12th International Congress on Mathematical Education
Program Name XX-YY-zz (pp. abcde-fghij)
8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

# IMPLICATIONS FROM POLYA AND KRUTETSKII

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*Enhancing mathematical problem solving abilities, George Polya gave tremendous contribution to mathematics educators. He identified 4 steps in the problem solving process; (1) understand the problem, (2) devise a plan, (3) carry out the plan, and (4) look back and check. For each step, Polya revealed many useful habits of thinking in forms of questions and suggestions. V. A. Krutetskii analysed mathematical abilities of school children, which suggest valuable implementation to many trying to develop effective ways of expanding mathematical problem solving abilities. Krutetskii's research was inspecting mathematical behaviour in 3 stages of information gathering, processing, and retention. He concluded that mathematically able students show strong trends to gather information in more synthetic way, to process information in more effective, economic, and flexible way and to retain indispensable information more than inessential.*

*Mathematical Heuristics, Mathematical Abilities, Elementary Teacher Education, G. Polya, V. A. Krutetskii*

## INTRODUCTION

Many mathematics teachers have been trying to enhance their mathematics teaching to be more meaningful and powerful in various ways. To teach mathematical concepts meaningfully, some teachers may provide activities to their students of constructing meaningful concepts through students' real life situations. In this case the Freudenthal's idea of mathematization should be very helpful. Some other teachers who want to enhance their students mathematical capabilities may provide activities to their students of drill and practicing to solve mathematics problems. On the other hand, many think it is important to enhance mathematical capabilities of teachers themselves.

M. A. Clements and N. F. Ellerton in Australia tried to reflect both ideas of Polya and Krutetskii on problem solving in school mathematics. Their efforts give much substantial help to mathematics educators (Clements, & Ellerton 1991). Now, such kind of efforts may be applied to pre-service teacher training courses. Let' s look at how such attempts might give shape to mathematics teacher education.

Today I will talk about my ideas about ways of enhancing mathematical capabilities of students and/or mathematics teachers of elementary school level. About ways of teaching mathematical concepts meaningfully, I will talk at another adequate opportunity.

*Wan Kang*

## ENHANCING TEACHERS′ CAPABILITIES TO TEACH MATHEMATICAL PROBLEM SOLVING

George Polya (1957) gave tremendous contribution to mathematics educators. He identified 4 steps in the process of problem solving; (1) understand the problem, (2) devise a plan, (3) carry out the plan, and (4) look back and check. For each step, Polya revealed many useful habits of thinking in forms of questions and suggestions.

According to Polya, for the first step of understanding the problem, such questions and suggestions as “What is the unknown?”, “What are the data?”, “What is the condition?”, “Is it possible to satisfy the condition?”, “Is the condition sufficient to determine the unknown?”, “Draw a figure.”, “Introduce suitable notation.”, and so on, are useful.

For the second step of devising a plan, such questions and suggestions as “Have you seen it before?”, “Do you know a related problem?”, “Could you restate the problem?”, “If you cannot solve the proposed problem try to solve first some related problem.”, “Did you use all the data?”, and so on, are useful.

For the third step of carrying out the plan, “Carrying out your plan of the solution, check each step.”, “Can you see clearly that the step is correct?”, and “Can you prove that it is correct?” are useful.

For the fourth step of looking back, “Can you check the result?”, “Can you check the argument?”, “Can you drive the result differently?”, “Can you use the result, or the method, for some other problem?” are useful.

These useful habits of thinking are very recommendable for every mathematical problem solver. It can be said that especially mathematics teacher should build up these habits for their students as well as even more for themselves. The right way of building up good habits is to act out.

As a professor of an institute of elementary teacher education, I have been giving a special course to students who are going to be elementary school teachers. In this course, my students should solve a mathematical problem of elementary school level a day during the semester. Of course, they are recommended to think in forms of Polya’s questions and suggestions to solve the problems. In addition to solving a problem, they have to note the mathematical solution with didactic analysis on the ways of helping elementary school children solving the same problem. The following Figure 1 and Figure 2 are examples of course notes of the students.

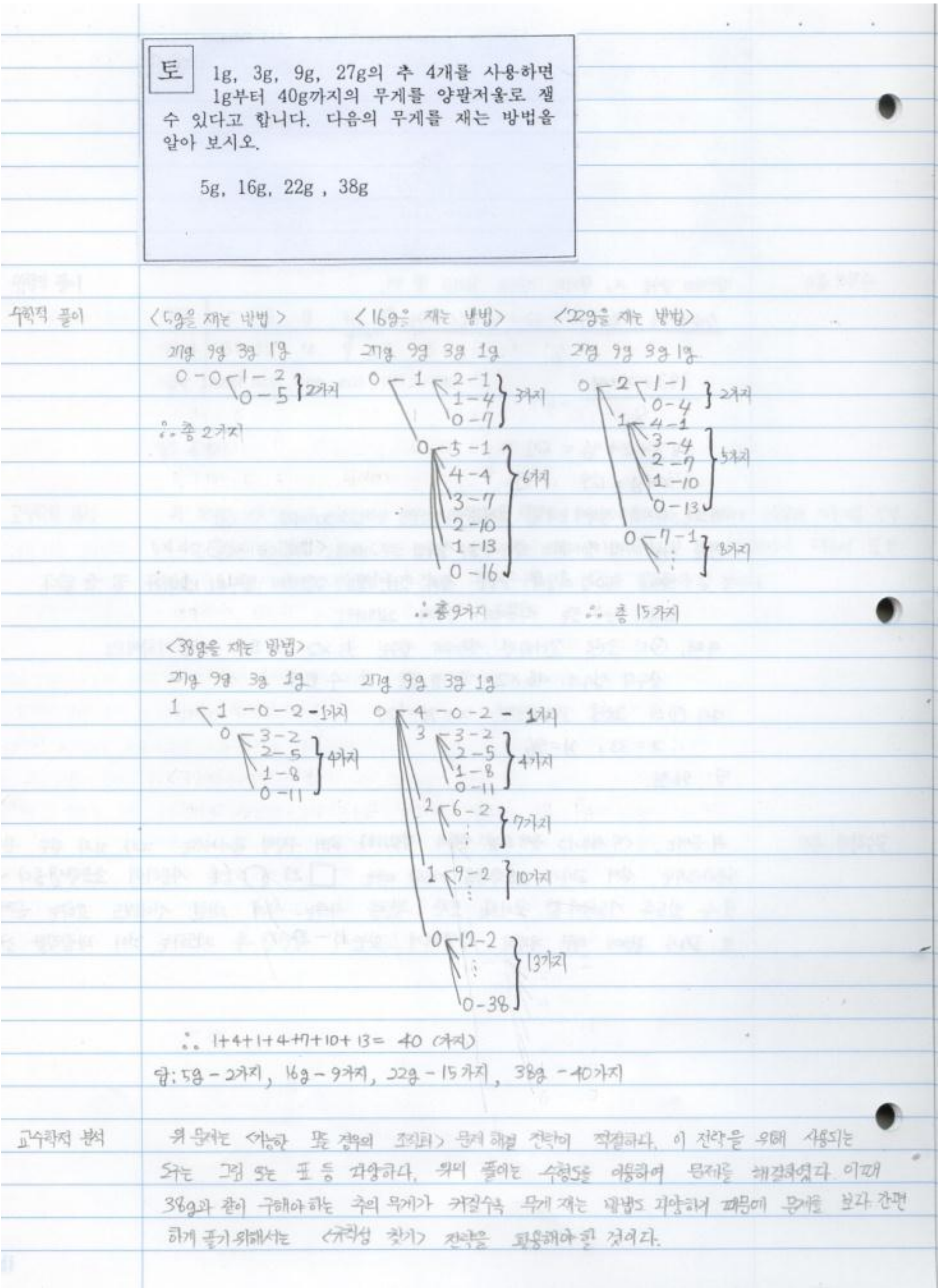



Figure 1. A Student's Note (Problem of Saturday)

*Wan Kang*

- In English -

[Problem of Saturday]

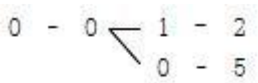
Using 4 poises of 1g, 3g, 9g, 27g, we can measure all the weights from 1g to 40g with a balance.  Explain the ways of measuring following weights:

5g, 16g, 22g, 38g

[Solution]

<Ways of measuring 5g>

27g   9g   3g   1g

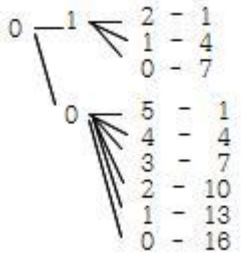



} 2 ways

∴ 2 ways in total

<Ways of measuring 16g>

27g  9g   3g   1g



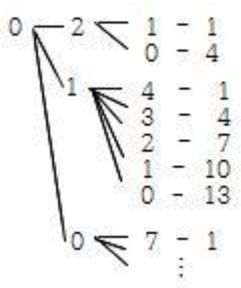

} 3 ways

} 6 ways

∴ 9 ways in total

<Ways of measuring 22g>

27g  9g   3g   1g




} 2 ways

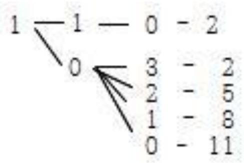
} 5 ways

} 8 ways

∴ 15 ways in total

<Ways of measuring 38g>

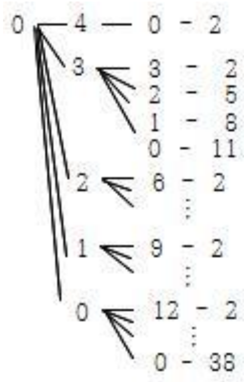
27g 9g 3g 1g



} 1 way

} 4 ways

27g 9g 3g 1g



} 1 way

} 4 ways

} 7 ways

} 10 ways

} 13 ways

∴ 1+4+1+4+7+10+13=40 (ways)

Answer: 5g – 2 ways, 16g – 9 ways, 22g – 15 ways, 38g – 40 ways.

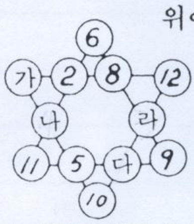
<Didactic analysis>

In this problem a problem solving strategy of <organizing all possible cases> is suitable. Devices for this strategy are various such as diagram or table. In the above solution the tree diagram is used. Because the heavier the poise is like 38g, the ways of measuring are more various, strategy of <searching patterns> should be used to find the solution easier.

DATE. PAGE.

금

각 선의 교차점에 있는 12개의 원에는 1에서 12까지의 수가 씌여 있습니다. 일직선 위에 놓인 4개의 원에 들어 있는 수의 합이 모두 같을 때, 나에 알맞은 수를 구해봅시다.




가 나 다 라
4 7 1 3

7

풀이과정 → 이 문제는 예상과 확인 전략을 이용해 풀수 있는데, 조금 더 체계적으로 풀자면

ㄱ. 먼저 아직 사용하지 않은 숫자는 1, 3, 4, 7이다

ㄴ. 가 - 2 - 8 - 12 → '가'를 제외한 일직선의합 ; 22

11 - 5 - 다 - 9 → '다' 〃 ; 25

6 - 8 - 라 - 9 → '라' 〃 ; 23

6 - 2 - 나 - 11 → '나' 〃 ; 19

ㄷ. 가장 합이 적은 '나'의 일직선에는 남은 숫자중 가장 큰 7을 집어넣고 총합이 26이 나오는것을 보고 나머지도 집어넣으면

ㄹ. 가=4 나=7 다=1 라=3

∴답) 7

문제의 수학적 분석 및 교수학습전략 →

이 문제의 유형은 덧셈과 뺄셈의 응용과 규칙성에 관한 문제로 파악된다. 보통 이런 문제가 나오면 대부분의 아이들이 숫자를 무작위로 대입하여 숫자를 맞추려한다. 그렇게 맞춘다면 학생들은 예상과 확인 전략을 통한 문제풀이이다.

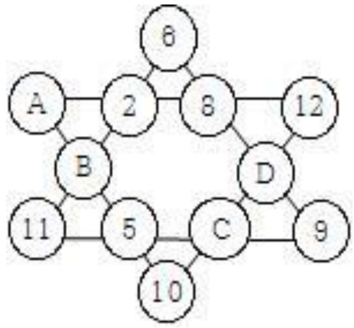
하지만 주어진 조건을 이용하여 식으로 쓴다면 더욱 조직화된 방법으로 쉽게 정답을 추리할수 있다. 먼저 일직선중 하나만 모르는 직선들을 골라 나머지부분의 합을 구해본다. 그리고 합이 작은 곳에 큰 수를 넣는 방법으로 풀면 쉽고 간단하고 빠르게 문제를 해결할 수 있다.

처음에 아이들이 이 문제를 접하면 아무 숫자나 넣으려 할것이다. 처음에는 그렇게 하다가 점차 스스로 규칙을 찾게 될 것이다. 어려워하는 학생들에게는 규칙을 찾도록 교사가 유도해 주어야 한다 "일직선에 모두 숫자가 나온 부분이있니?" "그렇다면 3개가 나온 부분이있니?" "그러면 일직선의 합을 이용하여 숫자를 넣어보자" 등의 발문이 적합하다.

Figure 2. A Student's Note (Problem of Friday)

[Problem of Friday]

Numbers from 1 to 12 are written in 12 circles which lie on the intersections of the lines. The sums of numbers in 4 circles on a same line are all equal. Find the number in the circle B.




[Solution]

This problem can be solved by using a strategy of guess and check. To solve systematically,

a. Numbers which are not used yet are 1, 3, 4, and 7.

b. Ⓐ-②-⑧-⑫ → sum of numbers except A : 22

⑪-⑤-Ⓒ-⑨ → sum of numbers except C : 25

⑥-⑧-Ⓓ-⑨ → sum of numbers except D : 23

⑥-②-Ⓑ-⑪ → sum of numbers except B : 19

c. On the line of B which has smallest sum, among the remaining digits, put the largest one 7 in B. Then the sum of numbers of a line becomes 26. So, we can put suitable digits in the remaining circles.

d. A=4, B=7, C=1, D=3

∴ Answer 7.

<Didactic analysis>

I grasped the type of this problem as <an application of addition and subtraction> and <a problem about pattern>. Usually for this kind of problem, most children try to match numbers by substitute numbers randomly. If they matched numbers in that way, students solved the problem using the strategy of <guess and check>.

However if you write expressions according to the given conditions, you can derive easily the right answer in more systematic way. First among lines choose a line which include the unknown [circle] and get the sum of the rest part [numbers in the other three circles]. And if you use the way of putting a large number in a place where the sum is small, you can solve the problem easily, simply, and rapidly.

At first children who come in contact with this problem would put an arbitrary number [in the blank circle]. Then gradually they would begin to find patterns. For the students who feel difficulty, teacher should guide to find patterns. "Is there a part on a line, where all numbers are shown?", "If not, is there a part where 3 numbers are shown?", "If so, put a number [in the blank circle] by estimating the sum of [the other three numbers on] the line." are useful questions and suggestions.

## ENHANCING STUDENTS′ MATHEMATICAL CAPABILITIES

Krutetskii's studies of mathematical abilities of school children gave me motivation to develop the ways of enhancing students' mathematical capabilities in elementary school level.

Krutetskii summarized factors of mathematical ability as below:

**Obtaining mathematical information**

- The ability for formalized perception of mathematical material, for grasping the formal structures of a problems.

**Processing mathematical information**

- The ability for logical thought in the sphere of quantitative and spatial relationships, number and letter symbols; the ability to think in mathematical symbols.
- The ability for rapid and broad generalization of mathematical objects, relations, and operations. (Generalization)
- The ability to curtail the process of mathematical reasoning and the system of corresponding operations; the ability to think in curtailed structures. (Condensation)
- Flexibility of mental processes in mathematical activity. (Flexibility)
- Striving for clarity, simplicity, economy, and rationality of solutions. (Being economical)
- The ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning). (Reversibility)

**Retaining mathematical information**

- Mathematical memory (generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem-solving, and principles of approach. (Structural memory)

**General synthetic component**

- Mathematical cast of mind.

These components are closely interrelated, influencing one another and forming in their aggregate a single integral system, a distinctive syndrome of mathematical giftedness, the mathematical cast of mind. (Krutetskii 1976, pp. 350-351)

Krutetskii's experiment clearly implies that the process of mathematical reasoning does not always coincide with the psychological process of children. In fact, mathematical abilities of gifted children, including ability of understanding generalized data, prompt generalization, reversibility, flexibility, condensation can be also developed by average students later. Considering differences of mathematical abilities, of course, does not mean ignoring logical aspects of mathematical contents in schools.

For example, in the aspect of reversibility, teachers sometimes present solutions only in one direction and ask students to solve problems that require reversibility of thinking process, assuming students can reverse without difficulties. Students can find $\sin 75^{\circ}$ after they learned

$\sin(\alpha+\beta)=\sin\alpha\cos\beta + \cos\alpha\sin\beta$. However, when $\sin 15^\circ + \cos 15^\circ$ is given, most students are puzzled because this requires reversibility of thinking process.

It may be suggested that students need thinking process disciplines considering their psychological characteristics before they fully develop their mathematical abilities. In other words, you may use a metaphor from baseball that a hitter needs to be trained in stands, swings, and concentrations in order to achieve higher hit potentiality and so do children in mathematics.

In this sense, types of learning activities enhancing students' mathematical capabilities can be listed as follows:

(1) Making problems with given materials

(2) Modifying problems with incomplete information

(3) Modifying problems by removing surplus information

(4) Developing insights of figures

(5) Classifying problem patterns

(6) Making problems by using given types

(7) Crypt-arithmetic

(8) Problems with several solutions

(9) Problems changing contents

(10) Recomposition of thinking

(11) Direct and reverse problems

(12) Logical inferences

Among these activities, <(1) making problems with given materials>, <(2) modifying problems with incomplete information>, and <(3) modifying problems by removing surplus information> aim to develop ability to recognize given facts in problems and their relationships. The activity <(4) Developing insight of figures> trains ability to derive geometrical factors from given figures and backgrounds. Activities <(5) classifying problems patterns> and <(7) crypt-arithmetic> help students to condense their deduction processes. The activity <(6) making problems by using given types> targets to sharpen generalized recognition of problems. Activities <(8) problems with several solutions>, <(9) problems changing contents>, and <(10) recomposition of thinking> pursue to develop flexibility of thinking process and elegant solutions. Activities <(11) direct and reverse problems> and <(12) logical inferences> are for improving reversibility and deducting ability, respectively.

Examples of activities <(10) recomposition of thinking> and <(11) direct and reverse problems> are presented as below:

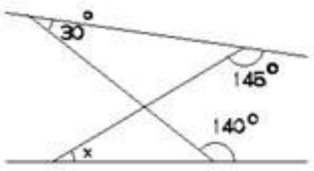
**Examples of activity (10) Recomposition of Thinking**

In case of repeating same types of thinking process, a habitual and fixed thinking process can be settled down, which is referred as “mental set.” Even if mental set may raise efficiency of thinking process, but it can jeopardize flexibility of thinking process when new patterns of problems are given. In general, most mathematics materials are focusing only efficiency of thinking process. In order to broaden thinking ability, disciplines of recomposing thinking process based on appropriate conditions are required.

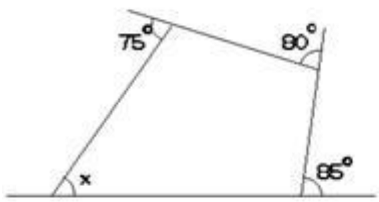
In the following examples of recomposing, five questions form a cluster of problems, representing categories (1) to (5). In order to solve question number (5), students need to produce different thinking process from question number (1) to (4). Therefore, after letting student solve five questions in order, the examiner are required to closely observe changes of student’s thinking process when he/she start to solve question number (5). And also, students by themselves need to find out difference between the solution of question (5) and solutions of the other previous questions.

[Example 1] Find angle x.

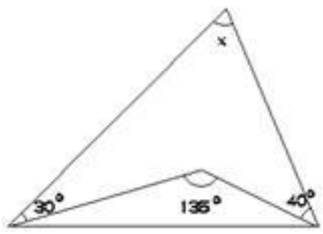
(1)



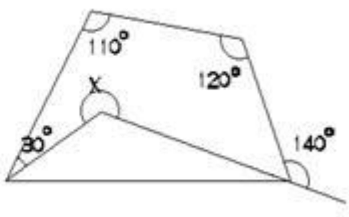

(2)



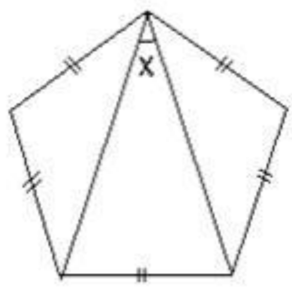

(3)




(4)




(5)




All questions from (1) to (5) require to find angle x using features of interior and exterior angles of polygons. However, the major difference is that no concrete measure of angles is given in question (5) while given in the other questions.

[Example 2] Formulate an equation between x and y.

(1) A bicycle with 38 cm wheels moves y cm after x revolution of wheels.

(2) A rectangular with area 200 has length x m and width y m.

(3) A sector with radius 5 cm has central angle x and arc y cm.

(4) X (the number of notebooks) notebooks can be bought with 2000 won (2$) when the price of each notebook is y won.

(5) Y L of seawater is required to gain 420g of salt when X g of salt can be obtained if 2L of seawater is evaporated.

From question (1) to (4), students directly find whether the relationship between variable x and y is direct proportional or inverse proportional with one concrete number is given. However, in question (5), students need to set up a proportional expression to find out the relationship between x and y with two concrete numbers are given.

**Examples of (12) Direct and Reverse Problems**

In many cases, mathematical principles and formula are formed by two-way thinking processes such as symmetric relations, operations and inverse operations, theorems and inverse theorems. However, most students tend to use principles and formula only in one way just as the way they learned. For advanced mathematical thinking, prompt reversibility of thinking process is required.

Examples of direct and reverse problems below are presented in pairs of direct problem A and reverse problem B. While reverse problems use similar materials as in direct problems, given data of problem A become the unknown of problem B, and vice versa. The examiner let student solve direct problem A first and right after turn to reverse problem B without any explanation on solutions. It is helpful to observe how and how fast the direction of student’s flow of thinking changes. At this point, the fact that problem B is actually reverse version of problem A does not need to be notice to student.

[Problem A] Worker A and B laid 500 and 700 bricks, respectively. The wage is proportional to the number of bricks they stacked. If the total wage of two workers is 180,000 won, what is the wage of A and B each?

[Problem B] The total wage of worker A and B is 48000 won after they stacked bricks. The wage is proportional to the number of bricks they laid. If worker A stacked 200 bricks and received 30,000 won, how many bricks were stacked by worker B?

Problem A can be solved by dividing total wage (180,000 won) in the ratio of the number of bricks (5:7), while problem B requires to find ratio of wages of worker A and B first, and multiply this ratio by the number of bricks that A stacked.

[Problem A] Anne, Tim, and Rachel divided 24 marbles in ratio of 3:5:4. How many marbles does each of them have?

[Problem B] Beth, Daniel, and Joshua have marbles in ratio of 5:7:3, respectively. If Beth has 15 marbles, how many marbles do Daniel and Joshua have?

Problem A demands ratio distribution while problem B is involved with the ratio of marbles that Beth has.

[Problem A] If a person was loaned 500,000 won with monthly interest rate 1.3% for four months, what is the total amount that he should pay back?

[Problem B] A person had deposited 150,000 won with annual interest rate 10%. When he withdrew, the amount became 195,000 won. How long did he deposit the money?

Problem A requires the amount with interest when original amount, interest rate, and period were given, and problem B demands the period when original amount, interest rate, and the amount with interest were given.

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