisons with TIMSS were briefly discussed, some results and outcomes were presented regarding the impact of PISA on mathematics education, and some limitations of PISA were also briefly explored. He concluded that PISA is now established as the major international comparative study of the mathematical ability of 15-year-olds. PISA's emphasis on assessing students' ability to *use* their mathematical understanding has led to many countries reviewing how their curricula "prepare their students for life". The PISA 2003 results, in which mathematical literacy was the major domain, are sure to precipitate further debate about the PISA approach and its relevance to universal mathematics education.

Paper presentations: Curricular developments and new contents

The first paper reported on the reform movement in mathematics education in People's Republic of China, focusing on curriculum at the compulsory levels. *Kwok-cheung Cheung* (University of Macau, China) argued that China's mathematics curriculum reform has been actively underway since the turn of the century. This paper sought to provide an introduction to new developments in mathematics education at the compulsory education levels (grade 1-9) based on the Mathematics Curriculum Standards (Experimental Version) released by the Ministry of Education. Basic concepts, design considerations, curriculum objectives, curriculum contents, curriculum implementation and recommendations on mathematical background knowledge were explicated in detail.

In a second paper *Guo Rong Xu* (London South Bank University, UK) discussed problems in Chinese education. Recently, the government initiated a mathematics education reform, influenced by Western educational ideologies and focused on changing the classroom practice to promote more active and creative learning. However, this reform, like previous reforms in Chinese mathematics education seems to be ineffective in implementing substantial changes in mathematics classroom practices. In her study in collaboration with *Stephen Lerman*, she looked at the actual impact of the reform on classroom practice and attempted to identify and analyse some factors that hindered it.

Maitree Inprasitha (Center for Research in Mathematics Education, Thailand) described the movement of lesson study in Thailand. Lesson study, a Japanese form of professional development, is a well-known approach to improve teacher practice. In his paper, he introduced how to use the lesson study approach for another purpose, namely to improve the recently launched 5-year program for educating mathematics teachers at all faculties of education in Thailand. In his concluding remarks, he stated that the lesson study approach has begun to have great influence on the reform program for professional development in Thailand. Furthermore, the National Commission on Science and Mathematics Education has incorporated the concept of lesson study into a national scheme of development of science and mathematics education.

Sofia Anastasiadou (Aristotle University of Thessaloniki, Greece) presented a research-based report on the perceptions, attitudes and conducts of Greek mathematicians towards statistics in secondary education. Sixty-three mathematics teachers responded to a Tatsp scale ("Teachers' attitude towards statistics and probability", a questionnaire with a Likert scale). These teachers generally had a wide range of teaching experience and knowledge of mathematics but not of statistics. Mathematics teachers showed both positive and negative attitudes towards statistics. The presenter suggested that the negative attitudes were a product of the long absence of statistical teaching, possibly creating repugnance, anxiety and disdain towards this science.

In a rather controversial paper, *Allan Tarp* (Grenaa International Baccalaureate, Denmark) looked for completely new ways to teach mathematics at the secondary school. Therefore, he introduced the concept of and operations with per-numbers. To solve the relevance paradox in mathematics education, he used post-modern "sceptical Cinderella" research. The presenter argued that the addition of per-numbers can be seen as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus.

Paper presentations: Learning from research and classroom practice

Also on behalf of *Dirk De Bock*, *An Hessels*, *Dirk Janssens* and *Lieven Verschaffel*, *Wim Van Dooren* (University of Leuven, Belgium) presented a paper related to the role of modelling and applications. Despite the increased attention for the modelling aspect in mathematics education, educational practice and research in the last decades uncovered many difficulties and systematic errors that may impede students' learning of a mathematical modelling disposition. The paper reported on a research-based teaching experiment with 8th graders aimed at remedying one of these errors, namely students' tendency to see and apply the linear model everywhere. Although the experiment was successful in improving students' performance on non-linear problems, it did not lead all students to a profound conceptual understanding of linear and non-linear relations, including the disposition to distinguish between situations that can and cannot be modelled linearly.

Athanasios Gagatsis presented a collaborative investigation with *Modestina Modestou* (University of Cyprus) on the predominance of the linear model in 12-13 year old Cypriot students, while solving non-proportional word problems involving area and volume of rectangular figures. Using three different kinds of tests, related to the context of the word problems presented, they attempted to identify a differentiation in students' responses. Two different statistical analyses were used on the data: Factor analysis and implicative statistical analysis. Both statistical analyses suggested the same grouping of students' responses and confirm the existence of improper proportional reasoning.

Also on behalf of *José Antonio Salvador* and *Pedro Luiz Aparecido Malagutti*, *Yuriko Yamamoto Baldin* (Universidade Federal de São Carlos, Brazil) presented the so-called Project Pró-Ciências carried out at their university in 2001 and 2002, in collaboration with elementary school authorities and governmental educational agencies. The project aimed at professional development of secondary school teachers, updating them with modern requirements of the school curriculum. The project grounded on National Curriculum Standards and focused on the understanding, planning and carrying out of interdisciplinary activities, connecting mathematics to other sciences and the real world.

In a last paper presentation, *Jiansheng Bao* (Suchow University, China) compared the old middle school mathematics syllabus to the newly published National Mathematics Standards in China. Numerous changes, both regarding the curriculum and the mathematical contents in China were noticed, leading to the following questions: What precisely are the differences between the new and the old mathematics textbooks? How do these differences affect the styles of mathematics teaching and learning? In order to answer these questions, the presenter used a self-developed model to evaluate the composite difficulties of new and old eighth grade maths textbooks using five factors of difficulty so as to highlight some initial findings.

Discussion papers

Dirk De Bock (University of Leuven, Belgium) outlined some major recommendations expressed in reform documents of the eighties making a strong plea for reforming mathematics education in all areas taking seriously into account the psychological aspects of teaching and learning processes and the societal demands and expectations with respect to mathematics. He wondered how these recommendations were implemented in school curricula and asked some questions about future trends that seem to appear in our field.

Masami Isoda (Graduate School of Human Comprehensive Science, Tsukuba, Japan) reported on "Mathematical Activity as a Human Endeavor" projects. Based on four basic ideas (mathematization, mediational means, theory of embodiment and hermeneutics), materials related to history and technology were developed for the project. He illustrated this approach with an example of studying an ellipse compass.

Elaine Simmt (University of Alberta, Canada), drawing from the papers presented, suggested that the illusion of linearity exists for more than just students of mathematics. In her view, this illusion is prevalent in teachers', researchers' and policy makers' interpretations and understandings of curricula. In her discussion paper, she proposed that new trends in secondary education discussed during ICME–10 challenged this illusion.

Athanasios Gagatsis (University of Cyprus) discussed the growing attention to the role of representations in learning and teaching mathematics. His paper was an attempt to exemplify the different roles representations can and should play in meaningful mathematics learning and in general mathematics education.

Finally, *Juan Antonio Garcia Cruz* (Universidad de La Laguna, Spain) reflected on the way mathematics and mathematics education is reported in the media and the mathematics classroom practice. He argued that we have to change teacher's attitudes and beliefs and also the way mathematical practice is perceived in our society.

The session ended with a panel discussion chaired and moderated by *Elaine Simmt*.

All papers presented can be downloaded from the TSG 2 ICME-10 website.

This report has been written by Dirk De Bock and Masami Isoda. They are happy to be contacted at dirk.debock@avl.kuleuven.be and msisoda@human.tsukuba.ac.jp for further information on the work of this TSG.

TSG 3: New developments and trends in mathematics education at tertiary levels

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Alexei Sossinsky, Independent University of Moscow, Russia

Introduction

Many of today's undergraduates cannot imagine life without cellular phones and laptops. How do these rapid technological changes in our society influence teaching and learning mathematics at the tertiary level? While many of mathematics lecture halls are still dominated by instructors "chalk and talk" and students' hasty note-taking, others engage in creative explorations, the use of technology and problem solving. But are there any identifiable developments and trends, either local or universal?

The aim of our Topic Study Group was to explore this issue of recent trends and developments from around the world. We adopted the format of presentations and discussions in an effort to aid this exploration. We attempted to balance our desire to provide a comprehensive overview of the state-of-the-art with an understanding that the time frame of the Group only allowed for "snapshots". With a considerable effort from all the team members we managed to assure the geographical variety of these snapshots. The presented papers illustrated work from Canada, Germany, Hong Kong, Israel, Japan, New Zealand, Russia, South Africa, the United States, and Uruguay. (See below for a detailed list of papers presented or distributed for TSG 3.) The work of the group laid a foundation to a special double issue of the International Journal of Mathematics Education in Science and Technology that appeared in 2005.

The topic of tertiary mathematics education is extremely broad not only because of our geographical variety, but also because of variety in content. "Tertiary" is more of a place – colleges and universities – rather than "level". Learning mathematics at tertiary level there are future research mathematicians taking advanced abstract algebra courses, future users of mathematics struggling with business calculus, liberal arts students relearning basic algebra to comply with "numeracy" requirements, and future teachers of mathematics, to mention just a few target groups and levels. In what way are the recent trends and developments similar or different for these groups of learners?

Selected points from the presentations

The work of our group at ICME 10 started with the presentation by *Annie Selden*, who put forward a provocative title for her paper: "New Developments and Trends? Or, More of the Same?" Selden's presentation served in *setting the stage* for the Topic Study Group. She identified four major topics of interest in undergraduate mathematics education. These were: the role of technology; the transition from secondary to tertiary education; the need to produce future mathematics teachers: and the potential impact of research into teaching and learning at tertiary level. She then pointed to a subset of these topics that was addressed at the congress and foreshadowed many of the forthcoming presentations, situating them in a broader context of tertiary mathematics education. In an attempt to cluster the work of the group in some reasonable way we identified several overlapping themes related to the theory and practice of mathematics education: issues of transition as related to both curriculum and pedagogy, teacher education, research in undergraduate mathematics education, technology and its influence on curriculum, pedagogy and research. We acknowledge that consideration of these themes overlaps with other working groups at ICME, in particular with those on teacher education and technology. However, considering the issues of concern to tertiary mathematics education, we find this overlap unavoidable.

The issue of challenges in *transition to undergraduate and advanced mathematics* was a significant focus in the work of the group. Lovric pointed to a trend that school graduates are less prepared in dealing with university level mathematics. This observation was based on comparing two cohorts of students: those with 5 years of secondary school education and those with 4 years, where the latter is the result of the recent imposed change in the province of Ontario, Canada. A similar argument was echoed by *Nishimori* based on the results of a survey among university faculty in Japan. This survey suggested deterioration in the problem solving skills of students as well as in their algorithmic fluency. Luk provided a personal account of transition, describing the challenges he faced both as a student and as teacher of undergraduate mathematics in Hong Kong. *Hockman* presented a concern of "watering down" courses in order to comply with the need to accommodate a larger amount of students and the lack of support from administration in South Africa. These papers raised a universal concern – the concern of deterioration – that was mirrored by several comments and reflections on the personal experience of participants.

Responding to the question presented by Selden – "New Developments and Trends? Or, More of the Same?" – we admit that the issue of the transition to undergraduate mathematics is not "new" to the field, but it has been reinforced and aggravated, and has received a new attention recently, considering the growing amount of students entering tertiary education and diminishing amounts of funds to support these students. Selden referred to the problem as "two contradictory trends": on one hand there is advocacy from the academic community for school graduates who are better prepared mathematically for both university and the work place, on the other hand there is evidence that legislatures and administrative bodies around the world are reducing requirements for both high school diplomas and university degrees.

There is an ongoing effort to develop curriculum and pedagogy to address better the *needs of all students* as well as of specific populations of students. These efforts were featured in the papers by *Paramonova*, who outlined the curriculum at the Moscow Independent University that serves to educate future research mathematicians, and *Safuanov*, who presented a view on pedagogical development implementing a "genetic" approach to teaching, that is, pedagogy that recognizes historical, logical and socio-development of the subject matter. Further, acknowledging that "understanding" is the ultimate goal of teaching mathematics, Kannemeyer provided an innovative instrument attempting to identify what such understanding entails in the context of a calculus course. The paper by *D'Arcy* presented strategies that may help students in memorizing the mathematical contents they are studying.

Teacher education was another important focus in our Group. After all, considering tertiary mathematics education, a significant part of it is education of future teachers of mathematics. The work of the group related to teacher education considered issues in curriculum, pedagogy and research. *Martinez Luaces* presented a case for the use of modelling in the curriculum for mathematics teachers and provided several examples of modelling activities that seemed to have a positive impact on the participants who engaged in these activities. *Wittmann* presented what he referred to as a "notion of operative proof", that is, a proof that introduces the ideas behind the mathematical argument without relying heavily on formalism and symbolism. He made an argument for using operative proofs as part of the pedagogical approach with pre-service teachers. The papers of *Zazkis* and *Liljedahl* presented reports on research conducted with the population of pre-service teachers. Zazkis investigated the ways in which students perceive irrational and prime numbers and pointed to common features of these two sets considering how these numbers can or cannot be represented in a standard algebraic notation. Liljedahl investigated the impact of successful mathematical discovery – referred to as an Aha! experience – on the beliefs and attitudes of pre-service elementary teachers. *Leikin*'s research considered both undergraduate and graduate mathematics education students and acknowledged the similarities in the interaction between a teacher and a student and between a mentor (supervisor of student-teachers) and a mentee. Leikin presented an example of connection between theory and practice of mathematics education, specifically, how a model of interaction developed by analyzing the work of in-service teachers can be used with pre-service teachers in order to raise the quality of their discourse about teaching.

Returning once again to Selden's question we believe that some novelty can be claimed here on two accounts. First is the relatively novel and growing attention to *research* in undergraduate mathematics education. Another is the attention to the *teacher*. The latter is in accord with the claim made by Anna Sfard in her plenary presentation, "There is nothing more practical than good research: On the mutual relation between research and practice in mathematics education", that identified the current decade in mathematics education research as "the decade of the teacher", while the previous two decades could be considered as the "decade of the curriculum" and "decade of the learner".

Technology is the theme that intertwines with all the areas on mathematics education. As a snapshot of the influence of technology on curriculum and pedagogy Hillel considered the case of Linear Algebra, he described in what way various computer based activities and assignments are used to further students' understanding and appreciation of the subject. Web-based or web-supported courses are a paradigmatic example of how technology influences the pedagogy of course delivery. *Engelbrecht* and *Harding* presented a classification of courses that rely on the internet in various ways – ranging from reference to the web for illustrative examples in a "standard" mode of delivery to a full course delivered electronically – and discussed the impact these courses have and may have on undergraduate education. As a snapshot from the possibilities offered by the world wide web, a paper by *Zgraggen* presented a programme that provides students with dynamic guidance to solving problems. As a snapshot from research that investigates the influence of technology on learning, *Gurevich*, *Gorev* and *Barabash* studied the impact of the use of various computerized tools on students' achievement in plane geometry and in analytic geometry.

So, again, is it "More of the same"? The "sameness" is in the idea that technology is one of the forces that is driving the change in curriculum, in pedagogy and in research. This is hardly surprising since some historians take a view that *any* societal change is due to the advancement of technological tools, be this the invention of printing technology or the invention of wireless communication. However, the novelty is in the kinds of technology and in the speed of the change.

In a summarizing presentation for the group we wanted to look into the future. However, rather than foretell the future – a task that would be impossible considering the rapid changes in technology – Holton in his concluding address "Tertiary mathematics education for 2024" presented a wish for the future. This wish included emphasis on the "creative" side of mathematics, rather than on its "created" side, that is, emphasis on the activity of doing mathematics rather than the focus on the artifacts of such activity of others. Technology appeared to be one of the means to this end. It further included an emphasis on research in mathematics education that will help understand better the learning process and in turn influence pedagogy. "That mathematics is seen to be something that is to be enjoyed and not feared" – was one of the aspirations put forward by Holton. And though this wish was presented in a rather personal tone, there was a sense in the group that it was shared by many.

List of papers presented or distributed for TSG 3
(available at www.icme10.dk)

D'Arcy-Warmington, Anne: Learning to make happy mathematical memories
Engelbrecht, Johann & Harding, Ansie: Taxonomy of online undergraduate mathematics courses
Gorev, Dvora; Gurevich, Irene; Barabash, Marita: How is the efficiency of the computer usage in geometry related to the levels of students' learning abilities?
Hillel, Joel: Trends in the teaching of linear algebra and the role of technology.
Hockman, Meira: Success at all costs or the cost of success?
Holton, Derek: Tertiary mathematics education for 2024
Kannemeyer, Larry: A reference framework for measuring student's understanding in a first year calculus course
Leikin, Roza: Professional dialog, its components and qualities: from graduate research on teaching to an undergraduate teachers program
Liljedahl, Peter: AHA!: The effect and affect of mathematical discovery on undergraduate mathematics students
Lovric, Miroslav: Transition from secondary to tertiary mathematics, McMaster University experience
Luk, Hing Sun: Gap between secondary school and university mathematics
Martinez Luaces, Victor: Teacher training for problem solving and modeling
Nishimori, Toshiyuki: The deterioration problem of university students' capacity to study mathematics in Japan from 1993 to 2003 and a recent inquiry
Paramonova, Irina: Mathematics syllabus innovation in Russia: The Moscow experience
Safuanov, Ildar: Design of the system of genetic teaching of some topics of algebra at universities
Wittmann, Erich Ch.: Learning mathematics for teaching mathematics: The notion of operative proof.
Zazkis, Rina: Representing numbers: Prime and irrational
Zgraggen, Bernhard: Interactive, generic, heuristic and dynamic step-by-step solutions to mathematical problems in the world wide web.

This report has been written by Derek Holton and Rina Zazkis. They are happy to be contacted at dholton@maths.otago.ac.nz and zazkis@sfu.ca, respectively, for further information on the work of this TSG.

TSG 4: Activities and programmes for gifted students

Team Chairs: ***Edward Barbeau***, University of Toronto, Canada
Hyunyong Shin, Korea National University of Education, Chungbuk, The Republic of Korea
Team Members: ***Alex Friedlander***, The Weizmann Institute of Science, Rehovot, Israel
Shialesh A. Shirali, Rishi Valley School, Chittor District, India
Emiliya. A. Velikova, University of Rousse, Bulgaria

Introduction

The program was organized by a committee consisting of the two Chairs and Team Members indicated above. The committee is particularly indebted to Dr. Velikova for setting up a special website for the study group, preparing and publishing along with Dace Bonka, Lasma Strazdina and Inese Berzina a volume of Proceedings (supported by the Universities of Latvia and Rousse), presenting the speakers with certificates of participation and providing pens to the attendees.

The registration of more than 120 participants from at least 36 countries for TSG 4 attested to the great interest in this topic. At least ninety attended each of the four sessions.

Session 1

There are a number of aspects involved in looking at programs for gifted students. The first task is to consider the characteristics of giftedness and how such students can be identified. A few papers touched on this area.

An important consideration is to provide sufficient stimulus to stave off boredom, as talented students who are not challenged can underachieve. Their natural curiosty needs to be stimulated. Gifted students appear to have the knack of making appropriate use of memory, working in productive and flexible ways, adapting to new settings and exploiting structure. They should be encouraged to be self-reflective. As they exhibit different styles of learning, their mentors need to be attentive to the means by which they will flourish. There is a potential for attaining more acute reasoning and deeper levels of abstraction that need to be fostered.

More than ever, it is a challenge for teachers to be tuned to the characteristics of students, to identify how they may best function and track down the resources that will provide each student with appropriate challenges.

Brenda Bicknell (New Zealand), Addressing mathematical promise in the New Zealand context
Bettina Dahl (Norway): How do gifted students become successful? A study in learning styles
George Gotoh (Japan), The quality of reasoning in the problem solving process
Djordje Kadijevich and *Zora Krnjaic* (Serbia and Montenegro), Is cognitive style related to the link between procedural and conceptual mathematical knowledge?
Borislav Lazarov (Bulgaria), Resulting effect of consecutive activities

Hye Sook Park, *Kyoo-Hong Park* (Korea), Analysis of the mathematical disposition of the mathematically gifted students in the middle school of Korea

Emiliya Velikova, *Svetoslav Bilchev* and *Marga Georgieva* (Bulgaria), Identifying of creative-productive gifted students in mathematics

Sessions 2 and 3

Having identified the group of gifted students, the next step is to consider how such students should be handled both inside and outside of the classroom.

Often gifted and non-gifted students are in the same classroom together, and it is a serious challenge for the teacher to make sure that all students are well-served by the situation. One can, for example, created orchestrated problem sets that start with very simple ideas and progress towards more difficult and significant mathematics. Students can be encouraged to work together on projects, presentations and publications, where each can make an appropriate contribution. Care needs to be taken to achieve a balance between first- and second-hand experiences, enough of the first to provide a solid base and enough of the second to encourage original enquiry.

Outside of the classroom, one can go further and make use of topics appropriate for youngsters than are not normally on the curriculum. This includes many topics in geometry and combinatorics, as well as recreational mathematics. With younger children, it is desirable not provide work that does not require much in the way of prerequisites. For example, it is known that many young children have a good intuition in solid geometry and this might be exploited.

Contests and rallies provide goals that gifted students can work towards and test their powers with. For the particularly well-motivated, there are many special classes, schools and clubs to which they may belong, and our speakers provided several fine examples of this activity. Finally, we must not forget that most modern of resources, the internet. The modern student not only has access to many resources but interactive activities on the web that will support the activity of individuals or of small groups and will permit contacts over large distances. Keeping up with all of this is an imperative for the modern teacher who wishes to be effective.

(a) Some of the presenters looked at regular classroom settings.

Carmel M. Diezmann and *James J. Watters* (Australia), Challenge and connectedness in the mathematics classroom: using lateral strategies with gifted elementary students

Victor Freiman (Canada), Mathematical giftedness in early grades: challenging situation approach

Elena Koublanova (USA), Teaching capable students in developmental mathematics classes

Mark Saul (USA): The unity of mathematics education

Bharath Sriraman (USA): Differentiating mathematics via use of novel combinatorial problem solving situations: a model for heterogeneous mathematics classrooms

(b) Others looked at other settings. These might involve the use of contests, journals or special classes, or even, institutions devoted specifically to gifted students.

Mariam Amit and *Alexei Belov* (Israel and Russia), Unlocking interlocking mathematical structures – an experiment at the Kidmatika Math Club

Dace Bonka and *Agnis Andzans* (Latvia), General methods in junior contests: successes and challenges

Anatolii Chasovskikh and *Yuri Shestopalov* (Russia), The advanced education and science centre of the M.V. Lomonosov Moscow State University – the Kolmogorov College

Ziva Deutsch, *Akiva Kadari* and *Thierry Dana-Picard* (Israel), "Alef Efes": Students create and publish a mathematical quarterly and an interactive site

Donco Dimovski (Macedonia), Mathematical schools, competitions

Kathy Gavin and *Linda Sheffield* (USA), Project M3: Mentoring mathematical minds

Kyoko Kakihana and *Suteo Kimura* (Japan), Activities in new curriculum for gifted students – trials in super science high schools in Japan

Peter Kortesi (Hungary), Self made mathematics

Kang Sup Lee, *Dong Jou Hwang* and *Woo Shik Lee* (Korea), Development of enrichment programs for the mathematically gifted: focussed on conic sections

Elena Levit, *Larisa Marcu* and *Orna Schneiderman* (Israel), Process of training and admission to a MOFET science class

Gregory Makrides, *Emiliya Velikova* and *partners* (Europe), European project: MATHEU – identification, motivation and support of methematical talents in European schools

Eugenia Meletea (Greece), Educational network communicating heuristic and sophisticated mental models of mathematical knowledge – developing pedagogical reasoning to support gifted/talented students in Greece

Dimitris V. Papanagiotakis and *Panayiotis M. Vlamos* (Greece), Web-based mathematical problem solving database for gifted students publications for primary and secondary school students in Macedonia

Emiliya Velikova (Bulgaria), Extracurricular work with creative-productive gifted students – program and activities

Session 3

The speakers in this session considered material that was put before gifted students, and discussed in particular, technology that might be of use. In our time, mathematics has evolved in many different directions, and not just those that make use of technology. Many of these, including some that are quite abstract, are relevant to the mathematical development of the young.

Alexandr and *Vladimir Chumak* (Ukraine): Algorithms and symbol-graphic language in mathematics education and using of last in the internet technologies

Hanhyuk Cho, Hyuk Han, Manyoung Jin, Hwakyung Kim and *Minho Song* (Korea), Designing a microworld: activities and programs for gifted students and enhancing mathematical creativity

Session 4
Finally, specific examples of problems and investigations were presented.
Andrejs Cibulis and *Ilze France* (Latvia): Work with gifted students in the investigations of polyforms
Alexander Soifer (United States): One beautiful Olympiad problem: chess 7 x 7
Sang-Gu Lee (Korea), Activity of a gifted student who found a linear algebraic solution to the Blackout puzzle

There were some contributions that were not part of the oral program, but which were included in the written proceedings.
Oscar Joao Abdounir (Brazil), Music and mathematics: relationships between intervals and ratios in mathematics education
Alex Friedlander (Israel), High-ability students in regular heterogeneous classes.
Risto Malcevski and *Valentina Gogovska* (Macedonia), The role of educational methods in the teaching of gifted and talented students
Nobuaki Kawasaki (Japan), Characteristics of Bulgarian mathematical education

All the speakers cooperated by giving well-prepared and brief talks so that there was about an hour available in the final session for a discussion that was quite free-ranging.

We hope that participants and others will continue the discussion. To this end, all are invited to surf the website of the TSG at www.icme10.dk, where, in particular, they will find the names and e-mail addresses of those who attended the sessions.

This report has been written by Ed Barbeau, who can be contacted at barbeau@math.utoronto.ca for futher information on the work of this TSG.

TSG 5: Activities and programme for students with special needs

Team Chairs: ***Sinikka Huhtala***, Helsinki City College of Social and Health Care, Finland
Petra Scherer, University of Bielefeld, Germany
Team Members: ***Ronnie Karsenty***, The Weizmann Institute of Science, Rehovot, Israel
Elisabeth Moser Opitz, University of Freiburg, Switzerland
Susan A. Osterhaus, Texas School for the Blind and Visually Impaired, Austin, USA

Introduction

The program of this TSG focused on different groups of students with special needs such as learning disabled or mentally retarded students, deaf children or students with visual impairment. Moreover, different aspects were discussed: questions concerning the curriculum, different mathematical areas (e.g. numbers, measures or geometry), diagnostic procedures, and results or specific activities and programs for initiating adequate learning processes.

Session 1

The first two sessions emphasized aspects concerning students with learning disabilities. The first session started with *Olof Magne* and *Arne Engström* (Sweden) who presented their study »Middletown Mathematics«, in which the total inventory of mathematical achievement of approximately 2000 students was carried out in three successive investigations in 1977, 1986 and 2002 respectively. By making comparisons between the three populations of the respective age cohorts it was possible to assess the changes of achievement in the course of time related to changes of curriculum and the age of students. It was shown that the differences as to mathematical achievement between the three dates were mainly insignificant. One important result was that the students in upper grades achieved lower results on grade typical items than did students in lower grades. The researchers' interpretation of the study was that the school system produces students' difficulties in mathematics and that the quality of low ability group classrooms has to be improved.

The second presentation, given by *Philippa Bragg* (Australia) was titled »Measuring the Consequences. Teaching Linear Measurement to Students with Learning Difficulties«. Large-scale testing by the US National Assessment of Educational Progress (NAEP), TIMSS, and Basic Skills Testing, in New South Wales primary schools showed that large gaps remain in student understanding of measurement concepts. While these tests reported an overall improvement in the basic skills of measuring, many students seem unable to apply their knowledge to tasks that require a deeper understanding of the concepts. Philippa Bragg introduced two studies: The first examined students' skills and understanding of linear measurement skills in grades 1 to 5 with particular emphasis on those students who find learning mathematics difficult. The second study looked at students in grade 6 who had completed their instruction on measurement. Both studies found that most low-ability students did not understand the important concepts of linear measurement even though they were able to use a ruler correctly. Suggestions were made for an instructional methodology that will go some way towards helping students learn important basic concepts about linear units and measurement. Getting a feel for usual household tasks may help forming a foundation for basic concepts.

Session 2

In the second session *Birgit Werner* (Germany) gave a talk titled »There is Something Wrong with the Hundred Square. Or: Observing and understanding mathematics classroom situations«. Based on the assumption that a situation in class is primarily one of interaction and communication, the understanding of communication turns into the main didactic and diagnostic issue. Thus, an adequate method of analysis for this situation becomes a necessity not limited to task fields of special-pedagogy. By means of a systems-theoretically oriented situation and communication analysis an instrument for paying attention to the moment of observation was introduced that answered both remedial-diagnostic as well as didactical-methodological questions. The observations of first, second and third order functioned as structuring devices, ultimately leading the way to a conclusion. Referring to these aspects Birgit Werner presented a case study, and concrete examples of activities with the hundred square were given.

Elisabeth Moser Opitz (Switzerland) went on with her presentation »Learning disabilities in grades 5 and 8: some results of a research project in Switzerland«. First of all, she referred to the many definitions that are used to describe children with learning difficulties in mathematics and the common sense that a major characteristic of learning disabilities in mathematics is falling behind the expected performance. She complained that most of the research does not describe more precisely what the nature of this falling behind is. Which are the specific competencies that the students are lacking? The project, carried out with 4000 children, addressed different questions: Which are the mathematical competencies of pupils with learning disabilities in mathematics in grade 5 and in grade 8? Is there empirical evidence that most pupils with learning disabilities in mathematics fail to understand basic arithmetical competencies like counting, place value concept, additive composition of number, efficient retrieval strategies etc.? Is there a difference in mathematical performance between pupils with learning disabilities in grade 5 and grade 8? Is there a difference in the attitude to mathematics between pupils with and without learning disabilities in mathematics? It was shown that students in higher grades failed in some basic competencies like counting or place-value.

Session 3

The third session started with *Ann Ahlberg* (Sweden) who spoke about »Children who are blind – Children with hearing impairment – Children without visual or hearing impairments – Experiencing numbers«. The overall aim of her study (in co-operation with researchers from Hungary and Norway) was to analyse the ways in which children handle and experience numbers. Three different groups of children – blind children, hearing impaired children, and children without these impairments – participated in the research. The main interest was to reveal the relations between the ways in which children handle numbers – their strategies – and their interpretation of meaning. The results showed that the children in all three groups handled and experienced numbers in various ways and that these were related to their sensuous and simultaneous experiences of the problem content. Some ways of handling were related to more than one way of experiencing numbers. The main findings in the comparative analyses showed that the sensuous experience of numbers to a great extent enables children to grasp numerosity, and that their simultaneous experience of different aspects of numbers contributed to their understanding. Furthermore it was shown that in spite of various sensuous experiences, children with visual or hearing impairment and children without these impairments are able to develop the same understanding of numbers on a group level.

The second presentation in this third session was given by *Akira Morimoto* (Japan) »On Mediation between Concrete and Abstract for the Hearing Impaired in Mathematics Classrooms«. The purpose of his study was to identify difficulties in mediating between concrete and abstract for the hearing impaired in mathematics classrooms. First, the framework was discussed in order to identify such students on the basis of the nature of perception in mathematics classrooms. According to this framework, two difficulties seemed to exist in mediating between concrete and abstract for the hearing impaired in mathematics classrooms. One is in seeing explicitly written mathematical symbols as a sequence of operations, not as a structure. The other is in selecting a category of concept and using a corresponding operation to make a structure in explicitly written symbols. Implications (e. g. discussing the individual perceptions more intensively) were drawn for communicating on mathematical idea in mathematics classrooms for the hearing impaired and the hearing.

Session 4

The final session started with the presentation »Accessible Math Technology for the Blind and Visually Impaired« given by *Susan Osterhaus* (USA). She gave an overview of the mathematical technology currently available, along with future projections, which should help teachers determine the most appropriate ways to teach their blind and visually impaired students and prepare them for success in the mathematics classroom, on standardized assessments, and in their chosen career. Various large displays, Braille characters, and talking scientific/graphing calculators were described, including their strengths and weaknesses. Moreover a variety of software and hardware products that assist in teaching math concepts or providing mathematical materials in an accessible format could be identified.

The last contribution »Math Education and Training for Autonomy in the Mentally Retarded Pupil« by *Antonella Montone* and *Michele Pertichino* (Italy) was presented by *Brunetto Piochi*. It was pointed out that a learning path addressing logic and mathematics for children with severe learning difficulties is often considered too difficult, or even impossible. Without denying or underestimating the hindrances, it was seen absolutely necessary to support the right of every child to learn as much as possible even in this field. Moreover, good mathematical knowledge is considered as an essential requirement to gain autonomy in life. In particular, the handling of relevant everyday and work place mathematics was discussed. During the talk some proposals for teaching and learning were given (e. g. to acquire a good knowledge of the concept of number, to be able to consult a calendar or a timetable; or to follow, show and draw paths). As personal autonomy represents an important achievement for every young person, the importance of the activities based on the methods presented was evident, since they are meaningful and useful to everybody.

In the four sessions of TSG 5 different aspects concerning mathematics education for students with special needs were illustrated. They provided evidence of the wide field of special education. Similarities as well as differences could be identified for the different groups of students. In the final discussion some key aspects concerning all different groups were pointed out:

- Future research concerning mathematics for students with special needs and empirical work should have a close connection to the didactics of mathematics.
- Research should combine theory and practice in a natural way so that the ideas and concepts may influence classroom practice.
- Some hypotheses about specific conditions for organizations supporting low achieving children were put forward, as for instance a change in the teachers' view of low achievers which may lead to a better understanding of individuals' thinking and learning.

This report has been written by Petra Scherer, University of Bielefeld, who will be happy to be contacted at petra.scherer@uni-bielefeld.de for further information on the work of this TSG.

TSG 6: Adult and lifelong mathematics education

Team Chairs: *Tine Wedege*[1], Roskilde University, Denmark
Jeff Evans, Middlesex University, London, United Kingdom
Team Members: *Gail FitzSimons*, Monash University, Melbourne, Australia
Marta Civil, University of Arizona, Tuscon, USA
Wolfgang Schlöglmann, Johannes Kepler University, Linz, Austria

ICME 10 2004

TSG

Topic Study Group 6

Introduction

In adult and continuing education there seem to be two parallel and combined processes going on: *an institutionalizing process*, where schools or colleges for adults become subject to the sorts of regulation already experienced by schools for children and adolescents; and *a de-institutionalizing process* with a focus on adults' learning processes outside schools, some of which may be accredited as more or less equivalent to formal qualifications. Both tracks were represented in the work of TSG 6 where the key words were: globalisation, exclusion, equity, participation, technological and economic development.

The terminological basis of our work was this: *Adults* are engaged in a range of social practices, such as working (or seeking work), parenting and caring for other dependents, budgeting and organising consumption, voting, etc. The term *lifelong* indicates that education takes place in all stages and spheres of life. By *mathematics* we mean multiple activities and knowledge, including academic mathematics, vocational mathematics, ethnomathematics, folk mathematics and adult numeracy. Regarding *education* we adopted the terminology of UNESCO (2000) as a point of departure: *Informal education* means the lifelong process whereby adults are learning mathematics in everyday life (e.g., work, family, leisure, society). *Formal education* refers to the adult educational system from adult basic education and vocational training through further and higher education. *Non-formal education* is defined as any educational activity organized outside the established formal system that is intended to serve identifiable learning objectives.

Adult and lifelong mathematics education has multiple dimensions and the approaches represented in our discussions embraced, besides mathematics, a range of disciplines (psychology, sociology, politics, pedagogy, anthropology and androgogy), and a spectrum of concerns about inclusion – along lines of gender, class, ethnicity, age and language group. The contributors came from all six continents, and a range of themes and issues was addressed, including:

Overviews of recent research and practice in adult numeracy and mathematics

After decades of neglect, adult numeracy and mathematics learning and teaching are coming to be recognised as worthy of serious research (Coben, 2003). Against this background, *Diana Coben* (UK) asked: What is specific about research in adult numeracy and mathematics learning and teaching? Are numeracy and mathematics distinct from each other? How do they relate to adult literacy? *Gail FitzSimons* (Australia) presented an analysis of "Adult and Lifelong Mathematics Education" utilising the theo-

1 Now at Malmö University, Sweden

retical framework of Basil Bernstein which offers a coherent set of principles for linking the institutions of mathematics and lifelong education, and the emerging but contested construct of adult numeracy. At the macro or institutional level, she analysed the positioning of mathematics and numeracy education and research in relation to each other and to discourses of lifelong learning. These are set within the arenas of knowledge production and distribution, influenced by policy formation at national and international levels. At the meso level, within the field of recontextualising where knowledge transmission takes place, FitzSimons discussed curriculum formation and conditions of teachers' work. At the micro level, within the field of reproduction, where knowledge acquisition takes place, she discussed examples of recontextualising texts and their possible impact on learners' identities.

Parents and community as intellectual resources

In her paper "Profaning the holiness of school mathematics", *Gelsa Knijnik* (Brazil) presented the theoretical basis, methodological procedures and results of a research project whose main goal was to discuss how cultural processes involving oral mathematics are produced and their curricular implications for the education of youths and adults in rural areas. This ethnographic study followed a group of 50 rural workers of the Brazilian Landless Movement, students in a teaching course belonging to this social movement, and illiterate youths and adults who participated in a workshop given by the course participants. *Marta Civil* (USA) presented two research projects on parents as adult learners of mathematics in a Latino/Hispanic working class community. The focus was on issues of content and learning environment: What mathematical content should we address when working with parents? What pedagogical approaches do they favour? The paper pays special attention to the voices of immigrant parents. Consistent with an emphasis on parents as intellectual resources, she addressed the need for adult education to build on all adults' experiences and knowledge. Under the title "Overcoming mathematics phobia in adults", *Vivek M. Wagh* (India) discussed some experiences of working with parents, guardians and community. His interaction had shown that that more than 90% of the parents of children facing difficulties in the learning of mathematics were found to have a phobia or repulsion towards mathematics.

Issues of affect, beliefs, motivation, resistance and anxiety in adult learners

Continuing education is experienced by adults as a field of tension between needs and constraints. *Jeff Evans* (UK) and *Tine Wedege* (Denmark) took this into account in their discussion of people's motivation and resistance to learning mathematics, as interrelated phenomena. While *Wolfgang Schlöglmann* (Austria) posed this question: 'Lifelong mathematics learning – a threat or an opportunity?' and made some remarks on affective conditions in mathematics courses where many of the adult learners are unemployed and where attendance and a certain type of performance are required. These adults have not chosen to participate in a learning program. *Dubravka Viskic* and *Peter Petocz* (Australia) presented their investigations into adult mathematics students' ideas about mathematics and learning, based on the students' written reflections on the process of carrying out projects as part of a preparatory mathematics university course.

Pedagogic resources and the dialogic approach

Javier Díez-Palomar, Joaquín Giménez Rodríguez and *Paloma García Wehrle* (Spain) presented the results of a case study, "Cognitive trajectories in response to proportional situations in adult education", about learning of proportional situations in a school for adults. The objective was to find ways of overcoming the forms of exclusion that occur in everyday mathematics situations that involve the use of proportions for decision making. Among other conclusions they found that perlocutionary speech acts can encourage learning, but can also create barriers when the speaker uses a position of power that breaks with egalitarian dialogue. In her paper *Marian Kemp* (Australia) noted that it is important for everyone to be able to engage with quantitative materials, in particular tables, to enable them to extract information and make informed decisions, and she presented a study with first year undergraduate students to evaluate the effectiveness of an intervention workshop designed to promote these aims.

Gender "mainstreaming"

Inge Henningsen (Denmark) discussed opportunities and challenges in mainstreaming that has been widely adopted by the international community as a strategy for equality. Mainstreaming of research on mathematics education means that gender, ethnicity, social class and other difference defining categories are involved consciously and explicitly in every research agenda. In mathematics education, curriculum, context, instruction and values have a gender dimension that should be acknowledged in research. Literacy surveys present intriguing instances of gender blindness – the paper points out how gender is a possible confounding variable in a number of comparisons. The paper contends that gender mainstreaming must be expected to play a positive role in the search for better research in mathematics education and a more inclusive teaching of mathematics.

Issues for pre-service teachers and for professional development of tutors

Terry Maguire and *John O'Donoghue* (Ireland) reported how grounded research has contributed to the development of a model of professional development for tutors of adult numeracy. The model incorporates a view that professional development is not a one-off activity, but something that allows for the development of a wide range of skills and knowledge, increasing complexity and specificity in the context of a tutor's own lifelong learning. *Miriam Benhayón* and *Mercedes de la Oliva* (Venezuela) described the steps followed to design a remedial course in mathematics for adult participants within the bounds of a university study program offered to practising teachers with no bachelor's degree. They emphasised that the contribution of this work is related to three main ideas: the social work it represents in a nation like Venezuela where there is a great number of non-graduate teachers; the opportunity to break paradigms and negative beliefs about the learning of mathematics; and the characteristics of the suggested evaluation instruments. *Sally Hobden* (South Africa) reported of some of the language, numeracy, emotional and learning management struggles experienced by preservice teachers in an initial one semester "Mathematics Literacy for Educators" module.

Roles for functional skills and understanding and commonsense

John Gillespie (UK) referred to the numeracy part of "The Skills for Life" surveys of adult literacy and numeracy in England that were carried out in 2002-3 for the Department for Education and Skills to meet their requirements. The findings confirm that for many, being "at a given level" is not meaningful for the individual, as notions of levels embody predetermined assumptions about progression and relative difficulty. In her analysis of "functional skills and understanding", *Lene Østergaard Johansen* (Denmark) distinguished between four different analytical domains in adults' lives (school, workplace, everyday life and democratic involvement) and considered functional skills from four different discursive perspectives (society/politicians, researchers, mathematics teachers, and the individual). The analysis emphasized that skills and understanding can be functional in one domain from one perspective and not functional in another domain or, from another perspective. *John O'Donoghue* and *Noel Colleran* (Ireland) reviewed their position on commonsense and adult problem solving and located their evolving understandings in the context of adult tutor training.

Methodological issues

These were raised in many of the papers. Here we can mention as examples: the possibilities and limits of survey research (*Gillespie*); the role of ethnography in researching oral mathematics (*Knijnik*); appropriate methods for the analysis of interaction and discourse (*Diez-Palomar et al.*).

Global issues

These were raised throughout the four group meetings, as can be seen from the summaries above. Throughout all the papers, there runs the thread of shared concerns and initiatives to promote social inclusion and social justice.

Reference

Coben D. et al. (2003) *Adult Numeracy: Review of research and related literature*. London: NRDC.
UNESCO (2000). *World Education Report 2000: The Right to Education: Towards Education for All Throughout Life*. Paris: UNESCO Publishing

All papers referred to are published on the TSG 6 website, www.icme10.dk – Programme.

This report has been written by Tine Wedege and Jeff Evans with support by the team members. They are happy to be contacted at Tine.Wedege@lut.mah.se and J.Evans@mdx.ac.uk for further information on the work of this TSG.

TSG 7: Mathematics education in and for work

Team Chairs: ***Henk van der Kooij***, The Freudenthal Institute, Utrecht, The Netherlands
Rudolf Strässer, University of Giessen, Germany
Team Members: ***Susan Forman***, City University of New York, Bronx, USA
Jim Ridgway, University of Durham, United Kingdom
Robyn Zevenbergen, Griffith University, Bundall, Australia

Aims and focus

The focus of this topic study group was to identify general characteristics of the nature of mathematics as it appears in or is needed for work, seen from the perspective of occupational standards ('broad occupational competences' versus 'highly occupation-specific skills', problem-solving skills, flexibility and quality, transfer) and the influences of Information and Communication Technology (artefacts, simulation, spreadsheets, data).

The second focus of the topic study group was to look into characteristics of the teaching and learning of mathematics at work, in classrooms and other settings, if teaching and learning are oriented to preparing students for work-place related situations. Those characteristics can be approached from the perspective of pedagogy (situated learning, situated abstraction, authentic learning) and quantitative literacy (mathematical literacy, overarching concepts (PISA), mathematics in context).

The overall aim of the TSG was to discus consequences of the above-mentioned characteristics for the nature of mathematics and mathematics education in and for work. Three sessions were organised on sub-themes and the final session was used for discussion. Even if the third session was planned to be on the use of modern technology related to workplace mathematics, this issue naturally came up also in the first two sessions. As a consequence, and reflecting the presentations and discussion in sessions 1 to 3, there is no specific report on workplace related use of technology.

Characteristics of mathematics for work

Quantitative literacy: An introduction
Henk van der Kooij, The Netherlands
"The contrast between mathematics in school and mathematics at work is striking. Mathematics in the workplace makes sophisticated use of elementary mathematics rather than, as in the classroom, elementary use of sophisticated mathematics. Work-related mathematics is rich in data, interspersed with conjecture, dependent on technology, and tied to useful applications. Work contexts often require multi-step solutions to open-ended problems, a high degree of accuracy, and proper regard for required tolerances. None of these features are found in typical classroom exercises." (Steen, 2001). Two statements for discussion were presented:

- The discipline itself should not define the mathematics program in vocational courses. Rather: the context of work should define the desirable mathematical activities.
- Abstraction and formalism should not be goals for mathematics education for the workplace; situated abstraction and 'mathematics in context' should.

Introduction on ambivalence of technology
Rudolf Strässer, Germany
A competent worker in a technology-rich environment should be in control of his workplace, should understand, not only handle, his instruments and should know about the mathematical models in use. Technology is hiding mathematics from the perception of its user (black box). In a technology-rich environment, mathematics becomes visible only in breakdown situations, when technology stops to function properly. But technology can also serve as a means to understand – helping to open – the 'black box', by simulations and 'what-if'-exploration.

Abstraction in workplace expertise
Celia Hoyles and *Richard Noss*, United Kingdom
Two studies – one with nurses in a paediatric hospital and the other (ongoing) in large manufacturing businesses – were presented. The findings from the first studies suggest that rather than being a set of disparate skills, mathematics used at work takes the form of well-connected situated abstractions, where abstraction is expressed by means of the tools and artefacts of the practice and relies on shared workplace knowledge and discourse. The latest research on 'techno-mathematical literacies', as used in computationally-rich modern workplaces, was presented too. "… the mathematical models involved appear different than conventionally-understood". They are "not mediated by formal mathematical symbol systems and artefacts, but by 'situated' techno-mathematical artefacts" (quote from the presentation in the Topic Study Group).

The numeracies of boat building
Robyn Zevenbergen and *Kelly Zevenbergen*, Australia
A case study of young boat builders was presented. What this case study has illustrated is that young people often approach the numeracy demands of their work in ways that are different from those of older staff. Their approaches tend to rely more on estimation, problem solving, holistic thinking and intuitive methods. As such, the study should not be interpreted to mean that young people do not have number skills. Rather, it suggests that the other skills take a higher priority in their approaches to working mathematically.

Mathematics at the workplace – the perspective of pedagogy

Mathematical knowledge of workers at South-African Cultural Villages
Mogege Mosimege, South Africa
Two studies of the making of artefacts in Cultural Villages were presented. Cultural villages are specifically created places to preserve the national heritage of pre-colonial South Africa from disappearance in a more and more global economy. A mathematical analysis of the various artefacts and activities at the villages provide an opportunity to explore the mathematical concepts that are used regularly by inhabitants (workers) at such villages. Mathematics is a useful subject for every one, it is both relevant and practical and is applicable to everyday life. Educators can help to close the gap between classroom activities and activities outside the classroom, ensuring that mathematical concepts learned in classrooms are not dealt with in isolation but take into account daily experiences of workers in various settings, including Cultural Villages.

Mathematics in Italian vocational schools
Brunetto Piochi and *Rosa Laura Ancona*, Italy
The case of a Vocational School for Tourism Operators was presented. Vocational schools in Italy operate on the basis of specific final "vocational profiles", which can be used as reference to identify the basic abilities required to transfer and use mathematical learning in the context of work-related projects. The teaching staff includes teachers specialized in the various vocational disciplines. It has therefore been possible to activate an analysis for some specializations, starting from the study of the vocational profile itself, to identify the mathematical knowledge and skills and to devise suitable activities bringing them to the forefront.

Constructing mathematical concepts. The effects of a writing workshop based on learner's own experience
Corinne Hahn, France
Students' practices in mathematics differ depending on whether they are solving a problem in class, or in a situation outside the classroom. They often have difficulties connecting school mathematics and out-of-school mathematics, which is a major problem in vocational education where students should be able to link professional experience to theory.

After presenting the conceptual framework, a system was discussed that is devised and experienced with sales managers-to-be in order to help them to connect knowledge learnt at school with business practices.

Mathematics needs of students in emerging technologies
Mary Ann and *Robert Hovis*, USA
The outcomes of CRAFTY-workshops, part of a bigger project, were presented. CRAFTY (Curriculum Renewal Across the First Two Years) brought together mathematics teachers, technical faculty people and people from industry. Most technical faculty believe that content should be addressed in ways that demonstrate connections between mathematics and other areas, as well as among mathematics topics. Students must be able to transfer the mathematics knowledge or skills to applications within their disciplines. Focusing on local businesses gives immediate relevancy to applications.

Mathematics faculty must help provide students with a variety of habits (soft skills) that will enable them to succeed in the workplace. The skills that employers want from their employees rarely include specific content. They want instructors to strengthen the student's ability to think, to communicate, and to be responsible.

A perspective on numeracy
Steve Thornton and John *Hogan*, Australia.
Numeracy certainly means more than having competence with a set of basic mathematical skills. This has serious implications for all teachers who are preparing young people for life, learning and the workplace. A Numeracy Framework was presented as a way of describing numeracy, diagnosing learning issues, supporting teachers' planning for teaching to students and workers so that they can choose to learn how to act numerately. Some practical ways of adopting this framework for use by teachers were briefly outlined.

Discussion

Based on the issues raised in the first three sessions, the final session was devoted to discussing some key questions about mathematics for work.

What is mathematics for/in the workplace?

It seems that a lot of discussion is focused on the question "can we call this mathematics or is it just general knowledge?" Why not just "coping with the quantitative/qualitative aspects of the reality (of the workplace) around us". Or should such aspects be incorporated in the definition of mathematical thinking and acting?

Given all the research that ends in "school mathematics and workplace needs don't fit together", what should a vocational mathematics program look like? What are the key issues to consider? (Steen, Hovis)

What about technology: What are 'appropriate uses'?

We need more insight into what technologies are important (use of spreadsheets, statistical quality control software). How deeply should these be understood so as to be manageable? We need more research on techno-mathematical literacy.

How about transfer?

One vision: Transfer to new situations is only possible after generalizing from the context-bound concrete problem situations and then apply the generalized knowledge to new situations.

This is the approach most often used in education. But the results are mainly very disappointing.

Another vision: Find a set of contextual and meaningful contexts which are not "too far apart".

Identify a structured situated abstraction. Will that enhance transfer to other contexts and situations? Not so much is known yet regarding this issue.

What status does research in vocational education have in the mathematics education research community?

Unfortunately the answer to this question is: hardly any.

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This report has been written by Henk van der Kooij and Rudolf Strässer with support from the team members. They are happy to be contacted at their respective institutions for further information on the work of this TSG.

TSG 8: Research and development in the teaching and learning of number and arithmetic

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Team Members: *Munirah Ghazali*, Science University of Malaysia, Malaysia
Joaquín Giménez Rodriguez, University of Barcelona, Spain
Wan Kang, Seoul National University of Education, The Republic of Korea

Aims, focus, and structure

This Topic Study Group brought together research developments from different countries relating to teaching number and arithmetic, and debated implications for classroom practices. The following three main topics were selected to provide a focus for the presentations and discussions: 1. Developing number sense; 2. Learning arithmetic through problem solving; 3. The role of contexts and models in teaching and learning about number and arithmetic. These topics were first addressed in the plenary talks that started each session. In addition there were refereed papers posted on the ICME-10 website for preliminary reading with authors contributing to discussions. These remain available at www.icme10.dk.

Plenary presentations

In the first plenary presentation entitled 'Developing number sense' *Alistair McIntosh* (University of Tasmania, Australia) elaborated on the ideas expressed in his seminal paper on number sense (McIntosh, A. (1992) A Proposed Framework for Examining Number Sense. *For the Learning of Mathematics, 12*(3):2-8), wherein he provides a conceptual framework for designing and assessing activities around number sense consisting of three components: knowledge of and facility with numbers, knowledge of and facility with operations, and applying knowledge and facility with numbers and operations to computational settings). Basically, McIntosh's paper involved a richly documented analysis of what goes wrong when teaching for number sense is largely absent in an elementary school mathematics curriculum, as well as giving several inspiring examples of instructional tasks and activities that favor the development of number sense. Although progress has been made worldwide, according to this plenary speaker, the basic ideas of the teaching for number sense are not yet (properly) implemented in most mathematics classes.

The next plenary lecture, 'Learning arithmetic through problem solving' by *Christoph Selter* (University of Heidelberg, Germany), distinguished two different types of goals in mathematics education curricula: process related goals (like conjecturing, describing, communicating) and content-related goals (like knowing the facts of the addition table by heart or adding three-digit numbers by means of the traditional written algorithm). According to this author, it is important that neither of the two dominates the other. In his talk he showed how both goals can be integrated in a systematic way. First, he showed why developing a mathematical disposition, including an ability and willingness to engage with mathematical thinking, should be considered as an important goal already in primary mathematics. Then he made some comments on how to learn basic skills (like the multiplication facts) in a meaningful and problem-based context. In the last section of his talk he sketched the approach of the Dortmund MATHE 2000 project, which succeeded in developing teaching/learning units wherein basic skills are not only practised, but also connected with the development of higher order thinking skills. In these units, the learning of the basic skills is treated as an opportunity for children to describe, conjecture and reason: in short, for developing a truly mathematical disposition already from a very young age.

In the third plenary lecture, 'The role of contexts and models' *Brian Greer* (San Diego State University, U.S.A.) took modeling to mean a correspondence established between some aspect of the environment and statements of arithmetic such that relationships within each domain have 'translations' to the other. He distinguished three forms of modeling. A first form where manipulatives and visual representations are models for arithmetic operations. Like several (socio-)constructivist authors e.g (Cobb, Gravemeijer), Greer drew attention to major problems with the use of such manipulations and pleaded for a more sophisticated view of the role of representations. The second form of modeling occurs when existing arithmetical knowledge is intended to be evaluated as an applicable model – or not – for a situation. Here, he pleaded that the essentials of this form of mathematical modeling can already be established early in children's development. Third, Greer introduced the term "developmental arithmetic modelling", to indicate an activity of mathematizing a situation in the course of which enhanced understanding of some arithmetic operation or structure emerges (as in the work of Freudenthal, Gravemeijer, Lesh, and others). This form of activity, stands in contrast with the two previous forms of modeling in which, in the former case, a pre-structured set of physical objects or a visual representation is presented, and, in the latter case, existing arithmetical knowledge is intended to be evaluated as an applicable model – or not – for a situation. According to Greer, this principle of reinventing mathematics through mathematizing a context 'rich and to be structured', as opposed to 'poor and structured', is deeply rooted in the theoretical constructions of Freudenthal and his followers.

Finally, *Julia Anghileri* talked on 'International perspectives on teaching and learning number and arithmetic and future directions'. She outlined some of the expectations in different national curriculum documentation and highlighted potential conflict in teaching for understanding while emphasis actually lies in teaching for fluency in computation. Although there has been a shift to encouraging informal approaches to calculating, she noted that students can undervalue these personal methods when so much time is given to teaching the compact traditional algorithms. She elaborated on research findings that show students need support in developing pencil and paper methods and discussed newer forms of algorithms that preserve number sense while providing a structure for written recording. The major question she identified concerns what we are trying to achieve in teaching calculating techniques for students in today's (and tomorrow's) technological society.

This TSG included also a number of papers refereed by the Organizing Team and circulated through the conference website. These provided a strong background for the discussions at the conference.

Papers discussed during Session 1:
Number sense

Informal strategies and adaptive expertise in proportional reasoning.
Silvia Alatorre and *Olimpia Figueras*, Mexico
Many different number concepts – or one integrated?
Ulrich Christiansen, Denmark
Building and stacking in a count and add laboratory.
Allan Tarp, Denmark
Exploring links across representations of numbers with young children.
Tony Harries and *Jennifer Suggate*, UK

Papers discussed during Session 2:
Learning arithmetic through problem solving

Reinvention revisited learning and teaching decimals as example.
Ronald Keijzer, Frans van Galen and *Lia Oosterwaal*, Netherlands
Two sides of a coin in teaching: An analysis of a lesson on comparing fractions.
Jinfa Cai, USA, *Ida Ah Chee Mok, Agnes Tak Fong Fung*, Hong Kong
The initial use of fractions on adults: the case of Enriqueta.
Marta Valdemoros Alvarez, Mexico
Why 25+4 might be 54: Children's interpretations of uncompleted equations.
Anna Susanne Steinweg, Germany

Papers discussed during Session 3:
The role of contexts and models in teaching and learning about number and arithmetic

Children's strategies for doing simple addition in an instructional environment that favors strategy flexibility.
Joke Torbeyns, Lieven Verschaffel, and *Pol Ghesquière*, Belgium
How basic arithmetic skills are obtained by children with learning difficulties?
Tadato Kotagiri, Japan
Multi-coloured natural arithmetic.
Jean-Noel Manouba, France
An experimental research on error patterns in written subtraction.
Carla Fiori and *Luciana Zuccheri*, Italy

Papers discussed during Session 4:
International perspectives and future directions

Teacher practice and student learning: An 'effective' mental computation lesson.
Ann Heirdsfield, Australia
Teaching Mental Calculation – how successfully are strategies being learnt?
Tom Macintyre and *Ruth Forrester*, UK
Narrowing the gap between mental computation strategies and standard written algorithms.
Ian Thompson, UK

In the final session, during a plenary discussion participants tried to identify the way forward for research over the next four years and looked at possible collaborations that may emerge from the conference. Some of the questions and issues that were addressed during this final discussion are the following:

How important is procedural flexibility or adaptiveness? What is the immediate and long-term effectiveness of aiming for this flexibility? Is it also feasible and valuable for young and weak children? Are there costs of aiming for procedural flexibility? What is the most appropriate way of teaching for procedural flexibility at the elementary school?

Is the problem-solving approach feasible and effective for all children? Do we all agree that a problem-solving approach, which focuses on encouraging students to (re)discover concepts and to (re)invent procedures, is the most appropriate approach to teaching arithmetic in the elementary school? What are the risks of this approach? Is the empirical evidence favoring this approach sound and convincing (even for the weaker students)? Is an evaluation of the value of this approach a (purely) empirical issue?

Is the 'emergent modeling' approach feasible and effective at the elementary school level? As argued convincingly by Greer, the term mathematical modeling is not only used as a synonym of 'applied problem solving'. Besides this type of modeling, which requires that the student has already at his disposal at least some mathematical models and tools to mathematize the problem situation, there is another kind of modeling, wherein model-eliciting activities are used as a vehicle for *the development* (rather than the application) of mathematical concepts. This second type of modeling is nowadays called 'emergent modeling' (Gravemeijer) or, as Greer termed it 'developmental arithmetical modeling'. Although deveopmental arithmetic modeling is getting more and more attention and approval among math educators, it also raises questions about its effectiveness and value. For instance: Do we *always* have to start from informal mathematical activity in real-world contexts in the elementary school?

Arithmetic: a subject for learning mathematics? Elementary mathematics can be viewed and taught as "basic" mathematics – a collection of procedures – or as "fundamental" mathematics. By fundamental mathematics, we mean that it contains already the basis of more advanced concepts and that it forms the foundations for children's further learning of these mathematical concepts. How can we ensure that current innovative approaches that cultivate informal and contextualized thinking, succeed in getting recognition as what we consider as the heart of mathematics (namely a search for patterns and relations) and the instructional attention that they deserve?

This report has been written by Julia Anghileri and Lieven Verschaffel. They are happy to be contacted at jea28@cam.ac.uk and lieven.verschaffel@ped.kuleuven.be for further information regarding the work of this TSG

TSG 9: Research and development in the teaching and learning of algebra

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Carmen Sessa, University of Buenos Aires, Argentina
Rosamund Sutherland, University of Bristol, United Kingdom

Aims and focus

TSG 9 investigated recent developments in the teaching and learning of algebra and provided participants with a forum for sharing and discussing their research endeavors, development projects, and experiences. The teaching and learning of algebra is a difficult area for study because across different countries, and even within countries, what is done in classrooms can be quite different. Against the backdrop of this challenge for international discussions of research and development in school algebra, the research group focused on:

- describing and understanding the variation of algebra in schools across the world; and
- the influence of technological developments on the algebra curriculum.

Session 1: International perspectives on algebra

In order to create common ground for conversations in our study group about algebra across the world, using the internet, prior to the conference participants in TSG 9 had access to three of the public TIMSS-R (Trends in Mathematics and Science Study, R for "repeat") videos: one each from Hong Kong, Japan, and Switzerland. During the first day of activity, researchers from these three countries commented on these videotapes in the context of outlining the nature of algebra in their society. They were joined by a colleague who presented a videotape from a Mexican classroom and members of the organizing team from the United States, the United Kingdom, and Canada. To orient discussion, as participants listened to these presentations, they were asked to consider the following four sets of questions:

- In your context, *what is algebra*? Is algebra seen as a central mathematical topic or as a less important one? Is it viewed as "the language of mathematics" or as a particular topic of mathematical study, or a set of methods to learn or problems to learn to solve? Is it taught as a separate entity or as part of some larger whole? Where would a visitor to your country find algebra in schools and how would the visitor know how to recognize it?
- Is the algebra curriculum in your context in *flux or change*? Are there tensions between what has been done and what reformers propose for school algebra? Are there different approaches to the content? What do you mean by an approach?
- Is instruction in algebra *similar to or different from other mathematical strands* in the curriculum? Is justification or argumentation different in the algebra classroom? If so, why? Are there patterns in the teacher role or in student participation that are peculiar to the algebra classroom in your context? Again, if so, how do people explain these phenomena? Is there a role for practice in the algebra classroom that is different from other arenas of the curriculum?

- In your context, is algebra thought of as *a difficult subject matter* for students to learn? If so, what are the explanations for this difficulty? When do students study this material? Do all students study this material? In your context, is algebra seen as a difficult subject matter to teach? If so, again why? Is students' motivation to study algebra seen as similar to or different from their motivation to study other aspects of mathematics?

Session 2: Theory in algebra learning

The second and third day of activities each focused on a particular topic. In session 2, *Barbara Dougherty* (USA) and *Toshiakira Fujii* (Japan) presented work involving the introduction of algebra to students. Barbara Dougherty presented a quantity-based approach to the use of algebraic symbols in grades 1-3. Discussion around her presentation involved the nature of the symbols that students used in the project. Toshiakira. Fujii advocated for the use of problems involving arithmetical identities that can lead to the use of numbers as quasi-variables. He suggested that students can come to use numbers as particulars that represent more general constructs, and they can identify and discuss algebraic generalisations long before they learn formal algebraic notation.

Session 3: Roles of technology in algebra curricula

In session 3, *Michèle Artigue* (France) and *Michal Yerushalmy* (Israel) gave presentations on the use of technology in algebra curricula. The presentations from these two authors related to each other closely. On the one hand, Michal Yerushalmy focused on the curriculum/technology nexus. She argued that there are key transitions in the nature of students' algebraic activity as they study algebra that can be used as a window into understanding how technology does or does not transform the algebra curriculum. She concluded that:

> "technologically-supported curricular change can lead to change in students' cognitive hierarchies, though such change may have as much to do with curriculum as it has to do with technology."

In a related vein, Michèle Artigue argued that whatever context in which students work algebraically, researchers must pay attention to the technologies with which they work. For example, she wrote:

> "the notion of algebraic literacy cannot be defined independently of technological considerations. It cannot be considered as something absolute and independent of technology. Each technology shapes algebraic thinking and activity in a specific way which depends on its affordances, constraints and limitations; each technology imposes specific mathematical needs, and a specific intertwining of mathematical and technological knowledge. Each technology shapes what has to be learnt in order to be algebraically literate and how it can be learnt."

Session 4: Short oral presentations and discussion

The final day of activity began with short presentations from colleagues from three more countries. *Charita Luna* (Philippines) presented a study in the context of College Algebra. *Jean-François Nicaud* (France) presented APLUSIX, a computer system for feedback on symbolic algebraic work. *Martin van Reeuwijk* (Netherlands) presented a game environment for the solving of equations.

The second half of the final session was devoted to a discussion with Michèle Artigue, Barbara Dougherty, and Michal Yerushalmy based on questions written by participants at sessions 2 and 3 in reaction to the presentations. The organizers took these questions and developed the following set of questions. Quick summaries of the discussions of each question are included.

1. *The emphasis in the presentations has been on the problem domain /use of technology. In what sense is this independent of the teacher?*

None of the authors feel that the problem domain or the use of technology is independent of the teacher. Barbara Dougherty reminded the audience that she is the curriculum developer and teacher in the project she described. Michal Yerushalmy and Michèle Artigue both underlined the importance of the institutional role of the teacher and the importance of understanding this role.

2. *What is an "approach to algebra" and can different approaches be combined?*

Michal Yerushalmy in particular called for researchers to be clearer about what they mean by an approach, whether it is a mathematical change to the curriculum, a change in pedagogy, or other changes. There was much discussion about where there is value in using terms like "approaches as a way to speak in a simplified manner to practitioners and policymakers about potential changes to the algebra curriculum. Michèle Artigue asked the audience to consider what is possible to change in a culture and to think about some work of researchers as attempts to "act on a culture." In her paper, Michal Yerushalmy also argues that research on curriculum might focus on the hypotheses that are made when a curriculum sequences student engagement in algebra in particular ways. Such research would be less focused on the effectiveness of a particular curriculum and more focused on the evidence supporting or undermining the hypotheses on which the curriculum is based. Barbara Dougherty's experience with her curriculum suggests that she needs to follow her students beyond fifth grade to continue to support the experiences they have had with the approach they have learned in earlier grades.

3. *If technology is not a magical wand for making the teaching of algebra easier, why should we endure the pain of change, why is it necessary and useful?*

Michèle Artigue argued against the assumptions in the question itself. Her paper suggests that technology is present whenever algebra is done. Martin van Reeuwijk suggested that technology may change who learns algebra. Ros Sutherland also suggested that some of the work on early algebra, like Barbara Dougherty's, might make work with computer technologies with older children less painful.

4. *How might the research on technology in algebra feed back into and inform more standard algebra teaching and bring new insights?*

Michèle Artigue suggested that technology can be useful for researchers by making changes in everyday classroom interaction. Such changes might allow researchers to see algebra classrooms from a new perspective and thus give insights that might not occur otherwise, not only into the use of technology, but also into standard algebra teaching.

She illustrated this contention with a number of examples of how computer algebra system (CAS) use brings in new forms of expressions that are potential windows on mathematical meaning. In her research, students' interactions with these new forms of expressions indicate they do not see the purpose of factorization. Such insights then have potential ramifications in classrooms not using CAS technology.

This report was elaborated by Daniel Chazan and Eugenio Filloy They will be happy to be contacted at dchazan@umd.edu and smmeef@aol.com respectively, for further information on the work of this TSG.

TSG 10: Research and development in the teaching and learning of geometry

Team Chairs: ***Iman Osta***, Lebanese American University, Chouran Beirut, Lebanon
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Team Members: ***Verónica Hoyos Aguilar***, National Pedagogical University, Mexico City, Mexico
David W. Henderson, Cornell University, Ithaca, USA
Ewa Swoboda, Rzeszów University, Poland

Aims and focus

The aim of this TSG was to examine and discuss recent research and developments in the teaching and learning of geometry at all levels of schooling from kindergarten to the university, and to present an overview of the current state-of-the-art in geometry teaching and learning. The four sessions held during the Congress, were the culmination of a year long process that we consider an integral part, probably as interesting and rewarding as the sessions themselves, and at the same time the most demanding and delicate. The exchange among the Organizing Team (OT) members and between the OT and the participants was very rich and challenging.

The OT members started by setting their premises: We wanted the TSG10 to be an open and refereed forum for exchange. A call for abstracts was launched by writing and widely distributing a reference paper, proposing themes of interest, and calling for contributions. Fifty-one colleagues responded. Abstracts were reviewed, then a call for full papers launched. A refereeing process was established, by which every paper was reviewed by two OT members. If the paper was to be rejected, a third opinion was sought. At the end, 29 papers were accepted and posted on the website.

The papers were classified under four themes, each theme was studied in one session during the ICME-10 Congress, under the leadership of one or two OT members:

1. Geometry outside the formal Euclidean mould (Leader: *David Henderson*)
2. New views on Dynamic Geometry Software use in geometry classes (Leader: *Iman Osta*)
3. From intuition to formal conceptions in the early grades (Leader: *Ewa Swoboda*)
4. Developing mathematical thinking and attitudes through secondary or college level (Leaders: *Verónica Hoyos* and *Harry Silfverberg*).

The theme leaders wrote, as well, papers setting the stage for the theme discussions and summarizing the papers corresponding to their theme. In order to allow for all papers to be presented, and for rich discussions, the OT opted for short presentations (5-6 minutes each) in 4 panels, followed each by 20-25 minutes of discussion. A note was sent to all participants, encouraging them to read all the papers (made available on the website) before the Congress. In addition to the theme leaders' papers, only 23 out of 29 accepted papers were actually presented. The other six papers' authors did not attend the Congress.

Following is a summary of the proceedings and the main ideas discussed in the four sessions:

Session 1: Geometry outside the formal Euclidean mould

In his paper "Geometry Outside the Formal Euclidean Mould", *David Henderson* (USA) looks into the origins of geometry, and identifies four historical strands, from which it emerged: Art/Patterns, Navigation/Stargazing, Building Structures, and Machines / Motion. The paper further traces the evolution of geometry and mainly the foundation of non-Euclidean axiomatic systems. It also traces Hilbert's and others' developments of more complete and abstract sets of axioms for Euclidean geometry, yet not ignoring "the tendency toward intuitive understanding". Challenging questions arose, such as "Do we still need to confine our geometry curricula to Euclidean geometry?"

Taro Fujita, *Keith Jones* and *Shinya Yamamoto* (UK and Japan) in their paper, "The role of geometrical intuition in the learning and teaching of geometry", argue that geometrical intuition is important – as a skill to create and manipulate geometrical figures in the mind, to see geometrical properties, to relate images to concepts and theorems, and to decide where to start when solving geometrical problems. Intuition is also identified by *Gloriana González* and *Patricio G. Herbst* (USA) in their paper, "Competing discourses defining the geometry Course: What was new in the twentieth century?", as they report that in the discourses in North America that defined school geometry courses there were four main arguments, including: "geometry as an opportunity to experience the work of doing mathematics, including the use of proof", a formal argument: "geometry as a case of logical reasoning", and a utilitarian argument: "geometry provides tools for future work and applications".

In the discussions, attempts to use simplified axiom systems in order to make the ideas more easily accessible than is possible using the full set of Hilbert's axioms for Euclidean geometry were presented. *Milan Hejný* and *Naďa Stehlíková* (Czech Republic) in their paper "Didactic simulation: Approaching deep ideas in geometry", investigate the use of "trileg mini-geometry", which has a simplified axiomatic structure, as a way to make these foundational ideas accessible to prospective teachers.

Among the papers connecting geometry to other mathematical disciplines, was "A geometry lesson using signed area" by *C.T. Zahn* (USA) who develops an extended geometry lesson that introduces middle school students to the connections between Euclidean geometry, computational geometry, analytic geometry, and algebra. *Nina Hayfa* (Lebanon), in her paper about the learning of vectors (one of the major concepts that link geometry to algebra), "Impact of the language on the conception of the vector", points out that the language used to describe vectors in textbooks and in the classrooms, causes confusion between bound vectors and free vectors.

Session 2: New views of Dynamic Geometry Software in geometry classes

Dynamic Geometry Software, DGS, had an important share in the discussions. Under the title "New views on DGS use in geometry teaching / learning", *Iman Osta* (Lebanon) opened the discussion by asking some questions that mark a shift in the type of issues related to DGS: How does DGS use in the classroom influence the debates between intuitive and formal geometry advocates? How are the DGS visualization capabilities affecting the necessity (or the opposite) of providing formal proofs? Does the use of DGS in the classroom create new types of geometric reasoning? How can DGS environments be compared to other mediating tools in the teaching of geometry? How would geometry curricula be modified to integrate the use of DGS?

Some of the papers which involved DGS issues also addressed the theme of geometry outside the Euclidean mould. Even though DGS was originally developed for the investigation of Euclidean geometry, they can also be used to study non-Euclidean, spherical, and hyperbolic geometries. *Bjørn Felsager* (Denmark) in "Introducing Minkowski-geometry using dynamic geometry programs" demonstrates that DGS can be used to give a non-axiomatic approach to teaching the non-Euclidean Minkowski geometry. Along the same line, *Margaret Sinclair* (Canada), in her paper "Adopting Cinderella and Spherical Sketchpad as exploratory tools: Some reflections on motivating factors", examines students' comments and assignments, in a graduate geometry course, who chose to utilize Cinderella or Spherical Sketchpad as additional exploratory tools outside of class.

DGS create computer microworlds with Euclidean geometry as the "embedded infrastructure". Nevertheless, *Francis Lopez-Real* and *Allen Leung* (Hong Kong) in their paper, "The conceptual tools of Euclidean and dynamic geometry environments" demonstrate that the function of "dragging" is a powerful tool in DGS that does not have a formal counterpart in Euclidean geometry. "Dragging" seems to be a conceptual tool that is, to the learner, as legitimate as the traditional Euclidean tools of compass and un-marked straightedge. The authors ask: Can we expand the usual formal Euclidean axiomatic system to include dragging?

Jeff Connor, *Laura Moss* and *Barbara Grover* (USA), in their paper "An obstruction to exploration with Dynamical Geometry Software" try to investigate whether or not students made effective use of DGS to explore the validity of geometrical statements, using Sketchpad. The analysis indicates that the way students regard the definition, whether as a 'dictionary definition' or a mathematical definition, affects the use of DGS. The effective use of DGS is also influenced by the ability to correctly parse a mathematical statement. *Thomas Gawlick* (Germany), in his paper "Restructuring dynamic constructions: Activities to stimulate the development of higher level geometric thinking", presents a sequence of tasks designed for student teachers using DGS. The aim is to reach higher level thinking. He relates DGS properties with the transitions through a revised version of the van Hiele levels, based mainly on an interpretation by Freudenthal.

Regarding teachers' use and attitudes toward DGS, *Lil Engström* (Sweden), in her paper "Examples from teachers' strategies using a dynamic geometry program in upper secondary school", presents examples of teachers' strategies when using Cabri. The assumption is that teaching strategies might depend on the teachers' definition of mathematics, on how they perceive the concepts of learning and knowledge, and on their experience of the computer program and experience of teaching.

Session 3: From intuition to formal conceptions in the early grades

Two of the main themes discussed in TSG 10 dealt with the learning of geometry at different levels, using various approaches. Reflecting on the teaching of geometry at an early age, *Ewa Swoboda* (Poland) stresses, in her paper "From intuitions to formal conceptions", the role of intuition in the teaching/learning of geometry, whether in the creation of the geometrical world that emerges from the real world, or in understanding space and relations between figures as a dynamic space organization. She identifies geometry as a way of persuing long-term aims related to the philosophy of mathematical thinking: from perception to definition and mathematical formulation; and for finding the general in the particular. *Mariolina Bartolini Bussi*, *Maria Alessandra Mariotti* and *Franca Ferri* (Italy) stress, in their paper "Semiotic mediation in the primary school", the opinion that the presence of artefacts does not mechanically determine the way in which they are actually used and conceived of by the students. They distinguish between (at least) two types of artefacts: a primary artefact (e.g. concrete instrument handled in the solution of problems) and a secondary artefact (e.g. text or system of signs). Their research hypothesis is that the intrinsic polyphony of the artefacts supports the production in classroom activities of the polyphony of voices (forms of speaking and thinking).

Based on Fishbein's "theory of figural concepts", *Edyta Jagoda* (Poland) conducts experimental work to investigate children's intuition of mirror symmetry in a plane. Her main research aim, as expressed in her paper "Perceiving symmetry as a specific placement of figures in the plane by children aged 10-12", is: How does the children's perception of the relationship between one shape and its transform shape in the plane, contrast with their noticing the dynamism of the transformation. Under the same topic of transformations, *Charlotte Bouckaert* (Belgium) describes a proposal (among others) to approach the notion of orientation of plane or space. In her paper "Some aspects of transformation geometry in primary school according to Michel Demal", she presents Demal's way of using a spiral curriculum in the spirit of Jerome Bruner. By comparing two figures and by using transparent sheets, children begin to become familiar with transformations.

Paola Vighi (Italy), in her paper "The geometry of squared paper" describes results of using squared paper for a task of drawing isosceles triangles. She found that, though the grid might help children, it might as well interfere with what the pupils have in mind with regard to isosceles triangles and act as an obstacle, thus making the task more difficult.

The emergence of the geometrical world from the physical world and from various activities was frequently visited. *Nancy Vezina* and *Lucie DeBlois* (Canada), in their paper "Geometry in context at the primary level: Using the environment as a starting point", suggest that the living environment can be used as a starting point for teaching about different geometrical shapes. They found that through different learning activities using the environment, children create and use a wide range of procedures that differ from those that are usually developed at school. In the same spirit, in the paper "Drama in teaching and learning geometry", *Asuman Duatepe* and *Behiye Ubuz* (Turkey) suggest that drama creates an environment in which students construct their own knowledge by means of their experiences. By using this method students build their meaning of a word, a concept, or an idea. In the paper "Geometrical pre-conceptions of 8 years old (third grade) pupils", *Carlo Marchini* and *Maria Gabriella Rinaldi* (Italy) use two visual representations for the concept "isosceles triangles": as a flag, and as a roof; saying more precisely that they use drawing "orientation" in the perception of "isoscelity" of triangles. They test how those orientations have an impact on children's mental representation for isosceles triangles.

We can see "reality" in a larger perspective. It could mean a comfortable environment, which can support mathematical thinking processes. For some students, a rigorous way for thinking, necessary for making mathematical proofs, is too difficult. They need to have some support by using very familiar facts, which they can imagine, draw, etc. *Michael Koren* and *Dan Amir* (Israel) in their paper "The rectangular approach – A royal road to Euclidean geometry in intermediate school" create such an environment for teaching Pythagoras' theorem and its proof.

Session 4: Developing mathematical thinking and attitudes through secondary or college level

Verónica Hoyos and *Harry Silfverberg* (Mexico and Finland) state, in their paper "Developing mathematical thinking and attitudes through secondary or college level", that many mathematicians consider geometry to be one of the branches of math least contaminated by rules, formulas, or algorithms in discovering and solving problems. Yet they recognize this feature as being problematic for the teaching and learning of geometry, especially at the level of deductive reasoning, a main area of current research. This paper identifies some of the major theoretical perspectives which serve as a background to current research in didactics of geometry (Piaget, van Hiele, cognitive science, and more recently socio-cultural approaches, using the theoretical constructs of Vygotsky and his followers). This line of research continues with the incorporation of new technologies, like dynamic software of geometry and internet chatting.

In his paper "Describing undergraduates' geometric thinking via an "object of thought" interpretation of the van Hiele model", *Stephen Blair* (USA) presents a review of the van Hiele model and its evolution through research works. He uses the "object of thought" interpretation of the model to describe undergraduates' geometric thinking and use of definitions across three different geometries (taxi-cab, spherical, and Euclidean), while documenting transitions across levels 3, 4, and 5 of the model.

Jaguthsing Dindyal's (Singapore) paper, "Students' thinking in school geometry: The need for an inclusive framework", raises issues about geometric thinking and the need to conceptualize it within a broad framework. The paper is the outcome of a piece of research investigating students' use of algebraic thinking in geometry, while solving problems which involve, among other things, the use of variables and unknowns, writing and solution of simple linear equations, and recall and use of formulae within geometry. Three aspects of algebraic thinking were investigated: the use of symbols and algebraic relations, the use of representations, and the use of generalizations within geometrical contexts.

Through placing students in a problem-situation, *Naim Rouadi* (Lebanon), in his paper "The development of geometrical thinking of Lebanese students aged 11 – 15", focuses on the geometrical thinking of Lebanese learners, taking into consideration van Hiele's five levels. The paper documents how the analytical perception of the problem allows the learner to move from drawing to figure (as a mathematical model) and between two registers: from geometry to arithmetic.

Oleksiy Yevdokimov's (Ukraine) paper "Skills of generalization in learning geometry. Are the students ready to use them?" tries to study the generalizing difficulties faced by students. It identifies three types of generalizations: generalization of definitions for different geometrical objects, generalization of geometrical object's properties by giving up certain features, and creative generalization.

Conclusion

The above survey of research studies, reflections and discussions raised in the TSG 10 on geometry shows the richness of this field, the various orientations and perspectives under which the teaching/learning of geometry can be studied. Richness, yes, but maybe fragmentation and lack of focus, as well. Beginning 2005, the NCTM's Research Committee presented a call for an Agenda for Research Action in Mathematics Education.

Now may be the time for us to define for ourselves some key issues for joint research and to consolidate the community that investigates the teaching and learning of geometry.

This report was written by Iman Osta and reviewed by the OT members of TSG 10. The author is happy to be contacted at iman.osta@lau.edu.lb for further information on the work of this TSG.

TSG 11: Research and development in the teaching and learning of probability and statistics

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Team Members: ***Dani Ben-Zvi***, University of Haifa, Israel
Manfred Borovcnik, University of Klagenfurt, Austria
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Aims and focus

While the teaching and learning of probability has been taking place in the context of mathematics for more than a century, the special challenges it poses when the audience is more broadly defined to include students (and, often, their teachers) at all ages and levels arise most sharply in the context of its application in the field of statistics. Statistics and statistics education are relatively newer disciplines. Statistics has only recently been introduced into the main stream school curricula in many countries. At the university level, there has been a steady increase in the numbers of statistics courses taught to fulfill the growing demand for students and professionals who can use and understand statistical information. Although the amount of statistics instruction at all levels is growing quickly, the research to support statistics instruction is advancing more slowly. The research literature in teaching and learning of probability and statistics is not well known and, hence, not often used. In fact, people working in this field still need to spread its influence – not only to achieve academic recognition, but also to convince others of its usefulness.

Teachers at all levels find that teaching statistics and probability is immensely challenging. Not only are there new developments in and approaches to the subject matter, but also there are opportunities afforded by access to new instructional materials and methods and more advanced educational technology. At the same time, the difficulties that students have in learning statistics and probability pose major problems for teachers. While developments in statistical software and hand calculators have eliminated much of the computational burdens associated with applying statistics and probability, the difficulties posed by the basic worldview inherent in those subjects are just as challenging as ever.

Mindful of these multiple contexts and challenges, we sought to encourage presentations at ICME-10 that would help us grow as professionals involved in this educational effort, that would represent the diversity of the work being done across the globe with students of all ages and contexts, and that advanced our knowledge of the possibilities and challenges facing us as educators. In general, we were successful in all of these goals.

The presentations made for TSG 11 at the meeting in Copenhagen were organized into four sessions:

Session 1: Exemplary work in statistics education

The session began with opening remarks by co-chairs *Joe Wisenbaker* and *Jun Li*, featured an invited address by *Jane Watson*, Australia, and presentations by *Iddo Gal* and *Dani Ben-Zvi*, Israel, and *Susan Starkings*, UK.

Jane's address provided an overview of much of the work that she has been involved with over the last decade. The work she spoke of, much of it in collaboration with other researchers, included explorations of students' statistical concepts, the products yielded from groups of students working collaboratively, the assessment of teachers' concepts related to statistics and chance, the effect of introducing cognitive conflict into discussions with individual students, students' understanding of variation, and formulating a broader model of statistical literacy. All of her work was presented as ongoing threads of inquiry woven in and around topics critical to the teaching and learning of statistics.

The work presented by Iddo and Dani centered on an examination of the ways that official statistics agencies have explicitly supported statistics education through their websites and a review of the kinds of materials that they have made available. Some part of their work actually revealed instances in which such potentially valuable resources have become more scarce in recent years, perhaps due to funding limitations facing those agencies.

Susan's presentation focused on a collaboration between the United Kingdom's Royal Statistical Society and the National Academy for Gifted and Talented Youth. It explored efforts to encourage some of the brightest and most able students to engage in the study of statistics and to consider careers in the field.

Session 2: Research on reasoning about variation and the use of technology in statistics education

The session featured an invited address by *Mike Shaughnessy*, USA, presentations by *Robert delMas* and *Yan Liu*, USA, *Dor Abrahamson* and *Uri Wilensky*, USA, and a discussion by *Maxine Pfannkuch* and *Dani Ben-Zvi*.

Mike's address centered on the work he and others have been engaged with in terms of students' incorporation of concepts of variability with respect to judgments about representations of distributions of data, both real and contrived. At issue in his work has been the extent and the ways in which students at the secondary level actively talk about variability as they interpret the meaning of distributions of data. This work forms the central theme in his ongoing research program funded by the U.S. National Science Foundation.

The presentation by Robert and Yan focused on students' understanding of factors affecting the value of the standard deviation in a collection of data. It was based on in-depth observations of and interviews with university students enrolled in an introductory course as they worked through a set of tasks based on graphical representations of data distributions that varied in their degree of variation.

Dor and Uri's paper explored the use of a rich, collaborative and interactive computer-based learning environment with which they have been involved for the last five years. It illustrated the ways in which they have exploited that environment and examined students' development of statistical concepts at the elementary level (12-13 year olds).

Session 3: Teaching statistics from multiple perspectives

The session featured an invited address by *Joan Garfield*, USA, presentations by *Robert Gould* and *Roxy Peck*,USA, *Alejandra Sorto* and *Alexander White*, USA, and a discussion by *Manfred Borovcnik*.

Because of health issues, Joan's presentation was actually delivered by her colleague, *Robert delMas* from detailed notes and discussions with her at the University of Minnesota just prior to ICME. It centered on the use she and her colleagues have made of the Japanese Lesson Study approach in thinking about statistics instruction, its goals for students, and ways in which lessons might be made more effective for them. Rather than using learners as the primary information source, this approach elicits ideas from the perspectives of experienced statistics educators with student feedback in the form of their success (or lack thereof) in learning from the various lessons planned for them.

Alejandra and Alexander's presentation focused on the issues surrounding requirements that mathematics teachers use to teach units in statistics to their students. Although reforms have done much to promote the teaching of statistics across the mathematics curriculum, the simple fact is that many practicing teachers lack the same skills and concepts as their students. They talked about their efforts to assess the content and pedagogical knowledge such teachers bring with them and the kinds of problems such teachers have with that material.

Robert and Roxy's presentation extended those ideas by focusing on their work in creating a new professional development program to help existing secondary mathematics teachers become more effective in teaching statistics. Their presentation illustrated the work they have done, an assessment of its effectiveness, and their plans for further development

Session 4: Exploring issues of reasoning about distribution, data and graphs

The session began with an invited address by *Koeno Gravemeijer*, The Netherlands, presentations by *Yingkang Wu*, Singapore, *Helen Chick*, Australia, *Carlos Monteiro*, Brazil, and *Janet Ainley*, UK, *Maxine Pfannkuch*, *Stephanie Budgett*, *Ross Parsonage* and *Julia Horring*, New Zealand, and closing remarks by *Joe Wisenbaker* and *Jun Li*.

Koeno's address focused on students' development of the concept of data distributions. He described a guided reinvention approach for instruction whereby students, using visualization tools, started with a set of measurement values and moved through ideas of data points and density towards understanding the concept of a density function. He argued that students should start with comparing data sets and that data sets should be tailored towards significant statistical issues.

Yingkang's paper examined Singapore secondary students' understanding of statistical graphs. The work was based on students' performance on a formal assessment of a variety of concepts based on a framework encompassing graphical reading, interpretation, construction and evaluation assessed from the standpoint of final answers and processes employed. It revealed both the level of students' current understanding and suggestions for modifying instruction to improve that understanding.

Helen's work looked at students (aged 11-13) and how they attempted to represent associational relationships in a set of data using graphs. Students' efforts were seen as clearly indicating some readiness for working with concepts generally not covered till late in high school. Students also differed considerably in the approaches they used suggesting the potential value in having these kinds of topics addressed through explicit instruction even for student as young as these.

Carlos and Janet's presentation focused on primary students' interpretation of graphs especially in the context of what has been termed 'Critical Sense' – the analysis of information beyond the initial assertions made by authors. In looking at the interpretations made by primary school student teachers, they concluded that encouraging a 'Critical Sense' of information will demand explicit efforts to focus students' attention on a wider array of considerations than only those that might be easily tied to material drawn from the context of everyday living.

Finally, the presentation made by Maxine of her work with colleagues in New Zealand examined difficulties encountered by 15-year-old students in interpreting data plots as part of the curriculum. Their work on creating a framework that teachers might use to embed the teaching of such concepts was seen as providing a context more encouraging of students' learning of formal inference.

There were several excellent papers chosen for presentation by distribution. Many of them were directly related to the themes around which we organized our sessions. Some added their own, unique perspective on the issues. Everyone is strongly encouraged to examine the submissions made by *José Carmona*, Spain, *Christine Duller*, Austria, *Sibel Kazak* and *Jere Confrey*, USA, *W. M. Luh*, *J. H. Guo*, China Taipei, and *J. M. Wisenbaker*, USA, *Mike Perry* and *Gary Kader*, USA, *Milo Schield*, USA, and *Ödön Vancsó*, Hungary. These contributions have been posted to the conference web-site on the main page for TSG 11.

The presentations were highly stimulating, generating a wealth of questions and comments that greatly exceeded the limited time available for the formal portion of TSG 11's part of the program. Discussions often spilled out into the hall and carried on with small, informal groups of presenters and attendees. Although no single meeting could possibly achieve all of the goals we had in mind as we organized the sessions, it was highly successful in promoting the discussion of important issues and, we hope, collaboration among the international workers who care so much about the teaching and learning of probability and statistics.

This report was written by Joe Wisenbaker and modified by Jun Li, Maxine Pfannkuch, Dani Ben-Zvi, and Manfred Borovcnik. They are happy to be contacted at wisenbak@bellsouth.net and lijun@math.ecnu.edu.cn respectively, for further information on the work of this TSG.

TSG 12: Research and development in the teaching and learning of calculus

Team Chairs: *Johan Lithner*, Umeå University, Sweden
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Ana Isabel Sacristán Rock, CINVESTAV, Mexico City, Mexico
David Tall, University of Warwick, United Kingdom

Aims and focus

In preparing for this working group, the organisers made a call for papers across a wide range of possibilities, including, but not confined to: the role and use of technology in the teaching and learning of calculus; the role and use of history in the teaching and learning of calculus; research about cognitive process in the learning of calculus; pre-calculus; from calculus to mathematical analysis (or) transition between secondary school and university; contextual approaches; graphic approaches; didactical engineering; misconceptions and epistemological obstacles in the learning of calculus.

The papers received covered a wide range of topics from which seventeen were selected for presentation on the ICME website with eight having short presentations in the topic group, preceded by two introductory lectures and a final session ending with a one hour round-table summarizing session.

The papers as a whole covered a range of different approaches to calculus for different target groups. Several papers focused on the nature of the limit concept, which in turn focused on the distinction between calculus and mathematical analysis. Most papers made reference to the use of new technologies, either through their increasing use in teaching or through research on their effects in learning. Several papers formed didactical structures that were used to guide the construction of curriculum goals and also as a basis for analysing the practices of learning and teaching.

To frame the discussions the organisers focused on three sets of questions to be kept in mind during the lectures and to form the basis of the final round-table debate. These questions were as follows:

1) *How can we differentiate the teaching of calculus or mathematical analysis according to the target public?* What means and criteria are important to realize a particular approach adapted for a specific public? How do we address the difficulty of the initial limit concept? Is it helpful to see the first approach to calculus through a blend of embodied meaning and symbolism and to postpone formalism to a study of mathematical analysis? How do we deal with the problematic transition between calculus and mathematical analysis?
2) *What is the role of technology?* How can we characterize and categorize more deeply the ways of using new technologies to teaching calculus and mathematical analysis? How do we take into account its use as a mathematical tool to solve problems, as a means of delivering the curriculum, and as a cognitive environment for learning?
3) *What do the various didactical theories bring to structure the preceding questions?* How do we evaluate the uses of new technologies in the learning of the students? What about the teacher's practices? How can we evaluate effects of these practices?

Summary of presentations

In the opening lecture, *David Tall* presented a framework for mathematical thinking that distinguishes three different modes of operation: the embodied world, "based on our sensory experiences and characterized by thought experiments," the symbolic world, "based on our use of symbolism to carry out calculations and manipulations," and the formal world that "relates to the building of formal theories based on definitions and proofs." This framework was used to formulate the growth of ideas in the calculus, including two significant discontinuities: the shift from finite processes in arithmetic and algebra to the potentially infinite limit concept and the shift from embodied thought experiments and symbolic calculations to quantified definitions and proofs. He suggested that the new technologies benefit the symbolic world by "performing calculations and symbol manipulations at a level of accuracy that would be difficult or impossible by hand." They also benefit the embodied world "in a more subtle way by providing an enactive interface (...) that allows the user to control and experiment with visual representations.. On the other hand, the formal world has the least benefit because of the gap between "the finite machine and the actual infinity of the theoretical limit process."

Isabelle Bloch, France, and *Maggy Schneider* presented a francophone view using the theory of the didactical situations of Guy Brousseau to analyse questions about learning and teaching calculus and analysis. They revealed a progression from situations to help students to contrast a view of mathematics about instantaneous velocities and curvilinear areas to manipulation of quantifiers in the production of graphs of functions that satisfy given conditions. Through the use of Brousseau's three connected types of dialectics of action, of formulation and of validation these situations formulate an analysis that has links with two or more of the three worlds theorized by David Tall. One important criterion which emerges from analysing new trends in teaching is the learning by adaptation created by an appropriate milieu (a term for the context of learning introduced by Brousseau): the students have to construct concepts that are mobilized by necessity in their own problem solving processes, rather than from the usual didactical technique (teachers' explanations) showing the objects of knowledge

Deploring the frequent lack of a proper but necessarily difficult approach to real numbers in high school or in university level calculus and the implicated misconceptions with the students, *Talma Leviathan* from Israel proposed "a project of a transitional course centred around number systems, where the real numbers wouldn't be introduced through an abstract set-theoretic definition but rather through the concrete notion of decimal expansion". This project articulates a geometric approach by "filling the gaps" in the "rational numbers ruler" and a more abstract approach where a real number is defined as a decimal expansion to which one associates its infinite sequence of truncations. On that basis the classical properties of the real numbers can be rigorously proved.

Laure Barthel, Israel, presented a computerized interactive module that familiarizes the students with decimal expansions of real numbers. With the help of this module, students are encouraged to investigate the properties of decimal expansions (e.g., periodic ones) to have a concrete introduction to several issues concerning the notion of limit.

Michel Helfgott, USA, who teaches calculus at college level in the USA, presented and illustrated five guidelines which to him appear to be important in the teaching of first-year calculus:

- Try to strike a balance with regard to what to prove and, what to accept without proof. For example, we can rely on geometric intuition in the case of the mean value theorem and prove an unexpected result such as the derivative of the product of two functions.
- Convey the idea that sometimes there is more than one way to solve a problem; for instance, to calculate some integrals by substitution or integration by parts.
- Discuss significant applications in the classroom, not relegating them to the end as optional materials; for example, problems about kinematics, optimization, physical work and so on.
- Place the subject in a historical perspective whenever possible, like summations linked to problems of integration across history.
- Use technology to supplement mathematical learning, not to supplant it; for example, having students learn how to build programs related to Newton's and Euler's method of approximation.

Victor Giraldo and *Luiz Mariano Carvalho* from Brazil described a qualitative study of the effects on learning caused by an approach to differentiation based on the embodied idea that a differentiable function, when magnified locally, 'looks straight'. Their approach uses computer technology including a software called *Best Line* and the symbolic manipulation software *Maple* to study both simple cases in which the graphic picture is a good representation of the numeric processes and also cases where finite computer arithmetic is compromised and produces pictures that fail to fit the expected theory of limits. The conflict between the finite world of computers and the perfection of human thought experiments is used to enrich formal meaning with a suitable pedagogical approach.

Erhan Bingolbali from Turkey showed the influence of the departmental affiliation of students (first-year undergraduate mechanical engineering and mathematics) on their developing conceptions of the derivative, based on several kinds of data: quantitative (pre-, post- and delayed post-tests), qualitative (questionnaires and interviews) and ethnographic (observations of calculus courses and `coffee-house´ talk). The findings reveal that mechanical engineering students develop a tendency to focus on rate of change while mathematics students develop an inclination towards tangent-oriented aspects. He argued that this difference cannot be solely attributed to the practice of the courses that the students had followed. He further suggested that departmental affiliation appears to have an influence on cognition, and plays a crucial role in the emergence of different tendencies between the two groups.

Yury Shestopalov and *Igor Gachkov* from Sweden described their method for using mathematical software in courses based on the "real-time-mode" for teaching university students in university. Demonstrations and computations are performed directly in the classes using calculators, PCs, desktops or workstations. The authors focused on an example concerning interpolations with natural splines. The main argument which supports this approach is as follows: the use of CAS programs is often reduced to pure illustrations of computing processes and is unsuitable for non-conventional mathematical courses which gain increasing popularity and which become indispensible for the modern engineer, such as coding theory, discrete mathematics and scientific computing.

Isabelle Bloch, France, and *Imene Ghedamsi*, Tunisia, brought to the fore factors of rupture between the secondary mathematical organisation of (pre-)calculus teaching and the university approach to the case of the concept of limit. They invoked various theoretical references: tools introduced by Aline Robert to distinguish between different functionalities of the limit and several sorts of knowledge (technical, or summonable, or available); the anthropological theory of Yves Chevallard who modelled mathematical activities in terms of tasks, techniques, technologies and theories; the distinction made by Anna Sfard between procedural and structural approaches; and Raymond Duval's notion of semiotic representational settings. These theoretical frames allowed the authors to identify the main didactical variables which measure the very important rupture between the two levels: the degree of formalisation, the setting of validation, the degree of generalisation, the number of new notions introduced in the limit environment, the type of tasks, the choice of techniques, the degree of autonomy of the students, the process or object status of the concepts and the nature of the transitions between the semiotic representation settings at issue.

Elfrida Ralha, Portugal, *Keith Hirst*, United Kingdom, and *Olga Vaz*, Portugal, presented a form of cooperative learning using Mathematica in the teaching of multivariable calculus for first-year university informatics' students in Portugal. The size of the traditional classes (200 in lectures and 65 in tutorials) was not changed, but a new methodology was introduced in tutorial classes, partitioning the allocated time into "students working on their own", "students sharing solutions" and "the lecturer summarizing the fundamental ideas for the session". Some of the examples treated offered the students the opportunity to realize misconceptions that can arise from graphics and to express their conceptual doubts. The aim is to engage them in mathematical/productive reasoning and to motivate them for the necessity of algebraic/analytical justification. A qualitative study as well as statistics show, among other things, an improvement in the pass rate for the course. There were also indications of a more positive attitude towards the course among students.

Salahattin Arslan, France, and *Hamid Chaachoua*, France, talked about the dominance of an algebraic approach to teaching differential equations in the upper secondary school in France and the lack of numeric and qualitative study of ODEs. They formulated the hypothesis that the limitation to the algebraic frame only for the treatment of differential equations can be the origin of difficulties and habits which students face with qualitative interpretation tasks, for example:

- difficulty in recognizing that the slope of the tangent can be calculated from the differential equation;
- difficulty in distinguishing a differential equation, in particular a non-linear one, from other types of equations.

The authors described a module on differential equations set up within a framework of training teacher-trainees to test this hypothesis, among other things. They explained reasons why one can hope to gain insight from the use of software, in particular dynamical software such as *Cabri*, because it can provide students with opportunities to discover new resolution tools.

In the final discussion session, Michael Thomas David Smith and De Ting Wu in their responses framed their presentations with regard to the initial three questions: relating to the variations in approach that may be appropriate for different target groups, the role of technology and the role of theoretical frameworks.

Michael Thomas, United Kingdom, focused on the role of technology and various theoretical frameworks, asking whether technology should be used to teach familiar ideas better or to teach new ideas appropriate to the new situations made possible by technology. He suggested that research and development to date has been either driven by practice or by theory but that there are signs that the two approaches are converging.

David Smith, USA, focused initially on the question of different approaches for different target groups and suggested that in his own university the decision was taken that there were only three major concepts to be studied: rate of change, accumulation, and the relationship between the two, so that all students should attend the same course, with the possible exception of mathematics majors who may require a more comprehensive understanding of analysis. He suggested to adopt a new theoretical framework by Kolb that emphasized learning activities that start with "concrete experience", then proceed through stages of "reflective observation", "abstract conceptualization", and "active experimentation". He related this to a neurophysiological theory of Zull that linked the way in which the Kolb cycle of learning relates to the use of different parts of the brain. He concluded by speaking about his own ways of using computer technology to teach the calculus.

De Ting Wu, USA, presented his professional opinion based on teaching central parts of the traditional college calculus and using new technology where this was appropriate. He acknowledged that technology is a powerful tool and a helpful aid in teaching and learning but affirmed that it would not replace the study of mathematics or the value of good teaching.

In the closing discussion, *Bronislaw Czarnocha*, Poland, questioned the strong emphasis in the discussion on the cognitive difficulties in understanding calculus and emphasized instead the power and beauty of the subject. Maggy Schneider argued that it was necessary to subordinate the analysis of teaching using new technologies to more general theoretical frameworks about learning and teaching relating to the nature of the educational institutions and teachers' practices.

The TSG provided a fruitful platform for sharing many ideas, with anglophone and francophone theories meeting in a constructive manner and research from many parts of the globe showing that the study of calculus remains vigorous and creative in many different ways.

This report has been written by the team chairs and members and by the paper contributors of TSG 12.
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ICME 10 2004

TSG

Topic Study Group 13

TSG 13: Research and development in the teaching and learning of advanced mathematical topics

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Aims and focus

The work of TSG 13 over the four days consisted of interactive panel discussions based upon several papers reviewed and accepted prior to the conference, reactions from mathematics education researchers, and group discussion. The work was focused on three intersecting areas:

1. Research on the teaching of advanced topics;
2. Research on the learning of advanced topics;
3. Development in the teaching and learning of advanced topics.

Given the widespread nature of curriculum in many countries, the organizers did not want to specify what advanced mathematical topics are but aimed to concentrate on topics taught post-16. This proved an important issue and was debated on the first day.

The main aim of the group was to assimilate, realize and discuss the current state of research and development in the teaching and learning of advanced mathematical topics and to present a broad outlook, and a potential research agenda, for new and existing researchers in the international community. The general format consisted of a preliminary introduction posing questions that had been disseminated prior to the conference, sharing thoughts, and, following the first day, reactions to work on previous days. Brief statements by the contributing authors, followed by discussion of the whole group in an interactive form with the contributors and team members, followed this introduction. Each of the first three days then concluded with a reaction from an invited, senior researcher to respond and extend the discussion. The format worked very well, and what was most noteworthy was the diversity of reactions and examples that the group shared with each other, coming from at least 15 different countries.

The final day concluded with a whole group discussion to bring participants' views together in a brainstorming effort that had several strands of debate and reactions to previous ideas. The sections which follow aim at summarizing the key contributions of each session, issues for debate and potential future work.

Session 1: Research on the teaching of advanced topics

The questions posed prior to the conference were:

- What skills are necessary to teach advanced mathematical topics?
- What types of links occur between pedagogy and learning, and of what consequences are they to our research?
- What modes of instruction are used in today's classroom?

[1] Caroline Lajoie, due to personal circumstances, was unable to attend the congress, but was involved in planning the programme of the TSG.

- How do these link to curriculum and innovation?
- How do we know what our students know/understand/apply in later work?
- What is effective teaching in undergraduate and graduate contexts?
- What techniques of assessment are used and how do we measure their effectiveness (links to technology)?

Ghislaine Gueudet (France), from the organizing team, led an interactive panel discussion highlighting the importance of the ICMI study edited by Derek Holton (2001) that summarizes theoretical perspectives of teaching and learning by prominent researchers such as Houston, Mason and Dubinsky.

A main theme through the first session was the introduction of advanced mathematical topics in earlier grades. *Tomoko Yanagimoto* (Japan) introduced her work with grade 6 students working with Euler circuits made out of rope, graph theory being proposed as one accessible course to young learners with suitable manipulatives. Using ropes, which could be twisted, allows students to directly construct and examine vertices or branches. This was said to help student address common misconceptions in logic early on in their learning. *Yury Shestopalov* from Sweden, discussed the potential of teaching applied mathematics, particularly numerical methods to pre-university students using natural contacts with 'pure' mathematics and internet resources. This work was based on the work of Kolmogorov (1964) who created an institute for pre-university education.

There was a genuine concern by the group of the role of, and particularly the need for, laying fertile 'seeds' early in children's education to prepare them for advanced mathematical topics later. In fact some were concerned with the paucity of the past 50 years of mathematics development in mainstream curricula today. *Olivier Gérard* (France) highlighted a project in France where 10-16 years olds are put into contact with professional mathematicians to discuss contemporary problems. Others were concerned with ideologies of contemporary students that contain mathematics to a series of finite, solved problems so that mathematics is finite, a major concern for university teachers.

Jean-Luc Dorier (France) offered some thought provoking reactions questioning whether logic was at the heart of mathematics, and extending the conversation with the question "what is ability in mathematical thinking?" Students can have good scores in traditional logic courses without being able to transfer logic to linear algebra, say. There is a difference between logical thinking and logic. He also cautioned the use of advanced mathematical topics for early learners. For instance, in the example of students using ropes to access core ideas in graph theory (first paper), the child reacts to the milieu first – the richness of the learning environment and the child's interaction with it (see Brousseau, 1997) – but for the child to progress, there needs to be a 'lifting' from the physicality of the situation to develop abstract reasoning. In this case, the students must be able to ignore the rope.

Session 2: Research on the learning of advanced topics

Our discussion began with the following questions:

- How far have we come in terms of theoretical models since the seminal collection of work edited by Tall (1991)?
- How do we conduct research in the development of mathematical thinking with students in college or university?

- What methods have we used?
- What data have we collected?
- How can this data be related to our theoretical claims?
- What are our main conclusions and remaining questions?
- What models or theoretical ideas are useful, and can we critique the claims they make about the learning of mathematics (for example, APOS theory, ideas of encapsulation and reification of a process and a mathematical object, theories of 2D and 3D visualization, metacognition, semiotics and the evolution of symbols systems)?

Two presentations focused on large scale or long term studies of university students learning advanced mathematics. *S. M. Hashemiparast* (Iran) discussed the results of a study of 30,000 students from five different state universities in Iran. Several subjects were analyzed and his conclusion, using various optimization techniques, was that learning abilities were not a function of the types of assessment or measurement used but, instead, of motivation and feedback. *Ildar Safuanov* from the Russian Federation described his new program for concentrated teaching of algebra and number theory. One main feature is "Linkage" which connects to student anticipation, concrete thinking and contrast with other forms of problem-solving.

David Tall (UK) offered some excellent reactions to the group. He questioned how far we can use these large studies, and thought we should be discussing advanced mathematical thinking. He outlined his own theoretical perspective to offer some grounding of the discussion on the development of concept images towards a "structure theorem" through "formal embodiment" – which structures our feelings into concrete and then formal theorems. One example he gave was: $a \sim b$ and $b \sim c \Rightarrow a \sim c$. This structure is fine when dealing with order relations such as ordering points on a line but we need to be concerened with what a student gives or brings to a mathematical context when using them more broadly as equivalence relations.

Session 3: Development in the teaching and learning of advanced topics

The questions discussed in this session were:

- Reform vs. traditional curriculum: what has actually happened?
- How does new work get introduced into a standard undergraduate curriculum? E.g. non linear dynamical systems.
- Are curricula connected?
- How are dynamic technological environments (e.g. *Cabri*, *Sketchpad*, *Fathom*, *SimCalc*, haptic and kinaesthetic devices) effectively used for developing mathematical reasoning and proof vs. the use of Computer Algebra Systems (e.g. *Mathematica*), or computer aided instruction, cognitive tutors, etc.
- If we change the infrastructure of the classroom, for example using wireless networks, do we change the nature of education, the effectiveness of teaching and the depth of understanding and learning advanced mathematical topics?

The participants added one extra question: How do we improve our teaching (by new approaches or developments) on traditional mathematics?

Alexander Khait from Israel discussed the use of algorithms based on computer languages (e.g. BASIC) in computerized environments to enable students to debug their misconceptions of mathematical ideas. Many nice examples are offered in his paper. Ghislaine Gueudet discussed her interests in courseware on the internet in the forms of exercises and technological environments and offered a general framework for the analysis of software (both in general and with respect to didactic issues). There was some discussion about communication in technology-rich classrooms, particularly on the role of dynamic mathematics technologies.

Luis Moreno (Mexico) offered some deep insights into the role of technology and the evolution of sign systems. He focused on dynamic geometry environments (for example *Cabri*) and discussed the establishment of a personal communication system through the holding and moving of geometric figures in such environments, which is often absent in the classroom. In reaction to the use of formal computer language to access mathematical ideas he contrasted it with certain functions of *Cabri* that act as mediators.

Session 4: Conclusions

Discussion had been wide and varied during the first three sessions, and the group went into the final session ready for discussion, and to hear from other participants. During the first three days the organizers and the reactants had tried to focus the discussion of the participants' papers on theoretical trajectories, whereas some participants thought there was a real need for practical implications for professional mathematicians. Some of the issues in forging links between these two professional communities (mathematics educators and mathematicians) was discussed. Many thought that the Topic Study and Discussion Groups had been thinned out too much and that collaboration with some of the other groups would have been beneficial.

The meaning of the term "advanced" was debated. Suggested meanings included:

- More abstract ways of thinking.
- Focusing on concepts rather than techniques.
- Making connections.
- Recent mathematics (say, from the 20th century).
- Mathematics ability.

Shlomo Vinner (Israel) gave an example of a graduate who thought a tangent was a line that cut a curve exactly once. For Shlomo this represented a failure to appreciate the role of definitions in mathematics, and so a failure to think in an advanced manner. For him this manner of thinking can well be independent of topics, whereas for Khait this example just represented a gap in the graduate's knowledge.

Juha Oikkonen (Finland) was invited to talk about the reform of the first year analysis course at the University of Helsinki. He described possible factors in the reform's apparent success. He highlighted efforts to take some of the pressure off the students (they were allowed to make up for poor test results, although not many availed themselves of this offer), help in finding study partners, as well as an instructional decision to spend more time on the hard topics rather than sticking to the usual strict order of development.

The group heard from a wide variety of international perspectives and mathematical experiences that gave rise to many discussions and issues. One future piece of work might be to develop an article contrasting the international perspectives discussed in TSG 13 with application to the teaching and learning of advanced mathematics.

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TSG 14: Innovative approaches to the teaching of mathematics

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Urs Kirchgraber, ETH-Zürich, Switzerland
Wong Khoon Yoong, NIE, Nanyang Technological University, Singapore

Aims and focus

The topic for this study group was both broad and deep: what is innovative for one teacher may be a regular way to teach for others, so as well as providing a showcase for new practices the group had to think about what 'innovation' really means. For example, recent interest in 'typical' Japanese methods could lead to practices which are innovative in North America, but are clearly standard in Japan. The organisers therefore decided to accept as 'innovative' ideas from all over the world which were novel for those who chose to write about them or make presentations about them. The organisers' intention was that the meetings at ICME would provide opportunities to reflect on the underlying issues of innovation, while also providing opportunities to learn more about what counts as innovative in a range of countries.

To frame our thinking about innovation, presenters were invited to represent the state-of-play internationally. In addition, refereed papers which had been submitted to the TSG were presented by distribution on the associated website. These were written from many perspectives, and there were tensions between maintaining an overall coherence of thinking about innovation and ensuring that the contributions were truly international. It was decided to keep the breadth which represented the full range of interest in the topic and to provide frameworks and discussion at each end of the programme to ensure meta-issues were also on the agenda. The accepted papers generated five subcategories of innovation:

- New ways to engage students affectively in mathematics
- New ways in which learning might take place
- New teaching methods
- Introduction of new topics and contexts into the curriculum
- Use of new technologies

The organisers posed a suite of questions for participants before the group met:

What questions, doubts and resolutions arise for you after reading one or more of these papers?

Is it the case that all these papers indicate improved, or different, learning as a result of innovation? If not, what has changed?

Are the methods and ideas presented here usable in all contexts, or only some contexts? What would be required for an innovation to be of generic usefulness or effectiveness?

Session 1

In the first session, *Laurinda Brown*, UK, presented ways of chanting using a Gattegno number grid. This acted as a timely reminder that unison chanting is not necessarily a mindless activity in classrooms, and that *what* is done is sometimes less important than *how* it is done, and how it engages the learner, consciously or unconsciously, in mathematical structure. *Gary Flewelling*, Canada, then introduced the thoughtful distinction between the 'knowledge game' and the 'sense-making game', showing that many practices which are claimed to be innovative are still concerned mainly with getting learners to acquire static knowledge rather than to become mathematically active, constructing and sense-making for themselves. He demonstrated how an intriguing geometrical dynamic display can be reduced to a sequence of instructions and closed questions, or presented as an arena for supported exploration. Assessment regimes influence the goals of teachers and of learners by focusing on knowledge reproduction. Modelling activities provide opportunities for learners to make sense, both of mathematics and of the context being modelled, and *Sol Garfunkel*, USA, presented examples of how this approach is successful in motivating exploration of mathematics with classes of learners who might not normally engage with the subject.

Session 2

In the second session, *Emily Shahan* and *Megan Staples*, USA, presented video research of classrooms in which teachers were encouraging a problem-solving approach, focusing on the importance of the dialogue between teachers and students. In their presentation, and their accompanying papers, it was clear that the quality and focus of interaction is crucial to the mathematical engagement of learners, rather than the problem-solving situation on its own. Teachers varied in their abilities to maintain collaboration, to get learners to ask questions and make connections for themselves, and to develop and use 'common ground' in their classrooms. The creation of a learning community was also reported by *Binyan Xu*, China, on the website, and *Luo Qiu Ja*, China, also discussed the importance of open, encouraging, interactions.

Anne Watson then presented two tasks, both of which offered opportunity to explore and act mathematically through generating data, conjecturing, and generalising. One of them was about turning a line of cups over in pairs and it was hard to see how this connected with any other mathematics, whereas the other involved school geometry and invited investigation using trigonometric formulae, calculus and geometric proof to find the largest quadrilateral which can be made with sides 7,8,9 and 10 units. This concern to generate a high level of mathematics content through investigation was also mentioned in a website paper by *Xu Liquan*, China,.

These two presentations illustrated the earlier suggestions that the kind of teaching is less important in the teaching and learning of mathematics than the quality of the classroom interaction, the active engagement of learners in sense-making, and opportunities to explore within mathematics as well as in other motivating contexts.

Marcos Cherinda then showed a glimpse of what is possible with a cheap paper weaving board, and the use of weaving as a context for exploring algebraic structure through two-dimensional pattern creation, symbolic representation and prediction. Particularly impressive was the way in which weaving could be used throughout school to create and express appropriate generalisations. Rather than doing 'weaving and maths' once or twice in their school career, learners used it regularly as a generic learning tool.

Session 3

The value of irregular use of motivating contexts was emphasised in the third session by *Joaquin Gimenez*, Spain, who presented a colourful and varied record of how Spanish learners are encouraged to relate their school learning to situations out of school which can be viewed mathematically. Methods included the use of mathematics trails, real (not 'realistic') problem-solving, liaison with industry and other employment, physical and mathematical model-making, and visits to fairground roller-coasters. The mathematics involved in many of these out-of-school links was not trivial and could include not only complex calculations but also geometrical, statistical and mechanical concepts. Clearly one issue which is important whenever innovative methods are imposed on teachers is the nature of their own mathematical knowledge and the experience they need to make the most of the method, rather than relying on the method to be automatically effective.

This issue was highlighted strongly in the presentation of *Sonoko Mori*, Japan, who told us how in Japan ICT is being incorporated into the mathematics curriculum. Latest data were given and there was a detailed description of problems to be faced, in particular, the training of all mathematics teachers in the use of ICT. The scale of the associated inservice education needs was obvious in this case, but in many cases of imposed innovation within mathematics teaching the inservice education needs of teachers are not obvious at all and new methods which work well with the teachers who created them can be ineffective for teachers who may not recognise the essential features.

The final presentation in the third session was from *Jogsoo Bae*, South Korea, who showed how the use of play materials and sweets, presented to young students by a clown, would motivate them to engage in mathematics. Children who wanted to demonstrate their arithmetic using the materials had to don clown outfits themselves. Bae showed many examples of activities in which he had used the affective and physical drives of children to get them to experience mathematics, such as pacing out huge shapes outdoors, making large models in groups, using colourful and textured materials, and so on. These are in contrast to working formally from textbooks or only with writing materials and the students clearly enjoyed themselves more than in their normal lessons.

Session 4

In the final session of this group, *Wong Khoon Yoong*, Singapore, offered the use of a 'multi-modal thinkboard' as a teaching, learning and assessment medium. The board is divided into several sections radiating from a central topic, and each section is for expressing an example in a different representation, such as diagrammatic, contextual, verbal, symbolic, practical, numerical, graphical and so on. The use of the board could act as a reminder for teachers to provide experience in all modes, as well as a tool for learners to develop complex and interconnected concept images. Use of multiple representations was also mentioned by *Malcolm Swan*, UK, on the website, and he also writes of the need for discussion and reflection to support sense-making. The think-board could scaffold the connections between experiential and formal mathematics which are hard for learners to make without pedagogic input. Indeed, one of the many unresolved questions raised in the final discussion was about how one builds bridges between the learners' experience in any mathematical activity and the growth of procedural, adaptive, formal and conceptual understanding which is the goal (in various combinations) of education.

Finally, *John Mason*, UK, brought several strands together by characterising learning and doing mathematics as making mathematical sense of phenomena, some of which may be normally experienced as mathematical, as in the case of the 'largest quadrilateral' problem, and some of which are not, as in the case of the weaving. His view was that it is the action of the learner which makes the phenomenon interesting and mathematical, not the task or phenomenon itself.

There were further papers on the TSG 14 website, www.icme10.dk. These included two papers from *Cecile Ouvrier-Buffet*, *Denise Grenier* and *Karine Godot*, France, about ways of teaching in which learners research mathematics for themselves. Buffet's work shows that learners can develop conceptual understanding by constructing their own definitions of concepts, and Grenier and Godot show that learners can, in carefully structured situations and with scaffolding materials, learn how to research mathematics for themselves.

Taking into account points made in the final discussion, and taking all the papers and presentations into account, the group had considered the following matters:

- a range of 'innovative' approaches can be used to engage learners' interests, emotions, physical drives and out-of-school knowledge, in learning mathematics
- there is a tension between the power of intriguing contexts to attract learners to mathematics and the features of the context dominating the learners' experience
- interest, engagement and mathematical activity are generated by the way a learner engages with a task; they are not intrinsic to a task or approach
- the quality of teacher-generated interaction in classrooms makes a difference to how learners engage with mathematics
- all ways of teaching mathematics can be turned into either knowledge-focused approaches or sense-making approaches by teachers, or by curriculum and assessment regimes
- there are tensions between learning to apply mathematics, achieving fluency and achieving understanding; it is not the case that any of these should always dominate, or always precede the others
- it is possible for learners to explore mathematics as if they are researchers, and to access higher-level ideas than those they are currently taught
- there is a need to know more about how to connect sensory experience of mathematical ideas, situated problem-solving, procedural mathematics, conceptual understanding, abstract mathematics, and understandings of structure
- in all innovative approaches the goal of giving students the opportunity a love of mathematics must be a crucial issue.

This report was written by Claudio Alsina and Anne Watson. They are happy to be contacted at claudio.alsina@upc.es and anne.watson@edstud.ox.ac.uk respectively, for further information on the work of this TSG.

TSG 15: The role and the use of technology in the teaching and learning of mathematics

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Introduction

Computer-based opportunities to connect and interact in a variety of ways with representations of mathematical knowledge continue to develop at an extremely rapid rate. As a result, numerous changes can be identified in the way we do mathematics and in the mathematics that we do. The rapid development of both hardware and software has also been accompanied by several educational trends, including increasing integration of technology into many different sectors of the educational world, a range of initiatives for the implementation of digital technologies into mathematics classrooms in different countries across the world, the design of a variety of environments to support a more experiential approach to mathematics learning and the search for new theoretical tools and methodologies that illuminate the learning processes associated with the presence of technology.

Each of these trends can be associated with a variety of questions for mathematics educators, related to issues such as the impact of ever-evolving tools on mathematical cognition and practices, understanding and evaluating the pedagogical approaches and classroom organizations that can be employed in technology-integrated environments, the design of tools that foster mathematical thinking, the epistemological impact of particular forms of representing and communicating mathematics and the challenges inherent in combining technological possibilities with curriculum demands.

Aims and focus

Within the complexity and diversity of this context, it was the intention that the TSG 15 would both serve as a forum in which mathematics educators could come together to discuss and to probe the major issues associated with the integration of technology into scenarios associated with mathematics teaching and learning and as a place to share ongoing work and perspectives. To maximise participation, the TSG programme was developed around: keynote addresses from *Richard Noss*, UK, *Luis Moreno-Armella*, Mexico, *Abigail Lins*, Brazil, *Federica Olivero*, Italy and *Nick Jackiw*, US; poster presentations; software demonstrations; and theme-based discussion groups. Contributions were invited to three inter-related topics.

- *Mathematical thinking, technology and the evolution of mathematics* – clarifying the reciprocal relationships.
- *Orchestration of mathematics teaching in the presence of technology* – understanding structure in the variation.
- *Key factors in the design of new technologies* such as classroom networks, new actions, new representations, and new devices, and the implications of these design factors.

The last of the four sessions was dedicated to a synthesis of the outcomes of the various activities, with reports from the group discussions and a summary of the major issues raised in the plenary activities, posters and demonstrations. A total of 54 contributions were accepted for inclusion, in some form or another, in the programme (available at the TSG 15 site at www.icme10.dk). The contributions brought a diverse variety of perspectives and interpretations. Various different technologies were covered, with nearly – but not all – participants interpreting technology as related to some kind of electronic digital instrument, be this hand-held calculators, stand-alone computers, display devices, class-based networks or aspects of the worldwide web. Through the issues raised by the keynote speakers, interactions during the poster and software demonstrations sessions and the group discussions, the three topics that had formed the basis for the contributions became transformed into three themes around which the outcomes of the topic group can be summarized: *the challenges of research in a constantly changing field, the co-evolution of mathematical knowledge and tools in activity* and *the need for a greater focus on teachers and teaching*.

The challenges of research in a constant changing field

Many of the contributions focused on challenge for both research and practice in an area changing at an exponential rate. As hardware and software evolve, it is not also easy to determine exactly which research questions and issues will disappear from view and those issue which will continue to be important when (and if) we reach the moment to look back on the technological revolution. The discussion around the use of the dragging facilities in dynamic geometry systems (DGS) illustrates various facets of this challenge and serves as a case in point. *Federica Olivero*, Italy, described in her presentation a range of different dragging modalities and how these might mediate the construction of proofs. In particular, in perhaps the most well known modality, the "drag-test", a figure "passes" if certain initial proprieties are preserved as it is dragged around the screen.

In early versions of DGS, the available construction tools were limited to those modeling the ruler and compass constructions of Euclidean Geometry, hence a figure which passed the drag-test, implied a construction procedure that would be valid in 'ruler and compass' geometry. *Gabriel* and *Andreas Stylianides*, USA point out that this is no longer the case in more recent versions of DGS. It is perfectly possible to build a construction to trisect an angle using the calculator tool, for example. Does this mean that previous research findings have been redundant with the changes to the software? Does it mean that we should refine the findings, distinguishing as the Stylianides' do between different types of figure? Or should we accept that the notion of figure cannot be separated from the medium in which is represented? On a slightly different note, when we change our lens from software and learner in research settings, to the mathematics classroom, *Abigail Lins*, Brazil, argued, we cannot assume that the drag mode – let only the drag-test – will be an essential feature of DGS for all users, and especially not for teachers with a history of mediating learning through more static representations.

Given, then, that we are in the still in the midst of the information revolution, *Nick Jackiw*, USA, argued in his keynote talk that the long term value of the work we are doing today may not be in terms of assessments of what will make for effective use and in enumerations of "do's and don'ts", as much as in documentations of technology-mediated change where and when we see it. He pointed to a series of evolution in research into technologies impact on mathematics over the past 25 years. Early research focused on the individual doing mathematics with software has gradually given way to research attempting to recognize the role of the teacher and of curriculum demands on the learner. More recently, it is being recognized that greater emphasis should be placed on the need to understand the mathematical practices that emerge in complex, self-organising, interacting systems, involving multiple learners and teachers using technological tools within and across a variety of settings.

While the first decade of research considered the potential of computers to transform the learning and the teaching of mathematics, the second decade has been characterized by a focus on how technologies also transform the mathematics that is learnt. An important contribution to the field of mathematics education as a whole has been the recognition that in addition to investigating the ways in which the tool, in the course of use, shapes the learner – the instrumentation process – we should also examine the complementary instrumentalisation process, by which communities of users can also shape the tool and hence the setting within which the interactions occur. These reciprocal relations emerged into a second theme discussed during the topic group meetings.

Co-evolution of knowledge and tools in mathematical activity

Luis Moreno-Armella, Mexico, stressed that the phenomenon of the co-evolution of knowledge and tools is not limited to digital technologies, but rather a characteristic of human development. By adopting a historical perspective, he focused primarily on the representational affordances brought by different notation systems which significantly altered the development of mathematical thinking and have become part of today's mathematical infrastructures. Looking back over the more recent history and present day use of digital technologies in mathematics education, various other contributors also provided examples of the ways in which technology shapes and is shaped by learners' mathematical activities. In this vein, both Luis Moreno and *Richard Noss*, UK, described the use of the qualifier "situated" in their attempts to develop theoretical frameworks. Moreno understands by "situated proofs" those expressed in terms of observation and actions permitted by the particular tools of an expressive media. Likewise, Richard Noss used the term "situated abstraction" in his illustrations of how tools and the ways in which they can be used within particular social systems represent an integral part of an individual's evolving conceptualization of mathematical knowledge.

Alongside the aspects of technology linked to its representational infrastructures, *Jim Kaput*, USA, brought into focus the communicational affordances of digital technologies. With advances in connectivity, he described how it is becoming possible for learners to interact alongside computational agents as well as other learners in mathematical explorations, bringing a new layer to what we understand by an experiential approach to learning mathematics – and another possibility with both epistemological and cognitive repercussions. The focus on representational and communicational affordances, like the examinations of the reciprocal relationships between tools, knowledge and thinking, to a certain extent however still leaves to one side what Jim Kaput referred to as the institutional infrastructures (schools, assessment systems, teacher education systems, curricula, etc.). The huge mismatch between the rate of change to representational infrastructures and to institutional infrastructures, respectively, motivated the third theme for discussion.

The need for a deeper understanding of teaching in the presence of technology

One unanimous point that emerged during the conference was a need for still more research that places the teacher as central focus. *Gail Burrill*, USA, reporting on the discussion of the on of the theme-based discussion groups (the group had focussed on teachers and technology), expressed the overall feeling that digital technologies have as yet made little systematic impact on mathematics as it is experienced in the great majority of the world's classrooms. During the third and fourth decade of research in this area, it is recommended that this area is given priority in research. As Abigail Lins pointed out, it may be more fruitful in future research to stress affordances, be they representational or communicational, in terms of the relationship *user-technology* rather than as a feature of the software itself, since her research suggests that teachers do not necessarily appropriate all the affordances attributed in the research literature to particular software environments. The process of instrumental genesis, by which an artefact becomes an instrument, presented by *Cristina Sabena*, Italy, was discussed as one theoretical approach useful in understanding the complexity of appropriating technology into practices. However, even within this framework research endeavours to date have concentrated far more on the integration of technology into mathematical practices than on its appropriation in teachers' pedagogical practices.

And here we can identify a somewhat paradoxical situation. Eight years ago, in summing up the discussion of the ICME 8 study group on computer-based learning environments, Jim Kaput predicted a continuing transition from "Doing (old) Things Better" to "Doing Better Things". The recognition of the transformation of mathematics by technology that permeated the TSG 15 discussions suggested that many participants are committed to the latter. However, there remains a problem with the mathematical legitimacy associated with the so called "better things". Tomorrow's technology might permit a mathematical discourse that differs substantially from that of today's curriculum, but unless changes in the institutional infrastructures accompany the changes in the representations and communication patterns supported by new technologies, the role of technology in school mathematics may continue as peripheral rather than central.

Postscript

Sadly ICME-10 will be *Jim Kaput*'s last ICME. He died in August 2005, following a traffic accident whilst out jogging near his Dartmouth home. Jim was a tireless contributor to the area of technology and mathematics education. He will be remembered, amongst many things, for his visionary insights, his enthusiasm in the face of innovations and his commitment to bringing a powerful and meaningful mathematics to all mathematics learners, exemplified in his work on democratising access to the mathematics of change. Characteristic to Jim's approach, it was his intention that the discussion during the TSG 15 meetings at ICME-10 would contribute to informing the longer term view on technology in mathematics education. This desire is evident in the way his own research programme has examined the new mathematics that particular technologies make possible, both from a historical perspective (looking back) and with an eye to future developments (looking forward). During the TSG meetings, Jim shared aspects of his recent work on classroom connectivity, fitting as he himself had a great gift for making connections – one of the reasons for which he will be missed by so many of us.

This report has been written by Lulu Healy. She is happy to be contacted at lulu@pucsp.br for further information on the work of this TSG.

TSG 16: Visualisation in the teaching and learning of mathematics

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Team Members: *Tania Maria Campos*, Pontifícal Catholic University of São Paulo, Brazil
Masataka Koyama, Hiroshima University, Japan
Michela Maschietto, University of Modena and Reggio Emilia, Italy

Aims and focus

It has come to be recognized that visualization and visual imagery are important aspects of mathematical understanding, insight and reasoning, and that visual presentations as well as attention to students' diverse use of visuals are essential to effective mathematics teaching. This topic study group held a discussion of visualization from multiple perspectives, addressing a variety of questions.

What are the roles of visuals and visualization in mathematics? How do visual forms and visual reasoning about mathematical ideas affect diverse mathematical fields, historically and at the present time? What are some of the classic and most effective examples that can illustrate these roles for students?

What is the psychological role of visual thinking, and related forms of representation (e.g., spatial and kinaesthetic representation), in the learning of mathematics? How do experts and novices "learn to see"? Do mathematicians, teachers and diverse students "see" different things when working with the "same" diagram or sketch? How can this be ascertained?

What do studies of cognition and diagrammatic reasoning tell us about visual representation in the human brain? How can we teach and learn to use visualization more effectively?

What relation now exists or should exist between visual forms and visual reasoning, and the mathematical curriculum? What kinds of external visual representations and internal visualizations (mental imagery) occur as children build concepts in relation to particular mathematical topics and processes, such as whole numbers, proportional reasoning or fractions? How can appropriate visualization increase mathematical power?

How does visualization relate to other ingredients of mathematical understanding, such as the use of symbolic notation? How do various visualizations relate to students' affect and motivation in relation to mathematics? What distinguishes effective from ineffective use of visuals in the classroom?

What are some of the most effective technology-based tools for mathematical visualizations: for example stills, sequences of stills, animated visuals, 2-D and simulated 3-D, passive and interactive? How are these used in the practice of mathematics? How can these be used most effectively in mathematics teaching?

The program included a balance of oral presentations and discussions, structured discussions of sets of distributed papers, and open discussion among participants. The papers whose descriptions follow were presented in person, for 15 minutes, with 5 minutes for discussion. Related papers were circulated for discussion, through the internet and at the sessions themselves.

In addition, the two co-chairs presented "book-end" initial and final talks, to introduce some conceptual terms and frameworks (Goldin) and to provide some summary reflections and suggestions (Whiteley).

Session 1

The first day, after initial introductions, *Gerald Goldin* opened with an overview of some aspects of visual imagery and cognitive representation, in relation to mathematical thinking. Key ideas include the notion of "pattern", internal vs. external systems of representation, and the different types of internal representation that are possible: verbal/syntactic, imagistic (including visual), formal notational (symbolic), heuristic and executive, and affective (emotional). Internal visual imagery can stand for (or represent) external objects or diagrams, as well as other kinds of internal configurations – e.g. words or formulas. Furthermore, representing can be a two-way relationship. Children display a rich variety of imagery, evident in their descriptions and drawings (Thomas, Mulligan, & Goldin, 2002; DeWindt-King and Goldin, 2003).

There followed a discussion of the related paper, "An investigation of the cognitive processes required for a Mathlet", *Ozlem Ceziturk*, Bogazici University, Istanbul, Turkey.

Session 2

The second day began with an animated display by Michela Maschiatto, anticipating questions to be raised in her talk. The session then featured two presentations, followed by open group discussion:

- "Can visualization promote causal thinking?", *George Malaty*, University of Joensuu, Finland. This paper describes some alternative forms for visualizing arithmetic processes involving fractions. A decomposition is presented of the usual single diagram into several steps, with corresponding arithmetic steps; and the ability of students to generalize the pattern is described.
- "Developing pictorial ideas in learning numbers and calculations", *Tadato Katagiri*, University of the Ryukyus, Japan. This paper describes a long-term study of the visual forms used by brain injured and other developmentally delayed students. One-on-one tutoring and encouragement to draw and play are used to develop arithmetic sense with these students. Examples of the evolutions of the diagrams are presented; more material is available on the TSG website, www.icme10.dk.

Session 3

On the third day, two additional presentations were followed by discussion of two additional related, posted papers:

- "Visual representation in the construction of mathematical meanings", *Michela Maschietto, Maria G. Bartolini Bussi, Maria Alessandra Mariotti*, and *Franca Ferri*. This paper, presented by Michela Maschiatto, University of Modena, Italy, presents a project in which students work on the mathematics of perspective drawings, using classical writings, reconstructions of classical apparatus for drawing ("intersecting the visual pyramid with the picture plane") and photo realistic computer animations illustrating such instruments. The project investigates students' ability to understand the process of drawing in perspective, and to describe the functioning of other instruments designed for perspective drawing presented in photo realistic animations.

- "Mental model training wheels: Scaffolding mental imagery with partial sensory support", *Glenn Gordon Smith* (USA) and *Jim Morey* (Canada). A series of experiments on spatial visualization, using specially designed software, were presented. Among other things, these studies showed that students who were actively engaged in an activity learned more about the specific task, than students who observed. Students who had alternating roles over the same time, learned the same amount as those active for the whole period. The most recent study investigated imagining the 'stamping' face of an illustrated cube which was rotated in pre-assigned steps then placed down onto the page. The variable was whether operations were better internalized when some of the visual cues were hidden – but no significant difference has yet been detected.

There followed discussion of two related papers: "Towards a theory of visualization by dynamic geometry software: Paradigms, phenomena, principles", *Thomas Gawlick*, (Germany) and "Students' development of geometrical concepts through a dynamic learning environment", *Isil Ustun* and *Behiye Ubuz*, Middle East Technical University, Ankara, Turkey.

Session 4

On the final day, three additional presentations were followed by a general discussion for the 30 minutes.

- "Dynamic geometry software as a simulation tool for algebra problems", *Stefan Halverscheid*, University of Bremen, Germany. This paper presents a project in which students can choose between using a CAS software (*Maple*) or a Dynamic Geometry Software to investigate and illustrate some problems of transforming and solving linear equations in two variables. In both situations, the students use pointing at, or referring the diagrams to express ideas and reasoning. With the dynamic diagrams, more active words were added to the descriptions. Some of the strongest students, realizing both tools can be used, select the dynamic geometry environment, for the interaction of the diagram with the algebra, where the CAS system offered only entry into the algebra to generate the diagrams.
- "The problem of misperception in mathematical visualisation", *John Malone, Daniel Boase-Jelinek, Martin Lamb, Sam Leong* (Australia). Presented by John Malone, Curtin University of Technology, Australia, and Martin Lamb, this paper describes a study of students' work on a task involving rotation and reflection of a plane figure, and identifiable patterns of misperception of the impact of such a transformation, something which is distinguished from misconceptions more traditionally studied in mathematics education. The impact (or lack of impact) of interventions with software and one-on-one work are also investigated, demonstrating the persistence of some of the identified misperceptions.

Some responses and reflections: *Walter Whiteley* extracted some key points and issues for further work from the authors' circulated papers, and from several of the regular presentations (Dörfler, Mason, Nuñez). Specifically highlighted were points such as the fact that different people see and attend to different things in the same diagram or sequence, as well the urgent need for teachers to be aware of possibilities for use of variable visualizations by themselves and their students and to be able to consider what various students are "seeing". The group considered the possible value of forming a community focusing on visualization, with some shared materials, assumptions, questions, etc.

In the concluding discussion, there was agreement that mathematicians at all levels do make effective use of visual representations and reasoning. For some, including those in modern research, the visual work is essential. Thus it is equally essential that the teacher have access to and value multiple visual forms. The current tendency to devalue visualization in comparison to symbolic computation was discussed.

Students have different strengths and weakness in learning styles, and equity requires that students have the option of building on their visual strengths and having these contributions valued as significant mathematics.

Finally an informal invitation was given for participants to join a continuing electronic discussion (organized by the second co-chair). Further invitation will be made to all participants who signed on to the Topic Study Group's list.

Other circulated and referenced papers:

Adrian M. DeWindt-King and *Gerald A. Goldin* (2003), "Children's Visual Imagery: Aspects of Cognitive Representation in Solving Problems with Fractions". *Mediterranean Journal for Research in Mathematics Education* **2** (1), 1-42.

Noel D. Thomas, Joanne Mulligan, and *Gerald A. Goldin* (2002), "Children's Representation and Structural Development of the Counting Sequence 1-100". *Journal of Mathematical Behavior* **21** (2002), 117-133.

Walter Whiteley, "Visualization in Mathematics: Claims and Questions towards a Research Program", 2004, www.math.yorku.ca/~whiteley.

This report was written by Gerald A. Goldin and Walter Whitley. They are happy to be contacted at geraldgoldin@dimacs.rutgers.edu and whiteley@mathstat.yorku.ca, respecttively, for further information on the work of this TSG.

TSG 17: The role of the history of mathematics in mathematics education

Team Chairs: ***Man Keung Siu***, University of Hong Kong, S.A.R., China
Constantinos Tzanakis, University of Crete, Rethymnon, Greece
Team Members: ***Abdellah El Idrissi***, Ècole Normale Supérieure de Marrakech, Morocco
Sten Kaijser, Uppsala University, Sweden
Luis Radford, Laurentian University, Sudbury, Canada

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Topic Study Group 17

Aims and focus

The aim of TSG 17 was to provide a forum for participants to share their teaching ideas and classroom experience in connection with the history of mathematics, in the spirit of the 10th ICMI Study on the role of the history of mathematics in the learning and teaching of mathematics (of the ICMI Study Volume titled *History in Mathematics Education: The ICMI Study*, edited by John Fauvel and Jan van Maanen, published in 2000), and to learn about work that has been done since then.

Roughly put, there are three aspects, which constitute closely related and yet separate issues:

(1) Doing research in the history of mathematics,
(2) Teaching the history of mathematics,
(3) Integrating the history of mathematics in the teaching of mathematics.

The four sessions in this group focused on aspect (3), in an effort to make clearer the meaning of a *historical dimension in mathematics education* and to deepen the understanding of its various aspects.

Programme

In addition to invited contributions there were also submitted contributions, which went through reviews. The final programme included 12 presentations scheduled in four one-hour sessions: 5 invited talks, 4 oral presentations and 3 presented by distribution contributions. Each presentation was followed by discussion among participants. In the last session an extra half-hour was devoted to a general discussion and a summary of the main points raised during the four sessions. Relevant material on the presentations has been made available on the TSG 17 section of the web page in the form of extended abstracts, full texts, related papers, or links to other web sites. Prospective participants were able to download material of interest to them and study it in advance. However, hard copies of material on some presentations were also available at the meeting. At least 64 people from 22 countries participated in this group.

Invited talks:

Chun-Ip Fung, Department of Mathematics, Hong Kong Institute of Education, China: "How history fuels teaching for mathematising: Some personal reflections"

Fulvia Furinghetti, Department of Mathematics, University of Genova, Italy: "History and mathematics education: A look around the world with particular reference to Italy"

Michel Helfgott, Department of Mathematics, State University of New York at Oswego USA: "Two examples from the natural sciences and their relationship to the history and pedagogy of mathematics"

Jan van Maanen, Department of Mathematics, University of Groningen, The Netherlands: "History in mathematics education: FAQ and facts"

Guillermina Waldegg, Departamento de Investigaciones Educativas, Centro de Investigación y de Estudios Avanzados del IPN, Mexico: "Problem solving, collaborative learning and history of mathematics"

Oral presentations:

Giorgio T. Bagni, Department of Mathematics, University of Rome "La Sapienza", Italy: "Prime numbers are infinitely many: Four proofs from history to mathematics education"

Marita Barabash and *Raisa Guberman-Glebov*, Achva Academic College for Education, Israel, "Seminar and graduate project in the history of mathematics as a source of cultural and intercultural enrichment of the academic teacher education program"

Daina Taimina, Deparment of Mathematics, Cornell University, USA: "History of mathematics and mechanics in the digital Reuleaux kinematic mechanism collection"

James Tattersall, Department of Mathematics, Providence College, USA and *Shawnee L.McMurran*, Department of Mathematics, California State University, USA: "Using the "Educational Times" in the classroom"

Papers by distribution:

Richard J. Charette, Central Connecticut State University, USA: "Integrating the history of mathematics in the teaching of mathematics"

Constantinos Tzanakis, Department of Education, University of Crete, Greece: "The ontogenetic development parallels the historical development: To what extent is this claim true, or false? Remarks and results from some case studies"

Oleksiy Yevdokimov, Kharkov Pedagogical State University, Ukraine: "Using material from the history of mathematics in learning by discovery"

Summary

Introducing a *historical dimension in mathematics education* involves three different areas: *mathematics*, *history* and *didactics*. Implicit in the presentations and discussions in this group was the key issue: to find and elaborate on a harmonious, balanced and effective interrelationship among these three scientific areas in a way that is enlightening and fruitful in mathematics education. More specifically, out of the discussion it becomes clear that the following two points are most needed in this context:

(i) There is a need to construct and develop appropriate relevant *didactical material* which can either be used directly in the classroom or constitute a resource for mathematics teachers. The material should aim to motivate and guide the teacher to improve the teaching approach or understand better students' difficulties or their idiosyncratic ways of learning mathematics.

(ii) There is a need to enrich *teachers' education* at all levels in this direction, both by introducing courses in (particular aspects of) the history of mathematics and its relation to other disciplines, and by letting them become acquainted with historically inspired material that can be, or has been, used in the classroom. In this way, teachers may hopefully begin to think of a historical dimension in teaching as a possible path for improving mathematics education at all levels, and may develop confidence and trust in this endeavour.

In this perspective, the presentations in this group can roughly be classified, as follows:

(1) *Presentations focusing mainly on introducing a historical dimension in mathematics teachers' education*

a) *Reports on specific courses in teachers' training*: *Waldegg's* presentation concerned junior high school mathematics teachers' collaborative work on problem-solving, based on historically motivated themes, using for that purpose excerpts from *The History of Mathematics* (Nuffield Foundation 1994). *Barabash* and *Guberman-Glebov* reported on a sequence of activities in prospective mathematics teachers' education that aims at making history an integral part of students' education programme and allowing them to profit from this knowledge in their teaching practice through the design and implementation of particular teaching units.

b) *Presentations reporting on the design of didactic material or its implementation in practice*. *Tattersall* and *McMurran* reported on the use of the (recreational) mathematical problems published in the *Educational Times* in the Victorian era. They talked about the origin of this material and gave a sample of examples they use in their teaching. *Taimina* reported on the rich didactic material (still under development) that comes out of a kinematic mechanisms collection developed by F. Reuleaux in the 19th century and its possible use to unfold the underlying deep mathematical ideas, concepts and methods emphasizing as well the relation between mathematics and mechanics. *Yevdokimov* presented some examples from an e-learning textbook on Euclidean geometry that elaborates on historical problems and contains related historical information.

(2) *Presentations focusing on integrating history into classroom teaching.*

Three speakers presented particular examples and the underlying rationale, aiming to illustrate how history may contribute to the improvement of mathematics teaching in one way or another – by exciting the students' interest, enriching their view of mathematics or deepening their awareness of what mathematics really is. *Fung* gave two examples enlightened by historical materials to illustrate the point of view that it is essential in mathematics education to design and investigate what E. Wittmann calls "substantial learning environments", where students are engaged in the process of mathematising (in H. Freudenthal's sense). His talk was accompanied by a short video on classroom activity, which was particularly illuminating. Through historical examples in optics (contributions of Heron, Fermat, Leibniz and Huygens to geometrical optics) and chemical kinetics (Briggs and Haldane's steady-state hypothesis) *Helfgott* illustrated the deep interrelation between mathematics and the physical sciences, and how rich and fruitful teaching ideas this can generate *Charette* outlined historically motivated teaching capsules on elementary geometry. One more paper by distribution came from

Alejandro R. Garciadiego, which is on how the history of modern, advanced mathematics, in this case the concept of a well-ordered set, can be illuminating in the context of undergraduate teaching. Unfortunately, the presentation was cancelled because the author did not participate in the Congress.

(3) *Presentations focusing on more general issues:*
van Maanen talked about questions from different quarters asking about the role of history in mathematics education, and classified them according to whom the enquirers are, what they ask and what can be the feedback from such questions to all those interested in integrating history into mathematics education. *Furinghetti* presented an outline of the different views of the role of history in mathematics education and identified two main lines of approach: (i) history as a vehicle to reflect on the nature of mathematics as a socio-cultural process, including the idea of history as a means to promote mathematics in the classroom in order to humanize mathematics; (ii) history as a possible way to conceive and understand mathematical objects, thus referring to the core of problems related to teaching and learning mathematics. Both these broad lines of approach include attempts to answer key questions like: "For a teacher or for a student, is it advisable to know the history of mathematics? How much history does one have to know? And how does one have to know history?" *Bagni* discussed some epistemological issues related to the historical analysis of a mathematical topic, necessary for achieving an effective and correct use of historical data in mathematics education, and presented some theoretical ideas that underline the prime importance of a correct social and cultural contextualisation. He used as an illustration different proofs of the infinitude of prime numbers offered at different 'historical periods' and in different mathematical settings. *Tzanakis* considered the quite old, but still unsettled, question of whether and to what extent the ontogenetic development parallels the historical development in mathematics? What kind of analogies are observed, and what can mathematics education research profit from investigating such analogies? The general ideas presented were supported by comparing the historical development with data obtained from empirical research in three case studies: (i) the order relation on the number line, the algebra of inequalities and the concept of the absolute value of a number (ii) the concept of plane in Euclidean geometry, and (iii) the introduction of basic statistical concepts and relations.

Full texts of these presentations were collected and refereed. Accepted papers appeared in a special double issue of the *Mediterranean Journal of Research in Mathematics Education*, vol.3, No1-2, 2004, published by the Cyprus Mathematical Society. During the final discussion two useful resources were mentioned: (i) a new online magazine of Mathematical Association of America (edited by V. Katz and F. Swetz) called *Convergence* (http://convergence.mtahdl.org) where mathematics, history and teaching interact, (ii) a forthcoming publication in a CD version from the Mathematical Association of America titled *Historical Modules for the Teaching and Learning of Mathematics* (edited by V. Katz and K.D. Michalowicz). Such resource materials will be of great interest to mathematics teachers who are enthusiastic about activities discussed in this group.

Final remark

History of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying. It is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying. In this wider context, the history of mathematics has yet a more important role to play in providing a fuller education of a person.

This report has been written by Man-Keung Siu and Constantinos Tzanakis and was approved by all the team members. The authors are happy to be contacted at mathsiu@hkucc.hku.hk and tzanakis@edc.uoc.gr, respectively, for further information on the work of this TSG.

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TSG

Topic Study Group 18

TSG 18: Problem solving in mathematics education

Team Chairs: *Jinfa Cai*, University of Delaware, Newark, USA
Joanna Mamona Downs, University of Patras, Greece
Team Members: *András Ambrus*, Eötvös Loránd University, Budapest, Hungary
Hideki Iwasaki, Hiroshima University, Japan

Aims and focus

The general aims of the TSG 18 were to provide a forum for those who are interested in any aspect of problem solving research at any educational level, to present recent finding, and to exchange ideas. The primary concerns were: (1) To understand the complex cognitive processes involved in problem solving; (2) To explore the actual mechanisms by which students learn and make sense of mathematics through Problem solving, and how this can be supported by the teacher; and (3) To identify future directions of problem-solving research, including the usage of information technology. In the second and third time slots available to the group at the congress, six sub-sessions were organized for researchers around the world to present their new findings.

A more specific aim of the group was concerned with determining the scope of problem solving. We perhaps can discriminate three major categories:

- Problem solving for developing general experience (e.g., non-standard tasks, open-ended questions and project work, modelling 'real-life' situations, setting tasks with impressive solutions for motivation);
- Problem solving specially designed for enhancing targeted conceptual development;
- Problem solving specially designed to stress reflection and valuation of solution paths, and the explicit development of techniques, heuristics and exploratory methods.

Having these three broad perspectives (along with many others that are more local), the organisers raised the question whether it is feasible or useful to talk about a single identity for problem solving. This theme was taken up by a round-table discussion that took place in the first time slot. Also, it is important to develop the field with all these perspectives kept in mind. This motivated us to organize, in the last time slot, two plenary addresses to consider the future directions of problem solving research.

First session: Roundtable discussion

In this session, the roundtable discussion took place, with the theme 'the identity of problem solving'. The rationale was to ask why there exists a particular sub-field called 'problem solving' within the areas of interest of mathematics education, when the phrase 'problem solving' would seem almost synonymous to doing mathematics anyway. Due to the time constraints, this issue was not illuminated much as such, but it provided a perspective that helped in 'coloring' some more specific themes in problem solving that were raised. The panellists invited were *Lucia Grugnetti* (Italy), *Kazuhiko Nunokawa* (Japan), and *Carolyn Maher* (USA). Each gave a short talk on themes suggested by the organizers, followed by questions and comments from the audience. The reactor was *Joanna Mamona-Downs* (Greece). A short summary is given below:

Lucia Grugnetti talked about constructivism in problem solving by describing the 'situation-problems' approach to mathematics teaching currently espoused in France, and talked about the importance of comparing peer solutions for students to appreciate formal explanations. Kazuhiko Nunokawa categorized four objectives in problem solving. i.e., enrichment of schemata, motivating students by exposing them to 'impressive' results, creating personal new mathematical knowledge, and giving experience to enhance general solving ability. In particular, he pointed out that it is important to identify the limitations of using problem solving in teaching whole mathematical theories. Carolyn Maher referred to a long-term project in which the same students were followed from primary school up to upper school. She focused on collaborative work, timely return to problems as students mature, building personal representations, monitoring one's own work. Further she made remarks concerning sense making, as well as affective issues, and the role of the teacher.

Second and third sessions: Presentations of refereed papers

The organizing team had a good response to its call for papers. In total 28 abstracts were submitted to the team before the deadline date, 18 of which were accepted. On the whole these submissions displayed a wealth and diversity of the material incorporated and the decision of acceptance/rejection was sometimes difficult. All authors of accepted abstracts were invited to present their work in a 15-minute talk during the group's sessions at the congress. These talks took place in the second and third time slots with activities being distributed over three different locations. Despite the resulting division of the participants attending the group, the sizes of the audiences were generally satisfactory, and the discussion lively.

Below we shall briefly outline each contribution. The organizing team gathered the papers into collections of two or three that shared some common characteristic. The description follows this grouping into the six ensuing sub-sessions. Full papers are available in the TSG's page in the congress's web site. The following people chaired the sub-sessions: A. Ambrus, M. Downs, H. Iwasaki, S.Leung, E. Pehkonen, and R. Speiser.

Sub-session 1: Broad issues and research projects in mathematical problem solving

The problem–solving agenda incorporates many differing concerns, so it is important to develop insights into how these concerns fit together. The three papers in this sub-session contribute to this in different ways. *Beth Southwell*, Australia, reported on an on-going project to develop a concept map of elements of problem–solving processes and to locate research that throws light on it. *Bernd Zimmermann*, Germany, discussed the use of historical material in various educational aspects of problem solving, such as explaining students' cognitive barriers, individual differences in strategies and the competence of teachers in making diagnoses. *Lucia Grugnetti*, Italy, *François Jaquet*, Switzerland, and *Daniela Medici*, Italy, described a broad international project where collaborative work, the role of designing problems, and the occurrence of unexpected student behavior are stressed.

Sub-session 2: Problem solving in an ICT environment

Researchers in problem solving show a strong interest in applications of Information and Communication Technology, as the following contributions illustrate. *Sergey Rakov*, Ukraine, demonstrated how a Dynamic Geometry package can contribute to problem posing, making hypotheses, finding evidence or counterexamples, and giving approximate solutions. *Ioannis Papadopoulos*, Greece, considered the advantages and disadvantages in using software for teaching the concept of area for primary school children. He pointed out that some activities (such as the cut and paste method) raised by the use of the computer in the end had more of a problem solving nature rather than contributing to conceptualization. *Tom Lowrie*, Australia, talked about a project where students were asked to think about some mathematics implicit in a popular computer game in order to motivate them to engage in mathematical thinking, and to link 'out-of school' and 'in class' activities.

Sub-session 3: Proof, modeling and teaching heuristics

These three themes deal with some specialized but central issues in problem solving. *Keith Weber*, USA, considered the problem solving aspects of forming proof, stressing the difference between obtaining a logical deduction (syntactic) and an argument that is meaningful (semantic). *Zemira Mevarech*, Israel, and *Bracha Kramarski*, Israel, reported a study showing that collaborative work is not sufficient for the enhancement of students' abilities in modelling. In addition special tuition in metacognitive processes is needed. *Murat Altum*, Turkey, and *Cigdem Arslan*, Turkey, gave evidence that teaching targeted heuristics can positively effect students' use of those heuristics.

Sub-session 4: Teacher development and questions of design in problem solving

If we want students to learn and make sense of mathematics, how should teachers design pedagogically sound problems for classroom instruction? The two papers presented in this sub-session addressed this question from experiences in Japan and Singapore. *Takeshi Yamaguchi*, Japan, and *Hideki Iwasaki*, Japan, put forward a view about the pedagogical design of problem solving based on the Dörfler's generalization model. *Kai Fai Ho*, Singapore, *Teong Su Kwang Teong*, Singapore, and *John G. Hedberg*, Australia, presented findings from a survey of 140 Singaporean 5th graders. In this study, students were asked to solve some problems, and to describe the reasons why they had difficulties in solving the problems. It was found that the students surveyed appeared to have limited knowledge of problem solving heurisitics, and the researchers suggested that Singaporean mathematics teachers need to enhance their instructional practices by explicating the processes of mathematical problem solving.

Sub-session 5: Sense-making in mathematical problem solving

Validating results has a long tradition as an important aspect of mathematics education, particularly in relation to problem solving. In this sub-session, *Maria de Hoyos*, UK, examined if and how students seek validations in their problem solving. She found that students do validate their results and do it by constantly seeking what has been called 'mathematical conviction' and 'cognitive reassurance'. *Ban Har Yeap*, Singapore, presented a study exploring how to prevent children from suspending their ability to make sense of mathematics when they perform mathematical tasks such as word problem solving, and, in the process, engage these children in critical thought. *Charita A. Luna*, The Philippines, and *Lourdes G. Fuscable*, The Philippines, examined the impact of mathematical symbolism on college students' problem solving performance. They found that mathematical symbolism has a powerful influence on students' problem solving performance especially in the translation of a word problem into an equation. Familiarity with mathematical symbolism enhances their problem solving ability.

Sub-session 6: Issues in mathematical exploration

In this sub-session, *Vic Cifarelli*, USA, and *Jinfa Cai*, USA, presented a conceptual framework concerning mathematical exploration. In this framework, they view mathematical exploration as a recursive process in which solvers determine goals of action as they formulate their problems, solve them, and reflect upon their solution activities to formulate new problems. They also presented some empirical data to support the conceptual framework. *E. Koleza*, Greece, and *M. Iatridou*, Greece, investigated the role of experimentation in problem solving. They examined and analyzed the mechanisms of experimentation by pre-service teachers, working in four person groups, whilst engaged with mathematical problem solving. *J. Piggott*, UK, presented the key aspects of mathematics enrichment and how the content and design of new resources would continue to build upon them.

Fourth session: Future directions for mathematical problem solving research

Kay Stacey, Australia, and *Edward Silver*, USA, delivered two plenary addresses to discuss the future directions of problem solving research. *Jinfa Cai*, USA, chaired the session. Stacey argued that while teachers around the world have had considerable successes with achieving various goals concerning problem solving, there is always a great need for improvement, so that more students get a deeper appreciation of what it means to do mathematics. This requires research with associated curriculum development directed to understanding the problem solving process for mathematics (in all its aspects), developing effective classroom practices, and designing suitable tasks. Silver argued that the work done to date has helped us gain important insights into how students might learn to solve problems but that it has paid too little attention to ways in which problem solving might be a core element in classroom instruction. He suggested that more work should be done that directly addresses the following central issue of importance to classroom teachers: What do the findings from research suggest about the feasibility and efficacy of teaching mathematics *through* problem solving?

An underrepresented theme

Problem posing is at the heart of mathematical research and scientific investigations. In fact, in scientific inquiry, formulating a problem well is often viewed as even more important than finding its solution. In mathematics education, there is a broad consensus of viewing mathematical problem posing as an essential and effective instructional practice. It is suggested that problem-posing activities not only lessen students' anxiety and lead to a more positive disposition towards mathematics, but also enrich and improve students' understanding as well as problem solving capacity. Given the importance of problem posing activities in both school and college mathematics, in recent years the mathematics education research community has began to investigate various aspects of problem posing processes. Despite the general interest of mathematical problem posing TSG 18 did not have any contributions devoted completely to this very theme. Hopefully, more attention to problem posing will be evident in the presentations at the next congress.

Final remarks

In the last three decades, there has been a great deal of educational research on mathematical problem solving which has deepened our understanding of the field immensely. As we reflect on the research trends on mathematical problem solving, we realize just how dynamic research on mathematical problem solving is. This is hardly surprising, when one considers some of the fundamental questions that the field has to address, such as: What is mathematical problem solving? What are the cognitive processes used in solving mathematical problems? What are the purposes of problem solving? What are the actual mechanisms in which students use to learn and make sense of mathematics through problem solving? What is the teacher's role in implementing problem solving in the mathematics classroom? The views of the mathematics education community on each of these questions have evolved over time and are still in flux. It is appropriate to periodically take stock of the field by examining how mathematics educators are currently looking at problem solving and seeing what issues currently have a need of further research. This TSG group served this purpose well. An indication of this claim is that the editors of the *Journal of Mathematical Behavior* solicited a selection of the work from TSG 18 for publication that resulted in the two special issues (Volume 24, issues 3 and 4).

This report was written by Jinfa Cai and Joanna Mamona Downs. They are happy to be contacted at jcai@udel.edu and mamona@upatras.gr respectively, for further information on the work of this TSG.

TSG 19: Reasoning, proof and proving in mathematics education

Team Chairs: ***Guershon Harel***, University of California, San Diego, USA
Sri Wahyuni, Gadjah Mada University, Yogyakarta, Indonesia
Team Members: ***Gudmundur Birgisson***, Iceland University of Education, Reykjavik, Iceland
Christine Knipping, Carl-von-Ossietzky University, Oldenburg, Germany
David A. Reid, Acadia University, Nova Scotia, Canada

This TSG included eight presentations given by:
Kirsti Nordström (Sweden), *Lara Alcock* (United States), *Susanna Epp* (United States), *Raisa Buberman* and *Marita Barabash* (Israel), *Virginie Deloustal-Jorrand* (France), *Alexander Khait* (Israel), *Takeshi Miyakawa* (Japan and France), and *Manya Raman* (United States).

Aims and focus

The aim of TSG 19 was to provide opportunity for TSG participants to share their research in reasoning and proving in mathematics education, with particular focus on:

- An international perspective on research on reasoning and proof, with particular emphasis on research conducted outside of Europe and North America
- Transition from informal argumentation to formal proof in mathematics classrooms, including classrooms where technology is used.

A summary of the eight presentations (listed in the order the papers were presented).

Lara Alcock, "Mathematicians' perspectives on the transition to formal proof".
This talk was based on interviews with mathematicians who teach a transition course at a large state university in the US. Based on their comments, Alcock identified four modes of thinking that are used flexibly by successful provers. These were: instantiation, structural thinking, creative thinking and critical thinking. Alcock illustrated each of these modes and identified an emphasis in current teaching on developing skills associated with structural thinking.

Kirsti Nordström, "A pilot study on five mathematicians' pedagogical views on proof".
Nordström has analyzed five mathematicians' pedagogical views on proof in order to test and improve the conceptual frame that she has created on the basis of the literature. The frame with three categories, conviction/explanation, inductive/deductive reasoning and aspects of formality, the level of rigor and the language, was developed to identify features in mathematicians' utterances and to consider the utterances in their social, historical and cultural context. During the process of analysis of the five interviews several new aspects emerged, for example the aspect of transfer Nordström considered proof as an artefact – a resource for mathematical learning. She argued that Lave and Wenger's concept of transparency captures a dual function of proof as a learning resource in mathematics: It needs to be both seen (be visible) and to be used and seen through (be invisible) to provide access to mathematical learning. Students' access to proof is a central issue for her study.

Alexander Khait, "Proofs as a tool to develop intuition".
Traditionally mathematics for non-mathematicians (e.g. physicists and engineers) is presented on an intuitive basis. Usually the formal side of mathematics is downplayed. If proofs are taught to this population at all, the teaching proceeds from non-formal explanations, banking on students' intuitive understanding. As a result of the computer revolution there is a large increase in individuals who are not inclined to study mathematics but need it for professional activities, typically connected to various computer applications. In this context the possibility of being satisfied with intuitive explanations and understandings is unacceptable: a computer can be engaged only in a formal talk. Computer professionals have to be comfortable with formal definitions. While programmers usually do not prove the correctness of their algorithms, to become a good programmer one has to develop intuition to create correct programs. This implies a new role for proofs, namely transitions from formal to informal proofs, teaching students to distinguish intuitively between true and false propositions. Khait discusses a teaching design aimed at achieving this purpose.

Manya Raman, "Key ideas in the context of a proof from collegiate calculus".
Raman's talk centers around a newly proposed theoretical model of mathematical proof. The model accounts not only for what a proof is but also how one gets created. This model grew out of empirical research in which college freshmen, graduate students, and mathematics faculty were asked to compare different proofs of a claim from college calculus. It appeared that both the novices and the experts distinguished between an essentially public aspect of proof (the formal, rigorous aspect) and a private aspect (the informal, intuitive aspect). The difference between those who were mathematically sophisticated and those who were not was that the former saw connections between the public and private domains while the latter did not. The link between the public and private domains is called the "key idea" of the proof. In the talk Raman defined "key idea" and gave examples across a fairly broad range of mathematical proofs. The next step in Raman's research is to explore ways in which the "key idea" could be used as a pedagogical tool in helping students understand given proofs and produce proofs of their own.

Susanna Epp, "The role of logic in teaching proof".
Susanna Epp's presentation argued that even simple mathematical proofs and disproofs are more logically complex than most mathematicians realize, and it discussed two possible reasons why so many students have difficulty with proof and disproof: differences between mathematical language and the language of everyday discourse, and the kinds of shortcuts and simplifications that have been part of students' previous mathematical instruction. It described research about whether explicit instruction can help students develop formal reasoning skills and suggested that such instruction can be successful when there is appropriate parallel development of transfer skills, such as the use of exercises to express statements both formally and informally and overt reference to logical principles in later mathematics instruction.

Raisa Guberman and *Marita Barabash*, "Improving reasoning abilities of 5th-6th grade pupils using a specially designed teaching unit in pre-formal logic".
Numerous researchers in mathematics education have referred to the need for intermediate stages towards formal proof and reasoning in mathematics. Guberman and Barabash assert that the idea of pre-formal proof must include pre-formal logic. In the course of school teaching and learning the necessary linguistic skills and thinking abilities are not being sufficiently developed. This in turn causes essential difficulties when a student arrives at the deductive stages of mathematical learning, e.g. in deductive geometry. Keeping in mind the purpose of developing the pre-formal logic in primary school pupils, a group of math educators from the Achva College in Israel have developed a teaching unit named "Learning with Alice to Think and to Reason" intended for the pupils of 5th – 6th grades of primary school. Based on this unit, Guberman and Barabash have planned an experiment to assess the effect of teaching in thus designed logic environment, on the development of pupils' ability to reason logically and to build logically valid argumentation. They presented the results of this experiment and some preliminary conclusions.

Virginie Deloustal-Jorrand, "Polyminoes: A way to teach the mathematical concept of implication".
Three points of view on implication were presented by Deloustal-Jorrand: a formal logic point of view, a deductive reasoning point of view, and a sets point of view. From the formal logic point of view, "if A then B" normally simply means "not A or B". From the deductive reasoning point of view, the statements "A is true" and "if A then B" force the conclusion "B is true". And from the sets point of view, "if A then B" means that the set defined by A is a subset of the set defined by B. Deloustal-Jorrand's study had as its research hypothesis that it is necessary to know and establish links among these three points of view to have a good understanding of implication and to use it correctly. She gave beginning mathematics teachers problems to solve that involved statements about very concrete situations – "paving" various types of polyminoes by dominoes. The first set of questions required students to make deductions; the second asked them to criticize proposed "proofs" about the polyminoes. Although the analysis of the results of the study is incomplete, the problems presented were clever and worth using as exercises in transition-to-higher-mathematics classes or in discrete mathematics classes.

Takeshi Miyakawa, "The nature of students' rule of inference in proving: The case of reflective symmetry".
Miyakawa gave some students two related problems about reflective symmetry in geometric figures. The two problems had opposite answers, and Miyakawa discovered that the students generated incorrect "rules" to justify some of the steps in their "solutions". The students themselves generate a rule of inference. For some of the students the backing they have for validating a rule of inference is construction (e.g., "if one can construct a figure which is accepted visually or perceptively, these properties can be a conditional statement of the rule of inference"). The construction is a way to validate the rule. However, one might raise the following problem: it is not certain whether the rule validated by construction will be accepted by the theory admitted at the beginning. Mathematically or theoretically the rule of inference should be accepted by the theory admitted at the beginning. But, are the rules of inference used in mathematics always

as such? It seems that there is not sufficient attention to whether the rule is accepted by the theory or not. This reflection poses an educational question: "What should students' rule(s) of inference rely on?"

Questions for further considerations:

- Alcock's four categories seem related to ways of thinking that have been discussed in other terms in the research literature. How do these categories differ?
- Nordström's work shows that mathematicians and students have very different ideas about what the other knows. What would be the effect if mathematicians knew more about students, or student knew more about mathematicians, before beginning university studies?
- Khait's presentation on proof for computer science students raises the question about "proof reading" versus "proof writing." A possible research question would be whether proof reading is conceptually prior to proof writing, and how one learns to read a proof.
- Raman's distinction between private and public proving is worth wider study. For example, being (personally) convinced and a fact being (publicly) verified are different but related events. Both are central to proving, but sometimes only one is considered, or the two are treated as equivalent.
- Epp's presentation brought out an important distinction between learning algebra and learning proof. Algebra can be seen as generalized arithmetic. But logic is not viewed by many as generalized speech. What implications does this have for teaching logic?
- Guberman and Barabash's presentation raised the issue of what is natural about logic. What is "common sense"? Logic is only generally functional within mathematics, in other contexts meaning matters more than logic. And yet logic is useful in culturally specified ways. How has logic evolved as a way of thinking in an illogical world? How might it develop in the worlds of children? How can the culturally specific use of logic in mathematics be taught?

This report was written by Guershon Harel with the support of Susanne Epp and David Reid.
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TSG 20: Mathematical applications and modelling in the teaching and learning of mathematics

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Team Members: *Toshikazu Ikeda,* Yokohama National University, Japan
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Introduction

The focus for TSG 20 was applications and modelling in the teaching and learning of mathematics and the scope included secondary or high school and university and college levels. The aims of the study group were to review the present state of our knowledge of how students learn applications and modelling, to hear from educational researchers of their very recent findings in this field of enquiry, and to discuss teaching innovations and interventions which are intended to enhance student learning.

It is right and proper that this theme should attract so much attention. Mathematics, of course, should be studied as of right for many reasons, but the application of mathematics and the technique of mathematical modelling, and the role these play in the world and in the lives of everyone must be included in the curricula of schools and universities. Furthermore, it is the contention of advocates of applications and modelling that the learning of mathematics itself is motivated, enabled and enhanced through the study of applications and modelling. This is testified to by many research studies, including some of those presented at TSG 20 and reported below. Mathematical modelling is a "way of life" for professional mathematicians (academic pure mathematicians excepted); it is akin to the scientific method employed by scientists, engineers and technologists; engaging in it enhances curiosity, an inquiring frame of mind and key life skills; it is employed in many areas of human endeavour, both social and scientific.

Invitations were issued through ICME-10 channels for authors to submit papers to the TSG. These were to be refereed and those selected would be "presented by distribution" via the TSG. They would be published on the ICME-10 website in advance of the conference and would be reported during the sessions allocated to the TSG. A considerable number were submitted and sixteen survived the refereeing process. These papers were to form the major part of the work of the TSG. The Organising Team agreed to use the first three sessions to report the papers and for brief discussion and to use the final session for further discussion and debate. The topics of the accepted papers fell into three categories – Sharing Good Classroom Ideas, Empirical and Theoretical Studies at Secondary Level and Empirical and Theoretical Studies at Tertiary Level. Rather than invite authors to present their paper in a short time, it was agreed to appoint an expert in each area to summarise and present the group of papers, and to have the authors in attendance to add further comments and to answer questions. This way of proceeding worked very well, the sessions did not over-run and there was lively interaction with the audience.

Details of each of the papers presented are included in the following sections.

Session 1 – Sharing good classroom ideas

Session 1 was presented by *Gabriele Kaiser* (Germany) and had six papers, two at secondary level and four at tertiary level. The two papers relating to secondary level mathematics were written by high school teachers from the USA. They both gave excellent ideas for bringing aspects of the real world into the classroom. *Ralph Farnsworth*'s (USA) paper was on "The Use of Geometry and Proportional Reasoning Techniques at the US Department of Agriculture". The mission of the Farm Service Agency is to help farmers to stabilize farm income, to conserve land and water resources and to help farm and ranch operations recover from the effects of disaster. These ideas were used in the classroom. For children living in rural communities, the agricultural setting of the problems presented by this work would make them very real indeed. Farnsworth gave examples of many situated problems that had all been tried and tested with his grade 10 pupils.

Steve Krevisky (USA) also presented ideas for high school classroom use, but this time inspired by the world of sport. His paper was titled, "The National Collegiate Athletic Association Basketball Championship Tournament – Statistics, Prediction and Analysis". Using recent historical data concerning the results of the NCAA annual basketball tournament, he worked out the relative frequencies of the different possible outcomes of the matches, and used these as predictive probabilities for the future. His pupils enjoyed this activity and it helped them learn some ideas in probability. These ideas could be used in many one-on-one sports such as the major international tennis championship tournaments.

The paper by *Hamid Chaachoua* and *Ayse Saglam* (France) – "Modelling by Differential Equations" – looked at the close relation between physics and mathematics. The historical development of modern physics is littered with models that involve differential equations. However, the situation in France, and in other countries as well, is that, while university students learn to solve the equations, they do not spend much time on the "modelling" that relates the equations to the physics. Thus they are not getting into the "way of life" of mathematical physicists. The paper gave several situations where the authors had observed this behaviour.

Another cross disciplinary paper from the USA – "Stealing from Physics: Modelling with Mathematical Functions in Data-Rich Contexts" – was submitted to this mathematics teaching conference when the author, *Tim Erickson*, and his colleagues realised that their work in physics was also very relevant to the learning of mathematics as well. University students were given data that arise in real world phenomena and they learnt to model these with functions. It was observed that the students of the two different disciplines had difficulties bringing the ideas of mathematics and the ideas of physics together. Erickson suggested that using their resource material and software tools helped students to overcome some of these difficulties, but he recognised the problem mathematics teachers have in finding time to let students carry out experiments.

Otávio Jacobini and *Maria Wodewotzki* (Brazil) used mathematical modelling in the university classroom to provoke some political thinking by their pupils to encourage greater "citizenship". They looked at the income tax laws in Brazil and speculated on how modifications to these might bring about social change. But more than this, they described how their pupils worked in a community project, teaching some teenagers who had committed some crime and were now in a rehabilitation programme. The students, by their peer teaching of the teenagers, also enhanced their own learning.

Finally in this section, from Iceland, *Thorir Sigurdsson* asked the question, "Could a Mathematics Student have Prevented the Collapse of the Atlanto-Scandian Herring?" Simple models to fit data were derived and analysed. This is yet another example of an investigation that is very relevant to the lives of the university students involved.

These six papers are published in *Teaching Mathematics and its Applications*, volume 25, number 1, 2006.

Session 2 – Empirical and theoretical studies – secondary

This session, presented by *Peter Galbraith* (Australia) saw the first airing of the results of six empirical studies, investigating questions of great interest to the community. There were three studies from Germany – *Gabriele Kaiser*'s study on the "Development of Mathematical Literacy", *Katja Maaß*'s study on "Barriers to, and Opportunities for Integration of Modelling in Mathematics Classes", and *Dominik Leiß*'s study on "Teacher Intervention versus Self-Regulated Learning."

Kaiser studied the development of mathematical literacy through an innovative teaching programme that relied heavily on applications and modelling. She used five levels of mathematical literacy, ranging from "illiteracy" – the inability to cope with relevant information, to "multidimensional literacy" – which incorporates contextual understanding and philosophical, historical and social dimensions. Kaiser found that the teaching programme encouraged great progress at the lower levels of literacy, but not so much at the higher levels.

Maaß's study aimed at showing the effects of integrating modelling tasks into the daily school routine. She particularly studied how students' mathematical beliefs changed through the course, and the connection between beliefs and modelling competencies. Beliefs were classified as understanding mathematics to be "process", or "application", or "formalism", or "scheme". Those students who had applications oriented or process oriented belief systems also had more positive attitudes towards modelling. Maaß concluded that modelling examples should be integrated into the early years of education to prevent barriers being raised in later life due to inappropriate beliefs.

Leiß reported that German students generally underachieved when faced with demanding tasks, and that German teachers had difficulties diagnosing and handling students' problems. He claimed that "work on tasks" in the mathematics classroom is all-important for students, and so the selection, design, handling and assessment of these tasks are all-important for teachers. The teacher also has to work out when to intervene and when to let the student work independently. Leiß's conclusions are worthy of study by teachers – students should put themselves mentally into the problem situation; authentic tasks require students to have some specialist knowledge of the situation of the task; students need to reflect upon their solution process, especially looking at all phases of the modelling cycle.

Jerry Legé (USA) in his paper "Approaching Minimal Conditions for the Introduction of Mathematical Modeling", considered two different instructional approaches – behaviourist and constructivist – to introduce modelling to students with weak content skills and no prior modelling experience. The students in the treatment groups were from two different high schools and the course was at pre-algebra level. The tasks related to aspects of contemporary student culture and included "planning a vacation" and "best rap artist". For the students, both treatments worked, but in different ways and both impacted strongly on the heretofore traditional nature of classroom instruction in these schools.

In Taiwan high schools, *Fou-Lai Lin* and *Kai-Lin Yang* told us, the environment is decidedly unfriendly to the teaching of modelling. The backgrounds of teachers and students, the examinations and the textbooks all militate against it. Nevertheless these authors ventured forth with a teaching intervention involving working on suitable modelling tasks, in an attempt to influence the prevailing culture. The tasks related to situations familiar to the students such as the design of furniture, the location of fire stations and the misuse of drugs. Lin and Yang found that, at the end of the project, students were beginning to exhibit the distinctive characteristics of the modelling process such as verifying the model and going round the modelling cycle.

Allan Tarp (Denmark) presented an interesting discussion paper on the importance of the words we use when teaching modelling. He distinguished between LAB- or laboratory words and LIB- or library words. Thus "Brahe, by observing and recording the motion of the planets provided LAB-data, from which Kepler induced LIB-equations that later were deduced from Newton's LIB-theory about gravity." He concluded that by replacing the authorised LIB-routines of mathematics with authentic LAB-routines solves the "relevance paradox" of mathematics, which comes from the "simultaneous objective relevance and subjective irrelevance of mathematics".

These five papers are published in *Teaching Mathematics and its Applications*, volume 24, number 2-3, 2005.

Session 3 – Empirical and theoretical studies – tertiary

There were four papers in this session, presented by *Chris Haines* (United Kingdom) including one by *Ros Crouch* and himself. Haines and Crouch, writing on "Applying Mathematics: Making Multiple-Choice Questions Work", discussed how some multiple-choice questions may be used to improve understanding, to develop and to assess modelling capabilities and as an aid to teaching. This is a development of ideas that they, and others, have been working on for several years and which is published mostly in the ICTMA series of books (ICTMA, 2004). Each question is designed to look at a single phase of the modelling cycle. These have proved effective in the uses mentioned above.

Djordje Kadijevich (Serbia and Montenegro), *Lenni Haapasalo* (Finland) and *Jozef Hvorecky* (Slovenia) asked pertinent questions about "Using Technology in Applications and Modelling", which is the title of their paper. What implications does the availability of technology have for the nature of the modelling problems that can be given to students? How does its use facilitate learning? When does it enrich learning possibilities? Can we do without it? All good questions, which the authors considered in detail, with examples, and which led them to not unexpected answers.

Thomas Lingefjärd and *Mikael Holmquist* (Sweden), in their paper "To Assess Students' Attitudes, Skills and Competencies in Mathematical Modeling", discussed the successes they had when using various forms of peer-to-peer tutoring and assessment with pre-service trainee secondary school teachers. Students voted 2:1 for peer assessment of their work, realising that it helped develop *self*-assessment. They also found that assessing mathematical modelling was much harder than they had anticipated.

Finally, *Dvora Peretz* (Israel) gave us an interesting novel concept of a model. Her paper is titled "Inverse Mathematical Model – Yet Another Aspect of Applications and Modelling in Undergraduate Mathematics of Prospective Teachers." Using the concept of an inverse model, she presented a useful way of helping students understand and teach ideas in elementary mathematics such as the division of one fraction by another.

These four papers are also published in *Teaching Mathematics and its Applications*, volume 24, number 2-3, 2005.

Session 4 and conclusions

Session 4 was a plenary panel discussion session, the panel consisting of the members of the organising team for TSG 20 and the three session presenters. Many of the issues raised in the papers were revisited and there was lively audience participation.

The organisers believe that TSG 20 worked very well. Fifteen papers by twenty-one authors from twelve countries over four continents were presented. This was a truly international study group. Clearly "applications and modelling" is taken very seriously across the globe.

The ideas presented in session 1 are worthy of replication by teachers at secondary and tertiary levels across the world; they may need some local "customisation" first!

The research findings of the secondary level studies presented in session 2 and the tertiary level studies presented in session 3 provide new insights into how students approach their learning and how the teachers involved managed this learning.

At the end of each of these three sessions and in Session 4, clarifying questions were asked of the authors, and useful discussion ensued. All of those attending expressed their appreciation of the authors and the session presenters for bringing their work to this TSG.

This report was prepared by Ken Houston who has now retired from the University of Ulster but is, nevertheless, happy to be contacted at sk.houston@north-circular.demon.co.uk for further information on the work of this TSG. Contact details for the authors are given in the appropriate issue of *Teaching Mathematics and its Applications*.

ICME 10 2004

TSG

Topic Study Group 21

TSG 21: Relations between mathematics and other subjects of science or art

Team Chairs: *Marta Anaya*, University of Buenos Aires, Argentina
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Team Members: *Helmer Aslaksen*, National University of Singapore, Singapore
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Joseph Malkevitch, City University of New York, USA[1]

Aims and focus

TSG 21 gathered a group of congress participants who were interested in the "Relations between mathematics and other subjects of science or art". The activities of the group included presentations, a keynote lecture and discussions of new trends and developments in research or practice related to this topic.

Keynote lecture

A keynote lecture was given in the last session by *Monica Wijers*, of the Freudenthal Institute, University of Utrecht, (The Netherlands), who spoke on *Connecting mathematics and other subjects.* In her lecture she referred to the position of mathematics in relation to other subjects according to one of the fundamental principles of the Freudenthal Institute, "Realistic Mathematics Education", pointing out the concern for more stress on connections between subjects in the current Dutch Secondary Education.

While describing the state-of-the-art of this educational level, Wijers referred to: other subjects as contexts, mathematics as a tool and mathematical modeling, showing several examples and reflecting on them. In reference to context use she emphasized the importance of

- not presenting the model but having students choose one and
- using more interesting examples and posing more interesting questions

Monica Wijers further spoke about the risks and benefits of context use, of looking at mathematics as a tool while also describing the corresponding risks and benefits of mathematical modeling. Here she pointed out that that "Emergent Modeling"[2] is a basis for mathematical modeling.

As a final conclusion, she returned to the question: "How to position mathematics in relation to other subjects?" and recommended a careful balance of the risks and the benefits according to the considerations previously exposed.

This lecture turned out to be motivating for further reflection for those present, as was expressed by some participants: "We need something like the track suggested by the keynote speaker".

1 Joseph Malkevitch participated during the process of organization of the activities of TSG 21 though he could not attend the conference due to personal reasons.

2 Emergent modeling refers to both the process by which models emerge, and the process by which more formal mathematical knowledge emerges. The model develops first as a *model of* the students' situated informal strategies and gradually, as a *model for* more formal mathematical reasoning. (K. Gravemeijer, 1999. How emergent modeling may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*. **1**(2), 155-177).

The TSG sessions
The group had four sessions which were organized with the purpose of providing both an overview of the current state-of-the-art in the topic and presentations of recent contributions to it, as seen from an international perspective. Specific topic areas of contributions included the following: "Mathematics and Arts", "Mathematics and Sciences", "Mathematics and Interdisciplinarity".

The Organizing Team agreed that the selection of papers for oral presentation would not depend only on the intrinsic quality of the submitted papers but also on the diversity of ideas and domains to be presented in the different time slots. In this line, the following papers were selected for oral presentations:

Mathematics and Visual Arts:
"Mathematics as underlying structure in the arts: A capstone course for preservice teachers", *J.Wanko*, USA

Mathematics and Philosophy:
"Re-creating the Renaissance", *B. Sriraman*, USA

Mathematics and Literature:
"Mathematics, Mathematicians, Literature and Art", *I. Safuanov*, Russian Federation

Mathematics and Music:
"Mathematics and music: some educational considerations", *P. Maher*, UK

Mathematics, Interdisciplinary Competences and Sciences *(with regard to teacher education):*
"Mathematics and science", *P. Baggett* and *A. Ehrenfeucht*, USA
"An Inter Disciplinary Approach To Math Teaching: Mathematical Ideas in Sciences Taught By Future School Math Teachers", *R. Guberman* and *M. Barabash,* Israel

Mathematics, Interdisciplinary Competences and Sciences *(with regard to upper secondary and tertiary education):*
"Interdisciplinary competences integrating mathematics and other subjects of science", *C. Michelsen, N. Glaargaard* and *J. Dejgaard*, Denmark
"Attaining mathematical competences via the use of other subjects in a first year mathematics course at an agricultural university", *T.V. Pedersen*, Denmark
"Courses on mathematical modeling with information literacy: successful attempts at Karlstads' university", *Y. Shestopalov* and *I. Persson*, Sweden
"Cultural astronomy and mathematics in art and architecture: Two general education courses at the National University of Singapore", *H. Aslaksen*, Singapore

These papers covered a wide area of topics relating to the teaching and learning of mathematics and its connections with arts and/or science at secondary, university and pre-service teachers level.

The main conceptions that can be extracted from the different works deal with the deep interrelation between mathematics, arts and science. The arguments for this are founded in the historical influence of mathematics in arts and sciences, in the common aspects that can be found between creative work in mathematics and art, and in regarding mathematics as part of human culture, and as part of mankind's struggle to understand the world.

The educational implication of this interrelation is the need for interdisciplinary activities, to provide a sense of meaning to student's mathematical learning by relating mathematics to different aspects of human culture. In a broad sense these activities might be related to visual arts, music, and literature or to the mathematical modeling of phenomena of life. Different responses to these needs are present in the selected works.

With respect to mathematics and arts, educational activities may include:

- Exploring some mathematical topics in music and visual arts and designing a course for prospective teachers. (Maher)
- Exploring ways in which different mathematical concepts (e.g., symmetry and asymmetry, patterns and randomness, ratios and proportional reasoning) can be identified in the arts and then focusing on the specific mathematics of those concepts (course for preservice teachers). (Wanko)
- Re-creating the Renaissance in a microcosmic way in the high school classroom gives an opportunity for realising and appreciating the underlying unity of the arts and sciences. (Sriraman)

With respect to mathematics and science, educational implications may include:

- Two competencies are to be emphasized: the modeling competence and the problem solving competence using examples taken from applications to train the selected competencies, also taken as a starting point for the presentation of the mathematical theory. (Pedersen)
- The development of meaningfulness in a student's mind by bringing him/her near phenomena of life by introducing inter-disciplinary topics like for example 'From the globe to a map – mapping projections' where mathematical concepts like proportion, similarity, scale, diagrams, axes and coordinates are used. Another example is 'Winds and sea streams' which relates these issues to geometric vectors. (Guberman)
- A program for in-service and pre-service teachers of mathematics. It allows one to combine mathematics and science much earlier that is usually done. (Bagget and Ehrenfeucht)
- Modeling activities are emphasized as an alternative of the traditional transfer method of education, to overcome students' difficulties in combining mathematics and science. (Michelsen et al.)
- Developing *information literacy* (information search in collaboration with others) (Shestopalo and Persson)
- Design of General Education courses based on group projects and homework tasks. Tasks related to astronomical observations are recommended. (Aslaksen)

Participants

Five teaching levels were represented in this group (with overlaps: 20% of the participants teach at primary level, 30% at secondary level, 5% at tertiary level, 70% at university level, while 20% were involved in preservice teacher education).

The poster sessions of ICME-10 included posters related to TSG 21. Many of the authors were active participants at the sessions of the topic study group. Also an active participant of the TSG 21 sessions was the lecturer of a regular lecture related to arts: *Intersection of mathematics and art* (Vera W. de Spinadel from the International Mathematics and Design Association, Buenos Aires, Argentina).

Inquiry

At the end of the last session a brief questionnaire of three questions was given to the 20 participants. The aim of the questionnaire was to enquire whether the participants included topics relating mathematics with art and/or science in their own teaching, and to get to know participants' personal views about including or not those topics in their classes, and also to detect if there were teachers who would like to do so, without having done it, and which were the reasons in either case. The questionnaire was:

> *Dear participant, the OT of TSG 21, will be happy to know about you:*
> Q1. Educational level you work at:
> Primary – Secondary – Tertiary – University – Preservice teachers
> *We would also like to know whether:*
> Q2. Do you include in your teaching topics relating:
> a) mathematics with art? Yes – No – Why? Write three reasons please.
> b) mathematics with science? Yes – No – Why? Write three reasons please.
> *Or:*
> Q2 Would you like to include in your teaching topics relating:
> a) mathematics with art? Yes – No – Why? Write three reasons please.
> b) mathematics with science? Yes – No – Why? Write three reasons please.

Three reasons were asked (though this was not expected to be fulfilled completely) to be given in both cases to allow for other reasons than eventual institutional or curricular prescriptions.

Results

Q2: Including topics relating mathematics with arts an/or science in the teaching of mathematics

The responses of the participants showed that topics of science are more often included in the teaching (80%) than topics of arts (65%). These responses can be summarized as follows:

Q2 Mathematics and art	**Q2 Mathematics and science**		
	Responses %	YES (80%)	NO (20%)
	YES (65%)	50%	15%
	NO (35%)	30%	5%

Topics relating mathematics with both arts and science are included by 50% of the participants. In this case the arguments presented referred to:

- Motivation and interest: 20% of the participants in both cases referred to motivation and context:

– "Looking for context for lessons. Trying to reach learners that are not mathematical. Fun and interest".

- Broadening students' perspective (15%). In this case the stress was put on the potential effects of this teaching as for example:
  Q2A) "Yes: Gives meaning to geometry. Helps students to develop visualization". Q2B) "Yes: Allow students to connect mathematics with other science. Allow work with real data. The tasks are mere formative -students need to read, understand the context, chose mathematics tools to use, decide the answers".
- Specific links between these two disciplines with mathematics, and to topics they related were mentioned. (15%)

Some participants specified the subjects they related to as for example:

> Q2A) "Yes: Geometry, landscape, paint". Q2B) "Yes: Biology, chemistry, physics".

Other participants stressed in both cases, the relationships that could be found between the disciplines and the context of teaching as for example:

> Q2A) "Yes. Some students are more interested in non-mathematical subjects (like art). The aesthetics of art and the aesthetics of mathematics are often related".
> Q2B) "Yes. Mathematics is often discovered as a way to explain science. I teach a course in mathematical modeling, and models from science are common examples".

Not including topics relating mathematics with art
35% of the participants stated that they did not include these topics in their teaching. Arguments were mainly based on students' or teacher's context, for example: "Not relevant for the students. No time". "It is not an essential aspect of my work. (It can be incidental)". "I'm a chemistry instructor at the university level".

Not including topics relating mathematics with science
20% stated that they do not include these topics in their teaching which is carried out at different levels. Arguments were presented in these responses referring to:

- Context (10%). Arguments refer to curricular demands or to institutional circumstances as for example: "Little call in my institution for *maths*. With science; some call for statistics with health sciences".

No interest is expressed regarding the inclusion of any topic of science.

- Lack of orientation and/or motivation (10%), as is shown in the response: "Haven't found the ways to for it in yet – But may do it this year. Others do this – I wanted to focus on something that's covered less".
  Another participant expressed the conviction of the relevance to include these topics: Q2b): "Yes. It is important to show *all* the relations between different topics".

Final remarks

Motivation and interest for students and/or teachers was found to be a major reason for including topics relating mathematics and sciences (45%) and/or topics relating mathematics and arts (30%). A response that might represent this emphasis on motivation is: "Because I like it, and what we teachers like helps us to teach better".

Some differences were found between the argumentations about the inclusion of mathematics and art topics and the inclusion of mathematics and science topics.

In reference to relating mathematics and arts some participants expressed: "Teachers and students find "lessons" involving art and mathematics engaging and motivating".

Other participants stressed an association to pleasant feelings for example: "It is colourful, fun, tactile, allows group interactions and opportunities for creative students to shine in a subject (mathematics) that they often don't".

These responses suggest that including topics of art might be associated with evoking pleasant feelings and to motivation.

In reference to relating mathematics and sciences some participants remarked other aspects: "Provides relevance to the topics". Also other participants expressed: "Helps maintain student interest better. Shows the interdependency of mathematics and other subjects (that mathematics as science cannot exist in isolation)".

These responses suggest that showing links of mathematics with science is perceived as a relevant argument. This argument was stated by the majority of participants (70%) for including topics of science, thus surpassing the motivation proportion (45%). This was not the case when considering the relations between mathematics and arts, where the corresponding proportions were 30% for motivation and 15% for showing links of mathematics with arts.

The context of teaching (stressing students' profile, curricular demands and institutional restrictions) seemed to be determinant for *not* including topics of other subjects in the teaching of mathematics.

Responses stressing "lack of material" and "discomfort with science" together with "fear of giving misleading information" should be considered as a call for attention with regard to future action of this TSG in order to play an assisting role for teachers. Against this background, the TSG organisers have decided to produce an independent booklet to account for the presentations in the TSG. The booklet is expected to serve as a reference on the new trends and developments in research or practice related to this topic.

In view of the growth of research in mathematics education over the last decades, it is remarkable that only very little attention has been paid to research on relations between mathematics and other subjects of art and science. Issues related to this topic are complex, because they comprise at least two different components, an extra-mathematical and a mathematical one. Further research is needed in this field. At one of the TSG sessions, a participants suggested the establishment of a network for senior researchers and graduate students involved in research on relations between mathematics and other subjects of art and science. Such a network might play an active role in information exchange and communication between mathematics educators interested in these relations and in the development of the theoretical framework that we are currently lacking.

This report has been prepared by Marta Anaya and Claus Michelsen. They are happy to be contacted at manaya@fi.uba.ar and claus.michelsen@dig.sdu.dk, respectively, for further information on the work of this TSG.

TSG 22: Learning and cognition in mathematics: Students' formation of mathematical conceptions, notions, strategies, and beliefs

Team Chairs: ***Terezinha Nuñes***, Oxford Brookes University, United Kingdom
Dina Tirosh, Tel Aviv University, Israel
Team Members: ***Farida Abdulla Khan***, Delhi University, India
Willy Mwakapenda, University of the Witwatersrand, Johannesburg, South Africa
Günter Törner, Gerhard Mercator University of Duisburg, Germany

Introduction

TSG 22 focused on four main aspects of learning and cognition in mathematics:

1. Models of mathematical thinking and understanding;
2. Learning and instruction: the role of technology;
3. Teachers' cognition and students' formation of mathematical conceptions;
4. Difficulties in learning mathematics.

Each session was devoted to one of these issues. We shall briefly describe the main themes that were raised in each session. A short summary of each of the ten invited talks are provided, as well.

Session 1. Models of mathematical thinking and understanding

In this session *Anna Sierpinska* (Canada), *Celia Hoyles* and *Richard Noss* (UK) and *Günter Törner* (Germany) presented and discussed three main current issues in mathematics education: The complementary roles of theoretical and practical thinking in mathematics learning, the notion of situated abstraction and its relevance to mathematical understanding and the epistemological dimensions for knowledge systems of teachers as reflected in the case of linear functions. The three presenters highlighted the need to develop new and more robust paradigms for thinking about the development of mathematical thinking and understanding, while emphasizing the following aspects: the nature of mathematics, the role of the teacher, the complex process of instrumental genesis and the connection of tool use and traditional techniques. This session provided the setting for sessions 2-4.

Sierpinska, in the beginning of her talk, "On the necessity of practical understanding of theory" raised a crucial issue: What is the use, for didactics of mathematics, of *general* models of mathematical thinking and understanding, which are not specific to some concrete mathematical contents? She took the example of the classical epistemological distinction between theoretical and practical thinking: a crude model of an extremely complex reality. Sierpinska argued that more refined models are needed, and even these could be uninformative unless complemented by analyses of the particular mathematical content. She presented examples of students' work in linear algebra, showing that successful students certainly have a sense of what it means to work within a theoretical system, but these students are also very "practical" in moving about in the theory, finding conceptual shortcuts, and picking exactly what is needed from the theory. Moreover, these students readily give up on rigor and generality of the solution, if this is not absolutely necessary for obtaining a solution. They solve the problem at hand; they do not develop a theory of solving all problems of a kind. Yet there is an undercurrent of generalizable techniques in their solutions. Sierpinska concluded by emphasizing that using only the theoreticcal/practical distinction does no justice to students' ingenuity and that it is necessary to get into the particular mathematical contents of the problem.

In their talk "Situated abstraction: mathematical understandings at the boundary", *Hoyles* and *Noss* argued that any model of mathematical thinking and understanding needs to take account of two consistent research findings: (1) mathematical knowledge is generally characterized by fragmented and pragmatic strategies focused on the task in hand, and (2) learning mathematics is neither necessarily cumulative nor is it necessarily portable to novel situations. They showed on the basis of research in two widely divergent settings (children's use of computational systems and adults' use of mathematics in workplaces in which the level of expected mathematical knowledge is rather ill-defined) that tools and representational infrastructures shape the nature of mathematical knowledge. Further, they illustrated how conceptions of mathematics were situated in terms of language and connectivity with context and, at the same time, abstract – in that the representations extended beyond the immediate to take account of more general mathematical structures. They discussed the notion of *situated abstraction*, which seeks to describe how the process of generating and expressing meanings with the available representational infrastructures tends to produce individual and collective understandings and ways of working that appear divergent from standard mathematics: the 'symbolic tools' used are *constitutive* of meaning and, by implication, of thought. The presenters also suggested that the notion of situated abstraction could be defined in a broader theoretical context, e.g., within activity theory and building on the notion of boundary object: For example, it is at the boundaries between activities that communication of meaning and 'transfer' can be problematic unless and until the different conceptualizations – and the language in which they are expressed – are brought into alignment.

Törner: "Epistemological dimensions for knowledge systems of teachers – the case of linear functions". Törner argued that even though an immense and very diverse literature exists about the knowledge of mathematics teachers, valuable classification systems, which are both universal and suitable for specific contents, are still lacking. He noted that such a multidimensional classification system should take advantage of Shulman's categories of subject matter knowledge and pedagogical content knowledge and integrate those aspects. He emphasized that, in order to present pragmatic, workable schemes, it is necessary to restrict the focus on the epistemological type of knowledge only. He then reported on investigations on linear functions (Grade 7) and the specific knowledge and belief systems of teachers concerning this area. The discussion was based on data gathered within a recent bilingual German/Dutch in-service-training project integrating video-lessons in the two countries and discussions between the teachers.

At the end of this session seven short papers were distributed among the participants. These papers focused on two main issues. Four papers dealt with students' formation of mathematical concepts, including whole numbers (*Liu Jing* and *Song Nai-qing*, China), fractions (*Suhaidah Tahir* and *Md Nor Bakar*, UK) and functions (*Pal Lauritzen*, Norway, and *Jonathan Stupp*, Israel). Three described and discussed issues of communication in mathematics (reading problems in mathematics (*Hak Ping Tam*, Taiwan) tacit-explicit perspective for the cognition in school mathematics (*Cristina Frade*, Brazil) and deaf children's concept formation in mathematics (*Elsa Foisack*, Sweden)).

Session 2. Learning and instruction: the role of technology

In this session *Ricardo Nemirovsky, Tracy Noble*, *Cara DiMattia* and *Apolinario Barros* (USA), *Colette Laborde*, (France), and *Kenneth Ruthven* (UK) presented their views regarding the role of technology in learning and cognition in mathematics. The researchers defined, discussed and contrasted psychological, philosophical, systemic and instructional issues related to the role of technology in mathematics education.

Nemirovsky et al in "Manipulatives, Limit Objects, and Mathematics Learning". addressed two questions: 1) If mathematics education aims at familiarizing students with abstractions, and abstractions cannot be directly touched, seen, heard, etc., why would bodily activity be relevant to learning about mathematical abstractions? And 2) How and why should the use of tools which engage eyes and hands in drawing, writing, manipulating, or touching be relevant to learning about mathematical abstractions? The presenters advocated a view of doing and thinking as woven in body activity in all its forms: eye motion, drawing, writing, grasping, gesturing, talking, and so forth. They showed a classroom episode that took place in a public high school in Boston. The students were using a "Drawing Machine" device with which two students can jointly draw a figure by hand-controlling the X and Y axes. This machine produces also a graph of the motion along the X and Y axes over time. The latter corresponds to two parametric functions for the figure produced. The presenters concluded that through tool-use students develop specialized and bodily sensitivities and goal-oriented responsiveness which allows them to imagine necessary conditions for a mathematically-defined trajectory.

Laborde's talk "New technologies as a means of observing students' conceptions and making them develop: the specific case of dynamic geometry" started from the hypothesis that solving mathematical tasks in a technological environment requires two kinds of knowledge, mathematical and instrumental. Most of the time, especially because ICT used in the teaching of mathematics embeds mathematics, both types of knowledge interact in the use of technology, giving rise to what Rabardel calls instrumentation schemes. This interaction can be used to favour mathematical learning. Sequences of tasks are designed in the computer environment. The interventions of the teacher are critical to establish a correspondence between the actions in the environment and the theoretical concept to be learned. These interventions can support an internalization process (in the Vygotskian sense) transforming actions made in the computer environment into mathematical knowledge. Examples showing the interplay between mathematical and instrumental knowledge in sequences of tasks were shown in the case of the dynamic geometry environment Cabri.

Ruthven's "The instrumediation of mathematical activity and capability: Thoughts on instructional adaptation and learning facilitation" analysed the ways in which upper-primary-school students made use of calculators in tackling a division problem in 'using and applying mathematics'. Drawing analogies with recognized parallel strategies of written division, the presentation provided examples of a number of important phenomena: how a shift from written to calculator computation can augment student capability and sustain task strategy; how the low cost of calculator computation can encourage trialling of plausible variants, but effective use depends on understanding of procedures and results; how primitive cumulate-and-count strategies are relatively vulnerable to irretrievable error in the absence of recording, whereas trial-and-improvement strategies are more robust; how trial-and-improvement strategies involve formulating (direct or inverse) relationships between variables, prefiguring the idea of covariation;

how tailoring calculator technique insightfully provides scope for developing mathematical understanding; how developing effective use of a calculator is linked to developing understanding of relations between number/fraction systems and division forms, and between dual-unit systems of measures/money and the place-value system of decimal numeration; and how there is currently no standard system of calculator division, comparable to recognized standard techniques of written division, to provide a cognitively efficient, socially recognized, and mathematically theorized system of techniques.

Session 3. Teachers' cognition and students' formation of mathematical conceptions, notions, strategies and beliefs

In this session the presenters *Tânia Campos* and *Sandra Magina*, (Brazil) and *Pessia Tsamir* and *Dina Tirosh*, (Israel) focused on various aspects of teachers' cognition and discussed its role in mathematics learning and instruction.

Campos and *Magina*: "Teacher's conceptions of fractions and their teaching strategies". The aim of the research presented in this talk was to investigate Brazilian primary school teachers' concepts of fraction and their teaching strategies. The main hypothesis was that teachers will be able to solve fraction problems in these different situations, but will display a limited range of teaching strategies when proposing ways of helping children overcome misconceptions about fractions. It is possible that their own knowledge of the invariants of fractions remains implicit in situations, that they have not explored in a more systematic way. The methodology used was to ask 70 primary school teachers to answer a questionnaire containing items where they solved some fraction problems and reacted to answers that were presented as children's answers, which displayed misconceptions that have been observed in previous research. The study concluded that these teachers displayed adequate concepts of fraction in most of the situations, but the majority showed some confusion about representing situations numerically through fractions or ratios. As expected, their main teaching strategy was to use concrete materials or drawings to facilitate perceptual comparisons. In the ratio situation teachers could solve the problem through ratios but most did not make a connection between ratio and fraction part of their teaching strategy.

Tsamir and *Tirosh* in their talk "What types of content knowledge are needed to teach mathematics: The layers' model" focused on one main issue: The types of mathematical knowledge that elementary school teachers need for leading rich mathematical discussions in their classes. They introduced a three-layer model of the subject matter knowledge needed for teaching mathematics in these grades, comprising: the *Core Layer*, the *Wrapping Layer* and the *Meta Layer*. A segment of a discussion in a sixth grade on the divisibility of sums of natural numbers was used to illustrate the approach. In this case, the *Core Layer* encompasses the *arithmetical knowledge* needed for understanding the various, related statements and justifications (e.g., knowledge about natural and operations with natural numbers, even and odd numbers, prime numbers, factorization). This kind of knowledge is quite straightforward and obviously necessary for evaluating arithmetic statements. The *Wrapping Layer* consists of a*lgebraic knowledge* that may grant a teacher a powerful tool to reach the correct conclusions regarding the validity of arithmetic statements that are made by the students in their class in a swift and efficient manner. The *Meta Layer* is the *validation-refutation knowledge*, i.e. the knowledge of appropriate ways to prove (or refute) a given statement (e.g., What is considered a correct "general proof"? When is an example or a counter example sufficient for proving or refuting a given statement?). Tsamir and Tirosh discussed the potential implications of this model for elementary teachers' education and suggested possible extensions to other mathematical domains.

Session 4. Difficulties in learning mathematics

The first part of this session was devoted to two presentations. The first, by *Herbert Ginsburg*, (USA) described children's understanding of multiplication as reflected in their work with hand held computers. The second (*Terezinha Nunes*, *Peter Bryant*, and *Ursula Pretzlik*, UK) discussed the role of schema and working memory in primary school children's mathematical difficulties.

Ginsburg: "I didn't know they knew that! Using hand held computers to investigate children's understanding of multiplication". An essential step in helping children with learning problems is to gain an understanding of their thinking. Children experiencing learning difficulties do not merely lack knowledge. Rather, they fail because they use faulty concepts and buggy algorithms, but may at the same time possess interesting informal ideas and personal strategies that can serve as a foundation for productive learning. The goal was therefore to develop a Personal Digital Assistant (PDA) that could help teachers to acquire insight into student thinking as well as performance. The system, developed with Wireless Generation, helps teachers to use a simple form of clinical interviewing to assess several aspects of mathematics, from block play to algebra; learn about children's strengths as well as weaknesses; understand development over time; develop deeper theories of children's thinking; and learn to assess on their own, without the PDA. In the case of multiplication, the PDA helps teachers to investigate children's number facts, motivation and meta-cognition, mental calculation, concepts (models), writing, alignment and place value, and written calculation. At present the system is functional and Ginsburg's team is investigating how teachers use it and what they learn from it.

Nunes, *Bryant*, and *Pretzlik*: "The role of reasoning schemes and working memory in explaining primary school children's mathematics difficulties". Two approaches to the explanation of children's difficulties in mathematics are currently used in psychology. One approach, based on ideas related to how the brain works, suggests that children's mathematical difficulties can be explained by a reduced capacity to retain information in memory while operating on it. The second, based on constructivist theories in developmental psychology, suggests that mathematical difficulties are related to children's specific problems in using logico-mathematical reasoning schemas. Previous evidence for the working memory view is based on correlational studies that use performance in arithmetic tasks as the measure of mathematical ability. They described two longitudinal studies where the outcome measure of mathematical ability included problem-solving tasks that went beyond arithmetic. The predictive power of an assessment of working memory was compared to the predictive power of an assessment of children's mathematical reasoning. Both studies showed that working memory was not a significant predictor of mathematics achievement in school whereas the assessment of reasoning schemas was, even after controlling for the children's general intelligence and knowledge of arithmetic. This finding was viewed positively because it is possible to improve children's reasoning but there are no good methods for improving their working memory.

The second part of this session consisted of short presentations of eight selected papers. Three papers described and discussed the formation of mathematical concepts and ideas: fractions (*Susan B. Empson*, *Debra L. Junk*, *Higinio Dominguez*, *Kevin LoPresto*, and *Erin Turner*, USA), proportion (*Olof Bjorg Steinthorsdottir*, *Kristjana Skuladottir* and *Maria Sophusdottir*, USA and Iceland) and geometry (*Alexandra Gomes* and *Elfrida Ralha*, Portugal), and five papers described various aspects of students' and teachers' conceptions of mathematics (*Rita Borromeo Ferri*, Germany; *John Francisco* and *Carolyn Maher*. USA; *Ok-Ki Kang*, Korea; *Willy Mwakapenda*, South Africa, *Mihaela Singer*, and *Cristian Voica*, Romania).

This report has been written by Terezinha Nuñes and Dina Tirosh. Dina Tirosh will be happy to be contacted at dina@post.tau.ac.il for further information on the work of this TSG.

ICME 10 2004

TSG

Topic Study Group 23

TSG 23: Education, professional life and development of mathematics teachers

Team Chairs: *Milan Hejny*, Charles University of Prague, The Czech Republic
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Team Members: *Sandy Dawson*, Pacific Resources for Education and Learning, Honolulu, USA
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Introduction

TSG 23 addressed both theory and practice in mathematics teacher education and teaching development. The organisers' main focus for the group was: The nature of being and of developing as a mathematics teacher or teacher educator.

27 papers were submitted to the group. All were reviewed by three reviewers. 20 papers were accepted. These were posted on the ICME-10 website for reading before the conference. A list can be found at the end of each section of this report. There were no oral presentations in the group in order to give as much time as possible to a consideration of concepts and issues. Papers were grouped according to four themes, based on questions related to the main focus of the group. Thinking of ourselves as mathematics teachers and/or educators we asked:

- How do we use collaborative processes in learning and teaching to enable effective learning experiences for students and teachers?
- How do we create or facilitate the learning of mathematics in our teacher education programmes and relate this to mathematical learning in classrooms?
- How can we relate understandings of mathematics with understandings of pedagogy to create effective didactic situations in classrooms?
- How do our theories and beliefs influence our work as teachers with pupils or our work as educators with teachers?

Four eminent scholars in the field, *Terry Wood*, *Romulo Lins*, *Jeppe Skott*, and *Tom Cooney*, were invited to respond to papers on each of these themes respectively. They were asked to read and comment on a set of papers and to raise critical questions and issues related to the papers for discussion in one group session. The plan for each of the working sessions was as follows:

- Brief introduction by one or two group leaders, setting up the focus for the session.
- Oral presentation by one respondent: an account of the ideas and issues raised by the set of papers; offering a critical response and raising questions for discussion.
- Discussion among participants, either in small groups or in plenary. Guidance for discussion to be provided by the group leaders.
- Brief synthesis, by the group leader, of the discussion in the session.
- Last session only: Summing up ideas from the four sessions.

1 In 2007 Barbara Jaworski's affiliation changed to the University of Loughborough, UK

We present now key elements from the presentations of the four respondents. Each respondent offered critical responses to each of five papers, but we do not have enough space here to include the detail of these responses.

Theme 1: What is the difference between collaboration and cooperative process in learning? Terry Wood's response

Forms of working together are defined as:

a) co-operative forms of interaction in which participants communicate about and share their work but do not develop ideas together, and
b) collaborative forms of interaction in which the work is the joint product of the participants.

Examined empirically these *collaborative inquiry approaches* are found to engage students in acquiring mathematical practices such as reasoning, representing and communicating and support the development of their conceptual understanding.

Teaching practices that enable effective collaborative learning experiences for students observe that

- the environment or culture of class matters and the teacher plays the major role in establishing and determining what this environment will be.
- tasks matter--they must be mathematically challenging, open-ended or problem-solving in order for students to have a need to collaborate.

A central premise drawn from papers is: Teachers at all levels of development (e.g., beginning; experienced), teacher educators and researchers work collaboratively to investigate, examine, revise and record their experiences to develop a shared knowledge base for mathematics teaching as a way to develop competencies.

Theme 1 papers:

Fiorentini, Dario; Freitas, M.T.M.; Miskulin, R.G. et al (Brazil): "Brazilian research on collaborative groups of mathematics teachers"

Barbara Georgiadou, Christos Markopoulos, Despina Potari and *Vassiliki Spiliotopoulou* (Greece): "Teachers' and researchers' collaboration: the development of common goals"

Alena Hošpesová and *Marie Tichá* (The Czech Republic): "Learn to teach via collective reflection"

Jana Kratochvílová (The Czech Republic): "Educator–teacher interaction(fragment of a case study)

Will Morony (Australia): "Teacher defined professional standards as a blueprint for professionalism in the work of teachers of maths"

Theme 2: How do we create or facilitate the learning of mathematics in our teacher education programmes and relate this to mathematical learning in the classrooms? Romulo Lins' response

First, it seems that the approaches suggested by the papers are guided primarily by assumptions (theoretical or not, implicit or explicit) about the learning of future teachers and by studies on what seems to be the qualities or competencies of a good teacher, apparently not taking a great interest in how this education is 'transposed' to the actual professional practice. Let me clarify this. I suppose we all agree that the contexts of particular classrooms are of great relevance for what happens there, be they social, cultural, economic or other. When we educate teachers according to a given design or approach, we do it in the conviction that we are helping them to develop the abilities we believe they need to work with their students in a given way – be it the ways suggested by, say, NCTM Standards or others.

But there are no ideal students in the classrooms, so, there are no ideal classrooms. Perhaps, being able to switch to traditional approaches is as important as being able to conduct lessons with open ended problems and investigations, and this is the kind of decision making ability that I do not often see being fostered in teacher education programmes.

I want to suggest that we need a carefully designed and conducted research agenda, to investigate how that 'transposition' happens, and the actual impact the whole process – teacher education/transposition – has on what happens in the classrooms. With that in hand, we might be able to add, to teacher education programmes, a component in which this process is discussed, including the examination of decision-making processes that involve possible radical changes in the teaching approach adopted in a given circumstance.

Theme 2 papers

Jian-sheng Bao, *Yun-quan Lu* , and *Yan Xia* (China):"A hypermedia video-case: A new tool for teachers' professional development"

R. Elaine Carbone and *Patricia T. Eaton* (USA): "A Clarion call: Changing CPD for secondary mathematics teachers"

Iben Maj Christiansen (South Africa): "Mathematical competencies and awareness in a teacher education practice"

Mona Fabricant and *Sandra Peskin* (USA): "Preparing tomorrow's teachers: The role of the community college (2-year college)"

Juan D. Godino, *Pablo Flores* and *Francisco Ruiz* (Spain): "Professional development for mathematics teacher educators through international cooperation the "Edumat-maestros" group"

Theme 3: The relationship between pedagogy and mathematics and the way that relationship may inform teaching practice. Jeppe Skott's response

I have raised four critical issues related to this theme. The first issue is that teachers need to become flexible and reflective curriculum makers, in the enacted sense of curriculum. I do not think there is much disagreement about this, neither in the papers for this session, nor in mathematics education research in general. However, I think the question of how we may better deal with this situation needs more explicit attention in teacher education.

My other main points may be much more controversial. One is that this new role of the teacher requires us to move beyond modelling good teaching in mathematics teacher education. Another point is that in order for teachers to become autonomous decision makers *also* in relation to their students' mathematical learning, teacher education needs to adopt a much broader view of pedagogy than one that is linked exclusively to mathematics. And my last critical comment concerns what I perceive as over-individualistic emphases in the larger part of research on teachers and teacher education.

There are two aspects to this. One is that teacher education needs to discuss what I call the limitations and opportunities arising from the cracks and openings in the social fabric of specific classrooms. The other is that even if we succeed in educating autonomous professionals who are able to take broader pedagogical concerns and the social structure of the institutional setting into consideration, there are limitations to what should be expected. The notion of implementation, if understood as wholesale incorporation of intentions developed elsewhere into schools and classrooms, does not do justice to the social complexities of school life.

Theme 3 papers

Jeanne Albert (Israel): "What leads to meaningful change in teacher' views of mathematics?"

Milan Hejný and *Darina Jirotková* (The Czech Republic): "The key role of tasks for the development of future primary teachers' teaching style"

Raimo Kaasila, *Erkki Pehkonen*, *Markku S. Hannula*, and *Anu Laine* (Finland): "Preservice elementary teachers' self-confidence in mathematics at the beginning of their studies"

Margaret L. Kidd (USA): "Factors that enable or impede a transformation of pedagogical style in secondary schools"

Wang Linquan (China): "What are the mathematics teachers' needs in their professional development?"

Theme 4: The impact of theories and beliefs on practice. Tom Cooney's response

The education of mathematics teachers has many facets including consideration of how our theories and beliefs interface with our practice. Were it the case that teachers, teacher educators, students, parents, and society in general agreed about what constitutes "good teaching" the practice of teacher education would be greatly simplified. But this is not the case and consequently there is a certain tension that exists among the various stakeholders in teacher education.

For whatever reason, the practice of mathematics teaching generally does not match the vision of mathematics teaching espoused by various scholars and reform-oriented proclamations. Teacher education is generally designed to promote a reform-oriented teaching style, one which takes advantage of technology and uses instructional strategies that are based on students' understanding of mathematics. Above all, reform teaching emphasizes the processes of doing mathematics, not just the accumulation of mathematical facts and algorithms.

Often we find that descriptions of teachers' practice stop short of linking that particular practice to the practice of many other teachers. In order to make that linkage, we need to have theories that help build bridges among the various cases. This is, perhaps, the most important way that theories shape our work in mathematics teacher education. If our practice of mathematics teacher education is to move beyond the isolation of individual descriptions, we must have some way of talking to one another about good teaching in a language other than the language used to tell our stories about good teaching. One of our tasks is to consider what and how theories can contribute to our thinking across cases and thereby influence both our research and practice in mathematics teacher education.

Theme 4 papers

Kim Beswick (Australia): “Factors preventing one mathematics teacher from changing her beliefs and practice”

Olive Chapman (Canada): “Facilitating reflection in pre-service mathematics teachers' education”

Eileen Fernández and *Mika Munakata* (USA): “Developing mathematical resourcefulness in middle school teachers”

Merrilyn Goos (Australia): “Learning to teach with technology: A sociocultural analysis”

Victoria Sánchez and *Mercedes García* (Spain): “Thinking about mathematics education for future teachers”

For full reports from the four respondents please contact b.jaworski@lboro.ac.uk, or contact the respondent directly.

Small group activity

Discussion in small groups, in each of the four sessions, was lively, and there was good feedback from participants that this discussion had proved valuable and enjoyable. Sadly time did not allow collecting of detailed comments from groups, but short feedback from groups was included in each of our sessions. Again, feedback suggested that time had been spent profitably within the group sessions. It was a pity that small group work had to be conducted uncomfortably in a tiered lecture theatre with fixed seats, but people suffered the discomfort cheerfully in order to achieve good communication.

This report has been written by Barbara Jaworski. She will be happy to be contacted at b.jaworski@lboro.ac.uk for further information on the work of this TSG.

TSG 24: Students' motivation and attitudes towards mathematics and its study

Team Chairs: ***Philip C. Clarkson***, Australian Catholic University, Fitzroy, Australia
Markku Hannula, University of Helsinki, Finland
Team Members: ***Astrid Brinkmann***, University of Duisburg, Germany
Gudbjörg Pálsdottir, Iceland University of Education, Reykjavik, Iceland
Tim Rowland, University of Cambridge, United Kingdom

Introduction

Affect in mathematics education has been studied for various reasons. Some researchers have been interested in the role of affect in mathematical problem solving, mathematical thinking, or in learning of mathematics in general. Some have been interested in the role of affect in the social interactions in the classroom. Affective variables are sometimes seen as indicative of learning outcomes, sometimes as predictive of future success. Affect is also often seen as a case or a consequence of gender differences. However, up until recently, few have argued that the effect of affect variables on students has a right to be considered as an important issue in its own right, and not only in their relationships to students' cognitive abilities.

McLeod (1992) identified three concepts to describe the affective domain in mathematics education: beliefs, attitudes and emotions. In his invited presentation at this TSG meeting he acknowledged that there are yet other important concepts within this field, such as values, motivation, feeling, mood, conception, interest, anxiety, and view, all of which he noted have been the subject of important studies in more recent years. He also suggested there is a growing interest in this area of study in mathematics education.

Aims and focus: Outline of contributions

As part of this growing awareness, the topic group that was studying the issue of affect at Third Congress of the European Society for Research in Mathematics Education (CERME 3 (2003)) suggested some new directions for research on affect in mathematics education (Evans, Hannula, Philippou & Zan, 2003). The TSG on Students' Motivation and Attitudes at ICME-10 subsequently addressed some of the goals identified in the aforementioned research agenda. One theoretical goal was to specify the different dimensions of affect and their relationships. This dimension was addressed in the papers presented by Hannula, Bikner-Ahsbahs (Germany), Rowland, Brinkmann, and Op 't Eynde and De Corte (Belgium). *Hannula* made an analysis of motivation as a concept and *Bikner-Ahsbahs* made an elaborated analysis of interest in mathematics. *Rowland* introduced a new concept, propositional attitude, and *Brinkmann* reported how students can see beauty in mathematical tasks. Instead of looking at specific aspects of affect, *Op 't Eynde* and *De Corte* had, in their study, approached students' affect in mathematics as a belief system.

Another theoretical goal was to understand the relationship between affect and cognition. A special emphasis was placed on problem solving and problem posing. In his mainly theoretical paper, Hannula looked at both affective and cognitive self-regulation of motivation, including the processes that are not usually conscious. Bikner-Ahsbahs' view of interest-dense situations encompassed both the relevance of the mathematics included and the affective processes of the individuals involved. The papers by Rowland and Brinkmann also fall within the area where cognition and affect meet.

A third theoretical goal addressed during the Topic Study Group sessions was the understanding of the role of affect in a social context. This was specifically addressed by Op 't Eynde and De Corte who reported how students' belief systems are affected by the social context in which they are situated. *Forgasz* (Australia) focused on one social variable, gender, and Rowland focussed on the language in the classroom.

Methodology

Regarding methodology issues in the study of affect, it was acknowledged that there is a deep need to develop better instruments for measurement of different dimensions of affect, and a need to use multiple methods. Op 't Eynde and De Corte have developed questionnaire instruments for different dimensions of belief systems, and Brinkmann for the aspects of beauty in mathematical tasks. *McDonough* (Australia) has developed interview methods to use with young students. *Uusimaki* and *Kidman* (Australia) used an elegant combination of multiple methods, including use of on-line questionnaires. Bikner-Ahsbahs even described a method for summarising several theories into a meta-theory. One of the conclusions of this TSG was that far more attention should be specifically given to appropriate methodological issues in this area of study, and to the inbuilt assumptions behind different methods. Although mixed method approaches can at times prove to be most insightful, the inadvertent mixing of incompatible theoretical frameworks can set up paradoxes that are not always obvious for beginning, and sometimes experienced, researchers. The responsibility to prevent this lies with the original researcher.

Practice

An important aim of this TSG was to address the needs of practice. Naturally, participants of this group see affect as an important factor in good mathematics teaching. Bikner-Ahsbahs' paper on interest-dense situations contributes to supporting a positive affective climate in classrooms. *Glendis* and *Strassfeld* (USA) reported a case study of a group of underperforming students with negative attitudes and low self-confidence who through an intervention were able to overcome many of their initial problems. Also Uusimaki and Kidman reported an intervention study, where math anxious teacher students were able to develop more positive views of themselves as mathematics learners and teachers. McDonough indicated ways in which her research approach has been adopted by teachers in the classroom. A summary position reached by the group was that this is a very worthwhile aim to pursue. Clearly the onus is on the researchers to work closely with teachers so that useful classroom strategies can develop, that in turn allow greater insight into the interplay of affective attributes of the students.

Conclusion

This TSG was able to address issues that had been identified as important for the development of this field, but – as was acknowledged in the closing session of the group – more research is still needed. Such research should be premised on a triple bottom line approach. Cognitive outcomes are clearly important in their own right. How students develop understanding of mathematical ideas, and the skills to process them, needs to be researched. However, the second bottom line of affective outcomes, and the third bottom line of the relationship between the cognitive and affective features, also must be seen to be as important and stand in their own right. In this era it is not good enough for a hypothetical student to be able to 'do' mathematics, if he or she 'hates' doing it.

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Appendix

The following were the papers presented at the meeting of TSG 24.

Beliefs
Peter Op't Eynde and *Erik De Corte*: "Junior high students' mathematics-related belief systems"
Helen Forgasz: "Year 11 students' beliefs"
Andrea McDonough: "Investigating children's beliefs"

Motivation
Markku S. Hannula: "Regulating motivation in mathematics"
Angelika Bikner-Ahsbahs: "Interest-dense situations and their mathematical valences"

Interpreting mathematics
Tim Rowland: "Propositional attitude"
Astrid Brinkmann: "The experience of mathematical beauty"

Changing attitudes
George Frempong: "Influence of practice on attitudes and confidence"
Sirkka-Liisa Uusimaki and *Gillian Kidman*: "Challenging maths-anxiety"
Margaret Glendis and *Brenda Strassfeld*: "Emotions and Motivation: Changing Goals"

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TSG

Topic Study Group 25

TSG 25: Language and communication in the mathematics classroom

Team Chairs: *Norma Presmeg*, Illinois State University, Normal, USA
Siegbert Schmidt, University of Cologne, Germany
Team Members: *Viviane Durand-Guerrier*, IUFM de Lyon, France
Linda Galligan, The University of Southern Queensland, Toowoomba, Australia
Carl Winsløw, University of Copenhagen, Denmark

Introduction

After a call for 12-page papers, all of the 29 manuscripts received were reviewed by three experts. On the basis of these reviews, three papers were chosen to be presented in joint sessions (one on the first and two on the last of the four days), 13 were allocated to three parallel sessions in subgroups (presented on the second and third days), and 12 were papers presented "by distribution", published along with the presented papers on the TSG 25 web site.

The three plenary papers were based on three different aspects of language and communication considered to be of strong significance and general interest to the mathematics education community. All three reported on empirical research, underpinned by three different theoretical frameworks. Each paper was followed by considerable discussion and audience participation. The presenters and topics were as follows.

Session 1: Plenary presentation 1

Bill Barton and *Pip Neville-Barton* (New Zealand): "Undergraduate Mathematics Learning in English by Speakers of Other Languages".

This opening plenary reported on three studies that investigated the dynamics of learning university mathematics taught in English, for students for whom English was an additional language (EAL). Two studies of first-year undergraduate students' learning were followed by a third study of proof and mathematical argumentation used by third-year students whose native language was Mandarin. Contrary to common assumptions that mathematics is a subject in which language will have less impact on learning, in the first study EAL students experienced a 10% disadvantage in overall performance through lack of textual understanding. EAL students unjustifiably relied on symbolic modes to try to compensate for their textual disadvantage. However, the complexity of the issues was reinforced in the second and third studies, in which EAL students who had recently arrived in New Zealand self-reported levels of understanding similar to those of native speakers of English, on common types of problems.

Session 1: Plenary presentation 2

Morten Misfeldt (Denmark): "Computers as Media for Mathematical Writing: A Model for Semiotic Analysis".

Using a semiotic framework to analyze the mathematical writing of research mathematicians and undergraduate students collaborating in groups, the presenter gave examples that focused on LaTeX and its role in the five functions identified for this mathematical writing, namely, (1) heuristic treatment, (2) control treatment, (3) information storage, (4) communication with peer collaborators, and (5) production of a paper. Analysis of examples by means of semiotic diagrams suggested a congruence between verbal mathematics and LaTeX code, while handwritten mathematics and standard mathematical code corresponded to the genre of writing in the *previewer* on the computer screen. The results offered insights into the use of different media at different stages of writing, and also into possible reasons why many mathematicians find LaTeX appealing.

Session 1: Plenary presentation 3

Norma Presmeg (USA): "Use of Personal Metaphors in the Learning of Mathematics"

The linguistic devices of metaphor (illumination of a concept or phenomenon by substituting a well known entity from a different domain for it) and metonymy (designation of a concept by means of one of its traits/attributes) have evoked interest in the mathematics education community in recent decades, but only a few studies have examined the roles of idiosyncratic personal metaphors in giving meaning to mathematical constructs. From an early study, examples of personal metaphors of secondary school students were used to illustrate source and target domains of metaphors, and their tension (elements common to both domains) and ground (differences). The role of associated imagery was discussed. A more detailed analysis was presented of later research on the personal metaphors of undergraduate students for *point*, *line*, and *plane*, in a university geometry course.

The presentations in three subgroups (numbered 1, 2 and 3) on the second day (1A, 2A, and 3A) and on the third day (1B, 2B, and 3B) are summarized as follows.

Subsession 1A:

Semiotic aspects of mathematics learning (chair and recorder: Carl Winsløw).

Silke Ruwisch (Germany) presented a paper entitled "Metaphors and metonymies and their impact in mathematical classroom discourses". She first defined metaphors and metonymies as general forms of reference (based, respectively, on *similarity* and *contiguity*). Then she discussed various uses of these concepts in the mathematics education literature and observed that in an educational perspective, *dynamic* aspects (how these forms of reference function and interact in learning and communication processes) seem to be mostly ignored at the expense of *static* aspects (particular occurrences in mathematical discourses). The subsequent discussion addressed, among other things, to what extent metonymies in the dynamic sense are linked to *abstraction*, such as when using letters to designate "arbitrary numbers" (which was an example of mathematical metonymy given by the speaker in her talk).

Hiro Ninomiya (Japan) combined two theoretical frameworks to analyse metacognitive forms of elementary students' note-taking in the context of working with decimal representation of numbers. The two frameworks were the Peircian theory of signs and nested chaining of signs, and Hirabayashi's notions of object representation, meta-representation and "other self". Ninomiya analysed and advocated the method of *reflective writing*, where the "inner self" (student interpretation of signification) is made explicit through metacognitive comments. In the discussion, some doubts were raised as to the appropriateness of identifying *object* and *representamen* directly with *questions* and *answers* in the analysed student work. Clarification was given as to the way in which the student writing had been brought about (namely, by students using a method exemplified by the teacher).

Related works by *Herbert Gerstberger* (Germany) (on cognitive transfer in understanding ratio) and *Filip Roubiček* (The Czech Republic) (on geometrization and semiotic representations) were presented by distribution.

Subsession 1B

Mathematics learning in an interactionist perspective (chair and recorder: Carl Winsløw).

Here, three connected papers were presented by researchers from J. W. Goethe University in Frankfurt/Main, Germany. The first paper, by *Götz Krummheuer*, outlined the research interests and theoretical basis for the work of the group. The author especially emphasised their interest in writing as a means for *externalisation* of student thinking, and for intensifying and enriching the participation of students in interaction about mathematics.

In this perspective, *Marei Fetzer* presented an empirical study using Toulmin's model of argumentation to analyse the structure of interactive problem solving, as observed in classroom interaction around certain mathematical writing tasks. Even students who are not attentive throughout such interactions – but who have access to their own writing pertaining to the problems – may participate meaningfully in specific parts of the oral argumentation process.

A more clinical type of study of students' mathematical writing was presented by *Christof Schreiber*. Using the software MS NetMeeting, the written and graphical interaction among pairs of 9-10 year old students around a simple word problem was monitored and recorded together with the oral utterances of each pair. Using Peirce's triadic sign model and Hoffmann's notion of "the general" (roughly speaking, elements of the context of signification), the data were analysed with a view to exhibiting the dynamic role of symbolic representations in communicating a solution from one pair to the other. The following discussion raised both technical and more general issues pertaining to the analysis of student writing and argumentation.

Related works by *Florenda Gallos* (The Philippines) (on students' private conversation) and *Cristina Tavares* and *Márcia Pinto* (Brazil) (on mathematics classroom discourse) were presented by distribution.

Subsession 2A

Bilingual learners of mathematics (chair and recorder: Linda Galligan).

The first presenter was *Linda Galligan*, (Australia) whose paper was entitled "The role of language-switching in bilingual students' processing of mathematics". This study investigated the language used by two Chinese-English bilingual beginning university students as they processed and solved various basic mathematics problems. The results showed uses of both Chinese and English at various levels of thinking. The results of the study aimed to begin to clarify the stages of interlanguage in mathematics for bilingual students, important for the teaching of mathematics to English Second Language (ESL) and Non-English Speaking Background (NESB) students at all levels of schooling.

The other presenter in the sub-session was *Lena Khisty*, (USA) (with Hector Morales). The title was "Discourse matters: Equity, access, and latinos' learning mathematics". This paper discussed multilingual classrooms and the issues second language learners encounter in reform-based mathematics that emphasizes talking to learn. Highlighted was the interaction between academic language proficiency and mathematics teaching and learning. Qualitative data from one primary and two secondary grade classrooms revealed the effects of this interaction. Results suggested that more attention needs to be paid to academic language proficiency development in the mathematics *context*. There were also two papers for distribution: *Serge Hazanov* (Switzerland), "Across the Language Border. Bilingual Mathematics for the International Baccalaureate" and *Soledad A. Ulep* (The Philippines), "Language practices in teaching and learning mathematics using English in a bilingual class in the Philippines".

Subsession 2B

Classroom communication (chair and recorder: Linda Galligan).

Michaela Kaslova (The Czech Republic) gave the first paper entitled "Communication and interpretation of the solution – developing didactic thinking". Students often know the answer but are not able to explain how to obtain it. The paper investigated the form of questions and solutions in open ended word problems – oral, phonetic, mathematical symbols, pictures, dramatizations; given to and by pupils of various ages and by adults (students and prospective teachers). Some elements of discussion raised by the analysis of primary and secondary pupils were given. The second paper was by *Rosa Ferreira* (Portugal) (with *Norma Presmeg*). The title was "Classroom questioning, listening, and responding: The teacher modes". This study traced how two student teachers evolved in their teaching modes, that is, their interrelated questioning, listening, and responding approaches in the classroom. The relationships between the participants' beliefs about mathematics teaching and learning and their dominant teaching modes were also investigated. This study raised several questions for future research and implications for teacher education. There were also two papers for distribution: *Minoru Ohtani* (Japan), "Symbolizing and Tool Use in Classroom Mathematical Activity: "Revoicing" as a Unit of Analysis", and *Michelle L. Wallace* (USA) (with *Nerida F. Ellerton*), "Language Genres in school mathematics".

Subsession 3A

Logic and language in mathematics discourse (chair and recorder: Viviane Durand-Guerrier).

The two presented papers shared the same theoretical framework, relying on first-order logic as an epistemological reference for analysing mathematical statements and reasoning in a didactic perspective, especially those involving quantification. *Viviane Durand-Guerrier* presented a paper entitled "Surreptitious changes in letters' status in mathematical discourse". After recalling three possible types of status of letters from a logical point of view, the author dealt in depth with a proof from a high school textbook in which there are numerous changes in the logical status of letters, without any indication of these changes. She then showed responses to a questionnaire attesting to students' difficulties related to the logical status of letters in understanding mathematical statements. For a conclusion, the author stated that this phenomenon is widely underestimated and gave recommendations to take care of this in learning and teaching mathematics at all levels.

The paper by *Imed Ben Kilani* (Tunisia) was titled "Negation of universal statements between the demands of Arabic language, French language and mathematical logic". The author presented the Tunisian context briefly, in which mathematics is taught first in Arabic and then in French. He then presented a logical and grammatical enquiry showing that Arabic and French languages on the one hand, French language and logical-mathematical language on the other, are not congruent in Duval's sense. As a consequence, difficulties in handling negation are likely to appear in the Tunisian school context, especially when a change in language occurs. This was confirmed by the current research of the author. It is noteworthy that this paper suggests that there are broader issues in the case of bilingual classes.

Related papers by *Faiza Chellougui* (Tunisia) ("Articulation between logic, mathematics and language", closely related to the two presented papers), *Filippo Spagnolo* et al. (Italy) ("Logical-linguistic questions in European and Chinese cultures") and *Leigh Wood* (Australia) ("Language of university mathematics") were presented by distribution, complemented by a two minute presentation of each.

A brief but effective discussion followed, pointing to questions of rigor, effective impact on students of difficulties highlighted in theoretical research, and ways in which researchers might help teachers in addressing these difficulties.

Subsession 3B

Associated challenges in doing and formulating mathematics (chair and recorder: Viviane Durand-Guerrier).

This session was more epistemological and philosophical in focus. *Paul Ernest* (UK) presented a paper, "The semiotics of mathematical texts and myths". The author proposed a metaphor of the hero and creation myths as representations of human agency in following a proof and defining a mathematical theory, respectively. Through a dialogue between Logos and Mythos, the author assumed that, like all mathematical and scientific knowledge, mathematical proof in particular is a discursive form, even a narrative, thus it is amenable to the tools of linguistics, semiotics and literary analysis. This led the author, in particular, to consider the reading of a proof as a journey that could be modelled by a circle, due to 'the Cyclic Pattern of Mathematical Proof'.

One paper originally for distribution, by *David Wagner* (Canada), "Facing Mathematics: looking at and looking through mathematical symbols", was offered a ten-minute presentation. A paper by *Allan Tarp* (Denmark) entitled "Pastoral power in mathematics education: A postmodern sceptical fairy-tale study" was presented only by distribution.

During the discussion, the main focus was on how mathematics discourses carry validity and address the reader in order to convince him or her as well as contribute to a successful appropriation of mathematical knowledge.

All in all, the presented and distributed papers manifested the manifold extent of the theme of the TSG. The topics ranged from problems of bilingual learners, by way of interactions in mathematics classrooms and the relevance of metaphors and metonymies for mathematics learning and teaching, to logical aspects of mathematical texts and the use of different media when thinking and writing mathematically. Discussions revealed that the issues were of vivid interest to participants.

This paper was written by Norma Presmeg and Siegbert Schmidt. They will be happy to be contacted at npresmeg@msn.com and siegbert.schmidt@uni-koeln.de respectively, for further information on the work of this TSG.

TSG 26: Gender and mathematics education

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Team Members: *Stephen Lamb*, University of Melbourne, Australia
Roberta Mura, Laval University, Québec, Canada
Ferdinand Rivera, San José State University, USA

Aims and focus

This Topic Study Group offered participants the opportunity to explore pluralism and multiculturalism in mathematics education from the perspective of gender, and the many ways in which they affect mathematical understanding, attitudes and participation. Altogether, 14 papers from nine countries were presented in the four sessions. The presentations were grouped according to content, also taking into account that the papers should illustrate the great variation in cultural, economic and other background factors that influence the formation of female and male differences in mathematics. An awareness and acceptance of such differences formed the basis for the study group. From this point of view it was essential that researchers from a great variety of countries contributed with their papers and in the discussions. Abstracts and papers from this study group are available at www.icme10.dk

Session 1

Three papers were presented in the first session as part of the theme *Mathematics and computers – male domains?* The first paper was "Mathematics – a male domain" by *Gerd Brandell*, *Peter Nyström*, and *Christina Sundqvist* (Sweden). They referred to the GeMa-Project which investigated whether students in compulsory and upper secondary school considered mathematics to be a male, female, or gender neutral domain. Based on the fact that there is a strong gender imbalance in recruiting students to mathematics in Sweden, the hypothesis was that if mathematics is considered to be a male domain, this might influence girls not to study the subject. If, however, mathematics is perceived as a female domain, girls' interest in mathematics may be positively affected. They concluded that mathematics is gender stereotyped in some aspects by Swedish students in both year nine in compulsory school and in year two in upper secondary school, and in accordance with studies from other countries older students stereotype mathematics as a male domain more than younger students do.

The second paper to be presented was "Computers for mathematics learning and gender stereotypes" by *Helen Forgasz* (Australia). She pointed out that computers and hand-held technologies are now very common in mathematics classrooms in Victoria, Australia. Graphic calculators are mandatory for some mathematics subjects in the final year of schooling, and soon computer algebra systems will become compulsory. The surveyed teachers strongly believed that computers helped their students' understanding of mathematics. The findings reported in this paper support the contention that males are perceived by teachers to have more suitable personal characteristics to benefit from using computers to advantage in the mathematics classroom. Compared to females, males were considered more confident and interested in using computers, and also more prepared to take risks and have a go at using the software.

The last paper in this session was by *Xin Ma* (USA): "Current Trend in Gender Differences in Mathematics Performance: An International Update". Based on data from the Organisation for Economic Cooperation and Development (OECD) Programme for International Student Assessment 2000 (PISA 2000), the paper aimed to analyse and describe from a global perspective the current status of gender differences in performance in mathematics literacy as defined in PISA 2000. Research questions that were addressed were to what extent within-school gender gaps in mathematics literacy vary across schools, whether schools with higher average mathematics literacy have greater within-school gender gaps in mathematics literacy, what school characteristics accounted for within-school gender gaps in mathematics literacy, and whether gender differences in mathematics literacy differ between OECD and non-OECD countries. The analysis included 27 OECD countries and 14 non-OECD countries. The results revealed consistent gender differences in favor of boys in mathematics performance in 29 out of 41 countries, but these gender differences were in general small. OECD countries were more likely to demonstrate gender difference in mathematics performance than non-OECD countries.

Session 2

The title of the second session, which included four papers, was *Affective factors among students and teachers*. First, two papers from South Africa were presented, one named "Demystification of the learning of mathematics: Analysis of narratives from feminist perspective" by *Sechaba MG Mahlomaholo* and *Maureen Mathamela*, the second called "Gender differences and black learners' attitudes towards Mathematics in selected secondary schools" by *Sechaba MG Mahlomaholo* and *MZ Sematle*. Both papers focused on problems related to the myth about mathematics as a masculine discipline. The first paper described and analysed narratives of women who have been successful in the study of mathematics. Based on interviews of three women, the study concluded that two major categories of factors are responsible for enabling women learners of mathematics to excel at the subject, namely social, contextual factors and intra-psychic motivational factors. The first category is external and it is authored by things and people outside the learners' self while the latter is internal and personal. The second paper examined and compared the attitudes towards mathematics of black male learners against those of their female classmates by interviewing 10 boys and 10 girls in four high schools located in the rural area of Phuthaditjhaba in South Africa. The study concluded that there are gendered attitudes towards mathematics as a result of socialization into varied gender roles. For change in these attitudes to occur, it is required that changes occur regarding socialization, hence gendering of human beings.

The third paper with the title "Pupils' Gender and Attitude Towards Mathematics in Mozambique" was presented by *Bhangy Cassy* (Mozambique). Mozambique has a ratio of 72% illiteracy, and although one main aim of the Education policy of the country is to promote gender equity in the access to all education levels, there are more females than males who do not benefit from this. This gender discrepancy increases over the education levels, being more evident at the tertiary level and particularly in mathematics and its related fields. Based on a questionnaire to secondary school students the study concluded that, from the beginning of the secondary school stage, females perceive their mathematical ability to be lower than that of males. Although girls did not strongly stereotype mathematics as a male domain, they believed much more than boys that mathematics is more appropriate for males than for females and this was particularly evident among the younger pupils. Girls agreed equally with boys that mathematics is useful. Gender differences found in attitudes were by themselves not large enough to justify the gender disparities in mathematics participation.

The last paper in this session was by *Riitta Soro* (Finland) with the title "Teachers' beliefs about girls and boys and equity in mathematics". In Finland there are only minor differences between girls' and boys' mathematics achievements in the evaluations of comprehensive school or in the matriculation examinations arranged in upper secondary schools, but females do not participate in advanced mathematics courses or in mathematics-related careers at the same level as males do. The focus of the survey study was to examine, on the one hand, teachers' beliefs about differences between boys and girls as learners of mathematics, and, on the other, teachers' beliefs about gender equity in mathematics and the means they used to promote equity. Even though many of the teachers did not express strongly stereotyped beliefs, a great majority held different beliefs about girls and boys and those differences favoured boys. A great majority of teachers did not believe that they had a responsibility to address gender equity and they did not pay any attention to the issue. Gender equity was considered self-evident and mathematics gender-neutral. Many teachers believed that they treated each student as an individual and not as a girl or a boy.

Session 3

The third session, whose theme was *Cooperative learning and mathematical experiments*, included two papers. *Huang Xiong* (P.R. China) presented shortly his paper "Mathematical Experiments; A Survey of Difference Between Girls and Boys in Middle School in China". Mathematical experiments were difficult to conduct in China before the eighth reform of courses. Both girls and boys liked doing experiments, but, contrary to girls, boys seemed to have the view that the more experiments the better.

The second paper was by *Mary Barnes* (Australia), "Student-student interactions during collaborative learning: How does gender influence participation?" Even if recent research on gender in many countries has focused on boys' underachievement and disaffection in academical studies in general, gender differences in mathematics in favour of boys still persist. This indicates a continuing need to focus on the role of gender in mathematics learning. In this study gender issues in a pedagogical approach called collaborative learning were explored, by observing senior classes engaged in collaborative learning. Each class was observed for two three-week periods in order to develop an understanding of classroom routines and to interpret nuances of meaning – unspoken assumptions, shared understandings, jokes and references to past events. Positioning Theory was used as a theoretical framework for analysing the complex interactions within collaborative groups. Student learning gains during small-group discussions arise from activities such as engaging with, and being supported in completing complex tasks, explaining and justifying their own thinking, and trying to understand and critically monitor other people's thinking. Optimal collaboration requires fluid positioning, with students able to move freely in and out of positions such as Expert, Critic, Collaborator and In-Need-of-Help. Exclusive occupancy of any position by one individual may have negative consequences for all.

Session 4

The fourth and last session included two themes and five papers. The first theme was *Gender equity in high schools and universities/colleges*. *Mohammad Hossein Pourkazemi* (Iran) presented his paper "Gender and Mathematical Education." The Nation-wide University Entrance Exams play an important role for students to continue their studies at state universities in Iran. The paper investigated the exam results of male and female students and showed that female students achieved better than male students in topics like Persian literature, Arabic language and religion studies, while the opposite was true in mathematics and physics. Also in chemistry female students achieved best. The overall position of the female students in high school and undergraduate studies is better than male students, but the male students hold better position than the female students in the Graduate Entrance Exam of the Mathematical Sciences. Then *Indira Chacko* (South Africa) gave her paper called "Going from TIMSS-R to the problem solution". This small scale qualitative study was prompted by the results in mathematics of TIMSS–Repeat South Africa. The study attempted to find out from high school students about their problems in learning mathematics and, indirectly, it also tried to identify the approaches used in teaching mathematics in Outcome Based Education (OBE). The results indicate that the approaches used in teaching mathematics in the OBE and the non-OBE curriculum were more or less the same. Most of the problems in learning mathematics were common for girls and boys. Provision of text books, committed teachers that are kind and patient and extra coaching after school were suggested by students as means to attract more students to mathematics. Girls in particular would like to see the content related to situations in real life where these could be applied. Girls in township schools seem to spend more of their out of school time on house hold chores, which could affect their studies.

The second theme in this last session was *Perspectives in research – actions for equity*. The first paper under this theme was "Emerging Perspective of Research on Gender and Mathematics: A Global Synthesis" by *Joanne Rossi Becker* with *Ferdinand Rivera* (USA). The paper was based on discussions from working groups at the last several meetings of the North American Chapter and the International Group for the Psychology of Mathematics Education, meetings and publications of the International Organization of Women and Mathematics Education, which meets every four years in conjunction with the International Congress on Mathematics Education, and other published research that relates to gender and mathematics. The paper examines perspectives used to investigate gender and mathematics in different countries and explores how new perspectives might allow us to un/re/think gender as it pertains to the teaching and learning of mathematics. Different perspectives and methodologies used to investigate gender and mathematics in different countries are examined and explored. The paper underlines that there is a need for alternative methodologies to the positivist framework. The emancipatory viewpoint that celebrates the qualities specific to females was discussed, as well as the deconstructive viewpoint which problematizes the basic notions of "gender" and "differences." The need for research exploring the relationship between class and gender, and especially research of gender and mathematics in developing countries, was pointed out.

The second paper, by *Heather Mendick* (England), was about "Objective subjectivities, subjective objectivities and guilty pleasures: exploring the possibilities of deconstructing the separated/connected opposition for thinking about gender and mathematics." Issues related to aligning separated-ness with masculinity and connected-ness with femininity were discussed in this paper. Even if this alignment has led to valuable interventions, we have to be aware of its limitations. This way of thinking may feed the oppositional binary patterning of our thinking and, in the final analysis, re/produce it. The author explores and argues for a more productive approach. Instead of re-inscribing the dichotomy of masculine/feminine and the location of mathematics within it, its demands are disrupted and refused, by deconstructing the two related oppositions. A main conclusion is that in understanding what is happening when people are learning mathematics, we need to be sensitive to all the varied things that students may or may not be doing when they do mathematics and to make space for a wider range of subjectivities in our classrooms, ones outside and beyond the traditional binary frameworks.

The last paper was presented by *Lynda R. Wiest* (USA): "The Critical Role of Informal Mathematics Programs for Girls." This paper described a mathematics and technology program for Northern Nevada middle school girls. The program consists of a one-week, residential summer camp with two full-day fall and spring follow-up sessions. The research reported here relates the impact this program has had upon its participants in three years of operation and the critical program features that have fostered the successful outcomes presented in these data. Both the girls who participated and their parents provided analytical perspectives on the program. This program demonstrates the continued importance of informal education, in the form of intervention programs, for underrepresented groups – in this case, girls – in mathematics. The single-sex environment – at least for an academically supplemental program such as this – was pointed out as having positively influenced girls' attitudes and performance. The data provided in the paper showed that a well-planned program that targets girls in mathematics and technology can have a positive impact on girls' attitude and performance in these domains both in and out of school.

This report has been prepared by Liv Sissel Grønmo and Hanako Senuma. They are happy to be contacted at l.s.gronmo@ils.uio.no and hanako@nier.go.jp, respectively, for further information on the work of this TSG.

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TSG

Topic Study Group 27

TSG 27: Research and development in assessment and testing in mathematics education

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Aims and focus

The purpose of TSG 27 was to investigate recent developments in assessment and testing in mathematics education and to provide the participants with a forum for sharing and discussing these developments. To organize the TSG two themes were followed:

Research and development in external assessments.
Research and development in classroom assessments.

The way the topic study group was organized gave an unique opportunity to connect two different ways of assessing mathematical understanding which belong to two different assessment worlds with each their own purposes and approaches, but which at a closer look turned out to have more in common than might have been expected. Having the possibility to connect these two assessment worlds worked out to be very enriching to our thinking about future developments in assessment.

External assessment

External assessments are tests designed by a source (e.g. state agency, test publisher, researcher) external to the mathematics classroom, and are administered via a prescribed set of procedures. Such tests are often written but they may be oral as well. There are at least three types of such tests that differ in terms of how the information about student performance is derived from their use, and how the information is used. First, *profile tests,* such as the Third International Mathematics and Science Study (TIMSS), or OECD's Programme for International Student Achievement (PISA), are designed to present policy makers with information about a population or subgroups of a population of students. No summary information about individual students is possible. Second, *research tests* are designed by researchers to gather evidence in order to verify some assertion, test a hypothesis, etc. Administration of such instruments may either be for gathering information about groups of students or about individual students. Finally, *objective tests* are designed so that the information derived can be used to make decisions about individual students. The information may be used for diagnostic purposes, meeting performance criteria, admission to programs, and so forth.

Max Stephens of the University of Melbourne (Australia) presented an overview paper on research on external assessments. This presentation examined four aspects of recent work. First, recent developments in external assessment at the end of high school, including assessments used for university entry have tended to blur the distinction between

[1] Today, MvdHP's affiliation is Utrecht University, The Netherlands, and Humboldt University, Berlin, Germany.
[2] Today, YS's affiliation is Tsukuba University, Japan.

internal assessment carried out by teachers and externally mandated assessment. Second, the growing use of technology in assessments is important because of the use of technology by students in externally prepared assessments of mathematics. It is also important as a means of delivering externally constructed assessment tasks to the classroom. This opens up new possibilities for various forms of profile tests that are mandated for use in schools by national or state or local education authorities. Technology delivered assessment challenges us to consider what is assessed and who is ultimately responsible for assessment in mathematics. Third, the impact of profile tests, including those that are mandated by educational authorities, as well as those that are part of external international assessment in mathematics, such as TIMSS and PISA was addressed. Fourth, the growing interface between instruments developed by researchers to assess students' mathematical understanding and the use of these instruments in system-endorsed programs of school improvement and teacher professional development was examined.

Nine papers were presented or made available for participants on this topic. The papers that were presented were by:

- *Dvora Gorev* (Israel). This paper dealt with how students handle a final exam in calculus that is based on tasks using the computer, and how a computer-based environment can improve low achiever's presentation of understanding.
- *John Threlfall* and *Peter Pool* (UK). This paper described the consequences of creating a computer environment to what is being assessed. Different from paper and pencil tests, computer items have the potential to offer a dynamic and interactive environment in which mathematical thinking may arise and can be assessed.
- *Murad Jurdak* (Lebanon). This paper described the analysis and development of assessment tasks on problem solving by means of action maps (a schematic representation of the organization and sequence of the actions of an activity).
- *Brian Doi*g (Australia). This paper dealt with the development of a formal assessment instrument for young children. The items of this instrument were presented orally in a 'lock-step' fashion (that is, all children were working on the same question at the same time, and advanced through the questions at the same pace). The children could answer the questions by ticking a picture or writing a number.

The papers that were made available were by:

- *M. Pedro Huerta*, *Eduardo Galán*, and *Ramon Grandell* (Spain). This paper dealt with the possibilities of using concept maps built by the students as an assessment tool.
- *Lázaro S. Dibut Toledo*, *Narciso R. De León Rodriquez*, *Eduardo Backhoff Escudero*, *José Luis Ramirez Cuevas*, and *Héctor León Velazco* (Cuba and Mexico). This paper described the development of an on-line university entrance exam via collaboration between faculty in two countries.
- *Signe E. Kastberg* and *Beatriz S. D'Ambrosio* (USA). This paper deals with what understanding students have when they solve mathematical context problems

from NAEP assessments in mathematics and other subjects and with the consequences for assessment design.

- *Göta Eriksson* (Sweden). This paper deals with assessing and teaching young special education children. It is argued that we are obliged to view children in need of special support as competent learners and logical human beings and that it is our responsibility to understand the child and never give up in finding the child's competence.
- *R. M. Dimitric* (USA). This paper deals with the development and testing of a diagnostic test for elementary statistics for university students.

These papers dealt with a rich variety of topics. Attention was paid to psychometric aspects related to the development of a standardized or a diagnostic test, the consequences and possibilities of using computer or on-line assessment, and the use of action maps and concept maps to improve the assessment from the perspective of the mathematical content that is being assessed. Looking at the trends that were mentioned in the overview paper, the nine papers gave support to the analysis that was made. The papers also made it clear that there is a blurring distinction between internal assessment carried out by teachers and externally mandated assessment, that there is an increasing use of technology in assessment, and, finally, that there is a growing use of external assessments as levers for both school improvement and the professional development of teachers.

Classroom assessment

Classroom assessments are methods used by teachers (or groups of teachers) to gather and document information about individual student performance. To monitor progress, to grade performance, and to modify instruction teachers use the information derived from such assessments in a variety of ways. Mathematics teachers have traditionally monitored their students' progress by giving quizzes and chapter tests, scoring answers, and periodically summarizing student performance in terms of a letter or a number grade. Although often items have been developed externally (e.g. by the text book authors, researchers and test designers) teachers are free to modify, adapt, or add tasks; to develop scoring rubrics; and to include information from observations of student work, or from interviews. Today, because of the reform initiatives throughout the world, teachers are expected to incorporate information from observations, interviews, project work, etc. in their judgments of student performance.

David Webb of the University of Wisconsin-Madison (USA) presented an overview paper on the research on classroom assessment. Although there has been a heightened interest in research on this aspect of classroom practice, researchers' varied perspectives on what constitutes classroom assessment appears to have left this potentially informative line of research languishing as an ill-defined and misunderstood topic. To move the discussion of classroom assessment forward three fundamental aspects of internal assessment were used to distinguish it from external assessment: (1) the influence of teachers' conceptions and experiences, (2) the norms and routines of school contexts, and (3) the central role of pedagogical decision-making. For each aspect the findings from several related studies that collectively position classroom assessment as an essential research context were examined. Given that internal assessment is contextualized and somewhat dependent on teachers' conceptions, norms and practices, it was important to take note of these factors when engaging in research, as well as interpreting research findings in this domain. The paper concluded with observations on future directions for research drawing upon intersections between classroom assessment, research on pedagogical decision-making, and reform initiatives in mathematics.

Eight papers were presented or made available for participants on this topic. The papers that were presented were by:

- *Ruhama Even* (Israel). This paper dealt with how teachers make sense of assessment data, and how they can be helped to adopt reform assessment procedures.
- *Marie Hofmannová*, *Jarmila Novotná*, and *Renata Pípalová* (The Czech Republic). This paper dealt with teacher-made assessment instruments in two schools where mathematics is taught in a foreign language.
- *Hari P. Koirala* and *Marsha J. Davis* (USA). This paper described the design of an assessment task for judging pre-service teachers' ability to assess high school students' mathematical understanding.
- *Lisa Björklund* (Sweden). This paper dealt with how year-5 teachers in Sweden assess and describe pupils' performance by means of an external instrument.

The papers that were made available were by:

- *Ilana Lavy* and *Atara Shriki* (Israel). This paper described how class discussion and portfolios could be used in teacher education to assess pre-service mathematics teachers' professional growth.
- *Nellie Verhoef* and *Harrie Broekman* (The Netherlands). This paper focused on views of the learning and teaching of geometry as a framework for designing classroom assessment materials by teachers.
- *Rosemary Callingham* and *Patrick Griffin* (Australia). This paper dealt with establishing the validity of external performance assessment tasks for year-10 students when administered by teachers.
- *Pi-Jen Lin* (Taiwan). This paper described procedures for assisting primary school teachers to design assessment tasks and analyze student responses.

In summary, because there are several fundamental aspects of classroom assessment that distinguish it from external assessment, four additional major issues were addressed in these papers. First, the influence of teachers' conceptions and experiences about how mathematics should be assessed and how those conceptions could be changed was examined for both pre-service and in-service teachers. Second, a particular concern was related to how to improve teachers' ability to hear from students and judge more than just answers, as they are involved in current reform initiatives in mathematics. In particular, the ways in which teachers' judgments are related to "learning lines" or "assessment trajectories" were deemed critical. Third, understanding the influence of assessment within the norms and routines of school contexts and teachers' didactical decision-making is critical. Finally, collaboration between teachers and researchers with multiple perspectives is important, given that classroom assessments are contextualized and somewhat dependent on teachers' conceptions, norms and practices.

The study group ended with a discussion of "visions for the future." Three questions were of particular concern. First, how can the vast amount of research on assessment become a more coherent body of information? Coherence is important because at present there are many "blind spots" such as – how do teachers come to know a student's understanding.

Second, how can assessments be better linked to instruction than is presently the case? In fact, the suggestions by several authors was that use of learning lines, trajectories, progress maps, and so forth can link assessment to what is worthwhile to teach. If so, it gives the field an opportunity to develop a didactical model for assessment design.

Third, how can external and internal assessments be linked? The suggestions included using more standard(ized) assessments in classrooms, as is being done in Sweden, and more teacher-based assessments in external assessment systems.

This report was written by Marja van den Heuvel-Panhuizen and Thomas Romberg. They are happy to be contacted at m.vandenheuvel@fi.uu.nl and tromberg@wisc.edu, respectively, for further information on the work of this TSG.

TSG 28: New trends in mathematics education as a discipline

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Introduction

The aims of TSG 28 were to portray some of the new trends in mathematics education research. This topic potentially includes many and diverse issues, far too many and too diverse to be treated in a single topic study group. The team has therefore decided to select two from among the potential issues:

1. Mathematics and cognitive science, with particular attention to theories of embodiment in mathematics education.
2. Combining quantitative and qualitative research methods in mathematics education.

This report will briefly summarize the proceedings for each issue separately and conclude with some speculation on connections between them.

The four working sessions were organized in the following way:

1. Opening session on topic 1 with two plenary presentations.
2. Opening session on topic 2 with two plenary presentations.
3. Parallel sessions on topic 1 and topic 2, with contributed papers and open discussion.
4. Concluding plenary presentations, one on topic 1 and one on topic 2.

In addition, brief discussions took place after each presentation. The presented papers and/or related papers by the authors are available from the website of the topic study group. (www.icme10.dk).

Topic 1: Mathematics and cognitive science

Three invited (plenary) papers and three contributed papers were presented. The invited speakers set themselves the task to stimulate the discussion on the main themes of embodied cognition, around questions such as "What are the bodily and biological mechanisms underpinning cognition?", and, particularly concerning mathematics, "What are the grounding metaphors used in the construction, systematisation and communication of mathematical thinking?"

The first plenary was given by *Nathalie Sinclair* (USA) on "Embodied and evolutionary perspectives in mathematics education". Nathalie posed the following questions: What is the role of bodily experience in our thinking? When we speak of theories of embodiment, what do we mean by that word, embodiment? Her presentation focused on these questions, using as a lens the interplay between embodied cognition and the use of dynamic geometry software. She identified three different ways of thinking about embodiment. The first relates to the way we embody different procedures, that is, learn to implement them in a way that is "automatic," without discursive mediation, that is, without thinking. Our body knows what to do, without conscious thought, or without computation. The second way lies in the origins of our ways of thinking: even the most abstract concepts have roots in our bodily experience. The third relates to the close coupling between motor and conceptual systems. On the one hand, when people conceptualise a category, they infer relevant actions that they could take on it; on the other, when people perform an action, it influences the construction of conceptual representations.

In the second plenary *Rafael Núñez* (USA) talked about "What embodiment for mathematics education? Issues and controversies from the perspective of cognitive science". Rafael opened his presentation by giving a brief introduction to the role cognitive science has played in mathematics education. He then focused on embodied cognition and discussed the meaning of the concept of embodiment in the cognitive science of mathematics. He stressed the importance of studying the bodily-based inferential organization of mathematical concepts "in themselves", and not only the bodily experiences that particular individuals may have in the process of learning. This can be achieved through 'Mathematical Idea Analysis', that is, the set of techniques for studying the inferential organization of concepts provided by implicit and largely unconscious cognitive mechanisms such as conceptual metaphors, conceptual blends, fictive motion, and gesture-speech production. Mathematics education could benefit from the study of these embodied mechanisms which play a crucial role in constituting the very fabric of mathematics.

Three contributed papers were presented. The first speaker was *Ornella Robutti* (Italy) on "The construction of mathematical knowledge through multiple perspectives". The presentation was aimed at showing various applications of the basic metaphor of infinity in teaching and learning activities, some of which are mediated by the use of artefacts. Ornella highlighted the possibility that the artefacts help students in conceptualising mathematical infinity at a cognitive level, in the same way as metaphors do. The analysis suggested that in some cases even metaphors can be produced through the interaction with an artefact, and these kinds of metaphors can be introduced ad hoc, both by teachers and by students. The crucial point is not how metaphors could be used to do mathematics, but to approach the mathematical concepts at a cognitive level.

Francesca Ferrara (Italy) presented a paper on "Bodily experiments, metaphors, gestures and artefacts in grasping the meaning of a motion graph: a case study". The presentation was aimed at analysing a learning activity, taking into account the contextual ingredients that shape the way 9th grade students interpret a graph arising on a symbolic-graphic calculator from a body motion in front of a sensor. The analysis reveals that a metaphor characterises the students' cognitive behaviour in the process of understanding. However, this metaphor arises in a complex *context*, in which it alone cannot adequately explain the students'cognitive processes. It is necessary to integrate the analysis with the consideration of the mediation role of the artefacts in use and of gestures that students need to represent and communicate ideas.

The third speaker, *Janete Bolite Frant* (Brazil), presented a study done with some collaborators (M. C. Barto, C. Dallanese, A. Mometti) on "Reclaiming visualization: when seeing does not imply looking". This study was part of a larger study that investigates meaning productions for calculus contents by mathematics teachers and professors. Janete focused the presentation on the role of visualization in understanding calculus and in producing meaning for the concept of derivative of a function. Based on a neuroscience perspective according to which seeing does not imply looking, she produced an analysis of two episodes taking place in a computer laboratory used for a calculus class.

Finally, *Marianna Bosch* (Spain) presented a concluding plenary lecture on "Mathematical cognition and the anthropological approach to didactics: the institutional relativity of knowledge". Marianna introduced the main tenets of the model of knowledge used by the anthropological approach. Then, she showed the main differences between the analysis provided, and the questions raised, by the embodied approach and those of the anthropological approach. She argued that the anthropological approach to mathematical cognition is a useful tool for mathematics educators to raise and solve educational problems. It is a theoretical tool that helps to better understand, but also a practical tool to progress in the teaching and learning of mathematics. Marianna concluded with a few short remarks about the role played by the body – and also by other kinds of material 'artefacts' – in the production and development of mathematical knowledge.

Topic 2: Quantitative and qualitative research

The issue

A judicious choice of research methodology depends on the aims of the research. Nevertheless, the potential benefits and drawbacks of qualitative, quantitative and mixed methodologies in educational research in general have recently become a common topic of discussion (see e.g., the November 2002 Theme Issue of the *Educational Researcher*). A common opinion expressed in this discussion is that quantitative research, when it is possible, yields results of greater validity than qualitative research but that because of the large influence of contextual factors and because of the ubiquity of interactions, quantitative research is often impossible in education. The time seems ripe for research in mathematics education to transcend the dichotomy of quantitative versus qualitative research and ask whether well designed combinations of quantitative and qualitative methodologies could yield results that would be more useful and more valid than those obtained from either type of methodology separately. The task set by the team for this part of the topic study group was thus to explore effective ways of combining quantitative and qualitative methods. Three invited (plenary) and two contributed papers were presented, each adding valuable experience and insight to the issues and questions under discussion.

The presentations

Kurt Reusser and *Barbara Vetter* (Switzerland) presented an invited paper entitled "Combining quantitative and qualitative analyses of lessons in (large scale) mathematics video studies. Insights from research and potential for teacher education". They presented a study aimed at explaining outcomes of instruction in terms of a large number of variables related to teaching. The study is set against the background of a complex multilevel mediational framework of instructional quality and effectiveness. For example, the multiple levels of student, classroom, school and system were taken into account when collecting and analyzing data. The novelty of the study lies in the method of data collection, which combined video survey (tapes of 156 lessons) with the best of ethnographic case studies. A number of non-trivial methodological problems had to be solved; these included decisions on the level of classroom actions to be coded as well as training the coders. One of the advantages of a study like this is the option to zoom in from the survey data onto a single teacher or classroom or lesson for a detailed qualitative analysis.

Jo Boaler[1] (USA) presented an invited paper entitled "Studying a complex practice – using multiple methods to capture the relationships between teaching and learning". She discussed methodological aspects of an intensive long-term study of three schools. Two of the schools had mainly traditional and a few reform classes; the third school had only reform classes. Low inference variables are relatively easy to measure using appropriate coding schemes; an example of low inference coding is whether teacher questioning is respondent to student actions. It turns out that it is more so in reform classes than in traditional classes but it also turns out that this difference does not explain the achievement gap between the reform and traditional classes. A deeper level of analysis concerns question types; for example, questions in traditional classes are almost exclusively factual whereas in reform classes they are varied. This difference had more explanatory power but it is more difficult to obtain quantitative data at such a deeper level. For even greater explanatory power, one needs interviews that are non-quantitative in nature: If teaching is to make meaning of how students think, then teaching cannot be measured quantitatively. On the other hand, the public domain and policy makers are being reached only by quantitative results. In summary, results at different levels need to be obtained and integrated.

Mi-Kyung Ju and *Oh Nam Kwon* (Korea) presented a paper entitled "Mixed methods: different ways of talking about students' views about mathematics". They made use of a mixed methods approach to evaluate the instructional design of an inquiry-oriented differential equations course in a university in South Korea. The questions steering their evaluation research required both quantitative and qualitative methods. They considered the development of reliable explanations as the strongest advantage of the mixed methods approach. Using their own research as an example, they showed that it is possible to increase this reliability by cross-checking whether explanations from different methods converge. They also pointed out, however, that their use of mixed methods led to a challenge, namely the development of divergent explanations of students' views about mathematics. In retrospect, they looked at the challenge as a critical learning experience in that it provided an opportunity to witness the intricateness of the phenomenon under inquiry and to develop a richer description and explanation through crosschecking and reflection. They admonish researchers to be cautious in selecting measuring tools consistent with the theoretical perspective of their research.

Peter Petocz, *Anna Reid*, *Leigh Wood*, and *Geoff Smith* (Australia) presented a paper entitled "On becoming a mathematician: an international perspective for future professionals in the mathematical sciences". They reported on their investigations of mathematics students' ideas about working as professionals in the mathematical sciences, and on the impact that these ideas have on the students' learning of mathematics. The research design used a three-phase combination of qualitative and quantitative approaches. Two phases of qualitative methods were carried out in order to carefully prepare a questionnaire that was then used to collect quantitative data. Thus the strengths of each methodology were exploited to increase the overall usefulness and researchers' confidence in the results.

[1] Today, JB works in the UK.

Jeremy Kilpatrick (USA) presented an invited paper entitled "Methods as ideologies: Is our research scientific or political?" About research in mathematics education, one can ask whether it is scientific and, if so, with regard to what science? One can also ask whether it is political and, if so, with regard to what politics? The history of research in our field shows that it began by emulating the natural sciences but over the past half century shifted rather drastically away from science even in the loosest sense. As that happened, and research became more qualitative, it got embroiled in controversies over "reform" and even found its way into partisan politics. All research involves comparison, whether explicit or implicit, and all data analysis is potentially quantitative. Begle's call for an experimental science at ICME-1, however, is far from being met. Though he underestimated the role of value judgments in our research, he did help us see the value of empirical work. A merging of qualitative and quantitative methods – as shown, for example, in some studies of embodied cognition – can help move the field forward both scientifically and politically. The answer to the question posed in the title of this talk is that our research is both, scientific and political.

Discussion

The presenters showed a wide variety of ways to combine qualitative and quantitative research methods. They drew attention to the great advantages in terms of explanatory power that the combination of methods brings with it. They also pointed to the large amount of resources that are needed in order to carry out a quantitative analysis of qualitative data. Specific methodological difficulties, such as determining the level of inference at which coding is both feasible and relevant, were also mentioned. In summary, it appears that we can get significant and reliable results from qualitative research by up-scaling it, but that there are methodological problems to be considered and that the amount of resources in time and money associated with such up-scaling is very large.

This report was written by Tommy Dreyfus and Domingo Paola. They are happy to be contacted at tommyd@post.tau.ac.il and domingo.paola@tin.it, respectively, for further information on the work of this TSG.

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TSG

Topic Study Group 29

TSG 29: The history of the teaching and the learning of mathematics

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Team Members: *Hélène Gispert*, University of Paris-Sud, Orsay, France
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Introduction, aims and focus

The invitation by the International Programme Committee to organise this TSG meant the impetus to establish for the first time an international network on this subject. The first step for the organising team was hence to elaborate a programmatic outline serving as common basis for its work. The following understanding was agreed upon:

"The history of the teaching and learning of mathematics is an interdisciplinary field of study. It constitutes a part of the history of mathematics, of the history of education and of sociology. The broad range of relevant topics includes the evolution of programs in a variety of countries, the status of mathematics as a teaching subject, the cultural and social role of mathematics, policy in teacher education, evolution of the profession of mathematics teachers, teachers' associations, journals on mathematics education, and textbooks. The history of the teaching and learning of mathematics is still a relatively underdeveloped field, and most studies deal with national histories. There are only few studies on international and comparative issues. Mathematics learning and teaching is not exempt, however, from the present tendencies towards internationalisation and globalisation. As international studies on evaluation like TIMSS and its follow-up study PISA show it is very important to develop categories which permit to grasp national specificities as well as overall and global trends in the evolution of mathematics teaching. The work of TSG 29 should contribute to gather the researchers working in this field, establish common patterns in the history as well as revealing differences, and develop research programmes which enhance international perspectives."

Since the field for TSG 29 is extraordinarily broad, given the range of topics, the number of states and cultures through history, and the different levels of school systems, the team decided that the focus will be on institutionalised forms of teaching and learning – in types of schools equivalent to primary and secondary levels. Higher education has been included in so far as it concerns mathematics teacher education.

The next step was to establish an international bibliography of publications in order to identify the main trends of present research and to know the persons active in related research. By common effort, it was possible to establish a significant bibliography of publications from an enormous number of countries. This first international bibliography constituted a main focus of the website of TSG 29. At the same time, this bibliography made it possible not only to identify the most active researchers in the field, but also to structure the field. Three dimensions have proved to constitute basic issues of research across countries:

1. Modernisations of mathematical curricula.
   Focussing on transmission and/or socio-cultural reform movements.
2. Aspects of teaching practice.
   Focusing on textbooks, methods, teacher training.

3. Cultural, social and political functions of mathematics instruction. Focusing on, for instance, practical/vocational versus formal/academic function.

The team decided to have no subdivisions of its sessions at the Congress in order to realise the intended joint discussion. Besides the final session, which was designed to present the state of the art as a synthesis of the contributions, nine presenters were invited for the three other sessions, each session being devoted to one of the three main dimensions of the field.

The invitees reacted with enthusiasm to the facilitation of communication offered by this TSG. In the end, two of them were unfortunately unable to attend the Congress due to lack of funds.

Two papers were accepted for presentation by distribution:
Taro Fujita (UK), *Keith Jones* (UK), *Shinya Yamamoto* (Japan): "The Role of Intuition in Geometry Education: Learning from the Teaching Practice in the Early 20th Century"
Marie Kubínová (The Czech Republic): "Teaching Mathematics in Czech Schools – Trying a Change"

The following papers were delivered during the sessions:

Session 1: Transition and modernisation of mathematics curricula.
Shinya Yamamoto (Japan): "The Process of Adapting a German Pedagogy for the Modern Mathematics Teaching in Japan".

The paper exemplifies the complex processes which transmitted concepts undergo in the recipient culture or country: The insistence of the German "Treutlein" on abolishing the strict separation between plane geometry and solid geometry was eventually reduced, in its Japanese reception, to a methodological reform in dealing exclusively with solid geometry.

Nikos Kastanis (together with *Iason Kastanis*) (Greece): "Transmissions of Mathematics into Greek Education, 1800-1840: From Individual Choices to Institutional Frames".

This paper studies the changes in the mathematical culture occurring during the transition of Greece from Ottoman rule – where there was no institutionalised schooling, the meagre elements of education being under the control of the Orthodox Church – to an independent state establishing an educational system of its own. Whereas in the former period isolated students studying abroad in Western Europe had brought back some elements of traditional elementary mathematics, without developing them further, though, the latter period was characterised by the transmission of more modern, up to date knowledge from France and Germany, and by developing mathematics on these bases within the new educational system.

Kristin Bjarnadóttir (Iceland):"From Isolation and Stagnation to 'Modern' Mathematics – A Reform or Confusion?"

This paper studies the transmission to Iceland of the movement which is nowadays usually referred to, often negatively, as the modern mathematics movement. Given the considerable time lag in the development of the school system and of higher education in Iceland, the introduction of "modern mathematics" coincided with a general social and educational change, developing for the first time a culture of mathematics education in Iceland.

Session 2: Teaching practice, textbooks, teacher education

Eileen Donoghue (USA): "The Education of Mathematics Teachers in the United States: David Eugene Smith, An Early Twentieth Century Pioneer"

The paper studies the development of preparing mathematics teachers for secondary schools in the United States since the 1890s, and in particular the model program established by David Eugene Smith, initially at Michigan State Normal School and, subsequently, at Teachers College, Columbia University. Smith's teacher education program is discussed and compared to a contemporary, but distinct program at the University of Chicago. It also considers how Smith influenced teacher education through his extensive international links, and his role as a prolific textbook author.

Harm J. Smid (The Netherlands): "Between the Market and the State: The Emergence of Mathematics Instruction and of its Teachers as a Result of State Initiative and of Pressure by the Market"

In 1815 and in 1826 the Dutch government undertook some measures for teaching mathematics in the so-called "Latin schools", and during the first half of the 19th century, mathematics became an important part of the entrance exams for military and engineering academies. But the real break-through in the Latin schools occurred between 1840-1845, when these were forced to modernize their programs and organization due to the heavy competition by the so-called "French schools", which were private schools with a much more modern program. The result was that when the state in 1863 at last introduced a Dutch version of the German *Realschule*, the mathematics curriculum, textbooks and teaching staff were easily available.

Session 3: Cultural, social and political functions of mathematics instruction

Livia Giacardi (Italy): "From Euclid as Textbook to the Gentile Reform: Problems, Methods, and Debates in Mathematics Teaching in Italy 1859 to 1923"

This paper provides an excellent case for studying the third dimension, identified above, where the relations between the various agents and instances are made clear and explicit – those between mathematicians, mathematics teachers, cultural traditions and their impact on school structure, and political movements and decisions. The salient feature of the Italian case is the split between various groups within the mathematical community.

Alexander Karp (Russia): "'Universal Responsiveness' or 'Splendid Isolation'? Episodes from the History of Mathematics Education in Russia"

Given the state of general underdevelopment of the teaching of mathematics and the sciences in Russia in the past, the energetic introduction of an educational system by Tsar Peter the Great meant a decisive modernisation of the country. For a long time, its evolution relied on the transmission of foreign science, and it took a long time until a significant national production in science began to take off. While the country was, for a long time, open and receptive to the transmission of mathematics, an isolationist stance gained momentum since the second half of the nineteenth century.

Mahdi Abdeljaouad (Tunisia): "Issues about the status of mathematics teaching in Arab countries – elements of its history and some case studies".

This contribution was distributed on the TSG's website. It presents the first synthesis of an extensive research on the history of mathematics teaching in the Arab civilisation. By its methodologically guided research, it deals with the fact that although mathematics experienced important developments in the classical Arab period, mathematics teaching seldom obtained more than a marginal status.

Session 4: Synthesis

In the final session, the Organising Team presented a synthesis, which evaluated the contributions to the three dimensions and developed methodological categories derived from the key issue of unravelling the function of mathematics teaching within the respective system and its context, across the various national histories, and presented characteristic specific features of the historical development.

In the subsequent general discussion, a high degree of consensus about the goals of the TSG and about the envisaged research approaches became evident. The success of the TSG, also expressed by the considerable number of participants, was also confirmed by the participants urging to continue this work.

In fact, a network of people interested in promoting research on this topic was established. This group will organise future activities and involve more scholars.

A first important outcome of this ongoing work will be the publication of the Proceedings of TSG 29.

This report was written by Gert Schubring and Yasuhiro Sekiguchi. They are happy to be contacted at gert.schubring@uni-bielefeld.de and ysekigch@yamaguchi-u.ac.jp, respectively, for further information on the work of this TSG.

ICME 10 2004

DG

Discussion Group 1

DG 1: Issues, movements, and processes in mathematics education reform

Team Chair: *Zalman Usiskin*, University of Chicago, USA
Team Member: *Bengt Johansson*, University of Gothenburg, Sweden

Aims and focus

Mathematics education can be studied through a variety of lenses. These lenses range from those that zoom in to allow us to see the veins of petals and leaves of individual lessons, students, and mathematical concepts to those that zoom out to allow us to examine the climatic phenomena influencing a country's mathematics education as a whole. The focuses of DG 1 required lenses that zoom out to discuss the issues, movements, and processes in mathematics education reform. With the help of the overall ICME organizers, the DG 1 organizers framed a set of questions to guide the thinking of those who might be interested in contributing to or attending this group. The first purpose of these questions, summarized below in this brief report, was to encourage papers dealing with the processes by which mathematics curricula are formulated and goals of mathematics are determined and announced, and the issues, forces, and interest groups that affect these developments. The second purpose of these questions was to serve as an organizing tool for cross-country discussion and comparison.

Organization

This discussion group had two organizers (from China and the USA) and three associate organizers (from Chile, Japan, and Sweden). Of these five, only Zalman Usiskin (USA) and Bengt Johansson (Sweden) were at the congress. Huang Xiang (China) was ill and the associate organizers Fidel Oteiza (Chile) and Eizo Nagasaki (Japan) both had to remain home because they were leading figures in mathematics reforms that needed attention in their countries even as the congress was going on. Bengt also was in the position of having to do work at home during the congress, but being from Gothenburg he was able to go back home and return during the congress. Thus one could argue that, for the most part, the unfortunate absence of these people was an outgrowth of the high positions they held, exactly the positions that made them appropriate to be organizers.

This expertise of the other organizers was matched by the expertise of many of the people who attended one of the three meetings of DG 1. A number of attendees at DG 1 were in charge of testing programs, curriculum frameworks, or development projects in their countries.

No formal presentations were allowed in the DGs. And although the organizers asked for papers to be sent to us before the conference, only one paper written for the conference from Margaret Kidd of the United States was received. Another person sent two papers written some time before the conference. The absence of papers on the web may have been a boon for DG 1, because unlike the TSGs and posters, everyone could participate without preparation.

Around 50 people attended one or more of the sessions; 45 at the first; and about 28 at the second and third. The discussions involved participants from 20 countries and all but a few of those in attendance. The organizers heard later that the leading ministry person in one Asian country attended the group but did not contribute to the discussion.

First session

At the first session, the following questions were raised before the whole group.

1. Who is mostly responsible for mathematics curriculum reform?
2. How do these individuals get together?

By "reform" the discussion centered around "re-formulating the mathematics curriculum", not necessarily tied to a particular kind of reform movement within mathematics education. Thus the discussion centered more on process than on the substance of the reforms (which operationally were defined as major changes). Most of the contributions were informational in direct response to the questions.

In most of the countries represented at DG 1, nation or state-wide ministerial committees are formed to lead the reform. Sometimes commercial publishers are involved, either because they write directly to the reforms or because they help stimulate some reforms. Rarely are professional organizations involved. An exception in this regard has been the involvement of the National Council of Teachers of Mathematics in the United States in encouraging reform initiatives and in steering the direction of reform. It was noted that the size of this organization (about 100,000 members) automatically gives its work a presence in that country.

The individuals responsible for working out the details of the implementation of the reform are usually mathematics educators, mathematicians, and mathematics teachers. Rarely are parents, students, users of mathematics, or the interested public involved. At times, people working anonymously behind the scenes become involved in the reform. A disturbing commonality appeared in the discussion: In a number of countries, consensus-building that has been carefully reached over an extended period of time (often a number of years), in committees whose members have been carefully selected to represent different viewpoints, is sabotaged by last-minute changes by people whose competence is questionable and whose identity may not even be known. The phenomenon seems to occur most often when there is a change at the a high governmental level and the new leaders in education want to place their own stamp on the reforms, or when the education leaders disagree with the consensuses that have been reached, or sometimes (it seems) when the leaders (old or new) are or wish to remain ignorant about what is done somewhere else than under their watch.

The phenomenon of the unknown reformer is not universal. In areas where education is separated from politics and where well-established procedures are in place for decision-making (e.g., Japan, where reform in the system follows a schedule planned years in advance), reform proceeds in a more orderly way.

Second session

In the second session, the large group was split into small groups of 6-16 to discuss the following questions:

3. What are the goals of mathematics education reform?
4. What developments in mathematics curriculum reform are currently being undertaken?
5. What forces inside the mathematics community have had significant effects on curriculum reform?

6. What forces outside of mathematics have had significant effects on curriculum reform?
7. What is the role of various kinds of documents in instituting reform?

Several participants noted that the opportunity to be in a small group was something that they had not experienced at previous ICME congresses, and they were quite happy to be able to sit with a few people they did not know before and discuss issues of common concern. Summaries of the discussions were presented to the whole group by representatives of the small groups and centered more around Questions 5 and 6 above than any others. Six forces inside and outside the mathematics education community were identified: professional groups of mathematics educators and/or mathematicians; politicians, often those dissatisfied with how schools are doing; new technologies, which influence both the content and approach; commercial interests, including publishing companies and electronic sources; business leaders, who desire an educated workforce; and teachers of mathematics, who can work both for and against change.

Third session

In the third session, the group met again as one body. Attendees were asked to identify reforms that they felt were working in their countries. A number of examples were offered, allowing the DG to end on a positive note. Among the mentioned reforms was the National Numeracy Project in England, a project whose main goal to get children in grades K-5 to think about mathematics (rather than to view mathematics as all memorization and rote) by working with their teachers. This project has been adapted in Australia under the banner "Count Me In, Too", and New Zealand. A reform in Singapore has been to raise awareness of social issues and society in mathematics classrooms. Again the mechanism for reform is to transmit ideas to teachers through workshops. In Sweden, the movement has been towards systemic reform, namely to consider the following aspects of mathematics education simultaneously: the public sector, the teacher's professional identity, the commitment of all participants in the process, institutional issues, time resources, curriculum content, and assessment. The group was also informed about the latest reforms in Spain and Japan.

This report has been written by Zalman Usiskin.

DG 2: The relationship between research and practice in mathematics education

Team Chairs: ***Luciana Bazzini***, University of Torino, Italy
Kenneth Ruthven, University of Cambridge, United Kingdom
Team Members: ***Kiril Bankov***, University of Sofia, Bulgaria
Nuria Gorgorio*, The Autonomous University of Barcelona, Spain
Cassius Lubisi*, Special Advisor to the Minister of Education, South Africa

Aims and focus

This DG examined relationships between educational research and professional practice in mathematics education. The call for background papers to stimulate discussion requested analyses of specific cases of work which illuminate interaction between educational/didactical research and professional practice/policy in mathematics education. It was envisaged that, in many cases, this would also involve consideration of the interaction between researchers and practitioners; but equally, cases where the same persons – such as teacher-researchers – fulfil both roles were of interest. The call for papers invited submissions to address several or all of the following questions:

In relation to the case(s) analysed:

- *What were the research and professional motivations for the work described?*
- *What form of interaction between research and professional practice did this work involve, and how was this interaction organised?*
- *How did this work build on existing research knowledge and/or existing professional practice?*
- *How did this work lead to the development of professional practice and/or of research knowledge?*
- *More broadly, through this work, what did researchers learn from practitioners, and practitioners from researchers?*
- *What were important factors affording and constraining this work?*

Considering the case(s) analysed as prototype(s) for wider diffusion:

- *Does work of this type provide models, artefacts, or theories which could be more widely used?*
- *How viable is work of this type as a means of improving professional practice?*
- *What contribution does work of this type make to advancing mathematics education research?'*

Papers for discussion

In response to the call, eight papers were accepted for presentation by distribution, as follows. (These papers were not delivered orally):

Transition of mathematics teaching: Action research in cultural and linguistically diverse classrooms

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* Unable to attend but contributed to shaping the programme of the DG

Current math reform calls for changes in math teaching and learning (NCTM, 2000). These changes are transitions involving fundamental shifts in reconceptualizing in both mathematical activities and the role of the mathematics teacher (Cooney & Shealy, 1997). Although there have been numerous research projects on mathematics teachers in transition (e.g., Fennema & Nelson, 1997), there has been little research on the connections between mathematics teachers' transition in beliefs and pedagogical content knowledge and the role of teachers' action research on their teaching practice. The notion of teachers' transition has many dimensions. This study only included teachers' beliefs and pedagogical content knowledge in mathematics teaching. The general purpose of this research was to investigate the impacts of mathematics teachers' action research on their transition in beliefs and their pedagogical content knowledge in cultural and linguistically diverse classrooms.

Researcher and teacher in interaction: The graphic calculator in the teaching of mathematics in Denmark

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This article describes a development project where four mathematics teachers co-operated with the author as a researcher. The co-operation process was part of the author's Ph.D.-project. The collaborators worked together on the development of teaching material designed to introduce the concept of Derivative using the Graphic Calculator as a teaching tool. The interactions between researcher and teachers were analysed using a model developed by Ole Skovsmose and Marcelo Borba, and the article also discusses this model. The author generalises from her own experiences both as a teacher and as a researcher and makes a few comments on how research and practice might enhance each other.

The construction of algebraic expressions as context for the interplay between theoretical and practical standpoints

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Luisa Bertazzoli, Scuola Media Statale 'G. Carducci', Brescia, Italy.

In this paper the authors approach two main issues as outlined in the DG discussion document, namely "How the interaction between research and professional practice build on existing research knowledge and existing professional practice" and "How did this work lead to the development of professional practice and of research knowledge?" Such questions will be addressed through the discussion of a research study on algebraic thinking, which has been carried out thanks to a close co-operation between university researchers and school teachers. More specifically, we will focus on a teaching experiment, which was carried out in grade 8 (pupils' age: 13-14) and aimed at promoting a functional approach to algebra.

Critical issues in researching cultural aspects of mathematics education

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In this paper the author refers to three significant areas of research which are currently presenting us with equally significantly issues – culturally-based mathematical knowledge, hidden cultural values, and culturally situated mathematics learning. More importantly, in the context of this ICME Discussion Group, as the paper is focussed on socio-cultural aspects of research in mathematics education, this immediately raises issues about practitioners. Who are they? Clearly not just teachers. There are several other practitioners within the educational field who are also involved – curriculum developers, teacher educators, policy practitioners, school principals etc. But the students are themselves some kind of practitioner, and their parents and the wider community also play roles in this field with their own specific practices. So perhaps this area of research will make us problematise and broaden the oft quoted researcher/practitioner dichotomy.

Putting research into practice: A case in mental computation

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This paper reports on a teaching experiment, conducted in 2003, which aimed at enhancing young students' mental computation performance through incorporating research (the researcher's own and that of others) into classroom practice. Research findings on students' mental computation performance were presented to two Year 3 teachers, along with practical ideas (web sites, readings, etc.) to form a foundation for a short instructional program. The researcher supported the teachers in developing the program, and the teachers took responsibility for implementing the program. This program was developed by the teachers and the researcher. Pre- and post-instruction individual interviews were conducted to monitor student progress and inform the instructional program.

A way of expanding the relationship between researchers and practitioners in mathematics education: The Interlink Network

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The case of the Interlink Network focuses on the interaction between researchers and mathematics teachers from the secondary and middle school (11 to 17 year old students). This network – constituted by teachers, future teachers and researchers – has been operating since 2000 and aims at facilitating the use of information and communication technology (ICT) in the mathematics education developed in some Brazilian public schools. Interlink takes the form of both face-to-face meetings and virtual meetings. In being engaged in a network the teacher may construct knowledge about the use of ICT and act as a result of the collaboration with others. Being an Interlink member has been a stimulus for many participants, in particular for the teachers. The paper points out that Interlink opened possibilities for student, teacher and researcher acting as multiplier. This is a direct interpretation of the idea that many different members could act as a "centre" of the network. This network is a space for personal and professional development.

Linking researching with teaching: Towards synergy of scholarly and craft knowledge

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This paper argues that coupling the creation of scholarly and craft knowledge can contribute to building a more powerful and systematic knowledge-base for teaching. This calls for an approach to knowledge creation in which the distinctive practices of teaching and researching accommodate to one another, through the co-operation of teachers and researchers, or through the co-ordination of teacher and researcher roles by teacher-researchers. The tendency in such collaborations has been to highlight – and privilege

– the creation of scholarly knowledge within the practice of researching, and its application within the practice of teaching. Yet, not only does the craft knowledge of teachers play an important part in converting scholarly knowledge into actionable form, but there is a significant – though largely tacit – process of knowledge creation within the practice of teaching. Equally, research processes can play a valuable part in eliciting and systematising this craft knowledge of teachers.

The relationship between research and practice in mathematics education: Can mathematics education be an evidence-based practice?

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This paper reports on a project involving classroom-based research activity by a group of primary school teachers in the U.K. The teachers' perspectives on the research process in which they were engaged are reflected on to examine the relationships between research activity, evidence, policy and classroom practice. Scepticism about teaching as an application of evidence-determined policy, in particular the current enthusiasm for educational research modelled on practices in medicine, is set against a model, developed through the work of these teachers, of classroom-based research that is directly focused on practical value in the researching teachers' own classrooms. It is argued that classroom research need not be concerned with generating generalisable findings in order to contribute to improving practice. The paper proposes a conception of research in teaching not as application but as realisation in a local context.

The Discussion Group met on three occasions, breaking down into smaller subgroups to facilitate interactive discussion. The first meeting examined, in a more general way, the issues raised in the call for submissions; while the last meeting was organised as a paper discussion session, allowing participants to join a discussion involving the author(s) of a specific accepted paper. The middle session focused on the work of the ICME Survey Team on 'The relations between research and practice in mathematics education', as reported by Anna Sfard in an earlier plenary lecture.

Few of the participants had read the material made available in advance. Consequently much of the discussion in small groups centred on exchanges about personal situations and experiences, notably about how to improve communication and coordination between different components of the mathematics education system – educational research, teacher education, classroom practice, and systemic policy. Particular reference was made to the value of roles such as teacher-researcher and mentor-teacher which involve teachers in working across two components of the system (classroom practice and educational research or teacher education respectively). The lively discussions exemplified and confirmed the crucial role of co-operation between research and practice in mathematics education. Such co-operation takes many different forms and occurs under very differing circumstances. The examples described in the preliminary reading and accepted papers witness the richness and potential of issues related to the theme. All the papers remain available at www.icme-organisers.dk/dg02/

This report has been written by Kenneth Ruthven and Luciana Bazzini. They are happy to be contacted at kr18@cam.ac.uk and luciana.bazzini@unito.it for further information on the work of this DG.

DG 3: Mathematics education for whom and why? The balance between "mathematics education for all" and "for high level mathematical activity"

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Team Members: ***Sol(omon) Garfunkel***, COMAP, Lexington, USA
Gerardus Polla, Bina Nusantara University, Jakarta, Indonesia
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Aims and focus

Worldwide, educational ministries, institutions and societies are trying to answer this hot and controversial headline theme in different ways. Discussion Group 3 recognised both the diversity of social, economic, political and cultural problems in the different countries and at the same time some similarities in hopes and aims for mathematics education. The debate was centred on five questions:

Question 1: *Who should receive what kinds of mathematics education, why, and with what goals?*
Question 2: *Is the dichotomy between 'mathematics for all' and 'for future mathematicians' genuine?*
Question 3: *How can 'mathematics education for all' embrace opportunities for 'high-level mathematical activity'? But also: How can 'mathematics for high level activity' embrace opportunities for 'mathematics education for all'?*
Question 4: *How can instructional practices support the development of highly motivated mathematics learners as well as mathematics for all?*
Question 5: *What is mathematical literacy? Must mathematical literacy be the same for all? If not, does mathematical literacy depend on socio-cultural factors? Why?*

Organisation

Discussion Group 3 organized its work this way: In the first part of Session 1, L. Lindenskov made a presentation in Power Point of: i) the purposes of the discussion group; ii) the answers given by panellists R. Askey, S. Carreira, Y. Namikawa and R.Vital, who in a plenary session at ICME-10 had expounded their points of view on the headline theme; iii) the questions asked by the Organizing Team and divulged through the ICME web page; and iv) the contents of the documents presented as materials in the same web page related to the questions asked by the Organizing Team and by G. Malaty, V. Freiman and B. Evans. In the second part of Session 1, the participants made groups freely, in order to exchange points of view with respect to the ideas expounded by the panellists.

In Session 2, S. Garfunkel synthesized what the DG 3 had advanced in the previous session, and the Team Chairs asked participants to divide into four groups to continue with the discussion and to answer the questions. Each subgroup handed in the result of their work at the end of the session. Having these products as a basis, M. Villavicencio and L. Lindenskov systematized the answers and elaborated on them in a work document.

In Session 3, this document was handed in. This document was presented by M. Villavicencio in Power Point and served as a basis for the discussion in the plenary group meeting. Owing to the lack of time to agree with all that was presented, the Organizing Team considered it appropriate to continue the discussions electronically among the participants after the congress through e-mails.

Discussions and recommendations

Following up, the main themes in the discussions – as seen by the organizers and contributors – with respect to the five questions and answers and recommendations for the formulation of policies were displayed. Elaborated results from the e-mail discussions will be displayed on the website.

Question 1: *Who should receive what kinds of mathematics education, why, and with what goals?*
Everybody should receive mathematics education, because they need thinking tools for work, everyday life and citizenship that can be developed by learning mathematics, and because mathematics gives them possibilities for enjoyment, creativity and for personal development. Mathematics makes use of a universal language to describe nature, human society, and so on, and it helps to train logical and abstract thinking; and given that it uses models, mathematics helps in learning systematically to understand things or to solve problems.

In order to ensure mathematics for all, unequal opportunities in mathematics education have to be overcome. That means that it is crucial to give more attention to and guarantee appropriate math education for:

- Female children – Such actions are necessary, for instance, because male-centred traditional customs usually guide girls, even though mathematically talented, to choose the college departments unrelated with mathematical fields. Parents and even teachers do not expect girls to learn mathematics as well as boys
- People in the rural areas – Particularly for native people who speak their mother tongue and have traditional cultural background, and for minority socio-cultural groups (e.g. immigrants). Generally, rural areas, compared with urban areas, have an educationally inferior environment in aspects of teaching, mathematical competency, parents' educational expectations and information, and so on.
- Those who have special needs – i.e. who are blind or handicapped should be given special attention.
- Children and adults, illiterates and other vulnerable groups in society.

The DG 3 recommended that an educational system should emphasize:

- Cultivating mathematical ability and curiosity, and not isolated skills and knowledge.
- Providing students with experiences that put emphasis on the mathematical problem solving and thinking abilities (reasoning and communication).

- Providing students with experiences that give a broad perspective to the mathematics content and structure and to the relations among the various topics, starting at a young age.
- Supporting teachers to overcome their own bad learning experiences.

Mathematics education – like other subjects – must support universal social values (solidarity, tolerance, openness, inclusiveness and attitudes to maintain a dialogue in our own social group and with others) seeking for the well-being of mankind.

Question 2*: Is the dichotomy between 'mathematics for all' and 'for future mathematicians' genuine?*

It seems to the organizers that, with question 2 and 3, the participants faced the biggest challenges in their efforts to interpret and understand each other's viewpoints. Some participants defended the viewpoint that a solid mathematical ground is a necessary prerequisite for engaging with any use of mathematics; others defended that learners can develop both areas simultaneously. The majority of the participants tended to give the following answer to question 2:

No, the dichotomy is not genuine. It is not genuine because high-level learners also need mathematical literacy. While everybody needs mathematical literacy, it is not needed that all people acquire high-level mathematics. But the scientific, technological and welfare development of the world needs a great many responsible mathematicians, who must be capable also in mathematical literacy.

Especially for adults, mathematics education must answer to their needs, expectations and intentions.

Question 3: *How can 'mathematics education for all' embrace opportunities for 'high-level mathematical activity'? How can 'mathematics for high level activity' embrace opportunities for 'mathematics education for all'?*

It might be a common belief that 'math education for all' should and could ensure the development of capabilities and high levels of performance for some learners. We share this belief, and in this sense, 'math education for all' can embrace opportunities for 'high-level mathematical activity' by teaching with challenging situations accommodations to different kinds of students.

The opposite direction is not so commonly demonstrated. In our view, however, 'mathematics for high-level activity' also ought to embrace opportunities for 'mathematics education for all' to ensure that high-achieving learners learn more than abstract de-contextualised math knowledge, also they should be given opportunities to acquire mathematical literacy by problem-posing and solving in authentic contexts.

This reflects on what mathematics education must be given, and which interesting ideas must be displayed? Which tools must be used? Which questions must be asked? How do we support an appropriate teacher's mathematics knowledge and their ability to create a meaningful learning environment in which each student would be given opportunity to realize her full potential? Also, particularly in the developing countries, what information and training must be provided for the teachers of different basic education levels?

Question 4: *How can instructional practices support the development of highly motivated mathematics learners as well as mathematics for all?*
Instructional practices can be supportive to all groups of learners by:

- Considering different learning styles and using a variety of instructional strategies and materials. Developing and nurturing mathematical critical and creative thinking is not possible solely with routine activities (say, arithmetical tasks and applying algorithms told how to be used by the teacher).
- Emphasizing a participatory role for learning. That means using mathematical language, oral discussion, and writing, listening and observing skills; creating mutual respect and equal treatment regardless of ability; expanding career and economic horizons; incorporating technology as a thinking and learning tool; and assessing performance through a variety of evaluation techniques.
- Valuing the learners' creativity and supporting discussions and reflections on different strategies and the use of different means.

Question 5: *What is mathematical literacy? Must mathematical literacy be the same for all? If not, does mathematical literacy depend on socio-cultural factors? Why?*
Mathematical literacy could be defined as "An individual's capacity to identify and understand the role that mathematics [practice and knowledge] plays [and could play] in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen" (Mathematical Literacy defined in PISA (Programme for International Student Assessment), www.pisa.oecd.org/pisa/math.htm, July 2004.)

This definition is valid for the human being as a citizen of the world, a world in an accelerated process of globalisation; and in this global village, mathematical literacy must be the same for all.

In actual practice, we are very far from this mathematical literacy as something which is the same for all. As the first step it might be appropriate for members of the mathematics education community to refer to a more local mathematical literacy that can be national or regional, according to the environment for which the person's mathematical capabilities are functional, that is, that permits him/her to respond to the needs of his/her current and future life as a constructive, responsible and reflective citizen in his/her country or region. Such necessities evidently vary from one community to another, and from one epoch to another, because, for example, the socio-economic and cultural reality of a European city requires that a person acts with knowledge and mathematical capabilities very different to those that an inhabitant of the Peruvian mountains needs to unfold with efficiency, efficacy and effectiveness in his own socio-cultural context; and the requirements of today's corresponding populations are different to those of fifty years ago. From this point of view mathematical literacy is relative; it depends on the demands of the persons' social, economic, and cultural reality in a given environment and time.

From the viewpoint of mathematics education being a means to enhance intercultural understanding, however, mathematical literacy in a broader sense could be realized by providing students from, say, European cities with knowledge of the mathematical culture of, say, Peruvian peers living in rural areas, and vice versa.

Conclusion

DG 3 seems to have succeeded in:

- giving room for an open and engaging exchange of different views
- formulating some answers and some recommendations.

Time did not allow the DG to focus on questions such as: Is there sometimes a tendency to say 'what not everyone can learn, nobody should learn'? Does every student need to take mathematics courses every year? What is the future of mathematics as an education subject in a changing world dominated by technology? Is more better, or …?

This report was written by Lena Lindenskov, (lenali@dpu.dk; lena.lindenskov@gmail.com) and Marta Villavicencio, (villavicencio.mr@pucp.edu.pe; villavicenciomr@yahoo.com) with the assistance of Sol(omon) Garfunkel, and Gerardus Polla. They are happy to be contacted for further information on the work of this DG.

ICME 10 2004

DG

Discussion Group 4

DG 4: Philosophy of mathematics education

Team Chairs: ***Susanne Prediger***, University of Dortmund, Germany
Maria Viggiani-Bicudo, State University of São Paulo, Brazil
Team Member: ***Paul Ernest***, University of Exeter, United Kingdom

Aims and focus

The aim of Discussion Group 4 was to explore the nature, role and state of Philosophy of Mathematics Education (PhoME) and particular themes focused on the perspective of PhoME.

The group met three times. The initial part of the first session was dedicated to an orientation with an introductory overview "What is philosophy of mathematics education?" (see Ernest, 2004, for a written version). The second session, "Strands and issues for discussion within PhoME", took place in smaller groups addressing different questions, followed by a synthesizing session. The prepared questions were: What are the conceptions of mathematics and mathematical knowledge underlying different learning theories? What roles do philosophies of mathematics play in the teaching and learning of mathematics? How do they relate to mathematics curriculum, teaching reforms and classroom practices?

Different perspectives – a first metaphoric approach

The group agreed on Paul Ernest's suggestion to consider the PhoME not as one single (perhaps dominant) position, but as an area of investigation (2004, p. 1). Beyond this most general description, there is a great variety of possible approaches. The discussion in the group was dominated by the experience that the question for the core of PhoME can be answered in many different ways, each of them interesting and with a totally different perspective.

In order to give a first intuition about the different possible perspectives, we will start the section by quoting the story told by Jean Paul Bendegem as his report from one subgroup. It illustrates the range of important questions in PhoME:

"Rather than presenting here a faithful reproduction of the discussion itself, I, as reporter, have taken an option to summarize our findings in the form of a story. The story runs like this.

As we know this conference started out from building 101. Now suppose that someone, let us call her the teacher, explained to us how to get to building 208. We, the pupils, are given a set of instructions to find 208. So we all wander out: some of us get their straight away, some get absolutely lost, some ended up in another building, some in wild nature and, actually, there was one person who found his way, although he did not hear the instructions.

Let us analyse this short story on the premises that we want to understand what is going on. What can you do? Well, you can start "bottom-up", of course. What one studies are the (specific) relations between teachers and pupils and one ends up with questions like:

- What are the philosophical ideas of the teacher?
- What are good road indications and how does his or her philosophy help to determine (if so) the instructions?
- How did the pupils understand the instructions? How does their cultural background interfere, or, if you like, what are the philosophical views of the pupils?

In a way what one does in answering these questions is to render the implicit explicit. This in turn raises a two-fold question:
- Are there several ways to make the implicit explicit, how does one justify a choice of methodology?
- At the same time, making the invisible visible can trouble one's view. In concrete terms: will it actually help teachers and/or pupils to know explicitly this implicit background? Will it necessarily constitute an improvement?

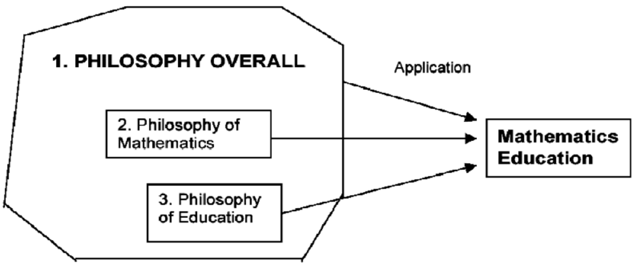
If one does not feel all too happy with this approach, a different route can be tried out: let us look at the problem how the instructions relate to the actual (I prefer not to use "real") situation. This perspective creates a different set of questions:
- Is there in fact a road from building 101 to 208? Or are there many roads and do we simply prefer (for whatever reasons) one particular road over all the other possibilities?
- Or, quite the opposite: as it turns out, there is no road. The instructions are in a sense an invitation to wander out and make or construct a road. Subquestions here would be what kind of roads we make in this way and, of course, how we do the constructing.

This metaphorical way of speaking refers to a large part of our discussion about the opposition between a structuralist view (sloganesque: "The language of road instructions is the language of set theory") and a more or less radical constructivism ("Roads are created collectively by people moving about in the open field".) However there is no need to stop here!
So far there has been no questioning of the fact why we all have to go to building 208 in the first place. And, of course, this leads us into a new set of "big" questions:
- Why should we all know where building 208 is?
- Who built the damn thing in the first place?
- What other buildings (if that is what they are supposed to be?) are possible, desirable, and accessible to few, many or all?

These questions invite us to philosophise about societal issues about mathematics education. And, for that matter, to be openly critical about it. And, finally, to wonder why all of a sudden we became so critical?
The story leads me quite easily to some observations in the guise of conclusions:
- At all levels and from all perspectives mentioned philosophy does enter into the picture. However the role philosophy has to play is quite different in each case. Perhaps part of the complexity of the problem of what a philosophy of mathematics education can or should be, resides in this fact.

- Is there a kind of "division of labour" imaginable? Can, e.g., questioning the existence of building 208 be done more or less independently from questioning the quality of the instructions? The view that "all is connected to all" is perhaps a philosophically pleasing view, but often one is left unable to act.
- And, finally, perhaps the most important one: who will listen to whom? Or better still: who is prepared and willing to listen and be listened to? To see the complexity of the issues involved, just ponder the following question: imagine that a philosopher somehow manages to show convincingly that building 208 does not exist, what is the poor teacher in building 101 supposed to do? If you know the answer to that, an important problem would have been solved." (Jean Paul van Bendegen at ICME-10)

Picking up the story systematically, we can see that the different questions to be posed are influenced each by a different understanding of the term 'philosophy' in 'philosophy of mathematics education'. It is obvious in theory but a challenge for communication in practice that the notion of philosophy and its relation to practice is understood quite differently by the different participants of the discussion group. Each understanding is one the one side influenced by the participants' culture and tradition in each country, and on the other side by the question on what exactly the philosophical focus is. This last question has been raised by Stephen Brown (1995) by posing a trichotomy.



Is the philosophical focus or dimension: Philosophy applied to or of mathematics education? Philosophy of mathematics applied to mathematics education or to education in general? Philosophy of education applied to mathematics education? The figure illustrates these alternatives diagrammatically in a simplified way. Each of these three possible 'applications' of philosophy to mathematics education represents a different focus, and might very well foreground different issues and problems. Far from trying to give a survey about all possibilities, we specify some issues that were dominant in our discussions.

Philosophy of mathematics and its impacts on mathematics education

René Thom's statement that "all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." (Thom, 1973, p. 204) is the classical starting point for studying impacts that views of mathematics can have on mathematics teaching. Steiner (1987) has also emphasized the other direction: every philosophy of mathematics includes implicit implications on instructional practices. Various empirical studies have provided evidence for both directions, even though the connections are not uni-causal dependencies (e.g. Thompson, 1984).

This has raised the normative question for the desired instructional practices and desirable views on mathematics (e.g., Ernest, 1994). In the course of these discussions, a simplistic opposition appeared between "the absolutist view" on mathematics and "the fallibilist" one, and these were also too directly connected with "transmission practices" versus "constructivist practices" in classrooms. Meanwhile, important contributions have been made to elaborate such (too) over-simplistic pictures into well-founded and multi-faceted accounts for the nature of mathematics. However, this can still be called a major task for PhoME (to which e.g. Meneghetti, 2004, contributed in the DG).

Envisaging the importance of personal philosophies for classroom practices, many authors in PhoME have concluded that changing instructional practices in mathematics classrooms can not only be a matter of new curricula or of providing materials, but also a matter of challenging traditional personal philosophies of teachers. That is why philosophical reflections become more and more part of teacher education programs (Lindgren, 2004, has contributed one example to the DG).

Beyond these activities is the conviction that if we acknowledge the impact of even implicit philosophies, the most important strategy is to make the underlying assumptions explicit. This idea of *making explicit the implicit* has become a leading idea for the whole discussion group.

The idea of making explicit the implicit philosophies is not restricted to teacher education or professional discussions of reform curricula. Instead, it should also be transported into the classrooms itself. Perhaps only if students can reflect on central questions in philosophy of mathematics themselves, they will develop a well-balanced and reflective knowledge about mathematics. However, empirical studies (like François & van Bendegem, 2004) show that there is still a big gap between this claim and classroom practices and even written curricula.

Philosophy as reflecting discipline with respect to mathematics education

Brown (1995) has claimed not to limit the discussion in PhoME to the philosophy *of mathematics*. If we understand philosophy as the reflecting discipline with respect to all aspects of mathematics education, the field becomes much larger, addressing all issues like "how does mathematics relate to society?", "What is learning (mathematics)?", "What is teaching (mathematics)?", and also "What is the status of mathematics education as knowledge field?" (as Ernest, 2004, suggested in much more detail). In all these areas we can apply the idea of making explicit the implicit and can hence do philosophy as a mode of making critical analyses and rigorous interpretation of the questions presented in learning processes.

There was a controversial discussion on the question whether this extensive way of understanding PhoME produces the problem that *all* mathematics education becomes subject of PhoME, hence whether PhoME tends to be reduced to a reflective basic attitude.

Starting from philosophy of education: Impacts on mathematics

A much more focused approach is to reflect on mathematics (education) against the background of a well-founded position in philosophy of education, especially on aims and rationales of general education and mathematics' contribution to it. One important example for this approach present in the DG was the "Philosophy of Critical Mathematics Education" (Skovsmose, 1994). In this approach, the aim of mathematics education is specified by the ability to critique the uses of mathematics and its "formatting power". For that, students need to engage in mathematics-based projects which focus on its social applications.

Following this pathway to its logical end, this approach to PhoME formulates claims for mathematics itself: If mathematics education aims at reflecting critically on mathematics, the discipline mathematics is responsible for providing mathematics in a way that it can be critiqued by laypersons. For that, it must be presented embedded in its aims and purposes, meanings and senses. This is the core idea of the philosophical program called General Mathematics (cf. Lengnink, Prediger & Siebel, 2001). If this is taken seriously, PhoME can have important impacts on the discipline of mathematics, and not only on the philosophy of mathematics.

Concluding remark: Concurrent or complementary?

In the end, which is the most important perspective? It is an easy (relativist?) first step to emphasize that all perspectives have their important aspects and since they are complementary, they should all be elaborated in future research. On the other hand, we cannot deny that they are clearly concurrent due to restricted time and resources in a research community. That is why, on the one hand, we will have to continue learning from each other and follow the different perspectives, and on the other hand, we cannot stop discussing on priorities of questions to be raised in the community.

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DG 5: International cooperation in mathematics education: Promises and challenges

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Introduction

The aim of Discussion Group 5 was to discuss some relevant issues concerning internationall cooperation in mathematics education. The Discussion Paper posted on the Conference website identified the following aims for the group.

- Sharing experiences and learnings by mathematics educators from around the world arising from their international contacts;
- Identifying benefits and problems arising from such contacts; and
- Developing recommendations for research and action towards making such contacts more socially just and more effective for achieving the interests of all participants.

The Discussion Paper raised some initial issues and questions, rather than take a stand on them, towards initiating the discussion and dialogue between mathematics educators. Six main questions were identified for consideration by the group.

1. *What are the goals for international collaborations?*
2. *What are the barriers to genuine and equitable international cooperation?*
3. *Should cooperation be regional or global?*
4. *What forms could such cooperation take, and how to organize them?*
5. *How can a cooperative preparation of researchers in mathematics education contribute to the development of a genuine and equitable cooperation?*
6. *Can international cooperation lead to excessive homogenization?*

The discussion sessions during the congress combined questions 3 and 4, and were planned to allow each participant the opportunity to discuss in small groups at least three of the 5 resulting questions. In order to give justice to the rich discussion at the conference, only three of the main questions will be reported upon here.

Two comments might be relevant about the constituency of the group. First, and perhaps unexpectedly, the majority of participants were teachers, teachers' educators, school administrators not particularly engaged in research, but interested in international cooperation concerning the teaching of mathematics. While the interests of both groups are not identical, and while the Discussion Paper aimed at discussion of the issues from researchers' cooperation perspective, the discussion had demonstrated that there were sufficient common issues that spanned both areas. Such opportunities may not have been provided to in past ICME conferences. In particular, there was general consensus about the fact that international cooperation in research can produce tools and perspectives that are useful to orient international cooperation concerning the teaching of mathematics. Some questions put in the Discussion Paper such as those concerning diversity and the necessity to avoid any kind of domination or homogenization in the research field, were recognized to have immediate parallel concerns with delicate issues concerning how to avoid that some orientations in the teaching of mathematics become globally dominant. Indeed it was observed that a dominant position in the research field can be used to support with "scientific" arguments the diffusion of teaching projects and teaching methodologies. Further, the participants shared the opinion that international cooperation in research can provide curriculum developers and teachers with tools and perspectives suitable to deal with that diversity in the local curriculum as well as classroom practices.

Second, the lack of attendance by many mathematics education researchers might be due to the overlapping of issues discussed here with other parallel groups running at the same time. In our opinion, this fact suggests an interesting question for the planning of the next congress: is it better to keep a plurality of groups that work in parallel on issues related to the organization and orientation of research in our field – or it would be better to select for each congress only one or two issues concerning policy of research, in order to avoid dispersion of people interested in that kind of topics?

Why international cooperation in mathematics education?

The Discussion Paper identified two, arguably conflicting, reasons for international cooperations. Firstly, mathematics education today is undoubtedly affected by the globalization trends of our new times. Increasingly, public funding to universities is based partially on the amount of money they attract externally and on the number of publications they produce. In many countries, international competitive publications are given higher value than local publications. For many universities around the world international projects, in forms of attracting international students, conducting international development projects, and international publication, are seen as highly lucrative revenue. Further, World Bank lending schemes have imposed similar changes on many developing countries. Hence, undoubtedly there are economic benefits to educators engaging in international cooperation.

Secondly, a striking feature of this increasingly globalized world is its inequality. Numerous reports from international organizations have pointed out that the gap between the "haves and have nots" has increased within many countries and between countries. The cost of such inequality for social, political and peace conditions around the world cannot be neglected. Arguably, such inequality in access to resources and funds is paralleled by the dominance by some countries of the agendas and voices in international cooperation in mathematics education. Traditionally, mathematics education has been isolated from discussion of its contribution to this inequality either as a vehicle to legitimate it, if not increase it, or as a potential contributor to its reduction. Several authors have challenged the prevailing image of mathematics as a neutral/apolitical body of knowledge that is isolated from social and cultural considerations. The Discussion Paper noted the curtailing of funds from international agencies towards developing countries making it more difficult to look for governments for improved international cooperation in mathematics education. The late Miguel de Guzmán, a past President of ICMI, called for an increasing role of cooperation between professional mathematics educators and their associations to work to improve mathematics education worldwide. Hence there are social justice dimensions behind international cooperation.

The discussion at the congress identified other reasons for cooperation between academics and teachers from different counties or cultures. Cooperation in research and teaching provides opportunities to foster mutual understanding towards resolving conflicts arising from power relationships and inequality. Some participants made the distinction between collaboration (in the sense that one is working for another) and genuine cooperation (in which both parties are working towards common aim). Cooperation leads to greater understanding of social and historic reasons behind the different experiences and practices of the different systems. It also allows for self-reflection on the otherwise hidden assumptions and values behind each participant's practices. Similarly, it exposes similarities as well as differences between the different participants. Finally, it enhances the preparation of the students, teachers and researchers for better success in a globalized world.

What are the barriers to international cooperation?

The Discussion Paper identified quite a few barriers to international cooperations. Among these were:

- *Financial*: The cost of attending international gatherings, or subscribing to international journals as a prohibiting for participation of some countries.
- *Language*: Educators from non-English speaking countries often feel excluded from some international activities that use English as language of communication.
- *Cultural norms*: Cross cultural cooperations often experience conflict due to lack of knowledge about appropriate manners of behavior and speech that extends beyond mere language.
- *Lowest Common Denominator*: Cooperation between cultures that are very different may lead the collaborators to compromise to levels that are lower than their individual interests and needs.
- *Conflicting agendas*: International projects may be regarded as a source of income for some countries (e.g. paid consultancies or international students) while other less affluent countries may need them as aid projects.
- *Voice*: Collaboration between educators with varying backgrounds, interests and resources may lead to domination of the voice of the more able and marginalization of the less powerful.

The discussion at the conference considered these and other barriers to cooperation. Considerable discussion was focused on the terms *equitable* and *genuine* in relation to international cooperations. Many participants warned against the naïve position towards the meaning of international cooperation that pretends that cooperation necessarily implies they are carried out among equals. Often, international cooperations are established among unequal participants with some participants positioned in a dominant role due to access to resources such as funds, technology or expertise in dominant modes of operation in research and/or teaching in mathematics education. In these contexts, *equality* in cooperation is built on a respect for the different *type* and not *quantity* of contributions of the partners, on the acknowledgement of the equal value given to the different knowledges the participants, and on the necessity to tackle problems of relevance for each of the parties involved. Moreover, genuine collaboration is one that is based on self-critical reflection by the different partners about their self interests and expected contribution to the cooperative activity, and on the transparency among participants in relation to their expectations, contributions, benefits and voice in representation of the results.

Similarly, further discussion arose about the role of language as a limiting factor for genuine collaboration. In addition to the dominance of English in many international cooperative activities, the problem of language is also a matter of particular professional jargon used in different national communities to refer to the objects of their practices. Problems of understanding emerge due to differences in the meanings of commonly used terms. For example, the phrase "didactics of mathematics" carries almost opposite meanings for a native English speaker and speakers of other European languages. Further, care must be given not to exclude some participants from having access to that technical language by oversimplifying it. Hence, genuine cooperation must include a process of communication in which, through languages (natural and specialized), the parties involved negotiate their meanings and intentions for action

Lastly, a discussion arose related to the attitude of the parties involved in the cooperation. There was an agreement on the fact that "missionary" attitudes, that aim to uncritically transport knowledge and learning from one place to another, on the part of some partners do not help the establishment of genuine cooperation. Those attitudes lead to a patronizing relationship, which does not respect and value the diversity of the parties involved. Instead, an attitude of humility and openness to learn from each others should be the basis of international cooperations.

Does international cooperation lead to homogenization?

The Discussion Paper stated that many authors have noted the similarities in mathematics education curricula and research in mathematics education around the world and raised the question whether international cooperation is leading to homogenization and standardization of the discipline. To aid the discussion on this question the Discussion Paper made the distinction between the two related terms "internationalization" and "globalization". The term "internationalization" refers to any activity that has participants from more than one nation. They can be either official at state-to-state level, or less formal interaction at a professional or even personal level; they may involve two or more countries; and they may be at a regional level or a more extensive international level. The term "globalization" refers to the receding boundaries between countries and the awareness that such boundaries are receding. In other words, it is the increasing awareness of the "world as one" or the realization of the "global village". In the minds of many, globalization is associated with transnational companies, multinational organizations, removal of barriers in trade and investment, and new forms of colonization of culture. However, the Paper made the distinguish between "globalization from above" and "globalization from below", where the latter are activities motivated by concerns about environment, human rights, diversity of culture, and seeking an end to poverty, oppression and violence.

The Discussion Paper went on to argue that while globalization "from above" might lead to homogenization, globalization "from below" is associated with diversification and differentiation rather than homogeneity and universality. Perhaps this is best illustrated by the example of the ethnomathematics movement in recent years. The Discussion Paper argues that even though ethnomathematics is a globalized movement in mathematics education, it rejects the universalization of mathematics and mathematics education and stresses local knowledge and difference.

The discussion at the congress pointed out several examples where international cooperation leads to contradictory patterns in homogenization and diversity. For example, some have pointed out how in the new world order, reforms in one country are transplanted, in many cases uncritically, to other countries. Some talked about the "Americanization" or the world curricula. However, many also argued that cooperation between different countries can lead to awareness of different approaches to both research and teaching methods that might increase variety at local level. Researchers in the discussion pointed out that the mathematics education literature in many countries reflects a greater variety in methodologies and theoretical stances now than fifty years ago. Hence, at the same time that trends in research and teaching are becoming homogenized at a global level, they are becoming increasingly diversified at a local level.

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ICME 10 2004

DG

Discussion Group 6

DG 6: The education of mathematics teachers

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Introduction

This discussion group was commissioned to consider issues such as:

- What mathematics should teachers in training and teachers already in service study?
- To what extent should mathematics teacher education be focused on pedagogical skills or didactical knowledge or mathematical considerations?
- What roles do mathematics teachers play and what roles should they play?
- What should teachers know about student learning?
- What practices seem to hold the best hope of reforming the teaching of mathematics and how are those practices best presented in teacher education programs?

A set of indicative questions was made available on the website under each of these themes. This was not to define the areas of discussion – that was to be done by the participants in their groups – but to attract papers from participants to support the work of the groups. The group was timetabled to meet three times, the sessions were either plenary or group sessions.

Since a discussion group is designed to discuss among a large variety of participants, we felt it important in this contribution to stress the questions that have arisen in our sessions, even if these questions have not received definitive answers. In fact, it would have been preposterous to think that a 6 hour session would have led to articulated answers to very important questions. Participants did find the questions in themselves stimulating: Sometimes it emphasised the fact that the question was not a personal or local one, but one of worldwide interest, and at other times the questions that seemed crucial for some participant was completely new for another.

The initial plenary element was used to introduce the team and to introduce the contributory papers which had been placed on the website after being approved by the organising team. Although there was no scope for participants to present their papers during the discussion group their contribution was acknowledged when they were introduced individually to the, whole group and an abstract of their paper was displayed for information. In all seventeen papers (from twelve different countries) had been posted on the ICME website (www.icme10.dk).

These papers generated four main areas for discussion in the group sessions which followed the plenary meeting.

A. Specific ‘problem’ areas for which time may or may not be prioritised (e.g. geometry, history) or topics in which trainee teachers often had problems (e.g. fractions and proportionality).
B. Maintaining the complexity of the teaching task whilst initiating beginners.
C. The nature and depth of trainee teacher’s knowledge and attitudes.
D. Effective practices and structures which help to underwrite them.

The second element of the introductory plenary was an outline of the training systems from France; Taiwan; Finland; Serbia and Montenegro; Brazil. These were intended to raise issues for discussions in the groups, each containing participants from a wide range of countries and each led by a team member. During this first session these groups continued to explore the outlines of training systems with participants giving brief descriptions of their training systems leading them to discuss the variations and to present the three most significant issues relating to their system.

The training institutions for primary or secondary school teachers varied for different countries. In some countries, such as United States, prospective teachers are trained by universities, whilst in others such as Malaysia, Denmark and Taiwan they are trained in separate pedagogic institutions such as teachers colleges In most countries the training lasts for a total of about four years after the conclusion of secondary schooling so that newly qualified teachers are usually at least 22 or 23 years of age, though in Iran, for example, the training is for two years and a new teacher may be just 20 years of age.

In primary school teacher programs, there are two different systems. One is where pre-service teachers need to take 3-4 year courses in an Education Faculty. The other is where the pre-service teachers study for their first degree in any subject matter and take one more extra year following a course in pedagogy institute. In secondary school teacher preparation programs, there is an assumption that pre-service secondary school teachers are required to have more mathematics content knowledge than those of primary school. There were again two training systems in the participating countries, the pre-service secondary school teachers in some of the countries need to accomplish a mathematics degree in university followed by a teacher training program, while those in some other countries mathematics and education are combined in their degree studies.

After the course preparation, there is some similarity in the teaching experiences required in most countries. Trainee teachers need to have at least half year equivalent of teaching practice before being certified as a qualified teacher. During the practice teaching, the trainee teachers are usually mentored by both school teachers and a supervisor from the training institution but there is often only limited interaction between mentors and supervisors. Sweden had developed a particular initiative in developing mentorship amongst teachers. Assessment practices varied, and whilst all have end of course assessment the French system requires trainee teachers to pass a pedagogic examination before acceptance onto a training course (requiring some pre-course study); whilst in England trainees have to pass centrally managed computer based tests of skills in literacy, numeracy and information technology.

General discussion about the recruitment and education of teachers

Most countries do not have great problems in recruiting primary teacher-trainees, but nearly all have some difficulty in recruiting secondary mathematics teachers, one exception being Korea which recruits secondary mathematics trainees well with only 10% of those completing the training course actually obtaining posts.

Difficulties with students which were identified by participants included poor knowledge of mathematical and pedagogical content knowledge together with poor mathematical vocabulary and poor communication skills. Trainee teachers frequently had low confidence and a negative attitude toward mathematics. Problems in the teacher education system included too little time to address mathematical knowledge, especially for primary teachers, and there was a perceived problem in the USA of primary teachers avoiding being seen as specialist teachers of mathematics.

In one group there was considerable discussion about how the isolation that teachers so often face can be overcome and how teacher educators can help teachers develop a more thorough knowledge of the mathematics they teach and of various ways of teaching that mathematics. A number of suggestions were offered, the most prominent of which emphasized the need for teachers to work in concert with other teachers in which problems are discussed in a professional atmosphere, forming a 'community of practice'. Individuals discussed how they worked with the entire mathematics staff within a given school; others talked about the formation of networks that provided a structure for teachers from different schools to communicate with one another. The role of technology (primarily email and the Internet) was explored as a vehicle to promote this kind of communication.

Another form of isolation discussed was that between theory and practice. How could tertiary mathematics programs become more relevant for those planning on becoming mathematics teachers. There was a perceived gap between university level mathematics and the student teachers' understanding of school mathematics. How could teacher educators enable student teachers to see the more empirical side of mathematics given that mathematical formalism dominated most of their mathematical training? The view was expressed that mathematical formalism was not helpful to generating effective teaching strategies for enabling students to learn mathematics. Student teachers needed to study less mathematics but more thoroughly, 'advanced mathematics from an elementary standpoint'; we needed to create more and better connections between tertiary mathematics and school mathematics.

New questions raised by the discussion

A. Specific problem area (e.g. the place and nature of geometry or the history of maths) or specific problem topics (fractions and proportionality).
   - How far does/should the training curriculum (or the inservice programme) reflect the current state of the school curriculum?
   - How do we 'fill in the gaps' in the trainee teacher's mathematics experiences in the time which is available?
   - Problem areas have been problem areas throughout their learning – what can we change to help them improve?
   - How do we help teachers/trainee teachers to identify weaknesses/areas for development without challenging their confidence and self image?
   - If practicing teachers themselves have problems with some particular concepts (e.g. ratio and proportion) how do we enable them to support trainee teachers to teach these concepts?

It was felt important that the training curriculum must go beyond the student curriculum to help teachers to model, represent in multiple ways and prove at different levels of justification. It was clearly important to achieve an appropriate balance between pedagogy and the subject (i.e. mathematics) curriculum. Much depended on the quality of mentoring by experienced teachers supporting the trainee teachers.

B. Teaching mathematics is complex – how do we support trainee teachers to recognise this and teach in this complex manner.
   - Images which capture the complexity.

- 'Simplifications' which still contain the complexity.
- Methods of analysis which retain the complexity.
- Maintaining the 'connectedness' of trainee teacher knowledge by providing rich networks of knowing.

This topic provoked much discussion as participants set out trying to describe the complexity. Complexity can be found both in pedagogy (for example dealing with unconventional or unexpected answers from pupils) and in the subject (connectedness, content, misconceptions). This complexity may also vary for primary and secondary teachers.

It is important that students recognise this complexity. Teacher educators must however be careful not to over-complicate (thus perhaps causing anxiety for elementary teachers) or to over-simplify (secondary teacher students need to know that their "brilliant explanations" may not be good enough to get pupils learn). It is important that students understand the depth of "simple" mathematics – echoing Klein's call for an understanding of elementary mathematics from an advanced standpoint.

Participants also pondered how we can prepare teachers to teach successfully in such a complex environment. It is important, that in pre-service training we should seek to reduce mathematical fear, to model best practice, to provide a variety of tools and to enable the possibility to gain real/realistic experiences (e.g. video case studies, real classroom activities, "What happens if …?" situations). It is also important that teachers continue pondering these topics through in-service training.

C. 'Borderlines' of acceptability and the nature of teacher/trainees knowledge.
   - Attitudes towards knowledge and the teaching act.
   - The nature of the trainee teacher's mathematical knowledge – the struggle between technical efficiency and relational understanding in learning mathematics.
   - Assumptions about learning and the learning act – trainee teachers assumptions about pupils.
   - Mathematics as distinctive or representative in the school curriculum.

   This was addressed by discussing four questions.
   - What mathematics do all students/teachers need?
   - Should we start the training by assuming an empty slate?
   - Should our students be able to explain underpinnings?
   - Are we intending to educate good apprentices or mediocre masters?

D. Effective practices and structures in teacher training courses.
   - Ways of working with trainee teachers.
   - The role of in-school experiences and the contribution of teacher-mentors.
   - Assessment and analysis of classroom situations which help students to access the complex classroom.

In responding to these questions it was important to recognise the need for trainee teachers to be faced with realistic situations which model good mathematics teaching. To help trainee teachers develop criteria to help analyse good teaching we must be able to define effective practice and provide opportunities to observe good teaching. They must feel part of a community of learners of teaching through participation in an effective "lesson study" model with immediate feedback, assessment and a chance for reflection, including the development of skills in analysis – What could I have done? What will I do next?

Feedback is of critical importance, including:

- Self-feedback (observe video).
- Mentor teacher feedback.
- Teacher educator feedback.
- Design/utilize a feedback system from schools to pedagogical institutions.

It is clearly important that universities and schools should work more collaboratively. Supervisors can help or impede trainee teacher's experience in a particular context and it is valuable to enable school teachers contribute to the pre-service program and for university staff to work in schools. Some schools hire 'coaches' – specialist support personnel - to work in this context betwixt and between theory and practice. The issue of providing quality mentoring was raised through asking the following question: 'How can a university mathematics teacher with no school experience have knowledge of the pedagogy of primary or secondary mathematics?' One answer to this rests upon the existence of a body of knowledge provided by research in mathematics education. Where this knowledge exists, a university educator can make him/herself aware of this knowledge and of its significance. What (s)he can then give to the trainee teachers is not a personal knowledge but – as is often the case for other professions – a useful cultural knowledge. Furthermore, this question can be reversed to ask 'what kind of knowledge can an experienced primary or secondary teacher transmit to trainee teachers?' To reflect upon one's own practice is a very difficult exercise, and to know how to transmit the major element of a successful practice is even more difficult. Sometimes the experienced teacher can only present his/her own personal solution to a trainee teacher's search for how best to teach a particular piece of mathematical knowledge. But how then can this particular professional solution be compared to another solution? To what extent is this solution really related to an outmoded traditional curricula or ways of teaching embedded in the experienced teacher's history? The group agreed on the importance of having a variety of educators interacting with trainee teachers to avoid the sense that there is one particular approach or solution to a problem.

It must also be recognised that assessment influences classroom practices and assessment practices should be ones which support and are sympathetic to developing the analytical skills and self awareness of trainee teachers.

E. To what extent is mathematics teacher education about enculturation or about reform?
   - What is the priority for mathematics teacher education?

Some raised the question whether enculturation and reform are really polar opposites. The point was made that teacher education is about facilitating student teachers' entrance into the profession of mathematics teaching but it is also about educating student teachers on the use of technology, problem-solving strategies, open-ended assessment techniques, and various instructional methods other than the traditional lecture method. Much of this discussion focused on how teacher educators could help beginning teachers incorporate these methods into their teaching in a successful way. The creation of communities of professional teachers was one of the primary means for enabling young teachers to be mentored as they entered the profession and to provide a support system for "reforming" the teaching of mathematics.

F. Inducting trainee teachers into the profession.
- How can we promote "communities of professional teachers"?
- How can we help teachers develop a better understanding of school mathematics?
- More generally, How do we prepare trainee teachers to work with colleagues?

Again a theme emerged regarding the need to provide a context for beginning teachers to feel a part of the teaching profession. The Japanese approach that places considerable emphasis on mentoring beginning teachers was mentioned as one example of what was needed. Part of the discussion dealt with the need for beginning teachers to appreciate the complexity of teaching and dealing with students' thinking about mathematics, yet enable them to develop confidence in their teaching ability. It was agreed that the beginning teacher is a rather fragile entity that needs nurturing and support.

Appendix: The papers
- Primary school teachers' perceptions about their needs concerning mathematics teacher education. *Solange Amorim Amato – Brazil*
- Training elementary teachers of mathematics: What are the essential components? *Mark Arvidson – USA*
- History of mathematics and didactics: Reflections on teachers education. *Giorgio T. Bagni – Italy*
- Synthetic Euclidean geometry as a didactic basis for primary and secondary school geometry. *Marita Barabash, – Israel*
- Transposition of didactical knowledge: The case of mathematics teachers' education. *Isabelle Bloch – France*
- What is the role of the university in influencing the behaviour of trainee teachers in the classroom? Theory and practice in teacher education. *Paul Dickinson et al – UK*
- Reflections on mathematics teacher education. *M. García and V. Sánchez – Spain*
- Constraints, coins and combinations: Working with teachers in South Africa. *Faaiz Gierdien – South Africa*
- Integration of didactical knowledge and mathematical content knowledge in pre-service teacher training. *Pedro Gómez* and *Luis Rico – Spain*
- An exploration into the mathematics subject knowledge of primary trainees. *Tony Harrie* and *Ruth Barrington – UK*
- On the mathematical and didactical content knowledge of prospective teachers: the case of the division of fractions and proportional reasoning. *Tapio Keranto – Finland*
- Didactical Analysis – A Plan For Consideration. *Milosav Marjanović- Serbia and Montenegro*
- Teaching mathematics student teachers in challenging contexts. *C.E.F. Monteiro* and *M.M.F.Pinto – Brazil*
- What mathematical and educational competencies should be developed on elementary prospective teachers? *Cecília Monteiro* and *Lurdes Serrazina – Portugal*
- Toward a Model for Teacher Professional Development in China. *Huang Rongjin* and *Bao Jiansheng – P.R.China*
- A study of middle school teachers' understanding and use of mathematical representation in relationship to teachers' zone of proximal development. *Zhonge Wu* and *Gerald Kulm – USA*
- Teacher Education in Iran. *A. Shahvaraniran – Iran*

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DG 7: Public understanding of mathematics and mathematics education

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Team Members: *Andy Begg*, University of Auckland, New Zealand
Jean-Michel Kantor, University of Paris 7, France
Torgeir Onstad, University of Oslo, Norway

Aims and focus

The aim of this discussion group was to provide a platform for the participants to discuss issues and problems relating to the public understanding of mathematics and mathematics education. The discussion was broad and covered five themes:

a) the meaning of public understanding of mathematics and mathematics education;
b) issues and problems associated with the prevalent public understanding of mathematics in culture and society;
c) public perceptions of the nature of mathematical literacy and its relation to learning of mathematics;
d) the roles of mathematics education community in promoting public understanding of mathematics and
e) strategies of popularizing mathematics.

The discussion group commenced with an introduction by Chris Budd. During the three days four papers were presented to introduce the work of the DG. Moreover, two demonstrations on strategies for popularizing mathematics were given. The papers were:

- Rethinking the image of mathematics by Andy Begg (New Zealand)
- Use of mathematics for national education and development by S.E. Anku (Ghana)
- Improving perceptions of mathematics education through political action by Jonny W. Lott (USA)
- Mathematics as social construct: Two examples by Adriana C. M. Marafon, Chateaubriand N. Amancio and Denizalde J. R. Pereira (Brazil).

Also Chris Budd and Steve Humble from UK demonstrated some fun mathematics activities such as mathematics magical tricks, games and puzzles with all the participants. Most participants of the discussion group were amazed and interested in trying out the fun mathematics activities demonstrated. Perhaps these are ideas and strategies that could be disseminated to more teachers and be tried out in school classrooms as well as with the public. [For more details, please see Humble, 1994, 2001, 2002 and Chris Budd's home page www.bath.ac.uk/~mascjb]

For every session, the participants were divided into subgroups to discuss the individual themes. Some of the major points that emerged from the discussion are:

(i) Mathematics does have an image problem and we need to work both with the public but also to improve teaching so that the perception of mathematics of the next generation will be different.

(ii) There is a need to consider a variety of approaches to improve public images of mathematics and public understanding of mathematics, and to ensure that mathematics emphasises creativity as well as applications.

(iii) Some issues and problems related to the negative public images of mathematics raised were:

a. Mathematics is hidden in most human endeavour, thus efforts are needed to make mathematics visible and the public aware about the relevance and the applications of mathematics in daily life as well as in the workplace.

b. Most policy makers and politicians of different countries tend to be non-mathematics graduates. Although they usually recognise that mathematics must play a central role in school, they often have a superficial understanding of the subject and tend not to be able to promote appropriate mathematics learning in schools. For example, there are politicians who believe that mathematics learning needs little language competence because it is made up of mainly numbers and symbols.

c. There is generally lack of coverage and promotion by the mass media about mathematicians and mathematical knowledge, hence the need to have more TV programmes, publications and coverage of mathematical discovery, mathematicians' life stories etc to promote the awareness and interest of the public about mathematics.

d. The need to change the tradition of procedural learning of mathematics in schools to more problem solving and project based mathematics learning.

e. The need to create parents' awareness, upgrade their knowledge and interest in mathematics, because parents play important roles in cultivating or developing a positive image of mathematics among the children (who are our future public).

(iv) Various strategies for popularizing mathematics were suggested. These include master classes, school talks, popular articles, mathematics contests such as Mathematics Olympiads, mathematics and science fairs at popular public places such as supermarkets and developing websites such as NRICH and PLUS. Both NRICH and PLUS are parts of the family of activities website in the Millennium Mathematics Project. NRICH or Enriching Mathematics www.nrich.maths.org/public/index.php provides mathematics problems, games and articles while PLUS http://plus.maths.org/ is an internet magazine published four times a year which aims to introduce readers to the beauty and practical application of mathematics.

(v) There is an urgent need to find ways that will intrigue the mass media and the politicians to raise the public awareness as well as understanding of mathematics. Some of the ways suggested are

a. encourage more science and mathematics graduates become politicians and policy makers;

b. have more science and technology related company to sponsor TV programme that promote or popularizing mathematics such as those discussed in (iv). Nevertheless, every one acknowledged that it is going to be a challenging task trying to intrigue the mass media and the politicians if mathematics is hidden and mathematics people remain silent about mathematics.

At the end of the last session, all participants were asked if we should continue this discussion group in the next ICME-11. All participants unanimously agreed that the issues discussed by this discussion group, that is the public understanding of mathematics and mathematics education remains an important issue and needs to be pursued further.

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DG 8: Quality and relevance in mathematics education research

Team Chairs: *Margaret Brown*, King's College London, United Kingdom
Rosetta Zan, University of Pisa, Italy
Team Members: *Cyril Julie*, University of Western Cape, Bellville, South Africa
Frank K. Lester, Indiana University, Bloomington, USA
Perla Nesher, University of Haifa, Israel

Aims and focus

Our aim was to share knowledge and experiences in order to deepen our own understanding of how the quality and relevance of mathematics education research is, and could be, judged. This would help us, individually and collectively, to improve the quality of our research and make it more relevant to practice. We thus based our discussions on our experiences of doing or reading mathematics education research.

The form of discussion group activity

Some of us felt that one of the weaknesses of previous ICMEs has been the lack of opportunity for extended discussion with small groups of people involved in similar fields, openly sharing information, views and problems between different countries and institutions in a relaxed way. We therefore welcomed the decision to incorporate discussion groups into the programme at ICME10 and decided that almost all the time available to the DG 8 should be spent in small group discussion. No plenary presentations were made, other than a brief introduction to the themes and discussion questions by one of the chairs. In our plan we suggested that the short third session should be spent reviewing some of the issues raised in the first two sessions. However in the event members of the group were keen to continue discussing the questions given in the previous sessions, together with some related issues raised in the plenary lectures, in the same format. Thus all three sessions were mainly devoted to informal discussion, with 6-7 people sitting round each table.

Perhaps because of the unusual organisation, and the overlap of the theme with other DGs, the attendance was small (about 25 at each session, with about 17 different countries represented), but of very high quality! Participants were given the choice of sitting in the same small group or changing group between sessions; most decided to stay with broadly the same group, although there was some change of membership as a number of people joined us from other DGs at the start of session 2 and about an equal number left us to sample others. There was little change in membership however between sessions 2 and 3.

Four papers (see details below) were made available beforehand on the website to inform the discussion, although inevitably it was not possible for all members to have read these.

A list of questions around which to structure the discussion was provided for each session (see below). The members of the organising team who were present distributed themselves among the groups and helped inform and structure the discussion. (Frank Lester was not able to travel to Copenhagen due to illness but as with some other members of the organising team had given valuable advice about the questions for discussion and the selection of papers, and indeed had on request from others contributed two of his own.)

For the first session each small group was asked to provide a brief written summary of key points which emerged, but the participants decided that since the focus was experiential and on oral discussion, there was no real need for written reports or for final feedback to the whole group. We are therefore not in a position to summarise the answers to any of the questions, nor even the main issues which arose in each group. This explains why the report of this group may be shorter and apparently more incomplete than those of other groups!

However the core of the participants attended loyally and seemed to feel that they had really enjoyed and benefited from the opportunity for loosely structured small group discussion sessions, and from the sharing of views and experience across different countries. Small group discussion may be a minority interest but it would seem nevertheless useful to retain it as an option at future ICMEs.

The content of the discussions

Session 1

This focused on identifying and comparing possible criteria of quality for research in mathematics education, and on identifying and discussing possible indicators for such criteria. This included matters of equity and power in the community. The questions provided for discussion were:

1. What pieces of educational research can we agree to be of outstanding quality? Which criteria do we each use for distinguishing outstanding pieces of research from more average research?
   – *Which views about the nature, purposes and methods of research in mathematics education are behind these criteria?*
2. How do we recognise weak research?
3. In the lists given in the literature, (examples of these drawn from the references on the website listed below were made available) are some criteria more important than others? If so, which ones and why?
   – *How may different views about the nature, purposes and methods of research in mathematics education lead to different criteria, different interpretations or different weightings for the same piece of research?*
   – *What are* indicators *for the criteria that we have selected as most important?*
   – *Is it possible to define a 'structure' in the set of criteria? For example: Which criteria refer to the entire research process? Which refer to some particular phases (choice of the problem, of the methods, conclusions, communication)? Which refer to the passage from one phase of the research to another (for example from the choice of the research problem to the choice of the methods)?*
4. What, if anything, makes these criteria especially relevant for mathematics education?
   Does mathematics education need a shared set of criteria of quality for research?
5. Is it reasonable to expect one set of criteria to be suitable for every type of research?
   What about different 'types' of research, such as qualitative and quantitative methodology; normative and interpretative paradigms, …?

6. Are the criteria used by those with power (e.g. editors and reviewers for journals, books, conferences) fair to all researchers?
   What are indicators for the criterion of a good communication?
7. How could we achieve more equity?

Session 2

This focused on the criterion of relevance in educational research, including differing definitions of relevance reflecting viewpoints of different stakeholders (researchers, practitioners, policymakers, learners, parents …), and on the relationship of relevance to other criteria.

1. Who are the different stakeholder groups in mathematics education and what do they each mean by relevance? Suggest some research studies in mathematics education which have successfully addressed different audiences.
   – Is it possible – and if so, how – to share a common meaning of 'relevance' among researchers and practitioners?
   – Must relevant research be useful? What research is useful?
   – See Schoenfeld's (2002) scheme for categorizing the impact of studies in education:
   ***Trustworthiness**: How well substantiated is each claim?*
   ***Generality**: To how wide a set of circumstances is the statement claimed to apply?*
   ***Importance**: What contribution does this paper make to theory, methods, or practice?*
2. What is the relationship between relevance and communication?
3. Are there different types of research for different audiences?
4. Should the set of criteria for quality of research be specific to the audience the research is meant to influence?
5. What are the tensions between different stakeholders and how do they affect status and funding?
6. What can be done to achieve more agreement?

Papers made available on the website

Four papers were made available on the website; the first two were related to the questions for session 1, and the second pair were related to the questions for session 2.

Zan R. (2004) The Quality of Research in Mathematics Education (This was an adapted English translation of the author's presentation to the Plenary Panel about Quality of Research in Mathematics Education, held in Santarem (Spain), 6-10 July 1999, for the first *Scuola estiva italo – portoghese – spagnola* in Mathematics Education.)

Lester F. and *Lambdin D.* 'The ship of Theseus and other metaphors for thinking about what we value in Mathematics Education Research', in J. Kilpatrick & A. Sierpinska *What is research in mathematics education and what are its results?* ICMI Study Publication, Kluwer, 1998

Lester F. and *Wiliam D.* 'On the Purpose of Mathematics Education Research: Making Productive Contributions to Policy and Practice', in Lyn D. English (Ed.) *Handbook of International Research in Mathematics Education*, Lawrence Erlbaum

Associates, 2002 (Frank Lester kindly supplied a condensed and slightly modified version for the website)

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DG 9: Formation of researchers in mathematics education

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Gerd Brandell, Lund University, Sweden
George Ekol, Kyambogo University, Kampala, Uganda

Aims and focus

In common with other Discussion Groups, a pre planned structure and series of issues to be addressed were placed on the web site in advance of the ICME-10 conference. Themes suggested for discussion included:

- What academic and professional backgrounds should be expected for individuals admitted to graduate programs in mathematics education?
- What should be the nature of the course work in higher research degrees in education and the most appropriate balance between time spent on this course work and the dissertation?
- What is the role of mathematicians as (co-) supervisors for dissertations in mathematics education?
- Should emerging researchers in mathematics education be encouraged to gain experiences in a wide range of geographic and educational contexts?, and
- Should there be an international "standard" for the training of researchers in mathematics education?

Once the participants' motivations for attendance at the Discussion Group were shared and the participants' specific interests were clarified, adjustments were made to the organization to optimise the relevance of the discussions to those present. A notable feature of the group – representing 13 countries from all continents – was the lively discussion throughout the three allocated time slots and the highly consistent rate of attendance.

The beginning

As part of the introductory activities, the currency and importance of the DG 9 topic were noted, and issues particularly appropriate for discussion were identified. Evidence and topics put forward, included:

- Identification of the topic by the ICME organisers as sufficiently important to mathematics educators to warrant a separate Discussion Group,
- The diversity of higher research degree programs in place, or being planned, at different institutions, and whether there was commonality among such programs,
- How much credence should be given to the diversity of pathways taken up by those who complete a doctorate in mathematics education,
- The continuing calls, from within and beyond the educational community, for reform in mathematics education and the extent to which researchers in mathematics education should be involved and can contribute to these developments,

- The increasing international cooperation among universities, within and beyond Europe. The Bologne Agreement[1] was given as an example of the former, the Cotutelle[2] arrangements among the latter.

It was decided to chose one of the papers (see below), placed on the web-site immediately following the first meeting of the Group, as a common basis for discussions in the second session.

The middle

The work to be read before the second session was "Preparing Mathematics Education Researchers for Disciplined Inquiry: Learning from, in, and for Practice" written by Jo Boaler, Deborah Loewenberg Ball, and Ruhama Even – one of the chapters in the *Second International Handbook* (published in 2003 by Kluwer Academic Publishers edited by A. J. Bishop, M. A. Clements, J. Kilpatrick, and F. K. S. Leung). This chapter covered issues such as: "What does it mean to consider research from the perspective of its *practices?*"; "What is involved in mathematics education research?", "What is there for new researchers to learn?", e.g., with respect to reading, to forming research questions, the collection and interrogation of data, pattern making and generalizing from data, the place of mathematics in research, and communication of research findings; and "Learning in and for the practice of research". We selected this work, out of many available, because of the authors' novel and insightful approach to "unpacking" the practice of research in mathematics education and drawing on this perspective to examine pathways for shaping the preparation of researchers in mathematics education.

Being able to draw on a common reading was both an advantage and a disadvantage. On the one hand it helped focus the discussion; but on the other the chapter content inevitably shaped the discussion and set apparent boundaries for it.

The following issues were explored in particular:

- What is meant by *disciplined inquiry* in relation to doctoral work?
- How can such an inquiry be reconciled with being *open minded*? And with *rigor*?
- What determines if a question is *researchable*? What emphasis should be given to its components: reading critically and insightfully, methodology, data gathering, analysis, and reporting – to different audiences and via different media?
- Should researchers in mathematics education be concerned with the link between *theory and practice* and explore the impact of *theory on practice*?
- What *level of mathematics attainment* should be mandatory for researchers in mathematics education?
- To what extent should/does *social context* shape the scope and direction of research?
- What views of research are currently being promoted?
- When should research preparation be an *individual* or a *group* activity?
- Is it best to rely on a single supervisor or a supervisory team?
- What can be taught/learnt from others and what only through *participation?*

- (What) can we learn from the preparation of other professions and disciplines?
- The predominant emphasis in our discussions is on research. But how many doctoral graduates become *researchers*? Does it matter?

The diverse views expressed provoked much debate. Even within our relatively small group, it became clear that the setting in which participants worked served as a powerful filter for the issues raised in the discussion. Institutional and Departmental conventions, traditions, and aspirations were seen as powerful constraints or motivators. Some questions such as "What *level of mathematics attainment* should be mandatory for researchers in mathematics education?" and "Is it best to rely on a single supervisor or a supervisory team?" had answers which were contextually driven. For others, including "What views of research are currently being promoted?" and "What can be taught/learnt from others and what only through *participation?*" group consensus was found more readily.

The ending

The findings and issues raised during the plenary session of Survey Team 3, who focussed on *Trends in Research*, were found particularly relevant by the group and subjected to further exploration. These included the team's findings that mathematics education research was now dominated by qualitative, small scale studies; that large scale studies are useful for generating theoretical developments and cross cultural case studies for comprehensive hypothesis testing; that longitudinal studies are needed to track developments over time; and that research carried out in English speaking countries dwarfed reports of research carried out elsewhere. Time was also put aside in the final session for participants to reflect on the wide ranging discussions. Some indicated they were well satisfied with the ground covered; others, as can be seen from the next section, thought that the debate was far from complete and had left a number of issues – critical to them – unresolved. It seems fitting to let the participants have the – almost – final word. Representative views are summarised in the next section.

What participants indicated they had gained from the discussions

- We are starting a new doctoral program. I gained much from the discussions and learning about problems faced, and sometimes solved, by others in the group.
- We are also creating a new academic pathway in our institution. The discussions here were helpful in deciding whether we are on the right track.
- Achieving a good balance between generic and specific issues was difficult. More time could have been spent exploring in detail "what should researchers do?" and restricting this discussion to mathematics education (and not policy makers, for example).
- Keeping a focus on individual differences was a critical aspect of our discussions. There are few solutions that will suit all. We need to consider how we can best help some budding researchers refine their very broad research question; others how to broaden their horizons beyond their far too specifically focussed question – from one too simplistic to one appropriate for a major research study.

- Did we ever define just what it meant to be a researcher? On what issues were we able to move beyond/stay blinkered within a personal perspective or that current in our institution? Having some common reading (see summary of the second session) was helpful in providing a focus and starting points for discussion.
- What must we do to avoid simply reproducing ourselves? I gained some useful pointers but am still searching for more answers. Discussions such as those we have just held are useful.
- Just what we, at our institution, can learn from the mistakes of others is what I'll be taking away from here.
- I am more convinced than ever that there is no unique prescription or recipe for producing researchers in mathematics education. Much can be learnt from experienced researchers, though it must be remembered that (successful) styles and approaches will differ.
- We could well have spent more time on skills needed by successful researchers, e.g., sharing information orally and in writing to different audiences.
- The bulk of the discussion focussed on what we can teach our students. Should we have focussed more on what we can learn from them? After all, many of those who chose the mathematics education research path are mature students, often successful in their previous work.

A final word

Multiple pathways currently exist for the training of mathematics education researchers. On the one hand, given the diversity of contexts in which this training takes place, this seems appropriate. On the other, some routes appear to be more beneficial than others. Institutional, external, and historical factors affect the viability of different pathways. Research techniques change and fall in and out of favour. Political pressure influence perceived needs and desired outcomes. Yesterday's answer is often tomorrow's problem. Let the debate continue …

This report was written by Gilah Leder. She is happy to be contacted at g.leder@latrobe.edu.au for further information on the work of this DG.

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[1]) In 1999, Ministers of Education from 29 European countries signed the Bologna declaration on higher education. Its long term goals are to enhance and facilitate student and teacher mobility and to raise the quality of higher education. Considerable progress has already been made on its implementation, with many European universities now fully in support of the process. In the Berlin declaration of 2003, the Ministers agreed to go beyond the present focus on two main cycles (bachelor and master) of higher education to include the doctoral level as the third cycle in the Bologna Process. The general structure aimed for is a 3+2+3 years system. The general process now involves 40 countries whose ministers have declared their commitment to establishing the European Higher Education Area by 2010. The aim is not to unify the current diversity of doctoral programs into one common program. Rather students will be able to tap into different programs in different countries as they complete their degree. (Details supplied by participants)

[2]) The concept of Cotutelle joint or double-badged doctoral programs was developed by the French government to promote partnerships between universities in France and in other countries. Candidature in the program is conducted under joint supervision with enrolment in both the "home" and "partner" institution. Students spend approximately $^2/_3$ of their candidature in their "home" university and the remaining $^1/_3$ in the "partner" institutions. On completion of their doctorate they receive degrees from both universities, with the testamur and official academic record indicating that the degree was obtained under Cotutelle arrangements. Doctoral regulations of both institutions must be fulfilled. Joint-badged doctoral programs have been broadened to other countries. (Details supplied by participants)

DG 10: Different perspectives, positions, and approaches in mathematics education research

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Team Members: *Jere Confrey*, University of Washington at St. Louis, USA
Marie-Jeanne Perrin-Glorian, IUFM Nord-Pas de Calais, France
Tatyana Oleinik, Skovoroda Pedagogical University, Kharkov, Ukraine

Aims and focus

The main issue discussed in the group was the difficulty of accumulating knowledge in mathematics education research (m.e.r. for short) in view of the existence of a diversity of approaches to mathematics education research that sometimes appear as passing fads or fashion waves. On the one hand, the diversity could be seen as an advantage because it promises a more complete picture. On the other, it causes fragmentation, which could be an obstacle to recognizing m.e.r. as a discipline, characterized by a coherent body of knowledge. Moreover, the diversity complicates communication: giving examples of concrete results is difficult without a lengthy presentation of the theoretical underpinnings. The fashion waves in m.e.r. also have their advantages; by focusing attention on a single aspect, they allow this aspect to be thoroughly examined. Too often, however, when the fashion fades, the deep results obtained during this period are forgotten. There is a risk of lack of real progress, and of missing the chance of laying a strong and lasting foundation of research for understanding of educational phenomena.

Written contributions

Three papers were accepted for distribution and publication on the web:
David Clarke, "Issues of voice and variation: The problematics of international comparative research in mathematics education" (Australia).
Bettina Dahl, "Can different theories of learning work together? Some results from an investigation into pupils' metacognition" (Norway).
Steve Lerman and Anna Tsatsaroni, "Surveying the field of mathematics education research".

The organization of the sessions

The first session started with an introduction to the theme of the discussion group and continued with a panel discussion. The panelists were: Gerald A. Goldin, Marie-Jeanne Perrin-Glorian, Lyn English, Anna Sierpinska and Tatyana Oleinik.

The aim of the panel discussion was to provide examples of approaches, theories, concepts that made particularly brilliant careers in mathematics education, and which have later been criticized and either abandoned or, on the contrary, transformed and developed. We decided that panelists were going to be much more convincing if they spoke from their own experience.

The second session was devoted to discussing, in two subgroups, the following two questions:

1. What can research in mathematics education tell us about the constraints that define the reality of teaching and learning mathematics and limit our possibilities of changing this reality, independently of the "approach" used in this research?

2. What are the possible research questions, methodologies and anticipated research results that stem from our answers to question 1?

One subgroup (15 people) considered the questions from the "macro-level" of curriculum studies, educational policy issues, research priorities issues, international studies of scholastic achievement, cross-cultural studies, and the "meso-level" of the organization of teaching of a particular mathematical content by a teacher in a classroom; design and evaluation of teaching materials and learning environments. The other group (33) concentrated on the "micro level" of studies of particular classroom interactions, learning difficulties related to a specific mathematical content, evaluation of teaching experiments, etc.

The third session started with presentations of summaries of discussions in the subgroups and continued with a plenary discussion of the issues raised by the group.

Some details of the discussions

Summaries of the panel contributions

Three panelists confirmed the experience of "fashions" in their lives and two others denied it by seeing the development more as expansion and building on the previous results.

G.A. Goldin: 'I spoke to the history of "paradigms", "fashions", and all-encompassing claims in mathematics education research from the 1960s to the present time. One example (which influenced me in my early work, but in my view proved insufficient) was the emphasis in the 1970s and 1980s on "artificial intelligence" models – the human thinker as essentially an information processing system, with computer simulations of human thinking processes as fundamental to the paradigm. A second example was the ascendance of behaviorism – today it is difficult to appreciate how predominant behaviorist ideas became in the early 1970s, as a reaction to the "new math". A third example was the dominance of radical constructivism during the1990s, from which our field is only now shaking loose. I addressed the need for a synthetic and eclectic approach that includes rather than excludes the many different, important constructs that have previously been viewed as mutually exclusive.'

L. English: English described how her early research career was strongly influenced by the cognitive movements of the 1980s. She referred to the computer metaphor for learning as a basis of her research on mathematical problem solving. Her work focused on the nature and types of knowledge, on higher-order thinking processes, and on the interactions between knowledge forms and thinking processes. There was also an emphasis on individual learning, with detailed analyses of individual children's mathematical reasoning during problem solving. No consideration was given to environmental issues in students' learning. In contrast, English's research now focuses on both the cognitive and social aspects of children's mathematical learning, together with the professional development of their teachers. Her analyses of learning include children's developments during collaborative problem-solving situations, with a focus on their mathematical modeling. The mathematical growth of their teachers (both content and pedagogy) is also a strong component of her research today.

A. Sierpinska: 'In my life as mathematics educator I have known at least these 'fashions' or, rather, focalizations in m.e.r.: focus on *mathematical theory*: pedagogical organization of mathematical material; – *mathematical epistemology*: epistemological studies of mathematical concepts, in abstraction from the socio-cultural conditions of learning at school; *Constructivism*: study of students' construction of mathematical concepts; *The body in the mind*, including *Instrumentation*: taking into account the physical body of the learner and his/her interaction with tools in learning mathematics. I have also witnessed the rise of some words as key words for research and then the ban of these same words as representing a backward philosophy (e.g. misconception, obstacle, understanding, reality). The rise and fall of "epistemological obstacle" is part of my personal story. There was a time in mathematics education when this was a fashionable concept, with special publications and conferences. This concept refined the notion of error in mathematics, turned it into something serious, and changed our attitude to students' errors. And then, in the mid-90s, from the post-modern perspective, epistemological obstacles became a bad word; the philosophy underlying epistemological obstacles was criticized for being "recapitulationistic and parallelistic", for not sufficiently taking into account the socio-cultural factors.'

T. Oleinik: Tatyana referred to her poster, co-authored with Victor Yevdokimov, about the traditional methods of conducting educational research in Ukraine. These methods favor a pluralistic approach, which is not considered as a sequence of passing fads, but as a way of taking into account as many factors influencing teaching and learning as possible in the design, implementation and evaluation of teaching approaches.

M.J. Perrin-Glorian: 'Change and continuity needn't contradict each other. As a researcher within the French school of didactics of mathematics, I see my own story as a process of enrichment of theories to take into account more of the classroom complexity. The research project remains the same: to find and study the conditions for the best possible mathematics teaching. I see three stages in my research history.

1975-1984: Design, implementation and analysis of teaching, based on initial versions of the Theory of Didactic Situations (Brousseau, 1997). and the theory of the Tool-Object Dialectics (Douady, 1987). The focus was on epistemology of mathematics, and the main problems were: To what extent can mathematical situations themselves trigger knowledge construction in students, thus reproducing the conditions of production of original mathematical knowledge?

1984-1993: Difficulties in implementing, in classes of low achievers, situations that worked well in other classes, gave rise to new questions and require complementary theoretical elements. To explain the discrepancy between the planned situations and what actually happened in the classes, I first used the theoretical frame of "metacognitive representations", concerning students' and teachers' ideas about mathematics and mathematical teaching At the same time, I was becoming acquainted with the developments in the anthropological theory of didactics, which was attempting to connect the micro and macro levels of didactic analyses and theories. The distinction as well as links between institutional and personal relationships to knowledge that this theory introduced, became, for me, a way to transform the notion of social representations in a way that was better adapted to didactical questions and more coherent with other theoretical choices. From my research in low achievers' classes, I was now convinced that a good "didactic transposition" of knowledge for the purposes of its teaching was not enough: trying to improve teaching by taking into account only epistemology and student's difficulties may produce worse learning than traditional teaching. Teachers' resistance to new practices, the overlapping of students' difficulties and teachers' choices showed that it was not a question of personal representations. Theory was needed to address questions such as: Do teachers have any choices? What are their choices?

Since 1993: I used the theory of didactical situations mainly to produce good teaching situations. But further developments of this theory, complemented by elements of the anthropological theory, allowed to apply it to the study of ordinary mathematics classes.

Thus, from my experience, research questions changed while theoretical frames were growing. I think that three reasons explain, in the French context, how theoretical construction preserves and enriches previous elements instead of replacing them: the will to construct a specific coherent theoretical framework, theoretical options allowing this coherence, and the existence of institutions supporting continuous exchanges and debates between researchers.'

Notes about the group discussions

As could be expected, the subgroup discussions and even the final plenary discussion diverged somewhat from the questions initially posed. There was a tendency to question the questions themselves. Also, expressions such as "conceptual change" and "evolution" were seen as better describing developments in m.e.r. than "progress" and "accumulation".

In the subgroup supposed to concentrate on the "micro level", the focus was on achievements of m.e.r.: participants were asked to name a result that surprised them personally and had a significant impact on their research or teaching practice. Many results were mentioned but, generally, evolution was seen in research which helped us better understand the boundaries of our freedom to change the reality of teaching mathematics according to a prevailing ideology of the time. Examples included problematization and study of aspects long taken for granted in the teaching of mathematics, apparent in concepts such as, e.g. "socio-mathematical norms", "Zone of Proximal Development", or "didactic transposition". Evolution was seen in an increased awareness of the differences between students' ways of knowing and the ways of knowing that teachers expect them to develop; and in the greater acknowledgement of the need to develop content specific didactic means to bridge the gap without eradicating these individual ways of knowing which are seen as potentially creative and fruitful. If anything, research on students' conceptions has overthrown the naïve belief that if only the teacher used the right words in explaining concepts, everybody would understand.

In the sub-group that addressed "macro-issues", five questions were highlighted for consideration:

1) The kinds of questions and problems addressed in mathematics education research.
2) The objects of this research.
3) The issues associated with different schools of thought.
4) Issues pertaining to methodology.
5) The driving forces behind mathematics education research movements.

This group commenced discussion with the question, "Why is there an attack on previous theories, methodologies, and movements?" In addressing this issue, comparisons were made between research in education and research in medicine, where it was pointed out that education tries to solve problems for the present time, whereas medical research attempts to solve problems for both now and the future. Furthermore, medical research builds on existing research; the extent to which education adopts this course of action was debated. Other points raised in addressing this question included: (a) there appear to be social, political, and academic rewards for attacking previous movements; and (b) the education research community does not have a common knowledge base on which to refute some of the extravagant claims made (it was noted that "extravagant claims require extravagant evidence"). It was concluded that, as a community, we are not giving adequate attention to knowledge accumulation; we have not been sufficiently willing to say which ideas, theories, studies, etc., are important. Our problem is not that we use different ways of proving the claims we make, but rather that we don't use them.

In addressing issues pertaining to methodology, the group agreed that we need to use all methodologies: qualitative and quantitative research answer different questions. The swing back to quantitative research methodologies was considered a real concern to our community. It was raised that one common yardstick for "measuring" students mathematical achievement is to present tasks that enable them to display their achievements, in contrast to performance based studies that highlight failure.

With regard to "objects of study", the group discussed the use of the mathematical construct as the object of study, rather than the student, classroom, teacher etc. In focusing on the mathematical construct, consideration could be given to: (a) how it has developed historically, (b) how the student understands it, and (c) how the teacher understands it. *However*, the point was made that, oftentimes, it is not clear what the object of study is.

In addressing issues pertaining to "schools of thought," the group considered there were serious impediments to creating synthesis in mathematics education research. It was noted that we need time to find agreement between the various schools of thought (which have been presented as opposing ideas, such as the socio-cultural perspective overtaking and ignoring the cognitive aspects). Different schools of thought have different kinds of roles, yet researchers have tended to use them as a platform for justifying their own approach (oftentimes, however, the school of thought doesn't support the researcher's study). The group agreed that we need to respect different schools of thought for what each has to offer and take the best of each one. It was also stressed that mathematics educators need to understand the *entire* school of thought, not just aspects of it.

Finally, the group concluded that there is a need for more theoretical development in our discipline; *but* the question of balance in theoretical perspectives was emphasized. The question of "what the next ideology will be" was raised as a means of warning us not to ignore multiple perspectives on mathematics learning and teaching.

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Jere Confrey was not able to come to the congress. Lyn Lyn English and Anna Sierpinska led the group sessions at the congress and compiled this report. They are happy to be contacted at l.english@qut.edu.au and sierpan@alcor.concordia.ca for further information on the work of this DG.

ICME 10 2004

DG

Discussion Group 11

DG 11: International comparisons in mathematics education

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Introduction

International Comparisons have had considerable impact to educational debates in the last years. What we can learn from these studies? Do they widen our perspectives towards mathematics education? These questions come with a lot of problems: How should we cope with the apparent different traditions when comparing students and/or countries? How can international comparisons foster further developments in mathematics education?

There are many ways to address the issue. The public interest in international comparisons is mainly bound to large-scale achievement studies such as TIMSS and PISA. However, in the mathematics education community also comparative studies of smaller scale are greatly appreciated, as well as other types than achievement studies. In the introductory remarks to the Discussion Group, the organisers exhibited several fields to which a comparative study could be related. E.g. there are

- studies on achievement,
- studies on lesson structures,
- studies on teaching materials,
- studies on beliefs,

to mention only those themes the group was going to discuss. Consequently, DG 11 gave space to a broad spectrum of international comparisons. Furthermore, it was also a permanent topic in the discussion, how the various kinds of international comparisons should be balanced, and what the respective benefits could be.

The invitation to take part in this group pointed to three strands that should be discussed:

- Overview on recent international comparisons in mathematics education,
- Overview on topics addressed in comparative studies, and the central question:
- What can we learn from international comparisons for the development of mathematics teaching and learning?

According to the guiding questions above, the three sessions each covered a special topic. The discussion always started from papers, arranged in such a way that similar questions could be discussed in one session. A total of 10 papers, distributed in advance of the congress, were considered for discussion, and these papers are described below.

Basic issues

The first session was devoted to questions of a basic nature. The discussion started with some arguments about why and at what level international comparisons are to the benefit of mathematics education. One can there distinguish three lines of argumentation:

1) International comparisons as a means for "benchmarking": This idea addresses questions like, where a certain country is placed in the world, to what degree an educational system is "effective" in relation to comparable others. Public discourses on education are likely to be influenced and to even be initiated by using the international data in this way.

2) International comparisons as an opportunity to reflect one's own way to organize and to achieve educational progress: Related questions in this line of argumentation could be the following. Are "others" better in some interesting respect? What do they do, and which of their ideas are adaptable to one's own situation? Essentially, international comparisons can be a means to see one's own system better, through the mirror of observing other systems. Such observations can even be an impulse for starting to act.

3) International comparisons as an instance to start with reflections on principles of education, teaching and learning: International comparisons are always exposed to the argument that their focus is biased, by selection of materials, posing contextual questions, asking questions not in the accustomed way, etc. Therefore, international comparisons are more strongly than national studies forced to make clear their assumptions and their theoretical conceptions. However, just by that critical attitude, such studies can contribute to the development of mathematics education. One can learn and critically discuss how the common ground for a comparison was defined in the studies, or in which ways the results were gained and communicated.

One of the really "basic" questions of any comparative study is: How to understand what's going on in a different culture's classroom, and therefore how to find a common ground to map the observed differences? It was exactly this problematique the Discussion Group started with. Two presenters opened the discussion.

Paul Andrews (Cambridge, UK) reported on a joint project in five European countries (Flemish Belgium, England, Finland, Hungary and Spain), the "Mathematics Education Traditions of Europe" (METE) project. To examine how learning was structured over sequences of lessons taught on a particular topic, video recordings of such sequences are made. However, to ensure the elimination of cultural bias from subsequent analyses, data collection was preceded by an extensive programme of live observations to facilitate the development of a descriptive framework for the analysis of videotaped lessons. Paul Andrews exhibited how observation schedules were constructed, facing the problem that even the simplest assumptions about the vocabulary of mathematics classroom activities proved to be far from unproblematic. An iterative process finally produced a schedule that all participants could use with confidence, understanding, reliability and which satisfied the project's desire to describe lessons in ways that highlight both similarities and differences among teachers and countries.

Permanently, these issues arise when setting up video studies, as *David Clarke* (Melbourne, Australia) pointed out. The very general theme of "Voice and Variation" dominated the first session. "Voice" in this context refers not only to the voices of the participants (teachers and students) in classroom settings, but also to the voices of the interpreting researchers, whose cultural affiliations inevitably contribute to the form of their analyses. Concern with "variation" relates to the need in international comparative research not to minimize variation. The simplistic aggregation of data at the level of nation, or the implicit imposition of a common international curriculum through international testing, or the aspiration to remove variation through the identification and advocacy of uniform internationally-applicable best practices were instances of such simplistic views. In contrary, it seems necessary to document and report variation in educational policy and practice in a manner that anticipates further variation in adaptation and application of such research.

Closing the first day, *Tibor Marcinek* from Slovakia gave some personal experiences while visiting, observing and analyzing mathematics education in US and his home country. To him, the personal experience itself, while conducting qualitative research, is a key to interpreting international comparisons. He posed the question, why a consistently top scoring country, his home Slovakia, nevertheless may call for rather radical reform of teaching mathematics. Other qualitative, personal experience-based international comparisons should be made to find a common ground on which outcomes may be communicated.

Teachers

The second session of the Discussion Group focused the discussion and collected contributions that dealt - in a wider sense - with issues of teachers in an international perspective. Teachers' knowledge, teachers' aids, teachers' beliefs were addressed among other issues. Three papers served as the starting point for the discussion.

Linda Haggarty and *Birgit Pepin* (Oxford, UK) could unfortunately not attend personally, but contributed with a background paper on an investigation of mathematics textbooks and their use in English, French and German classrooms. In particular they looked at the treatment of "Angle" in these textbooks, and examined teachers' mediation of those books. They observed and interviewed a small sample of teachers in those countries. An analysis of the data suggests that learners in the different countries are offered different mathematics and given different opportunities to learn that mathematics, both of which are influenced by textbook and teacher. So, who gets an opportunity to learn what?

Bracha Kramarski and *Zemira Mevarech* (Bar-Ilan, Israel) reported on a study, using PISA data, which proved the mutual relations of enhancing students' mathematics literacy on the one hand and their teachers' attitudes towards mathematics literacy on the other. PISA gave a good data basis since PISA's aim to assess students' literacy in a large-scale study is rather unique because the traditional approach focuses mainly on curriculum assessment. However, it is not clear at present to what extent teachers adopt this approach and teach accordingly. The present study addresses this issue by utilizing a PISA national option in Israel. About 150 mathematics teachers answered to a questionnaire that asked them about their attitudes toward mathematics literacy, emphasizing mathematics literacy in mathematics classrooms, and the extent to which they apply metacognitive and self-regulated learning (SRL) techniques in mathematics classrooms. The results, though showing only small effect sizes, indicated that teachers' attitude towards enhancing mathematics literacy plays a somewhat contradictory role. Students of teachers who believe that mathematics literacy is important, tend to score lower on the PISA test. The suggestion was made, and also the discussion sticks to this point, that teachers and students are to be exposed more intensively to literacy materials, particularly in mathematics and science, and that materials alone do not suffice as long the underlying principles are not made explicit.

An Shuhua (Long Beach, USA) compared teachers' knowledge in USA and Chinese mathematics education. The study addressed mathematics teachers' pedagogical content knowledge within the respective cultural context and tried to identify missing components in the teachers' knowledge bases. Eight "missing components" were found, among them problems like that of bridging from manipulatives to mathematical ideas, of approaching students' misconceptions by using probing questions, of engaging students in study-questions, and of seeing the whole picture of the knowledge network. The study's benefit for the development of mathematics education research was seen in that it provides insight in how to set up dimensions for further international comparative studies in teachers' knowledge.

Students

At the third session of DG 11, some ways students are taught in different cultures, and views on student's beliefs were taken into the focus. This topic was addressed under diverse perspectives, from extensive studies to local investigations, and even plans for further collaboration.

Sun Xuhua (Chinese University of Hong Kong) started with a culture-bound look at the differences in mathematics beliefs between Chinese and USA students. More than two hundred Chinese students, 10^{th} and 11^{th} graders, were surveyed to compare their mathematics beliefs with American students. The overall findings indicate that the mathematical beliefs are value-laden and culture-bound. In particular, some of the most salient differences came from the attribution of success or failure (Chinese students tended to emphasize effort and interest more than Americans, and: Chinese students emphasized the teachers` attitudes in grading more than Americans), reasons to learn mathematics (mathematics is a more mandatory subject for the Chinese students), value view of parents (Chinese parents tend to stress the importance of math much more than Americans), self-belief (Chinese students tend to be lower in self-evaluation). A causal model of Chinese culture-bound mathematics beliefs and achievement was given which comprises the findings.

David Clarke and *Carmel Mesiti* (Melbourne, Australia) presented the basic assumptions and conceptions as well as findings of the Learner's Perspective Study. Data generated in 8^{th} grade mathematics classrooms in Australia, Germany, Japan and the USA were shown: How are the classroom practices of different countries most usefully compared if our goal is the improvement of those classroom practices? The aspiration to compare at one level (international) implies an aspiration to typify at another level (national). Both processes, comparison and typification, should be subjected to scrutiny. Lesson structure provides one potential unit of comparative analysis. One argument for the utility of lesson structure as a unit of comparative analysis is its potential adaptability. However this deserves intensive further conceptualization in mathematics education.

The aim of *Jiangsheng Bao* (Suzhou, China) was to introduce and present to a wider audience a newly established Video Case Study. The video clips in that study are accompanied by the plans of lesson preparation, by extensive hints to critical issues in the intended learning processes, by questions for reflective thinking about the lessons, and by selected background data. Those videos and the materials can be used as a good and effective means to assist programs of teacher's further education. The video clips themselves and the accompanying hints are easily accessible by a hyperlink based technological platform. This platform allows both, a concrete look into the classroom, and a close tie to analysing and reflecting questions. Jiangsheng Bao presented his material, which is still in the phase of construction also with the desire to incorporate international data, in order to broaden the scope on mathematics teaching.

A more specialized topic was brought into the discussion by *Thomas Judson* from USA. In a small scale study he compared the ways, concepts, and skills, in the teaching of Calculus in USA and Japanese secondary schools. The study contained interviews and examination problems of two kinds, some probing conceptual understanding, some pointing to computational and reasoning skills. Little differences were found in the conceptual understanding if isolated, however when the examination question also contains the need of computational skills the Japanese students demonstrated much stronger capabilities.

Discussions and conclusions

The discussion several times and under various viewpoints came back to the basic distinction, already addressed in the introduction: How to cope with the two different levels on which international comparisons can focus, the systemic level and the individual level? While the aim on the systemic level is to draw conclusions in order to develop the educational system and the system of mathematics education as a whole, one searches on the individual level for understanding mathematical achievement. Both aspects could give impulses to each other, however often the two sides are not related. Several issues discussed in the DG 11 Group are sub-aspects of this basic question: Where is the place of large scale studies, and when are studies on smaller scales appropriate? On one side, the interpretation of large scale studies is bounded to an understanding of individual thinking, and vice versa, a small scale study can also draw upon concepts created for the bigger studies. But are these mutual dependencies really used enough? Apparently no, since the contacts between the researchers in the two paradigms are not as intense as they should be for close cooperation. The discussion in DG 11 once more revealed, that in mathematics education, we observe only few studies bridging the gap from the observation of the mathematical thinking of individual students to the mathematical achievement in bigger educational systems. Some of the instances shown in the topics discussed in the group however showed, that such bridges are necessary and fruitful, e.g. when one aims to compare lesson construction in different cultures or when one addresses teachers' or students' beliefs. Thus, a final conclusion of the discussions in the group could be to foster mutual exchange between the "two worlds" of international comparisons.

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DG 12: Assessment and testing shaping education, for better and for worse

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Aims and focus

This Discussion Group focused on the impact that educational assessments have on the teaching of mathematics, and was organized around three main themes: what mathematics should be assessed; the alignment of standards, curriculum and assessment; and how mathematics should be assessed.

What mathematics should be assessed?

One of the key areas of agreement in the three sub-groups (organized around the three themes) was that the intended goals and curriculum (the mathematics expectations or standards) should determine the way assessment in mathematics is done, rather than the other way round. Therefore, the connection between these should be clear to all – including students, teachers, parents, and other interested agencies.

The focus of assessment should be expanded to include both content and process aspects of mathematics (see paper by vom Hofe et al on the DG 12 web-site). Assessment of content alone without regard to mathematical processes (e.g. communicating, modeling, reasoning, showing rigor in approach, and connecting) is not sufficient. On the other hand, processes do not exist independently of curricular content – for example, if students are to communicate, they have to communicate about something. However, assessing mathematical process is not easy since exemplary problems that have the potential to promote the development of mathematical processes in students can, in the hands of some teachers, be reduced to a set of routinized skills. For example, in England, where the assessment of mathematical investigations was introduced into the national school leaving examination in the 1980s, in many classrooms this was taught as an additional piece of content, with students given a set of procedures to follow. Similarly in Ontario, Canada (see paper by Suurtamm and Lawson on the DG 12 web-site), a new curriculum emphasized instruction using challenging problems, the student construction of multiple solution methods, and mathematical communication and defence of ideas. However, the curriculum was operationalized through a detailed list of content standards that teachers were expected to cover, which militated against the rich curriculum envisaged.

Results from international attempts to assess mathematical processes reveal that worldwide the emphasis on processes is not as well developed as is needed for students to become users of mathematics in real contexts. Progress in this area can begin with helping teachers design and carry out assessments of mathematical process in their classrooms. Such work can also have the side effect of helping teachers develop their teaching practices and learn to use assessment tasks that promote process aspects of mathematics.

Of course, these process elements of mathematics should permeate all the content aspects of mathematics, but there is the separate, essentially strategic, issue, of whether these process aspects should be identified separately. In the development of the NCTM 1989 *Curriculum and Evaluation Standards for School Mathematics K–12*, the writing groups did articulate standards that were more like traditional content standards but added a set of standards – communicating, problem solving, reasoning and connecting – that were meant to convey the importance of these bigger more powerful competencies. Similarly, in England, the National Curriculum for mathematics specifies traditional content areas (number, shape and space, probability and statistics) as well as separate process aspects, and process aspects are also clearly identified within the PISA assessment framework. While separating content and process in this way does create difficulties, it also highlights the importance of process aspects of mathematics. Ideally, the process aspects should permeate all the content aspects of mathematics curricula, but initially at least it may be necessary to signal the importance of process aspects by making them separate.

Another issue that emerged in many discussions was that of curricular progression. In many countries, the connections between the curricula for different years or grades are very weak, so that the fundamental question of progression ("When someone gets better at mathematics, what is it that gets better?") is rarely addressed. This is exacerbated in countries, such as the USA, where teachers tend to teach only one grade or year, so that while a teacher may, for example, understand the fourth-grade mathematics curriculum well, they have only a hazy idea of how this builds on the third-grade curriculum, and how it leads on to the fifth-grade curriculum.

Alignment of curricula, teaching and assessment

In many countries, (e.g. France, Japan, Sweden) the mathematics curriculum is the responsibility of central government, while in others (Australia, Germany, United States) this is devolved to regional bodies (states or regions). In some countries, the mathematics curriculum is the responsibility of the district, the school, or even the individual teacher. However, even when the curriculum is clearly specified, in many countries the assessments are not well aligned with the curriculum. Two problems are particularly common. The first is *construct under-representation*. This occurs when the assessment systematically under-represents the construct of mathematics as defined in curriculum standards (for example when an assessment does not assess process aspects of mathematics even though this is an explicit component of the curriculum). The consequence of this is that where schools are under pressure to improve their students' results, those aspects of mathematics that are not assessed are neglected. As a result the curriculum does come to be aligned with the assessment, but only by changing the curriculum (instead of making the important mathematics measurable, we end up making the measurable mathematics important).

The second problem is that of *construct-irrelevant variance*. This occurs when the assessment requires students to have skills or knowledge beyond those that it is intended to assess. For example a mathematics test might be couched in complex language so that to succeed, the student would need to be good at mathematics and reading. When a student gets a low score, we do not know whether this is because they cannot do the mathematics, or whether they could not understand what they were being asked to do (in the latter situation, part of the variance in scores is attributable to factors not relevant to the construct being assessed).

From the discussions in DG 12, it appeared that lack of alignment between curriculum and assessment appeared to be a widespread, if not universal problem, although in some countries there were signs of progress. In France, new methods are being developed for assessing 'process' aspects of mathematics in the Baccalauréat such as the quality of exposition, accuracy of justifications and coherence in the reasoning process (see paper by Feurly-Reynaud on the DG 12 web-site). In Sweden there have been considerable changes in the mathematics tests given to 18-year-olds over the last 30 years (see paper by Jakobsson-Åhl on the DG 12 web-site). In 1973, the national test was abstract, placing great emphasis on decontextualised, axiomatic thinking (the result of the introduction of the "new math" in Sweden at that time). The 2002 test reflects more of an emphasis on student thinking, allowing more time and including three lengthy "story" problems. The exams were also scored differently, with the 2002 requiring complete solutions, rather than just answers. In the Netherlands, the development of investigative skills was introduced into the mathematics curriculum for lower secondary schools in 1993, but the national tests were traditional pencil-and-paper tests, which did not assess adequately the investigative aspects of the curriculum (see paper by Vos on the DG 12 web-site). To remedy this, the National Institute for Educational Measurement developed a series of practical tests so that even those teachers who saw their role as 'teaching to the test' had an incentive to incorporate investigative work into their teaching.

However, even when the assessments are well-aligned with the curriculum, both can be seriously out of alignment with the teaching in classrooms. In Portugal (see paper by Carvalho e Silva on the DG 12 web-site), two committees were established in the mid 1990s, one to work on assessment and one to work on curriculum. A subsequent evaluation of their work established that they did manage to align the assessment with the curriculum to a good degree, but that test items specifically aligned to the standards (e.g. using graphing calculators to find the intersection of 2 curves) were among those with the worst student performance. In other words, the assessment was well-aligned with the 'intended' curriculum' but not with what the students actually learn (the 'achieved' curriculum), showing that large-scale assessment does not necessarily drive practice. A similar outcome was found in South Africa where the new 'Common Task Assessments', which focused on students' ability to use their mathematics to solve 'real' problems, were perceived by teachers to be neglecting the more 'academic' mathematics the students had studied (see paper by Naidoo and Parker on the DG 12 web-site). Such examples show that aligning curriculum standards with assessment is only the first step. We need also to find better ways of communicating about these curricular standards to practitioners, in particular for the process standards. We also need better examples of problems and rubrics for assessing student performance.

How should we assess mathematics?

It was agreed that the predominant methods of assessing mathematics, such as timed written tests and examinations, do not assess adequately many of the aspects of mathematics curricula around the world. This is not to say that such tests and examinations are useless, but, especially where curricula give substantial emphasis to mathematical processes, other forms of assessment will also be necessary.

Since many of these process aspects of mathematics can be assessed only through extended pieces of work, two things follow. The first is that in order to justify the time taken for such extended pieces of work, the assessment tasks must also be tasks during which students learn (assessment *as* learning). The second is that if the costs of assessments are to be kept down, teachers must be involved in summative assessment, so that the assessments that teachers make as part of their normal work can contribute to the overall judgments made about the capability of students.

To achieve this, teachers will need examples of exemplary tasks, rubrics for scoring such tasks, and professional development focused explicitly on teaching practices that support the use of such tasks and the development of the appropriate mathematical competencies in students. The good news is that there is a range of models in use around the world that can provide some ideas about how to broaden the basis of assessment. Denmark, and many former Soviet bloc countries have a strong tradition of oral examination in mathematics. In Sweden, there are national tests in mathematics at age 18, but these are used only to guide the teacher's judgment of the student's grade, which is based on all the work done in the final years of upper secondary school (students who get low grades can also get into university on the basis of a special aptitude test). In Queensland, Australia, again, university entrance is based on teacher judgments but there is a 'core skills test' that is used to calibrate performance across different subjects. In the USA, practice is very varied, but assessment of mathematics appears to be conducted primarily for the purpose of holding schools to account for the performance of their students, so that the assessments tend to be low-stakes for the students but high-stakes for the schools. In order to keep costs down, most states use standardized multiple-choice tests, and while it is possible to measure some higher-order skills in this way, it seems fair to say that the tests in widespread use do not. Furthermore, many tests are kept secret so that teachers have only a hazy idea of the content of the tests their students will take.

Summary of DG 12 discussions

As might be expected, the group came to no clear consensus about the best ways to assess mathematics. Indeed, a clear finding of the group was that the assessment systems in place in each of the countries are inextricably linked to the local contexts in which the systems operate. Solutions that work in one setting are unlikely to work in another without some adaptation. However, there appears to be general agreement about some broad principles that should govern all assessment systems.

- Education goals and curriculum expectations should determine the way assessments are done.
- Teachers are a critical part of the assessment process.
- The process should be as transparent as possible, meaning that students, parents, teachers, administrators, and school boards should all be aware of what the tests will comprise and be supportive of the goals.
- There is a need for greater flexibility in the timing of exams.
- In order to test what we care about, including thinking and reasoning, we need more innovative assessment tools and accompanying text materials.
- We need to ask what the functions of assessment are. – Why assess?
- We need to ask what the domain of assessment is. – What to assess?

- We need to ask who has authority for assessment. – Who is involved and why?
- What is the target of assessment? – Who is assessed and why?
- What is the means of assessment? – How to assess?
- What are the constraints and the affordances of the testing process. – What does a test reveal to students and teachers that helps improve mathematics education?

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ICME 10 2004

DG 13: Evaluation of teachers, curricula and systems

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DG

Discussion Group 13

Aims and focus

All participants agreed that the theme raised key issues for policy and practice, locally, nationally and internationally, and that simple recipes for evaluation are to be avoided. Three sub-groups corresponding to the key words "teachers", "curricula", and "systems" then met and reported back on key questions and findings. It was agreed that focusing on the quality of students' learning is a necessary link across all three areas of evaluation.

Evaluation of teachers

Several countries are using professional teaching standards as a tool for individual and systemic evaluation of teachers of mathematics. Public, credible standards – and the assessment of individuals against these – are fundamental to the high esteem in which other professions are held. How these other professions practise self-regulation enhances their standing, and adapting this model has the potential to produce the same positive outcomes for teachers of mathematics (Bishop, Clarke, & Bennett, 2000; Ingvarson, 1998). Some significant claims have been made about the power of standards and their assessment:

> The National Board for Professional Teaching Standards is … dedicated to student learning and to upholding high standards for professional performance. We have raised the standards for teachers, strengthened their educational preparation through standards, and created performance-based assessments that demonstrate accomplished application of the standards.
> (NBPTS, www.nbpts.org/about/index.cfm)

However, teaching is a complex performance and there are many challenges in assessing accomplishment (Delandshire & Petrovsky, 1999). A number of tensions arise for teachers as they develop and present evidence of their accomplishments:

- *Comparability versus creativity* – Can I exercise creativity in presenting evidence of my accomplishments to assessors who need to show consistency in their judgements?
- *Meaningful versus manageable* – How can I give evidence of my accomplishment in ways that do justice to my achievements and yet avoid excessive amounts of time in assembling and presenting evidence? How much evidence do assessors really need? What do I tell them so that they fully understand the nature of my achievements?
- *Accomplishment versus ongoing critical inquiry* – To what extent can I use this process as a means of engaging in genuinely critical inquiry into my own teaching, rather than simply demonstrating my accomplishments?
- *Personal goals versus school policy* – Is it possible to demonstrate personally meaningful accomplishments beyond the professional development goals required by my school or those required by government policy?

The following questions guided group discussion on the evaluation of teachers.

- What are the "costs" and benefits of evaluating teachers? If all teachers are to be evaluated, does this encourage better teaching or distort priorities?
- What methods are available – what are their strengths and weaknesses?
- How might teacher self-evaluation be developed?

There was considerable discussion trying to answer the question "What is it that it is most important to evaluate?" The resulting list of four points has many similar themes to "standards documents" developed in, for example, Australia and the USA:

1. *Subject matter knowledge for teaching mathematics* – including both breadth and depth of knowledge, clear understanding of the connections between concepts, an understanding of potential difficulties students may have, use of appropriate representations and knowledge of typical misconceptions students may have.
2. *Knowledge of student learning* – including the need to provide opportunities for all students to learn, knowing how children learn, and what they have learnt as well.
3. *Professional growth* – including a commitment to their own learning and being open to reflection – to keep reassessing their own practice.
4. *Teaching practice* – successful planning and implementation of appropriate tasks, developing a culture and environment where learning happens.

In considering these four points, the group strongly agreed that any evaluation process should include evidence of *actual* student learning – have students learnt some important mathematics and does a teacher facilitate independent thinking of students?

As well as the complexities of evaluating teaching, there are the multiple agendas of those involved. Resolving or just managing the professional, the political and the systemic contexts is difficult. But a key point was the need to resist simplifying both a framework for describing teaching and also the process of evaluating it. Teaching is a complex activity and any evaluative process developed must both acknowledge and work within that complexity. While much of the initiative for teacher evaluation comes from systems, the voice of practitioners – teachers of mathematics – must lead the way in which standards for teaching are conceptualized, developed and implemented.

Evaluation of curricula

It insufficient to evaluate curricula by inviting comments from, for example, an expert panel. The discussion group argued for a more rigorous process of gathering and using evidence of the implementation of a curriculum; its impact on the work of teachers and students, including the extent to which students and schools can demonstrate achievement of standards embodied in an intended curriculum.

Evaluation is needed to provide feedback on the implementation of a curriculum, including the impact of curriculum reform, in a range of realistic contexts affecting school personnel and resources. This information is necessary to revise (or reform) a curriculum in the *next* stage by evaluating *current* school practice. Such feedback is also necessary to provide rich and reliable information to guide policy and decisions regarding textbooks; and to provide reliable insights into the quality of teaching and learning.

The group's view of curriculum includes not only statements of content and their relation to different year levels, but *also* recommendations on how the curriculum is to be taught; for example, through sample programs or advice to teachers. It was also argued that any evaluation of the curriculum must include the procedures that are used to assess students' performance either externally, or by internal classroom assessment.

Any rigorous evaluation of a curriculum must therefore attend to the full set of documents and procedures that are intended to explain and give effect to a curriculum. There will be room for interpretation, especially in the case of curricula that operate across a large school district or a whole state, and even more so for a national curriculum. While different implementations will be encouraged, some *givens* must be spelled out clearly and unambiguously in terms of what students are expected to know and to do.

The group argued that several other *givens* must be present. (1) A curriculum must contain a clear description of content to be covered, how that content is to be distributed across the various year levels, and how elements of content are linked together year by year. (2) A published curriculum has to spell out a comprehensive set of *standards* which relate to students' attitudes towards mathematics, the forms of mathematical thinking that are to be encouraged, the skills students are to have, how students are to represent mathematics, and what mathematical understanding they are to achieve. (3) It must be clear to those implementing a curriculum how they can be sure that all the intended standards have been covered. (4) Finally, any published curriculum must provide clear guidance for the development, appraisal and adoption of textbooks and other resources.

Evaluating an implemented curriculum must therefore have a clear focus on the *quality of learning* by students. Consequently, any well-designed curriculum must contain a range of teaching and assessment tasks exemplifying a range of formats and task demands. Actual student responses should be used where possible to illustrate the range and *quality* of successful performances, common misconceptions and errors. These are necessary to inform teaching, to guide the construction of assessment tasks, and so ensure that teaching and assessment address all aspects of the intended curriculum. They also show how implementation of a curriculum will be evaluated.

Evaluating the implementation of a curriculum can take place at the level of an individual school, at local or district level, or at state or national level. One example of a large-scale evaluation of the implementation of the curriculum is the extensive surveying of students' mathematical performance carried out periodically by the Ministry of Education in Japan. In evaluating implementation of a curriculum, it is possible to use innovative and experimental tasks that may not be appropriate for measuring school accountability or testing student achievement. Thus, the Japanese surveys include questions to elicit students' attitudes to mathematics generally and to particular topics. Two important results flow from evaluating a curriculum in this way. In the first place, tasks can be used to challenge current assessment and to change teaching in areas that may not be aligned with the intended curriculum. Second, performance standards can be established for students at specific year levels, and indicate how performance can be expected to change over time.

Evaluation of systems

Evaluation of systems is judging the value and worth of systems, their actions, and their conditions to produce quality learning in mathematics. The group agreed that a system includes all who have responsibility to provide education, students, materials, conditions, forces, and other influences. A system also includes interconnections and interactions between these elements for the general purpose of serving the mathematics education needs of students, consumers of education, the community, and society at large.

A system under consideration for an evaluation generally is a school or a school district, but it could be a classroom, a state, or even a nation. It was found helpful to identify four components in thinking about designing an evaluation of a system. These are Student performance, Program, Policy and Contextual factors. *Student performance* constitutes the main aim of any evaluation. The *Program* consists of teachers, their practice, curriculum, materials, professional development, and other factors intended to lead towards desired student performances. *Policy* comprises the decisions and directives made to guide the system. Policy can be made at all levels within the system. Finally, there are *contextual factors* that may not be under the control of those in the system being evaluated, but are still influential on what the system does and what it accomplishes. The needs of the work force, higher education entry requirements, and parent expectations are such factors. All four components need to be built into an evaluation plan of a system.

The discussion noted that three types of variables – enabling variables, target variables, and explanatory variables – should be part of any model for a system evaluation.

Target variables represent elements that are specifically identified to be changed by a system initiative. Improved student performance is generally the primary target. But other target variables can include curriculum, alignment, saturation, equity, and quality. These variables can be seen as means or pathways to achieving higher student achievement. Having a more aligned system with higher quality curriculum and professional development is thought to lead to improved student performance.

Enabling variables represent those conditions that can either inhibit or facilitate the attainment of the target variables. These variables or conditions are necessary for advancing student performance within the system. Capacity and sustainability are two examples of enabling variables. For a system to attain the desired student performance and to support a new initiative requires the system to have some capacity including resources, social capacity, and other factors that can deliver a desired change in the system. Any initiative needs to be sustained adequately to have an impact. For example, professional development needs to reach all teachers in need to have a full impact on the system.

Explanatory variables help explain why or why not the target variables reach the desired goal. The group noted that when a new curriculum is adopted, some *trade offs* are inevitable. When new topics are added to the curriculum, other topics are eliminated. Identifying *trade offs* can explain why student performance in one area can increase while performance on another topic decreases. Cost and resource allocations can help to explain why desired goals are not being obtained or why an initiative has been successful.

At a minimal level, the evaluation of a system must include some measures of student performance. This measure should include both an indication of student growth and attainment over time. Ideally the measure of student growth will be an assessment of the same cohort of students at the beginning of the school year and at the end of the school year. The students who are assessed can be a sample of the population. In addition, there has to be some measure of practices within the system including variables from all three categories. It may not be necessary or possible to measure all three types, but the more variables that can be measured, the more precise the evaluation information will be.

Three issues related to evaluation of systems were identified. The first is to explicate *what the system is* in terms of the four components noted above, and defining expectations for how the system should function. A second issue is *time* for measuring the impact of some initiative. Trying to measure the impact prematurely can produce discouraging results and cause the discontinuing of resources for an initiative. Such poor results could be a result of underestimating the time when impact can first be expected. A third issue is the problem of *attribution*. Since systems are complex it is difficult to isolate specific factors and attribute change in the system to these factors. If the evaluation is to produce information on reasons why the system has changed, the major challenge for any evaluation is to identify the variables or initiatives that have contributed to the observed change.

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This report was written by Max Stephens in cooperation with group members Barbara Clarke and Norman Webb who wrote draft reports of sections on “teachers” and “systems” respectively. Max Stephens can be contacted at m.stephens@unimelb.edu.au for further information on the work of this DG.

DG 14: Focus on the development and research of mathematics textbooks

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Aims and focus

DG 14 was prepared by the entire organising team. As Emanuila Gelfman was unable to attend the congress, the DG organisation at the congress was taken care of by the two chairs and the two team members present. The DG 14 was well-attended over all three sessions which indicate the interest in mathematics textbooks by congress delegates. This report provides an overview of the aim and focus of DG 14 and a summary of the discussion that occurred throughout the sessions.

As set by the organization team, the general aim of DG 14 was, in the international mathematics education community, to increase awareness of the importance of textbooks in the process of teaching and learning of mathematics, to promote exchanges and collaborations in the area of mathematics textbooks, and hence to raise the level of research, development, and evaluation of mathematics textbooks. More specifically, through its official program during the congress and other activities (including those before and after the congress), DG 14 was intended to provide an international forum for all interested parties (e.g., mathematics education researchers, curriculum specialists, textbook developers, and school teachers, etc) to:

- Share experiences in developing, using and evaluating mathematics textbooks;
- Disseminate findings from research about mathematics textbooks;
- Exchange ideas about mathematics textbook research, development, and evaluation, and
- Identify various issues concerning research in mathematics textbooks.

The focus of DG 14 was mathematics textbooks, which according to the organization team included mainly the core teaching and learning materials (both the printed textbooks in the traditional sense and hypertexts in electronic devices that can be read as texts), but also other teaching and learning materials (such as resources books, problem booklets, workbooks, etc.).

Five specific aspects along with a series of questions were identified and recommended for contributions and discussions. They are as follows:

1. The development of mathematics textbooks

How are textbooks developed in different countries and how should they be developed? Who are the authors of mathematics textbooks and who should be the authors? Should textbook development be experience-driven, research-driven, or market-driven, and what are the realities and restrictions in different countries? What role does technology play in the development of mathematics textbooks, and how does it affect the development of textbooks? What are the peculiarities of an electronic textbook?

What role do the government, mathematicians, mathematics education researchers, curriculum specialists, and classroom teachers play in textbook development? What are the interests and forces that drive the development of textbooks in different countries and how should different interests and forces be viewed and dealt with for improvement? How do different socio-cultural values influence the development of mathematics textbooks in different education systems? What lessons can we learn from the history of mathematics textbook development in different countries?

2. The relationship between mathematics curriculum standards/syllabi and textbooks
How should mathematics textbooks follow and reflect the intended curriculum standards/syllabi, if there are such standards or syllabi? To what extent are mathematics textbooks in different countries aligned with curriculum standards/syllabi? How can the gaps between mathematics textbooks and curriculum standards/syllabi be filled? How do textbooks serve as a means to transmit socio-cultural norms and values embedded in different education systems or national curriculum standards/syllabi?

3. The role of textbooks in the teaching and learning of mathematics
Are textbooks essential in the teaching and learning of mathematics, and under what circumstances? Should mathematics textbooks be written for teachers or students or both? Should textbooks be treated only as an information source or should they be regarded as an instrument of organizing student's educational cognitive activity? How do textbooks shape the teaching and learning of mathematics within and outside schools and classrooms, for worse or for better, and to what extend? How do teachers and students use mathematics textbooks (e.g., do they follow textbooks closely or just use them as one kind of information source among others)? And why do they use textbooks in a particular way? How can teachers and students benefit from having/using a textbook, and to what extent? What are the influences of textbooks on students' achievement in mathematics, and how can this be measured?

4. The evaluation of mathematics textbooks
How can judgments about the quality of mathematics textbooks be made, for research and for practical purposes? What criteria and constructs should we use in making such evaluations? What textbooks may be called "good" for students, or teachers, or even parents? How can the evaluation of textbooks be related to the adoption of textbooks? What are the current processes of decision-making for textbook adoption in different countries, and how can such processes be improved? Who (e.g., educational administrators, school principals, heads of mathematics departments, classroom teachers, students and parents) should be involved in the decision-making process, and how?

5. The research in the area of mathematics textbooks
What is the status of mathematics textbooks as a subject for disciplined inquiry in the international mathematics education community? How can awareness of the importance of textbooks in mathematics education research be increased? What are the important issues in this area? What methods should be used to conduct research centering on mathematics textbooks, in addition to the commonly used ones such as comparative study and document analysis? What can we do to raise the level of research in mathematics textbooks?

Nine papers were accepted for this DG, after being reviewed by at least two reviewers of the organising team and/or an external reviewer, and were made available on the DG's official website before the congress. They focused mainly on (theme 2) The relationship between mathematics curriculum standards/syllabi and textbooks ("An analysis of the representation of problem types in Chinese and US mathematics textbooks", by Zhu, Yan and Fan, Lianghuo, Singapore; "Differentiation in mathematics textbooks", by Anna Brändström, Sweden; "Fractions and ... fractions again?! A comparative analysis of the presentation of common fractions in the textbooks belonging to different didactical fractions", by Viktor Freiman, Canada and Alexei Volkov, France), (theme 3) The role of textbooks in the teaching and learning of mathematics ("A text-book as means for organizing students' cognitive activity", by Emanuila Gelfman, L. Demidova, and V. Panchischina, Russia; "Mathematics textbooks, opportunity to learn, and achievement", by Jukka Törnroos, Finland; "On the problem of typology and functions of school texts", by Emanuila Gelfman, A. Podstrigich, and R. Losinskaya, Russia; "Reading mathematical texts: Cognitive processes and mental representations", by Magnus Österholm, Sweden; and (theme 4) The evaluation of mathematics textbooks ("Characteristics and issues of China's primary mathematics textbooks based on the current curriculum standards", by Li, Zhongru, China; "The new edition of Chinese mathematics textbooks for primary schools," by Lu, Jiang and Wang, Yongchun, China). The organising team was encouraged by the good attendance of the participants during the congress, but a bit disappointed by the lower-than-expected contributions that were received, which suggests that mathematics textbooks are still under-researched and that more attention is needed to this important area.

Session 1

Focus: Development of textbooks in different countries

The first session was chaired by Lianghuo Fan and Stefan Turnau. After introducing the team members and providing an overview of the aim of DG 14, participants were invited to give an overview of textbook development in mathematics in their own countries. From this discussion, it was found that there are very different ways in which textbooks are developed in various countries. Summarised below:

Singapore – The National Institute of Education is the only teacher education university in the country, and textbook writers from this institution are approached by commercial publishers. All textbooks must be approved by the Ministry of Education before they can be published and used in schools.

France – A variety of textbooks are available to schools, but there is a central committee which determines whether a particular textbook is acceptable for use in schools. The committee then make a list of accepted texts available to schools.

Germany – 16 committees look into textbooks. They play a similar role as in France.

USA – Any author is free to seek publication of his/her textbook. The publisher's role is to market the text.

China – Textbook development was originally only done by the People's Education Press. The situation has changed. Textbook authors now need to apply and be approved by the Ministry of Education to write their proposed textbook, and then the textbook must be reviewed and accepted by a committee set up by the Ministry of Education for reviewing textbooks before they can be published and used in schools.

Poland – During communism, there was only one text approved by the government. Now, any person is free to write a textbook, but the book must be approved by government "experts". There is a perception that the market has been flooded with textbooks of varying quality as the guidelines appear to be very loose.

Finland – There are three publishers and a de-centralised curriculum. There is a lot of variation between books in terms of organisation, appearance and price.

Denmark – Most of the authors of textbooks are teachers, but this has progressively changed to include authors who are involved in teacher education and/or mathematics education research. Publishers pressure teachers to make their schools buy particular texts.

Through this discussion, the issue of pressure from publishers arose. There was general discussion around this. One participant from Sweden voiced the concern that teachers are too dependent upon textbooks and that students spend too much time doing textbook exercises rather than discussing mathematical concepts and issues. It was further noted that the most popular texts selected by schools were those that were the easiest for teachers to use.

One participant from the Netherlands stated that there seemed to be a big difference between primary and secondary school textbooks. This issue was marked for further discussion, but did not occur at this time.

A participant from Romania stated that there was tension between what teachers wanted and which texts contained tasks and activities that children could understand. Teachers preferred texts that provided challenging problems for the students, but sometimes students could not understand what was required by such problems.

Yeping Li asked the question: What should be placed in a textbook? What kinds of pedagogical features are used by textbook writers?

Stefan Turnau raised the issue of teacher reliance on textbooks. He posed the following question: Could we foster innovation in the classroom through textbooks?

During this first session, two people provided an overview of their research into mathematics textbooks. Yan Zhu spoke on "An analysis of the representation of problem types of Chinese and US mathematics textbooks" (co-author: Lianghuo Fan). Anna Brändström spoke on "Differentiation in mathematics textbooks" (content and use of textbooks with students of varying ability). The two papers were pre-reviewed and chosen by the organisers to stimulate discussion.

Session 2

Focus: The role of textbooks in teaching and learning

Shelley Dole introduced this session by outlining the use of textbooks in Australia (where any person is free to write a textbook, and the publishers market the books to schools), and also overviewed her research on comparison of two popular Australian textbooks and their presentation of a particular mathematics topics (ratio and proportion). From her research, Dole concluded that both texts provided a rather confused presentation which had few connecting elements to students' prior knowledge. Dole posed the following questions to the group:

- Are textbooks essential in the teaching and learning of mathematics?
- Should texts be written for teachers or students or both? (or parents?)
- Should texts be treated only as an information source or should they be treated only as an instrument for organising students' educational cognitive activity?

- Would reform in mathematics education proceed much quicker if textbooks were banned?

Prior to discussion of these questions, three participants from China Zhongru Li, Yongchun Wang, and Zaijin Tian outlined some of the features of the new editions of Chinese mathematics textbooks, and discussed some relevant issues. Then Jukka Törnroos presented an overview of his doctoral study into mathematics textbooks, opportunity to learn, and achievement. These overviews were all based on the papers submitted by the speakers. At the end of this session, there was some input from the participants as to the focus for the final session.

Session 3

Focus: Research in the area of mathematics textbooks

Lianghuo Fan lead this panel-discussion session with panel members Stefan Turnau, Shelley Dole, and Yeping Li. Turnau first provided an overview of the research focus of papers that had been presented to the DG 14 group for review:

- Literary-type research (analysing text-books)
- Direct-impact type research (impact of text on students)
- Clinical (or classroom-based) experiment
- Project type research on construction of textbooks.

He suggested that there were three other fields of research required:

- Project type research – verification of the effectiveness of textbooks on students' learning (school-based)
- Ethnographic – observations were carried out in classrooms of how texts were used
- Hands-on developmental research.

Discussion was then opened up to consider the impact of textbooks upon teaching and learning. It was generally agreed that mathematics textbooks had a role to play and that they were a valuable resource, though it was also noted that this was not always true and there were cases that students, especially in the primary level, in some countries (e.g., Australia) did not have textbooks in learning mathematics due to different reasons. There was general consensus that more research should be conducted on mathematics textbooks to inform both textbook writers and textbook users on how to make the most use of this resource for teaching and learning.

Concluding comments

Due to the good attendance of participants at DG 14 over the three days, there appeared to be a strong interest in this topic. As a new Discussion Group, attendance and interest warrants the continuation of DG 14 in future ICMEs. The organising team also feels there is a need to conduct an ICMI study in the future in order to promote the level of the development and research of mathematics textbooks.

The chairs will be happy to be contacted at lianghuo.fan@nie.edu.sg and sturnau@atena.univ.rzeszow.pl for further information on the work of this DG.

ICME 10 2004

DG

Discussion Group 15

DG 15: Ethnomathematics

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Team Members: ***Rex Matang***, University of Goroka, Papua, New Guinea
Daniel Clark Orey, California State University, Sacramento, USA

Aims and focus

According to the discussion document the DG 15 mostly aimed to provide a forum of participants to exchange their ideas and experiences in ethnomathematical research, particularly, those related to research on cultural aspects of mathematics education. Thus, the activities of DG 15 at ICME-10 aimed to discuss the following issues in the area of ethnomathematics:

1. What is the relationship between ethnomathematics, mathematics and anthropology, and the politics of mathematics education?
2. What evidence is there, and how do we get more, that school programmes incorporating ethnomathematical ideas succeed in achieving their (ethnomathematical) aims?
3. What are the implications of existing ethnomathematical studies for mathematics and mathematics education?
4. What is the relationship of different languages (or other cultural features) to the production of different sorts of mathematics?

The Organising Team was able to mobilise contributed papers from all five continents. More than 50 people from more than 30 countries participated in the DG activities. The discussion was mainly based on the 15 papers accepted by the Organising Team and made available to the public at www.icme-organisers.dk/dg15/. The papers can be roughly grouped, with respect to the above issues, in this way:

1. Maria Cecilia de Castello Branco Fantinato (Brasil): *Quantitative and spatial representations among working-class adults from Rio de Janeiro.*
   Gelsa Knijnik and Fernanda Wanderer (Brasil): *The art of tiles in Portugal and Brazil: Ethnomathematics and traveling cultures.*
   Hsiu-fei Sophie Lee (Taiwan): *Ethnomathematics in Taiwan – A Review.*
   Charoula Stathopoulou (Greece): *Mathematical cognition in and out of school for Romany students.*

2. Franco Favilli and Stefania Tintori (Italy): *Intercultural mathematical education: Comments about a didactic proposal.*
   Giuseppe Fiorentino and Franco Favilli (Italy): *The electronic Yupana: A didactical resource from an ancient mathematical tool.*
   Laura Maffei and Franco Favilli (Italy): *Piloting the software SonaPolygonals_ 1.0: A didactical proposal for the GCD.*
   Mogege Mosimege (South Africa) and Abdulcarimo Ismael (Mozambique): *Ethnomathematical studies on indigenous games: examples from Southern Africa.*

3. Pierre Clanché and Bernard Sarrazy (France): *Occurrence of typical cultural behaviors in an arithmetic lesson: how to cope?*
   Philip Clarkson (Australia): *Multicultural classrooms: contexts for much mathematics teaching and learning.*
   Maria do Carmo Santos Domite (Brasil): *Notes on teacher education: An ethnomathematical perspective.*
   Issic K.C. Leung, Siu-hing Ling and Regina M.F. Wong (Hong Kong): *Students' Mathematics Performance in Authentic Problems.*
   Jerry Lipka and Barbara L. Adams (USA): *Some Evidence of Ethnomathematics: Quantitative and Qualitative Data from Alaska.*
   Rex Matang and Kay Owens (Papua New Guinea): *Rich Transitions from indigenous counting systems to English arithmetic strategies: Implications for Mathematics Education in Papua New Guinea.*
   Daniel Clark Orey and Milton Rosa (USA): *Ethnomathematics and the teaching and learning mathematics from a multicultural perspective.*

Even if no papers were completely or mostly related to issue 4, in some of them the relationship between different languages and different mathematics was clearly pointed out.

Based on the assumption that the participants had made themselves familiar with the papers before the congress, the DG 15 activities were carried on as follows:

- As only the two Team Chairs of the Organising Team were present at the first session, it was decided to split the session in two subgroups. It was also decided to give an opportunity for the participants who had submitted a paper and were present at the session, to give a very short oral presentation of their paper, in order to initiate and enhance the discussion. The participants joined the subgroups in accordance with their specific interests. In these parallel subsessions and in the second one (see below), papers from each of the three groups of papers, mentioned above, were made available for the discussion.
- In the second session, the two subgroups met together and there were some more short oral presentations followed by a short discussion on each paper and finally the general discussion.
- The final session was completely devoted to reporting and general discussion.

The accepted papers for the DG and their short presentations raised a large variety of issues within the area of ethnomathematics, in general, and the issues set out by the Organising Team for the discussion, in particular, as shown by their summaries:

First group of papers

Fantinato's paper reports some of the results of a piece of ethnomathematics research developed with a group of low-educated adults, living in a poor neighbourhood of Rio de Janeiro. The research aims to understand quantitative and spatial representations built and used in different life contexts, as well as relationships between these representations and school mathematical knowledge. Results show a strong association between the use of mathematical skills in daily life and survival strategies to satisfy basic needs such as managing a reduced budget. This appears to be related also to emotional factors like protecting one's identity.

Knijnik & Wanderer's paper discusses some aspects of the relationship between Mathematics Education and art, focusing mainly on the study of Portuguese tiles, which were brought to Brazil in the colonial times. In Brazil they were re-appropriated in a special way and later on came back to Portugal influenced by that hybridized form. The paper shows the curricular implications that can be established through the links between pedagogical processes involving isometries and the fruition of art.

Lee' s review paper argues that multiculturalism having been a trend in educational reform around the world, Taiwan is not exempted from this trend. Indeed, multicultural curricula have been implemented in Taiwan from the elementary to college level of education. However, when compared to the concept and implementation of multicultural curricula, ethnomathematics appears to be an emerging new concept and has not been extensively studied yet.

Stathopoulou's paper presents a study in which first hand material, collected on the spot in a multicultural community in Athens, is used to demonstrate the relation between the mathematical cognition acquired by Romany people within their community and mathematics learning of Romany students in school context. The fact that the formal education contemns or ignores the special cognition with which Romany students come to school is connected not only with their low school aptitude but also with the preservation of their marginal role in school as well as in the society.

Second group of papers

In their paper, *Favilli* and *Tintori* argue that the practical implementation of the theories developed in the area of ethnomathematics research and culturally contextualized mathematics education does not seem to have been devoted much attention in some countries where multiculturalism is a relatively recent educational requirement. This paper presents some considerations made by mathematics teachers and their pupils after piloting an intercultural and interdisciplinary didactic proposal related to the construction of a *zampoña* (the Andean flute) and elaborated in the context of a European project.

Fiorentino and *Favilli's* paper introduces an electronic version of the *yupana*, the Inka abacus. One of the paper aims is to show that it is possible to make ancient mathematical artefacts attractive and usable, thus proving their present didactical utility. The electronic yupana represents an attempt to link tradition and modernity, indigenous and scientific knowledge, poor and rich cultures. It represents an educational environment, a friendly tool through which pupils can achieve the notion of natural number, compute basic operations, familiarise with positional notation and base change, and develop personal "algorithms".

The paper by *Maffei* and *Favilli* presents a didactical unit designed within a research project on arithmetic. The unit objective is to introduce the notion of the Greatest Common Divisor through *sona*, sand drawings from African culture, and their representation by an appropriate software. A brief description of the project framework, the practice of the *sona* and the guidelines of the didactical proposal, as well as a sketch of the main characteristics of the *SonaPolygonals_1.0* software are presented. The first findings of a pilot project at a few lower secondary schools are also discussed.

In their paper, *Mosimege* and *Ismael* present studies on a variety of indigenous games carried out in regions of South Africa and Mozambique. It studies the use of these games in the classroom, with a variety of activities for the purpose of learning mathematics. Such games are usually viewed from the narrow perspective of play, enjoyment and recreation. However, analyses of games reveal complexities, such as their origins, socio-cultural contributions to societal and national activities, mathematical concepts associated with the games, general classroom related curriculum development possibilities and implications.

Third group of papers

Clanché and *Sarrazy's* paper offers a topic for discussion. They present a reflection based on observations of an arithmetic lesson, in which certain typical cultural behaviours occurred. The study is based on the assumption that mathematics must now be understood as a kind of cultural knowledge, which all cultures generate but which need not necessarily look the same from one cultural group to another.

Clarkson's discussions paper calls for attention, particularly, to the issue that many if not most mathematics classrooms are micro-sites of multiculturalism. Hence notions of ethnomathematics are in play whether it is acknowledged or not. However, the fact that there are often multiple cultures and languages represented means that the learning and teaching carried out in these classrooms is more complicated than in mono-cultural classrooms. The notion of multiple contexts of classrooms, because of the variety of possible combinations of cultures and languages present, is emphasised as a potentially important factor that has not been recognised in research so far.

Domite's paper presents a reflection on a ethnomathematical perspective of teacher education. Based on one example of a teaching situation with a teacher in which a different approach to division was presented by a child, based on out-of-school experience. The aspect underlined in this paper might help student teachers to get inside the world of the students, thus indicating how student teachers could be educated in order to become aware of the mathematical background of their students.

Leung, *Ling* and *Wong's* paper on a preliminary research project carried out in Hong Kong, refers to a distinction between ethnomathematics and the formal way of learning mathematics in schools, stressing the richness of ethnomathematical activities. It presents the idea of linking mathematics and the real world through what the authors call authentic problems. Their result suggests that authentic problems could lead to a better learning environment in mathematics.

The paper by *Lipka* and *Adams* presents the first results of a project aimed at improving the academic performance of Yup'ik and other Alaskan students, and to alter the politics of exclusion by including elders' knowledge in the school math's curriculum. Quantitative data derived from a research design shows that the Alaska native students who engage in this culturally-based curriculum outperform comparably matched groups of students who use their school mainstream curriculum only. Further, qualitative data shows some of the teacher-student, student-student and school-community effects of this curricular effort. The study also shows that the culturally-based mathematics modules appear to motivate and increase students' interest.

Matang and *Owens'* paper explores the possibility of utilising and building on the counting and arithmetic strategies embedded in the country's 800-plus traditional counting systems. This is based on the commonly accepted educational assumption that learning of mathematics is more effective and meaningful if it begins from the more familiar mathematical practices found in the learner's own socio-cultural environment. Based on the basic number structures and operative patterns of the respective counting systems from selected language groups, the paper briefly describes how the rich diversity among these language groups can be used as the basis to teach basic English arithmetic strategies.

Orey and *Rosa* argue for a distinction between ethnomathemics and multicultural mathematics. They warn against using ethnomathematics as a lead into Western mathematics. Taking into account that the basic tenant of ethnomathematics is the sincere belief that all people use mathematics in their daily life, not just academic mathematicians. Yet, globally speaking, not all people have regular access to or attend school. Ethnomathematics as a program of study offers one possibility – allowing researchers to examine what and how mathematics is taught in the context of school, culture, and society.

The discussions gave the DG participants the opportunity to reflect upon additional issues related to those dealt with in the papers. These issues include:

- the relation between mathematics education in its cultural context and the theory of didactic situations;
- the relevance of using indigenous mathematics in the classrooms;
- the role of different counting systems in basic arithmetic;
- the importance of making cultural and historical mathematical activities, tools and artefacts available to the classroom in a modern way, respectful of the tradition;
- the interaction between the various languages used in an increasing number of classrooms (language of education, different mother tongues, mathematical language).

As a conclusion, it must be stressed that the richness of the contributions and the variety of issues provide evidence of the great vivacity of the international community of ethnomathematics, a field of study that keeps attracting an increasing number of scholars.

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DG 16: The role of mathematical competitions in mathematics education

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Team Member: ***Petar Kenderov***, Bulgarian Academy of Science, Sofia, Bulgaria

Aims and focus

This discussion group was organized by the two chairs, Peter Taylor and Frédéric Gourdeau with the collaboration of Petar Kenderov making up the third member of the team. André Deledicq (France) had originally been appointed chair with Peter Taylor, and helped design the program, but he had to give up his role before the congress and his place was taken by Frédéric Gourdeau.

The program had two two-hour and a single one-hour sessions The first session, with invited introductions by Andrejs Cibulis and Dace Bonka (Latvia) and Peter Crippin (University of Waterloo), focused on the range of competitions and related activities which are available. The second session, with an invited introduction from Andy Liu (University of Alberta) discussed the relation between competitions (and related activities) and the teaching and learning process. The last session summarized the previous proceedings with a view to writing a final report.

The discussion group was well-attended, not only by regular participants in World Federation of National Mathematics Competitions activities but also by many different people, mainly teachers and educators from countries in Europe. About 40 people attended each of the first two sessions and about 12 attended the last session. It is estimated that between 60 and 70 people attended at least one of the sessions. The following report has been prepared by the organizing team based on the discussions and after all participants who left their email address had had a chance to comment.

What are competitions?

In recent years the meaning of the word "competition" has become much more general than the traditional meaning of either a national olympiad, or more broadly based multiple choice question exams which have become popular in a number of countries. The World Federation of National Mathematics Competitions, the principal international body comprising mathematics academics and teachers who administer competitions, has formally defined competitions as including enrichment courses and activities in mathematics, mathematics clubs or "circles", mathematics days, mathematics camps, including live-in programs in which students solve open-ended or research-style problems over a period of days, and other similar activities.

In addition there are publications of journals for students and teachers containing problem sections, book reviews, review articles on historic and contemporary issues in mathematics in addition to support for teachers who desire and/or require extra resources in dealing with talented students, were also important activities related to competitions.

Competitions come in a number of categories, the elite national and international olympiads, the broader and popular inclusive competitions usually involving (regretfully) multiple choice questions, and special themed competitions, which sometimes involve teams rather than individuals. In some cases, these teams are composed of whole classes, giving a very different feel to the competition. Special note was made of project, or research based activities, in which students have a longer time frame to solve problems than normally permitted in an exam-based environment. In addition to purely mathematical competitions there also exist competitions focusing on mathematical modelling.

These activities all have in common the values of creativity, enrichment beyond the normal syllabus, opportunities for students to experience problem solving situations and provision of challenge for the student. Competitions give students the opportunity to be drawn by their own interest to experience some mathematics beyond their normal classroom experience.

Competitions are usually administered by teachers on a voluntary basis beyond their required duties. Administering bodies are usually independent of the standard curriculum and assessment bodies.

How competitions contribute positively to the teaching and learning process
There are many ways for this. Competitions provide, for example, a focus on problem solving, sometimes giving students an opportunity to be associated with a cutting edge area of mathematics in which new methods may evolve and old methods be revived.

Competitions provide opportunity for creativity and independent thinking, as students often solve problems in unexpected and innovative ways. The success of competitions over the years, particularly the resurgence in the last 50 years, indicates that these are events in which students enjoy mathematics. Different students derive different experiences, and it is exciting for students when they see how a problem can reach the same solution by two quite different techniques. Because competitions give students an opportunity to discover a latent talent, they provide a stimulus for improving learning.

Paul Erdös was reported to have said about competitions that the most important thing about them was the enthusiasm they generated. For many participants in popular contests, the aim is not to win, but to take part, thus taking up the challenges provided. Olympiads provide higher mountains for the more able students to climb.

Discussants had various attitudes towards competitions. Some preferred individual competitions, others felt it was positive for students to develop a competitive attitude. Many strongly supported team competitions and competitions involving interactivity.

Some time was spent discussing the creation of problems and the importance of creating problems with good structure that can capture the imagination.

Assessing negative images of competitions
A number of criticisms are often made of competitions. These include claims that competitions are only for the elite, they involve pressure and stress, widen the knowledge gap between students, are a negative experience for many students, and display a bias towards boys.

The discussion group did not engage in a detailed discussion of these criticisms as competitions vary and have different objectives and formats. Some participants argued that for competitions to have a positive impact, teachers must see the progress made by their students. In this view, the role of competitions is to develop a critical body of kids who can do problem solving: in a sense, this role is to get people interested. This suggests that a different view of competitions may be needed. For some, the suggestion that doing mathematical competitions had a negative impact on many students was not borne out at all by their experience of broad-based mathematical competitions. However, the International Mathematical Olympiad teams do contain predominantly more boys than girls. (Apparently, evidence shows that average scores of boys and girls are similar and that boys show a greater standard deviation.) In contrast, evidence from large, broad-based competitions indicates at least equal participation by girls, at least up to the age of about 15. This differing participation of male and female needs more research and better understanding.

Entry in competitions is usually voluntary; students' performance does not usually affect their regular school assessment, and, if anything, gives the student an opportunity to discover talent (as argued in the previous section). One teacher noted that elite students in mathematics often do not act elitely with respect to their peers and that there is much less social pressure in mathematicts than for instance in sport.

Collaboration and support for teachers

Finally there was much discussion about this theme. In particular Peter Crippin in his invited introduction mentioned that competition organizers are now focusing increased attention on various forms of support for teachers.

The competitions themselves, often available with solutions and grading advice, provide vast resources for class room discussion. Material available to teachers should not just include problems and solutions, but should be well structured, with good advice on practical use. Some competitions even provide didactical notes so that teachers can know what type of solutions to expect and how to use these in their teaching. Many organizations which run competitions also now run seminars and workshops for teachers.

This report was written by Peter Taylor, Frédeéric Gourdeau, and Petar Kenderov. They are happy to be contacted at pjt@olympiad.org, fredg@mat.ulaval.ca, kenderovp@cc.bas.bg for further information on the work of this DG.

ICME 10 2004

DG

Discussion Group 17

DG 17: Current problems and challenges in pre-school mathematics education

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Team Members: *Carol Aubrey*, University of Warwick, United Kingdom
Marj Horne, Australian Catholic University, Fitzroy, Australia
Ingvill Stedøy, Norwegian University for Science and Technology, Trondheim, Norway

Aims and focus

With participants from Australia, Canada, England, Sweden, Norway, and the United States in attendance, the group availed itself of this opportunity to build global perspectives on early childhood mathematics education. The central purpose of DG 17 was to support productive dialogue about important current problems, issues and challenges relevant to young children's mathematical development. Research of the last few decades has made important steps to clarify how young children think, behave, communicate, construct their worlds and reason differently than adults or even older children do. Further, the range of contexts in which younger children build ideas and learn, and the variety of adults and older peers who interact with them across these contexts, present important special features of their own. To make the most of the particular qualities, strengths and challenges that contribute to pre-school mathematics learning, the DG 17 organizing team drew from data and analyses that emphasize listening to and observing young children closely in everyday practice. Through this discussion group process, the group aimed to co-construct what such research might tell us about young learners' mathematical development, and what such research might imply for policy and practice. In particular, discussion was expected to be strengthened by the diversity of conceptual approaches being taken, across a wide variety of settings, with the potential to bring wider substantive and methodological issues to the foreground. In essence, the group set out to examine what we might know (or still need to know) about the focus questions, "where", "how", "who", "when", "what", and "why" of young children's mathematical engagement.

Session 1

Session 1 was opened with two striking instances to stimulate discussion on the first three focus questions. Carol Aubrey shared some examples of child-solitary math related speech and joint dialogue with a parent, focusing on the eighteen-month to three-year period. This related to recent work examining the contexts, early pedagogical strategies and linguistic inputs that pave the way for later mathematical development as well as foster expectations and attitudes to future learning (Aubrey, Bottle and Godfrey, 2003). Ann Anderson showed a video excerpt in which a three-year-old boy, using 7 flat sticks (i.e. sidewalks), constructed several different outlines of parking spaces while his mother supported his actions with general discussion of parking lots they had visited. This episode was one of many captured in a longitudinal study of supportive environments for mathematics learning in the home (Anderson, 2003). Once these two researchers had shared parts of their work, participants broke into small groups to reflect.

Where: the settings or contexts where pre-school children might think mathematically. For instance, when we speak of mathematics education of young children, it seemed best not to restrict attention automatically to young children's activities and communication in formal settings such as pre-school classrooms or day care centres. It also seems important to learn more about young children cared for in less formal environments, including such non-school settings as at home, or outside the home in museums, science centres, outdoor activities – in all situations where mathematical concepts can be an issue.

How: the ways in which pre-school children learn/engage with mathematics. For instance, mathematical or mathematics-related thinking can emerge for young children in everyday events (in play, through social interaction, informally, embedded) where children and adults may reason mathematically, yet not necessarily call such thinking mathematical. In the early years, play is central to how children live in and understand their world. But how much do we know, or still need to know about playful or informal mathematics? In particular, we might enquire more systematically about the influence of people around the youngsters, and about how such people can help children see mathematics in the world around them. One aspect of learning is actually attending to something, for example by drawing a child's attention to a potentially productive issue by a question or through an aspect of structure in a game or activity, without directly teaching. How important might this aspect be, and what might be its contribution?

Who: the important others (adults, siblings, peers or friends) with whom young children interact. How might better understanding of significant adults (including parents and a wide variety of caregivers and teachers) help us to support or assist young children most effectively? It seems helpful to learn more, in detail, about their mathematical education, knowledge and capacity, and about their understanding of young children's intellectual development. It may also help to understand, through comparative studies, more clearly in what ways might mathematics educators' access (or lack of access) to such adults affect the quality and strength of pre-school children's mathematical learning. Such research might clarify to what extent we have the capacity (including needed understanding) to educate adults more broadly, and suggest potential forms that such education might take. For example, could information disseminated by health care centres (such as the Swedish National Centre for Mathematics Education (NCM) project "Mathematics from the Beginning") be helpful?

As this session came to a close some commentary on the parking lot episode that arose in the small groups was debriefed. For example, there were several comments on the parent-child dyad that (i) emphasized the mother's capability to hold back and encourage her child's activity without directing it in any particular fashion; (ii) focused on the spatial understanding that the child demonstrated as the task unfolded and he parked cars both between, to the side of, and at the end of "sidewalks" so that formations divided space in multiple ways; and (iii) indicated that the boy also seemed to engage in goal-directed critical rethinking, which his mother seemed to welcome. Such comments seem especially important here because they emphasize the need to step outside the usual school and daycare settings, to consider the strengths that other caregivers, such as parents, might bring to helping children learn. Further, they suggest a power and richness in young children's thinking that may be greater than has been widely believed.

Session 2

Session 2 was opened with two striking instances to provoke discussion on the next three focus questions. Ingvill Stedøy shared experiences of her team at the Norwegian Centre for Mathematics Education, as they developed math clubs for five year olds. Two masters students have studied how the children developed their communicative competences about mathematical issues. They also pointed out the importance of letting the children's own thoughts and questions lead the communication. The aim for the clubs has been to help them discover mathematics in their own play, games and daily lives, to give them a view of mathematics that is different from the picture they may get from parents and older brothers and sisters. In this way it is likely that they will be more open for a variety of ways and places to learn mathematics. They will also think of mathematics as fun and natural, not scary or hard. Marj Horne shared interview footage of a young girl, age 5, counting a collection of over twenty small teddy bears. She clearly demonstrated the one-to-one correspondence aspect of counting and the idea of cardinality, but made an error in the rhyme, skipping the number fifteen. Over 1000 five year old children were interviewed on beginning school and over 40% of them counted a collection of over twenty teddy bears successfully. It would help to learn more about what kinds of past experience contribute to such knowledge, and about the related understanding of number that has developed in such children. Similarly it would help to learn more about other kinds of knowledge that they bring with them to school, and about the impact of such past experience on children's subsequent mathematical development in formal school. Such episodes as these reminded us of the diversity in children's strengths and of their confidence in solving interesting problems. Again, small groups discussed these striking instances in light of the focus questions provided for this session.

When: The time frames or age ranges that delineate young children's mathematics. When we speak of mathematics education for pre-school children, are we implicitly envisioning children who are 4 and 5 years old? Perhaps the answer varies across countries. In some places there are pre-schools attached to schools, and in other places not. If we include still younger children, including both non-verbal (birth through one year or so) and early verbal years (through, say age three) it would surely help to locate relevant special opportunities and challenges that might arise. For example, how might the choice of stimulating games and focus of communication help children prepare for later mathematical challenges before or in school? Responses to such questions may not depend just on place, but also on the different ways that programs for young children may be focused.

What: the nature of the mathematics in young children's own emerging worlds. We need to understand more clearly the nature of the mathematics (such as the implicit vs. explicit presence of quantity and space in everyday activities) that young children can introduce, work with, and explore in problem situations. In particular, how might we best conceptualize the particular strengths we find in younger children's successful mathematical activity? It seems quite unlikely that such mathematics (or perhaps emerging mathematics) will be limited simply to counting or pre-number tasks, as we often see suggested in the media. Considering the verbal and social skills of younger children (including potential opportunities, as well as challenges, presented by linguistic and cultural diversity) we recognize a need to understand more systematically the obstacles or opportunities that the current emphasis on talk in mathematics pose for children in this age group, and for adults who work with them, and to explore alternative perspectives that may be more helpful for young learners and adults who work with them.

Why: questions of motivation and engagement, both for young learners to engage with mathematics deeply, and for important others to see such mathematical engagement as important. Here we sensed a need to see more studies that seek, in a variety of settings, to clarify what significant, highly motivated mathematical activity by young children looks and feels like. What motivating factors (including the design of tasks or situations with which young learners engage) can be shown to support young children's rich, extended mathematical activity or exploration? What attitudes and beliefs do young children have about their mathematical learning, and how do adults and older children who work and interact with them regard such learning? In what ways do important others (whether age-mates, older children, or a variety of adults) contribute (or perhaps not contribute) to young children's growing mathematical engagement? It could be equally important to address how some adults and other children do not contribute to or do not stimulate young children's mathematical engagement.

To bring this session to a close, a representative from each small group shared highlights of their discussions. In essence, the following points were made. Interestingly, discussants found themselves posing further questions rather than proposing answers. For instance, in response to "what maths?" one group wondered "should we emphasize the structure? If so, how much should we do? Who should introduce the structure? When should we do this?" As they reflected on "Why bother with maths?" this same group offered "because the kids are interested; because it helps them organize their world." A second group began their discussions around "when to do maths?" and responded with "2-3-4-or 5. It depends on the child's current knowledge and the child's interest. Pre-schools without a formal "curriculum" have the opportunity to "do" the appropriate maths." A final reflection for this group was characterized as "big problem: How do we support pre-school teachers to do maths?". A third group responded to "when?" with ages and the philosophical stance, "birth to three and three to six curriculum should include mathematics; intertwining and progression – of tasks, of children, of mathematics, in different dimensions." As for "what opportunities and challenges?" this group simply (and profoundly) said "who knows … what and where to challenge". With respect to "why", they felt that "young children have an endless drive, energy and curiosity to explore with joy," and their "teachers' positive attitudes and beliefs confirm why we should bother with math." A final group posed two further questions that evolved from their discussions, namely "how do we draw mathematics out of authentic activities? And, what are authentic activities?" as well as "what can we learn from different cultures according to how children are viewed?" As the group debriefed further, participants were asked to visit the posters display to view the Swedish NCM project *A pilot project in pre-school: math for teachers and children aged 1-5* as examples of this project teams' experiences were arising in the discussions.

Session 3

The final session was opened with two striking instances to provoke discussion on our final focus question and two sub-questions. To what extent is it desirable to expose preschool children to structured or institutionalized mathematics teaching? What aspects of mathematics should be taken as significant for very young children? What ways other than formal curricula are there to organize thinking, practice and research to support young children's mathematical engagement? Herbert Ginsburg shared video excerpts of pre-school classroom episodes where *Big Math for Little Kids* activities (Belfanz, Ginsburg, & Greenes, 2003; Greenes, Ginsburg, & Belfanz, 2004) were in progress. He showed an excerpt in which a pre-school teacher engages her 4 and 5 year old children in counting from 1 to 100 during circle time. The method is to pause each time the children have reached a number ending in 9 (like 19, 39), and to encourage them to learn the subsequent decade number (20,40 in the example cited) and then to construct the next 9 spoken numbers by applying the rule "add 1,2,3, … 9 to the decade number". The teacher also encourages the children to relate each spoken number to the corresponding written number in a large hundreds chart in which each row ends with a number ending in 9. Ginsburg claimed that children learn from this activity that the spoken words are rule-governed in important and mathematically significant ways. Robert Speiser shared a project in which five year olds had been encouraged and supported to photograph patterns and shapes that they found interesting, using disposable cameras. Once these two examples had been shared, participants engaged with the presenters on them. For instance, looking at a selection of the children's photographs, participants commented on what they saw as a powerful geometric emphasis in many pictures, especially quite striking perspective effects. Again participants noted the richness of young children's spatial and perceptual experience. To bring some sense of closure to the discussions, the whole group attended to this session's focus questions: *Given the recent development of formal curricula for pre-school mathematics education in some countries, and given what we now might know about how young children think and learn, to what extent is it desirable to expose pre-school children to structured or institutionalized mathematics teaching?* This question is critical. Many adults who work with young children do not see many things as mathematical that we view as mathematical activity. *What aspects of mathematics should be taken as significant for very young children? Are there further ways, other than formal curricula, to organize thinking and practice to support young children's mathematical engagement?* Interestingly, group members found themselves, as a collective, rather dissatisfied with the wording of the first question, *"To what extent is it desirable to expose preschool children to structured or institutionalized mathematics teaching?"* and wrestled with its rewording. The group was happier with, *"Is it desirable for teachers (caregivers) to help expose pre-school children to a range of activities for children to develop and interact with focused mathematical ideas?"* The organizing team for DG 17 welcomes the enthusiasm participants exhibited about the possibilities and strengths that young mathematics learners offer us as educators. Although no consensus was sought (if even such a thing were possible!) the discussions seemed likely to inform participants' future steps individually and collectively, not simply as researchers and practitioners but also as policy developers, whose decisions might support young children's mathematics education significantly at local, national, or even international levels.

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Doverborg, E. (2004) *A Pilot project in pre-school: math for teachers of children, aged 1-5.* Poster presented at ICME-10, Copenhagen, Denmark.
Greenes, C., Ginsburg, H.P., & Belfanz, R. (2004). Big Math for Little Kids. *Early Childhood Research Quarterly,* **19** (1), 159-166.

One paper submission was distributed to participants in DG 17:
Anderson, J., Anderson, A., Moffat, L., Shapiro, J. (2004) *Co-constructing understanding of literacy and mathematics: An analysis of parent-child discourse while playing a board game.* (University of British Columbia)

This report was written by Bob Speiser and Ann Anderson who will be happy to be contacted at speiser@byu.edu and anders@interchange.ubc.ca for further information on the work of this DG.

DG 18: Current problems and challenges in primary mathematics education

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Aims and focus

The importance of primary mathematics education cannot be overemphasized. It is in the primary years that students from any part of the world learn number concepts and numeration, shapes and figures, and basic measurement skills, among other beginning mathematical skills. Yet, ironically, in the primary grades mathematics learning becomes more problematic than could be expected. Indeed, it is never true that teaching primary school mathematics is without effort.

DG 18, whose task was to address *current problems and challenges pertaining to the teaching and learning of mathematics at the primary level*, prioritized among the many important issues the following on which the group should focus:

1. *The social aspects of mathematics learning*
   How do the opportunities of pupils who enter primary school change in different parts of the world? What structures and facilities are available to students (buildings, classrooms, books, libraries, technologies, school buses)? Will the gaps among the First, the Second, and the Third Worlds be destined to widen more and more?
2. *The primary learner's motivation, strategies, understanding, individual development of mathematical thinking, curricular and pedagogical concern;*
   Children are naturally excited and interested in mathematics in, say, the first three years of school. Then many get disenchanted. Why? What is it about mathematics in the first years that keep them interested? How important are word problems for the development of mathematical thinking, on the one hand, and the ability to make use of mathematics in every day life, on the other? What mechanisms should there be to connect mathematical contents in the school curriculum over the years?
3. *Teacher preparation*
   How should teachers be prepared to be more effective in the classroom, or what are the characteristics of good teachers, irrespective of country?
4. *Technologies*
   How are different parts of the world coping with the challenges of embracing and using new technologies such as calculators and computers in the teaching and learning of primary school mathematics?

Papers posted for discussion

A few papers were posted on the DG 18 website to provide some impetus for a stimulating discussion of the important issues.

Ewa Swoboda proposed the inclusion of more geometry in primary education. In most countries, much of primary mathematics education consists of number concepts and very little geometry. Geometry is one area of study that could enhance children's intuitive thinking; it emphasizes the study of patterns, to which children seem to have a natural inclination. Moreover, geometrical signs and symbols are between the abstract and the real thing, making geometrical language much easier to understand.

Ed Wall's paper tackled the issue of how pupils in elementary school should behave. How can we teach students in a mathematics class to be respectful listeners and encourage them to pay attention to the different reasoning of their schoolmates? Some of the key ideas put forth are active listening, listening event, anticipation, appropriation, harkening, and attuning. Such ideas make one think of the many instances in the classroom when teachers could have spent more time trying to understand what really went on in the minds of students who appeared to be listening.

Results from a study in Germany by *Klaus Hasemann*, revealed that an abstract-symbolic approach to developing insight in word problems among low-achieving second-graders helps improve their performance in solving word problems and their arithmetical skills. Surprisingly, low-achieving second-graders who went through a program that focused on students' real-life action related behavior had the lowest success. Could this be a formula to help low-performing students succeed in mathematics?

Victor Polaki reported a wide gap between the mathematics curriculum contents on number concepts and operations as articulated in the documents written by the Ministry of Lesotho and the teaching materials that students have access to. This gap essentially revealed that students in Lesotho are, potentially, not able to learn much of the said contents in the intended curriculum. The implication is to strengthen the capabilities of elementary mathematics teachers and enable them to bridge the gap by developing more appropriate materials for their teaching.

A real novel idea to help elementary mathematics teachers overcome their insecurities about mathematics was discussed by *Virginia Keen*. She proposed that by giving preservice elementary teachers the experience of writing children's books that focus on mathematical ideas, they can overcome their fears of mathematics and even become more excited about it. What other ways can we use to help elementary teachers, who are often non-specialists, become more confident and competent in teaching mathematics?

Michaela Kaslova deals with approaching algebra in primary school and, mainly, working with letters and substitutions. The provisional conclusion of the research is that better results are achieved using letters than using pictures. Pupils tend to interpret the use of pictures as 'games', while they attribute to the letters a more 'important' role in a mathematical sense.

The sessions

In the course of the two sessions many of the proposed topics were discussed. As regards the four issues, the issue on social aspects was not discussed directly but was rather more implicitly referred to. The themes of the second issue were lengthily discussed, mainly those concerning mathematical language. Such a fact highlights the accord on its importance, mainly in the primary school, in the building of meaningful mathematical concepts.

Issue 2: The primary learner's motivation, strategies, understanding, individual development of mathematical thinking, curricular and pedagogical concern

On students: A point was made that a main objective in teaching should be to develop a mathematical disposition in students and good habits of mind. In this connection, a question was raised whether or not dispositions can, indeed, be taught (or are teachable).

On the curriculum

Geometry: Geometry is an important domain for exploration as well as for building up and developing important ideas to support the mathematical thinking of children by directing them to find general rules. On the other hand, it is a fact that geometry is more or less ignored in primary mathematics all over the world. One reason for this fact might be that arithmetic more easily lends itself to testing than geometry; even if a child sees a pattern in a mathematical situation this does not mean that s/he has already grasped a general rule or a mathematical concept.

Numbers: The difficulties of numerous children with mathematics are partly due to using numbers exclusively as calculating numbers. The weaker pupils already in their very first grades should particularly learn (and need the teacher's help) to shape relations between numbers, and not just be restricted to the conception of numbers as quantities and to actions with quantities. This argument is in accordance with Jeremy Kilpatrick's demand in the Plenary Interview Session to have a good balance between the power of the concrete, on the one hand, and the abstract and conceptualisation, on the other.

Mathematical language:

- It was concurred that *all* children must be given the chance to use the *mathematical language* as a powerful and typical human ability.
- *Metacognition and metalinguistics*: Many researchers think that the mediation of the natural language – written and spoken – since the first years of the primary school must precede the formalisation and the reflection on the systems of symbolic notation peculiar to mathematics. One could synthesise two views:
  - On the one hand, to favour a math education based on metacognitive (as a reflection on processes) and on metalinguistic aspects (as a reflection on languages) is considered a strategy of increasing importance in order to build *meanings* with students.
  - On the other hand, a usual sentence of the teachers is: "I think that the discussion in math classes can be very useful but I am afraid to do it because I know when it begins but I do not know when it finishes".
- Is a meaningful compromise possible between the two positions? Is *discussion* in mathematics a widespread practice? Is it considered important? Can mathematics be used – also in primary school – as a means of communication? Can mathematics be seen as a *social activity*, deeply concerned with *communication*? Is this a challenge in order to improve learning/teaching of mathematics? Is a meaningful discussion in a class of 30 or 40 pupils, lifting their hands in order to answer to a question of the teacher, possible? Are they learning *to speak mathematically*?

- *Linguistic approach*: A determining role is attributed to the *linguistic* approach and to research that affronts the didactical developments starting from the *concept of algebra as a language.* This role becomes even more significant if it is associated with the hypothesis of an *early initiation to algebraic* education beginning from the didactical readings of the relations between arithmetic and algebra. Research does demonstrate just how students' limited arithmetic experience becomes an obstacle when learning algebra. It is thought that an earlier approach can reduce this difficulty. It is only recently that interest has been shown towards an early approach to algebra, thus there is not yet much documentation regarding this area. Questions and answers are being formulated, such as: How early should *early algebra* be? What are the *advantages and disadvantages* of an anticipated start? How are the answers to these questions connected to *theories* of cognitive development and learning, and to the cultural and educational *traditions* of teaching algebra? Which algebra and algebraic thinking aspects should be part of an early algebraic education? What consequences would an early algebraic education have concerning *teachers and their formation*?

High-stakes testing: High-stakes tests exert undue pressure on teachers, forcing them to emphasise competition rather than conceptual learning of mathematics. The challenge is to turn these tests around so that they can support conceptual learning rather than detract teachers from it.

Issue 3: Teacher preparation
A challenge was given to the group, namely, how to help teachers to develop an open learning environment. It was stated that this should be a starting point for discussion and research. Reference was then made to the use of the interactive or digitized whiteboards in classroom instruction as a tool to encourage open discussions in mathematics classes. Another challenge that was posed is to review preservice teacher programs, but in what direction? One idea is to focus preservice programs on building metacognitive skills of teachers who in turn could help pupils develop their own metacognitive skills. The question of how much mathematics content knowledge primary mathematics teachers need to learn continues to challenge us as well. For many teachers what is needed is a long-term commitment to professional development activities such as the workshops that were run to teach the national standards in the US. Finally, a "burning" question was asked: What does it take to become a good teacher of mathematics? A good teacher needs to be involved in a number of hours of professional development so as to improve content knowledge – enough time and proper emphasis is needed. This is a continuing issue in mathematics education. Suggestions for holding conferences to discuss the issue further and compare achievements of countries in this area were strongly supported.

On parental involvement: More and more it becomes imperative for parents to become active participants in their children's primary school education. For poorer countries at least, the problem has more to do with both parents needing to work and therefore lack time and opportunity to get involved. In others, experiences by teachers have shown that how parents view the role of the school affects their level of involvement in their child's education. The challenge is how to get all parents to support and get involved in their children's learning? One recommendation that was well-received was to educate parents on how their children's educational success could translate to the families' potential economic progress.

Issue 4: Technologies (IT)
The role of IT in primary education: All participants agree that children have to learn mental calculation; however, the grasping of pattern as a basis for elementary algebra might be supported by the use of calculators and computers.

Access to computers: One participant pointed out that a major issue in his part of the world is the mere getting *access* to computers. They may be plentiful in some countries, or at least more plentiful than in most countries. But, there are countries in which students have no access to calculators or to computers in learning mathematics which indeed is a major challenge.

IT and geometry: A couple of examples were given on the use of technology, especially in teaching counting and geometry. Reference was made to a recent series of primary school textbooks in Germany [Wittmann and Müller] in which excellent software is used in teaching, closely aligned to the students' textbooks, to teach counting, estimation, and various operations. There seemed to be a sentiment in the group that technology has a great potential in teaching geometry and that geometry could very well be learned effectively in an "open" environment. The same can be said of developing students' number sense.

A final word

The discussion group realised that the challenges and issues that have been identified will remain to be such, at least for the next few years. At ICME-10, DG 18 was unsuccessful in putting a closure to the many questions that have been raised in the course of the discussions. The group, therefore, believe that a continuing dialogue among the many players in primary mathematics education would benefit our primary mathematics students. The team strongly suggests that succeeding discussion groups on primary mathematics education continue to focus on the same issues and questions that were outlined for ICME-10.

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DG 19: Current problems and challenges in lower secondary mathematics education

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Aims and focus

The following issues were selected by the organizing team:

1. *Mathematical literacy and "mathematics for everybody"*

How do we define "mathematics for everybody"? Is this what we could name a "minimum curriculum"?

- Does it just include applied mathematics?
- How can teachers foster students' ability to apply mathematics skills to different contexts?

2. *Relationships between different levels of knowledge*

In a constructivist approach, mathematics should be taught through activities that invite pupils to reason, explain and justify rather than simply to memorize and imitate, in order to construct mathematical understanding. Nevertheless, memorizing and imitating are parts of the learning process.

- Is it possible to find a "right balance"?
- What are the relationships between computational skills and reasoning or understanding?

3. *Different approaches to geometry*

- What kinds of geometrical reasoning do 11-15 -year- old pupils develop?
- How will dynamic geometry software (for example *Cabri*) change the teaching of geometry?
- Are there important differences between countries in the way geometry is taught? (Inductive and deductive reasoning, modelling, application of software)

4. *What is the role of algebra in lower secondary school?*

- Should algebra be taught to all students? What aspects of algebra are of value to everyone? What should a minimum curriculum consist of? How do answers to these questions relate to regional or cultural differences?
- What do we expect of an algebra-literate individual? What are the values of algebra learning for the individual, especially in view of increasingly powerful computing capabilities offered by ICT systems?
- How can we reshape the algebra curriculum so as to have more immediate value to individuals? Can we identify explicit examples in contexts meaningful to students in which algebraic ideas have a clear and unambiguous value? Are there undesirable consequences of such orientations to algebra?

5. *The role of technology and electronic tools*

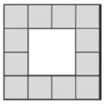
- How can the use of calculators and different software facilitate or – on the contrary – disturb mathematical learning? In which ways should such tools be used?
- Does the use of computers induce changes in curricula?

The two main challenges at the lower secondary level in many countries are the introduction of algebra and of deductive geometry. Therefore these topics were chosen as the main entries for the first and second 2-hour sessions.

First session

The question of mathematical literacy, mathematics for everybody, was posed as an introduction to the first session, followed by a workshop about the learning of algebra.

In order to launch the discussion on algebra, three tasks had been proposed for analysis. The first was the well known situation 'the border':



How to predict how many square tiles will be used to border any square?

This task represents an approach to algebra through its generalization function. It was pointed out that the problem may be set at different ages and levels, using material squares or letter symbols. Several algebraic expressions may be found by pupils, introducing the notion of equivalent expressions. The second task, entitled "tricks", has quite similar aims, although it is also connected with equations. It presents phenomena that can be explained by properties of operations and mastered by the use of letter symbols. For example:

Choose a decimal number
Calculate its double and its triple
Add the two results
Divide the result by 10
Given the final result, is it easy to guess the initial number?

The discussion centred on how this task could be used in the classroom. For example, after solving the task, pupils could be asked to create similar "tricks" and describe them by formulas.

A third task aimed at introducing the notion of equation.

Some participants were surprised to learn that the three tasks would be set at the same level in French classrooms (second year of secondary school, 12-13 year olds). The age for introduction of equations is different from one country to another, and this seems to be linked to different approaches to algebra. When generalization or functional perspectives are chosen for the introduction of algebra, the teaching of equations may occur later in the curriculum.

For lower secondary school, comparisons between countries are not easy because of the variety of conditions: lower secondary school may last from 3 to 5 years, the age of pupils ranges from 11 to 16, with different structures, one or several curricula, etc. There are countries where solving equations is not a part of compulsory teaching.

Second session

Time was shared between two topics: geometry and role of technology.

Maryvonne Le Berre, in a short overview, presented the problems linked to the introduction of deductive geometry, the relationships and opposition between practical geometry and deductive geometry, argumentation and proof, passing from drawings to figures, and the role of mathematization.

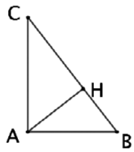
This was followed by a discussion about practices in proving the pythagorean theorem by means of the following questions:

Are the pupils given a mathematical proof, several proofs or only a demonstration by visualisation?
What kind(s) of proofs are used more often?
What are the reasons which lead to choosing one proof over another?

In the discussion the classification given by Christine Knipping [4] was used:

1) Proofs based on comparisons of areas,
2) Proofs based on calculation of areas,
3) Proofs applying theorems of similarity, like the following

$AB^2 = HB \times BC$
$AC^2 = HC \times BC$
Hence
$AB^2 + AC^2 = (HB + HC) \times BC = BC^2$



4) Proofs using visualisation of Euclid's proof, i.e. $a^2 = pc$ and $b^2 = qc$ (Turning squares into rectangles).

The four kinds of proofs seemed to be equally frequently used amongst the represented countries, but comparison of areas is more often used to present and explain the theorem (what is expressed in English by the word "demonstration") than for proving (in French "démonstration"). In contrast, in French instruction, this kind of proof may be chosen for the reason that pupils are able to master each step of the proof, one of the aims being to teach them how to prove statements. (Knipping [4] [5])

Some participants noted that pupils can't find any proof of the pythagorean theorem by themselves, and asked "Is this a good situation for proof?" A cultural perspective may lead to using different kinds of proofs in the classroom.

Finally, the discussion focused on *the role of proof in geometry. Again, t*he role of proof is different in different countries. Most people focus on understanding. Believing, knowing and proving are different levels that ought to be distinguished by pupils, but "knowing" is sometimes considered as sufficient.

In the second part of this session Merrilyn Goos introduced a set of issues on the role of technology of which the following were discussed.

Pedagogical issues:

- For what purposes is technology used?
- How does use of technology help or hinder students' learning?
- What teaching approaches are effective?

- How can tasks be developed that engage students with significant mathematical concepts?

Curriculum issues:
- How might technology change the content of curricula?
- What should be omitted? What should be added?
- What criteria should be used in making these decisions?
- Which students (and courses) should have access to technology?

Assessment issues:
- How can assessment be designed so as to recognise and test students' learning when technology is present?
- What is the role of 'technology-free' assessment tasks?

Participants responded with a range of comments and additional questions: For most curricula, it does not seem to be a case of "all calculator" or "no calculator". It seems to be fruitful to consider responsible uses of calculators. Students need some "no calculator" experience first. The question is: How much? Who decides what to leave out? Often it is people who already understand mathematics and learned it without calculators. What should be omitted? Using logarithm and trigonometry tables! What could be added? Studying iterative processes, complex experimental problems, exponential growth and suchlike! How might pedagogy change when the children have grown up with technology and computers in their environment? What learning and what teaching could come from electronic games? How can teachers capitalize on the abilities that pupils put to use in these games?

Third session

In the last one-hour session, there was a spontaneous discussion, from the definition of mathematical literacy given in OECD/PISA (OECD, 2003, p. 20):
Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays to the world, to make well founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.

Each participant was invited to propose answers to the question: *What should we expect of a mathematically literate person?* Responses included: Students should be problems solvers, and this involves communicating, reasoning, making connections, being creative. Students should be able to analyse situations, select data, pose mathematical questions from them, make investigations, evaluate, ask by themselves "what if?". Students should understand the news, the market place, measurement, basic probabilities. Affective issues are also important – what about enjoyment, confidence, perseverance, challenge?

Conclusion

In different countries, lower secondary school mathematics faces very different realities. This makes it sometimes difficult to identify and compare instructional choices.

In future ICMEs it might be relevant to create two discussion groups, one around the relation between primary and secondary school concerning pupils under the age of 13, and one discussing the relation between lower and upper secondary levels.

References

OECD (2003) *The PISA 2003 Assessment Framework – Mathematics, Reading, Science and Problem Solving Knowledge and Skills.* Paris: OECD.

The following papers are available from the ICME-10 website

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[2] Mathematics for Everybody; implications for the lower secondary school, *Steve Thornton University of Canberra, John Hogan, Redgum Consulting Pty Ltd., Australia*
[3] The differences between design intentions and implementation: The implementation of the Malaysian mathematics curriculum, *Noor Azlan Ahmad Zanzali, Faculty of Education, Universiti Teknologi, Malaysia*
[4] Towards a comparative analysis of proof, *Christine Knipping, University of Hamburg, Germany*
[5] Argumentations in proving discourses in mathematics classrooms, *Christine Knipping, University of Hamburg, Germany*
[6] How to deal with Algebraic Skills in Realistic Mathematics Education? *Monica M. Wijers, Freudenthal Institute, The Netherlands*
[7] Literal calculation and equations during French collège time, *Maryvonne Le Berre, IREM de Lyon, France*
[8] Misconceptions in Mathematics: Solving the Equation, *Suwattana Eamoraphan, Faculty of Education, Ghulalongkom University, Bangkok, Thailand*
[9] Affects and beliefs in school mathematics: gender differences, *Gard Brekke, Lise Wiik, Notodden, Norway*

This report was written by Maryvonne Le Berre. She can be contacted at iremlyon@univ-lyon1.fr for further information on the work of this DG.

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DG

Discussion
Group 20

DG 20: Current problems and challenges in upper secondary mathematics education

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Team Members: ***Gloria Stillman***, University of Melbourne, Australia
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Aims and focus

DG 20 provided a forum for participants to discuss current issues in upper secondary mathematics (USM) education. The team leaders proposed sample questions to reflect such issues, which were approved by other members of the organizing team (OT). These questions were grouped into four themes as follows:

A) *Research to practice and vice versa*: How do/can we make theoretical principles real in teaching USM? How can practice inform and develop theory? How can new theoretical trends influence practice? Are there any trends in preservice and inservice teacher education that can influence USM education research?

B) *Teachers and learners*: How does teachers' knowledge influence teaching and learning? What are valuable mathematical and pedagogical competencies of USM teachers? How do the different beliefs, values and cultural backgrounds of teachers or students affect teaching and learning? What are appropriate models of instruction and perspectives of learning?

C) *Tools and technology*: What are appropriate/meaningful uses of technology for USM? What can be the different roles of tools and technologies in the mediation of learning? How can the use of tools and technologies influence students' cognitive processes?

D) *Curriculum*: What are appropriate contents for students with different post-secondary goals? Can new theoretical trends influence school curricula? What are the new curricular trends recently developed in different countries?

Based on these themes, a call was made for papers. Four papers were received, reviewed, accepted and identified with the theme it best reflected. Each member of the OT was given the responsibility for further organizing and leading one of the themes. The activities of the group were structured as follows:

Session 1: The first hour was devoted to opening and overview of the DG. A panel of the OT of the DG addressed the themes. The DG was then divided on a voluntary basis into smaller groups according to the themes. About 60 participants representing a wide range of countries attended the DG – most choosing theme B or theme C. In the second hour, the smaller groups began discussion of their themes.

Session 2: In the first hour, the smaller groups continued discussion of their themes. In the second hour, representatives of the smaller groups highlighted the main points of their discussions in a large-group sharing, i.e., to the whole DG.

Session 3: This session was aimed at synthesizing the discussions and formulating statements about the possible common threads of issues in USM education.

There was no paper presentation in the DG. Papers accepted were posted on the DG website and participants were encouraged to read them prior to attending the DG.

Group activities by themes

Theme leaders were free to conduct their sub-groups in their own way and to determine issues for discussion. Thus each sub-group was unique in its activities as reflected in the following summaries prepared by each team leader as identified.

Theme A: From research to practice and vice versa (Gloria Stillman)

Making theoretical principles real in teaching USM: The teacher is the door to the students so teacher beliefs about the importance and relevance of theory are critical. Teacher-researcher projects may be a way to get started. These are highly practice based initially but then theoretical ideas and concepts mediate understanding of evolved changes in practice and the learning environment. Even when teachers are willing to incorporate theoretical principles into their practice there are obstacles such as high stakes assessment, which is externally controlled and set regionally or nationally. One view is that teachers *can do* when they *will it.* Alternatively, even if change is desired, the obstacles loom large. Student success in tests is uppermost in teachers' minds and this often drives practice. Researcher-driven projects need to be based in a genuinely collaborative environment where both theory and practice inform the research process and design. Teacher voices must be genuinely respected and supported in the research process. Possible solutions are: (a) school-based projects inspiring change instigated by outside research and curriculum experts or (b) formation of research-orientated teacher networks.

Practice informing and developing theory: Again, design research could be the answer here. This question gives rise to a further question that needs investigation: What methodological tools or research designs are necessary to *allow* this to happen?

Influence of new theoretical trends on practice: There are wonderful ideas about teaching in research and theory that might help teachers *see things differently* and allow them to think with these ideas; to see more insightfully; to acquire different tools for thinking and organizing their own experience and their own work than can be acquired from everyday life. However, there are lots of problems such as: (a) relevance of theory is not seen by many teachers, especially those with entrenched practices; (b) some theoretical ideas are not practical in classrooms; (c) teachers need to be in a place where they want to hear (i.e., where they have the desire to develop continually but also have a certain amount of current professional satisfaction); (d) teachers who take up research and are interested in applying theory in the classroom are seen as pioneers and allowed to go alone rather than find a ready source of collaboration amongst colleagues; (e) organizational structures maintain the status quo (e.g., no time to implement or money for ICT infrastructure).

Trends in preservice/inservice teacher education influencing mathematics research: (1) Opportunity to become teacher researchers, e.g., in Finland, where experienced teachers are given opportunity to research their own practice. These teachers are considered part of one research school but are doing degrees at several universities across Finland. (2) Postgraduate subjects where teachers read mathematics education research as consumers. (3) Researchers and lecturers from teacher education courses go to schools and cooperate closely with teachers bringing together researchers', lecturers' and teachers' viewpoints.

Theme B: Teachers and learners (Olive Chapman)

Participants of this sub-group were given the opportunity to determine issues of their own for this theme. In the first session, the sub-group formed three smaller groups and brainstormed responses to: "What are the most important issues/challenges pertaining to the teaching and learning of mathematics at the USM level?" Each small group identified key questions to share and to further discuss. In the second session participants worked in two groups and addressed: "How should these issues and challenges be dealt with?"

The group identified a unique set of issues that reflected the different professional contexts of the participants. These issues included:

- The conflict between covering content versus helping students to achieve true learning. There is a lot of curriculum content for the students to go through at too short a time.
- Teachers covering content based on textbooks and curriculum, and teaching for a test instead of considering what is mathematics and teaching for students' understanding.
- Teachers are not confident to take risks. They do not always feel sufficiently secure to allow students to take responsibility for their own learning. What kind of training do teachers need to take those risks?
- What are teachers to do with all the knowledge the students already have?
- Why should students learn "this" if a computer can do it? Do students need all of the mathematics they learn?
- How can we turn multi-cultural classrooms to our advantage, both mathematically and socially? How do we make it a positive learning environment for everybody?
- Encouraging/influencing students to take up mathematics at a higher level.
- Some teachers do not want to change teaching methods.
- Assessing for understanding.

Catherine Sackur's paper (DG 20 website) raised the issue of the challenge of making students responsible for the mathematics they learn.

There were no definitive answers to the above issues. However, some suggestions evolving from the discussions of actions that ought to be considered to address some of the issues are: More importance should be placed on process rather than product and on students' understanding of the mathematics rather than just learning a method. There should be increased emphasis on open-ended learning. Learning should be to promote active citizenship, i.e., to be able to form a coherent argument, to be critical. It is important to establish contact between teachers of all school levels. We should emphasise highlighting the usefulness of mathematics and not just passing exams as motivation of students; get students to write their solution process; and focus on quality not quantity.

Theme C: Tools and technology (Ornella Robutti)

Introduction to the theme focused on research results involving: (1) *Instrumental analysis* in which a device is considered with two interpretations: an artefact, i.e. an object constructed according to a specific knowledge, and an instrument, i.e. the artefact together with the schemes of use introduced by the user. (2) The distinction between the *symbolic-reconstructive* and the *perception-motor* ways of learning – the first based on mental reconstructions and decoding of symbolic messages, the second on activities in which doing, touching and perceiving are involved. (3) The notion of *mathematics laboratory*, not intended as opposed to a classroom, but rather a methodology. "In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together." (Robutti et al., DG 20 website).

The discussion of the subgroup, based on themes introduced by the participants, was very rich. During this discussion, the coordinator aimed only at supporting the participation of people and not at introducing new themes. The discussion centered on questions such as:

- How is it possible to help teachers not only to teach technological commands such as "push that button", "press a key", "write this number", but to teach also at a meta-level, transferring consciousness, awareness of the calculation, sequences, processes, …?
- Does the use of technology really change the mathematics we do at school?
- Does the use of technology promote curricular changes?
- What are the boundaries within which students can believe what they see with tools?
- Do teachers really change their ways of teaching with the use of technological tools?

For the first question, participants noted that what is important to construct is mathematical meaning, not command sequence. Students have to be aware of the fact that they are doing mathematics with the help of technology, and not vice-versa (see Nolli & Reggiani, DG 20 website). For the second and third questions, the crucial point is not the technology used or the algorithms implemented in it, but the way these are used at school. This implies the need for reconsidering methodologies, activities, instructional sequences and assessment. In fact, technology promotes educational and curricular change, for example, to diminish exercises based on rote manipulation, substituting them with problem-solving activities, or directing the attention towards graphs and representations, to a larger extent than what was done in the past. In this perspective, an important challenge for teachers enters the scene: every technology has its potentials and pitfalls. So, for the fourth question, students must be aware of those pitfalls, and learn how to manage them, mathematically checking the results given by technology. This involves new topics, e.g., estimation, discrete solution of equations, graphical representations. Therefore, technology may not only change the mathematics done at school, but also the way of thinking. It can call for discussion, conjectures, different feedback, and also theoretical knowledge, including proof. For the last question, maybe in some cases technology does not influence the *way of teaching*. There are teachers who use technology but teach in the same traditional way as they did without it. A possibility to improve the use of technology aimed at constructing theoretical thinking can be found in both preservice and inservice teacher courses.

Theme D: Curriculum (Carlos Vasco)

The USM curriculum seems to be very *homogeneous* in the countries known to the members of theme D-subgroup: algebra, geometry, trigonometry, analytic geometry, pre-calculus, and, for a few countries, calculus. Known variations seem to occur in five aspects of the curriculum: (a) amount of geometry (e.g., much in Japan, China, Russia and very little in most Latin-American countries); (b) fusion or separation of algebra and pre-calculus: more or less emphasis on functions in the second or third year of algebra or in the course on analytic geometry; (c) required status of calculus (for all students, apparently only in Colombia; for many students, as in the German "Gymnasium" and in the French "Lycée"; for few students, where only college-bound students take calculus as an elective subject); (d) in the introduction of descriptive statistics, and (e) in the substitution of something called "business mathematics" or "consumer mathematics" for those students who do not show potential for a solid pre-calculus or calculus course.

The USM curriculum seems very *stable*. It is very similar to what was found 50 years ago, except for the variations listed above and the widespread use of set-theoretic language during the last 30 years. Set theory as curricular content went in and then out in most countries, but the language stayed. In fact, the perceived stability seems so strong that it seems not to change much even by government curriculum reforms. Teachers' traditions, college-entrance examinations and textbook publishers manage to bring the taught curriculum back into place after a few oscillations. The only visible changes seem to be initiated not by academic or governmental decisions, but by the gradual introduction of ICT and pressure from television, fashion, parents, students, business, and journalists. The introduction of ICT does not change the content substantially, only the teaching strategies.

Finally, the low attendance of the curriculum subgroup raises questions about the conference participants' interest in this topic as a separate theme.

Conclusion

Given the uniqueness of each sub-group's discussions, it was difficult to synthesize and formulate statements about the possible common threads of issues in USM education during the one hour of the last session. Thus no overarching conclusions were reached outside of those drawn within each theme.

This report was written by Olive Chapman and Ornella Robutti with valuable support by the team members. They are happy to be contacted at chapman@ucalgary.ca and ornella.robutti@unito.it, respectively, for further information on the work of this DG.

DG 21: Current problems and challenges in non-university tertiary mathematics education

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History and common concerns

This Discussion Group was designed to provide a forum for mathematics educators from colleges of technology, junior colleges, community colleges, universities of technology, polytechnics, colleges of engineering, and other post-secondary institutions that are not traditional universities. Each of the types of institutions represented in DG 21 has functions unique to that type of institution and to the region it serves, but all serve the needs of students beyond secondary school in an environment unlike the university.

The international community of mathematics educators of such institutions came together as a program group (WGA 4, Fujita et al, 2004) for the first time at ICME-9 in Tokyo/Makuhari, Japan in 2000. The approximately 70 participants found many similarities in their institutions and many commonalities in their concerns. Regardless of the missions of these institutions, students are focused on immediate goals of an education relevant to the world of work and are often underprepared for their course of study.

Many issues and concerns about mathematics education at non-university tertiary institutions were raised during the presentations at ICME-9 and in the discussions that followed. In addition, all authors had been asked to submit a list of the three issues most important to them and their colleagues. The organizers collected these and compiled a master list that was distributed at the meeting in Japan.

Against this background the DG 21 organising team at ICME-10 called for papers that addressed the three most often mentioned issues: Faculty Development, Mathematics Curriculum, and Need for Research-Based Information. It was noted that faculty development can help faculty at their institutions keep pace with current trends in mathematics education and, also, have the capability to address the needs of their clientele and be aware of research and effective practices in teaching adult learners. The mathematics curriculum should have a strong content base and motivate students to engage in meaningful learning and prepare them for the workplace. Relevant research-based information can improve the educational system of two-year colleges and other non-university institutions. This includes research on curriculum, pedagogy, and workforce and student needs.

A forum for issues

While the above were areas of ongoing interest, several papers in ICME-10 were submitted on other topics. One such area was the process of transitioning students from secondary school to two-year or technical school to university. Many participants also had concerns about the level of students' preparedness and course placements.

The three time slots for DG 21 were used in the following manner:

Session 1: Sign in and introductions of the Organising Team and all participants; discussion led by Co-chairs of issues mentioned above and related concerns; topics to be discussed were outlined and abstracts and copies of papers under consideration were distributed.

Session 2: Organisers led continued discussions with more focus on curriculum and its relevance to students' lives and faculty preparedness to address the needs of their students.

Session 3: Summary of the discussions from the previous two sessions.

The need for research-based information and the transition from secondary institutions to college, college to university and college to work began with recalling a remark of a speaker at ICME-9 in regard to research in education. It was to the effect that it was difficult to do highly significant research in education because if you were researching anything worthwhile it was usually such a broad topic that it was not easy to control all of the variables but the experimental one. So we could do highly significant research on relatively insignificant minutia or do less well-controlled research on broad and significant topics.

Testing is receiving more emphasis in many countries. While the purpose of testing is principally to determine an individual student's level of preparedness, it also enables educators to learn more about how well students as a group are learning mathematics in a state, province, or country. Most mathematicians question whether the tests are measuring the mathematics that students need to know. In the USA, for instance, many colleges are using standardized exams to determine how much students have learned and also, to place students into college courses. The tests do a good job of testing what students have learned in the very traditional classes in high school but not necessarily what mathematics they need in college. Consequently students are allowed to enroll in courses for which they are not always well prepared.

It was noted that many countries represented in DG 21 do not use placement tests. In these countries all students start in the same course regardless of their level of preparation. In two-year colleges in the USA, students can start in the usual mathematics course or take a bridging course. Participants agreed that where students had no choice and all were required to start in a college level course, the success rate was much lower than in the case of students who were allowed to take bridging courses. In addition, students who take this course come out of the two-year college better prepared for the job market. Two-year colleges used to be the last choice for students in France. Because they offer students the opportunity to prepare themselves for college level mathematics and because they provide a more practical education, they are now the first choice. The system can accommodate only a limited number of students and not everyone can go to the two-year college where they have more options of preparing themselves for the regular college curriculum. So, many go to the university by default and fail.

It appears to be widely recognized that learning mathematics is not separate from social issues. Robert Moses, the great civil rights leader in the USA years ago began seeking equity for poor minority students through the Algebra Project aimed at getting more students on the college-preparatory track. ("Mississippi Learning", Jetter, February 23, 1993). This idea has been echoed by leaders the world over. Mathematics has been referred to by many as "the Gatekeeper" or "the Critical Filter," preventing many students from pursuing a college education or a future-oriented career. Students who are not adequately prepared in secondary school have difficulty transitioning to college.

Focus on transition

Measuring progress and readiness for classes (placement) are but two aspects of transitioning particularly from high school to college. Successful projects and aspects of successful projects were discussed by the participants. Many successful programs that aid entrance into college and help students adjust and be successful in college courses include workshops and special classes to answer questions and give guidance. Some of these occur in the summer or the term prior to the student's entrance to college and some are concurrent with the student's first term or year at the college or university.

Participants also discussed the reasons why the gap exists between what students are prepared to do and what they are expected to do. Reasons include:

- Teaching style in schools encourages learning disjointed facts and memorization.
- Measures of success in schools focus on computation, not conceptual understanding
- Secondary students have lots of simple problems to solve. There is little problem solving that requires sustained effort through multiple parts.
- There is a major jump from secondary school to tertiary education in the thinking level required. Secondary students may only need to give the right answer, not explain how they got it.
- Students are taught to solve problems out of context and not consider reality or check for validity.

Participants agreed that problems of students going from secondary school to tertiary appear to be universal. Many institutions offer bridging courses, called developmental, remedial, or bridging. However, many of these bridging courses do not really bridge the gap. The two main reasons are:

- The courses are too short in duration. Many bridging courses are just a few weeks or months long. During this time some students are not able to master the material usually covered at school for several years.
- The mathematical background of the students is often so poor that the emphasis on bridging courses must be on the fundamentals of mathematics: rules, techniques, manipulations, and algorithms. There is no time to teach students higher level of thinking (proofs, reasoning, etc.). So the gap in thinking is not bridged.

While these were the inadequacies of bridging courses described by most participants, one individual from Israel described a bridging course aimed at preparing students for advanced mathematical thinking.

One participant from Canada suggested a possible reason why so many students have difficulty making the transition from secondary to tertiary institutions. Most cultures have rites of passage in which people are given an introduction to the next phase of life. They know what is expected of them after marriage, bar mitzvah, etc. If students knew the rules of playing the role of tertiary student, they might function very well. McMasters University in Canada mails their expectations to students the summer before they come to college. The question educators need to ask themselves is "Who is responsible for making the transition – the faculty or the students?"

There are three kinds of students who have transition difficulty:

- Students who have just a few difficulties.
- Students who never had an opportunity to prepare themselves because of poor schools, poor career advisory services, etc.
- Students who had the opportunity to prepare but did not take advantage of it.

Faculty and institutions have devised many types of transitional courses and learned many factors about what contributes to student learning.

- One instructor starts with teaching the language of mathematics; the subject matter of high school is used but the language of mathematics is emphasized.
- In another program, all faculty take "duty" in the learning laboratory where they respond to student "call-ins," e-mails, and visits. Data kept on students indicate that students prefer face-to-face interaction and that those who came in for face-to-face tutoring did better in their classes. Sometimes getting students to come in for the first visit is difficult but those who do are much more likely to come in again.
- Supplementary instruction usually refers to sessions where students drop in to get extra help. They work individually or in groups. These sessions are not used to help them do homework problems but to solve more challenging problems.
- Students want to interact with each other and with instructors,
- Students want problems that relate to their individual or group experiences and interests.
- While it is helpful for students to be provided problems relating to their areas of interest, if the instructor is unfamiliar with these applications. It is better to seek further help from colleagues who have such experience.

The future of non-university tertiary institutions

The three sessions at ICME-10 served to further our knowledge of the institutions we represent. Institutions with similar goals, students, problems, and achievements exist in many countries in the world. One purpose of the group is to better understand the role our institutions play in our countries, our societies and our economies. It appears that, as we move further into the 21st century, non-university tertiary institutions are increasingly important in education. For the many students who did not leave their secondary institutions well-prepared for further education or the workforce, they provide a bridge. For adults who never had an opportunity for education because of political unrest, poverty, or social custom, they often provide that opportunity. But more importantly, these institutions are increasingly becoming the choice of many more students who desire a practical education, grounded in the realities of emerging technology and global economics.

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DG 22: Current problems and challenges in university mathematics education

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Aims and focus

This report contains the conclusions of the deliberations of the members of DG 22. On the first day there were short plenary introductions to the themes of the DG. It was decided to concentrate on three themes, "widened recruitment," raising the profile of teaching", "dissemination of research." The participants from 18 countries were divided into three working subgroups, each charged with one theme. The task of each subgroup was to come up with a concise report on its deliberations, with concrete recommendations on how to improve undergraduate mathematics education with respect to its theme. Here is the resulting final report.

Theme 1. *To address challenges originating from the fact that nowadays a greater proportion of the general population enters university and as a result, the background of students entering mathematics departments and those attending service courses in these departments has changed.*

There was agreement that in many of our countries elements of logic are not taught to the extent so that the senior high school students can tell a correct (or fallacious) proof when they see it or devise simple proofs of their own. In the United States, this problem is manifested when students move from computationally-oriented calculus courses to more theoretical, proof-oriented mathematics courses. In France it used to be the case that quite a bit of logic was taught at the secondary level because of the strong influence of Bourbaki. But the presentation was overly formal, and there was a backlash. Now very little logic is taught, and at the university level only a minimal amount about quantifier rules is given at the beginning of students' programs. The secondary school program in France has a big focus on geometry and students generally do well in this subject. But, nonetheless, when they arrive at university, they have difficulty recognising that when you have "If A then B" and you know B then you might or might not have A. In Algeria, students typically take calculus in secondary school but have a lot of trouble with the kind of quantified statements that occur in university coursework. For example, students have difficulty in proving that if $A \subseteq B$ then $f(A) \subseteq f(B)$, or that the limit of a composite of continuous f and g at a point a is f of the limit of g at a. Much of the problem is students' inability to work with quantified statements.

Recommendations. Students should be required to take a "transition-to-higher-mathematics" course. This is a course that introduces ideas of logical reasoning and proof but uses simpler topics than those in a theoretically-oriented calculus or analysis course. Another recommendation is to include proof techniques s part of a high school curriculum in order to improve students' logical abilities. A good geometry course prepares students to improve their logical reasoning abilities if careful use of definitions, theorems, and proof are emphasised. The role of semantics, in addition to syntax, needs to be emphasised at all levels.

What works to motivate students? The mathematical needs of non-mathematics-majors may vary significantly with the university and academic major. Too often in the past have we funnelled non-majors into some form of calculus course. In many instances this choice fits poorly the needs of our clients. We should endeavour to design service courses that actually fit the needs and values of our clients. Service courses should be regularly re-examined to ensure that they are kept timely and continue to meet the needs of the departments and students that they are intended to serve.

A variety of assessment practices can be used to motivate students. Project work, both individual project work and group project work, is used widely. Writing can be used not only to check how much students know but also to motivate their learning mathematics. For instance, writing an autobiography about their K-12 mathematics experience can be used to motivate students' learning. Well-designed forms of assessment can help students to see that mathematics is alive and active in many contemporary contexts.

Theme 2. *To consider ways of raising the profile of the teaching component of an university career in order that it receive greater attention from researchers, universities and society at large.*

It is generally accepted that teaching excellence, in comparison to research achievements, generally plays little role in academic promotions. For example, the Dearing report in the UK in 1997 (downloaded from www.leeds.ac.uk/educol/ncihe/) asserts that the phenomenon of unbalanced rewards for research and for teaching was very serious for the quality of teaching throughout higher education. In the USA the faculty rewarding system is also a national issue (see www.aahe.org/initiatives/facultyroles.htm). Surveys in many countries have shown that in every category of staff and in every type of institution, there was widespread agreement that more emphasis should be placed on teaching than was currently the case. Next we cite some examples of practices that have been used with various levels of successes in various places.

Student participation in teaching evaluations. It is quite common practice for students to express their opinion of a class through questionnaires and written feedback. It is not clear, however, to what extent this practice is effective (e.g., is the questionnaire taken seriously? Does it differentiate between the students that like the teacher as a person and the students who think that he or she is a good teacher?) An interesting example of this type of assessment is the Student Evaluation of Educational Quality (SEEQ), of Curtin University of Technology. SEEQ is an instrument used to obtain student feedback on teaching and to develop teaching quality through reflective practice by the teacher. SEEQ recognises the complex and multidimensional nature of teaching and aims to provide feedback about teaching rather than content (see http://lsn.curtin.edu.au/seeq/index.html).

Helping young faculty in their teaching duties. An interesting example in the USA is Project NExT (New Experiences in Teaching) (see http://archives.math.utk.edu/projnext/). Project NExT is an MAA professional development program for new and recent Ph.D.'s in the mathematical sciences (including pure and applied mathematics, statistics, operations research, and mathematics education). It addresses all aspects of an academic career: improving the teaching and learning of mathematics, engaging in research and scholarship, and participating in professional activities. It also provides the participants with a network of peers and mentors as they assume these responsibilities. In other countries initial teacher training programs for newly hired academic staff exist (for instance at most British universities as well as at Danish and Norwegian universities).

Team teaching. At Hanoi Technological University, at Greek universities, and other institutions, teams teach large classes and meet weekly to review class progress. In Gloucestershire University, teams of five teachers, from across different courses are formed, and these teams meet regularly to informally review each other's courses and teaching. In this way teaching issues and problems come under a broader departmental view and have an explicit focus.

Peer evaluation of research and teaching. At the University of Nebraska in the USA, every year each academic will sit down with the Head of Department and together they will agree on the proportion of time the person will spend on research, teaching and administration. Other faculty review each of these agreed upon aspects at the end of the year. The faculty member is rated from 1 to 5 on each aspect of work and this produces a score. Similar schemes are employed at the University of Maryland and Utrecht University.

Teaching awards and centres. For an excellent example of rewarding excellence in teaching see www.ncteam.ac.uk/ilts/publications/excellence.pdf. Another approach is the creation of centres for teaching development. Centres are places where faculty can find information, supporting material, consultancy etc. Such examples are the Derek Bok Centre for Teaching and Learning at Harvard (see http://bokcenter.fas.harvard.edu) and the Centre for Science Education at the University of Copenhagen (see www.naturdidak.ku.dk).

Theme 3. *To explore ways of disseminating the findings of research on undergraduate mathematics education to mathematicians – in particular, to promote learning from the theoretical advances in K-12 mathematics education research.*
This subgroup broadly explored the relationship between professionals in mathematics and professionals in mathematics education, with an eye toward examining ways to improve this relationship. For the purpose of this summary, Ms (mathematicians) refers to those professionals engaged in producing new results in mathematics and/or those who are primarily tertiary mathematics teachers, and MEs (mathematics education professionals) refers to those professionals in mathematics education producing research in tertiary mathematics education. Two small groups consisting of Ms and MEs from five different countries discussed three focus questions. The focus questions and the discussion of these are summarised below.

1. *What are the present relationships between professionals in mathematics and mathematics education professionals, and what made the present relationships as they are?*
The idyllic past: Mathematics as a discipline itself was less compartmentalised. Mathematicians from various specialisations somehow could understand each other. The atmosphere within the departments was more of the "liberal art" type, with small or no involvement from the outside. From the 1970's to the present, however, there has been more pressure on mathematicians to publish, more pressure to evaluate teaching, and move to mass university education. At the same time, MEs were professionalizing and specialising in primary, secondary, and more recently tertiary education. This move to specialisation may have inadvertently fostered a splintering between Ms and MEs.

Ms recognise the need to improve collegiate mathematics education. In that respect, they recognise mathematics education as an important and useful (if not necessarily scientific) discipline. Many Ms, however, do not accept MEs, while on the other hand Ms goals and beliefs are often not sufficiently considered by MEs. A common critique of reform-oriented K-12 mathematics programs is that teachers' beliefs, goals, and experience are de-valued or ignored. If Ms beliefs and goals are at variance with MEs beliefs and goals, then there will be difficulty in their interpretations of MEs results. In particular, MEs tend to focus on process (e.g., how a point is argued) rather than on content (e.g., what algorithms/concepts are taught). If Ms cannot appreciate this distinction, they will fail to see the point, value, and legitimacy of ME work.

2. *Why don't some professionals in mathematics consider research in mathematics education to be "scientific" or of value? To what extent are these and related views justified?*
Some Ms feel MEs lack the credentials to do the work that they do. How can someone who has not taught real analysis tell an Ms how to teach real analysis? How can an ME tell someone who proves for a living about the epistemological nature of proof? Mathematics Education, along with other social sciences, is less precise and less objective than Mathematics and Sciences. If you insist on defining, as many Ms would tend to do, "scientific" by being rigorous, systematic, and having exacting standards, then mathematics education is less scientific than Mathematics and Sciences. Ms are by training, and perhaps inclination, going to be more comfortable with quantitative research. Today MEs work is primarily qualitative. As a consequence, when MEs results are described in a way so that they are accessible to Ms, they are perceived to be common sense.

Another way to address this question is as follows: Even mathematics disciplines must go through a lengthy period before they are accepted by the mathematical community. For instance, Cantorian set theory and, more recently, category theory. It is only natural that mathematics education should have to go through such a period before being accepted as a legitimate discipline by Ms, especially since it is so different from other mathematical disciplines.

3. *How can the relationship between professionals in mathematics and professionals in mathematics education be improved?*
One suggestion is to try to improve the relationship at a local rather than a global level. It is natural in any discipline that journal articles are tough to read (they are densely packed, employ jargon, and are trying to move a field forward) while face-to-face interaction moves more smoothly. The group envision changes in Ms views of ME coming department by department, or mathematician by mathematician. In short, the group want to change the interactional paradigm between mathematicians and mathematics educators. Five ways of doing this were proposed:

1) Have mathematics educators as faculty members of mathematics departments, perhaps through joint appointments.
2) Have MEs "revamp" courses that they have had experience teaching with input from mathematicians.
3) Offer colloquia discussing teaching between Ms and MEs.
4) Hold seminars in which Ms and MEs each read articles on mathematics education and then discuss them.

5) Develop reports of qualitative research accompanied by high quality videotape, when available. Videotape more sharply illustrates the value that innovative mathematics education courses can do in a way that transcripts cannot.

At the global level, it is useful to work more broadly by establishing pedagogical centres/didactical units in which Ms and MEs can interact and work on problems of teaching and learning together, with each bringing their own expertise.

Ms must see a need for MEs. This need must be both internal and external. Internally, it is critical for Ms to perceive something as problematic in their teaching practice. Externally (from administrators and from the public at large) Ms are beginning to receive pressure to seriously consider aspects of their teaching, to make improvements to their teaching, and to demonstrate impact on student learning and attitudes toward mathematics. Both internal and external pressures can promote a mutual need between Ms and MEs.

The products from MEs (both theoretical and practical products) must be of real use to Ms. Products from MEs should be adaptive, rather than prescriptive. Three additional suggestions for continued improvement are (1) to develop publication(s) of MEs products specifically for Ms. Ms are not interested nor do Ms have the background to read original research reports written for other MEs; (2) to develop collaborations between MEs and Ms centred around problems and challenges of tertiary education; and (3) to explore possibilities of adapting a "lesson study" type collaboration between Ms in collaboration with MEs.

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DG 23: Current problems and challenges concerning students with special needs

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Aims and focus

Prevention and intervention in early childhood is viewed widely in the mathematics education community as important for increasing the opportunities of children at risk of poor learning outcomes, and for ensuring the educational success and general wellbeing of young people (Doig, McCrae, & Rowe, 2003; McCain & Mustard, 1999; Ochiltree & Moore, 2001; Shonkoff & Phillips, 2000). This contention, as it relates to mathematics education, underpinned the work of DG 23 which gathered congress participants interested in exchanging ideas, and exploring and discussing substantial issues and dilemmas related to students with special needs in mathematics. The particular focus of the discussion group was primary and lower secondary students who have a specific difficulty learning mathematics rather than children who have a general learning disability, although the interests of participants extended to adolescent learners, adult learners and learners with visual disabilities. The group examined recent research and developments in the diagnosis and teaching of students, and the early identification of students needing special programs to enhance their learning of mathematics. The specific aims of the group were to

- gather information on current diagnostic procedures in identifying students with special needs in mathematics
- collect research outcomes on successful programs that help students with special needs in mathematics
- exchange information on how the problem of students with special needs is handled in different national and institutional contexts
- encourage participants in future common research development activities
- formulate recommendations to relevant desiderata.

Enhancing the learning of students who are vulnerable in learning mathematics

There are many factors that influence children's learning of mathematics. For example, participants in the discussion group noted the effect of attitudes, motivation, confidence, anxiety, persistence, language, culture and parental attitudes. These influences need to be considered when working with children who are vulnerable in learning mathematics. Another consideration is the content of mathematics programs for students who are vulnerable. Participants expressed the view that program content needs to emphasise the "big" ideas in mathematics and be relevant to children's life experiences and interests. The role of metacognition in learning mathematics was also highlighted. In conducting intervention programs with students, participants believe that it is important for teachers to identify and build on the existing mathematical knowledge of students, and create a bridge for the construction of new knowledge and skills. This is dependent on suitable assessment tools and instructional programs.

Classroom teachers' perspectives

The large number of classroom teachers participating in this discussion group highlighted many important issues and concerns in relation to children who have difficulty learning mathematics. These issues need to be considered when approaching the development of assessment tools for identifying 'at risk' students, and when developing intervention programs for students. The key points raised were:

- mathematics anxiety is high for many vulnerable students;
- children who are vulnerable in learning mathematics tend to have poor number sense;
- relevant contexts for learning mathematics are important, particularly for those students who are vulnerable;
- some 13-year-old students still use counting strategies to solve number problems and use few reasoning-based strategies;
- early identification of students who are vulnerable in learning mathematics is essential. This involves assessment of cognitive abilities, working memory, and disposition towards learning and mathematics;
- there is great need for classroom teachers to gain professional knowledge about learning pathways in mathematics and how to identify where students' current knowledge lies in reference to these pathways, and in how to diagnose students' difficulties;
- there is a need for professional development of teachers that focuses on strategies for identifying students who are vulnerable in learning mathematics, and effective instructional practices for assisting these students; and
- there is a need for specialist mathematics intervention teachers in schools. Extra teaching materials are not enough to assist students; specialist teacher knowledge is required to effectively support mathematical learning for vulnerable students.

Approaches to identification of vulnerable students

An important issue for participants was how to effectively accomplish the early identification of students who are vulnerable in learning mathematics. It was noted that many countries use formal national testing in mathematics, but participants argued that this approach was insufficient for informing teachers about the specific instructional needs of students, and did not enable the early identification of vulnerable students. Diagnostic tasks are used in Sweden for developing profiles of students' mathematical strengths and weaknesses. Participants noted that this form of assessment can be used by teachers to plan classroom instruction, but is insufficient for identifying students who are 'at risk.' A form of assessment that participants believed offers promise for the early identification of vulnerable students is the clinical interview. This form of assessment is widely used in Australia, and in association with a framework of growth points, may be used by teachers to identify children who are vulnerable in particular domains of mathematics.

Approaches to intervention

During discussions, many participants described 13-16-year-old students who rely on counting strategies for solving arithmetic problems. They noted that the ability gap widens as students become older, and argued that strategies to prevent this situation are needed urgently. In discussing this issue, three particular approaches to intervention were described. In a German project, Lorenz and colleagues (Lorenz, 2004) tried to identify those cognitive factors in the preschool years that are relevant for (later) mathematics learning in the primary grades. The study showed that visual factors and certain factors of language reception significantly contribute to success in arithmetic learning. The findings suggest that understanding of prepositions, space and time relations, and the ability to operate with visual stimuli correspond to mathematics success or failure. Further, the study demonstrated that it is possible to identify students at risk as early as age 4 and 5, i.e., in kindergarten.

In a second study, Lorenz and his group (Kaufmann, 2003) gave those first graders who were identified as "students at risk" in the sense above at the beginning of schooling additional training units in the regular classrooms. The remedial units comprised training for the development of (deficient) cognitive factors as well as certain prenumerical tasks. The study showed that those students who were included in the remedial program not only had sufficient gains but did not differ significantly from their average classmates. In some mathematical areas they even showed results comparable to the high achievers. The students at risk who did not receive additional remedial units differed from all other groups and lagged behind in all areas. Thus it is possible to identify students at risk in the early weeks of schooling and prevent the development of mathematical learning difficulties.

The second approach to mathematics intervention discussed in DG 23 was an Australian intervention program *Extending Mathematical Understanding* (Gervasoni, 2004). This program was first implemented by specialist teachers in 24 Early Numeracy Research Program trial schools (Clarke, Cheeseman, Gervasoni et al., 2002). Children were identified for the program on the basis of their mathematics growth point profiles that were developed following a clinical interview. The *Extending Mathematical Understanding* (EMU) program comprised daily 30-minute sessions for between 10 and 20 weeks. Sessions focused on children's learning in the domains of Counting, Place Value, Addition and Subtraction, and Multiplication and Division. Teachers worked with groups of three or four students or with individual students. The program targeted children in both Grade 1 and Grade 2 (7- and 8-year-olds) and was not remedial in nature, but was built upon constructivist learning principles. The EMU program was successful in accelerating the mathematical learning of participating students.

Another approach to enhancing the mathematical learning of 'at risk' students was developed as part of the Early Numeracy Project in Canada (Kelleher & Nicol, 2002). This project investigated ways to best enhance numeracy learning for young learners, particularly those at-risk in the area of mathematics. It had been recognised in Canada that although teachers knew that some children were having difficulty with learning mathematics, they did not know how to determine what type of difficulties children were experiencing, or what type of instructional practices would help. This dilemma is one that participants in DG 23 noted was common in all countries. Significant outcomes of the Canadian project were four tools for teachers: assessment items which teachers use to determine numeracy strengths and weaknesses; helpful suggestions for ways to address early difficulties in numeracy, including activities to support small group intervention; helpful activities and suggestions for whole class support; and activities to support the development of children's numeracy at home.

Recommendations for further research

The participants in DG 23 firmly believe that international research programs and dialogue are urgently needed to provide clear advice to education stakeholders and classroom teachers about effective strategies for the early identification of students who are at risk of poor learning outcomes in mathematics, and effective mathematics intervention strategies for enhancing the learning of these students. In particular, the following research recommendations were endorsed:

- identifying effective assessment tools for identifying students who are 'at risk' in mathematics. The group highlighted the promise of clinical interviews for this purpose;
- developing effective diagnostic tools for mathematics that are teacher friendly, culturally appropriate and are not dependent on psychologists for administration;
- identifying effective mathematics intervention programs for enhancing mathematics learning for vulnerable students;
- identifying effective classroom-based strategies for enhancing mathematics learning for vulnerable students;
- identifying the type of professional knowledge, teaching strategies and professional learning courses that lead to accelerated mathematical learning for 'at risk' students;
- identifying environments that motivate and encourage students to learn mathematics; and
- identifying appropriate curriculum and teaching strategies for learning the 'big ideas' in mathematics.

Recommendations for key education stakeholders and governments

- fund programs of ongoing monitoring and support for 'at risk' children;
- educate and employ specialist mathematics intervention teachers in schools;
- introduce dedicated courses in universities that focus on supporting the learning of children with different levels of mathematical ability;
- introduce smaller class sizes to enhance learning opportunities for all students;
- produce materials for parents with children in the early years of schooling that suggest strategies for parents to assist the numeracy development of students; and
- fund research programs addressing the specific research recommendation listed earlier.

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DG

Discussion Group 24

DG 24: Current problems and challenges in distance teaching and learning

Team Chairs: ***Alexander Afanasiev***, The Russian Academy of Science, Moscow, Russia
David Crowe, The Open University, Milton Keynes, United Kingdom
Team Members: ***Ryosuke Nagaoka***, University of the Air, Chiba City, Japan
Merilyn Taylor, University of Waikato, Hamilton, New Zealand

Aims and focus

DG 24 (which was attended by almost fourty participants) met three times to discuss current general researches and particular experiences in the field of distance teaching and learning. The group focused on the new powerful means and techniques that mitigate the problem of distance in the processes of teaching and learning as well as on perspectives of developing the software products in the field. Particular focus was paid to the following issues:

- Common principles and differences in the various modern concepts of distance teaching and learning (DTL).
- New technologies to be used in DTL in the nearest future.
- Successes and difficulties in present DTL researches and experiences.

The first session began with a round table discussion, with each participant describing his/her background and interest in the field. It turned out that there was a very broad spectrum of interests covering all sectors of the mathematics curriculum. Debra Woods then gave a lucid account of her experience of the distance learning programme based on the software *Mathematica* at the University of Illinois at Urbana-Champaign. As one of the first initiatives to use computer algebra technology this generated much further discussion.

The second and third sessions concentrated on three themes

- How to organize content and manage the process of education?
- How to organize remote on-line interaction between student and teacher?
- How to organize a full-scale mathematics course run through the internet?

The main thrust of all the discussion concerned technological questions.

One of the most popular topics turned out to be learning and teaching geometry via the internet. Contemporary e-technologies provide great potential in this area, and present challenging opportunities to modern high school education (both theory and practice). Three topics related to this geometrical theme were discussed in depth:

- the application of existing e-educational standards to planning, creating, packaging and delivering geometry e-courses
- the organization of distributed storage of e-learning materials to enable on-request search, and customizable assembly of e-courses by remote internet access
- the provision of an "e-environment" that fosters and maintains distributed collaboration among remote students and tutors when studying geometry.

The first topic concentrates on applying the so-called SCORM (Sharable Courseware Object Reference Model) standard (together with other related e-learning standards) to the requirements of e-learning design. To comply with SCORM, any e-learning system will consist of a Learning Management System (LMS) processing learning e-courses (the content). A student acquires knowledge when interacting with the content under the supervision of the LMS, so that the latter plays the role of a tutor in the learning process. The SCORM-compliance guarantees inter-operability between any LMS and any e-course regardless of the underlying technology. SCORM compliance is an important requirement of LMSs. Unfortunately a present-day LMS often looks like rather a "mechanical piano". It answers the question "what to learn?", by defining content, but says next to nothing concerning the "how to teach?" question (which is a much more complicated and intellectually challenging one). New approaches to this problem were discussed.

The second topic concerns principal questions which are traditional for distributed systems and, as a consequence, important in distance learning too: how should registration and storage of e-learning courses be organized in order to provide efficient search and fast structural access to learning materials? Many important aspects of e-learning depend on such distributed organization of the e-learning content. For instance; can one find the desired learning content using a set of indirect descriptions – perhaps formulated in terms of learning metadata? Or is it possible to dynamically create an on-demand e-course whose structure is influenced by individual e-learning history of an individual student?

The last topic focuses on support of the distributed collaborative "learning by doing" process. As a specific example, a student can learn by solving a geometrical problem being driven by remote tutor, who manages the same pictorial constructions and demonstrates correct approaches to obtaining the result. Such processes imply not only distributed resources, but also a built in functionality of interaction in both synchronous and asynchronous modes. Unfortunately, current SCORM compliant systems (and other existing commercial e-learning systems that the authors are aware of) provide only partial solutions in distributed environments, mainly because there is no effective standardized support of the interaction capability. So, in summary, the prospect of standardizing interactivity and building it into existing standards for distributed systems is attractive and potentially very powerful. However to date the results are sparse and fragmentary and this is certainly an area that will repay future investigation.

There were many contributors to the debate, several of whom showed prototypical systems that have been used in a distributed context. Some particularly notable examples were as follows.

Luiz Carlos Guimarães (Brazil) presented two DGS (Dynamic Geometry Software) computer programs currently being developed by group in the Laboratório de Matemática Aplicada of the Federal University of Rio de Janeiro.

Tabulæ is a program to facilitate dynamic plane geometry, which has, at the time of writing, been a year in development. Entirely written in Java, in its current version it displays facilities similar to those available in other DGS systems such as Sketchpad and Cabri. However there is a greatly increased capability for communication and interaction between remote users. However the system still lacks a few basic features such as a proper scientific calculator interface, and a macro facility.

Mangaba is a tool for solid geometry. Its main features include building primitives, transformations and resources available for visualisation, object editing, import/export of code and the feature of construction sharing. It could be used in elementary and secondary schools. Once again there is a facility for communication and remote interaction.

TT2K. Valery Krivtsov (Russia) presented the geometry e-learning environment TT2K developed by the Department of Distributed Computing of Institute for System Analysis of Russian Academy of Sciences. The TT2K is a distributed NG e-learning system possessing advanced functionality which can be accessed either locally or by the internet. It can

- treat any SCORM-compliant learning course
- analyse a current student's past achievement and dynamically form an individual route through the learning course
- create mini-courses on user demand
- store and retrieve individual results of learning.

The TT2K prototype at present works in the context of school geometry (but is extensible). It handles an electronic version of "Geometry 7", a Russian school textbook by Igor Sharygin. In order to foster the learning of geometry TT2K provides advanced functionality for supporting live geometrical sketching and real-time collaboration between student and teacher (via the internet). Such synchronous communication is important, for instance, when teaching teenagers. Many students cannot keep their interests and activities if response time exceeds some minimal value. Synchronous mode systems allow response times that are effectively zero. Collaboration between learners is also of great importance and valuable learning seems impossible without it. The computer can assist a teacher, but is not a substitute for him/her.

Educational applications and design aspects of a generic and heuristic step-by-step problem solver

The discussion was not confined solely to geometry. Bernhard Zgraggen (Switzerland) presented educational applications and design aspects of a generic and heuristic step-by-step problem solver. Providing step-by-step solutions to mathematical problems through the internet plays an important economic and didactic role in distance education. A project at the Distance University of Applied Sciences of Switzerland deals with the development and deployment of programs interactively generating detailed and dynamic step-by-step solutions to typical problems in higher mathematical education using *Mathematica*.

Already developed are software modules for advanced mathematical topics like computation of limits, determination of extremal values, analysis of functions, series approximation, computation of differentials, solving linear equation systems, linear optimization and Lagrange optimization problems. These have been tested among students and tutors in a distributed context. The software modules can be accessed and controlled via the internet (www.webmath.ch)

AiM (Assessment in Mathematics)

The final session had an extremely important discussion of techniques for assessment of mathematics by computer. Chris Sangwin (United Kingdom) presented a system, widely used at the University of Birmingham) that deals with assessment in mathematics. (Further details at http://web.mat.bham.ac.uk/C.J.Sangwin/aim/index.html). Interesting aspects of this system include the generation of random versions of questions that are:

- well-posed in a mathematical sense
- fair, when used in assessment
- progress through a scheme of different cases, not just "as random as mathematically possible" and the generation of worked solutions. (These cases can be either from randomly generated problems, or from questions asked by students.)

This work is novel in that a computer algebra system is used to support open-ended questions, eg "give me an example of a function with the following properties". The possibilities are far greater than with conventional multiple-choice tests.

At the end of the group's final session it was agreed that a platform should be maintained on the internet for ongoing discussions concerning distance education.

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Closing Session

The Young Danish String Quartet
Carl Nielsen: String Quartet no. 4, F major, 1st movement

Ole Björkqvist, Master of ceremonies
We have listened to The Young Danish String Quartet which consists of students of the Royal Danish Conservatory of Music in Copenhagen. They have already played together for several years and won several competitions – at home and elsewhere in Europe. This year they have also made their debut in New York. We have heard them play the 1st movement of Carl Nielsen's String Quartet no. 4 in F, opus 44.

My name is Ole Björkqvist and I have the honour and pleasure to serve as the master of ceremonies at this closing session. I am a member of the International Programme Committee for ICME-10 and I represent Finland. On the podium we have some of the persons who were present during the opening session. They seem rather more relaxed now. I guess we all have the impression that the congress has been successful.

The closing session will now follow a format that is fairly similar to the ones used at previous International Congresses on Mathematical Education. I invite Bernard Hodgson, Secretary General of ICMI, to give the Secretary's Closing Remarks.

Bernard Hodgson, Secretary General of ICMI
Introduction
Mr. Ambassador, distinguished guests, ladies and gentlemen, dear colleagues and friends, participants in this 10th International Congress on Mathematical Education.

We are now on the final day of an international event which, for all those who gathered during the past week on the campus of the Technical University of Denmark, proved to be in turn intense, hectic, exhausting, hot – in spite of the refreshing Nordic climate –, stimulating, exhilarating, rewarding, friendly, and much more. An enthusiastic assembly of 2394 participants from 94 different countries – mathematicians, researchers in mathematics education, teacher educators and school practitioners from all regions of the world – met in Copenhagen / Lyngby-Taarbæk during the last week to bring to life this tenth congress of a series which was initiated 35 years ago in Lyon. The quadrennial ICME congresses are held on behalf of and under the auspices of ICMI, the International Commission on Mathematical Instruction. And it is in my capacity as the Secretary-General of the Commission that I have the duty and the honour of addressing you today in this closing session of the congress.

As I have just mentioned, ICME-10 was organised on behalf of ICMI – but *not by* ICMI, the direct role of the ICMI Executive Committee being the selection of a site and the appointment of an International Programme Committee (which includes representatives of the Commission). It was the responsibility of an *ad hoc* committee to bring this project to fruition and my first words are directed to all the colleagues who have made this extraordinary event possible and who have welcomed all of us here with their warm and friendly hospitality. As you all know, the organisation of ICME-10 was the responsibility of a consortium of the Nordic countries – Denmark, Finland, Iceland, Norway and Sweden – who jointly formed the so-called Nordic Contact Committee to support the preparation of the congress by fostering co-operation between the five countries. Such a collaborative effort between neighbouring countries is a first in the life of ICMI and I wish to stress both the originality and the fruitfulness of such a model.

The indubitable success of the present Congress rests on the sustained efforts of a very large number of people involved in the various committees and sub-committees of the organisational structure of ICME-10, many of whom have worked invisibly and unknown to the participants. While it would have been well deserved to thank personally every one of those who have devoted time and energy towards the success of this quadrennial gathering, their number makes this clearly impossible. But these hard-working and good-humoured hosts and organisers should all be assured that their dedication and efficiency have not remained unnoticed. Please join me in expressing our warmest thanks to the entire team of ICME-10.

Very special thanks should however be addressed to some of the organisers. On behalf of the Executive Committee of ICMI – and, I am convinced, of all of the participants – it is my great pleasure to express our deepest gratitude and appreciation to four great Nordic colleagues and friends who played key leadership roles towards the success of the congress:

- The Chair of the Nordic Contact Committee, Professor Gerd Brandell of Lund University; Sweden.
- The Administrative Secretary of the Local Organising Committee and to the International Programme Committee, Ms. Elin Emborg, from Roskilde University.
- The Chair of the Local Organising Committee, Professor Morten Blomhøj, Roskilde University.
- And finally, but maybe firstly if I may dare say, a long-time friend of ICMI, the Chair of the International Programme Committee and in many ways the heart of the ICME-10 adventure, Professor Mogens Niss, also from Roskilde University.

The General Assembly of ICMI, which constitutes the formal body of the Commission, met two days ago on this campus and a resolution was then moved, and unanimously approved by a round of applause, to request the President of ICMI to officially convey to the organising committee of ICME-10, and especially to the chairs of the three main committees I just mentioned, the gratitude of the General Assembly for the exceptional quality and the greatly innovative character of the congress, especially as regards its scientific programme, and for the gracious and exceptional hospitality offered to the participants.

Our Nordic hosts should also be thanked for the remarkable support provided to some of the congress participants. It has been indicated in the Second Announcement of ICME-10 that, following the tradition started at ICME-8 in 1996, the organisers have adhered to the general policy of ICMI of forming a solidarity fund established by setting aside 10% of the registration fees for grants. These grants aim at facilitating a balanced representation from all over the world, among presenters as well as among general participants, by assisting delegates from non-affluent countries to attend the congress. I have been informed by the Local Organising Committee that the amount actually spent through the ICME-10 Grant Fund even exceeds this engagement, as more than 11% of the total registration fee income has been devoted to the Grant Fund, allowing to support a total of 175 participants from 55 different countries.

The Chair of the International Programme Committee, in his comments on the opening day, compared the congress to a kind of supermarket intended for every member of the mathematics education community, rather than a specialised conference concentrating on a specific theme. This was reflected in the extraordinary diversity and richness of the scientific programme offered to us during the past days, with its eight plenary activities, the more than eighty regular lectures, twenty-nine Topic Study Groups, twenty-four Discussion Groups, five National Presentations, Sharing Experiences Groups, Workshops, etc., as well as the Thematic Afternoon, where five mini-conferences were offered in parallel.

A few ingredients on the scientific programme of ICME-10 were of a highly innovative nature. Such is the case for instance of the Plenary Interview Session moderated by Michèle Artigue and starring four capital figures of the mathematical education historical landscape, an activity which was a truly magnificent moment and a very cordial way of looking at the past, the present and the future of our field.

Another innovative feature of this congress was the creation of five so-called Survey Teams, each having as a mandate to survey the state-of-the-art with respect to a certain theme or issue, paying particular attention to the identification and characterisation of new knowledge, recent developments, new perspectives and emergent issues. The reports from these Survey Teams were presented in two plenary and three regular lectures and, according to the informal comments I have gathered, were extremely well received by the Congress participants. This overview of selected aspects of the field turned out to be a most appropriate and efficient way of gaining a better appreciation and understanding of some central issues in today's work in mathematics education. Having witnessed from a distance the long-term preparation and the extremely wide coverage implied by these surveys, including some intensive fine-tuning sessions held at rather unduly late hours over the last few days in hotel lobbies or elsewhere, I wish to express my gratitude, and I am sure that of all ICME-10 participants, to the members of the five Surveys Teams for their efforts and for the quality of the outcome of their deliberations.

I would also like to mention, among the new initiatives set in place for ICME-10, the organisation on the opening day of the congress of a Newcomers Welcome Programme offered by the Nordic Contact Committee. More than 450 first-time participants registered to this activity, a strong indication of its usefulness. While its logistics may need to be partly revised, the Newcomers Welcome allowed those attending their very first ICME congress to better understand the philosophy and the content of the programme, while facilitating establishing links with other ICME participants. Hopefully this was instrumental in helping all newcomers quickly feel part of the "ICMI family".

A personal testament

It is today for me, almost certainly, the last time that I have the privilege of addressing the closing plenary session of an ICME congress in my capacity as Secretary General of ICMI, as this is my second term in that position – at ICME-11, in 2008, my successor will be presenting the closing remarks on the analogous occasion. I would thus like to take advantage of this opportunity to raise a deep personal concern about our field that, I know, I share with many colleagues. I would like to introduce this concern by offering a basic question: *why are we all here at this moment?* More precisely, I mean: why did each of us spend the last week on the DTU campus amongst the couple of thousands other participants? I agree there may be something a bit silly in asking such a question. But still allow me to consider some possible, if not plausible, answers.

Fundamentally, I am tempted to say, very many, if not all, of us, would say they are here because of their love for mathematics. A love for solving problems, for proving theorems, a love for doing mathematics, for talking mathematics, a love for teaching mathematics.

Others may say that their presence at an ICME congress is connected to a large extent to the size and the scope of the congress which, in spite of some tendency to gigantism, allows for an exceptionally wide possibility of choices. And here one can either try to explore, if I may use an analogy from Nordic gastronomy, the full diversity of "det store kolde bord" – of the "smörgosbord", if you prefer to think in Swedish rather than Danish gastronomical terms – or one can aim at building some personal strands on which to concentrate inside a very diverse programme.

What may be other reasons? Well, many of us possibly came to ICME-10 to talk to people, to enjoy meeting with old friends and with the hope of making new ones. The human forum offered by the congress proposes an exceptional concentration of members of the mathematics education community, thus facilitating direct informal contacts… provided the density of the schedule or the innumerable time constraints allow some open windows. Possibly the wonderful surroundings of the Nordic nature was as well a source of motivation for others.

While some may express their attraction to a scientific congress such as an ICME through the objective of knowing about "what's new in the field", a common interest to all participants attending the International Congress on Mathematical Education is clearly the desire to gain a better understanding of our field and of its evolution, so as to be in a better position to contribute to its improvement. These quadrennial gatherings may serve as kinds of milestones allowing to better integrate mathematics education as a practical and academic domain of growing complexity, where the main outcomes of our endeavours are usually not stated as theorems or even well-defined results based on conclusive evidence, but rather as stages in an ongoing process in which it is not at all unusual that a question apparently answered needs to be revisited some years later, because of some new understandings or insights.

Whatever the merit of the possible reasons for attending ICME-10 that I have just mentioned, there is still, I strongly believe, one much more fundamental and deeper motivation, which can be formulated as follows: deep in our hearts, we know that we came to this congress because we are convinced that mathematics education is essential to everyone in all segments of any society, because we know that mathematics education can play a fundamental and unique role in addressing the many equity issues still facing our modern world.

To use the allegory proposed by a plenary panelist earlier this week, mathematics education has much to do in order to have the "dorsal spine of modern civilisation" – namely mathematics – develop in such a way as to build a better body, to build a better civilisation, to build a better society. Far from me the pretension of daring to attempt matching the eloquence or emotion conveyed by Ubiratan D'Ambrosio in his closing remarks, during the exceptional Plenary Interview Session we have enjoyed a few days ago. Nonetheless allow me to add my voice not only to those who express concerns about issues of equity within mathematics education, but more to the point to all those, like our dear Ubi, who see mathematics as playing a crucial role in the improvement of social justice and the betterment of societies as well as, ultimately, of mankind.

It is generally recognised that mathematics plays an essential role in the development of the active citizenship required for a truly democratic society. But what needs to be acknowledged is that the challenge goes far beyond this basic but essential level. Let me borrow again from Ubiratan D'Ambrosio, in a paper he presented at the Symposium organised in 2000 to celebrate the centennial of the journal *L'Enseignement mathématique*, the official organ of ICMI.[1] He was asking the question: what can we offer to the future generations, in order that they live in a better world than the one which our and the previous generations before us have constructed? His answer has to do with the capacity of developing a critical view of our current model and of the knowledge system in which it was built. And this is where mathematics comes into play, as it is recognised as central to this knowledge system.

Gila Hanna remarked, during the Plenary Interview Session, that a lot of progress has been made about gender issues in mathematics education. To a certain extent the content of the scientific programme of this congress, or even the composition of the current ICMI Executive Committee, may be seen as some reflection of this progress. But still more progress has to be made on that account, as indicated by Gila Hanna, and also in relation to new challenges such as the exclusion of men in some contexts. However gender inequities, in spite of their importance, are not the only inequities, and probably not the main ones. It was striking to see that many of the contributors to this congress explicitly referred to the crucial role of mathematics and mathematics education in issues of social justice and equity. This was the case in the plenary presentation of the outcomes of some of the Survey Teams, for instance the one on "The professional development of mathematics teachers", where comments were made about the dominance of research from English-speaking countries and the danger of a blurring of the distinction between local and global issues, when thinking of the main problems facing our community.

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[1] See D. Coray, F. Furinghetti, H. Gispert, B.R. Hodgson and G. Schubring, eds., *One Hundred Years of* L'Enseignement Mathématique: *Moments of Mathematics Education in the Twentieth Century*. 2003, L'Enseignement Mathématique, Monograph no. 39. ISBN 2-940264-06-6.

Concerns of social justice were also reflected in other components of the programme, including Regular Lectures and Discussion Group or Topic Study Group activities.

That mathematics education as a research field has reached a level where it is capable of a critical reflection on its practice and achievements as regards issues of justice and equity is definitely an important sign of maturity. As a matter of fact, it does take some maturity to be able to look at oneself, at one's past, and to admit that there are some matters that are still going wrong, or at least not as well as they ought to be. Issues of inclusion remain at the heart of our work. We may be accustomed to drawing strength from collaborative work, but such collaborations will often exist in rather "comfortable" environments, if I may say so. We also need to draw from collaborations that deeply and truly involve our differences: different gender, different cultures, different values, different viewpoints, different backgrounds, more or less connected to mathematics or to didactics, even different languages. Neither our field, nor our world for that matter, should aim at uniformity. But this clearly does not prevent some form of unity among us.

So, going back to the question I raised a few minutes ago: "*why are we here now?*", I see as the fundamental answer that we are here so as to try to be better mathematicians, better researchers in mathematics education, better teacher educators, better teachers, in order to try contributing to the rising of a better society, of a better civilisation. We are here because we believe in mankind, in its betterment and in the role that mathematics and mathematics education play in this connection.

But what about the International Commission on Mathematical Instruction in this grand project? While the means of ICMI are somewhat limited, in particular as regards financial matters, there is nonetheless a wonderfully great richness and potential inside the ICMI community. As a matter of fact the strength of ICMI is essentially based on people. People like you, active contributors to activities of the Commission such as the ICME congresses, the ICMI Studies or the ICMI Regional Conferences. People like the members of the various committees organising these activities, who accept to contribute their time freely and considerably to help the Commission set these activities. People who collaborate with ICMI in participating and contributing to the improvement of the field. People who believe in the role and impact of mathematics education in the betterment of our societies.

What is ICMI?

I would now like to review with you briefly what ICMI is and what its main activities are. In many of these activities you may identify some components related to the equity issues I have just been discussing. I will be rather brief in most of my next remarks, and those wishing to get more information are invited to consult the many reports appearing in the June 2004 issue of the *ICMI Bulletin* or to contact me directly.

As was mentioned during the Award ceremony on the opening day of this congress, ICMI has a long history, as it was established in 1908 during the International Congress of Mathematicians held in Rome, with Felix Klein as its first President. As a matter of fact the celebration of the centennial of the Commission, in 2008, is now under preparation and the Executive Committee is grateful to the Italian mathematicians and mathematics education communities for having accepted the task of hosting a symposium to be organised on this occasion. The International Programme Committee in charge of this symposium will be appointed shortly.

After an interruption of activity between the two World Wars, ICMI was reconstituted in 1952, as an official commission of the newly formed International Mathematical Union, IMU. This has as an effect that IMU is responsible for formal aspects of ICMI such as the Terms of Reference of ICMI or the election of the Executive Committee of the Commission. Under pressure from its own General Assembly, IMU is currently planning major changes in the election procedure of its Executive Committee, with consequential changes as regards the election of the ICMI Executive Committee. It is in such a context that the current ICMI Executive Committee has recently been involved in intensive discussions with the Executive Committee of IMU about the procedure for the future elections of the ICMI EC. A report was presented earlier this week at the ICMI General Assembly about the ongoing discussion between the ICMI and the IMU ECs, especially as regards the proposal that the election of the ICMI Executive Committee would in the future be made by the General Assembly of ICMI. Such a development, still unexpected up until recently, can be seen as a very positive sign as regards the future of ICMI and the implication of the mathematics education community in all aspects of its organisation. The current agreement between the ICMI and IMU Executive Committees still needs to be approved by the General Assembly of IMU, to be held in August 2006 in Santiago de Compostela, Spain.

The new role eventually to be played by the General Assembly of ICMI entails a need to improve the infrastructure of the Assembly and the links with each ICMI member country Adhering Organisation or with the ICMI Representative in each country. Contacts are currently being established on that account, each member of the ICMI Executive Committee having taken responsibility for re-establishing or reinforcing the links with eight to ten member countries of ICMI.

a) ICMEs

The most extensive among the activities of ICMI is, undoubtedly, the International Congresses on Mathematical Education. You have just been taking part in the 10th such ICME and it is now time to reflect on it, on the way it was organised, on the ingredients of its scientific programme or on any other aspect of the congress. A new team of collaborators has already started the preparation of the next ICME and it would be important to provide them with your observations, concerns and proposals on both the structure and the content of the ICME congresses. You are thus cordially invited to convey your comments and suggestions to me at your earliest convenience, by letter, fax or e-mail:

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A major decision recently made by the Executive Committee of ICMI was to accept the invitation received from Mexico to hold the ICME-11 congress. A little later during this closing session you will be receiving from Mexican delegates an official invitation to the whole mathematics education community to gather in Monterrey on July 6 to 13, 2008, to celebrate the 11th ICME. At this moment I would just wish to stress how important this first Latin-American ICME congress is as regards aspects I have discussed earlier, such as equity issues or culture differences. It is the sincere hope of the ICMI EC that, with the support of all interested parties within Mexico and elsewhere in the region, this congress may serve as a catalyst for the promotion of mathematics education in Mexico and Latin America, as well as internationally, and have a long-term impact in the region.

With ICME-11 already on its way, it is now time to start thinking of the following congress, ICME-12, to be held in 2012. The ICMI Executive Committee is thus launching to all its member states an official call for bids to host the twelfth ICME. The task of organising an international congress of the size of an ICME becoming increasingly immense, complicated and demanding, it is hoped that a formal decision about the site of ICME-12 could be made not too late in 2007. We would thus like to propose the following schedule:

- a declaration of intention of presenting a bid should reach the Secretary General by November 1, 2005;
- formal bids should be presented to the Secretary General by November 1, 2006.

A written call for bids will be presented in the December 2004 issue of the *ICMI Bulletin*, which will also include indications about the kinds of issues a bid should address. Those wishing to be informed of those guidelines sooner should contact me.

b) The ICMI Studies

For close to 20 years, the Commission has conducted a series of so-called ICMI Studies devoted to crucial current themes or issues in mathematics education. It is not appropriate here to describe in detail the general functioning of a Study or the state of ongoing activities in this regard. I refer those interested to a survey of the first eleven Studies that appeared five years ago in the *ICMI Bulletin* (No. 46, June 1999), and also to the June 2004, No. 54, issue for information on the latest ones. Here let me just mention briefly that:

- a new ICMI Study volume is due to appear very soon, as a result of the 12th ICMI Study on *The future of the teaching and learning of algebra*;
- editorial work on the Study volumes for Study 13, the so-called ICMI comparative Study (*Mathematics education in different cultural traditions: A comparative study of East Asia and the West*) and Study 14 (*Applications and modelling in mathematics education*), is now ongoing;
- three Studies are now at the stage of getting close to a Study conference, namely

  Study 15 (*The professional education and development of teachers of mathematics*), whose conference will take place in Águas de Lindóia, Brazil, in May 2005;

Study 16 (*Challenging mathematics in and beyond the classroom*), whose conference will take place in Trondheim, Norway, in June 2006;
Study 17 (*Digital technologies and mathematics teaching and learning: Rethinking the terrain*), whose conference will be held around December 2006.

The ICMI Executive Committee is contemplating the launching of a new Study soon. Final decisions on this are yet to come but among the topics currently considered are *The role of proofs and proving in mathematics education* and *Statistics education in school mathematics*.

Among the other topics also envisaged by the EC for a future ICMI Study are for instance:

- Mathematics and physics education
- Integration of mathematics education and science education at the primary school
- Primary school mathematics education
- Connection of mathematics and other discipline (from primary to university)
- Innovative teaching in constrained conditions
- History of mathematics education
- Relation of mathematics education to general education
- Mathematics for and from the workplace
- Diversity in the teaching/learning of mathematics
- Values in mathematics education.

The ICMI Study volumes currently appear in the so-called New ICMI Studies Series published under ICMI patronage by Kluwer Academic Publishers, now Springer. I wish to remind you that individuals buying a Study volume for personal use are entitled to a 60% discount on the hardbound edition. Information on how to obtain such a discount appears regularly in the *ICMI Bulletin*. The Executive Committee of ICMI is fully aware that such a discount, although substantial, does not solve all the difficulties related to the cost of the Study volumes, especially when considered from an equity point of view, and is working on identifying ways to improve the situation.

c) Affiliated Study Groups

Over the years, five international study groups have joined the Commission as so-called Affiliated Study Groups of ICMI. These groups are neither appointed by ICMI nor operating on behalf or under the control of ICMI. They stage activities of their own and they are also offered slots inside the programs of the ICMEs. The ICMI Affiliated Study Groups produce quadrennial reports presented to the General Assemblies of ICMI. The ICMI Executive is particularly pleased that a new Affiliated Study Group has joined the Commission as of 2003, namely ICTMA, the International Study Group for Mathematical Modelling and Applications. Thus, the five current ICMI Affiliated Study Groups, with their year of affiliation to ICMI, are:

HPM: The International Study Group on the Relations between the History and Pedagogy of Mathematics (1976)

PME: The International Group for the Psychology of Mathematics Education (1976)

IOWME: The International Organization of Women and Mathematics Education (1987)
WFNMC: The World Federation of National Mathematics Competitions (1994)
ICTMA: The International Study Group for Mathematical Modelling and Applications (2003)

The spirit and functioning of these groups are surely familiar to many of you. But allow me to take this opportunity to comment briefly on the activities of one of them, WFNMC. The importance of mathematics competitions for mathematics education should not be judged solely from the point of view of the direct improvement of every day's class room work. More important is the general impact on many factors that indirectly influence education, including public awareness and appreciation of mathematics. Some of you may have the perception of competitions having to do strictly with math olympiads. While such an activity is definitely within the scope of the Federation, it is clearly not the only nor the main framework of action for WFNMC. Large-scale competitions such as the *Kangourou mathématique* in France are probably a better reflection of the spirit of the Group. Such competitions aim at enlarging the horizons and removing limitations in education. Let me quote here from the comments made by André Deledicq earlier this week, when receiving the Erdös Award of WFNMC from the hands of ICMI President Hyman Bass: "One of the most beautiful things to experience is a child thinking hard, looking for solutions and the moment when his or her face suddenly shines and when his or her mind shouts out."

In some circles competitions may be seen as a way to attract some bright young people to become professional mathematicians. While such an objective is not to be neglected, it should not be considered as the leading philosophy within this field, at least not from ICMI's perspective. This is basically what stands behind the 16th ICMI Study I mentioned earlier, where the idea of competition has been enlarged to challenging mathematics, in and beyond the classroom, including math clubs, math camps, exhibits, museums, etc.

d) ICMI Regional Conferences

The Commission's "raison d'être" as an organisation is to offer a forum promoting reflection, collaboration, exchange and dissemination of ideas and information on all aspects of the theory and practice of contemporary mathematical education, as seen from an international perspective. Despite this international nature of its position and role, ICMI from time to time lends its name to a variety of regional conferences on mathematics education, primarily in less affluent parts of the world. These so-called ICMI Regional Conferences are supported morally by ICMI, and sometimes financially as well.

Since ICME-9, six ICMI Regional conferences were held, while four others are currently in the planning stage. More information about these activities can be found in the *ICMI Bulletin*.

EM 2000 – Espace mathématique 2000; Grenoble, France, July 2000
All-Russian Conference on Mathematical Education; Dubna, Russia, September 2000
ICMI-EARCOME-2 – Second ICMI East Asia Regional Conference on Mathematics Education; Singapore, May 2002
LASHEM – Latin-American School on History and Epistemology of Mathematics; Cali, Colombia, November 2002
XI-IACME – 11th Inter-American Conference on Mathematics Education; Blumenau, Brazil, July 2003
EMF 2003 – Espace mathématique francophone 2003; Tozeur, Tunisia, December 2003
First Africa Regional Conference of ICMI; Johannesburg, South Africa, June 2005
ICMI-EARCOME-3 – Third ICMI East Asia Regional Conference on Mathematics Education; Shanghai, China, August 2005
EMF 2006 – Espace mathématique francophone 2006; Sherbrooke, Canada, June 2006
ICMI-EARCOME-4 – Fourth ICMI East Asia Regional Conference on Mathematics Education; Penang, Malaysia, 2007

e) The Solidarity Program and Fund

In 1992 ICMI, under the impulsion of its President Miguel de Guzmán, established a Solidarity Program in Mathematics Education. The overall objective of the Solidarity Program is to increase, in a variety of ways, the commitment and involvement of mathematics educators around the world in order to improve the situation of mathematics education, in particular in those parts of the world where the economic and socio-political contexts do not permit adequate and autonomous development.

An ad hoc committee was set up in 1999 by the Executive Committee of ICMI to review the functioning and impact of the Solidarity Fund, after its first years of existence, and to bring recommendations to the EC concerning its orientation and development. Unfortunately this ad hoc committee was not able to complete its task, but a new committee has recently been appointed, which is chaired by Professor Alan Bishop, of Australia. A preliminary report has been received by the ICMI Executive Committee and presented at the General Assembly two days ago. Issues of equity and social justice are clearly at the heart of the action of the ICMI Solidarity Fund and it is the objective of the ICMI EC to reinforce the presence and impact of the Fund among its set of activities. Comments and suggestions about possible actions or orientations of the ICMI Solidarity Fund should be sent to a member of the Executive Committee, or to the chair of this ad hoc committee, Alan Bishop.

f) Miscellanea: ICMI Awards; logo; website

This brings me to the final part of my closing remarks, in which I would simply like to review briefly three additional bits of information

First a word about the ICMI Awards. The official establishment of the ICMI Felix Klein and Hans Freudenthal Awards was definitely a peak item in ICMI life of the last four years. The first recipients, Professors Guy Brousseau and Celia Hoyles, were presented with their medals during the opening session of this congress.

The design of the medals accompanying the Klein and Freudenthal Awards has brought forward the need for ICMI to finally adopt a logo, which was also officially presented at the opening session.

Finally I would like to mention that the ICMI website is currently undergoing a substantial rethinking and redesigning. It is the aim of the ICMI Executive to make a much wider and up-to-date use of the website. We hope to be able to present in a not-too-distant future a much better tool for communication and dissemination of information.

4. Conclusion
This brings us, dear friends and colleagues, to the conclusion of this session. The ICME-10 congress has provided us with an overwhelming richness of presentations and activities of all kinds. We have spent at the DTU a most intensive week, during which we have clearly worked a lot, probably discussed a lot, hopefully learned a lot, possibly laughed a lot – and presumably not slept a lot. Well, this is probably what congresses such as the ICMEs are about, after all.

With its many innovative ingredients and the high quality of the programme, this congress has brought the standard of excellence of our quadrennial gatherings to new levels and is yet another memorable milestone in the life of the mathematics education community. Our Mexican colleagues are now eager to welcome us in 2008 for the pursuit of our never-ending journey.

It is now my duty to declare the 10th International Congress on Mathematical Education, held from July 4 to 11, 2004, officially closed.

I wish all of you a pleasant journey back home and I look forward to seeing you all again in Monterrey, Mexico, in July 2008, for the 11th ICME. Thank you. Goodbye! Au revoir! ¡Hasta luego!

Musical interlude by The Young Danish String Quartet
Hans Abrahamsen: Preludes 1-5 (out of 10)

Ole Björkqvist
After these preludes by Hans Abrahamsen, I call upon Gerd Brandell, Chair of the Nordic Contact Committee to speak on behalf of the Nordic Contact Committee.

Gerd Brandell, Chair of the Nordic Contact Committee

Dear colleagues and friends,

After seven intense congress days ICME-10 is now approaching its end and it is time to say goodbye.

The preparation for the congress brought about a lot of collaboration among people involved in mathematics education in the Nordic countries, not only researchers but also teachers and policy makers. Obviously many contacts and much cooperation on various levels existed at the time when we made the bid to host an ICME congress. However - the network has been enlarged and reinforced during the process. The collaboration during these five years even went beyond preparation for the congress.

One example of a new project is the KappAbel competition. The competition started in Norway and was spread to the other Nordic countries through the Nordic Contact Committee for ICME-10. Some of you saw the final on Tuesday and Wednesday and could witness the joy and feel the spirit of this competition that involves several tens of thousands of pupils and a large number of teachers all over the Nordic countries.

We had set ambitious goals for the Nordic participation. We are happy that so many teachers and researchers from our countries have contributed to the program, and wish to thank you all, especially teachers and young researchers who shared their experiences with a large international audience for the first time at ICME-10.

We are pleased to have seen so many participants from our countries at the congress. We had hoped for even more participants from the Nordic countries. We have reasons to believe that the time of the year – teachers have their summer vacation right now - may have been one reason for choosing not to go to ICME-10. Anyway, we hope that all those who have been here will share your experiences from the congress with your colleagues as much as possible. There will be plenty of opportunities to spread material, and even show videos from the excellent plenary sessions – Morten will explain this further – and I hope you will use these opportunities.

The Nordic Contact Committee and the Local Organising Committee as well as the programme committee have made efforts to create gender balance at ICME-10. We are happy that these efforts have yielded results, and that gender balance has been apparent at all levels in the programme.

To those of you who are going to PME in Bergen or to the HPM conference in Uppsala or perhaps on a tour to visit our countries I wish a happy tour and good luck. To those who are returning home I wish a safe journey and a happy return.

Thank you for coming and thanks to all of you for making the congress such a wonderful experience!

Finally – we are about to leave our Danish hosts. I would like to share with you a stanza from the Hávamál, words of wisdom from the Viking age. This stanza is about visiting friends and about leaving them.

How to preserve friendship:
Go you must.
No guest shall stay
In one place forever.
Love will be lost
If you sit too long
At a friend's fire.

Thank you!

Ole Björkqvist
Our next speaker is Morten Blomhøj, Chair of the Local Organising Committee.

Morten Blomhøj, Chair of the Local Organising Committee
Dear friends and colleagues.

ICMI-10 has now come to an end and it is time for us to reflect on what we have accomplished. This I shall do against the visions for ICME-10 set up four years ago by the Local Organising Committee and the Nordic Contact Committee.

In my judgement ICME-10 has proven to be a well organised congress. However, we are very well aware of the occasional problems with the technical facilities, which in a few cases were so serious that they destroyed some sessions. Please accept my sincere apologies for these incidents. We have, I must admit, underestimated the difficulties of operating a system for uploading and downloading presentations with which many presenters are not familiar. Also on the first day we experienced some problems with the logistics of lunch and happy hour. However, it is our hope, in the LOC, that all the good experiences and the many exciting programme sessions will wipe out these incidents of mis-organisation from your memory of ICME-10.

With regard to attendance, the number of participants from outside the Nordic countries, has actually met our expectations and our budget. Moreover, it is our judgement that ICME-10 has in fact attracted a varied attendance of mathematics teachers and mathematics education researchers from all over the world. The statistics show that ICME-10 has been a truly international congress.

As far as equal access is concerned it does appear that at least the composition of the IPC and the group of invitees display a gender balance which has been considerably improved compared to the previous ICMEs.

As I mentioned at the opening session our budget was originally planned for about 3000 participants so we had to cut down on all variable expenses in the last phase of the planning process. This is the reason why, we could not in all cases, maintain the standards we would have liked to offer.

We have received quite a few requests for a DVD of the plenary sessions. Therefore, we have decided to produce and sell such a DVD. We expect the price to be approx. 250 DKK which more or less equals 40 USD. However, this scheme will only be realised if we receive more than 200 orders before September 1, 2004. So if you are interested, please sign up.

On behalf of the Local Organising Committee, I will express my deeply felt gratitude to all contributors to the ICME-10 programme, and to all ICME-10 participants. Personally, I would like to thank all members of the Local Organising Committee and of the Nordic Contact Committee for their extensive and effective work in the planning process of ICME-10. And once more, maybe twice more, I would like to warmly thank all parties who have sponsored and supported the congress financially.

On behalf of the Local Organising Committee I wish you all a safe journey home and a successful future in your professional life as mathematics teachers, teacher educators or researchers in mathematics education.

Thank you very much and goodbye!

Ole Björkqvist
Mogens Niss, Chair of the International Programme Committee will speak on behalf of the committee.

Mogens Niss, Chair of the International Programme Committee
Dear distinguished guests, dear colleagues, dear friends, dear participants.

Thank you for flying ICMI-10! It is my pleasant duty to express my sincerest and warmest thanks to all those who have been involved in the organisation of the scientific programme and its surroundings, committee members, organising teams, contributors, speakers, and participants in the different sessions. The possible success of this congress is all yours.

As has already been demonstrated we are not very strong on technical matters. I hope that the incidents that you have experienced will appear to you as mainly minor. There is, however, one major thing that we have not been able to deal with properly. That is the weather. Claudio Alsina, my dear friend from Spain, who was in my shoes for ICME-8 in Sevilla, said that we strongly need a new curriculum in Denmark, according to the following motto "Teaching in the rain". I hope that despite the possible drawbacks of different sizes you have enjoyed the scientific programme and have found ways to benefit from it in a multitude of different fashions. As I already said at the opening, and as Bernard was saying on behalf of the ICMI Executive Committee, we would very much like to have your feed-back on the scientific programme. Please fill in the questionnaire, return it to us or contact us in other ways. We would very much want to provide a heritage of inspiration and deliberations to our Mexican successors.

There will be a proceedings after this congress. That is part of the registration package that participants have signed up for. The proceedings, I very much hope, will be out in a couple of years; that is our ambition at least. All invited speakers and all organising teams have been given a deadline, 15th October 2004, to submit their reports and papers. Deadlines and details have already been e-mailed to you, and I very much hope to receive all the material in due course. And then it just needs a little bit of polishing up – and the proceedings is ready; perhaps with a little delay due to the review process.

We will continue to maintain the web-site which you have been consulting widely, I suppose. We will maintain it for the coming four years up till the Mexico congress in July 2008. And we invite all teams to continue using the web-site as a platform for exchange and for further discussion and elaboration on what they have been accomplishing in the different groups. So, please look at the web-site from time to time and use it as a source of information on all news pertaining to the aftermath of the congress.

Well, it is then my pleasant duty to thank you all for having come. It has been an intensive week for all of us – for some it has been more than a week. I wish you a safe trip back, farewell, "ha' det godt", as we say in Danish, and be well. We are very much looking forward to meeting you again in all sorts of ways, in all sorts of places, if not before then in Mexico at ICME-11. Thank you!

Musical interlude, The Young Danish String Quartet
Swedish hymn from Dalarne

Bernard Hodgson, Secretary General of ICMI
We are now in a transition process from ICME-10 to ICME-11 and there are two steps which need to be taken in this direction. First, we have the feeling, that although ICME-10 is officially closed the congress is not quite over yet, so in order to have a proper end to this wonderful event I now invite ICMI President Hyman Bass to come to the podium to take care of an essential ingredient of this session.

Hyman Bass, President of ICMI
It takes many, many people to make a congress and for this truly wonderful and enriching congress we owe warm thanks to an army of hard working and good humoured hosts and organisers. On behalf of ICMI Executive Committee, I want to take a moment to pay special tribute and more tangible thanks to the four people mentioned by Bernard who played key leadership roles in ICMI-10, Mogens Niss, Chair of the International Programme Committee, Morten Blomhøj, Chair of the Local Organising Committee, Elin Emborg, Administrative Secretary of the Local Organising Committee and the International Programme Committee, and Gerd Brandell, Chair of the Nordic Contact Committee.

Gerd successfully led the committee that created and sustained the impressive collaborative effort among the five Nordic countries, Denmark, Finland, Iceland, Norway, and Sweden. We present her with a silver necklace with an amber pendant as a token of our appreciation.

Morten and Elin led the committee that provided the incredible energy and management that steered this awesomely complex event through to a successful completion. As a small token of our gratitude for this work we present them each with a giftcard for a dinner for two at the Brasserie Degas here in Copenhagen.

Finally, we pay a special honour to Mogens Niss. As chair of the International Programme Committee he provided the primary vision of the scientific programme of the congress and implemented it with great skill and sensitivity. Beyond the congress Mogens has played many leading roles in the mathematics education community with intellectual depth, rigour and eloquence. He is perhaps the most influential figure in ICMI in recent years having been a member of the Executive Committee for four years, Secretary General for eight years, and chair of the International Programme Committee of this congress for five years apart from many other services. When I asked Morten what qualities best characterise Mogens, he said "perfection". Mogens is a perfectionist. And he said, Mogens would likely agree saying, "yes, perfection suffices when it is perfect it is ok". We present Mogens with a fine ceramic vase produced by the well known artist Bodil Richard Manz.

Bernard Hodgson, Secretary General of ICMI
Second, on behalf of the Executive Committee it is now my pleasure to pass the bâton, if I may use an Olympic analogy, from Denmark to Mexico. I now invite Professor Carlos Signoret, past President of the Mexican Mathematical Society, Sociedad Matemática Mexicana, and chair of the committee which prepared the successful bid to host ICME-11 to come to the stage to take charge of the last segment of this session.

Carlos Signoret, Chair of the Invitation Committee for ICME-11
Distinguished colleagues from ICMI and from the organising committees, señor Ministro, dear colleagues from all around the world, dear friends.

Three years ago the Sociedad Matemática Mexicana organised in the Mexican city of Morelia, a joint meeting with the American Mathematical Society. It was at that time that we began to think about the possibility of placing a bid for hosting the ICME. Then we started wondering about the convenience of having the largest mathematics education congress in the world in our country. The community was consulted and the opinions we got back were almost unanimous. Hosting the ICME would be very beneficial to the mathematics education communities in Mexico and in Latin America. That was the idea we had in mind when we initiated the process, when we formed the pre-organising committee, when we placed the official bid, when we received our colleagues from ICMI, now our friends, for inspection visits, when we received the official "yes" from the EC and when we prepared the participation of Mexico in this ICME-10. And it is the main idea we have in our minds now that we are facing the tremendous responsibility and challenges that hosting ICME-11 represents. We are facing these challenges with a mature attitude knowing that the next four years will be full of intensive work. At the same time we are sure that the benefits that result from ICME are worth it, and therefore, we will remember this in every effort we make.

The Sociedad Matemática Mexicana, leader of the Mexican initiative has integrated the very heterogenous pre-organising committee in which every institution, every working group, every organisation related to mathematical education in our country is represented. Moreover, every institution or person who is interested in helping the organisation of ICME-11 for the sake of mathematics education is welcome. It is our perception that one of the multiple benefits that ICME brings to a particular community is the strengthening of unity and tolerance among its members. I am sure that many joint programmes with Latin America will emerge as a consequence of this congress. On behalf of the Mexican Mathematical Society I want to thank all the organisations that have made the Mexican bid a successful reality, CONACYT (Consejo Nacional de Ciencias Y Tecnología), SEP (Secretaría de Educatión Pública), CINVESTAV, UNAM, UPN, Academía Mexicana de Ciencias, many provinces and university cities in Mexico, and the OCV Monterrey. And I want to express a special word of gratitude to the Mexican Embassy in Denmark, especially to the ambassador, Minister Vasconcellos Cruz, for his invaluable support to the Mexican National Presentation at this congress. Also many thanks to the Ministry of Foreign Affairs of Mexico for its support and help in bringing to Denmark most of the Mexican material for the National Presentation. I thank also Professors Mogens Niss, Morten Blomhøj, Elin Emborg, Henrik Nielsen, and the Local Organising Committee for their support to the Mexican delegation. Finally, I would like to thank the Executive Committee of ICMI for its positive answer to the Mexican bid. This shows confidence in our country in organising ICMI-11. You will not be disappointed. In particular, I want to thank Professors Hyman Bass, Bernard Hodgson, Michéle Artigue and Frederick Leung who visited our country, for their invaluable comments and questions that enriched both our bid and our scope of ICME. Exactly four years from now we will be in a closing ceremony like this one but in Monterrey. And I am sure we will be celebrating the most exciting meeting we have ever known: the Mexican ICME. Thank you!

Now let me introduce Professor Marcela Santillan, head of Universidad Pedagógica Nacional, to say some words.

Marcela Santillan, Rector of the National University of Pedagogy

Dear colleagues and attendees to the 10th ICME.

In the early 70's Mexico started its work in mathematics education. Since then strong efforts to launch different programmes have been made to strengthen mathematics education in Mexico and in some of the Latin American countries with the participation of the main institutions in different countries. Another important feature of our Spanish speaking community has been the development of research and establishment of study groups in different areas. As a final remark I want to mention the growing interest of the Ministry of Education in having these groups as a reference to improve mathematics education in our country.

These are some of the reasons why, as mentioned by our president of the Mexican Mathematical Society, all the mathematics educators who have been invited to be part of the 11th international congress see this as a very important opportunity for Mexico and the other Latin American countries.

Thank you very much to the ICMI Executive Committee and to the Local Organising Committee for the work at this ICMI-10 and for the support to ICME-11 that we have already received. They have joined us for quite some time so as to let us know what will be expected of us for the next four years. Once again, thank you very much to everyone. We hope very much to see you in four years in Monterrey in Mexico.

Finally, let me call upon Mr. César Ocaranza of the Mexican Embassy, Denmark.

César Ocaranza, Mexican Embassy, Denmark

Good afternoon everybody! Distinguished members of the committee, ladies and gentlemen.

On behalf of the Mexican government it is for me an enormous honour and pleasure to extend to you a very warm invitation to ICMI-11 from 6th to 13th July 2008 in Monterrey, Nuevo León in Mexico. Every year Mexico receives millions of visitors from all over the world because of its landscapes, traditions, and people. Mexico is also a land of opportunity. On behalf of the Mexican government, I hope you will accept this invitation and join us in Mexico, Monterrey. You are very, very welcome. Thank you very much!

Musical final, The Young Danish String Quartet

Carl Nielsen: String Quartet no. 4, F major, 2nd movement.

Ole Björkqvist

I would like to remind everyone of the farewell gathering which will start immediately after this closing session, in this building. A Danish local Mexican band will play.
Thank you!

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Education and research institutions

DTU	Technical University of Denmark (DTU) – Campus and technical equipment placed at the disposal of the congress during the congress period. In addition to this DTU provided meeting facilities during the planning process.
RUC, IMFUFA	Roskilde University, IMFUFA – Support to the congress in the form of manpower for the secretariat, the chairs of the International Programme Committee and of the Local Organising Committee, and support to the meetings in the International Programme Committee.
Centre for Research in Learning Mathematics (Danish University of Education, Roskilde University and Aalborg University)	Payment of wages for the secretary of the Local Organising Committee and the International Programme Committee.

Norwegian Centre for Mathematics Education

The congress was supported by all relevant universities, departments and professional associations for teachers from all educational levels in Denmark and in the other Nordic countries. Many parties supported the congress directly by paying travel expenses for members of the Nordic committee and the Local Organising Committee and by means of manpower.

Foundations and Organisations

ICMI

Carlsbergs Mindelegat for Brygger J.C. Jacobsen

Abelprisen

International Mathematical Union

Public agencies and bodies

The European Commission, the Socrates Programme

UNDERVISNINGS MINISTERIET Danish Ministry of Education

Danish Ministry of Science Technology and Innovation

Nordic Council of Ministers

Ministry of Education and Research in Norway

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