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HISTORY, APPLICATION, AND PHILOSOPHY OF MATHEMATICS IN MATHEMATICS EDUCATION: ACCESSING AND ASSESSING STUDENTS' OVERVIEW & JUDGMENT

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The paper addresses the three dimensions of history, application, and philosophy of mathematics in the teaching and learning of mathematics. It is discussed how students' overview and judgment – interpreted as 'sets of views' and beliefs about mathematics as a discipline – may be developed and/or changed through teaching activities embracing all three dimensions of history, application, and philosophy. More precisely, an example of such a teaching activity for upper secondary school is described along with a method for both accessing and assessing students' overview and judgment. Examples of data analysis are given based on a concrete implementation of the teaching activity.

History, applications, and philosophy of mathematics; overview & judgment; students' beliefs, views, and images of mathematics as a discipline.

INTRODUCTION

Recalling Imre Lakatos' introductory statement to his *History of Science and Its Rational Reconstructions* from 1970, "philosophy of science without history of science is empty; history of science without philosophy of science is blind", I intend in this regular lecture to address interrelations between the two dimensions of history of mathematics and philosophy of mathematics in the teaching and learning of mathematics and further relate this to the dimension of applications of mathematics in mathematics education.

Besides my personal interest in history, application, and philosophy of mathematics in mathematics education, my academic motivation for wanting to address interrelations between history, applications, and philosophy in mathematics education is twofold; one from the international scene and one from the national.

Research on history, application, and philosophy in mathematics education

Internationally speaking, with the creation of the ICMI affiliated *International Study Group on the relations between History, Pedagogy, and Mathematics*, known as HPM (http://www.clab.edc.uoc.gr/hpm/), the four past decades has offered a vast amount of literature on the inclusion of a historical dimension in the teaching and learning of mathematics (e.g. Jankvist, 2009a). Although the first couple of decades mainly provided literature of an advocating or descriptive nature, often drawing on teachers' own positive experiences, a shift towards actual research studies, including empirical studies, has occurred in the past decade or so (Jankvist, 2012a), and this in particular since the publishing of the ICMI Study on *History in Mathematics Education* (Fauvel & van Maanen, 2000). One initiative which has been taken is that of setting up a working group at the *Congress of the European society for Research in Mathematics Education* (CERME) specifically devoted to research on history in mathematics education.

Regarding application – and modeling – equally much, if not more, has also been done, in particular by the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), which has as its declared purpose "to promote Applications and Modeling (A&M) in all areas of mathematics education – primary and secondary schools, colleges and universities" (http://www.ictma.net/). ICTMA has had biennial meetings since 1983 and in 2003 it became the last of the five affiliated study groups under ICMI. As HPM, ICTMA is also present at general mathematics education conferences with topic study groups, discussion groups, working groups, etc. and an ICMI Study has been devoted to Modelling and Applications in Mathematics Education (Blum, Galbraith, Henn & Niss, 2007).

In comparison to the number of studies on history and applications/modeling of mathematics found in international journals of mathematics education, including special issues, etc., studies on a philosophical dimension in the teaching and learning of mathematics are close to non-existing. When one performs a search on 'philosophy' and 'mathematics education' the hits found almost all concern the use of philosophy in developing mathematics education theory, not its inclusion in the classrooms, etc. (For a list of the few exceptions I am aware of, see Jankvist, preprint). Now, why is this so? Is it the case that a philosophical dimension has nothing to offer mathematics education? To the best of my knowledge, the answer to this question is 'no'. To illustrate it, I turn to the second part of the motivation; the national one.

"Overview and judgment"

In Denmark in 2002 a report was published as the final product of a government funded project on *Competencies and Mathematical Learning*, edited by Mogens Niss and Thomas Højgaard (Niss & Højgaard, 2011, English edition). Besides listing and discussing eight (1st order) mathematical competencies (mathematical thinking; problem tackling; modeling; reasoning; representation; symbols and formalism; communication; and the tools and aids competency), the KOM-project also lists three (2nd order) competencies, known as three types of *overview and judgment* (OJ). These are:

- OJ1: the actual application of mathematics in other subject and practice areas;
- OJ2: the historical evolvement of mathematics, both internally and from a social point of view; and
- OJ3: the nature of mathematics as a subject.

Where mathematical (1st order) competencies comprise "having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role", or in other words a kind of "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge", the three types of overview and judgment are "'active insights' into the nature and role of mathematics in the world" (pp. 49, 73). Niss and Højgaard state that "these insights enable the person mastering them to have a set of views allowing him or her overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture" (p. 73). As we shall see below, the three types of overview and judgment largely resemble the dimensions of history (~OJ2), applications (~OJ1), and philosophy (~OJ3) in mathematics education.

The first type of overview and judgment (OJ1) concerns actual applications of mathematics to extra-mathematical purposes within areas of everyday life, society, or other scientific disciplines. The application is brought about through the creation and utilization of mathematical models of some kind. As examples of questions to be considered in relation to this, Niss and Højgaard (p. 74) mention: "Who, outside mathematics itself, actually uses it for anything? What for? Why? How? By what means? On what conditions? With what consequences? What is required to be able to use it? Etc."

The second type (OJ2) should not be confused with knowledge of the history of mathematics viewed as an independent topic (as taught *per se*). The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and the motivations and mechanisms responsible for this development. On the other hand, the KOM-report says, it is obvious that if overview and judgment regarding this development is to have any weight or solidness, it must rest on concrete examples from the history of mathematics. Examples of OJ2 questions are (p. 75): "How has mathematics developed through the ages? What were the internal and external forces and motives for development? What types of actors were involved in the development? In which social situations did it take place? What has the interplay with other fields been like? Etc."

The third type (OJ3) concerns the fact that mathematics as a subject area has its own characteristics, as well as the characteristics themselves. Some of these, mathematics has in common with other subject areas, while others of them are unique. As examples of OJ3 questions Niss and Højgaard (pp. 75-76) mention: What is characteristic of mathematical problem formulation, thought, and methods? What types of results are produced and what are they used for? What science-philosophical status does its concepts and results have? How is mathematics constructed? What is its connection to other disciplines? In what ways does it distinguish itself scientifically from other disciplines? Etc."

Narrowing down the problématique

As can be seen from the set of example questions above, the third type of overview and judgment (OJ3) embraces quite a few elements related to aspects of philosophy of mathematics. Clearly, due to the lack of studies discussion a philosophical dimension in the teaching and learning of mathematics, not much has been said about the interrelation between such a dimension and the dimensions of history and application, respectively.

One observation which may be made in regard to the use of history, application, and philosophy in mathematics education is that these dimensions may play either the role of a *means* for improving the usual mathematical instruction in one way or another, or they may play the role of an *end*. Regarding application (and modeling), Niss (2009) argues that, on the one hand, such a dimension may serve as a means to support the learning of mathematics, either by providing interpretation and meaning to mathematical ideas, constructs, argumentation, and proof, or by motivating students to study mathematics. On the other hand, it may be seen as an end in itself that students become acquainted with the use of mathematics in extra-mathematical contexts (and in relation to modeling, also become able, themselves, to actively put mathematics to use in such contexts). As discussed by Jankvist (2009a), the same situation applies to the historical dimension: history may be used as a *tool* for teaching and learning mathematical ideas, concepts, theories, methods, algorithms, ways of argumentation and proof; and history may be used as a *goal*, meaning that it is considered a goal to teach students how mathematics has come into being, the historical development of it as well as both human and cultural aspects of this development, etc. Not surprisingly, the role of a philosophical dimension subordinates to a similar categorization (Jankvist, preprint), where philosophy as a means/tool would embrace arguments stating that philosophy may assist students in their sense-making of e.g. mathematical argumentation and the notion of mathematical proof, including also how and why we argue and prove, mathematical ideas and constructs, etc., philosophy as a goal would include arguments stating that it serves a purpose in its own right for students to know something about e.g. the epistemology and/or ontology of mathematics and its concepts and constructs, the philosophical foundations of mathematics as a discipline as well as questions of why mathematics is constructed the way it is, its science-philosophical status, etc.

Looking at the above description of the three types of overview and judgment, it is clear that these concern history, applications, and philosophy in the role of ends/goals. However, what is not clear, nor from the KOM-report's normative description of overview and judgment, is:

1. How may teaching activities be designed in order to assist students in their development of the three types of overview and judgment?; and
2. How may students' possession and/or development of the three types of overview and judgment be both accessed and assessed?

In respect to these questions – which shall make up the research questions of this paper – it is worth noticing that the KOM-report talks about a person who is able to master his or her active insights in relation to the three types of overview and judgment as being equipped with a *set of views* regarding mathematics and its role in relation to nature, society, culture, and the world in general. Without entering into the long discussion of the difference between knowledge and beliefs, it seems fair to say that such a set of views is related to a student's beliefs about mathematics as a (scientific) discipline as well as his or her knowledge. Thus, I shall address these issues next.

BELIEFS ABOUT MATHEMATICS (AS A DISCIPLINE)

One recent definition of beliefs, although given in the context of teacher education, is that of Philipp, who in the *Second Handbook of Research on Mathematics Teaching and Learning*, describes beliefs as "lenses through which one looks when interpreting the world" and:

> Psychologically held understandings, premises, or propositions about the world that are thought to be true. [...] Beliefs might be thought of as lenses that affect one's view of some

> aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. (Philipp, 2007, p. 259)

Thus, what is knowledge for one person may be belief for another. Regarding beliefs, people are generally aware of the fact that others may believe differently, and even that their stances may be disproved. Concerning knowledge, on the other hand, people find "general agreement about procedures for evaluating and judging its validity" (Thompson, 1992, p. 130).

Various attempts have been made to try and organize people's beliefs about mathematics. One of the more recent categorizations of students' mathematics-related beliefs is that of Op't Eynde, de Corte and Verschaffel (2002). They review four earlier categorizations of students' beliefs (due to Underhill, 1988; McLeod, 1992; Pehkonen, 1995 – also to be found in Pehkonen and Törner, 1996; and Kloosterman, 1996), and provide a new more comprehensive framework of their own, structured under three different topics:

> 1. *Beliefs about mathematics education*: a. beliefs about mathematics as a subject; b. beliefs about mathematical learning and problem solving; c. beliefs about mathematics teaching in general
>
> 2. *Beliefs about the self*: a. self-efficacy beliefs; b. control beliefs; c. task-value beliefs; d. goal-orientation beliefs
>
> 3. *Beliefs about the social context*: a. beliefs about the norms in their own class (a1. the role and the functioning of the teacher; a2. the role and the functioning of the students); b. beliefs about the socio-mathematical norms in their own class (Op't Eynde et al., 2002, p. 28)

I shall not go into a detailed discussion of the components of these three categories of beliefs, only mention that in the context of this framework of students' mathematics-related beliefs it is point 1a, students' beliefs about mathematics as a subject, which is closest related to the KOM-report's 'set of views' in relation to overview and judgment. However, due to the embeddedness of point 1a in an educational context, it is fair to argue that there is a dimension missing in the above categorization – one we may call *beliefs about mathematics as a discipline*. In fact, this dimension is dealt with more independently by Underhill (1988) and Pehkonen (1995), but Op't Eynde et al. (2002) play it down significantly. Nevertheless, the shortage may be remedied by adding the extra dimension – as in figure 1 (right). (For further discussion of this, see Jankvist, forthcoming).

Such a dimension about *mathematics as a discipline* of course *embraces a student's 'set of views' in relation to the three types of overview and judgment*. The reason for placing the dimension outside the triangle in figure 1, i.e. not just turning this into a square, has to do with the fact that mathematics as a discipline is rather different than mathematics as a subject, as included under beliefs about mathematics education. However, if students are to obtain an image of and develop beliefs about mathematics as a discipline through their teaching and learning of mathematics, then this can only happen in the interplay between their social context (class), their mathematics education, and their self, which is to say the triangle making up the base of the tetrahedron.

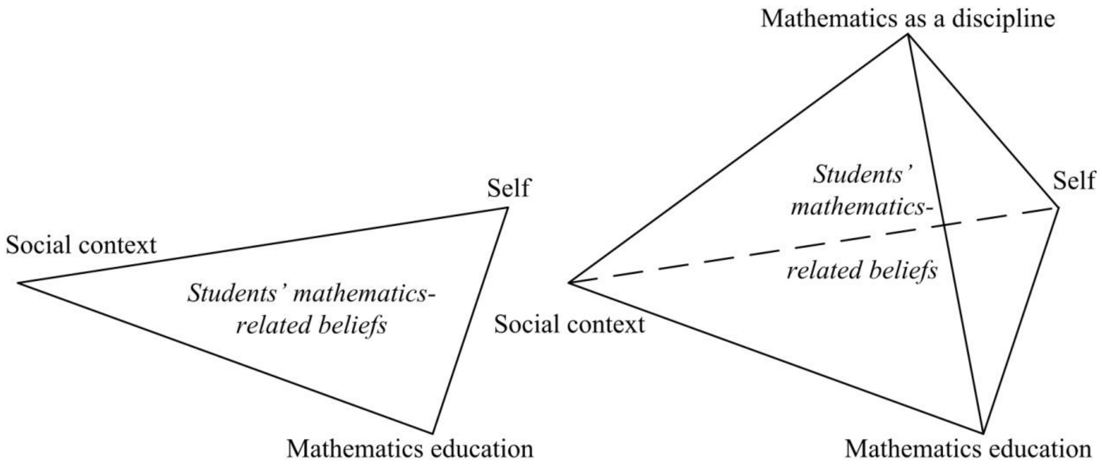

Figure 1. Left: "Constitutive dimensions of students' mathematics-related belief systems" illustrated by a triangle with the corners: *mathematics education*, *social context* (the class), and the *self* (Op't Eynde et al., 2002, p. 27, figure 2). Right: An expansion of the left hand side triangle to a tetrahedron, the dimension above the triangle illustrating *mathematics as a discipline*.

Thus, one way of trying to create a setting in which students' beliefs/set of views about mathematics as a discipline may be developed is to design an activity to be carried out in class (social context) as part of the students' regular mathematics program (mathematics education), during which the students also are exposed to both individual questionnaires and interviews (self) in order to access and assess their overview and judgment. I shall explain this in detail in the following sections.

DESIGNING A HISTORY, APPLICATION, AND PHILOSOPHY (HAPH) MODULE

As stated by Niss & Jensen (2011), it is clear that if overview and judgment regarding the historical evolvement of mathematics (OJ2) is to have any *weight or solidness*, it must rest on *concrete examples* from the history of mathematics. Although the KOM-report only states this in relation to OJ2, it is equally clear that something similar applies to OJ1 and OJ3 as well: to know about actual applications of mathematics in other subject and practice areas one must be exposed to concrete examples; and to know about the nature (and philosophy) of mathematics as a subject or discipline, one also needs concrete examples in order to hold one's beliefs/set of views more *evidentially* (Green, 1971) or more *knowledge-based*.

Mathematical problems: Euler paths, shortest path, and minimum spanning trees

The main idea of the design to be described is to have the students read and work with one original source for each of the three types of overview and judgment, all of them adhering to a common mathematical theme and/or topic. I shall illustrate this by describing a concrete teaching module, which was implemented in a mathematics class in first year of Danish upper secondary school in 2010. The class consisted of 27 students of age 16-17 years.

The three texts (in Danish translation) included in the teaching material for this module were:

- LEONHARD EULER, 1736: *Solutio problematis ad geometriam situs pertinentis*
- EDSGER W. DIJKSTRA, 1959: *A Note on Two Problems in Connexion with Graphs*

- DAVID HILBERT, 1900: *Mathematische Probleme – Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900* (the introduction only).

The overall theme was *mathematical problems*, which was what Hilbert addressed in general terms in the introduction of his lecture from 1900. To make Hilbert's general observations a bit more concrete, the students were first to read the two other texts, each of which addresses a mathematical problem. Euler's paper from 1736 is on the *Königsberg bridge problem*: how to take a stroll through Königsberg crossing each of its seven bridges once and only once – and today the paper is considered the beginning of mathematical graph theory. Two centuries later, with the dawn of the computer era, graph theory (and discrete mathematics in general) found new applications. *Dijkstra's algorithm* from 1959 solves the problem of finding shortest path in a connected and weighted graph, and today it finds its use in almost every Internet application that has to do with shortest distance, fastest distance or lowest cost. Furthermore Dijkstra also discussed a method for finding *minimum spanning trees*, a problem relevant for the building of computers at the time, but also highly relevant today.

Because original sources often are difficult to access, the presentation of these were supplied with explanatory comments and tasks along the way – a so-called 'guided reading' of the sources, inspired by the format developed by Barnett, Lodder, Pengelley, Pivkina & Ranjan (2011) and others related to the group at New Mexico State University who consider the use of original historical sources in the classroom. Practically no mathematical requirements were needed beforehand on the students' behalf to study the text of Euler – a major reason for choosing this text initially – and many of those needed for the Dijkstra text were introduced in commentaries along with the Euler text, thereby also bringing the students somewhat up to date with modern notation, etc.

The students' way into the first original text was by looking at Euler's diagram of landmasses and rivers in Königsberg (figure 2, middle) and then verify that this is in fact an accurate representation of (or model for) the Königsberg bridge problem by comparing with a picture of the town (figure 2, left). Afterwards the students were told that in modern graph theory, landmasses are represented by vertices (or nodes) and links between them by edges. Students were asked to transform Euler's diagram into such a modern graph individually and then compare their own representation to the students next to them, this illustrating that graph representations can look different. The idea was to have the students adapt more and more schematic representations of the Königsberg bridge problem until arriving at something looking like figure 2 (right), gradually increasing the level of abstractness.

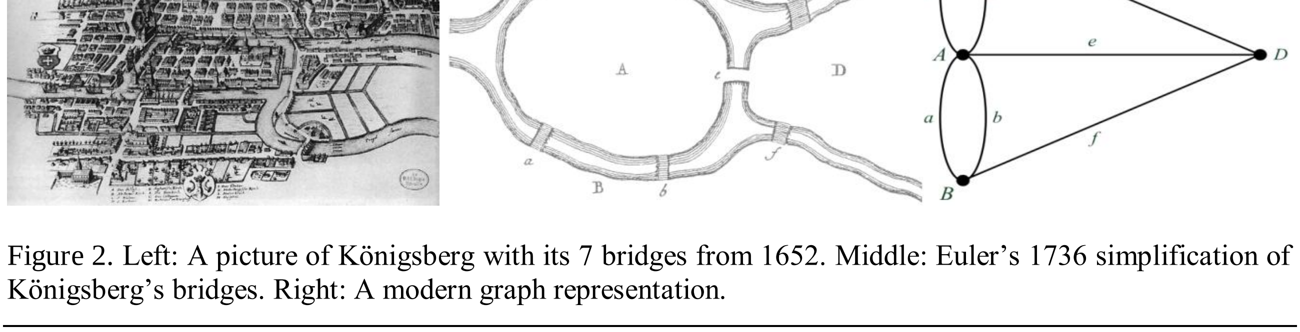

Figure 2. Left: A picture of Königsberg with its 7 bridges from 1652. Middle: Euler's 1736 simplification of Königsberg's bridges. Right: A modern graph representation.

Once being familiar with the modern representation of a graph, the students were introduced to the problem of representing *multiple edges*, such as for example the two edges between vertices A and B in the Königsberg graph. These cannot be represented by only their pair, (A,B), since this causes ambiguity (which also is why Euler named them a and b, respectively). To illustrate a formal and general way of dealing with this to the students, they were provided with the following modern definition:

> A graph **G** is a set of vertices V(**G**) and a set of edges E(**G**) together with a function ψ, which for every edge e ϵ E(**G**) assigns a pair, called ψ(e), of vertices from V(**G**).

The students were then asked to write up the sets $V(\boldsymbol{G})$ and $E(\boldsymbol{G})$ for the Königsberg graph and the seven function values of $\psi(e)$. On the one hand, the idea of this was to enable them to perceive the definition of a graph as a triplet $\boldsymbol{G}=\{V(\boldsymbol{G}), E(\boldsymbol{G}), \psi_{\boldsymbol{G}}\}$, and on the other hand to have them realize how the above definition in a general fashion resolves the problem of ambiguity when two vertices in a graph have multiple edges.

As Euler himself in his text introduces various constructs, the students were introduced to the somewhat equivalent modern terminology in the intermediate commentaries, e.g. *route*, *path*, *Euler path* (open and closed), *subgraph*, *degree* of a vertex as well as a few small theorems which Euler explicitly or implicitly uses, such as for example the so-called *handshake theorem*. At the end of his paper, Euler states his three main results (Euler, 1736, p. 139 in Fleischner, 1990, p. II.19, numbering is mine):

> (i) If there are more than two regions with an odd number of bridges leading to them, it can be declared with certainty that such a walk is impossible.
>
> (ii) If, however, there are only two regions with an odd number of bridges leading to them, a walk is possible provided the walk starts in one of these two regions.
>
> (iii) If, finally, there is no region at all with an odd number of bridges leading to it, a walk in the desired manner is possible and can begin in any region.

The students were first asked to formulate these three results using the modern terminology and notation they had been introduced to. Next they were provided with a modern definition of a *connected graph*, i.e. that there exists a route between every pair of vertices, a property Euler does not state explicitly. Using this property, the three results may be reformulated as:

> If a connected graph **G** has more than two vertices of uneven degree, then it does not contain an Euler path.
>
> Let **G** be a connected graph, then **G** contains an (open) Euler path if and only if **G** contains exactly two vertices of uneven degree.
>
> Let **G** be a connected graph, then **G** contains a (closed) Euler path if and only if all vertices of **G** have even degree.

Most of Euler's efforts goes into proving his first result (i), and regarding the third (iii), which today is considered the main theorem of the paper, he only proves it in one direction. To introduce the students to the notion of if-and-only-if theorems, they were to consider result i as being of the form $\mathbf{P} : \mathbf{A} \Rightarrow \mathbf{B}$, and then identify **P**, **A**, and **B**. After having the students prove that **A** ⇒ **B** ≡ ¬**A** ⇐ ¬**B** (by means of a truth table), they were asked to write up ¬**B** ⇒ ¬**A** for result i, i.e. formulating the contrapositive of this theorem, which states that

> If G is connected and has an Euler path (open or closed), then **G** has two or less vertices of uneven degree.

Since Euler has shown, in his own context of course, that a graph will always contain an even number of vertices with uneven degree, we may distinguish between two different cases: when ***G*** has exactly two vertices of uneven degree and when it has none, i.e. when all vertices have even degree. These cases correspond to the ⇒–direction in results ii and iii, respectively. Thus, by looking at Euler's original text again, the students would be able to deduce that the missing parts of the proofs are the ⇐–directions for results ii and iii. For result iii this is ascribed to Carl Hierholzer (published posthumous in 1873), and the students were shown this proof. Then they were asked to prove the ⇒–direction for iii and both ways for result ii using modern terminology.

While employed at Mathematical Centrum in Amsterdam in 1956, Dijkstra was asked to demonstrate how powerful the center's computer, the so-called ARMAC, was. He did so by devising an algorithm for finding shortest path between two nodes in a connected, weighted graph – today known simply as *Dijkstra's algorithm*. Dijkstra's description of his algorithm appeared in 1959 in a paper which also described an algorithm for finding minimum spanning trees in connected, weighted graphs. Unlike Euler's text the text by Dijkstra is short and builds on a large apparatus of existing graph theory. In fact, the text is only a few pages long. Also, Dijkstra only provides the description of his algorithms and he gives no examples of running these and no proofs of their correctness either, only a few remarks about running time. Thus, this text needed some 'unpacking' for the students in the form of explanatory comments, additional examples, tasks, etc. For example, the students were provided with definitions of a *weighted graph*, a *tree*, and a *spanning tree*:

> A connected graph **T** without any subgraphs that are circuits is called a tree, and a tree that for some graph **G** contains all vertices of V(**G**) is called a spanning tree.

And to illustrate that finding a least spanning tree is not trivial, they were asked to look at the Königsberg graph (figure 2, right) and find the number of different spanning trees that can be constructed from this and then explain their method for finding the answer. (The answer, which is 21, may be calculated using the so-called (Kirchhoff-Trent) *Matrix-Gerüst-Satz*. Deleting the *i*'th row and column of this matrix and taking the determinant of the one dimension smaller matrix reveals it. But the students had to do it by systematic inspection.)

In fact, Dijkstra's motivation for devising an algorithm for finding minimum spanning tree had to do with a very specific problem related to the construction of the ARMAC computer. The massive size computers at the time required vast amounts of expensive copper wire to connect their components. Finding a minimum spanning tree corresponds to leading electricity to all electric circuits while using the least amount of expensive copper wire. (A few comments were of course made about the earlier discoveries of the algorithms by Jarník, Borůkva, Kruskal and Prim, respectively.)

Having worked through the Dijkstra text, the commentaries and examples to this, and a modern proof of the shortest path algorithm's correctness, the students got to the third text by Hilbert; the introduction of his 1900-lecture in which he discusses 'mathematical problems'. Paraphrasing Hilbert roughly, he states that often some mathematical development is spurred on by a problem in the extra-mathematical world. Then it is drawn into mathematics and rephrased so that it is hardly recognizable anymore and embedded in a much more general context. Years later, when this has grown into a mathematical discipline, what often happens is that it may then again be used to solve some new extra-mathematical problem:

> Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. [...]
>
> But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. [...]
>
> In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. (Hilbert, 1902, quoted from the 2000-reprint, p. 409)

In a certain sense, the case of graph theory illustrates this: first, spurred on by the Königsberg bridge problem, which Euler generalized so that the answer to the original problem falls out as a small corollary to his more general results; and next, two centuries later when we have a much clearer idea about the discipline of graph theory, Dijkstra solves the extra-mathematical problem of shortest path (and also considers minimum spanning trees) in this graph theoretical context.

Three student essay-assignments

For the students to realize this connection between the three original texts, and thus the three dimensions of history, application, and philosophy, they were asked to identify the criteria that Hilbert proposes for a good mathematical problem (e.g. that it must be explainable to laymen and that it must be challenging but not inaccessible, etc.) and see to what degree the problems treated by Euler and Dijkstra fulfill these, and then relate these cases to Hilbert's comments on the development of mathematics in general. The context in which they were asked to do so was as part of a so-called *essay assignment*. In a previous study I found that having groups of students prepare small essays was a good way of bringing them to work with the history of mathematics (Jankvist, 2009b; 2010; 2011). So the same approach was taken to bring in the two other dimensions of application and philosophy. The module included three essay assignments, each addressing different aspects in relation to overview and judgment. The first essay was on the just discussed topic of *mathematical problems*, linking the three texts by Euler, Dijkstra, and Hilbert together.

The second essay was on *mathematical proofs* and first dealt with different kinds of proofs and proof techniques as well as the use and need for new signs and symbols (both arithmetical and graphical) in the development of new mathematics (concepts, definitions, etc.), something that Hilbert also addresses. The students were asked to discuss this with relation to Hilbert's text and try to draw connections to the two cases, in particular the advantages Dijkstra had in 1959 with a fully developed graph theoretical and conceptual apparatus at his disposal as compared to Euler who had to start from scratch in 1736. In the end, this essay moved into Hilbert's actual discussion of proofs and their role in solving mathematical problems as well as the role of rigor in mathematical proofs. On the overall, the idea of this was to spur some reflections on the students' behalf regarding the epistemological development of the notion of proof.

The third essay was about *mathematics' status as a (scientific) discipline*, in its own right and in comparison to other disciplines, e.g. physics. Based on their readings of Hilbert, and the two texts by Euler and Dijkstra, the students were asked to try to point out some characteristics of mathematical problems, methods, and ways of thinking as well as to say something about the types of results mathematics delivers and what they may possibly be used for. They were invited to discuss this by comparing mathematics to other academic disciplines. Then they were asked to identify what Hilbert says about the differences and connections between mathematics and other disciplines, and then discuss to what extent they agree or disagree.

ACCESSING STUDENTS' OVERVIEW AND JUDGMENT

In order to access the upper secondary students' overview and judgment, and the development of this, they were given an 'overview and judgment questionnaire' and a selection of the students (half the class) was interviewed about their answers. This questionnaire included three sets of 8 questions, each set connected to a type of overview and judgment. The first set, which was connected to OJ1, asked about application and sociologically oriented aspects of mathematics as a discipline:

(a1) *Do you believe it to be important for* you *to learn mathematics? If 'yes', why? If 'no', why not?*
(a2) *Do you believe it to be important for people in general to learn mathematics? If 'yes', for whom is it then most important and why? If 'no', why not?*
(a3) *From time to time you hear that mathematics is used in many different contexts. Can you mention any places from your everyday life or elsewhere in society where mathematics is being applied, either directly or indirectly?*
(a4) *Not counting the ordinary types of calculation (the four basic arithmetical operations, calculation of percentages, etc.) where do you then find mathematics applied in your everyday life and society in general?*
(a5) *Do you think mathematics has a greater or lesser influence in society today than 100 years ago?*
(a6) *Is mathematics a science? If 'yes, about what? If 'no', what is it then?*
(a7) *If you believe mathematics to be a science, is it then a natural science? Why or why not?*
(a8) *What do you understand by a mathematical* model *or that of carrying out mathematical* modeling*?*

The word *science* in a5 and a6 is to be understood in the more broad sense of Scandinavian *videnskab* and German *Wissenschaft*, including natural science, social science and the humanities (see Jankvist, 2009a for further explanation). The second set, related to OJ2, asked about historical and developmental aspects of mathematics as a discipline:

(b1) *How do you think that the mathematics in your textbooks came into being?*
(b2) *Why do you think it came into being?*
(b3) *When do you think it came into being?*
(b4) *From when does the coordinate system, as we know it, originate do you think?*
(b5) *When did Euclid live (approximately)?*
(b6) *What do you think a researcher in mathematics (at universities and the like) does? What does the research consist in?*
(b7) *When do you think the negative numbers were (formerly) defined in mathematics (in the Western world): approx. 2000 BC; approx. 300 BC; 15th century; or 19th century?*
(b8) *When do you think mathematicians (and others) began using negative numbers (in the Western world): approx. 2000 BC; approx. 300 BC; 15th century; or 19th century?*

And the third set, for OJ3, asked more philosophically oriented questions of mathematics as a discipline:

(c1) *Do you think that parts of mathematics can become obsolete? If yes, in what way?*
(c2) *Do geometrical figures, e.g. triangles, exist independently of us humans or are they human constructions?*
(c3) *Does the number 'square root of 2' exist independently of us humans or is it a human construction?*
(c4) *Are the negative numbers discovered or invented? Why?*
(c5) *Do you believe that mathematics in general is something you discover or invent?*
(c6) *What is a mathematical problem?*
(c7) *Can you give a short description of how an area of mathematics is built?*
(c8) *Why do we prove mathematical theorems?*

(For a more thorough discussion of a selection of the above questions, see Jankvist, 2009b).

During a two-year period the students were asked to answer the above questions three times, intervened by two HAPh-modules; the one on mathematical problems, graph theory and shortest paths described above and another on the unreasonable effectiveness of mathematics, Boolean algebra and Shannon circuits (see Jankvist, 2012b; preprint). HAPh-module 1 ran over twelve 90-minutes lessons and module 2 over seven 90-minutes lessons (due to practical constraints). After each module, the students were also given a test on the content of the module including its related tasks and essays. For a timetable of the study, see table 1.

Table 1: Timeline for complete empirical study.

Year	Dates	Activity
2010	February 8th	1st O&J questionnaire
	February	Follow-up interviews (round 1)
	April - May	Implementation of HAPh-module 1
	May 8th	1st test questionnaire
	May	Follow-up interviews (round 2)
2011	May 4th	2nd O&J questionnaire
2012	September - October	Implementation of HAPh-module 2
	October 12th	2nd test questionnaire
	November	Follow-up interviews (round 3)
	March 8th	3rd O&J questionnaire
	March - April	Follow-up interviews (round 4)

ASSESSING STUDENTS' OVERVIEW AND JUDGMENT

When assessing the students' possession and development of overview and judgment, I shall regard this as a kind of *reflected image of mathematics as a discipline*. This requires an explanation – or two to be more precise.

First, with reference to the literature describing beliefs (see earlier), beliefs are considered as something rather persistent. Therefore, to state that any developments and/or changes in beliefs are in fact permanent is difficult, if not impossible, based on a two-year study as the one described here. Of course, following a class of students for two years should make it more possible to verify observed developments and/or changes than if only following them for one year – but still. However, what we may say is that observed developments and changes occur in the students' views (and 'sets of views'), if we think of (and define) *views to be something less persistent than beliefs, but with the potential to develop into beliefs at a later point in time*. In this respect, we may also consider (and define) *students' images of mathematics (as a discipline) to be made up of their beliefs as well as their views* (Jankvist, 2009a; forthcoming), including of course also their knowledge, e.g. in the form of evidentially or knowledge-based beliefs/views.

Second, what then are we to consider as a student's *reflected* image of mathematics as a discipline? The definition of this will be based on empirical findings from a previous study, which concerned the use of history in mathematics education (as a 'goal') and students' development of their image of mathematics as a discipline, in particular in relation to OJ2. In this study it was found that upper secondary student' beliefs/views of mathematics as a discipline developed in (at least) three ways, namely in terms of (Jankvist, 2009a):

- a growth in *consistency* between a student's related beliefs/views;
- the extent to which a student sought to *justify* his or her beliefs and views; and
- the amount of provided *exemplifications* in support of the beliefs and views a student held, i.e. that the beliefs appeared to be held more evidentially or knowledge-based.

Of course, a prerequisite condition for this is that the students are explicit about their beliefs, i.e. that they are able to express them, but the continuous rounds of questionnaires and interviews provided a setting for this – in the previous as well as the present study.

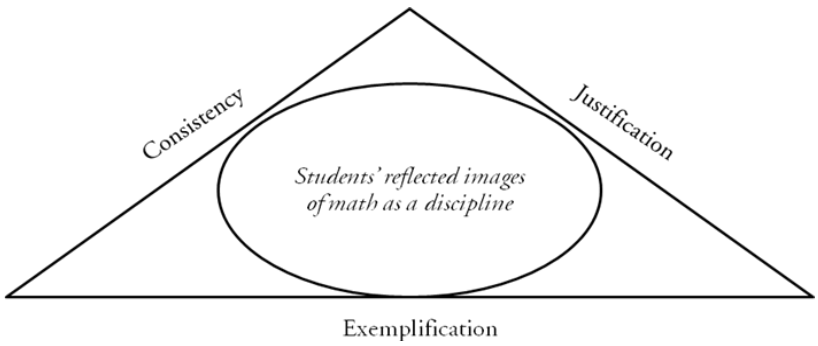

Figure 3. Students' reflected images of mathematics as a discipline as consisting of three dimensions on a basis of explicitness: consistency, justification, and exemplification (evidence) (Jankvist, 2009a).

On the one hand, the three observed dimensions of consistency, justification, and exemplification may be used define what is to be understood by students' *reflected images of mathematics as a discipline* (held on a basis of explicitness) – a definition which also provides some depth to the previously introduced fourth dimension of "students' mathematics-related beliefs" in figure 1. The situation is illustrated in figure 3.

On the other hand, we may consider figure 3 as a model of students' reflected images which tells us that in order to understand the degree of reflection in their images we need to consider each of these three dimensions. That is to say, the model in figure 3 may provide us with a way of assessing the students' development of overview and judgment – and, so to speak, operationalize the KOM-report's normative description of overview and judgment.

EXAMPLES OF DATA ANALYSIS

In order to illustrate how the three O&J questionnaires and the follow-up interviews from the above presented study can be used to access students' beliefs and/or views about mathematics as a discipline and thereby assess their possession/development of overview and judgment, I shall make some illustrative 'downstrokes' in the data and questionnaire questions, taking an approach of tracking changes on an individual student basis.

Accessing and assessing: Consistency

One of the places where consistency, and a growth of such, often shows is in the philosophy oriented questions c2 through c5. As illustration of how to search for consistency in the answers to these questions we shall look at two students, Samuel and Larry.

In the 1st questionnaire, Samuel answered the following to the question about geometrical figures and square root of 2, c2 & c3: "I guess it always existed, but we have defined it." To the question of the negative numbers, c4: "I don't think they are discovered. Should they have been discovered on a rock? Neither were they invented, we had to think to find them." And regarding mathematics in general, c5: "Neither. You can't discover mathematics in the ground, like fossils, for example. You can't invent mathematics, because then it'd be concrete – like a pencil, for example. It has always been there, but we had to think it, not discover or invent. In a way it is quite philosophical." What we may notice here is that Samuel's answers are both contradictory and somewhat inconsistent, because on the one hand mathematics has always been there, but on the other hand humans had to think it and define it. The statement that mathematics is neither a discovery, nor an invention, may be explained by the student not counting immaterial things as something which is subject to discovery or invention. One year later, in the 2nd questionnaire, Samuel provides the following answer to c2: "Human construction – nothing is coincidental" and refers to this answer in the other questions. Two years later, in the 3rd questionnaire, he answered to c2: "It is a human construction. Let us say that triangles can be found in the nature, but humans have defined them, i.e. a human construction." In c3 he referred to this answer also, and in c4 and c5 he answered: "It is something which has been invented." Now, even though Samuel states that maybe triangles can be found in nature (many students said so with reference to the shape of mountains, etc.), his answers in both the 2nd and the 3rd questionnaire are very consistent; in all answers he believes 'invented'. Thus, there is a growth in consistency from the 1st to the 2nd and 3rd.

Larry provided the following answers to c2 in the 1st questionnaire: "Can't think of places in nature where 'perfect' geometrical figures exist, so (even though there may be places where they do) it is human made." And for the square root of 2 in c3: "Can't give a qualified answer." Regarding negative numbers in c4, he wrote: "In the beginning invented. But physics have now proven the energy of the universe to equal 0, because of negative forces and quantum mechanics oscillation (which was believed to create everything), so now it is also proven." And for mathematics in general, in c5: "First invent (including providing arguments) and then proven by means of physics, etc." As the majority of students, Larry had never given thought to the question of invention versus discovery of mathematics before. But once the question had been raised, it kick started some thought processes. Already in round 1 of the follow-up interviews it was clear that he had thought about this, and perhaps altered his view a bit:

> Well, it depends on how you define 'invention'. I mean, if it is something where we say it's like that, and it is something that humans invented themselves. I mean, we know that there are some connections between things. [...] It is just the way we describe nature, by formulas. [...] It is our way of writing it. On the one hand, it is discovered, but at the same time it's ourselves who have invented it... [...] All things considered, it may be discovery.

In the 2nd questionnaire, Larry answered to c2: "They exist, we just defined them." To c3 he provided a conditioned answer: "Does it describe a relationship which exists in nature? If not, it is a human construction." To c4 he said: "Discovered. Things can have negative energy (electrons), i.e. something exists on the other side of the spectrum." And finally regarding mathematics in general in c5: "Discover through understanding of earlier mathematics." Clearly, there is a shift in Larry's answers from the 1st to the 2nd questionnaire, somehow reflecting his considerations from round 1 of the interviews, although there still appears to be some ambiguity present, e.g. in his answers to c2 and c5. By the time of the 3rd questionnaire, however, Larry's answers appear more consistent. c2: "Yes, a lot of math exists independently of us, the figures mentioned here." c3: "If the constant is used as for example π, then independently." c4: "Discovered." c5: "Discover, most often."

Accessing and assessing: Exemplification

In order to illustrate how exemplification can play a role in the development of a student's overview and judgment, let us stay with Larry. In round 3 of the interviews, after the HAPh-module 2, Larry said the following about discovery and invention:

> Larry: Well... There can be connections in mathematics which we discover. For example the equation with Euler's number in the power of π times i minus or plus 1 equals 0 [$e^{i\pi} + 1 = 0$]. These are some interrelations which we have not made ourselves. It is a lot of independent things which we have found and which then fits together and reveals a beautiful connection. [...] I think it is a good example of something which we just discover. As far as I know these [π, e, and i] were not that associated. But that they fit together in this

	way, it kind of shows... that there must be a... that no matter what, we did something right.
Interviewer:	Yes?
Larry:	So regarding invention or discovery in mathematics, I think... I think that some things are invented and some discovered. I will risk claiming that.
Interviewer:	Alright. Can you give some examples?
Larry:	Well... for example our way... in graph theory, to translate bridges into numbers and the way of writing it all up. That is something we've made. While things as... what is a good example? Things as π is something we discovered. [...]
Interviewer:	Okay. Is it possible to say if one precedes the other? Does discovery precede invention or does invention precede discovery?
Larry:	In most cases it must... well, not necessarily... With π, for instance, I guess that discovery was before invention, because... If we say that we invented, that we set a circle to 360 degrees. But when we calculate π [...] then we don't use the 360 degrees, as far as I recall. [...] It is different within different areas of mathematics, but with π I think we discovered that there was a connection first, and then we built on that. But it's quite related; when we choose something we quickly arrive at some further discoveries.

Worth noticing here is that Larry, in his process of going from believing mainly in invention (1st questionnaire) to mainly in discovery (3rd questionnaire), is able to provide several examples of one or the other. To illustrate discovery he first mentions Euler's identity – on the interrelation between the additive identity 0, the multiplicative identity 1, the base of natural logarithms *e*, the imaginary unit *i*, and the number π – as something which he finds it unlikely to have been invented. And he then carries on to elaborate on the number π in relation to discovery (see also answer to c3 in 3rd questionnaire). As an example of mathematics which is invented he refers to the cases of the HAPh-modules, as seen above when mentioning graph theory and Euler's way of solving the Königsberg bridge problem, but also elsewhere in the interview when he refers to Boolean algebra as a human construction and invention. Such exemplification helps Larry to hold his beliefs and views more evidentially. And it also assists him in justifying his beliefs and views as well as the development and changes in these. However, as we shall see below, justification does not always include exemplification.

Accessing and assessing: Justification

In the 1st and 2nd questionnaires the student Jean justified his answers to question a5, on the influence of mathematics in society today as compared to earlier, as follows: "I think it has greater influence today since mathematics is used to more and more and many things are about numbers." "I think it has greater influence since it is used for so many things." As compared to the students who would just answer either 'greater' or 'lesser', Jean clearly tried to justify his answers. But he did so without any exemplification whatsoever, which may be illustrated by his answer to a5 in the 3rd questionnaire: "Greater. Everything today is based on numbers and models. In particular computers and cell phones, which we couldn't live without, are based on numbers and do thousands of calculations per second." The mentioning of computers and cell phones is an exemplification made to support the justification that mathematics has greater influence today. Whether there is a connection between Jean's mentioning of computers and the HAPh-modules is not clear, although it was emphasized that Dijkstra's algorithms made up the basis of software used practically everywhere and that Shannon's circuit design ideas among other places were used in computer hardware.

Another illustration of the development of overview and judgment and the role of justification is that of the student Salma and her answers to questions a5 and a6 on mathematics being a scientific discipline and if it belongs to the natural sciences. In the 1st questionnaire she answered to a5: "I don't know if I'd call it so. But it is a tool for the other scientific disciplines within the natural sciences." And to a6: "A means for describing and understanding natural science." A year later in the 2nd questionnaire she said for a5: "It is a scientific discipline as well as a tool for understanding other sciences." And for a6: "Yes, it is a natural science since it is primarily used within the natural sciences. But at the same time it is also used within the social sciences, and for that reason it is difficult to 'classify'." And yet a year later in the 3rd questionnaire, she answered to a5: "Yes, I would say it is. In mathematics you show a theorem's validity through proofs, so yes." And for a6: "It is its own scientific discipline which may be used within the faculty of natural science as well as that of social science." Salma's answers to the questions a5 and a6 illustrate well how her image of mathematics as a (scientific) discipline gradually becomes more and more reflected over the two year period – a finding which is supported both by the four rounds of interviews and by the essay assignment hand-ins from HAPh-module 2, where one essay concerned the five faces of mathematics: a pure science; an applied science; an educational subject (both taught and studied); a system of tools for societal practice; and a certain kind of platform for gaining aesthetic experiences (Niss, 1994).

RECAPITUALTION AND FINAL REMARKS

As the reader will have noticed by now, this paper is mainly concerned with method:

- method for designing teaching modules bringing out aspects of all of the KOM-report's three types overview and judgment; and
- method for accessing and assessing students' possession and development of overview and judgment (through considerations of their beliefs and/or views of mathematics as a discipline)

But let us briefly recapitulate these methods as presented so far in order to provide possible answers for the paper's two research questions.

The way of addressing the first question on how to design activities which can assist the development of overview and judgment has been one of 'answer by example'. By this I am of course referring to the presentation of the HAPh-module on early graph theory in the form of Euler's solution to the Königsberg bridge problem, Dijkstra's algorithms, and Hilbert's discussion of mathematical problems. Through the description of this module and the essay-assignments that the students were to work with it was illustrated how a setting can be created in order for students to develop their overview and judgment regarding mathematics by challenging their existing 'set of views'. As pointed to earlier, in order for students' overview and judgment to have weight and solidity they must be provided with concrete examples. And that was what the module sought to do; provide examples in regard to the three dimensions of history, applications and philosophy of mathematics.

Regarding the second question on how to access and assess students' overview and judgment, the above may be rephrased to state that the HAPh-modules sought to provide the students with 'evidence' in order for them to make and hold their beliefs and views more evidentially and knowledge-based. As we know from Green (1971), we cannot expect students' to change or alter their beliefs and views – and consequently their overview and judgment – are they not provided with concrete evidence to 'measure' these against, the reason being that

> Not until students have access to evidence – or counter-evidence – are they likely to criticize rationally, reason about, and reflect upon their beliefs, and possibly accommodate and change them, should they find it necessary. (Jankvist, 2009a, p. 257)

This was illustrated in particular by the student Larry when he used the mathematical cases from the HAPh-modules as evidence for examples of invention and Euler's identify and the number π as evidence of discovery of mathematics. Thus, by means of examples Larry was able to justify by exemplification his gradually more consistent views regarding the question of invention versus discovery; illustrating the more reflected image of mathematics as a discipline, which he ended up with. (Something similar could be argued for the student Salma by a further display of data.) Of course, the HAPh-modules are not the only things which may cause the students to alter and accommodate their view of mathematics as a discipline – Larry's example with Euler's identity was not part of the modules and neither was Salma's example with mathematics being used within the social sciences. But whether the students' possible changes in beliefs/views can be linked to the modules through the choice of examples or not, the study illustrates that an approach through students' beliefs with repetitive questionnaires and follow-up interviews does appear to be one sensible way of accessing students' overview and judgment about mathematics and the development of such. Furthermore, basing the assessment on the presence and growth of consistency, exemplification, and justification is an approach which reveals some insight regarding the students' beliefs and views (and knowledge) of mathematics as a discipline. Due to the connectedness of the definition of students' mathematics related beliefs regarding mathematics as a discipline (cf. figure 1) and the KOM-report's three types of overview and judgment, it is clear that the students for whom it may be concluded that they have come to possess more reflected images of mathematics as a discipline also are the students who have developed their overview and judgment about mathematics. Again, Larry and Salma served as examples of such students.

Having now described and discussed the proposed methods for accessing and assessing students' overview and judgment as well as a design method for including the three dimensions of history, application, and philosophy in the upper secondary school mathematics program, let us return to our point of origin; the Lakatos quote. Thus, with apologies to Lakatos for abusing the quote, I end this regular lecture by stating that:

history and/or applications of mathematics (as well as other concrete, clarifying cases) can assist in making a use of philosophy of mathematics in mathematics education less empty;

applications of mathematics can assist in making a use of history of mathematics in mathematics education less blind; and

philosophy (and/or philosophizing) can assist in making uses of history and/or applications of mathematics in mathematics education less blind.

Additional information

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References

Blum, W., Galbraith, P. L., Henn, H.-W. & Niss, M. (eds.) (2007). *Modelling and applications in mathematics education. The 14th ICMI Study*. New York: Springer, New ICMI Studies Series 10.

Dijkstra, E. W. (1959). A note on two problems in connexion with graphs, *Numerische Mathematik*, *1*, 269–271.

Euler, L. (1736). Solutio prolematis ad geometriam situs pertinentis. *Commentarii academiae scientiarum Petropolitanae, 8* (1736/1741), 128-140.

Fauvel, J & van Maanen, J. (2000). History in Mathematics Education – The ICMI Study. Dordrecht: Kluwer Academic Publishers.

Fleischner, H. (1990). *Eulerian Graphs and Related Topics*. Amsterdam: Elsevier Science Publishers B.V.

Green, T. F. (1971). *The Activities of Teaching*. New York: McGraw Hill Book Company.

Hilbert, D. (1902). Mathematical problems, *Bulletin of the American Mathematical Society, 8*, 437–479. (Reprinted in: *Bulletin (New Series) of the American Mathematical Society, 37*(4), 407-436, Article electronically published on June 26, 2000).

Jankvist, U. T. (2009a). Using history as a ‘goal’ in mathematics education. Ph.D. thesis, IMFUFA, Roskilde: Roskilde University. http://milne.ruc.dk/ImfufaTekster/pdf/464.pdf

Jankvist, U. T. (2009b). A categorization of the ‘whys’ and ‘hows’ of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2010). An emprical study of using history as a ‘goal’. *Educational Studies in Mathematics, 74*(1), 53–74.

Jankvist, U. T. (2011). Anchoring students’ meta-perspective discussions of history in mathematics. *Journal of Research in Mathematics Education*, *42*(4), 346-385.

Jankvist, U. T. (2012a). A first attempt to identify and classify empirical studies on history in mathematics education. In B. Sriraman (ed.), *Croosroads in the History of Mathematics and Mathematics Education* (TMME Monographs 12, pp. 295-332). Charlotte, NC: Information Age Publishing.

Jankvist, U. T. (2012b). A historical teaching module on “the unreasonable effectiveness of mathematics” – The case of Boolean algebra and Shannon circuits. *HPM 2012* in Daejeon.

Jankvist, U. T. (forthcoming). Changing students' images of mathematics as a discipline. (Under review).

Jankvist, U. T. (preprint). History, Applications, and Philosophy in mathematics education: HAPh – A use of primary original sources. *Science & Education*. (Accepted).

Kloosterman, P. (1996). Students' beliefs about knowing and learning mathematics: Implications for motivation. In: M. Carr (ed.): *Motivation in Mathematics* (pp. 131-156). Cresskill, NJ: Hampton Press.

Lakatos, I. (1970). History of science and its rational reconstructions. *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*. Boston studies in the Philosophy of Science VIII, pp. 91-136.

Niss, M. (1994). Mathematics in society. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelman (eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 367-378). Dordrecht: Kluwer Academic Publishers.

Niss, M. (2009). Perspectives on the balance between application and modelling and 'pure' mathematics in the teaching and learning of mathematics. In M. Menghini, F. Furinghetti, L. Giacardi & F. Arzarello (eds.), *The First Century of the International Commission on Mathematical Instruction (1908–2008) – Reflecting and Shaping the World of Mathematics Education* (pp. 69-84). Roma: Istituto della Enciclopedia Italiana fondata da Giovanni Treccani.

Niss, M., & Højgaard, T. (eds.) (2011). Competencies and mathematical learning – ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde University. http://milne.ruc.dk/ImfufaTekster/pdf/485web_b.pdf

Op't Eynde, P., de Corte, E. & Verschaffel, L. (2002). Framing Students' Mathematics-Related Beliefs. In: G. C. Leder, E. Pehkonen, and G. Törner (eds.), *Beliefs: A Hidden Variable in Mathematics Education?*(Chapter 2, pp. 13-37). Dordrecht: Kluwer Academic Publishers.

Pehkonen, E. (1995). *Pupils' view of mathematics: Initial report for an international comparison project*. Research report 152. University of Helsinki, Department of Teacher Education.

Pehkonen, E. and Törner, G. (1996). Mathematical beliefs and different aspects of their meaning. *Zentralblätt für Didaktik der Mathematik (ZDM), 28*(4), 101–108.

Philipp, R. A. (2007) Mathematics Teachers' Beliefs and Affect. In: F. K. Lester, Jr. (ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Chapter 7, pp. 257-315). Charlotte, NC: Information Age Publishing.

Thompson, A. G. (1992). Teachers' Beliefs and Conceptions: A Synthesis of the Research. In D. A. Grouws (ed.): *Handbook of Research on Mathematics Teaching and Learning* (Chapter 7, pp. 127-146). New York: Macmillian Publishing Company.

Underhill, R. (1988). Mathematics learners' beliefs: A review. *Focus on Learning Problems in Mathematics, 10*(1), 55–69.