12th International Congress on Mathematical Education
Program Name XX-YY-zz (pp. abcde-fghij)
8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

NUMERICAL ANALYSIS AS A TOPIC IN SCHOOL MATHEMATICS

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Concerns about the divide between school mathematics and the discipline of mathematics are known in math education circles. At the heart of the debate is the sense that imperatives in school mathematics differ from those in the discipline of mathematics. In the former case, the focus is on remembering mathematical facts, mastering algorithms, and so on. In the latter case, the focus is on exploring, conjecturing, proving or disproving conjectures, generalizing, and evolving concepts that unify. It is clearly of value to find ways to bridge the divide. Certain topics offer greater scope at the school level for doing significant mathematics; one such is the estimation of irrational quantities using rational operations. This problem is ideal for experimentation, forming conjectures, heuristic reasoning, and seeing the power of calculus. The underlying logic is easy to comprehend. It would therefore be very worthwhile if we could make such topics available to students in high school.
Keywords: Numerical analysis, school mathematics, discipline of mathematics, estimation, irrational quantity, rational operation.

SCHOOL MATHEMATICS AND THE DISCIPLINE OF MATHEMATICS

Concerns about the divide between school mathematics and the discipline of mathematics are well known in math education circles. At the heart of the debate is the sense that imperatives in school mathematics differ in a fundamental way from those in mathematics as 'done' by practicing mathematicians.

In the former case, the focus is on the concrete, measurable and reproducible: on remembering mathematical facts (formulas, theorems, etc); mastering algorithms; reproducing proofs and derivations; answering questions in tests; and (typically) being answerable to an authority figure. In the latter case, the focus is on exploring; conjecturing; testing conjectures; proving or disproving them; establishing theorems; generalizing; creating; inventing; evolving concepts that unify; and so on.

Anne Watson writes in (Watson, 2008):

> In this paper I argue that school mathematics is not, and perhaps never can be, a subset of the recognized discipline of mathematics, because it has different warrants for truth, different forms of reasoning, different core activities, different purposes, and necessarily truncates mathematical activity. In its worst form, it is often a form of cognitive bullying which neither develops students' natural ways of thinking in advantageous ways, nor leads obviously towards competence in pure or applied mathematics as practiced by adult experts. … For me, the starting point [in this debate] is what it means to do mathematics,

> and to be mathematically engaged. In the discipline of mathematics, mathematics is the mode of intellectual enquiry, and effective methods of enquiry become part of the discipline – so much so that mathematics theses do not have chapters explaining methodology and methods. … 'Doing mathematics' is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalization, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. It can also be about creating methods of problem-presentation and solution for particular purposes, and it also involves, after all this, proving theorems. …

Here is Ramanujam in (Ramanujam, 2010):

> There are several ways in which mathematics in school classrooms misses elements that are vital to mathematicians' practice. Here, we wish to emphasize processes such as selecting between or devising new representations, looking for invariances, observing extreme cases and typical ones to come up with conjectures, looking actively for counterexamples, estimating quantities, approximating terms, simplifying or generalizing problems to make them easier to address, building on answers to generate new questions for exploration, and so on. In terms of content area and the methodology of content creation, it may be hard to mirror the discipline of mathematics in the school classroom, but we suggest that bringing these processes into school classrooms is both feasible and desirable. This not only enriches school mathematics but can also help solve problems that are currently endemic to mathematics education: perceptions of fear and failure, and low participation.

And here is what the widely cited NCF document (National Council for Educational Research and Training [NCERT], 2005) states:

> What can be leveled as a major criticism against our extant curriculum and pedagogy is its failure with regard to mathematical processes. We mean a whole range of processes here: formal problem solving, use of heuristics, estimation and approximation, optimization, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication. Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematisation of thinking and memorizing formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.'

Is there a way out?

The passages quoted above describe the problem eloquently. Is there a way out? Can we bridge this divide in any way?

Bridging the divide would imply helping students learn the ways that mathematicians think about problems, enabling them to experience the process of creating, conjecturing and exploring. If we do not attempt to do this, then we help perpetuate the divide.

Many writers have commented on the 'discontinuities' or 'transitions' that occur during a student's growth years; Klein has even talked of a 'double discontinuity' (Siu, 2008). These singular points clearly need to be taken note of and factored into our teaching methodology. The most crucial of these transitions is perhaps in the area of problem solving: the fact that "the answer, if there is one, is not the end of the process" (Ramanujam, 2010).

Is it possible to build bridges across these transition points? Help students learn how to mathematize alongside their learning of content? Let students in on the process of experimenting, creating and conjecturing? In this paper we suggest that this is indeed possible, and that numerical analysis is a convenient topic for enabling the passage through this transition. We report on a real classroom experience.

It may be true that in the ultimate analysis, it is the mode of transacting a class that is of greater importance than the subject matter itself: the manner in which the teacher opens up an issue for exploration or reflection; engages students as a group or as individuals; draws them into reflecting on a question or a problem; and turns even the tiniest of opportunities into avenues for learning. But there are some topics where this becomes easier to do, inasmuch as opportunities for asking open, accessible questions are greater, as also opportunities for using platforms such as computer software.

Two such topics which in our view offer succor and which do not find a place in the regular high school mathematics curriculum are *Elementary Number Theory* (ENT) and *Numerical Analysis* (viewed as a subtopic of mathematical modeling). The reasons for these choices are simple. In both cases it is possible to do many of the things described above – experimenting, looking for patterns, conjecturing, looking for counterexamples, and so on – in a way that is accessible at the school level. Both are well suited for computer based exploration. In (Shirali, 2010) the present author explored the possibilities offered by ENT. In this article we explore the possibilities offered by Numerical Analysis, which is a subtopic of mathematical modeling.

We list some desirable features of school level mathematical modeling activity: (i) The activity should use school level algebra, geometry, coordinate geometry and calculus, (ii) it should require working with computer software (e.g., *GeoGebra*, *Derive* and *Excel*, or any of their multi-platform equivalents), (iii) the methodology should be: exploration, data collection or generation, analysis of data, followed by theoretical investigation to understand the data.

A TOPIC FOR EXPLORATION: ESTIMATION OF IRRATIONAL QUANTITIES

The specific topic in numerical analysis that we take up is: *Estimation of irrational quantities using rational operations*. This simply stated problem contains a veritable wealth of opportunity. It is ideal for experimentation using software for computer algebra and dynamic geometry, and it provides a wonderful context in which we can form conjectures and reason heuristically, using elementary algebra and calculus. Finally, it provides a fertile ground for engaging with a historical perspective, because such kinds of reasoning go far back into human history.

Example: Estimating the square root of 2

Consider the problem of finding good rational approximations to $\sqrt{2}$. Numerous heuristic approaches can be envisaged, but here is one that is very straightforward.

We start with a known reasonably good approximation to $\sqrt{2}$, say $7/5$. We now make use of the fact that $|7-5\sqrt{2}| < 0.1$. From this we deduce that $|7-5\sqrt{2}|^2 < 0.01$. Expanding the expression on the left side we get:

$$99 - 70\sqrt{2} < 0.01, \qquad \therefore \sqrt{2} \approx \frac{99}{70}. \tag{1}$$

This yields an estimate for $\sqrt{2}$ which is accurate to 4 decimal places. We can easily carry this further. Squaring the expression $99 - 70\sqrt{2}$ we get

$$\left(99 - 70\sqrt{2}\right)^2 = 19601 - 13860\sqrt{2} < 0.0001. \tag{2}$$

From this we get the approximation

$$\sqrt{2} \approx \frac{19601}{13860}. \tag{3}$$

This is accurate to 8 decimal places.

Note the following important features of the underlying process:

- It needs an *initial good approximation*. If the starting estimate is not so close to the true answer, then the process will not yield good results.
- It is *iterative* in nature: starting with a close approximation, we get successively closer approximations.
- It may be applied to find a good rational approximation to the square root of any rational number, and if greater accuracy is required, the process allows for it; thus, it is *generalizable*. At the same time it is not infinitely generalizable: it can be applied to find an estimate for $\sqrt{3}$ or $\sqrt{5}$, but not π.
- The mathematics involved is simple; indeed, in this example, nothing more than tenth grade algebra was used. This is the case with many estimation problems – the underlying logic is elementary in nature, yet the different elements combine together to give a result that is highly satisfactory.

RATIONAL APPROXIMATIONS TO IRRATIONAL FUNCTIONS

Consider the problem of finding rational functions that yield close approximations to a given irrational function $g(x)$ in the neighbourhood of $x = 0$. Simple heuristic reasoning can lead us to find the first several terms of the Maclaurin series for this function. The reasoning proceeds thus: If we want two curves to stay close together in the neighbourhood of a point P where they meet, then surely we must ensure that they have equal slope at P; else the curves will quickly draw apart as we travel away from P.

Next, if their second derivatives differ at P, then this will lead to a steadily widening gap between the slopes and hence to a widening gap between the curves; so we would do well to make the second derivatives coincide too. Continuing, if their third derivatives differ at P, then this will lead to a widening gap between their second derivatives; and so on. So to achieve significant closeness of the graphs of two functions in the vicinity of a given value of x, we should try to ensure that the two functions coincide in their first several derivatives at that value – as many as possible.

Such 'kitchen' logic appeals readily to students, and the fact that we can actually test the resulting formulas using a hand-held calculator is most reassuring; a bonus in fact. As we show below, we can even reconstruct the famous Bakhshāli square root formula by arguing this way.

RATIONAL APPROXIMATIONS TO THE SQUARE ROOT FUNCTION

Let $f(x) = \sqrt{1+x}$, defined for $x \geq -1$. The first few derivatives of $f(x)$, evaluated at $x = 0$ (and starting with the zeroth derivative, which is f itself), are:

$$1, \quad \frac{1}{2}, \quad -\frac{1}{4}, \quad \frac{3}{8}, \quad -\frac{15}{16}, \quad \frac{105}{32}, \quad \ldots \tag{4}$$

Note that these numbers are rational. We wish to generate rational functions $g(x)$ of x that closely approximate $f(x)$ in the neighbourhood of $x = 0$; naturally, we want all their coefficients to be rational. The simplest such functions are the *polynomials with rational coefficients*, and the polynomials of successively higher degrees which agree with $f(x)$ in its successive derivatives at $x = 0$ are simply the partial sums of the Maclaurin series of $f(x)$ about $x = 0$:

$$1, \quad 1+\frac{x}{2}, \quad 1+\frac{x}{2}-\frac{x^2}{8}, \quad 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}, \quad \ldots. \tag{5}$$

The graphs of these functions, for $-0.5 \leq x \leq 2.5$ are shown in Figure 1, together with the graph of $\sqrt{1+x}$. Their closeness in the vicinity of $x = 0$ is very visible, as also the growing error when x goes beyond 1.

Next in line are the *rational functions with rational coefficients*; they have the form p/q where p and q are polynomials with rational coefficients and low degree. Here we consider the following three kinds of rational functions:

$$\frac{1+ax}{1+bx}, \quad \frac{1+ax}{1+bx+cx^2}, \quad \frac{1+ax+bx^2}{1+cx}. \tag{6}$$

By equating their successive derivatives at $x = 0$ with the respective derivatives of $f(x)$ we find the values of the coefficients. We get the following rational functions which we call, respectively, $r_{1,1}(x)$, $r_{1,2}(x)$ and $r_{2,1}(x)$:

$$r_{1,1}(x) = \frac{1 + \frac{3x}{4}}{1 + \frac{x}{4}}; \qquad (7)$$

$$r_{1,2}(x) = \frac{1 + \frac{5x}{6}}{1 + \frac{x}{3} - \frac{x^2}{24}}, \qquad (8)$$

$$r_{2,1}(x) = \frac{1 + x + \frac{x^2}{8}}{1 + \frac{x}{2}}. \qquad (9)$$

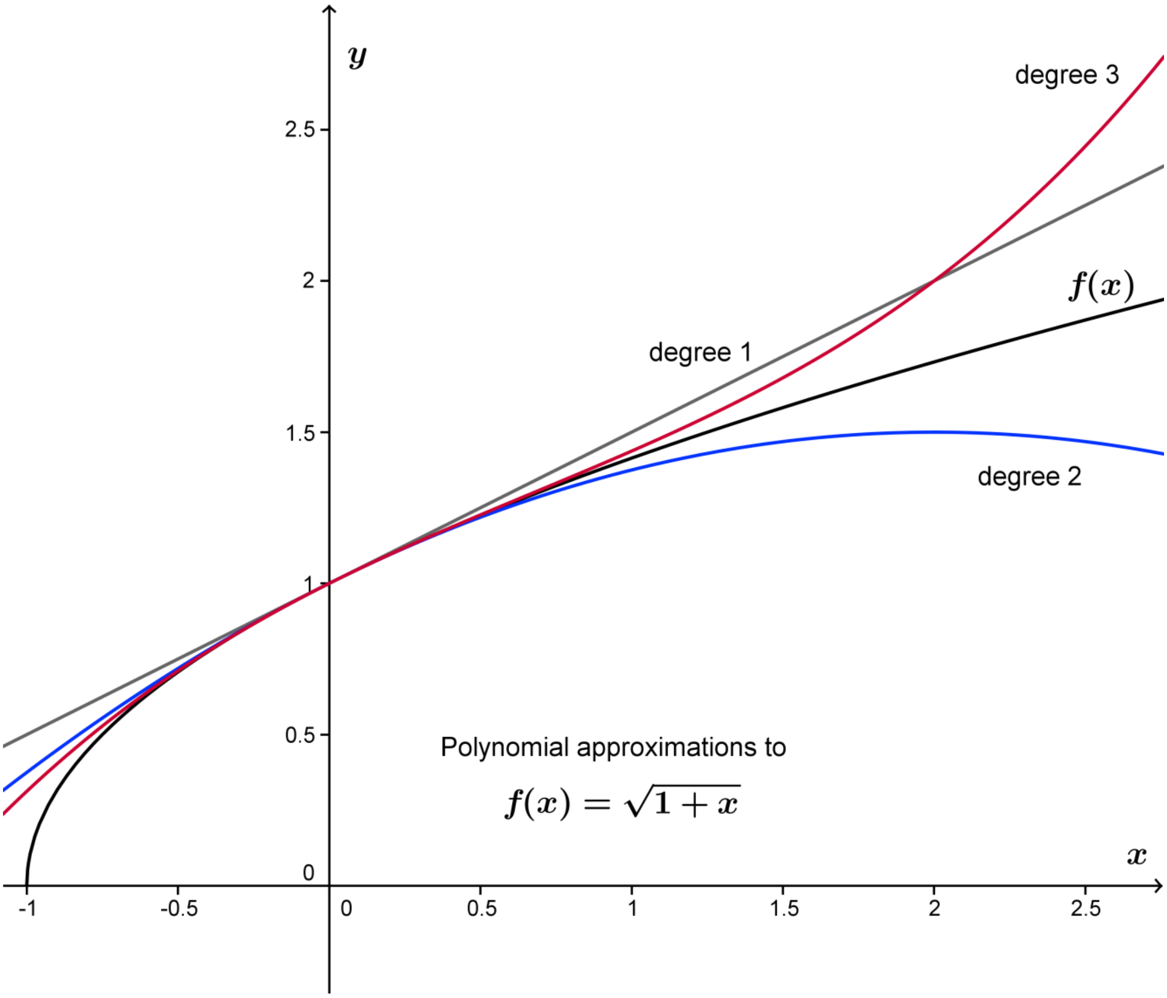

Figure 1. Graphs of $\sqrt{1+x}$ and the first few partial sums of its Maclaurin series about $x = 0$

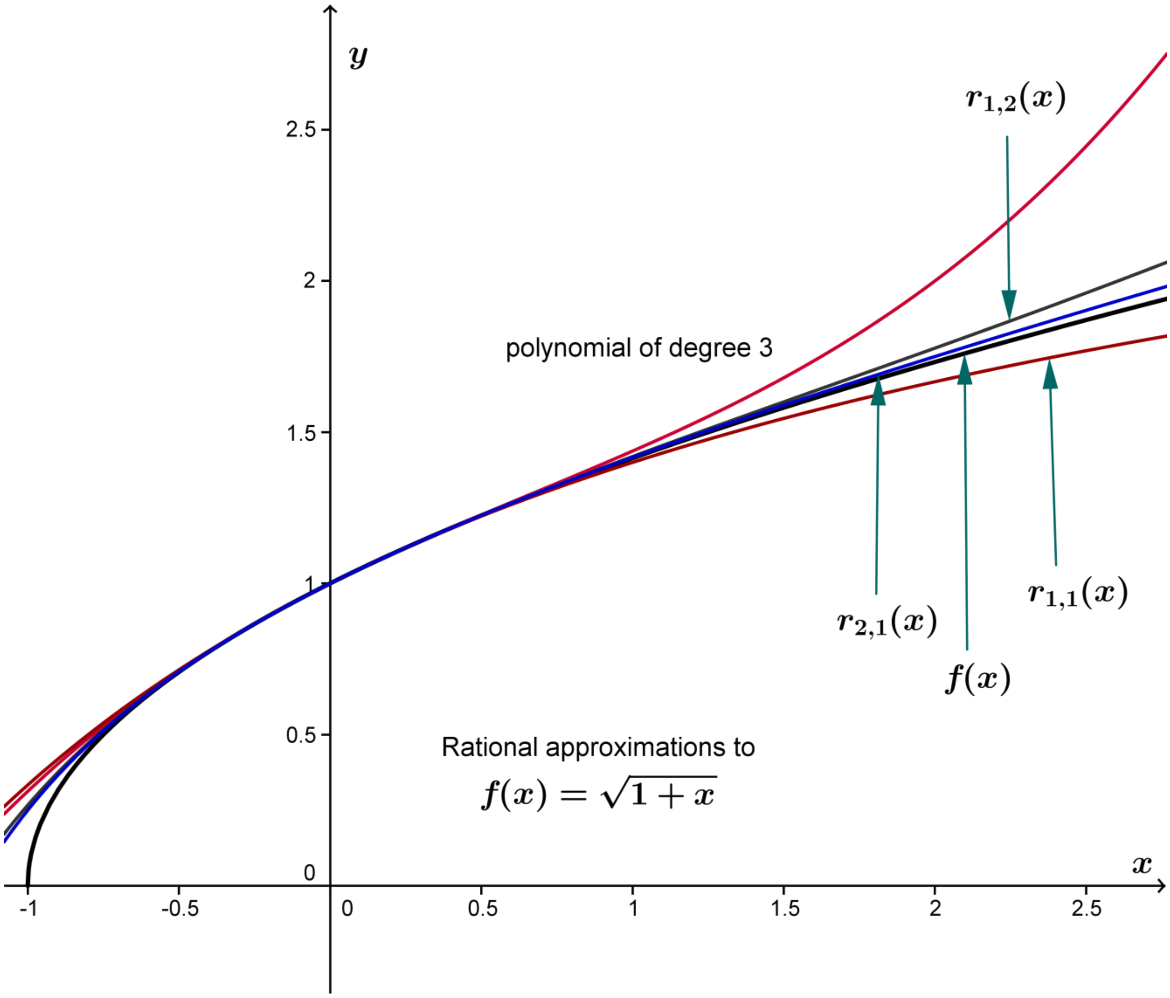

Figure 2. Graphs of $\sqrt{1+x}$ and the functions $r_{1,1}(x)$, $r_{1,2}(x)$ and $r_{2,1}(x)$

The graphs of these functions over $-0.5 \le x \le 2.5$ are shown in Figure 2. We see that the curves stay much closer to the curve $\sqrt{1+x}$ than do the polynomials obtained from the Maclaurin series.

We also see that of the three functions considered, the one that best approximates $\sqrt{1+x}$ is $r_{2,1}(x)$. Now a simple manipulation yields the following:

$$\frac{1+x+\frac{x^2}{8}}{1+\frac{x}{2}} = 1+\frac{x}{2}-\frac{\left(\frac{x}{2}\right)^2}{2\left(1+\frac{x}{2}\right)}. \tag{10}$$

The form given on the right side is the famous *Bakhshāli approximation* — a formula whose origins go back to the seventh century or earlier (the formula appears in a birch bark manuscript found in 1881 during an excavation in the village of Bakhshāli, in north west Pakistan). It is expressed in the following form:

> In the case of a non-square number, subtract the nearest square number, divide the remainder by twice this nearest square; half the square of this is divided by the sum of the approximate root and the fraction. This is subtracted and will give the corrected root.

In other words:

$$\sqrt{A^2+b} \approx A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}. \tag{11}$$

(To see the connection between this and the formula obtained above, put $x = b/A^2$.) Most unfortunately, there are no clues as to how the people of Bakhshāli found such a formula. For more on the historical background of the formula, see (Bakhshāli manuscript, 2012).

We see from this account how a fairly simple development of ideas has uncovered a formula that is not only very impressive but also connects with the distant past, allowing us to engage with questions about the people who lived in those regions, questions about anthropology and history, and questions about the history of mathematics itself.

RATIONAL APPROXIMATIONS TO THE TANGENT FUNCTION

Here is an approach to finding a rational approximation to the tangent function, as described in (Cheney, 1945). Let $f(x) = \tan \pi x/4$. We seek a rational function $g(x)$ that closely approximates $f(x)$. To this end we impose the following conditions:

- $g(0) = f(0)$ and $g(1) = f(1)$;
- $g'(0) - f'(0)$ and $g'(1) - f'(1)$ are very small.

We have the following values:

$$f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = \frac{\pi}{4}, \qquad f'(1) = \frac{\pi}{2}. \tag{12}$$

We search for a candidate $g(x)$ by using the approximation $\pi \approx 22/7$ and demanding that

$$g(0) = 0, \qquad g(1) = 1, \qquad g'(0) = \frac{11}{14}, \qquad g'(1) = \frac{11}{7}. \tag{13}$$

Since

$$\tan\frac{\pi x}{4} = \frac{\sin \pi x/4}{\cos \pi x/4} \approx \frac{\pi x/4 - \pi^3 x^3/384}{1 - \pi^2 x^2/32} = \frac{\pi x}{4} \cdot \frac{1 - \pi^2 x^2/96}{1 - \pi^2 x^2/32}, \tag{14}$$

we look for functions $g(x)$ of the following form:

$$g(x) = \frac{ax(b - x^2)}{c - x^2}, \tag{15}$$

where a, b, c are real numbers. On setting up the equations and solving for a, b, c we get

$$g(x) = \frac{x}{7} \cdot \frac{(22 - x^2)}{4 - x^2}. \tag{16}$$

This very simply produced approximation yields three accuracy to three significant figures for all values of x between 0 and 1. For example:

$$g\left(\frac{1}{2}\right) = \frac{29}{70} \approx 0.4143, \qquad \tan\frac{\pi}{8} = \sqrt{2} - 1 \approx 0.4142; \tag{17}$$

and

$$g\left(\frac{1}{3}\right) = \frac{197}{735} \approx 0.26803, \qquad \tan\frac{\pi}{12} = 2 - \sqrt{3} \approx 0.26795. \tag{18}$$

RATIONAL APPROXIMATIONS TO THE SINE AND COSINE FUNCTIONS

In much the same way, we may search for rational approximations to the sine and cosine functions. The exploration not only brings forth some striking results but also connects with an extraordinary formula dating from the seventh century. Unlike the Bakhshāli formula this one is well documented; it occurs in the text *Mahabhaskariya*, written by Bhāskarā I, who belonged the school founded by Āryabhatā. (Here too we see no sign of a justification or rationale given for the formula.)

We consider the function $f(x) = \cos \pi x/2$ and look for rational approximations of low degree to $f(x)$, over the interval $-1 \le x \le 1$. Since $f(x) = f(-x)$ and $f(\pm 1) = 0$, we look for approximations of the following kind:

$$g(x) = \frac{a(1 - x^2)}{a + x^2}. \tag{19}$$

The form chosen ensures that: $g(0) = 1 = f(0)$, $g(1) = 0 = f(1)$, $g'(0) = 0 = f'(0)$. We find a using the condition $g'(1) = f'(1)$.

From $f'(1) = -\pi/2$ and $g'(1) = -2a/(a+1)$, we solve for a and get $a = \pi/(4-\pi)$. Using the approximation $\pi \approx 22/7$ this yields:

$$\cos\frac{\pi x}{2} \approx \frac{11(1 - x^2)}{11 + 3x^2}. \tag{20}$$

On testing this out we find that it yields only two decimal place accuracy. Can we do better? It turns out we can. The analysis given below, from (Shirali, 2011), shows how.

Since the graph of $f(x)$ over $-1 \le x \le 1$ is a concave arch passing through the points $(\pm 1, 0)$ and $(0, 1)$, a first approximation to $f(x)$ over the same interval is the function $1 - x^2$, whose graph shows the same features. But this function consistently yields an overestimate (except, of course, at $x = 0, \pm 1$); see Figure 3.

In order to fix the overestimate, we examine the following quotient more closely:

$$p(x) = \frac{1 - x^2}{\cos\frac{\pi x}{2}}. \tag{21}$$

Figure 4 shows the graph of $p(x)$ for $-1 \le x \le 1$. (At $x = \pm 1$ the indeterminate form $0/0$ is encountered, but if we require p to be continuous at $x = \pm 1$ and use L'Hospital's rule, we get $p(\pm 1) = 4/\pi \approx 1.27$.)

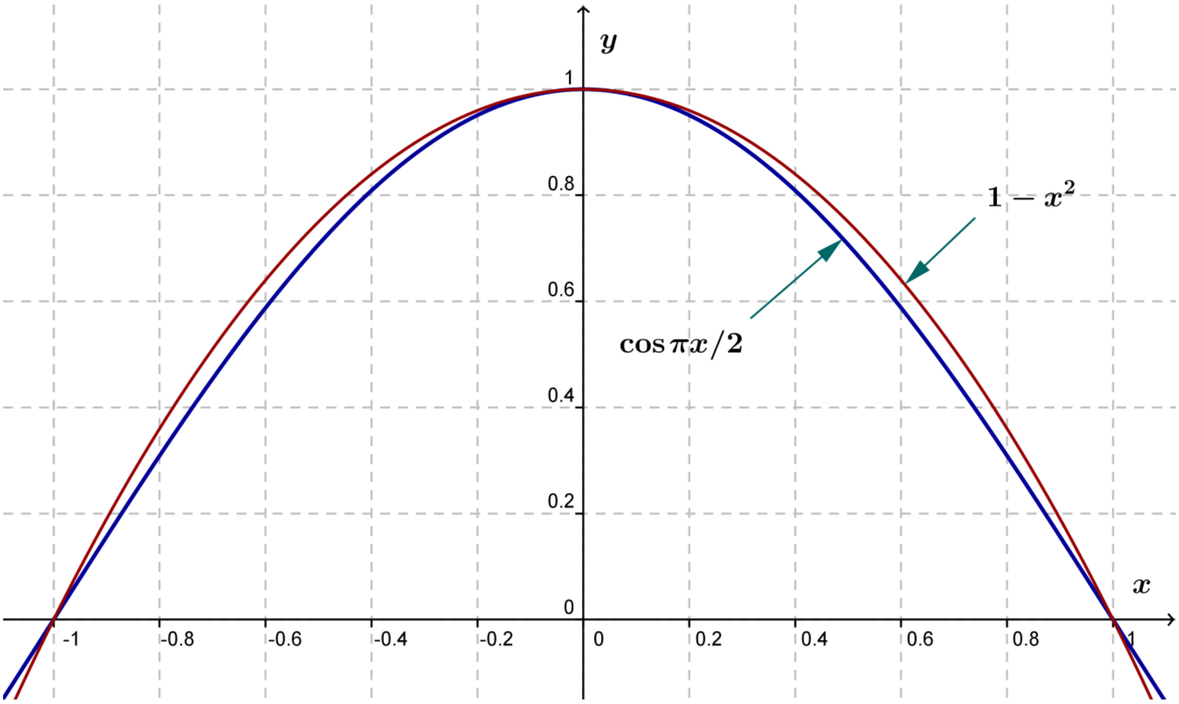

Figure 3. Graphs of $1 - x^2$ and $\cos \pi x/2$

The shape is suggestive of a parabolic function, so we look for such a function to fit the data. To this end we mark three points on the graph: $(0,1)$ and $(\pm 2/3, 10/9)$; rather conveniently for us, points with rational coordinates are available. For the parabola $y = d + e\,x^2$ to pass through them we must have $d = 1$ and $d + 4e/9 = 10/9$, giving $e = 1/4$. So the desired parabolic function is $y = 1 + x^2/4$, and we have the approximate relation

$$\frac{1 - x^2}{\cos {}^{\pi x}/_{2}} \approx 1 + \frac{x^2}{4}. \tag{22}$$

Hence:

$$\cos\frac{\pi x}{2} \approx \frac{1 - x^2}{1 + x^2/4} = \frac{4(1 - x^2)}{4 + x^2}. \tag{23}$$

All the coefficients in this approximation are rational numbers. Note that it is of the type considered earlier, $a\,(1 - x^2)/(a + x^2)$, with $a = 4$; earlier we had got $a = 11/3$. But the change of coefficient works wonders. The above approximation turns out to be extremely close.

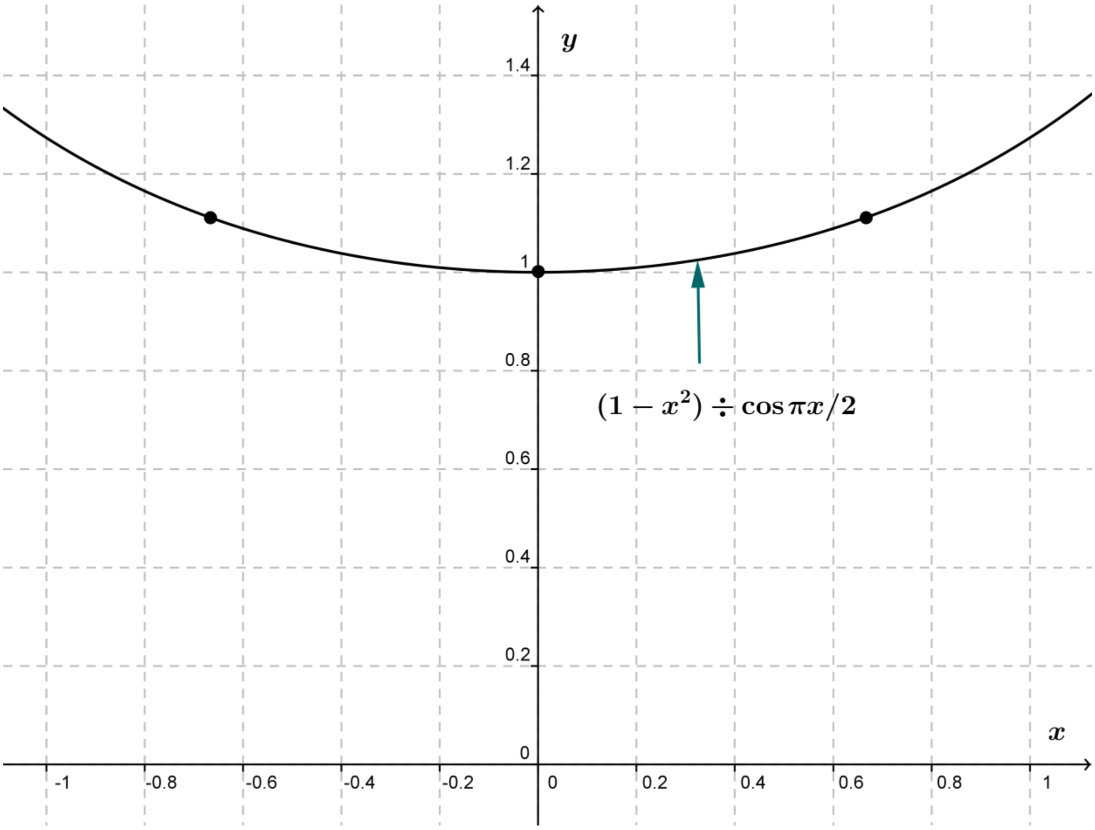

Figure 4. Graph of $(1 - x^2) \div \cos \pi x/2$

Let us now change the unit of angle from radian to degree. Since $\pi x/2$ radians equals $90\ x°$ the above relation may be written as

$$\cos 90\ x° \approx \frac{4(1 - x^2)}{4 + x^2} \qquad (-1 \le x \le 1). \tag{24}$$

The replacement $x \mapsto x/90$ yields:

$$\cos x° \approx \frac{4(8100 - x^2)}{32400 + x^2} \qquad (-90 \le x \le 90). \tag{25}$$

Finally, the replacement $x \mapsto 90 - x$ yields:

$$\sin x° \approx \frac{4x(180 - x)}{40500 - x(180 - x)} \qquad (0 \le x \le 180). \tag{26}$$

This is the approximation given by Bhāskarā I. Call the function on the right side $B(x)$. Here is a comparison of the values of $\sin x°$ and $B(x)$, given to three significant figures:

x	0	15	30	45	60	75	90
$\sin x°$	0	0.259	0.5	0.707	0.866	0.966	1
$B(x)$	0	0.260	0.5	0.706	0.865	0.965	1

It is evident that $B(x)$ yields a very good approximation to the sine function over the interval from 0° to 180°.

See (Plofker, 2008) and (Shirali, 2011) for more on the Bhāskarā approximation.

CLASSROOM STUDY AND STUDENTS' RESPONSES

The author held three one-hour classes on these topics for a group of twelfth graders in his school (Rishi Valley School in A.P., India), and asked the students to write about their experiences at the end. Here are some of the problems posed, which were used as a basis for class discussion:

1. A well known 'rule of thumb' in banking, relating to the number of years it takes for a fixed deposit to double in size, is the following: *If the interest rate is r % then deposits take $72/r$ years to double in size*. How do you justify this 'rule'? How accurate is it?
2. Examine the values given here of the tangents of some angles close to $90°$, in each case to two decimal places, and explain the pattern you see in the values.

$$\tan 89° \approx 57.29\,, \qquad \tan 89.9° \approx 572.96\,, \qquad \tan 89.99° \approx 5729.58\,,$$
$$\tan 89.999° \approx 57295.78\,, \qquad \tan 89.9999° \approx 572957.80.$$

3. (This finding was reported to me two decades back by a student of the 11th standard. It led to a wonderful exploration. But we'll leave that story for another day.) We wish to find a good approximation to $\sqrt{2}$. We now use the following observed fact: *If $x > 0$ is close to $\sqrt{2}$, then $(x+2)/(x+1)$ is still closer to $\sqrt{2}$*. Using this iteratively, we get a sequence of steadily closer approximations. We stop when we feel we have come close enough to $\sqrt{2}$. How do you account for this strange 'rule'?

(There was also a question on the Bakhshāli formula, which we do not repeat here.)

As all this was done in an informal way, in an interactive, problem solving mode, the students were relaxed; there was no 'exam pressure'. Here are a few of the comments they turned in (not categorized in any way):

> "Everything in maths is interconnected." "Calculus is powerful in answering questions whose solutions may not be intuitively apparent." "We now see how different areas in mathematics, learnt as separate chapters – calculus, binomial theorem, series and sequences – come together. It leaves in us a sense of wonderment about mathematics." "It shows the power of addition, subtraction, multiplication and division!" "We now see how by using just the basic functions $(+,-,\times,\div)$ we can get really good approximations to complicated functions." "I did not expect even for a moment that calculus could be used to find out the square root of a number." "It was nice to see things beyond our syllabus!" "The history of mathematics and mathematicians is itself an enticing topic."

We feel there is enough in this experience to suggest that dwelling on such topics in an enabling, problem solving environment will enhance students' mathematical maturity and their understanding of mathematics, and also their appreciation of the culture and history of mathematics; and to some measure help chip away at the Great Divide.

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