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WHAT DOES IT MEAN TO UNDERSTAND SOME MATHEMATICS?

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Mathematical activity involves work with concepts and problems. Understanding mathematical activity in mathematics education is different for the policy maker, the mathematician, the teacher, and the student. This paper deals with the understanding of a concept in mathematics from the standpoint of the student, that is, the learner's standpoint. We make the case for the existence at least five dimensions to this understanding: the skill-algorithm dimension, the property-proof dimension, the use-application (modeling) dimension, the representation-metaphor dimension, and the history-culture dimension. We delineate these dimensions for two concepts: multiplication of fractions, and congruence in geometry.

Curriculum, mathematical understanding, fractions, congruence, mathematical concepts

INTRODUCTION

To understand mathematics *as a whole* would entail a discussion of the roles mathematics plays in everyday personal affairs, in schooling (e.g., as a sorter), in occupations, in other fields such as physics, and in its existence as a discipline studied for its own sake. In contrast, this paper is primarily concerned with what it means to understand *some* mathematics, which generally means to begin with a bit of mathematics and to subject it to detailed analysis, usually from the perspective of the learning of that bit.

In taking on this task, I realize full well that I have entered an area on which a very large number of very talented individuals have trod. Indeed, these pages could be entirely filled by a reference list of works that include some analysis of mathematical understanding. However, my perspective is slightly different from many who have written on the subject. It comes from the standpoint of a curriculum developer, from decades of writing materials for students that attempt to lead them to understand the mathematics they are being asked to learn. This perspective falls somewhere between Freudenthal's *Didactical Phenomenology of Mathematical Structures* (1983) and Hiebert and Carpenter's chapter on learning and teaching with understanding in the *Handbook of Research on Mathematics Learning and Teaching* (1992).

UNDERSTANDING SCHOOL MATHEMATICS

Throughout the 20th century psychologists and mathematics educators wrestled with what it means to "understand" a bit of mathematics. When I was a student in the 1960s, one definition of "learning" was "a change in behavior", which meant that understanding was allied with certain actions. This represented a behaviorist view of learning. Indeed, there is a common saying attributed to Confucius, which in English is sometimes translated as: "I hear and I forget. I see and I remember. I do and I understand." Yet we often hear it said that students can "do" certain mathematics but not understand what they are doing. This roughly parallels the difference between what in psychology are sometimes called *behaviorism* and *cognitivism*. We in education act both as behaviorists and cognitivists. We view "understanding" as something that goes on in the brain without external actions yet we want students to exhibit their understanding by responding to tasks we present before them. Specifically, as behaviorists, we want students to answer questions correctly and sometimes do not care how they got their answers. As cognitivists, we want to know what students are thinking as they work with mathematics and we ask students to show their work.

In the 1970s, Skemp (1976) wrote on this subject with the phrases *instrumental understanding* and *relational understanding* essentially meaning *procedural understanding* and *conceptual understanding*. He wrote, "I now believe that *there are two effectively different subjects being taught under the same name, 'mathematics'.*" Skemp's dichotomy is, I believe, now the most common broad delineation of what is meant by mathematical understanding.

In this paper, I offer a view of understanding that evolved in my mind rather independently of Skemp. You will see that I agree with Skemp that instrumental and relational understanding are different but I do not agree that they are different subjects. I view them as different aspects of understanding the same subject. You will also see that I think there are more than two aspects or types of understanding, as different from each other as Skemp's two types, but all different aspects of understanding the same subject. For reasons I explain later, I call these aspects *dimensions of understanding*.

Since we are speaking of *mathematical* understanding, in discussing the subject, it is necessary to have some idea of the extent of the subject. For our purposes, mathematics is an activity involving objects and the relations among them; these objects may be abstract or abstractions from real objects. The activity consists of *concepts* and *problems or questions*: mathematicians employ and invent concepts to answer questions and problems; mathematicians pose questions and problems to delineate concepts. So a full understanding of mathematics requires an understanding both of concepts and of problems and what it means to invent mathematics.

It is natural for mathematics educators to view the understanding of mathematics from the standpoint of a person's *learning* of mathematics, whether that learning is for use in life, for use on a job, for personal enjoyment, or for a test. But the *full* understanding of mathematics in schools requires more than the learning perspective. It includes the understanding of mathematics also from the standpoints of *educational policy* and the *teaching* of mathematics. For mathematicians, the understanding of mathematics includes an understanding from the standpoint of those who *invent* or *discover* new mathematics.

Educational policy towards mathematics includes the selection of content to be covered in school, who should encounter that content, and when. In the selection of content, a fundamental question concerns what constitutes mathematics. Is statistics mathematics? Is physics mathematics? Is formal logic a part of mathematics? In general, when if ever does applied mathematics cease to be mathematics? Should telling time be a part of the mathematics curriculum? What about reading tables of data or locating one's home town on a map of a country? What about doing a logic puzzle such as a Sudoku puzzle? What about a discussion of lucky numbers and favorite numbers and unlucky numbers? Is computing using a calculator *doing* mathematics or *avoiding* it? Is conjecturing *mathematics* or is it *proto-mathematics*, that is, not the real thing but leading up to the real thing. These questions bring out differences among us in what we think mathematics is, and differences in what we think is *real* or *good* mathematics. A full paper could be devoted to these questions but it is not the focus here.

The understanding of a mathematics concept or problem from the teacher's perspective overlaps the learner's perspective but is not the same. I return to the teacher's perspective at the end of this paper.

The understanding of the invention or discovery of mathematics from the mathematician's perspective has been the subject of many books, of which the classics by Hadamard (1945), Hardy (1940), and Polya (1962) are probably the most well-known, at least in the West. The understanding of mathematical invention would not be complete without consideration also of the inventors, mathematicians themselves, through the many biographies that are available. The recent book by Reuben Hersh and Vera John-Steiner (2010) also falls into this broad category.

On problem solving, Polya's *How To Solve It* (1957) has long been a seminal work. It is significant that the first of his four steps of problem solving is *understanding the problem.* Polya and others since have treated this subject in such detail that I have nothing significant to add. For this reason, in this paper, I concentrate on the understanding of concepts.

There is a vast array of concepts which might be considered, ranging from very general concepts, such as number, function, point, linearity, or structure, to specific concepts such as the mean of a set, the Pythagorean Theorem, the solving of a linear equation, and many concepts in between. For the purposes of this paper, I have picked two dissimilar concepts as examples: *multiplication of fractions*, an arithmetic operation, and *congruence*, a geometric relation.

Finally, as the last demarcation of the topic to be discussed here, let us indicate what does *not* constitute understanding. We say that someone *does not understand* a bit of mathematics when that person acts *blindly* to the prompts in the situation, or acts *incorrectly* to the prompts.

FIRST EXAMPLE: UNDERSTANDING THE MULTIPLICATION OF FRACTIONS

Vocabulary

Mathematics is, among its many other attributes, a language of discourse. It is both a written language and a spoken language, for – particularly in school mathematics – we have words for virtually all the symbols. Familiarity with this language is a precursor to all understanding. You cannot begin to understand multiplication of fractions unless you know what a fraction is and what it looks like, and that multiplication is an operation which, given two numbers, produces a third. The vocabulary of fractions is interesting and not at all trivial.[1] In general, dealing with the written and spoken vocabulary of a concept is an essential part of its understanding that transcends all aspects of that understanding.

Skill-Algorithm Understanding of Multiplication of Fractions

If a random person on the street is asked, "Do you understand the multiplication of fractions?, a typical response might be, "Yes, you multiply the numerators and denominators to get the answer." To the world outside academia, understanding is often equated with getting the right answer. Knowing how to get an answer is the essence of the procedural understanding of the multiplication of fractions or any other concept. Because algorithms are often done (and supposed to be done) automatically, we often view applying a procedure as the opposite of understanding.

However, there is much more to procedural understanding than merely applying an algorithm. With regard to the multiplication of fractions, the procedure seems very simple. If we are confronted with calculation (1),

$$\frac{2}{3} \text{ X } \frac{4}{5}, \qquad (1)$$

we merely multiply numerators and denominators to obtain the product $\frac{8}{15}$.

However, the values of the numerators and denominators can alter what we do. In (1), change the 4 to a 6, as shown in (2),

$$\frac{2}{3} \text{ X } \frac{6}{5}, \qquad (2)$$

and we may divide the 3 and 6 by 3 and thus get $\frac{2}{1}$ X $\frac{2}{5}$, and now multiply numerators and denominators to obtain the product $\frac{4}{5}$. Or we may divide the 3 into the 6 and write 1 and 2. Some people cross out the 3 and 6 in the process. These variants of the algorithm used in (1) are different enough to require days of instruction in a typical classroom.

Change the 4 in calculation (1) to a 3, and we think about it even another way.

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[1] For instance, the word "fraction" itself has many different meanings in English that are all used in classroom discourse: (1) a number between 0 and 1; (2) a number that is not an

$$\frac{2}{3} \text{ X } \frac{3}{5} \qquad (3)$$

We ignore the 3s (some people cross them out) and just write down $\frac{2}{5}$.

Change the $\frac{4}{5}$ to 4 and there is another algorithm.

$$\frac{2}{3} \text{ X } 4 \qquad (4)$$

I multiply the 2 by 4 and write down $\frac{8}{3}$. Some students feel the necessity to replace the 4 with $\frac{4}{1}$ and then they treat the problem as if were of type (1) and multiply numerators and denominators to obtain $\frac{8}{3}$.

Change the $\frac{4}{5}$ to 60 and there is still another algorithm.

$$\frac{2}{3} \text{ X } 60 \qquad (5)$$

Now we may divide 3 into 60 and then multiply the quotient 20 by 2. Or, since the numbers are so simple, you might multiply 2 by 60 and then divide by 3.

Change the $\frac{4}{5}$ to $\frac{3}{2}$ and there is still another algorithm.

$$\frac{2}{3} \text{ X } \frac{3}{2} \qquad (6)$$

We recognize that these numbers are reciprocals and immediately write down the product 1.

Change the $\frac{4}{5}$ to $1\frac{4}{5}$ and there is still another algorithm.

$$\frac{2}{3} \text{ X } 1\frac{4}{5} \qquad (7)$$

The mixed number (what a term that is!) $1\frac{4}{5}$ needs to be changed to the improper fraction (another unfortunate term!) $\frac{9}{5}$ and then the algorithm used in (1) above is applied.

If there are more than two fractions to be multiplied, combinations of these strategies are applied.

Multiplication of fractions is perhaps the simplest algorithm in all of elementary school arithmetic. And yet the skillful arithmetician has at least seven different ways of multiplying two fractions, depending on the numbers involved in the situation.

Skill is sometimes thought of as a lower order form of thinking. Accordingly, procedural understanding is often viewed as not as deep an understanding as conceptual understanding. I would like to argue that the understanding of procedures is not so lower-level at all. Those who are skillful make all sorts of decisions while performing the skills. They possess *skill-algorithm understanding*. You and I exhibit skill-algorithm understanding of the multiplication of fractions when we do it and get the right answer. We exhibit a higher form of this same type of understanding when we know many ways of getting the right answer (that is, we know different algorithms) and choose a particular algorithm because it is more efficient than others. Many people possess this type of understanding because we spend so much time working on the skill.

Property-Proof Understanding of Multiplication of Fractions

For many people, understanding has a completely different meaning than obtaining the correct answer in an efficient manner. You don't *really* understand something unless you can identify the mathematical properties that underlie *why* your way of obtaining the answer worked. "Understanding" is contrasted with "doing". This kind of understanding is often found in courses in mathematics for elementary school teachers.

For example, for the multiplication problem $\frac{2}{3}$ X $\frac{4}{5}$, we wish to justify the rule

For any numbers a, b, c, and d with b ≠ 0 and d ≠ 0, $\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$,

by showing how it follows logically from a set of simpler properties.

$\frac{a}{b} \bullet \frac{c}{d} = (a \bullet \frac{1}{b}) \bullet (c \bullet \frac{1}{d})$	(definition of division)
$= a \bullet (\frac{1}{b} \bullet c) \bullet \frac{1}{d}$	(associative property of multiplication)
$= a \bullet (c \bullet \frac{1}{b}) \bullet \frac{1}{d}$	(commutative property of multiplication)
$= (a \bullet c) \bullet (\frac{1}{b} \bullet \frac{1}{d})$	(associative property of multiplication)
$= (ac) \bullet (\frac{1}{bd})$	(uniqueness of multiplicative inverse; each is the inverse of *bd*)
$= \frac{ac}{bd}$	(definition of division)

This aspect of understanding, *property-proof understanding*, is obviously quite different from skill-algorithm understanding. From this mathematical derivation of the rule, a student learns that multiplication of fractions is not an arbitrary rule but a property that can be deduced from more general properties of multiplication and division. The derivation also shows the importance of thinking of the fraction $\frac{a}{b}$ as being equal to $a \bullet \frac{1}{b}$, the relevance of reciprocals, and so on.

Technically, the proof above that $\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$ only accounts for the algorithm in case (1) mentioned above, a multiplication involving two fractions with no common factors in numerators and denominators. To justify case (2), in which the denominator of one fraction is a factor of the numerator of the other fraction, as in $\frac{2}{3}$ X $\frac{6}{5}$, we must show that $\frac{a}{b} \bullet \frac{bc}{d}$ $= a \bullet \frac{c}{d}$. A full justification of the ways in which we multiply fractions requires proofs also for cases (3) through (7). Thus the complexity of the algorithms is matched by a complexity of the mathematical underpinnings.

I have never seen anyone take the seven different cases of multiplying fractions shown above and subject each of them to careful mathematical analysis – and for good reasons – the seven cases are the tip of an iceberg of infinitely many variations. So such an analysis would always be incomplete, and the payoff in increased proficiency that would come from a large number of proofs is unlikely to reflect the extra time that it takes to handle all of the cases.

Some believe that if you understand these sorts of mathematical derivations and use language correctly, then you will be more skillful. This was the hope of many who worked with the "new math", that era that lasted in the United States roughly from about 1960 to 1974. But it was found that the transfer from understanding properties to understanding skill was not automatic. Skill requires practice, and also requires flexibility to choose among various possible algorithms, a quality that the mathematical derivations do not convey.

It is still the case today that some people will say that a person does not *really* understand arithmetic until he or she knows the mathematical theory behind it. It is also the case that when people contrast *procedural understanding* with *conceptual understanding*, they are often contrasting skill understanding with the understanding that comes from mathematical properties. However, there is more to understanding than these two facets.

Use-Application Understanding of Multiplication of Fractions

A person may know *how* to do something and may know *why* his or her method works, but – particularly to people who use mathematics in their daily lives and on the job – a person does not fully understand the multiplication of fractions unless he or she understands *when* to multiply fractions. I would like to call this the "use-modelling understanding" but the word "model" has too many meanings and could be confusing. So I call it *use-application understanding*.

This type of understanding is different from both skill-algorithm understanding or property-proof understanding. Many people who can multiply fractions do not know of any place where they could use it. Students can even know both algorithms and mathematical properties associated with a concept without knowing its uses; this is a common situation in mathematics classrooms worldwide.

That there are students who can multiply fractions who cannot use them tells us that understanding of applications does not come automatically. We have only begun to realize that uses can be taught and that must be taught before most students realize what to do. Here are some examples of situations that might lead to that multiplication.

(1) A rectangular region on a farm is $\frac{2}{3}$ km by $\frac{4}{5}$ km. What is its area?

(2) If an animal travels at a rate of 2 km in 3 hours (i.e., at $\frac{2}{3}$ km per hour), how many miles will it travel in 48 minutes (i.e., $\frac{4}{5}$ hour)

(3) If two independent events have probabilities $\frac{2}{3}$ and $\frac{4}{5}$, what is the probability both will occur?

(4) If a segment on a sheet of paper is $\frac{4}{5}$ inch long and is put in a copy machine at $\frac{2}{3}$ its original length, what will be its final length?

(5) If something is on sale at 1/3 off (i.e., at $\frac{2}{3}$ its original price) and you get a 20% discount (to $\frac{4}{5}$ of the sale price) for opening a charge account, your cost is what part of the original price?

That these five situations represent five different *types* of applications, not merely five different application contexts, can be seen by examining the units of measure (or lack of units) involved.

(1) A measure is multiplied by a measure: $\frac{2}{3}$ km X $\frac{4}{5}$ km = $\frac{8}{15}$ square kilometer.

(2) A rate, a measure with a derived unit, is multiplied by a measure:
$\frac{2}{3}$ km/hour X $\frac{4}{5}$ hours = $\frac{8}{15}$ km.

(3) Two scalars[2] are multiplied: $\frac{2}{3}$ X $\frac{4}{5}$ = $\frac{8}{15}$.

(4) A scalar is multiplied by a measure: $\frac{2}{3}$ X $\frac{4}{5}$ inch = $\frac{8}{15}$ inch.

(5) A scalar is multiplied by an unknown measure; then the product is multiplied by a scalar: $\frac{2}{3}$ X $\frac{4}{5}$ X original price = $\frac{8}{15}$ X original price.

The conception most people have about applications is that they involve a higher order of thinking than skill. The evidence is overwhelming that this is not the case, that applications involve a *different* kind of thinking, but not necessarily one that is higher or more difficult.

We spend large amounts of time teaching arithmetic paper-and-pencil skills, including weeks on multiplication and division of fractions alone. We spend relatively little time teaching students how to apply these operations with fractions. As a result, performance on

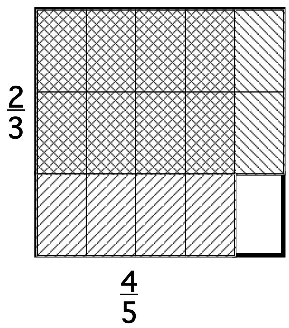
[2] A scalar is a number without a unit. Percents and probabilities are always scalars. Fractions may or may not be unitized.

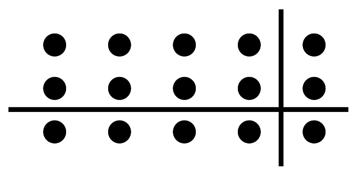
application is lower than performance on skill and we are led to believe that application is more difficult than skill. I would argue that some application is harder than some skill, but some skill is harder than some application.

Representation-Metaphor Understanding of Multiplication of Fractions

Even these three types of understanding do not encompass the entire scope of what it means to understand a mathematical concept. To cognitive psychologists with whom I have discussed this topic, the three types of understanding discussed so far do not convey the real true understanding of mathematics. From psychology we obtain the notion that a person does not really understand mathematics unless he or she can represent the concept in some way. For some, that way must be with concrete objects; for others, a pictoral representation or metaphor will do.

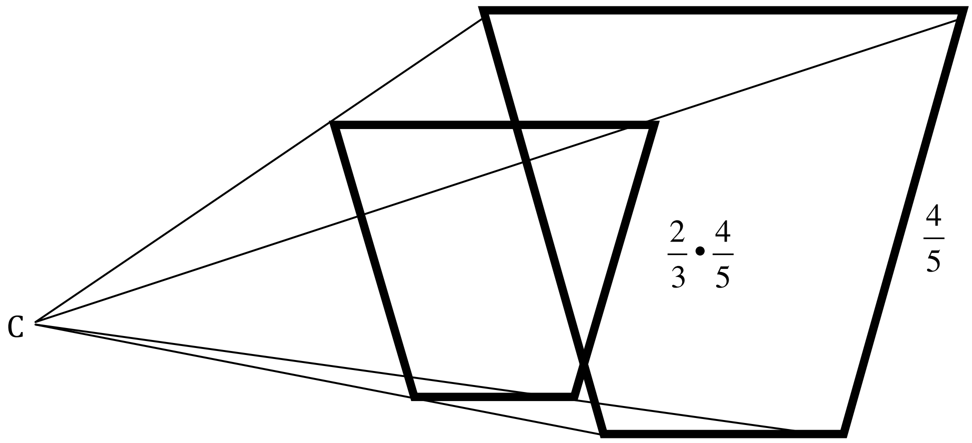
How can we represent $\frac{2}{3}$ X $\frac{4}{5}$? One obvious way, given the application situation (1) just stated above, is with area. In this representation, $\frac{2}{3}$ and $\frac{4}{5}$ are side lengths of a rectangle and the product is the area. We represent $\frac{2}{3}$ by splitting the square horizontally into 3 parts and shading the top two parts. We represent $\frac{4}{5}$ by splitting the same square vertically into 5 parts and shading the 4 parts on the left with a different shading than used for $\frac{2}{3}$. This splits the square into 15 rectangles, 8 of which have both shadings, a picture of $\frac{8}{15}$.





A discrete version of the area representation is with an array of dots. Above, $\frac{2}{3}$ is represented by putting 2 of 3 dots above a horizontal line; $\frac{4}{5}$ is represented by putting 4 of 5 dots to the left of a vertical line; and the product consists of the 8 of 15 dots that are both above the horizontal and to the left of the vertical.

A third representation is quite different and views the $\frac{2}{3}$ and $\frac{4}{5}$ not as equal partners but the $\frac{2}{3}$ as operating on the $\frac{4}{5}$. In this representation, suggested by application situation (4) above, we begin with any geometric figure (below, the larger trapezoid) on which there is a segment of length $\frac{4}{5}$. Then we draw segments from some point (below, point C) to the vertices of the trapezoid. Then, on each segment from C, we pick points $\frac{2}{3}$ of the way to the vertices of the trapezoid. Connecting those segments results in an image trapezoid whose sides have $\frac{2}{3}$ the length of the corresponding sides of the original trapezoid.



Although this representation seems like a lot of work for such a simple arithmetic operation, it has wide applicability.

A person can have a rather deep knowledge of multiplication of fractions even though the person has never seen these representations. Millions of youngsters have acquired skills, learned the mathematical underpinnings, and developed the ability to apply mathematics without touching any concrete materials or seeing any sort of representation of a particular piece of mathematics. Thus concrete or pictoral representations precede acquisition of the other types of understandings, even though some people appeal to psychology to advise us in the same way that we were told about knowing properties; if students are brought carefully to understand (in the representational sense) what they are doing, then they will ultimately be better at skill.

One of the principles advocated by some mathematics educators today is to delay the study of certain aspects of "formal arithmetic" in the elementary school, until these concrete and representational understandings are established. The rationale given for all this is that before these ages students either cannot or do not *really* understand what they are doing; that they need conceptual buildup before they can understand. Little is done to define what is meant by a concept or what is really meant by understanding. Little is done to acknowledge the vast numbers of students who gain other understandings without going through these stages. Little is done to analyze the possible effects of such practices if the theory is wrong. It may be as unfruitful to wait for this kind of understanding to move on as it is to wait for skill understanding to move on.

The understanding that comes from representations can be quite useful. Statisticians know the effects of a good graph. Mathematicians often use graphical and diagrammatic representations. More and more graphing is used in algebra and higher mathematics because graphs convey so much information; and with function graphing technology we do not have to work so hard to obtain this understanding.

It is not difficult to apply these four dimensions of understanding to other arithmetic topics and to the understanding of topics from algebra and analysis. So, for the second example, I have picked a concept that is quite different, congruence and congruent figures in geometry. The four dimensions of understanding will still fit, but a fifth dimension of understanding, quite different from the other four, will appear.

SECOND EXAMPLE: CONGRUENCE IN GEOMETRY

We begin with the vocabulary of congruence. We think of a *figure* as referring to any set of points. Congruent figures are often informally described as figures that have the same size and same shape. The types of sets of points being discussed ranges from discussions that apply only to triangles to discussions that apply to all figures. For this discussion, let us restrict ourselves to congruence in the plane.

In the early grades, the discussion is often quite general. Two figures are *congruent* if one can be placed on top of the other. This approach might be termed *dynamic*, for it speaks of a movement of one figure to another. In contrast, in higher grades in the United States, the approach in the later grades is *static*, and it is common to have individual criteria for one or more specific types of figures:

> Two segments are congruent if and only if they have the same length.
>
> Two angles are congruent if and only if they have the same measure.
>
> Two triangles are congruent if and only if there is a correspondence between their vertices with corresponding sides having the same length and corresponding angles having the same measure.
>
> Two circles are congruent if and only if they have the same radius.

Both static and dynamic approaches can be obtained by employing the language of geometric transformations, that is, functions that map one set of points (the preimage) onto another (the image).[3] Here is a characterization of congruence that is dynamic.

> Two figures are congruent if and only if one can be reflected, rotated, and/or translated to the position of the other.[4]

Transformations also provide static models for dynamic actions by ignoring any intermediary positions of the figure and concentrating only on the points of the preimage figure and the points of the image figure.

> Two figures are congruent if and only if there is a distance-preserving transformation (isometry) that maps one figure onto the other.

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[3] Some people restrict transformations to be 1-1 functions of the plane. This issue need not concern us here.

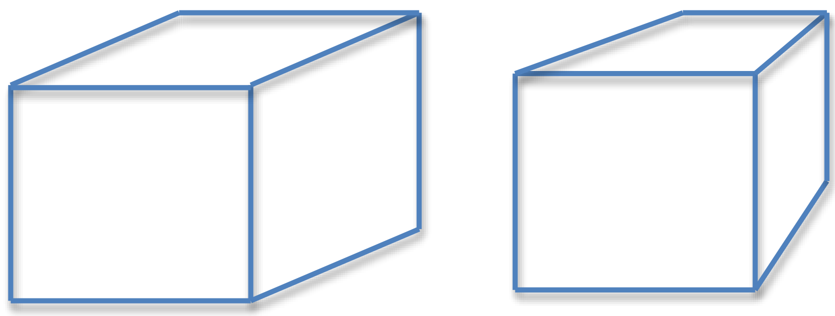
[4] It is understood that composites of these transformations are allowed. In particular, congruent figures may be related by glide reflections.

Skill-Algorithm Understanding of Congruence

The skills of geometry relate to drawing and visualizing and are as old as any other aspects of formal geometry. The first theorem in Euclid's *Elements* is the construction of an equilateral triangle, and the steps in that construction constitute the first algorithm of many in this genre. Specifically with respect to congruence, a child's first skill-algorithm understanding is likely with tracing a figure or using a stencil to create a congruent one. Dealing with the puzzle found in some newspapers and magazines where two complex drawings are shown and a student needs to locate the differences is part of this understanding. An older child may draw or construct reflection, rotation, or translation images of figures, or combine congruent figures to create a tessellation of the plane. Given two congruent figures, a student might be asked to describe in visual terms which transformation maps one onto the other.

This particular aspect of understanding is more difficult in three dimensions than in two. There visualization plays a greater role. What does a figure look like when viewed from a different angle? What are the possible plane sections of the figure?

Artists and designers often understand this aspect of geometry better than any other. Whereas it is common in mathematics textbooks to draw a stylized cube, in which the front and back look congruent to the viewer, an artist will use perspective to draw the cube the way it would look in the real world, with the back of the cube smaller than the front because it is farther away.



cube without perspective   cube with perspective

Skill-algorithm understanding in geometry is similar to that in arithmetic and algebra in that technology today is now commonly used when accuracy and speed are desired. The use of software such as dynamic geometry programs and computer-assisted design (CAD) falls under this dimension of understanding geometry.

Property-Proof Understanding of Congruence

The property-proof dimension of understanding is the aspect of congruence that is given most priority in secondary schools. It includes the side-angle-side and other conditions that cause two triangles to be congruent, and the use of these conditions to deduce properties of lines, angles, triangles, quadrilaterals, and other polygons. This dimension also includes the pro-perties of each of the isometries, the properties of symmetric figures that result from the congruence of a figure with itself, relationships between the various isometries (e.g., the composite of two reflections over intersecting lines is a rotation). We derive the basic formulas for the areas of triangles, special quadrilaterals, and regular polygons from the fact that congruent figures have the same area. These and other derivations are also part of the property-proof understanding of congruence.

Advanced aspects of this understanding involve such things as the statements about congruence need to be assumed to form a complete postulate set for Euclidean geometry, and the notion that in hyperbolic non-Euclidean geometry the only similar figures are congruent ones. The Banach-Tarski paradox in 3-dimensional geometry exemplifies the difficulties that can arise when the idea of congruence is applied to very complicated figures.

Use-Application Understanding of Congruence

Uses of 2-dimensional and 3-dimensional congruence abound. In textbook discussions of geometry, we often see pictures of railroad trestles or buildings using congruent triangles, congruent rectangular window frames, and other examples of congruence of the particular types of figures that are studied. But congruence is far more ubiquitous. Printing makes use of congruent letters and symbols, and copies of pages are congruent. Mass production of machines makes their parts congruent and simplifies repairs or replacements. From eating utensils to chairs, tiles to bricks, congruent objects are everywhere in our lives, so common we neglect to mention them.

Each of the types of isometries have their own applications. Reflections model mirrors. Rotations are intimately connected with turns. Translations can be thought of as slides. Glide reflections connect consecutive footprints a person might leave when walking. Problems of optimal packing bring into play areas and volumes of congruent figures. Since the measurement of angles, lengths, areas, and volumes involve congruent figures, the applications of measurements also fall into this dimension of understanding.

Representation-Metaphor Understanding of Congruence

Geometry is the study of sets of points and visual patterns. In mathematics, these points may take on many forms: idealized locations, as in Euclidean geometry; ordered pairs, 3-tuples, or n-tuples, as in coordinate geometry; data points as in statistics; nodes as in networks; dots as in the pixels on computer screens or some paintings. The variety of these uses of points underlies the geometric representations of so many arithmetic and algebraic concepts.

Sometimes we represent one type of geometry by another, as when we describe locations on a subway system by a diagram, or the Königsberg Bridge Problem by a network as Euler did, or a translation by a vector.

In school mathematics, representations of geometric ideas are often algebraic. We describe lines and some other figures, such as the conic sections, by equations. We place 2-dimensional geometric figures on a coordinate plane. For younger students, we may use a representation on a geoboard.

With respect to congruence, there is a formula for the coordinates of the image (x', y') of a point (x, y) under any isometry. This formula can be described algebraically or represented by a matrix. Equivalently, we can place a figure on the complex plane and describe any isometry by a formula involving complex numbers. These are not simple representations, but they are powerful and they are used in describing movements of figures we see on computer screens.

A TAXONOMY OF MATHEMATICAL UNDERSTANDING

The four dimensions of understanding detailed above have certain common qualities. Each dimension of understanding has supporters for whom that dimension is preeminent, and who believe that the other dimensions do not convey the real essence of the understanding of mathematics. Each dimension has aspects that can be memorized. Skills, names of properties, connections between mathematics and the real world, and even work with representations can be memorized. They also have potential for highest level of creative thinking: the invention of algorithms, the proofs that things work, the discovery of new applications for old mathematics, the development of new representations or metaphors.

The four dimensions of understanding are relatively independent in the sense that they can be, and are often, learned in isolation from each other, and no particular dimension need precede any of the others. Some believe mathematics should begin with real world situations; others with skills; others with concrete materials; and still others believe you should develop the mathematical theory first and let everything else come from that.

It is because of the relative independence of skill-property understanding, property-proof understanding, use-application understanding, and representation-metaphor understanding from each other, that I believe that the understanding of mathematics is a multi–dimensional entity, in the sense that there are independent components that constitute what might be called "real true", "complete", or "full" understanding.

There is, I believe, at least one other dimension to this understanding, one that is not usually found in school mathematics but is a part of the "real true" understanding of mathematics. It is the *history-culture dimension*. How and why did a certain bit of mathematics arise? How has it developed over time? How is it treated in different cultures? Those who study the history of mathematics or cross-cultural mathematics obtain an understanding of mathematical concepts that is different from any of the understandings we have discussed so far. It is a fifth dimension.

History-Culture Understanding

With respect to congruence, the history-culture dimension includes understanding the work of Euclid and its significance, and also the work of Fermat and Descartes to describe figures with coordinates, the Erlanger Program of Felix Klein, and Hilbert's use of the SAS Postulate in his *Foundations of Geometry*. Much of this is found in books on the history of mathematics and is familiar to us. Also, at least in the history of school mathematics, we might include the artwork of Maurits Escher. Escher's tessellations that brought home to many of us the idea that the notion of congruence can be applied to figures such as fish, lizards, and birds, and in doing this Escher expanded the applications of congruence and tessellations.

All of this history is from Europe because I am not familiar with the history of these ideas elsewhere. It is a gap in my history-culture understanding of congruence.

I also do not know much about the history of the *multiplication* of fractions, but most of us do have some cultural-historical understanding of fractions themselves. The first fractions were those for halves, thirds, and fourths. Over 2000 years ago, the Egyptians represented other fractions as sums of unit fractions. Simon Stevin, in his invention of decimals in the late 1500s called them *decimal fractions*, and some places still use that term. The first use of the bar for fractions seems to be among Arab mathematicians well over 1000 years ago but their common use did not appear until the 16^{th} century (Flegg 2002, pp. 74-75; Cajori 1928, p. 310). A sign very much like the slash for fractions first appeared in Mexico in the late 1700s (Cajori 1928, p. 313). Even today the symbols are not the same everywhere. In some places, the fraction a/b is represented by a:b, while in other places the symbol a:b represents a ratio that is mathematically not identical to a fraction. For mathematics education, the cultural history of fractions represents a dimension of understanding of the concept that is considered particularly important to those who believe in a genetic approach to learning, that is, a progression of learning activities that parallels the historical development of the subject. The cultural history of a mathematical concept is also central to ethnomathematics.

What is a "concept"?

As mentioned earlier, the word *concept* is often used as a counterpart to *skill*. In this paper, "multiplication of fractions" is identified as a concept. Why do I consider multiplication of fractions to be a concept and not a skill? The reason is that I believe a *concept* is something that lends itself to be analyzed by these dimensions of understanding. A concept has associated skills, properties, uses, representations and history. A concept is, in the language of this paper, multi-dimensional.

In contrast, an algorithm or a proof or a model or a representation is not by itself a concept. However, by connecting the various dimensions of understanding, one can take any of these and turn it into a concept. For example, the long division algorithm is not a concept, but if one analyzes this algorithm for its mathematical underpinnings, finds uses for it beyond just obtaining answers to division problems, represents it, and discusses its history and variants in different cultures, then long division becomes more than an algorithm; it becomes a concept.

Because concepts involve all of these dimensions of understanding, it is often the case that a view of a concept does not neatly fit into one of the dimensions. For instance, the proving that the algorithms for multiplying fractions mentioned earlier are valid, or justifying the straightedge-and-compass construction of a triangle congruent to a given triangle might be viewed as straddling the skill-algorithm and proof-properties dimensions. But carrying out the algorithms is so different from justifying them that it seems clear that they involve quite different dimensions of understanding.

There is thinking that does not fit this multi-dimensional conception of understanding. A notion that a student needs to be able to perform multiplication before he can understand how to use that multiplication, or that a student must be able to draw reflection images before being able to use them is in effect saying that these two things, doing and using, are in the same dimension of understanding, with using more advanced than doing. Likewise, the notion that a student must see concrete representations of ideas before learning the theory also does not fit this multi-dimensional conceptualization. Ordering ideas or concepts in terms of difficulty is only appropriate if these items are in the same dimension.

This multi-dimensional approach to understanding also conflicts with uni-dimensional Rasch models of evaluation, where items of all different kinds are placed on the same linear scale. It also brings into question statistical reliability tests that are used to throw out items that do not act like other items, such as items that higher-scoring students answer incorrectly in greater numbers than would be expected by their scores. It is quite possible that such items are merely in different dimensions of understanding and some students understand those aspects better than others.

This multi–dimensional approach to understanding also conflicts somewhat with the organization of knowledge found in Bloom's *Taxonomy of Educational Objectives* (Krathwohl, Bloom, and Masia 1964). In the taxonomy, there are six levels - from lowest to highest: knowledge, comprehension - a synonym for understanding, application, analysis, synthesis, and evaluation. At the level of knowledge would be the ability to do multiply fractions; at the level of comprehension are the mathematical underpinnings and representations; at the third level would be applications. I believe there is no such ordering. The experience that led to this belief was an attempt I had made in the middle 1970s to do a first-year algebra course in which the mathematics developed from real-world applications, not from the field properties. Students using these materials often knew how to apply algebra before they had the paper-and-pencil skills to carry out the application. Bloom (a colleague of mine in the same department) wanted to call such application higher-order thinking, at the third level of his taxonomy. I argued with him that something could be changed from higher order to lower order if you worked on it every day. It made me realize that one goal of mathematics instruction is to change higher-order activities into lower-order ones. This is why it is so difficult to teach problem-solving and proof and the invention of algorithms and new models.

On the other hand, a common view of "understanding" is that understanding involves connecting ideas, and so it should come as no surprise that for many decades we have known that applications and concrete representations can increase the learning of skills, that skills and properties taught together are better than either taught alone.

Applications of the Multi-dimensional framework

For over a quarter century, we have been using this multi-dimensional framework to help guide our development of the University of Chicago School Mathematics Project materials for secondary schools. Items on tests at the ends of units are identified with one of the first four dimensions (skills, properties, uses, or representations) – the cultural-historical dimension is not tested. Chapter review questions are also sorted into one of the dimensions. Some of the authors who work on our writing teams are stronger at instructing about the pure mathematics of properties and proof, while others are better at one of the other dimensions of understanding. By keeping a watchful eye on these dimensions, we feel that our materials become richer and reach more students than if we did not implement this broad perspective.

"Understanding" mathematics is also important in the new (2010) Common Core State Standards for Mathematics (CCSSM) in the United States. The words "understand", "understands", "understanding", and "understandings" appear over 250 times in the document. The standards for the multiplication of fractions involve several dimensions of understanding. Here is a part of one of the standards at grade 5:

> *Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.*
>
> *Interpret the product (a/b) × q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a × q ÷ b. For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.)*

Here we interpret the standard for asking for the dimensions of understanding as included in parentheses:

> *Interpret the product (a/b) × q as a parts of a partition of q into b equal parts* (Vocabulary); *equivalently, as the result of a sequence of operations a × q ÷ b* (Property-Proof). *For example, use a visual fraction model to show (2/3) × 4 = 8/3* (Representation-Metaphor)*, and create a story context for this equation* (Use-Application). *Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.* (Skill-Algorithm or Property-Proof).

We can conclude that students are asked to have a broad understanding of fractions.

Here is the standard dealing with congruence at grade 8.

> *Understand congruence using physical models, transparencies, or geometry software.*
>
> *1. Verify experimentally the properties of rotations, reflections, and translations* (Skill-Algorithm or Property-Proof)*:*
>
> *a. Lines are taken to lines, and line segments to line segments of the same length.*
>
> *b. Angles are taken to angles of the same measure.*
>
> *c. Parallel lines are taken to parallel lines.*
>
> *2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations* (Property-Proof); *given two congruent figures, describe a sequence that exhibits the congruence between them* (Skill-Algorithm) .
>
> *3. Describe the effect of dilations, translations, rotations, and reflections on two-*

dimensional figures using coordinates (Representation-Metaphor).

Although this analysis of the standard suggests that the standard does not include the use-application dimension, the mention of physical models at the top of the standard could certainly be interpreted as including applications of each of the three transformations that are mentioned.

Understanding Mathematics from the Teacher's Perspective

The teacher is an applied mathematician whose field of application involves the classroom and the student. Like other applied mathematicians, in order to apply the mathematics, the teacher needs to have a good deal of knowledge about the field itself as well as about mathematics. Thus, the understandings that a teacher needs involve more than the understandings the student needs. The teacher also must take into account students, classrooms, teaching materials, and the necessities of explaining, motivating, and reacting to students.

In 2005, the Mathematical Sciences Research Institute in Berkeley held a conference on the mathematics a teacher needs to know. In advance, the organizers, who included Deborah Ball and Hyman Bass, identified eight tasks they felt required a knowledge of mathematics. I have added to their list and provide the following as four realms of understanding mathematics from the teacher's perspective. I call these realms and not dimensions because operationally they are clearly very much interrelated.

For the first realm, there is a substantial literature. The phrase that identifies it, *pedagogical content knowledge*, was introduced by Shulman (1976).

Pedagogical content knowledge:

designing and preparing for a lesson

analyzing student errors

explaining and representing ideas new to students

responding to questions that learners have about what they are learning.

The second realm deals with applying the understanding of mathematical concepts.

Concept analysis:

engaging students in proof and proving

choosing and comparing different representations for a specific mathematical procedure or concept

choosing and using mathematical definitions

explaining why concepts arose and how they have changed over time

dealing with the wide range of applications of the mathematical ideas being taught.

The third realm deals with the understanding of problems and problem-solving.

Problem analysis:

examining different student solution methods

engaging students in problem solving

discussing alternate ways of approaching problems with and without calculator and computer technology

offering extensions and generalizations of problems.

The fourth realm integrates the other three.

Connections and generalizations to other mathematics

comparing different textbook treatments of a mathematical procedure or topic

extending and generalizing properties and mathematical arguments

explaining how ideas studied in school relate to ideas students may encounter or have encountered in other mathematics study

realizing the implications for student learning of spending too little or too much time on a given topic.

It is clear that teachers need understandings that go far beyond those of students.

SUMMARY

Understanding a piece of mathematics from the standpoint of mathematics education is different for the policy maker, the mathematician, the teacher, and the student. The policy maker needs to understand the importance of that piece to the student at a given time and place. The mathematician needs to understand the potential for the invention of new concepts, the consideration of new and previously unsolved problems, and the discovery of new results. The teacher needs to have a variety of understandings related to pedagogy, concepts, problems, and connections and generalizations of what is done in the classroom.

Mathematical activity consists of *concepts* and *problems or questions*: we employ and invent concepts to answer questions and problems; we pose questions and problems to delineate concepts. The central person in mathematics education is the student, and, primarily because there exist well-known treatises on the understanding of problem solving, this paper has mainly dealt with the understanding of a concept in mathematics from the standpoint of the student, that is, the learner's standpoint.

We view there to be at least five aspects to this understanding. In this view, a person has full understanding of a mathematical concept if he or she can deal effectively with the skills and algorithms associated with the concept, with properties and mathematical justifications (proofs) involving the concept, with uses and applications of the concept, with representations and metaphors for the concept, and with the history of the concept and its treatment in different cultures. Although these aspects are obviously connected when attached to a particular concept, we call them *dimensions of understanding* because each aspect can be mastered relatively independently of the others.

All but the last of these dimensions are important in the teaching and evaluation of mathematics learning. We have delineated these dimensions for two concepts: multiplication of fractions, and congruence in geometry. An actualization of this framework in a full curriculum for grades 6-12 can be found in the materials of the University of Chicago School Mathematics Project.

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