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VISUALISATION – THE KEY ELEMENT FOR EXPANDING GEOMETRICAL IDEAS TO THE 3D-CASE

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In this paper some interesting observations concerning the behaviour of university students in the 3D-Dynamic Geometry Environment (DGE) Cabri 3D are described. Two fundamental constructions in 2D are analysed and problems occurring by using comparable constructions in 3D will be discussed. Furthermore, an example in which visualisation in Cabri 3D fails is given. Reasons for these misconceptions on the basis of authentic data and the theory of basic ideas (Grundvorstellungen) are mentioned in connection with the instrumental genesis of the new tool Cabri 3D. In a further step, the advantageous visualisation potential of 3D-DGE is discussed to generalise adequately fundamental construction ideas to the 3D-case.

INTRODUCTION

In a dissertation project different dragging modalities in 3D-Dynamic-Geometry-Environments (3D-DGE) in exploratory and construction tasks were analysed and several user types were identified (Hattermann 2011). During these qualitative studies, I could -besides the research focus- observe several participants having specific construction problems in 3D that will be described in more detail. According to the idea of basic ideas (Grundvorstellungen) these misconceptions are explained with the help of conceptions generated in 2D. In a further step some ideas are given to master the transition of fundamental construction ideas from 2D to 3D, keeping in mind the instrumental genesis of new tools according to Rabardel's instrumental approach (Rabardel 1995).

THEORETICAL FRAMEWORK

2D-DGE are one of the best researched topics especially within the PME-group (Laborde et al. 2006). About four years ago researchers began to analyse 3D-DGE as Archimedes Geo 3D and Cabri 3D (e.g. Kimiho et al. 2007, Knapp 2011, Mithalal 2011). In a dissertation project some adapted and expanded theoretical terms developed in 2D-DGE were elaborated to describe students' utilisation of the drag mode in 3D-systems in an appropriate way (Hattermann 2011). During these qualitative studies different construction problems were analysed and students' misconceptions, which will be presented in the following section, were identified.

Basic Ideas (Grundvorstellungen)

These misconceptions can be explained by the theory of *basic ideas* (*Grundvorstellungen*, Kleine et al. 2005(1), 2005(2); vom Hofe 1998). *Basic ideas* can be divided into *primary* and *secondary ideas*. *Primary basic ideas* exist before mathematical instructions and are developed by dealing with material objects and daily routine, whereas in contrast *secondary basic ideas* are developed during the time of mathematical instruction.

> "Grundvorstellungen are neither fixed, nor used universally but are dynamic and develop within a networked mental system. The necessity for the development results in a varying range of validity: If 'Grundvorstellungen' are sustainable in one mathematical area, they must be extended in another area" (Kleine et al. 2005(1), p. 228).

If these *basic ideas* are fixed and no more flexible, they can't be adapted or extended to new areas and contexts. Fishbein talks in this context of *tacit models* (Fishbein 1989). An arithmetic example is the following: The idea of sharing (dividing two natural numbers) implies the association that the result is less than the dividend. This basic idea does not hold for rational numbers and has to be extended in the new context.

Instrumental Genesis

Especially in computer environments the development of new or extended basic ideas has to be regarded with respect to the utilisation of tools (Computer-Algebra-Systems, DGE, or special tools as the drag mode in DGE), which may be potentially new for the learner. In new environments the handling of tools can interfere with the extension of basic ideas. Nonetheless, problems in handling the software can be used by the learner to get an explanation for false constructions instead of reflecting his mental conceptions especially his basic ideas. Rabardel's theory of *instrumental genesis* deals with the development of different *utilisation schemes* to use tools adequately. He distinguishes the terms *artefact* and *instrument* to stress the importance of the *instrument* that has to be built by the user. An instrument is a complex system that depends on the user and on the given task that has to be solved.

> "An instrument cannot be confounded with an artefact. An artefact only becomes an instrument through the subject's activity. In this light, while an instrument is clearly a mediator between the subject and the object, it is also made up of the subject and the artefact" (Béguin and Rabardel 2000, p.175).

According to Rabardel the evolution from an artefact to an instrument is only possible by the development of mental schemes, in short: instrument = artefact + scheme. This evolution is called instrumental genesis, a bidirectional complex process that requires time and involves the subject (i.e. the learner) and the artefact. For details see Drijvers & Trouche (2008) and Rabardel (1995).

Visualisation

Visualisation in mathematical learning has been the subject of much research especially in geometry (e.g. Arcavi 2003; Kadunz & Sträßer 2001; Kadunz & Sträßer 2004).

Arcavi defines visualisation in the following way:

> "Visualization is the ability, the process and the product of creation, interpretation, use and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings" (Arcavi 2003, p. 217)

Furthermore he gives concrete examples of visualisation processes in three different contexts:

> "[…] a) as support and illustration of essentially symbolic results […] b) as a possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions […] c) as a way to help us re-engage with and recover conceptual underpinnings, which may be easily bypassed by formal solutions […]" (Arcavi 2003, p. 223).

Interrelation of theories

The actual paper will discuss some situations in which visualisation processes can help to overcome the difficulty of fixed basic ideas from the 2D-case besides formal reasoning or symbolic computing. Authentic data shows that the transition from the 2D- to the 3D-case and the transformation of basic ideas grounded in 2D are complex and difficult processes. So visualisation will be utilised in a geometrical context and a reflective manner upon pictures in technological tools (Cabri-3D) to enable students to reconceptualise their existing basic ideas of geometrical objects. As it is described by Arcavi an advanced understanding of geometrical ideas can be reached by students, considering the interaction of basic ideas, the instrumental genesis of new tools and guided visualisation processes. The observance of the instrumental genesis of new tools should not be neglected because students have to struggle with both the conceptual understanding in the sense of (extended) basic ideas and the development of mental schemes to handle the new tool appropriately. Sometimes students do not know which of these problems they are dealing with at a concrete moment so that they interpret a misunderstanding caused by an inadequate basic idea possibly as their mistake in handling the software.

EMPIRICAL STUDY

In this section the methodology of the dissertation project will be stated in short and two basic constructions in 3D that caused problems to some students and a situation, in which visualisation and the 'authority of the computer' lead students to a wrong statement, are described: the construction of a circle, the construction of an orthogonal line g to a given line h through a given point P on h and the question, if a rectangular-isoceles triangle exists as an intersection figure between a plane and a cube. These results will be analysed by using the theory of basic ideas and some questions will be discussed to foster the reconceptualisation of existing basic ideas with the help of visualisation processes.

Methodology of studies

For the dissertation project a qualitative approach was chosen because in 3D-DGE nearly no research results existed concerning probands' behaviour. The decision for pre-service teachers as probands was justified by their previous knowledge in 2D-systems, a reasonable requirement to work with Cabri-3D. During the solution process the participants have been working in groups of two members. The actions on the screen were recorded by utilising the screen-recording software 'Camtasia'. Furthermore, a webcam and a microphone were used to record students’ voices and interactions.

The construction of a circle in 3D

Students who failed to construct a circle several times and over a long period of time (i.e. three different working sessions in the space of 12 weeks) were observed. They utilised the icon in Cabri 3D to construct a circle in 3D and clicked on the center and a point on the desired circumference. As a result, Cabri 3D did not construct the desired circle and students were confused, sometimes they even assumed a bug inside the software environment. They did not think about the parameters that had to be given in 3D to define a circle uniquely.

The construction of an orthogonal line

Students who failed the construction of an orthogonal line *g* to a given line *h* through a given point *P* on *h* were observed. These students were not aware of the fact that an endless amount of straight lines with this feature exists and for this reason Cabri 3D does not display one line with the desired feature.

The rectangular-isoceles triangle

Students were asked if a rectangular-isoceles triangle exists as an intersection figure between a plane and a cube. Two students utilised Cabri 3D and the drag mode to construct a draggable plane and a cube (see figure 1). They measured an angle of the intersection figure and dragged one point until Cabri 3D displayed an angle of 90° (see figure 2). They analysed the figure and claimed that a rectangular-isoceles triangle exists as an intersection figure, not being aware of the fact that numerical values are rounded by the software.

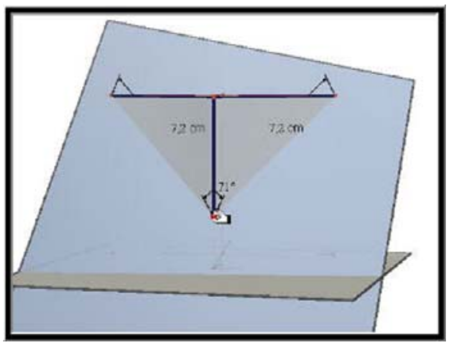

Figure 1. Construction of a draggable plane

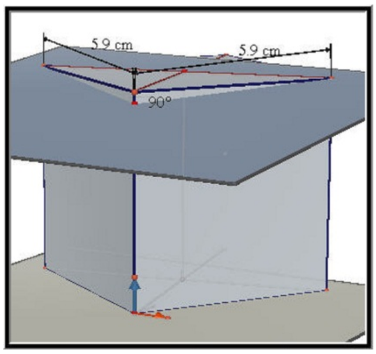

Figure 2. “Rectangular” triangle

During the solution process the following conversation between the two students took place.

Student 1:	The task is to construct a triangle that is isoceles and perpendicular… so the angle has to be 90°... (student 2 drags a point)… it doesn’t work... if only... it works only when the plane is on top of the cube.
Student 2:	90° (students create figure 2 in the software environment)
Student 1:	...when it is identical to the top surface of the cube.
Student 2:	Yes, we have it ! I will measure the distance of the segments one more time...we have one more task.
Student 1:	(is muttering)

…

Student 1:	But it works only (the angle of 90°) if it is identical to the top surface of … the cube....if not 90° are not possible.
Student 2:	No, it's right, it works,…
Student 1:	Ok, let's save it.

ANALYSIS

In the following the above mentioned problems will be analysed according to the theories of basic ideas and the instrumental approach. In doing so it can be shown that each of these theories is important to explore and explain students' behaviour in 3D-DGE. The visualisation of special constructions and a profound discussion about them can help to overcome misconceptions and to promote the instrumental genesis of tools. The empirical data shows that students are often unable to cope with the transition from 2D- to 3D-systems and the composition of mental schemes to work with new tools at the same time.

The construction of a circle in 3D

Table 1: The construction of a circle

	Basic ideas of a circle in 2D
formulae	$x^2+y^2=1$
definition	A circle is completely and uniquely described by its center M and a point P of the circumference.
features	A circle is used to measure equidistant distances.
	Every point P on the circumference has the same distance (r: radius) to the center M. dist(M,P)=r.
	The diameter d of the circle is a straight line through the center M of a circle connecting two points on the circumference. d=2r.

On the one hand it can be stated that students have to expand their basic ideas of a circle to the 3D-case and begin to think about the fact that spheres are more appropriate to measure equidistant distances in 3D. Furthermore, a circle is not described uniquely with a given center and a point on the circumference in 3D, thus the user has choose a plane. Alternatively a circle can be described uniquely by its axis and a point on the circumference. It is important to talk about these conceptions and ideas during the transition from 2D- to 3D-geometry and it can be assumed that the potential of visualisation, as it is described by Arcavi (see the theoretical framework) in 3D-DGE, can facilitate this transition. Besides, it has to be considered that beginners of utilising 3D-systems are at the outset of the instrumental genesis of several tools and functions of the software. That implicates their struggle with different problems which influence each other. On the one hand students are uncertain by using the software, on the other hand their existing basic ideas are not adapted to the 3D-case so that they are not able to identify faults and misconceptions in their constructions.

Ideas for practical intervention

The existing basic idea is represented in figure 3, in which different visualisation processes are involved to create an image (as it is described in Kadunz & Straesser (2004) that points to the underlying basic ideas from the 2D-case. To extend this basic idea different possibilities seem to be appropriate, only one of them is the utilisation of 3D-software as Cabri-3D. In this environment questions dealing with uniqueness of objects and dimensions can be answered in an explorative environment where the following questions could be helpful:

How many circles exist in a plane with a given midpoint M and a given point P on the circumference? Define an arbitrary plane E in Cabri 3D and define a point M in E and a point P in E. How many circles exist with M as midpoint and P as a point on the circumference? Define an arbitrary point M in Cabri 3D and an arbitrary point P. How many circles exist with M as midpoint and P as a point on the circumference? Imagine you created a circle in Cabri 3D and you know the coordinates of the midpoint and the coordinates of a point on the circumference. Can your classmate create the same circle on his computer only with the information about M and P and without having a look at your construction? Compare with the construction of a circle in 2D.

These questions and further discussion could help to utilise visualisation in a technological context to rethink about underlying basic ideas concerning the construction of circles and to reach an advanced understanding of this concept. It must be mentioned that advanced users in the handling of the software are presumed to answer the proposed questions, beginners in the handling of Cabri 3D would not be able to solve these problems because of their lack of adequate mental schemes to work with the software. In a learning environment situations have to be avoided in which inadequate basic ideas encounter undeveloped mental schemes for the utilisation of the tool. So mental schemes should be built in familiar mathematical contexts, whereas the visualisation process regarding the extending of basic ideas can only be successful when learners are familiar with the special tool.

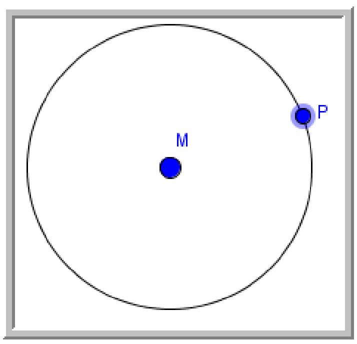

Figure 3. Uniqueness in 2D

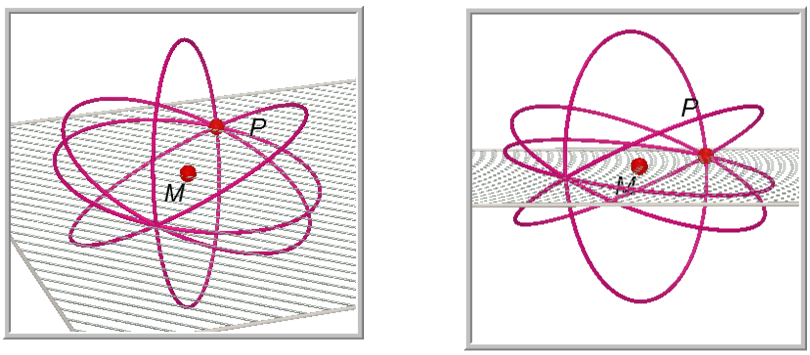

Figure 4. Several circles with midpoint M and P on the circumference

The construction of an orthogonal line

Table 2: The construction of an orthogonal line

	Basic idea of an orthogonal line g through a point P on h in 2D
idea	Uniqueness, 90°-angle between g and h
construction	With ruler and compass or with a set square

The problems occurring during this construction are comparable to the construction of the circle. The underlying basic ideas are not expanded to the 3D-case and at the same time these misconceptions are interrelated to the development of mental schemes to work with the new tool. The visualisation of the 2D-construction and a discussion about the transition to the 3D-environment can help to overcome this twofold difficulty concerning basic ideas and instrumental genesis. The best way to reconceptualise basic ideas in 3D with the help of technological tools is to work with complete and developed instruments.

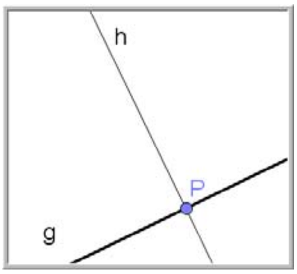

Figure 6. Unique perpendicular line to h through P

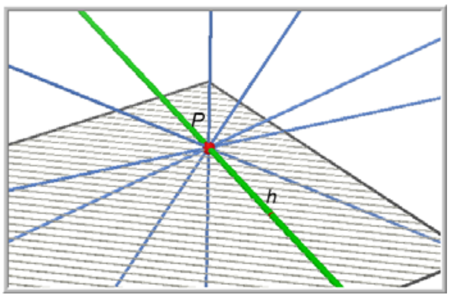

Figure 7. Different perpendicular lines to h through P in 3D

Ideas for practical intervention

Some questions comparable to the given ideas of the construction of the circle could help students to think about the uniqueness of the given construction in 3D.

Construct an arbitrary line h and a point P on h in space without using a computer software. Then imagine an orthogonal line to h through P and explain how many of these lines exist in 3D. Your classmate claims that there exist two perpendicular lines to h through P instead of one perpendicular line in 2D. What is your reaction to this statement?

By dealing with these questions the handling of the software is a further problem and the instrumental genesis of the tool has to be on an advanced level to handle these problems in the computer software. In Cabri 3D the user is able to construct a perpendicular line to h through P, but he has to indicate Cabri in what plane this perpendicular line should be. That is another example in which the instrumental genesis and the reconceptualisation of basic ideas encounter and eventually interfere with each other. To create several examples of perpendicular lines the user has to know that in each plane containing h there exists a perpendicular line to h through P, so the extension of the underlying basic idea to the 3D case must have been built up before. If the basic idea was not extended to 3D, the construction of several perpendicular lines to h through P would probably not take place. In this situation the drag mode in Cabri can help to find some perpendicular lines with the desired property by experimental proceeding. By defining an arbitrary point A in space and the construction of the line k(A,P) the point A can be dragged in space until the desired angle between h and k is 90°. This experimental procedure can be repeated several times to find different perpendicular lines. In the following the handling of the software to construct such lines can be discussed and in a final step these tool competence can be utilised to realise that there is an endless amount of perpendicular lines with the desired property in 3D. With the help of an experimental approach using dragging in 3D the encounter of unextended basic ideas of perpendicularity in 3D and undeveloped schemes by the user in the handling of the tool (Cabri 3D) can be avoided. So the drag mode and an experimental approach seem to be an appropriate way to separate problems of unextended basic ideas from insufficient user competence of the tool.

The rectangular-isoceles triangle

This example shows that visualisation with the help of technological tools can also fail. The students' conversation shows that they work with the software and try to benefit from the visual representation provided by the software environment. But the students do not know enough about the software to handle the results adequately. Otherwise they should have known that measured angles have to be rounded on given decimals. They discuss the problem and student 1 doubts his colleague's statements but the visual effect (a triangle can be seen as an intersection figure) and the display of 90° convinces him of the wrong statement. So the authority of the computer was dominant and the correct arguments of student 1 were invalidated. By describing this example it should be shown that visualisation can help to expand basic ideas, but nevertheless it is no guarantee for success, even more when students construct examples in which they only see what they want to see. Therefore, this is a good example to show that undeveloped mental schemes in combination with visualisation effects of the computer environment have a really strong influence on students. Although the argumentation of student 1 is correct and he refuses at first the existence of the perpendicular-isoceles triangle as an intersection figure between a cube and a plane, the authority of the computer combined with undeveloped schemes on the user side convinces the students of the wrong statement.

CONCLUSION

The theory of basic ideas is discussed profoundly in arithmetical and functional contexts (Kleine et al. 2005(1), 2005(2). Because of the experiences in 3D-DGE it can be assumed that the idea of basic ideas in connection with geometry can help to identify and overcome well-known problems in 2D-geometry (e.g. distance, construction of the inscribed-circle,…). Furthermore, these identified basic ideas in 2D (see the example of a circle) can help to master the transition to the 3D-case, especially in DGE with the help of visualisation. Because the transition from 2D to 3D in a DGE depends on both the expansion of existing basic ideas and the instrumental genesis of several tools, it has to be seen as a complex process that requires time and must not be underestimated by teachers and researchers. It seems to be relatively easy to assign tasks for students dealing with the extension of basic ideas when the instrumental genesis of the new tool is almost finished. Otherwise undeveloped basic ideas and partially built mental schemes for the handling of the new tool affect each other and create new problems to the user. In this respect an experimental approach using the drag mode in 3D can help to find some examples for correct solutions. In the next step the instrumental genesis of the utilised tool must be accomplished so that these mental schemes can be utilised in a last step to extend basic ideas to the 3D-case with the help of Cabri 3D. In order to generalise existing basic ideas to the 3D-case as it is shown in the examples of the circle and the perpendicular lines some theoretical work has to be done. The theory of basic ideas should be elaborated to 2D- and 3D-geometry to build a theoretical framework for further research concerning the encounter of basic ideas and the instrumental genesis of the tool. Moreover, it should be focused on the interdependency of the potential of visualisation in 2D and 3D-DGE, the basic ideas in the geometrical context and the instrumental genesis of different construction tools.

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