12th International Congress on Mathematical Education
Program Name XX-YY-zz (pp. abcde-fghij)
8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

MATHEMATICS AT UNIVERSITY: THE ANTHROPOLOGICAL APPROACH

Carl Winsløw

Department of Science Education, University of Copenhagen, Denmark

winslow@ind.ku.dk

Mathematics is studied in universities by a large number of students. At the same time it is a field of research for a (smaller) number of university teachers. What relations, if any, exist between research and teaching of mathematics in universities? Can research "support" teaching? What research and what teaching?

In this presentation I propose a theoretical framework to study these questions more precisely, based on the anthropological theory of didactics. As a main application, the links between the practices of mathematical research and university mathematics teaching are examined, in particular in the light of the dynamic between "exploring milieus" and "studying media".

Key words: university mathematics, tertiary, anthropological theory of the didactical

A PERSONAL INTRODUCTION

To make my choice of subject and viewpoint in this lecture more transparent I will start by introducing my background briefly, not because it is particularly interesting or unique, but for two other reasons: (1) I feel it could and should become more common for mathematicians to specialize in didactics; (2) to explain the choices I have made as a function of my background.

In short, I am a mathematician. Mathematics is the scholarly field in which I learned to do research, and in which I have continued to work even if my subjects and the scope of my work have evolved over time. It should be noted up front that unlike what is common in social and natural sciences, research in mathematics is usually not concerned with what is commonly understood as observations, experiments or data. My thesis was about automorphisms on inclusions of von Neumann algebras (so-called *subfactors*), which can neither be observed or measured. And indeed, developing, combining and exploring mathematical theories and methods is what I have learned to do professionally and what I continue to do.

Mathematicians do more than private thinking. They communicate and in particular, they publish and teach. As an article author, conference presenter and instructor, and later university professor, I have had (as have any mathematician) to think about mathematics from the point of view of didactics, that is: how to communicate mathematical ideas (my own and those of others). And as a university teacher, my job is not restricted to convincingly transmitting my ideas to other mathematicians, but it also includes the task of "empowering" students to do mathematics (of some kind) in a more or less autonomous fashion. This turns out to be a challenge which most mathematicians find interesting, pleasant and difficult. And it is at the heart of the deep relationship between mathematics and its didactical realm.

The relation between mathematical scholarship and mathematics teaching is so fundamental that Brousseau (1997) begins his seminal book on the "theory of didactical situations in mathematics" with explaining the relationship between the work of the mathematician, the work of the student, and the work of the teacher: the teacher has to *reconstruct* situations of learning for the students, given (often very old and transformed) *products* of what is, essentially, the mathematicians work (namely, mathematical knowledge in generalized and decontextualised form). And, while this implies that there is a kind of inversion problem at the centre of the work of the teacher, such problems are by no means foreign to what mathematicians themselves have to do all the time, not only when they act as teachers but also in their function as researchers, since one may say that

> 80 percent of mathematics research consists in reorganizing, reformulating, and "problematizing" mathematics that has already been "done", by the researcher himself or by others (Brousseau, 1999).

In fact, mathematics is – perhaps more than any other science – one in which important progress may be based on simplifying, generalizing, combining or even reformulating previous work, and in which, therefore, there is no sharp boundary between "presenting" and "developing" knowledge, or between the learning of the mathematician and the learning of the student (even if the former is certainly supposed to go beyond what is presently known by the community at large or by his students). We will return to this last point later in this paper.

I was fortunate enough to discover Brousseau's work around 1997, during one of my first searches for relevant knowledge on teaching. It was important for me to realize that teaching mathematics is essentially about reconstructing and developing situations in which mathematical ideas and methods become operational for students – and that solving these *didactical tasks* has an important and non trivial mathematical dimension. To solve them requires (hard) thinking about the mathematical content. At the same time it also requires observational knowledge about the students who are supposed to learn the content, and this evidently requires an empirical dimension which, in terms of research, goes beyond the mathematical domain.

In fact it also requires theoretical tools which are not in themselves mathematical, but which allow human activity and institutional phenomena to be modeled. This need for non-mathematical tools leads many didacticians to draw on human and social sciences such as psychology and anthropology. But these sciences have not developed the sharp tools needed to capture essential parts of a mathematical activity in a teaching context in a way that will bring about functional knowledge and not just general beliefs. Thus, "applying anthropology" (and for other reasons, psychology) poses several severe problems; we refer to Chevallard (1990) for an interesting review of early work in this direction. Here, we simply abandon it, and turn instead to outline some elements of an anthropological theory of the didactical (ATD), developed by Chevallard and others, in order to deal with these problems, with a specific view on the institutional context of the university.

THE ANTHROPOLOGICAL THEORY AND UNIVERSITY MATHEMATICS

Mathematicians and mathematics teachers alike have little or no precise means to describe mathematical activity (and hence the essence of what they do, and teach). Moreover, it is highly questionable whether "mathematics" makes sense as a homogenous whole, smoothly developed over time and across institutional and cultural boundaries which are, to some extent, obvious to the practitioner. Most evidently, school mathematics appears to be *different* in essential respects from the way mathematics is thought of at universities – for instance, numbers and shapes usually appear somewhat naturalised in the first setting, while they are presented and treated as theoretical (for instance, axiomatised) structures in the second. Clearly, these remarks are far from precise and satisfactory descriptions. The historical event of "new mathematics" in the interface between university and school mathematics contributed to a spurt of interest in the difference and resulted in sharper descriptions of it.

Institutions, knowledge and didactic transposition

Even natural numbers are no longer "natural" (or God-made, as Kronecker reportedly said) to the contemporary mathematician. Explicit constructions or axiomatizations characterize the way in which familiar numerical objects such as $27, \sqrt{2}$ or e are situated in the wider mathematical universe, along with triangles, Hilbert spaces, and so on. Most mathematicians are not overly concerned with the philosophical issues surrounding the foundations of their subject (e.g. Davis and Hersh, 1981). But even if the importance of formalization varies from one subfield to another, it is unquestionable that contemporary mathematics has developed impressively precise and explicit descriptions of its objects through formalizations and abstractions which are conceptually far from common sense notions of numbers, shapes and so on. It can also be argued that this difference has grown immensely over the past two centuries – where mathematics has become both a rapidly expanding academic field, and a school subject taught to entire populations worldwide. This reflects the progressive growth – and separation – of two kinds of *institutions*: universities and schools, each established to circulate and develop *knowledge*.

The anthropological theory begins with clarifying what "knowledge" and "institutions" mean – and in particular what they mean to the teaching and learning of mathematics. First, *knowledge* is taken in a quite wide sense, roughly "shared human ways to act and react to specific challenges" – including practical techniques, modes of explanation, theoretical structures, and so on. We return to a more detailed model of mathematical knowledge in the following subsection. As the term "shared" used above indicates, it is fundamental to human knowledge that it is developed and circulated in communities which possess a great deal of regularity, even if the members of the community change (think, for instance, of a community of research mathematicians or of researchers working on a subfield of mathematics). These "invariant communities" serve as habitats of knowledge as they develop, sustain and transmit the knowledge. And such habitats of knowledge are what we call *institutions*. Notice here that the definitions of *knowledge* and of *institution* are 'solidaric' in the sense that one depends on the other: in particular knowledge attains its status as knowledge *relative to an institution in which it is shared and developed*.

In contemporary educational systems, a selection of knowledge from institutions "outside the school" is specified as "knowledge to be taught" in school institutions – and it is subsequently transformed into "taught objects" by the school institution. This two-step process is what is called *didactic transposition* in the anthropological theory (Chevallard, 1991, p.39). The word transposition means that the process involves "moving" knowledge between institutions – and thereby, inevitably, adapting and modifying the knowledge according to the constraints of the receiving institution. As a case study, Chevallard and Johsua (1991) provide a detailed account of the transposition of the modern mathematical notion of *distance* (as defined in metric space theory) into French secondary school of the early 1970's.

Mathematical organisations

Up to now we have used rather broad and soft terms. We now come to what I perceive as the core of the anthropological theory of didactics, namely its tools to model mathematical and didactical knowledge (cf. Chevallard, 1999 and 2002, which we outline and interpret in the sequel). Let's consider what is loosely called the "notion" of *distance*, as just mentioned. And what does it mean to "master" it? Well, first of all, it means that you recognize a distance when you see it – that is, you can determine whether a given entity is one. More broadly, you can solve *tasks* involving distances – that is, you can apply certain *techniques*, defined as means to accomplish tasks. A technique usually solves a whole family of tasks – a *type of tasks* T, defined as those tasks which can be defined by a given technique τ. The couple (T, τ) is what is called a *practical block*; the two elements define each other. This is the minimal entity of *practical knowledge*. In the context of distance, we might for instance think of T as "compute the distance between two points in the plane", which – when given the coordinates – can be accomplished by the technique τ associated with the usual distance formula. In more elementary contexts, the points might be simply marked on a piece of paper, and the technique could be to execute a measurement with a marked ruler. And in more advanced ones, we might have different practice blocks, corresponding to other distances, such as the L^2-distance on the function space $L^2(0,1)$. The notion of *context* for a practical block can also be made more precise: it consists of some *technology*, a discourse about the techniques, which explains how to apply and distinguish a whole set of techniques – for instance, to distinguish and explain the two techniques for computing distance which applies in the Euclidean plane and in $L^2(0,1)$. At a higher level of discourse, technologies are developed, explained, related and justified in and by a *theory*, which in our case could be the theory of metric spaces and in particular include the definition of metric distance. For a given set of practical blocks, a technology θ and a theory Θ form together what is called a *theoretical block* (θ, Θ). Together, the practical and theoretical blocks form a *praxeology* $(T, \tau, \theta, \Theta)$. These form the "atoms" of mathematical practice and discourse; whenever faced with a task, the mathematician – or the student – will seek to identify it with a type of task and hence with a technique, which is then applied to solve the task. He might go on to explain and justify his choice of technique within a technology, and he might even be able to explain how this technology can be explained and justified within a theory. Of course, none of these derivatives of the task are universal or uniquely defined by the task, but in a given institution some technique, some technology and some theory may appear natural, privileged or even optimal to users. This, certainly, would be the case for the distance formula in many secondary schools, when faced with the task of finding the distance between two points given by coordinates.

Institutions are habitats of *praxeologies*, and these do not occur as independent atoms. Because technologies (e.g. explaining how to compute distances in different types of tasks) may serve to relate several practical blocks, mathematical practices are unified by technologies; a *local mathematical organisation* is a family of praxeologies defined by sharing one technology. Similarly a host of local mathematical organisations may be unified by one theory, to form a *regional mathematical organisation.* The power of modern mathematical theories resides exactly in unifying a host of previously unrelated local organisations, so that for instance the two inequalities

$$|a| - |b| \leq |a - b| \qquad (1)$$

(holding for real numbers *a* and *b*) and

$$\sqrt{\int_X |f|^2 d\mu} - \sqrt{\int_X |g|^2 d\mu} \leq \sqrt{\int_X |f - g|^2 d\mu} \qquad (2)$$

(valid for L^2-functions *f* and *g* on a measure space (X, μ)) could be justified from one and the same principle (the triangle inequality in the definition of a metric). Notice that the two inequalities – with explanations of their range of validity – could themselves appear as technologies to explain and justify calculation techniques, corresponding to a type of tasks, arising in more or less distant practices. The progressive development of still more encompassing theories and technologies – and with them, of regional mathematical organisations - is not only an essential part of the history of mathematics but also of the curriculum of mathematics students.

To make such developments meaningful and functional to students, they need to be (or in fact, remain) rooted in tasks which are somehow simplified or at least related through the unification of the practice blocks in which they live. Of course, historical developments are often more complex than what one can (or wants) to let students experience in an undergraduate course. An important recent area of didactical design is the use of *instrumented* techniques (based on computer algebra systems) as a means to facilitate the access of students to coherent theoretical blocks of real analysis (see Gyöngyösi, Solovej and Winsløw, 2011).

Didactic organisations and didactic co-determination

Associated with any mathematical organisation which students have to learn, we have *didactic tasks* for the teachers, which can be of various kinds and types but which are always linked to the challenge of establishing conditions and experiences which allow students to become familiar – at some level – with a specific mathematical organisation. Didactic tasks are solved by *didactic techniques* (giving rise to *types* of didactic tasks) and also when we explain, relate and justify those techniques, we are using *didactic technologies and theories*. In short, *didactic organisations* arise in close association to a mathematical organisation to be taught to a group of students. The theoretical level may, indeed, be less formalised and well articulated than in the case of mathematical organisations; to improve that is part of the vocation of didactics as a scientific field. At this point, we content ourselves to give one example, and to explain what is meant by the *co-determination* of mathematical and didactic organisations.

Consider again the notion of distance. In a first course on metric space theory, as the one studied and developed by Grønbæk and Winsløw (2007), it is a basic challenge to allow students to master the regional mathematical organisation, unified by the theory of metrics and relating technologies associated with (apparently distant) techniques as reflected in (1) and (2) above. An important type of didactic task is, then, to device (or choose) *mathematical tasks* which somehow require the student to make use of the abstract notion of metric distance, as well as its justification on a number of particular functions, to be shown to satisfy the axioms. One technique is to simply give examples of spaces M, familiar to the student, together with a real valued function d on $M \times M$, and ask to verify the axioms; this, indeed, is a type of mathematical task. A less immediate technique is to build such tasks into mathematical tasks where the validity of the axioms can be used as a technique – perhaps even before the axioms are formulated in general. Indeed, (1) and (2) can be viewed as a (slightly) concealed and (clearly) contextualised form of the triangle inequality which, together with other forms, might serve to develop the students' appreciation of its importance in general. The explanation and justification of different didactic techniques in this case would depend on wider theoretical ideas – or results, or beliefs – about how to teach the mathematical organisation in question, perhaps unifying a number of other didactical organisations as well. However, to be useful, a didactic theory would need to apply – and hence be directly associated with – the concrete didactic practices the teacher could implement, and not least with the mathematical organisations this (theoretically or empirically) enables students to develop. As an example, Grønbæk and Winsløw (2007, Appendix 1) developed a hierarchy of specific competency goals, related to the mathematical content and specific ways in which it should be mastered by students, to articulate the distinction between mathematical tasks whose accomplishment corresponds to just one such goal, and more advanced tasks which accomplish (parts of) several goals.

This brings us to the final theoretical point: while it is clear that didactic organisations make no sense independently of specific mathematical organisations *to be taught*, it is also clear that the mathematical organisations *actually realised* with students depend crucially on the didactic organisation developed by the teacher. And, as it develops, the students' mathematical organisations are observed by the teacher and the didactic organisation may be adapted according to these observations. There is, thus, a fundamental *co-determination* between mathematical and didactic organisation in a teaching setting, and this co-determination is an essential mechanism of the didactic transposition. First of all, the "knowledge" which is transposed is now modelled much more explicitly by mathematical organisations, with its four levels ranging from tasks to theory. Moreover, the transposition as it is realised *within an institution* – through teaching – involves two distinct forms of mathematical organisation: the one to be taught, and the one actually taught and learned. The means of this part of the transposition – called the *internal didactic transposition* – is modelled as a didactic organisation, involving both practices and theories which, moreover, is *co-determined* with the mathematical organisation developed by students and teachers together, and observable – in principle – in the space-time of their interaction.

UNIVERSITY MATHEMATICS AND SCHOOL MATHEMATICS

While the anthropological theory makes sense, and was developed, for the study of mathematics teaching in school institutions (for a general public), we now turn to its bearings on the two fundamental questions in this paper: (1) what, if anything, distinguishes university mathematics teaching from the teaching of mathematics in school? (2) assuming that important differences exist, what are their consequences for the didactical organisations to be developed in a university institution?

Didactic transposition in universities

Just as school institutions vary from one society to another, there is a considerable difference in how universities operate both in different places, as well as over time (cf. Madsen and Winsløw, 2009, 741-742), and in particular how decisions are made on what mathematical knowledge is to be taught in different programmes. However, as a collective within the institution, university teachers appear to have a much higher autonomy than school teachers, in terms of deciding the contents and methods to be used in a given teaching unit. This autonomy, however, may not be so much bigger in local, daily practice as it is principle. For instance, one of our respondents in a recent study (Madsen and Winsløw, 2009) on university teachers' praxeologies in research and in teaching, who is a senior mathematician, claimed that

> I think we should emphasize the [mathematical] work method more, I think, you can easily go through especially the bachelor programme without learning the work method, just learning mathematics in a mechanical way (…) There is a syllabus, a description of the course, this we must teach, also because other courses can see, we build on this course, and there they learned this and that, in our system it doesn't work if students in different years learn different things in the same course.

Within an undergraduate programme, course units – within and outside pure mathematics – build upon each other, and the result is often that the teacher finds himself obliged to teach a large – perhaps too large – number of more or less disconnected local mathematical organisations. So also in this case we can encounter the phenomenon of *thematic autism*, identified by Barbé et al. (2005) in the context of the Spanish high school, which can be described as a didactic organisation in which several local mathematical organizations are treated, in rapid succession, but with few or none of the mathematically relevant bonds (at the level of theory, sometimes even at the level of technology) between them. Indeed, such didactic organisations can easily be found also in universities, especially in introductory courses such as calculus, linear algebra and stastistics, which are catering to a large number of students, and respond to requirements of several study programmes.

So even if internal didactic transposition is in principle more comprehensive in universities, it has some of the same effects as the constraints which schools have imposed from outside (e.g. the requirements of a national curriculum). However university teachers may react collectively to malfunctions of the mathematical organisations to be taught in a given context, for instance a perceived overload in a course unit, inefficient sequencing of the organisations, etc. In most universities, they also have a larger command over the *assessment apparatus*, that is, the means to evaluate students' command of the target mathematical organisations. In practice, this gives considerable freedom in terms of didactical organisations and it is often possible for the teacher to align new elements in didactical organisations (e.g. teaching techniques, new types of mathematical tasks for students, etc.) with new methods in assessment; this in fact was crucial in the ambitious redesign of real analysis teaching presented by Grønbæk and Winsløw (2007). For this reason, I believe it is both appropriate and useful to consider evaluation techniques, along with their technological and theoretical components, as part and parcel of didactic organisations. This is particularly relevant in university mathematics where evaluation techniques can be described with high precision, even if more advanced techniques (involving more complex qualitative judgments) are more difficult – and therefore more important – to describe and justify, especially to students.

The teaching-research nexus

However we cannot understand the institutional conditions of university didactical organisations without paying attention to the presence in universities of another kind of praxeological organisation, namely those arising from *research tasks* – such families of praxeologies are called *research organisations* (Madsen and Winsløw, 2009, p. 747). While the disciplinary core of these research tasks is of course related to the mathematical organisations taught in university, didactic tasks and research tasks are different in many respects, some of them obvious and general (e.g. reward structures, time perspective etc.). At the same time they are commonly carried out by the same people – university faculty. An impressive literature exists on the general conditions for the interplay between didactic organisations and research organisations (for references, see Madsen and Winsløw, 2009). As in this literature, we shall refer to the highly complex interplay between research and didactic praxeologies as the *teaching-research nexus*. It can be considered at hugely different levels, from the individual faculty member to worldwide trends (e.g., in the correlation between quantitative measures of universities' performance on research and teaching). The general "higher education" literature does not ignore that the teaching-research nexus can be expected to evolve quite differently, depending on what discipline is taught. But the term "general" also means exactly that it does not study this specificity in a systematic way and so the occasional pertinence of its results, for the case of university mathematics teaching and research, remains accidental. As a matter of fact, the comparative case study of Madsen and Winsløw (2009) demonstrated that the teaching-research nexus in this field, and at the level of the individual faculty member, is hugely different from what is found in a relative scientific discipline (physical geography).

As a university discipline, mathematics is indeed quite special. It can even be questioned if it is one single discipline. Besides study programmes on pure mathematics, which are in fact quite similar in contents and methods across the world, we have a huge variety of other study programmes in which more or less wide mathematical organisations are taught – engineering, business, natural science, and teacher education, to name but the most important. The research organisations which are relevant to consider for those other study programmes are not always concerned with pure mathematics, and indeed the teachers often have other fields – related to the character of the study programme – as their research specialty. At the institutional level, this can give rise to overt or latent conflicts of interests, since allocation of resources to research in a given field is very often tied to the volume of teaching carried out by researchers of that field. These issues are certainly of importance when it comes to explaining some of the important constraints that weigh on university mathematics in general, and what we said above about the disciplinary specifity of the teaching-research nexus could be expected to hold also when it comes to teachers of similar (or identical) mathematical organisations, but with different research fields. However, to my knowledge, we have little research evidence, for instance, of how the nexus differs in the case of physicists and mathematicians teaching calculus to physics students, and how this affects the didactic organisations and the research organisations of those teachers.

In my own research, I have mainly considered the classical case of university mathematics teaching delivered by researchers of pure mathematics to students whose field of study is pure mathematics. But at least at the bachelor level, this does not imply that the students are necessarily aiming to become researchers in pure mathematics – and in most contexts, this will be possible only for a small minority. Also, at the undergraduate level, the mathematical organisations taught (e.g. from calculus or linear algebra) will not often be very close, and almost never identical, to the mathematical organisations which the researcher develops in his research. This means that the research organisation of the teacher cannot be used as a clear guideline for the design of the mathematical organisations to be taught, or for the associated (co-constructed) didactic organisations. At the same time, there is a widespread belief among mathematicians (evidenced in the study by Madsen and Winsløw, 2009) that an *implicit nexus* may still exist, at the level of didactic and research practices. What this means is hinted at by the term “work method” used by our informant in the quote given above; one could use other terms like “modality”, “approach” or “style” to indicate the idea that the teacher strives to develop didactic techniques that allow students to work with mathematical organisations in ways that are “similar” to the work of the researcher. Moreover to develop such didactic techniques is sometimes a “similar” experience to research work. Of course, this is clarified by examples, such as the following explanation of a mathematician who is teaching first year calculus (Winsløw and Madsen, 2008, 2384f):

> We give them relatively open tasks. (…) That is, where it is not just an exercise, with a question and a specific point in the text book to refer to and a unique answer. (…) For instance (…) we ask them to compute π by using the formula for a function, like arcus tangens or something, which gives π in some point, and then use the Taylor series at a point where they know it [*the value of the function*]. That I would say is a completely standard exercise. They also learn to estimate the error. But then we can go on and ask, can you with certainty find the first 100 decimals in π, using this method. Or when can they be certain that they found the first 100 decimals. (…) And that we give them as an open task which we don’t even ourselves completely, well of course we could, but we don’t even consider beforehand if we know precisely what the perfect solution would be, because we are not after the perfect solution, we are after *them* thinking about what are the problems involved in this task. [*explains the central difficulty of evaluating the error term*] (…) if they just explore this *problématique* we feel they have achieved a lot. And that we think reminds us of our own research, as for the processes (…) that type of exercises we give a lot. (…) We

> take a standard exercise, and open it up a bit (…) to see how far can you go with this type of task. (…) they need to get the experience that here they have to explore a domain by themselves. (...) That in a way you could call a kind of research. (…) One thing is to prove a theorem with all available mathematical methods. But to ask yourself, can you prove this theorem by just using the following methods, it is in itself a mathematical question. (…) [*similarly*] we ask ourselves, can we solve this *without* using this or that result.

We notice here that a didactic technique, related to the construction of mathematical tasks for students, is central to the teachers' explanation of how his teaching relates to his research. The phrase "take a standard exercise, and open it up a bit" becomes part of en emergent didactic technology where the central aim and justification is to get the students "thinking about what are the problems involved in this task". More "open" tasks are ones for which the student has not been given a standard technique – as in the case of a "standard task" – but where the student has to "explore" the problems it contains, and presumably associate some of those with known techniques. It is also interesting that the teacher needs not to have solved such a task "completely" but could consider the possibility of working along with the students "to see how far you can go on this type of task". The expectation that teachers know "the perfect solution" in advance is indeed common in mathematics teaching, at least at school level; but it is of course a radical difference from the mathematical practice of the researcher. It is clear that one cannot just ignore the expectations – and responsibilities – related to didactic organisations; indeed all informants stress a number of constraints (examinations and syllabus volume in particular) which excludes any excesses in having students "explore a domain by themselves". But the quote indicates at least one direction that university teachers of mathematics pursue, in view of developing didactic techniques to induce students into "research-like modes" of mathematical work.

Another important point in the quote is that the construction of the mathematical task – using the didactic technique just discussed – is considered, by the teacher, "a kind of research". This in principle brings us back to the "task of the teacher" explained by Brousseau as the "inverse" of the task of the researcher, namely to establish a problem situation for the student, corresponding to established knowledge. In fact, for the didactic technique considered above, there is a fine dynamic between the way you "open" a standard exercise, and the difficulties of solving it *using the constrained set of techniques the students know.* And indeed, it is fully acceptable in mathematics research journals to prove a new proof of a known result, especially if the new proof is more elementary or "direct" in the sense of building on a smaller set of technical prerequisites. This kind of mathematical research practice may thus be particularly interesting also as a model for "the teachers' work" to construct problems for students, even if the average school teacher will not often develop entirely new ones. It also shown that the implicit nexus very often comes down to differences in *degree* and *domain*, rather than in *principle*, between the mathematical experience of researchers and school teachers.

The difference in principle between the tasks of research organisations and didactic organisations is, evidently, the criteria for solutions: *novelty and interest* to the international research community, and *efficiency* in teaching a mathematical organisation to students. The implicit nexus gives up both criteria for research products, and focus on processes which can, indeed be similar, as explained.

However, there are also important differences in those processes which have not been mentioned above, and which relate to the ways in which *established* knowledge is used and studied in the researchers' work; these will be discussed in the next section. We stress that the problems of transition to and within university mathematics teaching cannot be considered solely on the base of work modalities, but need to be addressed through a study (and design) of the exact mathematical praxeologies of students, as explained above.

THE DIALECTICS OF MEDIA AND MILIEUS IN UNIVERSITY MATHEMATICS

The term "research" was used above to indicate the work of the mathematician, aiming at constructing mathematical knowledge which is *new* in the sense that it has not been previously published, and which is of interest to the international research community. Certainly this involves thinking about challenging tasks, combining and developing new techniques etc. In the introduction, we stressed the importance, in this activity, of consulting existing works – publications etc. While this part of the mathematicians work may seem, at first, much closer to what students do, there are also crucial differences in common practice.

The term "research" is sometimes used in a different sense, to indicate an activity of problem solving, carried out without consulting literature or other resources not given with the problem. To avoid confusion we shall avoid to use of the term research in this sense here, and focus on establishing a clearer distinction between two fundamentally different kinds of "arenas" for the search for knowledge (for students and researchers alike), called *media* and *milieus*. Both are external resources – material or immaterial – which an individual may use for developing his knowledge, e.g. to solve a task. Chevallard (2007) defines the difference by the presence of an intention (to inform, represent, etc..) in the media, and the absence of such an intention in a milieu (which operates like a kind of "mathematical nature"). Here the intention is understood as relative to the specific knowledge which the person seeks. For instance, in a didactical situation, the answer to the problems at stake are not given in the milieu, it has to found by the students themselves, by adapting their knowledge to the milieu (Brousseau, 1997, 40). By contrast, the teacher – as well as books, Internet pages and fellow students – may act as media, to the extent they provide the solution to the problem.

This definition is admittedly a bit vague but the distinction is rather operational in practice. For example, a student may stumble upon the formula $\tan(x/2) = (1 - \cos x) / \sin x$ in an Internet forum (a media, in which indeed intentions to instruct etc. are present). If the student wants to test if this holds, it may include watching a few cases with his calculator (milieu), try checking with paper and pencil (milieu), access other Internet pages or books (media), etc.

To seek knowledge in a milieu is therefore fundamentally different from seeking it in a media, because in the first case, the person has to adapt his knowledge to the *problem* in order to construct a solution, and in the latter case, he adapts his knowledge to an solution which is offered. As we have already noted in the introduction, both modes of mathematical activity form part and parcel of the work of the mathematician. How does it occur in university teaching of mathematics?

First of all, both are certainly present in any university course. The student will study media, typically a text book, in which a lot of instruction is given. During lectures, the teacher acts similarly, as entirely media, even he places himself in small milieus (with knowledge to be found), for instance to demonstrate a solution method. At the other end of the scale, the teacher can create a challenging milieu for students without any indications of a solution. More typically, the teacher offers very restrained media and relatively small milieus (like "End of chapter exercises", where the key techniques can be found in the preceding chapter).

Two radical propositions: the Moore Method and Undergraduate Research

There are many attempts to improve the quality and scope of milieus presented to students in didactic organisations of university mathematics. Besides offering more challenging and “open” exercises or problems, as mentioned above and in many other didactical designs (e.g. Grønbæk and Winsløw, 2007), we have also ambitious formats to teach mathematical theory as a kind of problem solving. One remarkable example is the “Moore method” (se e.g. Chalice, 1995). A classical Moore course just provides students with key definitions and results, while everything else – in particular proofs – have to be constructed by the students. To Moore, the exclusion of media appears as a principle of primary importance:

> Moore encouraged competition. Do not read, do not collaborate - think, work by yourself, beat the other guy. Often a student who hadn't yet found the proof of Theorem 11 would leave the room while someone else was presenting the proof of it - each student wanted to be able to give Moore his private solution, found without any help. Once, the story goes, a student was passing an empty classroom, and, through the open door, happened to catch sight of a figure drawn on a blackboard. The figure gave him the idea for a proof that had eluded him till then. Instead of being happy, the student became upset and angry, and disqualified himself from presenting the proof. That would have been cheating - he had outside help! (Halmos, 1985, p. 258)

The Moore method promotes, in an original way, certain similarities with the work of the mathematician, namely the experience of establishing formally, by brute force, a result whose validity is ensured. However, even more than the average university course, it omits completely the experience and familiarity with consulting relevant media.

Today, the device of “undergraduate research” is probably more widespread in the USA than the method of Moore. The programs of different universities differ, but generally it is an activity which teachers (as directors of the “research”) and students undertake voluntarily, during some months or a whole year. For students, working individually or in small teams, the format involves choosing a mathematical subject or “problem” (more or less open), exploring it in the literature, and then producing a product (paper) which reflects the work done. The shape of these texts is normally that of an article by researcher, complete with a bibliography that reflects an activity of study, motivated by the selected problem. There are many online journals that publish only such items, either from a single university or national publications such as the Pi Mu Epsilon Journal, which regularly publishes research subjects (e.g. Ahlin & Reiter, 2010). Conferences are held where students present and discuss their work in addition to the experience, which is also important in mathematicians’ research practice.

In many mathematics programs, at least in Denmark, the master thesis has features which resemble the American undergraduate research format. It requires students to present a topic based on journals and research monographs. This could, a priori, appear as a mere task of study and reconstruction of media. In reality, it is not so. Let me share my personal experience. For my Master's thesis, I was asked to produce a presentation of key findings in an article by Connes (1973). The proofs in this text are terse, at least for a student (with a lot of "we easily see that", etc.), so I often found myself in a situation similar to the students of a Moore course, having essentially the result and the task to prove it. And when I had to resort after all to my thesis supervisor, he often ended up doing for me that I had not managed to do: construct a proof directly without worrying about what the text said, except that the result was true. That is to say that what appears to be a study of media often transforms into the search for knowledge in a (terse) milieu. It is almost a general rule that the study of mathematics journal papers involves autonomous construction of solutions, which may indeed lead to new knowledge (such as alternative proofs). In other words, the study of professional "research" media in a didactical organisation almost automatically leads to a dynamic of media and milieus which is, locally, close to the work of the mathematician authors of those media. Of course, the mathematical organisations exposed in research journals are typically beyond the reach of undergraduate students.

An emergent challenge for university teaching of mathematics, in view of the examples we have just considered, is therefore on the side of the study of media, more than on constructing challenging milieus. Except for optional and advanced contexts, it seems that the autonomy of students with respect to accessing and choosing media remains very low, and so in that sense, the students' work is rarely close to mathematical research practice.

An example from an undergraduate mathematics course

To illustrate that the dilemma is one of entire programs, rather than individual courses, allow me to mention a recent experience from teaching a course of the capstone type, destined to ease the transition from university mathematics to high school teaching. The example is related to a topic which has recently become important in the Danish high school mathematics curriculum, but which is barely present in the undergraduate mathematics programme, namely *linear regression* as a tool for empirical model of data. The reason why linear (and other types of) regression has become more common in the secondary curriculum is, besides the uses in extra-mathematical contexts, the ease with which one may compute the "best fit" model from data, using technological devices (handheld calculator or a personal computer). However, the computations are still based on a mathematical method to find the optimal model. In the case of linear regression on a set of data $(x_1, y_1),...,(x_n, y_n)$, the "best" linear model is defined by the line $y = ax + b$ where a and b are chosen such that the square sum error

$$S(a,b) = \sum_{k=1}^{n} (y_k - ax_k - b)^2$$

is minimized. The task for the students of the course was to produce an explanation of the formulae for a and b in terms of the data, accessible to high school students. The formulae do appear in some Danish high school text books, without explanation. Most of the students used hints or proofs from university text books, all based on finding the (unique) critical point of *S* and proving that it is a minimum, using the Hesse matrix and an argument based on the sharp Cauchy-Schwartz inequality. To find a solution which is accessible to high school students requires a more intensive and determined study activity. Depending on the media identified, this has to be combined with autonomous work to simplify and explain the method (an elaborate "completion of the square", similar to the proof of the quadratic formula).

Institutional constraints to the dialectics of media and milieus

In addition to professional mathematics journals, a wide variety of mathematical resources, at all levels and within all subjects, is available on the Internet. An obstacle to make use of them appears in the programs and more specifically, in the institutional contract, which implies that the contents of a university program are defined as pieces of text (typically extracts from text book). The effects of this contract depend heavily on the institutional importance of student evaluations in universities. Unlike what is true for the mathematician researcher, student success is measured frequently and most often by individual examinations (written or oral), where the use of external media amounts to an act of fraud.

We cannot deny the fundamental difference between success criteria related on the one hand to the production of new and interesting knowledge, and on the other hand to individual performance within a few hours. The university institution has an obligation to society to be able to affirm the capacity of each graduate, and the modular organisation implies that this affirmation is ultimately based on the evaluation of student performance within each module. Students, on their side, have a personal interest in ensuring that these assessments are based on transparent and affordable criteria. The common model is well known: a written exam where the challenge in terms of milieus (to what type of task does this exercise belong) is as limited as the study (on which page in the manual can I find the corresponding technique). But for the maintenance of an internal institutional contract between students and institution, the absence of strong links with the work of the mathematician is not a problem.

It is perhaps not so in terms of external contract between the university and society. Especially with regard to the mathematical discipline, it can be maintained that the reason for keeping research mathematicians as teachers must be based on the contribution to teaching of their knowledge and experience as mathematicians. Under the direction of the mathematician-researcher, students should be able to venture far beyond notes or textbooks – even though textbooks dominate the vast majority of current university courses in mathematics.

New efforts and ideas are needed to teach students, from the undergraduate level on, that mathematical knowledge is not only communicated through textbooks, and that research is not limited to acts of individual force before a piece of white paper. To achieve this, we must also establish new institutional contracts – as well as new didactical designs – to facilitate the integration of devices like "undergraduate research" in the regular curriculum, even for prospective teachers (for the latter, perhaps the work of the research didactician is as relevant as that of the mathematician). At any rate, without substantial contributions of knowledge production (research) to university teaching, the university becomes just a pretentious school.

References

Ahlin, A. & Reiter, H. (2010). Problem Department. *Pi Mu Epsilon Journal* 13 (2), 559-560.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Dordrecht: Kluwer.

Brousseau, G. (1999) Research in mathematics education: observation and . . . mathematics, in: I. Schwank (Ed.) *European research in mathematics education*, vol. 1, pp. 35–49. Osnabrück : Forschungsinstitut für Mathematikdidaktik.

Chalice, D. (1995). How to teach a class by the Modified Moore Method. *American Mathematical Monthly* 102 (4), 317-321.

Chevallard, Y. (1990). On mathematics education and culture: critical afterthoughts. *Educational Studies in Mathematics* 21, 3-27.

Chevallard, Y. (1991). *La transposition didactique: du savoir savant au savoir* (second edition; the first edition dates from 1985). Grenoble: La Pensée Sauvage.

Chevallard, Y. and Johsua, A. (1991). Un exemple d'analyse de la transposition didactique: la notion de distance. In Chevallard (1991), pp. 125-198.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques,* 19(2), 221-266.

Chevallard, Y. (2002). Organiser l'étude 1. Écologie & régulation. In Dorier, J. L. et al. (Eds.), *Actes de la 11e école de didactique des mathématiques*, pp. 41-56. Grenoble : La Pensée Sauvage.

Connes, A. (1973). Une classification des facteurs de type III. *Annales scientifiques de l'école normale supérieure,* tome 6, 133-252.

Davies, P. and Hersh, R. (1981). *The mathematical experience.* Boston: Birkhäuser.

Grønbæk, N. and Winsløw, C. (2007). Developing and assessing specific competencies in a first course on real analysis. In F. Hitt, G. Harel, & A. Selden (Eds.), *Research in collegiate mathematics education VI*, pp. 99-138. Providence, RI: American Mathematical Society.

Gyöngyösi, E., Solovej, J. and Winsløw, C. (2011). Using CAS based work to ease the transition from calculus to real analysis. In M Pytlak, E Swoboda & T Rowland (Eds), Proceedings of the seventh congress of the European society for research in mathematics education European Society for Research in Mathematics Education, Rzeszow, s. 2002-2011.

Halmos, P. (1985). *I Want to be a Mathematician. An automathography.* New York : Springer Verlag.

Madsen, L. and Winsløw, C. (2009). Relations between teaching and research in physical geography and mathematics at research intensive universities. *International Journal of Science and Mathematics Education* 7 (2009), 741-763.

Winsløw, C. and Madsen, C. (2008). Interplay between research and teaching from the perspective of mathematicians. In D. Pitta-Pantazi & G. Philippou (Eds), Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education, pp. 2379-2388 (ISBN - 978-9963-671-25-0). Larnaca: University of Cyprus.