12th International Congress on Mathematical Education

Program Name XX-YY-zz (pp. abcde-fghij)

8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

YOUNG STUDENTS INTERPRET MATHEMATICAL VISUAL DIAGRAMS

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*Visual representations can enhance children's learning of mathematical concepts; however its structure based use is to be cultivated exclusively in classroom interaction. To learn more about factors influencing »visual structurizing ability«, the CORA project investigates which frames can be reconstructed in young children's interpretations of mathematical diagrams. To analyse and describe these frames, a theoretical construct called »Fric« (**fr**ame based **i**nterpreting **c**ompetence) is to be developed. Twenty clinical interviews with eight year old pupils were videotaped and transcribed before and after a series of ten mathematic lessons in which a variety of mathematical aids were employed. This paper presents theoretical background and first elements of the analysis grid »Fric«.*

frames - mathematical representations – qualitative analysis – visual structurizing ability

INTRODUCTION

The project CORA (epistemological study of **co**ntext and f**ra**mes) - supported by the Ministry of Research and Education, Germany - is based on an empirical research study (Söbbeke, 2005). Söbbeke investigated how far primary school children succeed in constructing abstract structures into a visual mathematical diagram. Based on theoretical research results from mathematics education and psychology and on careful case studies »Four Levels of **Vi**sual **S**tructurising **A**bility (ViSA)« could be distinguished (fig.1). These levels characterise the interpretations in a range between concrete, material based interpretations and relational and structural interpretations (Söbbeke, 2006). Qualitative analysis of clinical interviews showed that children do not spontaneously gain ViSA by using mathematical diagrams in everyday lessons. ViSA has to be specifically developed by introducing young students into a special »culture of using, thinking and speaking about structures and ambiguities in visual representations« (Söbbeke, 2005). We suppose that ViSA is influenced not only by the mathematical knowledge, but also by the approaches and by the handling of visual aids - which are culturally acquired. That means sociocultural factors are a constitutive elements to the structuring competence (Steinbring, 2005; Radford, 2010). To learn more about these influences, CORA's aim is to carefully elaborate the theoretical construct »Fric« (**fr**ame based **i**nterpreting **c**ompetence) in order to analyse the *contexts* and *frames* (Goffman, 1974; Krummheuer, 1984), which are supposed to have an effect on the children's interpretation. Thus, central research questions in CORA are: Which frames that children adopt to interpret visual mathematical diagrams, can be reconstructed (during the interviews and before and after the intervention)? In how far and in what ways do these frames influence children's flexible use and understanding of visual mathematical representations?

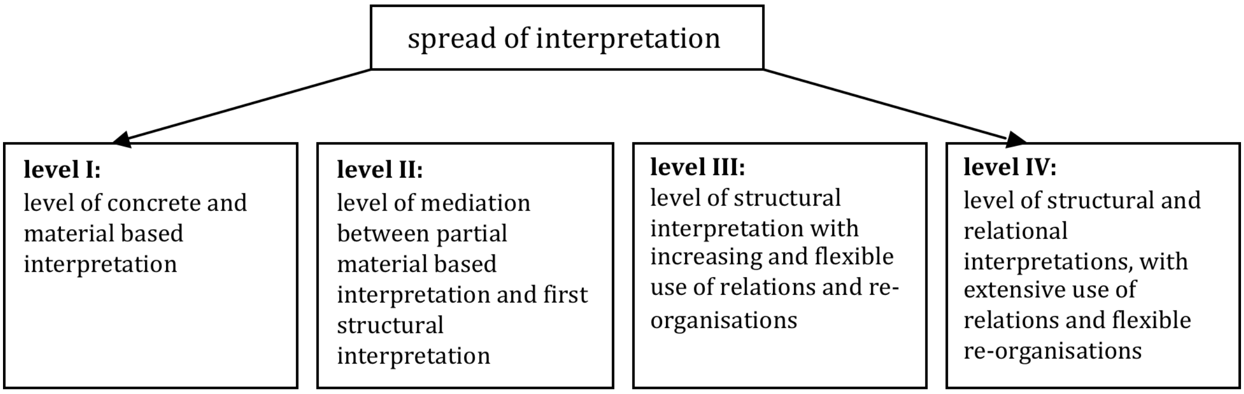

Figure 1. Four levels of »Visual structurizing ability« (Söbbeke, 2005)

DESIGN OF THE CLINICAL INTERVIEW

To investigate these research questions, CORA was planned as a qualitative intervention study conducting 20 clinical interviews at elementary school (third grade students). The same half-structured interviews were performed before and after a series of ten mathematics lessons, in which visual structuring ability was enhanced. During the interviews, pupils worked on tasks to the commonly used visual aids number line and hundred board. In order to demonstrate the interview procedure, a short example is given below:

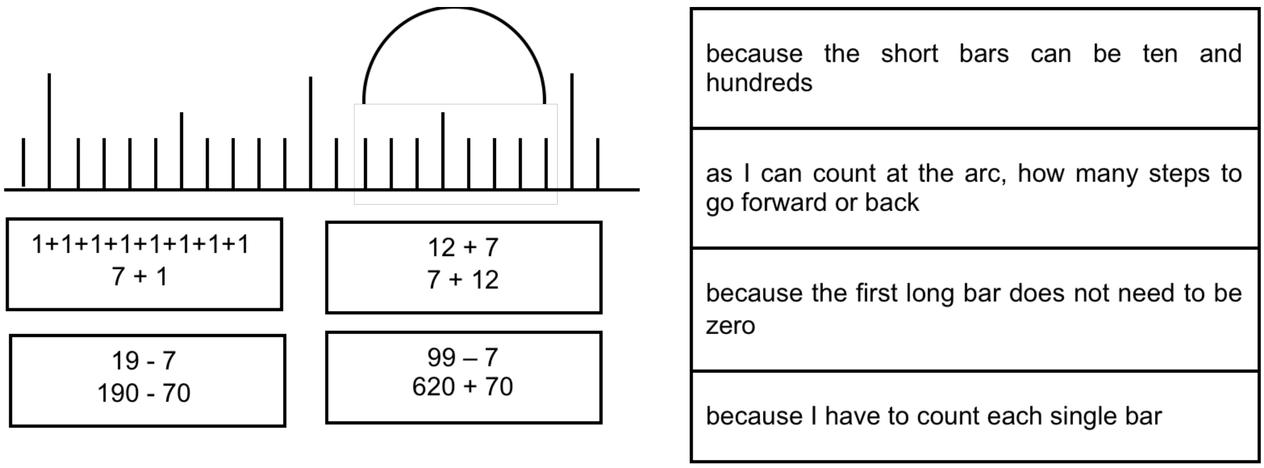

Figure 2. Problem cards (left side) and explanation cards (right side) concerning the number line with arc

First of all, the pupil is asked to choose one of the depicted problem cards that fits exceptionally well to the number line. After that, the student has to explain his / her choice and is asked to draw the chosen item into the scheme. Then, the pupil chooses one of the depicted explanation cards (fig. 2) and gives reasons for his or her choice. Finally, the student is asked to comment on why the other problems and explanations do not fit as well as the chosen ones.

THEORETICAL BACKGROUND

Next to »Visual structurising ability« which characterizes children's interpretations (Söbbeke, 2005), two further aspects configure the interpreting process (fig. 3): The object that has to be interpreted, that is the mathematical diagram with its symbolic elements and the interpreting subject, who interprets within an individual frame. The depiction might suggest a chronological sequence, but we suppose that in a process of interpretation these three dimensions alternately interact in a complex system. In the following, the analysis dimension *context* and *frame* will be explained in more detail.

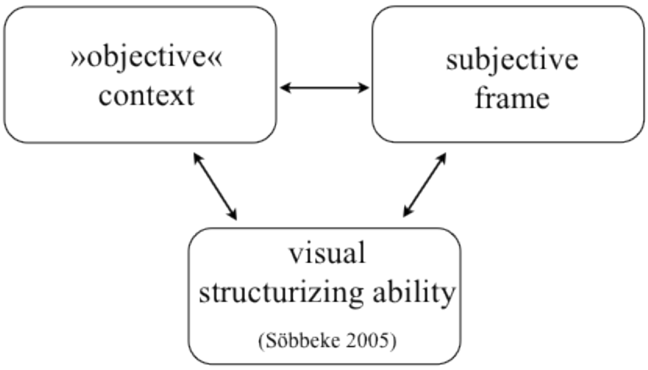

Figure 3. Analysis dimensions in the CORA project

Context elements

The term »context« is borrowed from cognitive psychology which argues that the spatial and situational context of an illustration influences the result of the interpretation (Hoffman, 2000; von Glasersfeld, 1987). According to Hoffman and von Glasersfeld, seeing is not only a stimulus-response mechanism, but a complex and active process of construction. Even within mathematical interpreting processes the spatial and situational context has an impact on the perception of visual diagrams. For instance, the same diagram will be interpreted completely differently in mathematics lessons than during an art lesson (Radatz, 1986). The mathematical diagrams children deal with during the CORA project, differ from »everyday-images«, because they are mathematical symbol systems which have to be interpreted in a specific mathematical manner. Hence, in this project the term »context« defines the given elements of a visual diagram that influence its interpretation. From a mathematic didactical perspective relevant context elements concerning the number line in figure 2 are:

- the single scaling bars
- the length of the different scaling bars
- the first long scaling bar
- the arc
- the basic unit (distance between two neighbouring scaling bars)

These context elements can be used differently and in a range between material-based and relational interpretations (Söbbeke, 2005). Within a »mathematic-didactical« frame for instance, the first long scaling bar could be seen as »500« and the arc as an operator »+70«. By relating the single elements to each other, the term »620+70« might be constructed as a fitting addition to the number line in figure 2. Taking into account the diagram's ambiguity, there are further structures that could be construed, like »690-70«, »69-7« and so on. These flexible interpretations are not self-evident to primary school children at all. For instance, they might see the first long scaling bar as the second bar to count, or as the »fixed beginning« of the number line, that always starts with zero. First results of CORA show that the *frame* within which a young student interprets the diagram influences decisively his or her use and interpretation of the context elements.

Frames

According to the theory of Symbolic Interactionism (Blumer, 1973) people act towards things based on the meanings they ascribe to them. Interpreting each other's actions creates these meanings. Thus, human actions are a kind of symbols or signs that have to be interpreted to become meaningful. The American sociologist Erving Goffman (1922-1982) partly shared positions of Symbolic Interactionism. According to his theory people constantly ask themselves: »What is going on here?« That means everyone has to interpret and to define a situation to coordinate his or her actions. To describe this essential organisation of everyday experiences, Goffman borrowed the term »frame« from Gregory Bateson (1904-1980). In his work »Frame analysis: An Essay on the Organization of Experience« Goffman (1974) describes frames as socially learned interpreting schemes people adopt to give meaning to a situation: »I assume that definitions of a situation are built up in accordance with principals of organization which governs events […] and our subjective involvement in it; frame is the word I use to refer to such of these basic elements as I am able to identify« (Goffman 1974, p. 10). Goffman distinguishes between »primary frames« and »modulated frames«. A primary frame is the most primordial type, spontaneously taken up to define a situation. This primary frame can be transformed by »modulation« which describes a set of conventions through which people alter and expand their interpreting schemes to »modulated frames«. In a theatre play presenting a crime novel for instance, the audience adopts a modulated frame (»play on a murder«) instead of the primary frame (»murder«) to answer the question »what is going on here?«

Krummheuer adopts Goffman's theory of interaction as a theoretical approach to develop »a theory of interaction of teaching and learning mathematics in regular classroom settings« (Krummheuer 1984, p. 285, translated by the author). According to this interactionist approach learning mathematics is understood as a social negotiation of meaning

(Bauersfeld, 1982; Steinbring, 2005). In this context Krummheuer uses the »frame-concept« to describe and analyse subjective interpretation processes in classroom interaction. Based on detailed interpretative analyses of transcribed small group discussions in mathematics classrooms, Krummheuer identified four content-specific primary frames to the topic »term-transformations«: »algebraisch-didaktisch« (algebraic-didactic), »geometrisch-schulmathematisch« (geometric-schoolmathematical), »alltags-geometrisch« (everyday-geometric) and »algorithmisch-mechanisch« (algorithmic-mechanical) (for further information see Krummheuer, 1983, p. 19). In classroom interaction teacher and pupils usually interpret situations within different frames. For instance, the teacher is commonly the only one who has a good grasp on the »algebraic-didactic« frame. In this context, Krummheuer describes learning as approximation of frames through »modulation«, thereby constituting new meaning (Krummheuer, 1983, p. 26). This is relevant for the handling of mathematical diagrams, too, particularly concerning the context elements, because the mathematical structure that is actively construed into a visual aid is the result of a corresponding structured frame and not the result of the visual aid itself (Krummheuer, 1992, p. 188). Thus, not only the context elements, but even the frame within which a child interprets a visual diagram, influence the resulting interpretation decisively. This relationship between the two analytic dimensions »objective context elements« and »subjective frame« will be illustrated in the next section.

Relationship between »context elements« and »frames«

As mentioned in the beginning, two aspects are decisively important to the children's interpretation processes of mathematical diagrams: the object that has to be interpreted as well as the interpreting subject. Whereas the objective context, the elements of a diagram, activates a special interpretation scheme, these interpretation schemes influence the perception and interpretation of the diagram. Bipin Indurkhya, professor of cognitive and computer science, comments on this dialectical reciprocity between subject and object. According to his interaction theory of cognition and metaphor (Indurkhya, 1994), he distinguishes between *ontology* and *structure* of the world. Whereas ontology, defined as »the set of objects or actions in terms of which we experience the world and act upon it: tables, water, trees, cows, front, back, walking, swimming etc. « (Indurkhya, 1994, p. 106) is subjectively and actively created by humans, »[…] the structure of the world depends on the mind-independent external reality« (Indurkhya, 1994, p. 106). Indurkhya illustrates this correlation with a simple example:

> »As another example, consider the lines of latitude and longitude. There is no doubt here that they are created by our cognitive apparatus. Yet, once the ontology is created, the structure, which determines whether two given 'places' have the same latitude or not, is no longer an arbitrary matter, but is determined by the world of things-in-themselves« (Indurkhya 1994, p. 115).

In other words: The subjectively created ontology stamps an objective structure on the perceived phenomena. This perspective shows the reciprocity between context elements and frames in a new light: Somehow, the subjective frame determines the structure between the context elements. Children naturally seldom expatiate their frames - especially because in most cases frames are taken up unconsciously. But first results in the CORA project reveal, that the structure the student construes can serve as an indicator to reconstruct the special frame, the child interprets within. Hence, to carefully develop the construct »Fric«, the analyses in the project CORA firstly investigate »Which context elements does the student use? Which structure does he or she construe?« to get indication for the adopted frame.

EXAMPLE: ANNE INTERPRETS THE NUMBER LINE IN PRE- AND POST-INTERVIEW

The following short analyses of a small pre- and post-interview episode shall illustrate a first element of the analysis grid »Fric« that is used to analyse the student's use and interpretation of the context elements (fig. 5). The interviews were conducted with Anne (8 years old), who actively participated on all of the ten intervention lessons. In both short episodes Anne has to work on the question: »Which problem-card fits exceptionally well to the number line« (fig. 2)?

Pre-interview episode

This short episode is taken from the pre-interviews, which were conducted during the data collection *before* the intervention lessons took place. About three minutes had passed, before Anne suggested to choose »12+7« and finally created »her own« fitting addition »14+7« (fig. 4).

1 Anne: …Ehm, perhaps I would take twelve plus seven *(taps with the pencil on the problem-card »12+7«)*, because ehm these *(moves the pencil under the number line back and forth)* are twelve then, perhaps. (.) But (..) *(moves the pencil shortly from the first long to the second long scaling bar)* hm, no that doesn't work at all *(looks to the problem-card »12+7«, purses her lips)*. Hmm *(6 sec)*.

2 Interviewer: Why doesn't it work?

3 Anne: (…) Because these are more than twelve (*skips with the pencil between the first and the second long scaling bar back and forth)* One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen *(counts with the left finger each single line from the first scaling bar at the left to the 14th scaling bar under the curve)*. Thus, one must have fourteen plus seven *(looks to I)…(draws »14+7« into the number line).*

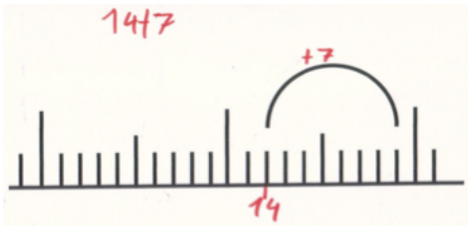

Figure 4. Anne's notation during the pre-interview

To reconstruct Anne's frame, firstly we analyse: Which context elements does she use to interpret the diagram? Which elements are relevant to her? In this short episode Anne's gestures and utterances show that she uses the *single scaling bars*, the *first long scaling bar* and the *arc* to interpret the diagram. She ignores the *different length of the scaling bars* and

does not comment on the basic *units*. She counts in steps of one, but her gestures reveal (l 3), that she counts the single *bars* and not the *units* as distance *between* the bars.
In a second step we analyse: How does she interpret the used elements? Does she use them in a concrete, material way or does she construct relations and structures between the elements of the diagram (Söbbeke, 2005, fig. 1)? In line 3, Anne uses the scaling bars as discrete *material* objects for counting to interpret the diagram. She counts each bar, in small single steps from the left to the right. Here, she interprets the context-element *scaling bar* concretely without considering the different lengths of the lines. The *first long scaling bar* is interpreted *materially*, too, as the second bar to count. With the interpretation of the *arc* as »+7« she even shows a first structural understanding by coordinating the context-elements »arc« and »scaling bars«. The table in figure 5 is a first element of »Fric«, which is used for illustrating and analysing this use and interpretation of the context elements.

used element	interpretation of the element	
	material	relational
single scaling bars (SB)	●	
lenght of the SB		
first long SB	●	
the arc		●
basic unit / increment		

Figure 5. Use and interpretation of context elements

All in all, a view on concrete single countable *unambiguous objects* dominates Anne's interpretation in this short episode: Spontaneously she tries to establish a relation between the item »12+7« and the depicted number line. But she fails, because she cannot »find« and show precisely the first summand in the diagram. Instead of that, she counts each of the fourteen single lines before the arc and summarizes: »*Thus, one must have fourteen plus seven*« (l 3). Anne's explanatory statements and her dominating approach »*counting*« show, that she is trying to find out and to verify the diagram's *unambiguous* well-defined properties (Steinbring 1994, p. 14). This preliminary named *»counting-frame«* determines in some way Anne's interpretation, because within this *»counting-frame«* the fourteen counted single lines are binding to Anne. This becomes obvious at another point of the same interview. Here Anne explains:

> »If there weren't all of these lines *(moves her finger back and forth from the first scaling bar to the beginning of the number line)* then I would take this one *(points at »12+7«).* But there <u>are</u> all. And that's the problem.«

This example clearly demonstrates that in this situation the »counting-frame« is constraining a flexible and relational interpretation. This illustrates the reciprocity between frame, context and visual structurizing ability (fig. 3).

Post-interview episode

The following short episode is taken from the post-test interviews, which were conducted *after* the intervention lessons took place. Here, Anne works on exactly the same task. After looking through the problem cards for 10 seconds, she chooses *two* given items on different problem cards »99-7« and »12+7« (fig. 6). For choosing the latter, Anne gives the following reasons:

1 Anne: Mmh. Twelve *(points on the short scaling bar under the left end of the arc*). It's just twelve then, that are ten twelve, yes *(points from the first long to the second long scaling bar, pauses for a while and then points again on the short scaling bar under the arc)*. Ehm, then here is zero *(writes "0" under the first long line)* then these are twelve *(draws an arc under the number line from the first long line to the short line under the arc)*, like this.

2 Interviewer: Mmh.

3 Anne: These are twelve *(writes »12«under the self made bow)* and then plus seven *(writes »+7« above the arc)*.

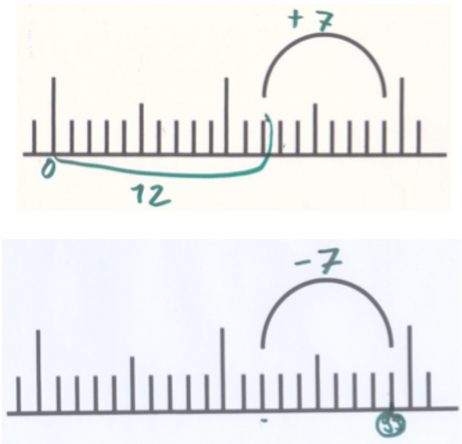

Figure 6. Anne's notation during the post-interview

As we can see in the transcript, Anne uses the same context elements as in the pre-interview. Even here there are no indicators specifying that she takes into account the basic unit. But in contrast to the previous episode, Anne considers the *different length of the scaling bars*. Her gestures *(points from the first long to the second long scaling bar, pauses for a while and then points again on the short scaling bar under the arc* l 1) demonstrate that she uses the structure given by the different lengths to define the ranking number »12«. In this episode Anne does not count, but relates the scaling bars to each other to establish a relation between »12+7« and the given number line. Hence, *single bars* and the *length of the scaling bars* are used in a *relational* way. The fact that Anne even selects »99-7« indicates that she interprets the *arc* as representation for an addition *or* a subtraction. Additionally, her selection implicitly points out, that the *first long bar* is flexibly seen, as »0« or »80« and interpreted in relation to the other scaling bars (fig. 7).

To sum up, Anne uses each context element in a more relational and flexible way: In this episode Anne creates *different* meanings by coordinating structure units demonstrating her basic understanding of an *ambiguous* number line. She presents *relational* strategies and arguments that resemble »if-than-sentences« (l 1). That offers a »hypothetic and implicative use of the number line« (Steinbring 1994, p. 13). This *»relational frame«* enables her to construct a more complex and flexible structure in the post-interview. Accordingly Anne starts to use the concrete context elements as parts of a mathematical *symbolic system*.

used element	interpretation of the element	
	material	relational
single scaling bars (SB)		●
length of the SB		●
first long SB		●
the arc		●
basic unit / increment		

Figure 7. Use and interpretation of context elements

Summary

These short analyses show that Anne's interpretations are more relational and ambiguous in the post-interview in comparison to the pre-interview. While trying to »decode« pre-given unambiguous and correct solutions in the pre-interview, she does something different in the post-interview: She actively allocates numbers and relations to the number line benefitting by the diagram's intended structures. Hence, from pre- to the post-interview, Anne has modulated her frame from a »counting frame« (focusing on discrete objects to count) to a more flexible »relational frame« (focusing on relations and not on the objects themselves) in the post-interview. Previous context and frame analyses in the CORA project reveal, that each student interprets the diagram within a specific individual frame, thus all frames reconstructed until now differ in some detailed aspects. In a first approach, these specific frames can be categorized in a range between *»unambiguous object - oriented«* frame types and *»ambiguous system – oriented«* frame types. Within the prior frame children focus on unambiguous objects (as Anne did in the pre-interview), within the latter the child focuses on the ambiguity and relations in a symbol system (as Anne started to do in the post-interview). To refine the construct »Fric«, these provisionally formulated frame types will be complemented, and elaborated more explicitly by further detailed analyses.

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