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DESIGNING FOR VISUAL INQUIRY IN SCHOOL MATHEMATICS

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In this paper and presentation, I will introduce the approach guided us in the development of the VisualMath curriculum. I will start with the principles that guided our organization of the domain, and use this organization scheme to describe our vision of classroom instruction and then proceed with technological innovations, the design of learning resources, and the design and facilitation of instructional patterns.
Key words: Algebra, Function-based, Curriculum, Technology

THE CHALLENGE AND THE VISION

VisualMath (Visual Math CET, 1995) was designed to challenge traditional notions of what school mathematics is and how it can be taught and learned. For over two decades, this curriculum has been implemented in a variety of settings in Israel. As a product of academic laboratory development, VisualMath aims at addressing future innovations and analyzing the potential of new technologies. Innovative resources were designed at a time when personal technology, e-books, and mobile phones were still in their infancy. Developing innovative instructional materials in an environment in which technology is continually changing involves a risk of letting the technological aspirations drive the design of the educational engine. Developing with and for schools, which for the last three decades have been undergoing worldwide educational reforms aimed at standardization of skills and centralization of assessment, means coping with societal and curricular constrains that at times are in tension with innovative visions. The best example for our visionary classroom is in Lakatos's Proofs and Refutations (1976). It demonstrates the evolution of mathematical knowledge as a process centered around conjecturing. Lakatos did not intend to present a class. He used the situation of teacher, task, and students as "…a sort of rationally reconstructed or 'distilled' history" (p. 5). Lakatos's essay presents a case study that challenges mathematical formalism in order "to elaborate the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of theorems but the improvement of guesses by speculation and criticism" (p. 5). Lakatos provided what he defined as a "simple pattern of mathematical discovery – or of the growth of informal mathematical theories" (p. 127), consisting of the following seven stages: (a) a primitive conjecture; (b) proof (a rough thought, experiment, or argument); (c) global counterexamples to the primitive conjecture; (d) proof re-examined; (e) primitive conjecture improved; (f) other theorems examined to analyze the newly proved concept that “might be lying at the cross-roads of different proofs;” and (g) analyzing accepted consequences and turning counterexamples into examples of a new field of inquiry.

This is my simplified picture of the process. This pattern provoked two essential questions, and our attempts to solve them led to the design of the Geometric Supposer (Schwartz & Yerushalmy 1995/1990), and later to the design of the VisualMath curriculum. One question concerns the challenge of the first required step, that of “[having a] primitive conjecture;” we wondered what could be regarded as “primitive” for young learners, and whether it could be developed to be considered important mathematics. The second question concerned the quasi-empirical process of reasoning: what would motivate mathematics students to argue, refute, and revise conjectures, and could such prominent habits of mathematical reasoning form the routine structure of school mathematics. Naturally, learners need scaffolding in order to perform a mathematical process that would provoke doubts, questions, and the motivation to solve the problems.

VISUALIZATION OF THE DOMAIN

To support the vision of Lakatosian inquiry for all students within the school system, we had to resolve the tension between curriculum constrains and classroom conditions on one hand, and the personal constructions required for learning meaningful mathematics on the other. For us, the first step in resolving this tension was to map “the field,” that is, design sets of mathematical objects and actions on the objects in order to define the major coordinates of the field and the relations among them.

The first design action toward representation of the domain was to outline our mathematical intentions stating distinctions of “functions.” The second design action toward representation of the domain was to map the field of functions: central objects and actions on one hand, and the objects and the content of school algebra and calculus courses on the other. Our most important challenge was to do it in a way that provides as simple as possible a map that (a) suggests how the terms of the objects and actions of a function reorganize the known school requirements; and (b) makes visible the variety of alternative routes for navigating the domain. The representation acquired the shape of a 2D matrix (Figure 1). The mathematical objects, i.e., the functions that are listed across the top and the columns, mark mathematical actions performed with these objects. This matrix can be imagined as having a 3rd dimension that marks the leading representation of the object and actions.

Operation / Objects	Represent	Modify	Transform	Analyze (function change)	Operate (with 2)	Compare
Generic						
Linear						
Quadratic						
Polynomial						
Rational						
Absolute Value						
Irrational						
Periodic						
Exponential						

Figure 1. Conceptual map of function-based algebra

To identify the central terms and skills of the traditional algebra curriculum on this map, it is necessary to clarify the analog function terminology. In Yerushalmy & Shterenberg (2001) we describe the visual course of learning algebra as derived by this map. Equations and inequalities are comparisons of two functions. Solving could be driven by symbolic operations, but the graphic representation can take the lead in Solving an equation or inequality as geometric transformations (e.g., reflection, translation) of the graphs of given functions are another way to produce new equivalent comparisons. Constructing algebraic models of phenomena (traditionally referred to solving contextual problems) is led by analysis of how a function changes, evaluating its rate of change as graphic object (stair), or attempting to approach calculus and derivatives by looking at the difference between two functions, one being a translation of the original one.

Representing the domain of linear, single-variable word problems had been modeled using a sketch representing two comparisons of two functions (Yerushalmy & Gilead 1999): two intersecting lines that have the same inclination and two that have opposite inclination. We identified eight various problem situations derived from this representation (Figure 2). With some limitations, this organization allows the mapping of most algebra word problems, linear or other, that involve a situational structure of comparison of two functions. This design makes it possible to reduce the number of problem types to be taught, and makes the study of the differences and similarities between problems mathematically interesting and manageable.

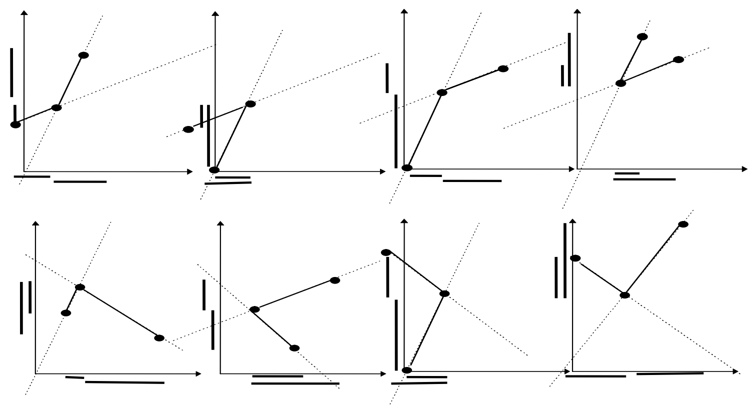

Figure 2. Eight problem situations (adopted from Yerushalmy & Gilead 1999 p. 194)

The map is different from the mapping that presents organization of the mathematics shown in the above matrix. It is also different from cognitive maps attempting to represent personal meta-cognitive processes. This map is an iconic representation of the mathematics of relations between linear functions. It functions as a tool for teachers planning the course and guided the design of tasks. Tasks go beyond solving or specific problem towards inquiry of the domain of temporal models including analysis of Problems' Similarities (as in figure 3 where different problems form a cluster of similar situations all generated by the same interactive diagram.)

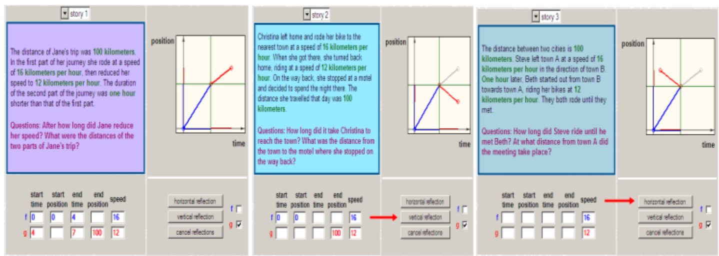

Figure 3. Reflections on Trips
http://www.cet.ac.il/math/function/english/line/applets/Applets/motions.html [1]

EXPLORATIONS WITH MULTIPLE REPRESENTATIONS TOOLS

Over the 25 years of the development, the hardware and affordances have been changing almost continuously. Newer designs reflect new technological capabilities and different human-computer interfaces, but the meta-design principle of visible coherence remains a leading design goal. In reviewing major challenges of adopting technology to support mathematics' education many would point on hyper technological expectations and ambiguous intentions regarding the roles of digital mathematical tools, being two serious obstacles to current attempts to improve math education. Pimm, for example, points that "There has been and continues to be concern, confusion and resistance over the use of hand held calculators, even some 15 years after their introduction to primary schools" (Pimm 1995, p. 82). Similar arguments have been made in regard to the role of symbol manipulators in the learning of algebra. Questioning the appropriate image of software tools became a major component in

[1] Another example of a task designed to produce a meta-level mapping is http://www.cet.ac.il/math/function/english/line/applets/Applets/motions.html

transformational designs: Should the software serve as "an intellectual prosthesis" as Schwartz argued (Schwartz & Yerushalmy 1987) or would the "scaffolding" image better describe its use? What of the many images borrowed from work with tools, artifacts or instruments in different cultures and domains could best contribute to characterize a design?

It was our attempt to contribute to this discussion by designing software tools that can be used in many ways upon different situations, but its role in supporting explorations would be evident. For example, software for explorations does not attempt to make complex algorithms easy, nor does it use sophisticated procedures for manipulation. Rather than looking for ways of improving computational algorithms, of presenting slick symbolic syntax, or of attempting to interpret users' actions and intentions, we must invest in designing an artifact that reflects the organization of the course (e.g. the terms in the maps), that mediates between bodily actions and semiotic activities and between people collaborating and participating in group work (Figure 4).

Figure 4.1: Artifact that visually reflects the organization of the course

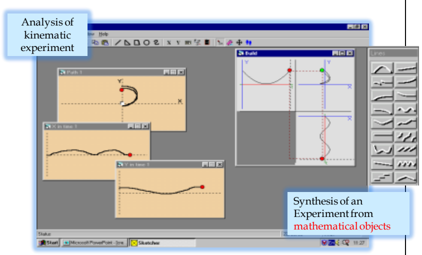

Figure 4.2: Mediation between bodily actions and semiotic activiti

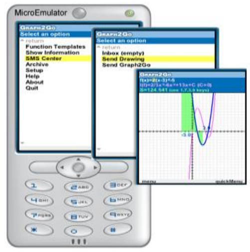

Figure 4.3: Collaborating in visual discussion in mobile learning sessio

THE MULTIMODAL INTERACTIVE TEXTBOOK

Technological artifacts never intended to replace other instructional materials but to challenge the traditional formats of these materials. Textbooks provide guidance to teachers about the content and how the content should be learned, and they organize the content for the student. Therefore, they remain essential mediators between the vision and the teaching. We envisioned the textbook as a mediator that prompts engagement: of the teacher to plan the instruction in a variety of ways that reflect the empirical processes followed by the students; and of the students, by providing and demanding interactivity through hypertext, multimodal reading, and interaction with embedded dynamic diagrams and tools. Years before hypertext and digital texts were available, we sought designs that provide incentives for multi-modal and non-sequential reading. The engagement with tools was then designed to be part of the book, and tasks used a variety of formats to support conjecturing and to organize conjectures.

Although the VisualMath Function eBook (Yerushalmy, Katriel & Shternberg 2002/4) was designed to be used as a formal text for secondary algebra students, the appearance is that of a less formal milieu: a museum. A leading image in designing the VisualMath Function eBook was the distinction Kress and van Leeuwen (1996) made to describe linear and nonlinear texts. They compared the linear text to "an exhibition in which the paintings are hung in long corridors through which the visitors must move, following signs, to eventually end up at the exit," and the non-linear text to an "exhibition in a large room which visitors can traverse any way they like… It will not be random that a particular major sculpture is placed in the center of the room, or that a particular major painting has been hung on the wall opposite the entrance, to be noticed first by all visitors entering the room" (p. 223). Opening the VisualMath eBook is like entering a hall that leads to various galleries, each gallery consisting of separate dedicated rooms (Figure 5).

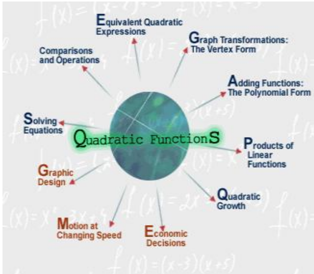

Figure 5. The hall of quadratics   http://www.cet.ac.il/math/function/english/square

There are probably too many galleries and too many rooms in each gallery for the visitor to become engaged with each one of them. There are different ways of visiting the exhibition hall, and museum guides provide different tours. Most visitors may required assistance, although there are those who would prefer to study the art by wandering around on their own and reading the information that appears next to the exhibits. There may be others who have already heard and read about the recommended tours or are revisiting the museum and would like to do it their own way. After a gallery has been chosen, the guide may or may not leave the visitors alone for various periods of time. The guide may leave the central piece at the entrance to be explored later, and lead the visitor directly to some other rooms. The guide may also suggest that before visiting the exhibit, guests should enter the tools shop to try out the interactive environment that would later help them understand the art. I would use examples from the Quadratic unit (http://www.cet.ac.il/math/function/english/square) will demonstrate a few design intentions that attempt to make the "museum image" visible. Using this interactive book teachers are engaged in designing and planning the course and they may personalize the plan for their students.

GUIDING VISUAL INQUIRY

The organization of the algebra field and the design of dedicated tools, books, and tasks were the necessary mediators. But all of this was not sufficient to answer the FAQ: "How does your envisioned class look?" Designing the methods for guiding classroom inquiry remained an enormous challenge. Whole-class guidance was not the only type of interaction. We had more definite answers related to the explorations of individual students or small groups, as the designed tools and tasks guided the work. But the process of change that teachers had to undergo when considering to replace lecturing with guiding was the most challenging and fascinating one. We sought a systematic, long-term documentation of secondary schools teaching VisualMath in Israel to identify analytic frameworks that would be helpful in turning the specific views into methods for guiding secondary school math inquiry. We devoted a few years to the systematic recording of lessons taught by a pioneering group of middle and high school teachers in schools of different socio-economic levels and norms. We initially recorded the lessons with the intention to capture teaching with different didactic styles at different points in the sequence. I would end with glimpses into this work, highlighting foundational patterns of guiding visual inquiry.

		
Documenting students' mathematics	Conflicts around a controversial task	A Public group meeting

Figure 6. Guiding patterns of visual inquiry

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