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THE TEACHING-LEARNING OF MATHEMATICS AS A DOUBLE PROCESS OF INTRA- AND INTER-INTERPRETATION: A PEIRCEAN PERSPECTIVE

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We consider the teaching-learning of mathematics to be a complex process of interpretation in which teachers and students jointly participate. We argue that, during formation, students' mathematical conceptions, constructed with the guidance of teachers, follow a long and corrective process of intra-inter-interpretation. We also emphasize that the teachers' multiple awareness of the evolving nature and refinement of both their own activity of intra-inter-interpretation and, especially, those that take place in the students, is essential to maintain a collaborative and dynamic teaching-learning signifying practice.
Intra-interpretation, inter-interpretation, objectification.

INTRODUCTION

We consider the teaching-learning of mathematics to be a signifying practice, one that is framed in a complex socio-mathematical classroom that functions as an *extended semiotic system*. Being embedded in this system, the mathematical discourse of teachers and students is mediated by a variety of mathematical, linguistic, paralinguistic, and social SIGNS. In the classroom signifying practice, teachers and students interpret and give meaning to mathematical notations, logical arguments, quantitative statements, numerical diagrams, graphs, etc. All forms of mathematical expression have their own intrinsic meanings and their own inner workings (Rotman 1998, 2000; Ernest, 2006), and they also direct the evolving meaning-making process of the participants.

Under the lens of the Peircean triadic theory of SIGNS, we look upon *classroom interpretation* as a progressive, ever changing *mental signifying process*. During this *signifying* activity, *a person* interacts with the self (Intra) and with *other people* (Inter) and collaborates in refining *cycles of objectification* by means of intentionally constructed *sign-interpretants.* During this meaning-making activity, SIGNS are encountered in the mathematical and in the socio-cultural *semiotic systems* that constitute the *worlds* that teachers and students inherit and activate in the classroom.

This paper is divided into three sections. In the first section, we sketch what we call a clarifying adaptation of the main components of Peirce's triadic theory of SIGNS: sign-object, sign-vehicle, and sign-interpretant. Essential to our model is his classification of the sign-object into *immediate* (*io*), *dynamic* (*do*), and *real* (*RO*); his classification of the sign-vehicle into *sv-icon*, *sv-index*, and *sv-symbol*; and his classification of the sign-interpretant into *immediate*, d*ynamic*, and *final*. In the second section, we use Peirce's triadic theory of SIGNS to present our view of teaching-learning of mathematics, when it is seen as a *double process of interpretation* in which both teachers and students actively participate. We examine *classroom interpretation* as a concomitant process of *intra-interpretation* and *inter-interpretation*. Each of these processes is examined as a sequence that consists of a triangular cyclic process of objectifications: (1) decoding-objectification, (2) abstracting-objectification, and (3) encoding-objectification. In the third section, we infer some pedagogical benefits that follow from taking into account the teachers' and the students' semiotic process of interpretation.

PEIRCE'S TRIADIC SIGN

Signs, in the broadest sense, were seen as mediating entities that prompt thought, that facilitate the expression of thought, and that embody original and conventional thought. Signs themselves were believed to have intrinsic meanings independent of the interpreter, meanings that were realized when signs were translated into other signs. Most of the time, signs were considered to have two components, to be *dyadic entities*, often called (sign)(object), also (signifier)(signified) (Nöth, 1990; and Vasco, Zellweger, & Sáenz-Ludlow, 2009). Note that, as indicated in Figure 1, if we start with the two components contained in the dyadic conception of signs, this leaves us with only one bidirectional relation (A) between the signifier (*sv*) and the signified (*so*).

When Peirce introduced the sign-interpretant as the third component of the SIGN, he transcended the dyadic notion of sign (signifier)(signified) and replaced it with his triadic relation of SIGN (sign-vehicle)(sign-object)(sign-interpretant) or, (signifier)(signified) (sign-interpretant). Consequently, we need two levels to represent the triadic relation: one level for the triadic unity called SIGN, located at the peak of the tetrahedron in Figure 1, and the second level for the three components in the base (*sv*)(*so*)(*si*). Note that, when Peirce added a third component, he also introduced two new bidirectional relations: (B) between sign-object (*so*) and sign-interpretant (*si*), and (C) between sign-vehicle (*sv*) and sign-interpretant (*si*).

Even though we support and follow Peirce's triadic theory, we acknowledge that Peirce himself uses his own terminology in such a way that sometimes leads to ambiguity and confusion. For example, Peirce uses the word "sign" to refer not only to the triadic relation itself but also to the sign-vehicle component of the triad. Avoiding this ambiguity has been a driving force behind our efforts to select vocabulary that will present a simple, direct, and clarifying adaptation of Peirce's triadic theory. We do this by being careful about how we label the four vertices of the tetrahedron in Figure 1. As shown at the peak vertex of the tetrahedron, the word SIGN, used only in upper case, stands for the unified and undividable totality that identifies a triadic relation as such, which is seen as a fundamental and defining property in Peirce's semiotic system. The other three vertices in the base of the tetrahedron, *always expressed in lower case*, sign-vehicle, sign-object, sign-interpretant, refer to the three components that constitute this triadic conception. We will use the expressions (*sv*), (*so*), and (*si*). Concisely, taking off from our clarifying adaptation, we will enter Peirce's system by way of the vocabulary that goes with the four vertices of the tetrahedron in Figure 1.

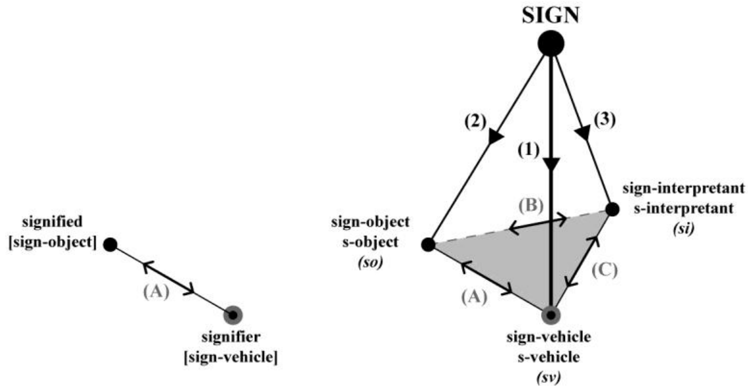

Figure 1. Dyadic and triadic conceptions of signs.

It was only about a century-and-a-half ago that Peirce was able to define *SIGN* as a *triadic relation* treated as a unique and undivided totality, one that, on another level of analysis, has three components. He argued on the one side that the only thought that *a person* can cognize is thought *in SIGNS*, and on the other side that thought can be known *between people* only by external *sign-vehicles* of some kind. This "one side, other side" is at the heart of the double semiotic process. In what lies ahead, "a person" will refer to an agent of self-sign-interpretant formation that takes place during intra-interpretation and "between people" will refer to the agents of other-sign-interpretant formation that takes place during inter-interpretation. The same distinction will also tie closely to what we later say about a common ground that exists between Peirce and Vygotsky.

By considering the SIGN as the triadic entity, the meaning of SIGNS is located in two worlds—the world of *intended meanings* and the world of *interpreted meanings.* The latter meanings seek to converge toward the former and, in this semiotic activity, the cognitive and epistemic *person-object relation* becomes established. Such a convergence emerges from different discursive contexts in which individuals interact with others and with themselves. This tells us that the meaning of SIGNS, specifically, what is *encoded into* sign-vehicles and what is *decoded from* sign-vehicles, emerges through *repeated* exchanges and interpretations that prompt the formation and evolution of inter-intra sign-interpretants.

One might think that the *object of the SIGN* is completely *encoded into* only one sign-vehicle and that it can be *decoded from* that sign-vehicle all at once. However, three difficulties emerge: (1) that the sign-vehicle cannot completely indicate the sign-object, (2) that the process of interpretation when *a person* generates, at different times, different sign-interpretants may or may not come close enough to the *intended* sign-object that was *encoded into* a given sign-vehicle, and (3) that signs-vehicles cannot be classified as *sv*-icons, *sv*-indexes, or *sv*-symbols without reference to the purposes of their users within particular contexts.

A SIGN stands for *something* other than itself. That *something*—a material, a conceptual, or an imagined Object—needs to be represented. In Peirce's theory, this Object can be represented *in* different SIGNS, each of which *presents* only a particular aspect but never all of the aspects of the Object at once. The particular aspect represented *in* the SIGN is what Peirce calls the *ground* of that representation, which is the sign-vehicle. Thus, to comprehend an Object, different but interrelated SIGNS, that is different and interrelated sign-vehicles, are necessary. A sign-vehicle functions as a proxy of the Object, which is equivalent to say that a sign-vehicle *serves* its Object as long as it helps to unveil one of its traits. This Object, which can be represented in one or more SIGNS, is what Peirce calls the *Real Object*. He argues that "it may be more convenient to say that that which determines a sign-vehicle is the Complexus, or Totality of Partial Objects" (Peirce, 1909, p. 492). Peirce also explains that it is proper to distinguish the *Real Object* from its *immediate* and its *mediate* aspects.

The *immediate object* (*io*) is about *the immediate aspects* of the *Real Object* (*RO*) of a SIGN. It is that object as the sign-vehicle represents *it* and whose existence is *dependent upon* that representation *in* the sign-vehicle. The *immediate object*, Peirce argues, is the "Object *within* the Sign [sign-vehicle]" (1977, p. 83). In other words, the *immediate object* is the object "as the Sign [sign-vehicle] itself represents it, and whose Being is thus dependent upon the Representation of it *in* the Sign [sign-vehicle]" (CP 4.536, italics added). Thus, the *immediate object* is a *mental representation* of the Object of a SIGN, whether this object actually *exists* or not (Nöth, 1990), and it lives as a *spur* to sign activity or semiosis (Corrington, 1993).

The *mediate* ,or *dynamic object* (*do*), is about the dynamic aspects of the *Real Object* of a SIGN: It is that object that can only be implicitly indicated by the sign-vehicle. The *dynamic object* is the "Object *outside* the Sign [sign-vehicle]" (Peirce, 1977, p. 83), or that object "which, from the nature of things, the Sign [sign-vehicle] *cannot* express, which it can only *indicate* and leave the interpreter to find out by *collateral experience*" (CP 8.314), or that object "which is the *reality* which by some means contrive to determine the Sign [SIGN] to its Representation" (Peirce, 1906). Thus, the *dynamic object* is that object determined by a collateral succession of new experiences and that, in the long run, comes to coalesce with the *immediate object*. While the *immediate object* (*io*) participates of a certain generality, it also calls for the specificity of the experience into focus. Peirce also insists that *inquiry* allows the *dynamic object* (*do*) to develop and get closer to the *immediate object* (Corrington, 1993).

The distinction and the complementarity between the *immediate object* and the *dynamic object* of a SIGN have implications for semiosis. This activity appears to be not only confined to the self-reference of a SIGN, but it is also expanded outward into the fields of the personal, the inter-personal, and the social experiences. These fields are relevant to a particular semiosis even though they could be only virtually semiotic with respect to that semiosis. For us, this complementary gives rise to the emergence of the mathematical conceptions of a person that come to approximate the *Real Object* of mathematical SIGNS, or a Mathematical Concept. This happens when there is sufficient coordination *within* and *between* those systems of mathematical SIGNS that represent that Mathematical Concept and other interrelated Concepts. These three Objects play an essential role in our view of classroom interpretation as a double semiotic process that is *intra* and also *inter*.

The *systems of mathematical SIGNS* consist of carefully connected collections of sign-vehicles extending across an extremely wide range of analogies, metaphors, vocabularies, notations, models, algorithms, proofs, arguments, diagrams, figures, graphs, tables, etc. Such *coordination* appears to be not only confined to the self-reference of SIGNS, but it is also expanded outward into the fields of the intra-personal and the inter-personal social experiences. These fields are relevant to a particular mathematical semiosis, even though they could be only virtually semiotic with respect to it.

As has been noted, the mathematical sign-vehicle serves as a *mediator* between the sign-object and the sign-interpretant. The sign-vehicle plays the role of a cognitive tool (psychological tool in Vygotsky's terms), one that is *determined* by the immediate sign-object and one that *determines* many possible sign-interpretants. This *double determination* calls for two acts of interpretation: the *decoding* of Mathematical Objects *from* sign-vehicles and the *encoding* of Mathematical Objects *into* sign-vehicles. This happens when the *dynamic* sign-object (*do*) is constructed and reconstructed to come closer to the *immediate* sign-objects (*io*), which, in turn, comes to approximate the *Real Mathematical Object (*Mathematical Concept). This *double determination* is aided by collateral observation, collateral experience, and collateral knowledge. Here collateral means the observations, the experience, and the knowledge that is not triggered by the sign-vehicle itself but that is brought into play based on the interpreter's insights and prior mathematical experiences.

MATHEMATICAL

CONCEPT-*RO*

$$(SIGN_k) \xrightarrow{k\rightarrow} RO$$

$$(do_j) \xrightarrow{j\rightarrow} io_k \xrightarrow{1\text{–}1} sv_k \xrightarrow{1\text{–}1} SIGN_k$$

Interpretant-constructed conceptions

Figure 2. From *conceptions* to *Concepts* by means of *intra-* and *inter-interpretation.*

The mathematical sign-vehicles play an special role as *mediators* between the *Real* Mathematical Object *RO* and a *person* who interprets them. Mathematicians *encode in* sign-vehicles (mathematical notations, or sv-symbols) certain aspects of the Real Mathematical objects. The particular aspect is considered the *immediate* sign-objects of the sign-vehicle. Teacher and student decode the sv-symbols and construct *dynamic* sign-objects (*do*)s. Over time, these sign-objects come closer and closer to *immediate* mathematical objects (*io*) that have been *stored in* and *carried by* mathematical sign-vehicles, namely, in what mathematicians have *encoded into* mathematical sign-vehicles, and in what teachers and then students *decode from* mathematical sign-vehicles. The third component of the SIGN, the sign-interpretant (*si*), also plays an essential role because it is the one that *decodes* and *constructs* dynamic mathematical sign-objects or conceptions (*do*)s.

It is important to note that the any *io* is in a 1-1 correspondance with its related *sv,* which is also in a 1-1 correspondence with its related SIGN. After sufficient *coordination* of different mathematical SIGNS, a *person* comes to approximate the Real Mathematical Object *RO* (Figure 2). Such coordination should be the goal of the processes of intra- and inter-interpretation.

Especially important is a consideration of the interpretations that take place *between people*, when the Mathematical Object is in the mind of one *person* (the mathematician M or the teacher T) and the sign-interpretant is in the mind of *another person* (the student S_i or S_{i+1}). That is, we need to consider not only what is determined in the mind of the sender (intentional sign-interpretant) but also what is determined in the mind of the receiver (effectual sign-interpretant). Peirce (1906) argues that, for communication to take place, reaching an agreement, *cominterpretant,* or *commens,* is a necessary condition. He describes the *cominterpretant* as whatever is expected to be commonly understood between sender and receiver, in order for the SIGN to fulfill its discursive function. The *cominterpretant* is, in essence, the communicative invariant of the SIGNS, that is, those meanings that transcend subjective interpretations and that tend toward the meaning of the *Real Mathematical Object*. This is to say that even though agreement may not come in its complete totality, senders and receivers should agree, at least, on some of the essential elements of the *Real Mathematical Object* that have been *encoded into* sign-vehicles. Also the *cominterpretant* involves, implicitly or explicitly, all of the social, the cultural, and the conceptual connections inherent in the use of SIGNS.

CLASSROOM INTERPRETATION

During communication, which is nothing more than the double process of intra-interpretation and inter-interpretation, discursive sign-interpretants (intentional, effectual, and communicational) play an important role in the activity of senders and receivers alike, with the ultimate goal to attain some sort of consensus. Given a sender with an intentional sign-interpretant in mind, what that sender *encodes into* sign-vehicles is an event, a fact, an idea, a concept, or any other sign-object, real or fabricated. When the sender *encodes* a sign-object *into* sign-vehicles with a particular intention in mind, the receiver is expected to *decode* it *from* the sign-vehicles and to produce an *effectual sign-interpretant.* This effectual sign-interpretant produces an action upon the inner world of the receiver that may or may not lead to a mental or a physical action. The receiver, in turn, becomes a sender, and the cycles of semiosis will continue until some type of communion is achieved between them. In Peirce's terminology, what is attained by both sender and receiver is a *quasimind, a commens,* or a *cominterpretant*.

Intra-interpretation

We consider intra-interpretation to be a triangular and cyclic activity of objectification (intra-decoding-objectification)(intra-abstracting-objectification)(intra-encoding-objectification) (Figure 3). Each of the classroom participants, (M), (T), (S_i), and (S_{i+1}), goes through their own cycles of intra-interpretation. They are represented in the triangles of Figure 4. In fact, Figure 3 is an essential part of Figure 4.

In intra-interpretation, the mathematicians either create their own mathematical objects by means of intra-abstracting-objectification or they *decode* existing mathematical objects *from* standardized (*sv*)s by means of intra-decoding-objectification. In this way, they construct their own dynamic mathematical sign-objects, *do*(M), and refine them so that they cohere with the logic of broader mathematical systems. This is done through repeated hypostatic abstractions, which eventually lead to the construction of new and better Mathematical Objects, *RO*(*M*). Finally, the mathematicians *encode* certain aspects of what they take to be the *Real Mathematical Object*, *into* novel or standardized mathematical sign-vehicles that are then communicated to others.

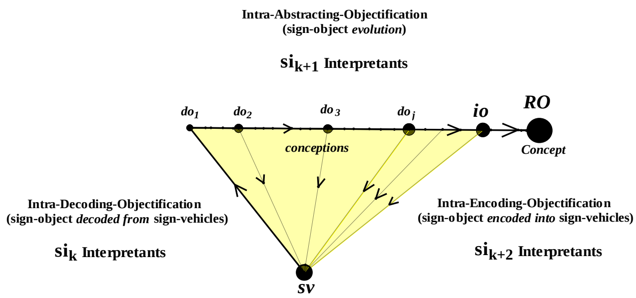

Figure 3. The process of intra-interpretation of each of the classroom participants.

In intra-interpretation, teachers and students *always* start with standard mathematical (*sv*)s and *decode* them to construct their own dynamic mathematical sign-objects, *do*(*T*), *do*(S_i), and *do*(S_{i+1}) (mathematical conceptions). Usually, at the beginning, these conceptions are very different from what the mathematicians intended when they *encoded* their mathematical objects. To construct their own mathematical concepts, teachers and students continue to modify and refine their dynamic mathematical conceptions, *do*(*T*), *do*(S_i), and *do*(S_{i+1}), in order to attain the construction of the immediate sign-objects, *io*(*T*), *io*(S_i), and *io*(S_{i+1}), that will better approximate *RO* (Mathematical Concepts).

In sum, intra-interpretation transforms the first interpretations, although vague they may be, by means of the triangular and cyclic activity of objectification (Figure3). Following the construction of successive (*do*)s, the immediate objects (*io*)s of the mathematical SIGNS, *encoded into* and *decoded from* (*sv*)s, become more general and abstract. When the classroom participants, (*M*), (*T*), (S_i), and (S_{i+1}), produce their own cycles of objectification, thus their own cycles of signification, their mathematical meanings tend to achieve a certain degree of objectivity as they approach *RO*. Intra-interpretation is nothing more than a personal process of intra-signification that will also depend on the collaborative interaction among the classroom participants. This is to say that intra-interpretation anchors the mathematical activity of the classroom participants. This collaborative and constructive interaction brings us to the process of inter-interpretation, which is the focus of the next section.

Inter-interpretation

We consider intra-interpretation to be a triangular and cyclic activity of objectification (intra-decoding-objectification)(*intra-abstracting-objectification*)(intra-encoding-objectifica tion) (Figure 3). Each of the classroom participants, (*M*), (*T*), (S_i), and (S_{i+1}), goes through their own cycles of intra-interpretation. These cycles are represented in the triangles of Figure 4. The triangles of inter-interpretation have one common side with the triangles of intra-interpretation, and their linkage and continuity can be followed in Figure 4.

Intra-interpretation and inter-interpretation coexist synergistically, and they can be separated only for the purpose of analysis. We also consider inter-interpretation to be a triangular cyclic process of objectification (inter-decoding-objectification)(*intra-abstracting-objectification*) (inter-encoding-objectification). Thus, intra-interpretation and inter-interpretation have a common component, namely *intra-abstracting-objectification*, as indicated by the horizontal side of each triangle in Figure 4.

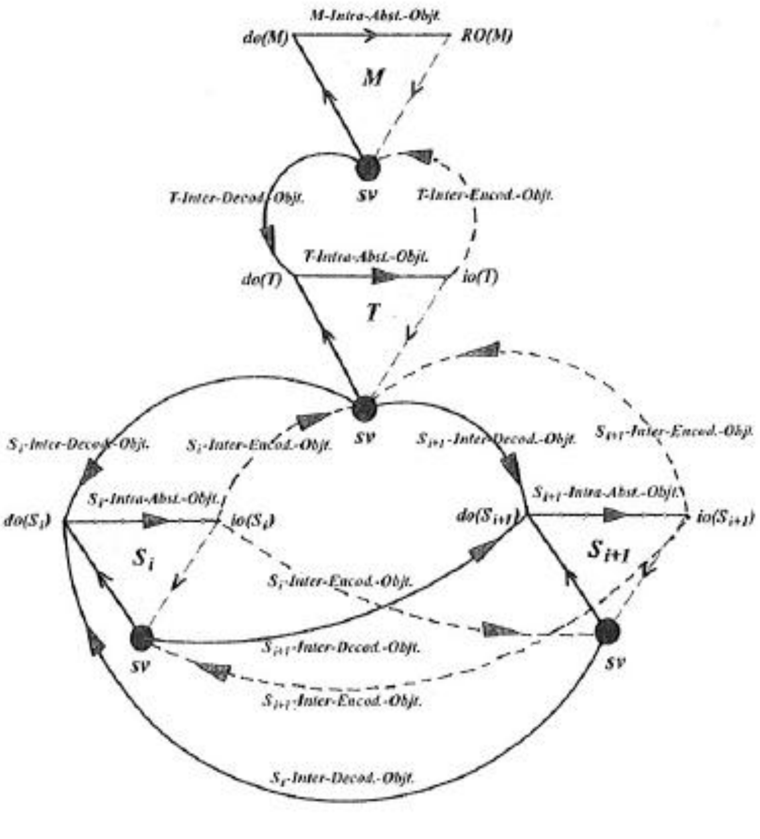

Figure 4. Intra-interprepretation and inter-intepretation of the classroom participants.

Figure 4 presents the interrelation between intra- and inter-interpretation. In this figure, the teachers' inter-decoding objectification (T-inter-decoding-objectification) is indicated by the directed bent segment starting at the mathematicians' (*sv*)s and ending at the vertex that indicates the teacher's dynamic sign-object *do(T)*. This objectification links mathematicians and teachers, and it is the first step in the process of inter-interpretation. Teachers' inter-decoding-objectification is followed by their own intra-interpretation (T-intra-interpretation). In particular, the *T-intra-abstracting-objectification* sustains the transformation of the teachers' $do(T)$ into the intended immediate sign-object $io(T)$ that, when coordinated with the (io)s of other mathematical SIGNS of the same concept, will approach *RO* (Figure 2). It is important to note that the teacher's intra-interpretation is the starting point of their interaction with the students. As the teachers interact with the students, they encode their $io(T)$ into standard mathematical (sv)s when they convey mathematical meanings to the students. Teachers' mathematical (sv)s are, in turn, decoded by the students, (S_i-inter-decoding-objectification) and (S_{i+1}-inter-decoding-objectification), when they engage in constructing their dynamic mathematical sign-objects.

In Figure 4, students' inter-decoding-objectifications are indicated by: (1) the directed bent segments starting at the teacher's (sv)s and ending at the vertices $do(S_i)$ and $do(S_{i+1})$ and (2) the directed bent segments starting at the students' (sv)s and ending at the vertices $do(S_i)$ and $do(S_{i+1})$. These inter-decoding objectifications bring about intra-abstracting-objectifications, (S_i-intra-abstracting-objectifications) and (S_{i+1}-intra-abstracting-objectifications), to transform $do(S_i)$ and $do(S_{i+1})$ into $io(S_i)$ and $io(S_{i+1})$. What follows is the students' inter-encoding objectifications indicated by the bent dashed segments that start at $io(S_i)$ and $io(S_{i+1})$ and end at the (sv)s of either the other student or the teacher. It is important to emphasize that it is the students' intra-abstracting-objectification that anchor their cycles of intra- and inter-interpretation. Accordingly, these cycles continue in order to approximate the Real Mathematical Object, *RO*.

CONCLUSIONS

Our view of interpretation as a double semiotic process accounts for not only the teachers' semiotic process of interpretation, not only the students' semiotic process of interpretation, but also the teachers' interpretations of the students' interpretations. Including all three of these concomitant semiotic activities of interpretation would improve not only the standard teaching practice, but it would also improve the learning conditions available to the students.

We come now to the common ground that exists between Peirce and Vygotsky. As expected, this calls for a social setting that puts inter-interpretation first in time and importance because it makes possible what will emerge in intra-interpretation. Vygotsky (1986) defines internal activity in terms of semiotically mediated external social activity. For him, this is the key in understanding the emergence of internal functioning. According to Vygotsky, "[E]verything internal [Intra] in higher forms *was* external [Inter], that is, for others it *was* what it now *is* for oneself" (as quoted in Wertsch, 1985, p. 62, italics added). Note that this also ties closely to Sfard (2008), who holds that thinking by "a person" is essentially a continuation of the practice obtained from previously held conversations "between people." When we consider *interpretation* to be a *process* and, in agreement with Peirce's conceptualization of the third component of the SIGN, namely the sign-interpretant, we can infer that the internal and the external processes of interpretation are semiotically mediated, intimately interrelated, and essential to *internalization*. In other words, Vygotsky's view of internalization makes possible all that we have said about *objectification*. Thus, *objectification* in the teaching-learning of mathematics is a subset of *internalization* in general.

Consequently, the classroom mathematical activity can be seen as a *semiotic activity* during which *systems of mathematical SIGNS* are interpreted, mathematical conceptions are constructed and refined, and habits *of mathematical thinking* are formed. This *semiotic activity* is grounded in the larger *socio-mathematical semiotic system*, and it is the manifestation of the formation of a living and dynamic social system, a larger more complicated semiotic system that combines *systems of mathematical SIGNS*, *systems of social SIGNS*, and classroom practices. Such a *semiotic activity* is based on the past mathematical activity of mathematicians, teachers, and students and, at the same time, guides their present and future mathematical activity.

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