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THE MEANING OF WRITING

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Writing about one's own doing of mathematics is a topic regularly found in papers on mathematics education. Fewer authors focus in their research on the act of writing when doing mathematics. The following paper concentrates on this kind of writing. What is the meaning we can give to writing - or preferably inscriptions[1] *- when we are learning mathematics. Considering the breadth of the field the statements presented here offer only a short view on the relation of speech and the written when doing mathematics. Building on various linguistic theories between speaking and writing and a case study, the aim of this paper is to stress the suggestion that the written form is more than simply a visual substitute for the spoken word. Using this for the learning of mathematics, I will argue that when doing mathematics new ideas can emerge from the written.*
Visualization, writing, semiotics, geometry.

INTRODUCTION

Literature on mathematics education offers a series of research results on the meaning of writing about doing mathematics, "post-process". Examples of these results and their lively and sometimes controversial discussion can be found in Doherty, 1996, Morgan, 1998, Porter 2000, or Pugalee, 2004. The majority of these studies view the written as an instrument of secondary importance. They investigate the use of texts students write on their own learning of mathematics. Writing about one's own doing of mathematics offers a chance to learn mathematical concepts. Therefore, students create their own texts on mathematics (Eigenproduktionen in German, Maier, 2000). Here we find mathematical diaries and similar extensive descriptions of mathematical activities. With their help the language of mathematics may become part of the student's language. When reading these papers very little information is presented about the written itself as a means of learning mathematics.

In the following I will present a particular – in some sense complementary – view on the written when doing mathematics. I will argue that the written is more than just materialized speech, that it is, so to speak, more than something that follows the spoken. In explaining my

[1] For the use of „inscription“ see Latour, 1990 or Roth, 2003. I use inscription to describe anything that is written on paper, blackboard, computer screen etc. An interesting research question would be finding similarities and differences between the use of the *inscription* and *image schemata* as presented by Mark Johnson (Johnson, 1987). In my paper I will not investigate this question as I concentrate on *materialized* mathematics being always visible to our eyes whereas "image schemata operate at a level of mental organization that falls between abstract propositional structures, on the one side, and particular concrete images, on the other." (Johnson, 1987, p. 29).

ideas I will start by presenting a video-based case study, which reports on two students' mathematical activities while solving a geometrical task. Afterwards I will refer to some valuable suggestions from media theory (linguistics) and its view on writing. So what is the significance of the written[2] in learning mathematics when new knowledge – knowledge in *statu nascendi* - comes into existence? I will argue that in some cases the written itself can become a source of new ideas when learning mathematics.

METHODOLOGY

To illustrate my view I will use a video-based case study[3]. It shows two 14 years old pupils solving a problem. Their activities together were captured by two video cameras. For the purpose of supporting the evaluation, the two video pictures were incorporated in a single picture (see figure 1). One camera was fixed in one position, while the observer focused the other on interesting details. The students were given 90 minutes time to answer the question presented to them. The video was taken in the afternoon when classes had finished.

CASE STUDY PART 1

Figure 1 shows both students and the object they had to investigate. The students had been asked to describe the movement of the given object (a surface of revolution) on the table. They were to use their mathematical/geometrical knowledge. The question seems to be formulated in a completely unrestricted way. This openness was intended. The researcher's aim was to establish a context where both students could themselves feel like researchers. The study was designed in such a way that they should write down all their attempts without looking for an algorithmic solution. A more narrowly formulated task would have resulted in such a strategy. The number of tools both participants of our case study were allowed to use also mirrored the openness indicated above. Besides their tools for doing geometry (ruler and compass), they could also use in addition different measuring tools (a tape measure, vernier calipers which is a tool for measuring the diameter of a circle) or computer software (spreadsheet software and software for dynamic geometry). Observing the video the viewer can easily recognize two courses of action. In both lines inscriptions – i.e. written forms – were invented and widely used. However, these inscriptions differ greatly from each other.

Figure 1

The first line starts from the observation of the movement of the surface of revolution on the table. Several times the participants in our case study, like young children playing with a toy, pushed the object to roll on the table and observed this rolling with great attention. Thereby

[2] Research results on similar questions – inventing and using inscriptions – can be found in diSessa, 2000.

[3] I am grateful to M. Katzenberger – mathematics teacher at Gymnasium St. Paul/Carinthia - for making this video available to me. He produced it in spring 2005.

they focused their interest on the points of contact where the rolling touched the table. Therefore, they could imagine a closed curve of these points. However the question of how to record the history of this rolling remained, and of how to fix the contact points? A clever strategy, invented by both students, was to bring them a step further. They took several sheets from a stock of paper to let their object roll on a "soft plane". While rolling the object on the sheets one student pressed it down hard onto the paper. The result was the complete and visible trace and impression of the movement on the sheets. In this way the history of the movement was documented as shown in figure 2. All subsequently constructed traces of the objects rolling - in this first action line – were built using this impression technique, which itself is nothing other than a special kind of inscription. It took only a few moments for both students to suppose that both curves are circles with a common centre. This point was found by means of elementary geometry. A circumscribed square was drawn around the greater "circle" using the impression. The diagonals of the square immediately led to the centre of the

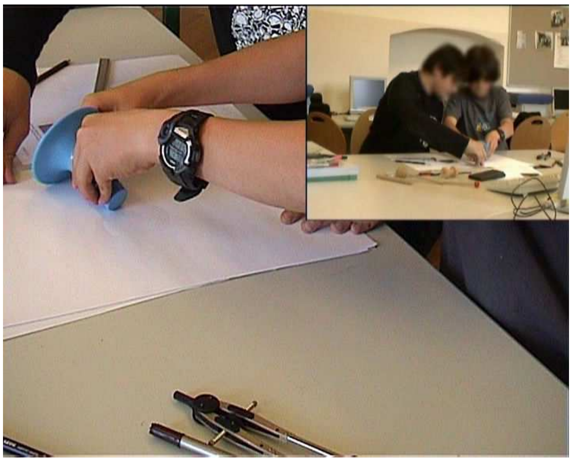

Figure 2

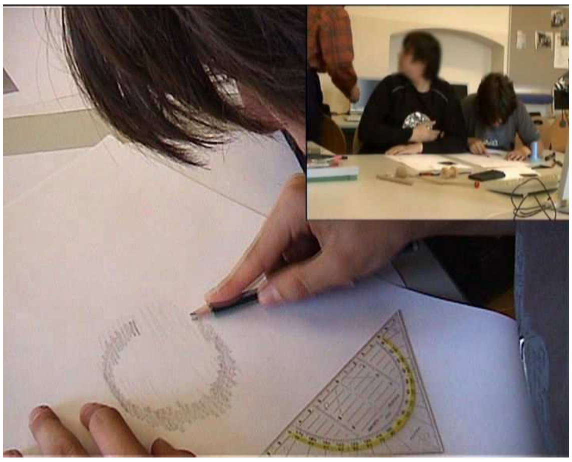

Figure 3

circle. It is worth recording that, although they had learned it in their geometry lessons, our students never used the theorem about the circumcircle of a triangle.

When watching the video one recognizes that many steps on the way to a solution were heavily influenced by inscriptions which the students had already produced or which they invented and drew 'as they worked'. Starting with a virtually unsystematic playing with the surface of revolution, they followed a strategy which enabled them literally *to feel the curves they were looking for*. To strengthen this first tactile impression and to make it more utilizable for their visual senses one student coloured the impressed curve with his pencil (figure 3). From the seen and the felt, both students conjectured that the curves they were looking for had to be circles. This paved their first way to a solution to the given problem. Thus our students used their inscriptions en route to their goal. The written – in this case study a geometrical construction with all its peculiarities - did not follow the spoken. On the contrary, – at least in my view – the written was the precursor to formulating the next step towards the solution, i.e. the writing comes first then the speech follows.

THEORETICAL APPROACHES

A review of research reports on mathematics education indicates quite different tools for interpreting students activities. Some of them focus on investigating relations between internal or mental representations and external or physical representations (Goldin and Kaput, 1996; Goldin, 1998, 1998a). Others pay particular attention to the language used when doing mathematics. They offer very thorough interpretations of students activities applying methods from hermeneutics (e.g. Krummheuer, 1997).

In 1987 Latour wrote: “Before attributing any special quality to the mind or to the method of people, let us examine first the many ways through which inscriptions are gathered, combined, tied together and sent back. Only if there is something unexplained once the networks have been studied shall we start to speak of cognitive factors.” (Latour, 1987, p. 258). We can also follow Latour’s suggestion using means from semiotics (Dörfler, 2005; Hoffmann, 2005; Kadunz, 2006;). In this paper I will offer another approach.

If we look into the history of western thinking, we notice that numerous philosophers, linguists or semioticians did not follow this view on the writing-speech relationship. The linguist Roy Harris (Harris, 2001) describes in detail this relation from a linguistic and a historical point of view. The only example I wish to take from Harris, is his reference to Aristotle’s strict separation of the written from the thought. He argued that the written (grammata) is inferior to the spoken as the written is ruled by convention[4]. To think and speak comes first; writing is only in second place. At least during the 20th century several interesting texts were published showing the relation of the written and speech in a new light (Harris, 1986, 2001; Leroi-Gourhan, 1993; Krämer, 2003). Harris as well as Leroi-Gourhan argued that the roots of writing could be found neither in an image like doubling of facts nor in a linear doubling of human speech. One step beyond this is Harris’ claim (Harris, 1986) that before man was able to use letters he learned to use numbers. Man became “numerate” before he became “literate”. Similar ideas on the history of counting can be found in Schmandt-Besserat (1997) or Nissen (1993).

A useful position on investigating writing is presented by authors like Sybille Krämer (Krämer, 2003) or Wolfgang Raible (Raible, 2004). Both of them can be seen in the tradition of Harris and Leroi-Gourhan. One of Krämer’s questions in her paper from 2003 asks whether the only job of writing is fixing the spoken. Furthermore, she asks whether the order of writing follows the order of the spoken. If we answer this last question positively, then “…it is only the presence of the graphic-visual dimension that is admitted to the writing” (Krämer, 2003; p. 159; my translation). It is this graphic-visual property of writing we can recognize in the two dimensions of a written text, which are a source of viewing structural aspects to the reader and the writer as well. Modern literature (poetry) makes use of this. There is no equivalent of this aspect in spoken language. Krämer’s conjecture that, by writing, the order of our thoughts can become visible is significant for me. We find examples

[4] “A possible clue lies in the fact that Aristotle was born about twenty years after an important orthographic reform: the official introduction of the Ionic alphabet to Athens (403 BC), replacing the previously used local Attic alphabet. Naturally, documents and inscriptions in the old Attic alphabet did not disappear overnight. Every Athenian of Aristotle’s generation was perfectly well acquainted with the two systems, and therefore with the following facts. … the possibility of changing alphabets shows that there are no intrinsic links between grammata and sounds: grammata can be invented, borrowed or adapted to suit any needs.” (Harris, 2001, p.35)

in the table of contents of a book. By looking at such a table for just a second, we get an impression about the importance of the parts of the book. We do not need to read the table of contents sequentially. Other examples which present aspects of our thoughts in the written are the use of italics or footnotes.

“What is presented in a text is not the phonetic event (Lautgeschehen) but structural facts such as grammatical categories and relations between thoughts and structures of arguments” (Krämer, 2003, p.160; my translation).

Krämer offers these ideas as a basis for an alternative theory of writing. Thereby she investigates writing as a medium, a symbol system and as *Kulturtechnik*. It would be far beyond the scope of this paper to present Krämer’s ideas in detail. There is only one point to which I wish to refer. In her deliberations about writing as a symbol system Krämer writes about the construction of “cognitive objects“(Wissensdinge).

“A phonographic understanding of writing is based on the assumption that writing refers to speech. In contrast to this position, we presume that the reference for all writings are abstract things, more or less theoretical entities, which are not visible. If this assumption holds then the power of notational iconicity lies in the fact it brings everything we can think, and which is thereby invisible, to the register of perception.” (Krämer, 2003, p. 164; my translation).

I stress that this capacity of writing is in Krämer's view not a capacity with which we can see the invisible – whatever this may be – behind the visible. Rather, she asks whether this bringing to the register of perception is by itself already a form of creation of that which is offered to the visual sense.

Beside the structural aspects as a result of the two dimensions when looking on a written text there is another aspect of the written when doing mathematics. The written can be seen as a means for performing operations or as a system for doing operations. Following Krämer we call it operative writing (“operative Schrift”). This kind of writing does not concentrate on spoken language and so it does not serve communication immediately. What are the profits we can expect from using the written as “operative Schrift”? I mention two: Exploration and cognition. If we take notice of exploration then mathematical writing offers the opportunity to transform mathematical signs following very strict rules. While transforming there is no need for considering the semantics of the signs. The simple addition of natural numbers is an example for using operative writing as we only need to know an algorithm for adding and a multiplication table we have learned by heart in primary school. When solving linear or quadratic equations we also need no semantics. We just stick to the algebraic rules. If there is no need for interpreting the activities done then we are free to concentrate on the beginning and the end of a “calculation”. We can change the given parameters to explore their impact. The following case study part 2 will present the example of one of our students searching for an error he had made using an algebraic equation.

I mention a second aspect. As operative writing frees us from time consuming interpretation of (mathematical) signs we gain the freedom to interpret the results of a calculation. In front of our eyes - metaphorically speaking - new ideas can come into existence. In this sense operative writing serves our cognition. With these brief indications that writing itself can construct the new, I shall now return to the students and to their activities.

CASE STUDY PART 2

The video data I will present now offers a new solution of a very different kind. The route to this solution can be seen from three positions. From the first we see free hand drawing plays a crucial role. As a second I mention the collaboration between the students where they use one drawing together and from this drawing develop the main solving strategy. As a third we will find in the students' activities different kinds of inventing and using writing and drawing, in particular the rule-governed transformation of an algebraic equation.

After finding their solution empirically, the observer encouraged both students to look for an alternative method of solving the task using the measuring tools offered. They suggested the vernier calipers to be the best tool for this purpose. The diameters of the base circle, the diameter of the top circle, the height and the distance between points of these circles were measured with these vernier calipers. Finally, these measuring activities were the source for the inscription shown in figure 4. To be able to judge the creation of this inscription it is necessary to explain the mathematical-geometrical background of our students. During class seven and class eight both were members of a course named "Geometrisches Zeichnen" which means geometric drawing. Geometric drawing (Technical drawing in England) is a subject of instruction taught in Austrian academic secondary schools (Gymnasium) and in lower secondary schools (Hauptschule) as well. The main topic of this subject is to learn how to draw a plane or spatial object following the laws of geometry. Computer software is widely used. Beside this, they develop drawing skills for producing sketches with and without measured values. Furthermore, our students' mathematics teacher always encouraged them in their mathematics lessons to produce sketches when they had to solve a mathematical problem. Therefore, inventing and using sketches with measurements was part of their mathematical life. The sketch we can see in figure 4 became the starting point for a new method of solving the given task. After this student, I will call him B, had finished his

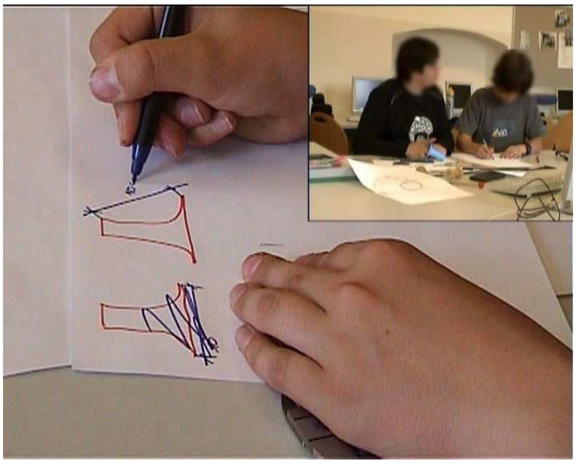

Figure 4

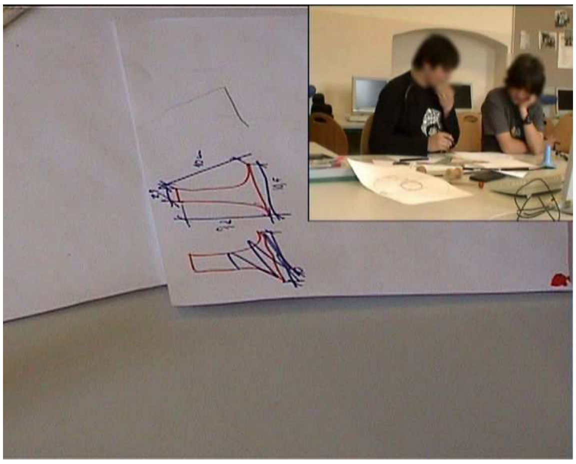

Figure 5

drawing he started to label it with measured values obeying labelling rules he had learned in school. During sketching and labelling our three- dimensional object became an object in the drawing plane. A problem from geometry in the three-dimensional space was transformed into a problem in plane geometry (figure 4). After observing the drawings in figure 4 the other student, I will call him A, started to draw a right- angled triangle. We can see the faint drawing in figure 5. Then he stopped and both students compared the given object with their drawings. After several minutes student B started a further attempt. The three sketches – figure 5 – motivated student B to look for similar triangles to calculate the cone which encloses the given surface of revolution. To fulfil this plan he had to calculate the length of a segment from an arbitrary point of the base circle to the unknown top of the cone. In figure 4, we see one part of this segment, which was measured with the vernier calipers. The idea of employing similar triangles developed not only from the sketch in figure 4 but also from the sketch student A had drawn (figure 5). Let's hear what student B said after two minutes of carefully observing all sketches.

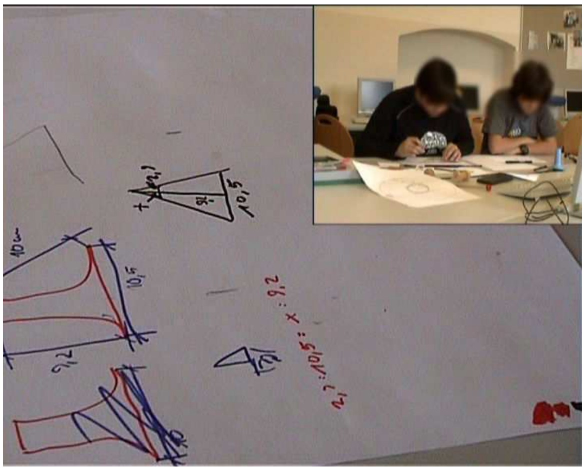

Figure 6

B: Just wait a moment.
In the meantime student A had begun to draw his vertical projection.
B: Now let me draw. Do you know what I have thought? It is the intercept theorem that represents the relation!
A: (looks doubtful)

B started his explanation with the aid of his labelled sketch. Then he began to draw a new inscription. He labelled it with all the measurements (figure 6) and used this inscription as a means to establish an algebraic equation.

In his first equation, he made a mistake. As he compared his solution with the already existing "engraving" solution he recognized his error. So he made another attempt using the intercept theorem. He labelled A's faint drawing of a right-angled triangle – not shown in figure 6 – with measurements and obtained a second equation from this drawing. This equation led after some transformations to another numerical solution which fitted the "impressed" solution.

INTERPRETATION

Viewing this video and taking Krämer's ideas into consideration, we can say that "initial" ideas – the ignition so to speak - and their verbal formulation often started immediately after a new sketch was finished. Both the ways of arriving at a solution that I have presented here support my view on writing and drawing when doing mathematics.

If we remember the first data I presented in part 1 the successful idea for finding a solution started from rolling the object on the table. The students had already used this kind of movement when they investigated other bodies of revolution. Finding their interesting strategy of pressing the object into the sheets of paper emerged from a rather chance observation. The students' achievement was their connecting of the impression and the given task. This impression was just a necessary requirement for finding the first step to the solution. Memorizing the colouring of the impressed curve we can say that the first solution was determined by their senses. Hand and eye the sense of touch and the visual sense organized the student's activities.

Compared with the data given in the case study part 1, the data from part 2 seems to be more profitable for my enterprise as I will show now in this precise description. After having made a series of measurements B started to draw a sketch from the axial section of the object which he labelled carefully. The labelling with all its details was an easy job for student B. This ability has its root in his geometrical socialization. On the other hand this construction of the sketch was in some sense like “mechanical” activities. How did student B invent the idea of using the intercept theorem? To begin with we could suggest that B could read this theorem from his drawing. But B could not, and he needed support from his colleague, student A. Similarly to the impressed solution something unintentional was the source of a successful idea. We can find this source in student B’s activities when he labelled his sketch with measurements. Labelling a sketch or any other geometric drawing was a well-known practice for both students. Student A did not see just a section of the given object when he looked at the given sketch. His engagement with the given object and observing the measurement labels led A to have the idea of drawing a right-angled triangle. We can say that A’s unorthodox action “abused” these measurement labels. When (ab)using these labels A always had the context in mind as he referred all sides of his right-angled triangle to the given object. Video data show that in the meantime student B had followed A’s activities very carefully. Now two drawings were drawn on the paper. There was the right-angled triangle as the result of A’s “abuse” and B’s own sketch. If one lays the first drawing over the other and additionally knows the intercept theorem then it is conceivable that a person would gain the idea of using this theorem. This is exactly what B did. With Krämer we can say that the idea for solving this problem developed from the drawn and the written.

The remaining activities can also be seen in the light of Krämer’s view on the written, as presented, or more precisely on the operative use of the written. Video data shows that student B formulated an algebraic equation. He used it to explore his solving strategy and to prove it empirically. As B had deduced the interception theorem with the aid of two geometrical drawings this theorem had to pass the test. But this did not happen. In his first attempt student B made a mistake when establishing his equation. However, as the calculation of one variable was the only task B had to fulfil he could easily test his calculated result against the already existing “impressed” solution. We can say that the rule-governed transformation, the operative use of the written, supported the exploration.

CONCLUSION

The case study presented has shown the importance of inscription (the written and the drawn) when solving a mathematical problem. Constructing and using drawings and the written as well can be seen as a possible source of new knowledge. By using well-known inscriptions or by inventing new ones, allows mathematics to happen right in front of their eyes. On this basis they may be able to use these writings successfully. In some cases, as illustrated through video recordings, spoken language only came after the written.

My deliberations should not challenge the importance of the spoken when learning mathematics. Similar as in the introduced view on the written, where inscriptions may bring something – which is not part of the spoken – to the eyes of the learning student, there are elements of the spoken language, which cannot be expressed by the written (e.g. gesture, facial play).

Beside answering the research question it was also of importance for me to offer some arguments that the relation between the spoken and the written is not a hierarchical one. An example of destructing such a relation was introduced by Jaques Derrida (1967/1997).

Further research could follow at least two directions. Following the first one we could compare the results of interpretations of similar - or the same - empirical data using other theoretical approaches (See footnote 1 and Kadunz 2006). A second direction could carry on Derridas idea of destructing hierachical relations where the relation between the written and the spoken is just one example. Learning mathematics always means inventing rules and following rules. Is the relation between inventing and following a hierarchical one? Some hints to answer this question may be found in Wittgensteins deliberation on language and mathematics (Wittgenstein, 1984; Krämer, 2002).

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