12th International Congress on Mathematical Education

Program Name XX-YY-zz (pp. abcde-fghij)

8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

VISUALISATION IN MATHEMATICAL DISCOVERY, GENERALISATION, AND COMMUNICATION

Caroline Yoon

The University of Auckland

c.yoon@auckland.ac.nz

Students are often advised to visualise when they are stuck on a problem as a way of generating new ideas. However, visualisation can also help students generalise mathematical discoveries and communicate mathematical ideas. This paper describes how the nature of students' visualisation can change depending on the purpose for which it is used. A pair of participants used embodied visualisation methods while trying to discover maxima and minima in an antiderivative problem. Their visualisation techniques changed to encourage more local analysis when they began generalising a rule for discovering maxima and minima. Then, they developed a simple visualisation tool to communicate their rule to layperson clients. The study highlights the need for students and teachers to be aware of the different mathematical purposes for which visualisation can be used, and the kinds of semiotic systems that are appropriate in each case.

Visualisation; calculus; mathematical discovery; generalisation; communication.

INTRODUCTION

Visualisation is often considered to be a generative activity that helps us develop new insights. Students solving mathematics problems are often encouraged to visualise during the initial stages of their problem solving activity by using diagrams, graphs and gestures to discover mathematical patterns and relationships (Tall, 2004). When it comes to generalising their discoveries and communicating their findings, students are usually encouraged to use more formal kinds of semiotic systems such as written language, mathematical symbol notation and algebra. Yet visualisation can play a productive role in these latter stages of mathematical problem solving, just as it does in the earlier stages.

This paper describes a case study of a pair of participants, who use visualisation to discover, generalise and communicate a rule for determining whether an x-axis intercept on a gradient graph of a function corresponds to a maximum or a minimum on a graph of the original function. As the reasons for visualising changed, so did the kinds of semiotic systems they employed. The case study demonstrates that the nature of students' mathematical visualisation can vary according to the purpose for which it is used.

VISUALISATION IN CALCULUS

Visualisation has been promoted as a powerful tool for enhancing students' conceptual understanding of calculus. Researchers caution that overemphasising algebraic procedures in calculus can lead to students applying rules without understanding their meaning (Aspinwall, Shaw & Presmeg, 1997; Thompson, 1994). Modern calculus reform movements (for example, Hughes-Hallet et al., 2002) advocate teaching calculus using graphical and visual representations in addition to the algebraic and numerical representations that are often heavily favoured in practice. Eisenberg and Dreyfus (1991) argue that the overemphasis on algebraic techniques in teaching has led to students being reluctant to use graphical or visual techniques to solve calculus problems, even when the problems are presented visually. Instead, students typically resort to more familiar algebraic methods, despite the problem being much simpler to solve through visualisation.

Two studies have explored students' visualisation of antiderivative graphs from functions presented graphically. Berry and Nyman (2003) found that students who exhibited an algebraic symbolic view of calculus had trouble constructing antiderivative graphs within the graphical domain. However, when they experienced the "physical feel" of "walking" a displacement time graph using graphic calculators, they developed a deeper understanding of the relationships between a graph of a function and a graph of its antiderivative. Another study (Haciomeroglu, Aspinwall & Presmeg, 2010) presented students with derivative graphs, and observed the visual and analytical techniques that the students used to sketch the corresponding antiderivative graphs. They found that students who synthesised both visual and analytical approaches were more adept at the task than those who favoured one approach exclusively. These studies suggest that visualisation can help students to understand relationships between a function and its antiderivative in the graphical domain.

THEORETICAL FRAMEWORK: SEMIOTIC SYSTEMS

This paper uses the framework of semiotic systems (Ernest, 2006; Arzarello, Paola, Robutti & Sabena, 2009) to analyse students' visualisation. Ernest defines a semiotic system as consisting of three components:

> First, there is a set of signs, each of which might possibly be uttered, spoken, written, drawn, or encoded electronically. Second, there is a set of rules of sign production, for producing or uttering both atomic (single) and molecular (compound) signs… Third, there is a set of relationships between the signs and their meanings embodied in an underlying meaning structure. (Ernest, 2006, p. 69)

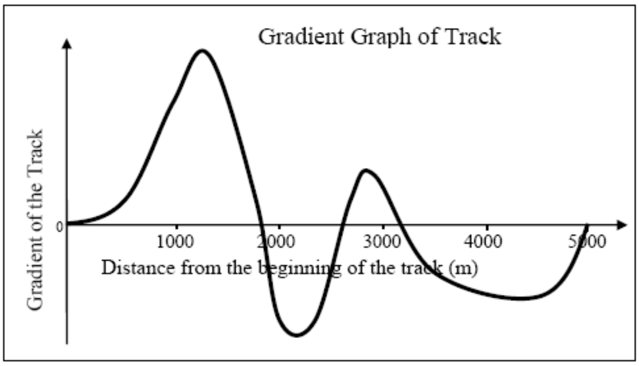
The above excerpt emphasises that a semiotic system is more than just an individual sign taken in isolation—it is a system of related signs that are linked in their production and in the way they give rise to meaning. Arzarello et al. (2009) point out that a semiotic system may draw on many different modes, including:

> …words (orally or in written form); extra-linguistic modes of expression (gestures, glances,…); different types of inscriptions (drawings, sketches, graphs,…); various instruments (from the pencil to the most sophisticated information and communication technology devices); and so on. (p. 97)

Furthermore, a semiotic system may also include multiple linked mathematical representations, such as tables, graphs, formal symbols, diagrams and so forth. Thus, the framework of semiotic systems requires one to examine the connections between different sets of signs produced in different modes, and within different mathematical representations. When applied to analyses of students' visualisation activities, the framework encourages researchers to focus not only on semiotic systems produced through modes and representations commonly associated with visualisation, such as gestures, diagrams and graphs, but also on the written and spoken words and symbols with which they are linked.

THE ANTIDERIVATIVE TASK

The antiderivative problem used in the study was set in the context of tramping (a term used in New Zealand to describe hiking), and was created using Model Eliciting Activity design principles (Lesh, Hoover, Hole, Kelly, & Post, 2000). Accordingly, it begins with a newspaper article that describes how a person died while tramping a dangerous track because he was misled by the track's vague difficulty rating, and calls for tracks to be described more carefully in terms of how steep they become. Afterwards, participants are given warm-up activities in which they calculate gradients of a given distance-height graph of a tramping track, and then sketch the gradient graph (derivative) of the track. This sets up the problem statement, which asks participants to design a method that can be used to find the distance-height graph of any tramping track from its gradient graph. Students are asked to explain their method in the form of a written letter to hypothetical clients, the O'Neills, and to apply their method to find features of a specific track whose gradient graph is given in Figure 1.



The O'Neills need your help!

Design a method that the O'Neills can use to sketch a distance height graph of the original track (like the one given in the warm up question). You can assume that the track begins at sea level.

Write a letter to the O'Neills explaining your method, and use your method to describe what the tramping track will be like on the day. In particular, you must clearly show any summits and valleys in the track, uphill and downhill portions of the track, and the parts of the track where the slopes are steepest and easiest.

Most importantly, your method needs to work not only for this tramping track, but also for any other tramping track the O'Neills might consider.

Figure 1: The tramping problem and the graph of the tramping track's gradient

This task is mathematically equivalent to creating instructions for finding the graphical antiderivative of a function presented graphically. However, the task doesn't mention the term "antiderivative", nor does it require formal prior knowledge of antiderivatives. In order to encourage students to reason visually, the gradient graph is intentionally given without a scale on the vertical axes, so that students are not inclined to compute the values of the distance-height graph numerically. Similarly, the function of the gradient of the track is presented graphically without an algebraic formula, and is not an immediately recognisable function (such as a quadratic, cubic, or sin or cosine function). These measures were taken to discourage students from solving the question algebraically.

METHOD

The participants in this study were two female New Zealand secondary school mathematics teachers: Ava and Noa. Both teachers had studied calculus, but neither had taught it at the Year 13 level (the last year of secondary school in New Zealand). In this study, Ava and Noa functioned as students working on calculus tasks for their own professional development, rather than as teachers teaching in a classroom. The data were collected as part of a larger study that looked at students' construction of calculus concepts (see acknowledgements).

Ava and Noa worked together on the tramping problem for an hour in the presence of a researcher who clarified the task instructions but refrained from directing them mathematically. Their work on the problem was videotaped and audiotaped to produce verbal transcripts, which were then annotated to include the gestures and nonverbal cues that Ava and Noa performed during the problem. The annotated transcripts were then coded to identify: the semiotic systems Ava and Noa created using drawings, symbols, graphs, speech, written language, and gestures; the mathematical relationships, objects and properties that Ava and Noa described through these semiotic systems; the goals Ava and Noa set for themselves during the course of the problem. The author performed the initial coding. Two research assistants then used the same coding schemes to code parts of the annotated transcript independently (see acknowledgements). These were compared with the author's codes, and consensus was reached on how to revise the coding.

The tramping problem led Ava and Noa to engage with a number of mathematical concepts such as maxima and minima, points of inflection and the area under the curve and its relationship to vertical displacement. In order to compare visualisation in three stages of problem solving, I restrict my focus to the first of these concepts, which was the development of rules for finding maxima and minima in antiderivative graphs. The task called for students to deal with maxima and minima properties of antiderivative graphs in three ways. First, the students were required to describe the tramping track (the antiderivative)—consequently, they had to discover the maxima and minima for the given track. They were also required to generalise their method for finding graphs of any tramping tracks and identify the maxima and minima from any gradient graph. Finally, they were asked to communicate their generalised method in the form of a nontechnical written letter to layperson clients.

VISUALISATION IN THREE STAGES OF MATHEMATICAL PROBLEM SOLVING

Ava and Noa engaged in visualisation during three stages of mathematical problem solving: while discovering, generalising, and communicating maxima and minima properties of antiderivatives. In each case, Noa initiated the semiotic systems that were associated with visualisation, and Ava copied and appropriated them.

1. Visualising while discovering maxima and minima

Noa (and later Ava too) used gestures to visualise the slope of the tramping track by tracing with her right hand finger along the curve of the gradient graph (in Figure 2a) and performing gestures with her left hand to describe the corresponding gradient of the tramping track (see Yoon, Thomas & Dreyfus, 2011 for more detail on the gestures). Figure 2b shows the trace of Noa's gestures, which has been superimposed onto the snapshot image. The following excerpt describes how Noa takes into account the direction and changing steepness of the slope indicated in the gradient graph to gesture the shape and slope of the track. This excerpt only describes the gestures that correspond to sections 2-4 on the gradient graph (in Figure 2a), but Noa had previously performed similar gestures for the first section also.

Noa: It's quite gentle gradient, getting steeper gradient to the point where it is the hardest gradient and then it starts levelling off and getting easier again (*traces finger along section 2 in the gradient graph shown in Figure 2a and gestures corresponding negative gradients*) until you get to like a bottom (*points to 3 on Figure 2 and gestures a flat gradient*). And then we start going back up (*traces finger along section 4 on Figure 2 and gestures a positive gradient as shown in Figure 2b*).

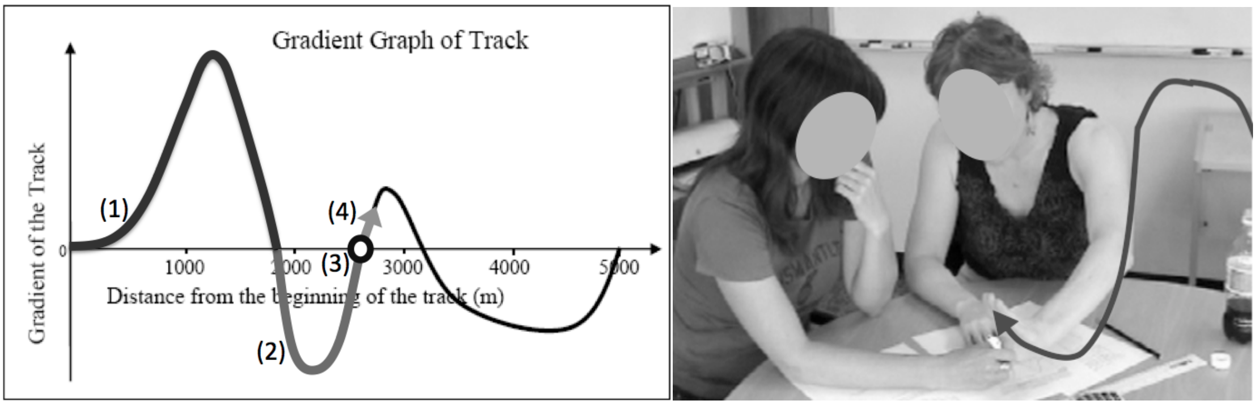


Figure 2: (a) Noa traces her finger along sections of the gradient graph. (b) Noa gestures the slope of the tramping track

As the excerpt shows, Noa describes the journey along the tramping track from the point of view of walking the track herself. She takes into consideration the continuously changing y-values of the gradient graph, and varies the tilt of her hand smoothly to reflect the corresponding changes in the track's gradient. While visualising the track thus, she encounters maxima and minima on the tramping track—not by deducing their presence from basic principles, but by noticing them as they occur, much like trampers would notice the summits and valleys while walking the terrain of the track. Ava and Noa then convert their gestures of these maxima and minima into a graph of the track shown in Figure 3.

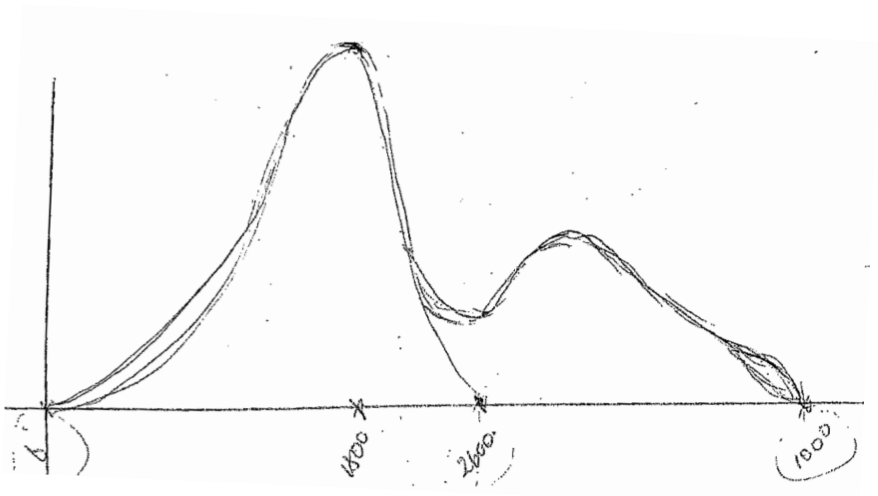


Figure 3. Ava and Noa's graph of the tramping track

2. Visualising while generalising a rule for finding maxima and minima

After drawing the graph, Ava asks Noa "how are we going to generalise that?" At this point, Noa shifts her attention from discovering maxima and minima by encountering them while visualising the track as a whole, to examining more closely the mathematical properties that give rise to maxima and minima. She compares the incidence of maxima and minima in the graph of the tramping track they have drawn, with the behaviour of the gradient graph immediately before and after the corresponding x-axis intercepts.

Noa: I was just wondering if we were going to be able to say, the first one, you know, when it goes from a positive (*points to positive segment above point 1 in Figure 4a*) to a negative (*points to negative segment below point 1*), that's going to be a maximum (*points to point 1*), when it goes from negative (*points to negative segment below point 2*) to positive (*points to positive segment above point 2*), that's going to be a minimum (*points to point 2*).

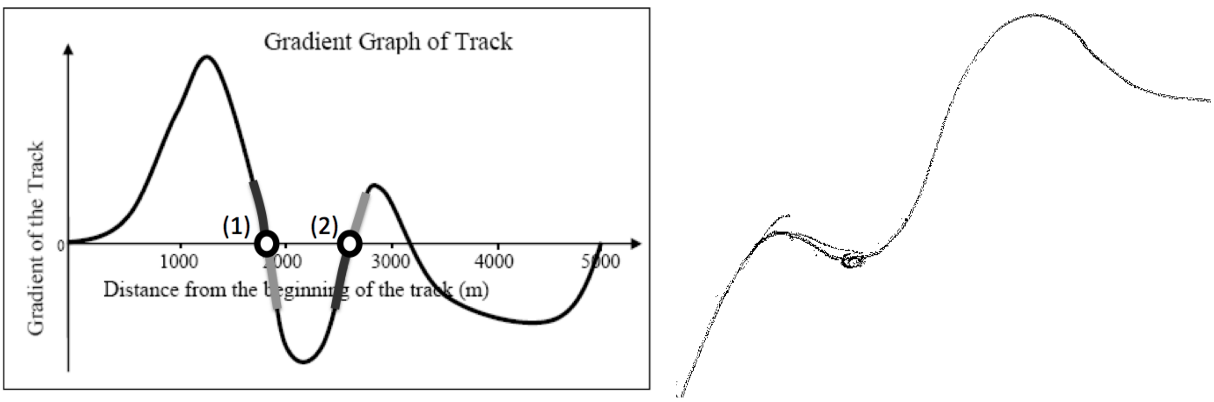


Figure 4: (a) Noa points to the positive and negative segments of the graph adjacent to the two x-axis intercepts. (b) Noa's drawing of a curve.

Noa notices that in the gradient graph they were given, a change from positive to negative y-value corresponded to a maximum in the track, whereas a change from negative to positive y-value corresponded to a minimum in the track, and she wonders whether this pattern applies to any gradient graph. She then draws a new curve with two local maxima and one local minimum (Figure 4b) to explore the behaviour of peaks more generally. She reasons:

Noa: For that to be a peak (*points to first maximum in the curve in Figure 4b*) and not just a variation in the positive gradient – an actual peak, it has to have a downhill (*points to the downhill section to the right of the first maximum in Figure 4b*). So you will have to cross over (*traces with her finger, a curve crossing over an x-axis on the gradient graph in Figure 4a*)… from above to below.

Here, Noa generalises a rule: A peak in a graph will occur at the corresponding x-value where its gradient graph changes from positive to negative y-values. Interestingly, she describes the change from positive to negative y-value in the gradient graph as an embodied experience of *crossing over* the x-axis, similar to the embodied experience of walking the track. Ava agrees, and applies the generalisation back to the gradient graph they were given, saying, "if it goes from positive to negative then it's a maxima", while analysing the behaviour around point 1 on Figure 4a. Ava and Noa's focus shifts away from considering the continuous variation in the y-value of the gradient graph to considering the discrete changes in the y-value (positive to negative) and the corresponding change in the track's direction.

3. Visualising while communicating a general method for finding maxima and minima

Towards the end of the activity, Ava and Noa switch back and forth between generalising the method and communicating their method to the O'Neills—the layperson clients described in the problem statement. Initially, Ava writes the following instruction to the O'Neills: "Find where the graph cuts the horizontal axis. This is either a valley or a peak!" She reflects on the vagueness of this instruction saying, "That's very helpful [*laughter*], you're either up high or you're down low. You'll know when you're there." Her sarcasm suggests that she is aware of the inadequacy of these instructions, and that they need clearer directions on how to identify peaks and valleys from the gradient graph, rather than relying on the O'Neills identifying them experientially.

Noa constructs a visual tool for communicating to the O'Neills the maxima/minima rule that they had previously generalised. She suggests they tell the O'Neills to draw a rough sketch of the track first, showing uphill, flat and downhill portions of the track. She demonstrates this by drawing straight lines on the graph of the tramping track they had drawn (see Figure 5), which correspond to positive, zero, and negative gradients. This rough sketch only describes the direction of the track's slope, and ignores many of the other features they had considered previously, including the variation in the track's steepness and the height of valleys and summits. Nevertheless, the simplicity of the sketch makes it a compelling, visual tool that the O'Neills can use to identify the location of summits and valleys easily.

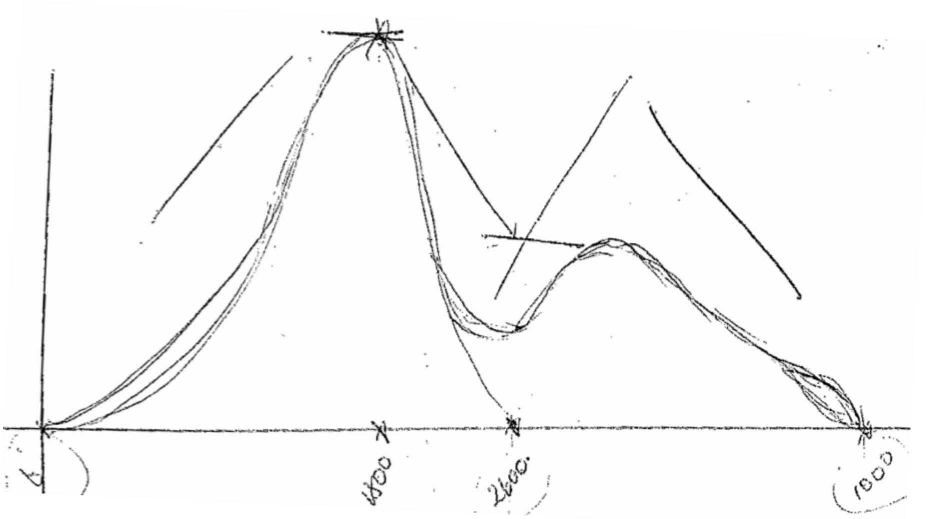


Figure 5. Noa draws straight lines on the graph of the tramping track, indicating positive, negative and zero gradients.

DISCUSSION

Although Ava and Noa drew heavily on visualisation throughout the activity, the nature of the visualisation they used varied depending on the purpose for which it was employed.

While they were engaged in discovering maxima and minima on the antiderivative graph, their visualisation took the embodied, contextual form of gestures, which simulated the experience of actually walking along the track. These gestures enabled them to imagine the physical exertion involved in walking up or down a hill, to notice the continuous variation in the incline, and to encounter summits and valleys along the way. This “physical feel” is similar to what Berry and Nyman (2003) described when observing students use graphic calculators to make sense of gradient graphs in the context of speed. In both instances, the students’ visualisations enabled them to imagine and experience the physical feelings associated with the contexts in which the gradient graphs were set.

The nature of Ava and Noa’s visualisation changed when they began to generalise a rule for determining whether an x-axis intercept in a gradient graph corresponded to a maximum or a minimum in the original graph. Previously they had visualised the entire tramping track, but their attention turned toward the behaviour of the gradient graph (and the tramping track) immediately before and after the x-axis intercepts. In order to focus their attention on this narrower set of mathematical objects and relationships, they drew a portion of a tramping track (Figure 4b) that emphasised maxima and minima, and began pointing at small regions of the gradient graph surrounding x-axis intercepts. These semiotic systems enabled them to shift their structure of attention (Mason, 2004) from the global graph to the local behaviour of the track and its gradient graph in these smaller regions. In doing so, they focused less on the continuous variation in the gradient, and more on the discrete change in the sign of the gradient, which formed the foundation of their rule for determining maxima and minima.

Ava and Noa continued to visualise discrete components of the tramping track when their goals changed from generalising a rule for determining maxima and minima, to communicating that rule to layperson clients. The sketch of uphill, downhill and flat sections constituted a simple tool for visualising the peaks and valleys in the track. Again, the semiotic systems used are indicative of the shift in attention: whereas the gestures used during the discovery phase led to a graph of the complete track (Figure 3), the rough sketch in the communication phase (Figure 5) distilled only those components of the track that would help the clients identify maxima and minima. Both semiotic systems are graphs of the track, but they focus on different mathematical features of the track, due to the different purposes for which they are built. In the first instance, the detailed accuracy of the drawing (and gestures) was important for discovering maxima and minima in the track, as it was the faithful simulation of the experience of walking the track that counted as evidence of maxima and minima in the track. Once the general rule had been developed, however, that faithful accuracy was deemed unnecessary for convincing clients of the presence of maxima and minima. It became more important to develop a simple way of visualising the maxima and minima, and the rough sketch was more appropriate for this goal.

CONCLUSION

This case study supports calls for visualisation to be used in calculus teaching and learning to provide an alternative way of making sense of calculus concepts. It demonstrates that visualisation can be used for reasons other than simply helping students generate or discover mathematical ideas—visualisation can also help students generalise and communicate those ideas. However, this doesn’t mean any kind of visualisation will be successful automatically in each case. Rather, different semiotic systems can be constructed and manipulated to allow for different types of visualisation, depending on the desired goal.

This paper showed that initially, Ava and Noa used complicated gestures to engage in a kind of embodied visualisation, which helped them make mathematical discoveries. When it came to generalising those discoveries, however, Ava and Noa turned to local pointing gestures and simple diagrams that facilitated a visual analysis of key mathematical objects and relationships. In order to communicate their generalisation, Ava and Noa created a simplified graph that could be used as a visualisation tool by layperson clients. Ava and Noa's case study should not be interpreted as suggesting a trajectory of visualisation approaches, or a guide for matching semiotic systems or modes to visualisation needs. Instead, it highlights the importance of being aware of the reasons for which students may engage in visualisation, and challenges educators to evaluate critically the kinds of semiotic systems that they use and encourage in each case.

ACKNOWLEDGEMENTS

I acknowledge with thanks the contributions of Tommy Dreyfus and Mike Thomas, who gave me permission to use in this paper data that we collected together in a joint project. I am also grateful to Tessa Miskell and Joseph Nelson, who assisted by checking and corroborating the coding of the data.

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70(2), 97-109.

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*, 301–317.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' understanding of the calculus. *Journal of Mathematical Behavior, 22*, 481–497.

Eisenberg, T. & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmerman and S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics* (Vol. MAA Notes Series, pp. 25-37). Washington, DC: MAA Press.

Ernest, P. (2006). A semiotic perspective of mathematical activity. *Educational Studies in Mathematics*, *61*, 67–101.

Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning, 12*(2), 152-176.

Hughes-Hallett, D., McCallum, W. G., Gleason, A. M., Pasquale, A., Flath, D. E., Quinney, D., et al. (2002). *Calculus: Single variable*. Danvers, MA: John Wiley & Sons.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Lawrence Erlbaum Associates.

Mason, J. (2004). *Doing ≠ construing and doing + discussing ≠ learning: The importance of the structure of attention*. Regular lecture at the 10th International Congress of Mathematics Education, Copenhagen http://www.icme10.dk/proceedings/pages/side01main.htm downloaded 10th Jan 2012

Tall, D. O. (2004). Building theories: The three worlds of mathematics. *For the Learning of Mathematics, 24*(1), 29–32.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*, 229–274.

Yoon, C., Thomas, M. O. J. & Dreyfus, T. (2011). Grounded Blends and Mathematical Gesture Spaces: Developing Mathematical Understandings via Gestures. *Educational Studies in Mathematics, 78*(3), 371-303.