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THE CHALLENGES OF PREPARING A MATHEMATICAL LECTURE FOR THE PUBLIC

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As public curiosity and interest for science grow, mathematicians are invited more often to address a public that is not a classroom audience. Such a public talk should certainly convey "mathematical ideas", but it obviously differs from the classroom lesson. Preparing for such a talk offers therefore new challenges. I give examples from recent public lectures given by prominent mathematicians and by myself that try to tackle these challenges. I also reflect about how these efforts have changed my behavior in the classroom.

Key words: mathematical lecture, public awareness, public interest for mathematics, science awareness, mathematics communication.

INTRODUCTION

Over the last centuries, some learned societies have felt the responsibility to foster public awareness of their field. In Canada, the Royal Canadian Institute in Toronto has held public lectures for more than a century. Their lectures touch upon all sciences and mathematics. With the creation of many mathematics research institutes around the world, the last ten years have seen the launch of a few lecture series for the public specifically on mathematics. Still public lectures on mathematics remain rare and not many mathematicians or mathematics educators have had the opportunity to explore this way of communicating mathematics.

The advantages of mathematical lecture series are numerous. They may present mathematics as a living discipline very much in development and share with the public the intellectual adventure of research. By the range of problems covered, pure to concrete, they can show to students and their parents that scientific activity may translate into career opportunities. They may also show that mathematics and science are useful to attack complex issues, but they do not necessarily provide definitive and clear-cut answers. Public lectures are therefore a fruitful addition to "teaching mathematics" in its broadest sense, one that has an impact on citizens, governments and, of course, scientists.

It may seem that the question of how to prepare such public talks is faced by only a limited community, that of professional mathematicians. But a question, just slightly modified, touches a much wider audience than this limited group: How do we tell people around us what we do? Students in science and math teachers from high schools to universities are asked on a regular basis to explain the purpose of their work. Their successfulness might have an impact as diverse as fostering interest in science among teenagers or stopping the erosion of the number of hours devoted to mathematical activities in curricula.

In this short paper I would like to share my experience of public talks on mathematics. This experience is limited, but yet covers the two distinct aspects of listening to and giving a talk. I have attended to several around the world and I have the privilege to be the organizer with my colleague Christiane Rousseau of the *Grandes Conférences* series held by the Centre de recherches mathématiques based in Montréal. I have also prepared several and given them, some numerous times, to diverse audiences. The point of view expressed here is resolutely personal. It is not based on any scientific assessment of the impact of these talks. I will not even attempt to define a set of goals at which such series should aim. Still I think this personal point of view might be of interest. The public of the *Grandes Conférences* has changed significantly over the six years of the series existence. The audience, constituted at the beginning largely by the community of the departments of mathematics of the four universities in Montreal, has now evolved to a point where at least half comes from outside academia. Slowly this series is reaching its target. And the public is becoming more discerning: After the lectures, the participants will not hesitate to express their view to my co-organizer or to me and their comments almost always coincide with my personal view. It seems that what is a good public talk for the layman is also a good public talk for me. As an organizer, I find this is reassuring.

The next section identifies the difficulties of preparing such talks and proposes ways to tackle them. The next two sections are devoted to examples, first to the overall structure of the talk, second to specific examples of what I think might be efficient sharing of mathematical ideas. Concluding remarks follow.

PITFALLS AND CHALLENGES

There are several challenges facing someone about to design a public lecture. Most come from the practice that every good teacher has honed over the years in her or his classes. Indeed many skills developed for the classroom interfere with what is called for to design a good public talk. Here are some of the pitfalls.

Who are they?

A good teacher is always aware of the inhomogeneity of the students in the classroom: They have different background, talents, interests and dedication to the topic taught. But, compared to the audience of a public talk, these differences are minor. After all, most of the students in a trigonometry class will have mastered the basic rules of algebraic manipulations. Their levels of virtuosity will definitely vary, but it should be easy to remember them that such or such manipulation can be used at a given point. In a public lecture, the audience will be in general *very* inhomogeneous. One could imagine that the background could spread from high school mathematics to that of a researcher in mathematics. Moreover even those with a university degree in mathematics might not have used this knowledge since a long time and have forgotten most of it. Such a spread in backgrounds is indeed common. A lecture of general interest, say about mathematical modeling of biological phenomenon, will pique the curiosity of the widest crowd. Often these public lectures are given within a professional mathematical meeting; it is then aimed, at least in the official discourse, to the participants' accompanying family and the public of the city where the meeting takes place. But a substantial part of the audience will be the mathematicians attending the event. To whom should the lecturer speak?

The large spectrum of backgrounds is not the only difficulty stemming from the inhomogeneity of the audience. The varying interest level is another one, the practice with long scientific explanations still another. People with a keen interest in science, but without a mathematical training, will find it hard to follow long mathematical arguments. Who has the ability to concentrate on an hour-long difficult reasoning?

Once exposed these pitfalls are obvious. Still I feel they need to be stated. It makes the challenge ahead clearer. When about to prepare a public lecture, the speaker should *(i)* decide to whom the conference will be targeted, *(ii)* evaluate honestly the background of this audience and *(iii)* break down the lecture into pieces of various difficulties making sure that most if not all of them are accessible to the chosen public.

A story is being told, a question being answered

For each level of teaching, for each course in a program, a syllabus is imposed or agreed upon. The validity of an education or a diploma is seen in the educators' mind as tied to the goals of this syllabus being reached. In a classroom, the flow of ideas is slow, each argument being substantiated and explained. A logical argument may overlap two classes and the properties of triangles in Euclidean geometry will be covered during several weeks. It is therefore normal that a student at the end of, say, high school Euclidean geometry might fail to see that she or he was introduced to the basic geometric tools and their use and, more importantly, to the foundations of logical reasoning. These overall goals are often obscured by the details of the syllabus.

Here are a few titles that would raise my interests: *Leonardo's architectural works*, *Global warming and Kyoto protocol*, *The international community and the Arab-Israeli conflict*, *Popular music in the 18th and 19th centuries*. They circumscribe a topic well, but they could be titles for a public talk *and* for a one-term course. For the latter, an exhaustive point-of-view will be taken and many detailed aspects will be studied. For the former, a few outstanding points will be chosen for their significance and they will be tied together in a way that informs, makes a point and, if possible, entertains. Often a story will be told or a question will be raised at the beginning so that the lecture has a sense of direction, a goal, a unity. This is surely not the only way to construct an interesting lecture. But there is a lesson for mathematicians here. Mathematics often grows out from concrete questions, it is useful to our society, it is taught and developed by human beings, and its breakthroughs often come after years of conjectures and fruitless efforts. Even though mathematics is indeed a deeply human enterprise, our teaching rarely shows it that way. There is no reason to hide these human aspects, but for the lack of time in our classrooms. Why couldn't a mathematical public lecture tell a breathtaking story? Or solve a question so intriguing that it enthralls the public for one hour? The pitfall here is clear. The planning of a public lecture should not proceed as that of a lecture for the classroom. The challenge is to find a sense of direction that will carry the mathematical ideas and make them inescapable.

The first thing that comes to mind while preparing a public lecture is obviously the mathematical ideas that one would like to communicate. But, from then on, the path is different from that of preparing for the classroom. I believe that the next steps should be *(i)* to seek elements complementing the mathematical concepts that relate to the genesis of the latter and show their importance within and without mathematics and *(ii)* prepare a plan, or more precisely, a *script* or a *storyboard* of the whole talk orchestrating all the parts. The complementing elements can be related to human, historical, sociological and even political aspects; they can reveal hidden connections with other parts of science and unexpected applications; they can raise open questions and, ideally, they should be entertaining. As for the script, should it be a thriller (like the efforts over almost two thousand years to prove the fifth Euclidean axiom from the first four), a drama (like the birth of logic in the 20th century with its disillusionments, the suicides of some main characters, and Gödel's final coup de théâtre), or a historical epic (like the history of estimating the age of the earth, as seen in the next section)?

The lure of rigor

Rigor is one of the outstanding characteristics of mathematics. Among all human intellectual endeavours it is the only one that can claim such an excruciating level of precision and rigor. Many mathematicians feel that an argument has been presented only when it is made rigorous beyond the slightest doubt. A serious pitfall is the temptation of bringing this level of exactness to a public hall. This can manifest itself is several ways, either by giving too much details to make sure all bases are covered or by organizing one's talk in a long chain of arguments where A implies B and then B implies C and so forth. It is unlikely that a general public will have the will or interest to go through such a grueling exercise. But, if proofs should be downplayed, will there be any mathematics left?

The challenge here is to construct a talk that both keeps mathematics as the central focus of the event *and* captures the attention of the audience from beginning to end. I feel that the following guidelines might be of use: *(i)* choose a handful of mathematical ideas, three or four maximum, that can be linked logically and whose union will solve satisfyingly the key question raised by the talk, *(ii)* think of the best way to explain each of these mathematical ideas in a convincing way (graphics, animation, even metaphors could be used), *(iii)* make sure that each explanation is accessible to the targeted audience and that, if one is missed, the following ones can still be understood and *(iv)* organize the whole so that the logic between these ideas stand out.

The audience is likely to have done science courses before and experienced the thrill of the "ah!" moment, this magical moment when a scientific idea suddenly makes sense, is adopted and tied to one's existing scientific knowledge and everyday life experience. If this audience comes to such an event outside the classroom, it is probably looking for living such an experience again. The care put in point *(ii)* above is therefore intimately tied to the success of the event.

THE SCRIPT OF A MATHEMATICAL LECTURE

The remaining part of this article is devoted to examples: In this section, of the overall structure of mathematical public talks and, in the next, of ways to communicate a well-circumscribed mathematical idea.

The public interested in intellectual quests is likely to have been exposed to various conferences, documentaries and movies with a significant scientific content. She or he will have seen Al Gore's *An inconvenient truth* (2006), or NOVA television programs like the adaptation of Brian Green's *The elegant universe*, or *The Proof* about Andrew Wiles's proof of Fermat last theorem. These large-budget popular successes have been written by scientists together with professionals of movie scripting and editing. The single mathematician does not have these means. But she or he might want to learn from them. As argued in the previous section, a good script mingling purely mathematical ideas with elements with a broader scope but still related to them seems a sound way to a successful public talk. I chose to discuss two scripts of public lectures that I think reach this balance.

When the earth was to young for Darwin, a public lecture by Cédric Villani

Cédric Villani is a French mathematician who won one of the four 2010 Fields Medals, the single-most prestigious award given to mathematicians under forty years of age. He was invited to give a public lecture in the series *Grandes Conférences* of the Centre de recherches mathématiques based in Montréal, Canada. His lecture was entitled *Quand la terre était trop jeune pour Darwin* and told the story of the clash between two titans of British science of the 19th century. Charles Darwin, the first titan, in his *On the origin of the species*, argues that species transform, appear and disappear, and that what is observed at one time is the consequence of a long evolution process that still continues. One of his assumptions, largely based on geological knowledge of his period, is that the age of the earth is larger than 10^8 years and allows for such an evolution. Lord Kelvin, the second titan, is the physicist who set out to use the recent developments in mathematics launched by Jean-Baptiste Fourier to compute the age of the earth. His best estimates placed the age of the earth between 20 to 40×10^6 years, far too short for Darwin. Both scientists were aware of the other's prediction and a heated debate ensued, one that the educated society of the time followed with great interest. The topic cannot be better for a public lecture: It offers drama, suspense and humour, a vivid look at science in progress, a surprising resolution and even a sociological commentary. A good script is the last element required to make the story a great mathematical lecture. Villani's was essentially faultless. The main parts of his lecture are the following.

- Presentation of William Thompson, Lord Kelvin: short bio, major works, e.g. involvement in the construction of the first transatlantic cable;
- the estimates of the age of the earth, from the beginning of the written word to the 19th century: James Usher and the old testament, then George-Louis Leclerc de Buffon, his estimate (one of the first based on a scientific argument) and his prudent stance toward the Church;

- Joseph Fourier and his treatise *Théorie analytique de la chaleur* that creates a new chapter of mathematics; a short description of ideas behind the new tools;
- the equation describing heat propagation, Kelvin's hypotheses and computation of the age of the earth using the temperature within the earth but close to the surface;
- Charles Darwin and the geologists estimates for the age of the earth and the debate that follows the two conflicting predictions;
- Ernest Rutherford observes that the recently discovered radioactivity violates one of Kelvin's hypothesis (that the earth does not internally generate heat);
- John Perry suggests that the center of the earth is liquid which increases convection;
- recent results (post-1960) and conclusion.

The two first points above are mostly historical; they both inform and engage the audience. The discussion of Buffon's efforts also shows the power of the Church and how scientists of the 18th century were still constrained by religious and political forces of their time. Villani spends some time explaining Buffon's needle, a problem tied to probability theory that offers a mechanical way to compute the number π. This mathematical curiosity is not important for the following, but the simple argument acts as an easy mathematical warm-up and probably provides a sense that the level of the lecture is really for the participants.

The real science starts with the next two points on Fourier's new theory and Kelvin's computation. Here Villani relies mostly on drawings to remind his public of trigonometric functions and to show how they can be used to approximate other functions. Of course no formal definitions are given, nor theorems or general statements. The relationship between trigonometry and Fourier analysis is explained elementarily, but again its necessity for understanding the next step is minimal. The heat equation requires mathematically advanced concepts usually covered during the last two years of a BSc. This is clearly beyond such a lecture. To capture the meaning of this complex equation, Villani has recourse to everyday experience: A hot object whose temperature is uniform throughout is brought in a cool room. As it cools down, the external layers of the object will have a lower temperature than the inner ones. This is why one stirs a bowl of soup or a cup of tea. The temperature of the various layers of the object being cooled can actually revealed how long it has been loosing heat. This is one of the uses of the heat equation and this is how Kelvin estimated the age of the earth. The description of these ideas takes about ten minutes of Villani's talk. The trigonometric functions, their role in Fourier analysis, the notion of variation of the temperature of a cooling mass like the earth, and the general meaning of the heat equation are the main mathematical points discussed by Villani. To a great extent, they are independent and, if a participant misses one of them, she or he may hope to catch the next.

In the next point, Villani introduces the second main character, Charles Darwin, and discusses the disagreement between Darwin and Kelvin's estimates. It is interesting to see how the personalities of both characters are revealed by their respective reactions to this clash. A few contemporaries' comments are also quoted; they show awe for the scientific debate in progress.

The remaining parts of Villani's talk are devoted to the resolution of the clash. The discovery of radioactivity and Rutherford's observation that it weakens one of Kelvin's crucial hypotheses are presented. They rest on the understanding of Kelvin's hypotheses presented earlier, but are otherwise independent of the previous mathematical parts and surely easier to understand. But Rutherford's observation cannot yet bring together Darwin and Kelvin's windows for the age of the earth. The crucial proposal, that the inner core of the earth behaves for practical purposes as a liquid, was made by John Perry. But it took more than half a century for this idea to be accepted and for models to include it properly. The ideas in science do not progress in a line and their acceptance is often slowed by the prejudices or the inertia of the scientific community.

By telling the history of the ideas and of the human efforts leading to the accepted age of the earth, Villani's script provides a breathtaking drama whose main characters are probably science and mathematics, more than Darwin and Kelvin.

***The archaeology of the tabla,* a lecture by the author**

The *tabla* is a traditional Indian percussion instrument. It is constituted of two drums, a small one called *dayan* played by the dominant (right) hand and a larger one called *bayan*. The small one, also referred to simply as tabla, has a remarkable property: It has a recognizable pitch, one that a musician can easily identify and sing. Drums used in occidental orchestras, kettle drums, timpani and others, produce a sound whose pitch is difficult to identify and even though composers of the classical tradition write in their scores the note the timpani should play, many musicians find it hard to sing the notes played. The remarkable harmonic character of the tabla was first noticed by two Indian physicists, C.V. Raman and S. Kumar (1920). The former is the first Indian (and first non-European- or American-trained scientist) to ever win the Nobel Prize for Physics. Raman and Kumar suggested that its remarkable sound was due to the black patch in the middle of the membrane, known as the *gab*, that is constituted of flour, water, iron powder and other ingredients and added by the artisans. The density and the radius of this patch are very uniform over all known tablas. But, how the Indian artisans could possibly devise this musical instrument and find the proper density and radius? These questions are very similar to those asked by the subfield of archaeology known as *experimental archaeology*. How civilizations responsible for an implement, a monument, a technique could achieve these results? In the case of the tabla, an "experimental archaeological" explanation was proposed recently: Gaudet, Gauthier and Léger's argument (2006) is based on a mathematical analysis of the natural vibration modes of the membrane with the added patch.

This tentative archaeological explanation provides the backdrop of a mathematical lecture that I developed for students registered in Quebec cegeps. (In Quebec, the cegeps are education institutions offering a two-year program to high-school graduates. Their diploma is necessary to register to Quebec universities.) The topic does not have the drama of Villani's script, nor does it offers personalities like those of Darwin and Kelvin. (The role of Raman is limited to the observation that the sound of the tabla is unlike any occidental drum.) Even though it might lack in intrigue, the topic leads to a scientific understanding of what a musical sound is and how mechanical objects produce it. It is therefore intimately related to an everyday life experience of these young adults. Moreover the homogeneity of the targeted audience allowed me to be a little more ambitious for the mathematical concepts presented. As for Villani's talk, here is the breakdown into approximately equal-length parts of the script.

- What is experimental archaeology? Some examples: the mystery of Inca quarrying and stonecutting and Protzen's work (1985) and the mystery of the steel of Damas with Juleff's discovery (1996) and the computational work of Tabor, Molinari and Juleff (2003) on Sri Lanka furnaces;
- Raman and Kumar's observation: The sound of the tabla is remarkable. An example of traditional music played on the tabla;
- Experiment 1: what is a musical sound? what distinguishes it from noise?
- Experiment 2: how physical objects "choose" certain frequencies?
- Experiment 3: can one write any function in terms of a sum of trigonometric functions? The analysis of the result of Experiment 1 using this mathematical tool;
- Experiment 4: what are the natural frequencies of the vibrating string? of the drum? Chladni patterns;
- consonant and dissonant tabla as a function of the density and radius of the patch. An archaeological hypothesis to explain the mystery of the tabla.

Like Villani, I prefer, almost instinctively, to start with a story. To define what experimental archaeology, I give first the example of the incredible work of Inca masonry and Protzen's efforts to reproduce the stonecutting techniques. This example does not bring in any mathematics, but my second example does. The quality of the steel of Damas was observed by Europeans as soon as during the Crusades. Only recently furnaces in Sri Lanka were discovered that might explain the mystery. Indeed numerical computation showed that these furnaces could use wind to bring temperatures high enough to produce such a high quality steel. This introduction is, I believe, entertaining and uses minimal scientific knowledge. Most students at this level will know that airplane wings are designed through numerical computation and will understand that the internal working of a furnace might be understood by similar methods. The observation of Raman and Kumar and the description of the tabla is at the same level.

The scientific part starts next with the understanding of what is so special about a musical sound. The four next parts are described above as "experiments" and I actually ask a member of the audience to join me in front to do each of them. For the first, I call for a musician. I ask the volunteer to listen to musical extracts and to sing the pitch of one note in particular. The extracts chosen are of string instruments, of timpani and of the tabla. Most volunteer will succeed in singing correctly the notes of the string and of the tabla and fail (or even refuse) to sing that of the timpani. This demonstrates the particular nature of the tabla. (I also record the volunteer's voice for ulterior use.) The second volunteer is asked to produce a repetitive pattern with a heavy chain. After some exercises, the volunteer is usually able to produce the three first natural modes of this chain, showing that physical objects vibrate at natural frequencies. The third volunteer, chosen for his or her ability at video games, is asked to match to curves on my laptop. This experiment will be described in more details in the next section. Its goal is to have the audience realize that the graph of a function can be approximated by a sum of well-chosen trigonometric functions. This is again a consequence of Fourier analysis, the same tool Villani described in his lecture. Very often the volunteer, after playing with the small simulation, will guess correctly that this approximation can be as good as desired by adding more terms to the sum and that the choice of the terms is in fact unique. At this point, I stop the experiments and explain that this mathematical tool, known as Fourier analysis, can be used for many types of functions, including the graph drawn by musical waves. I then proceed to make the Fourier analysis of the volunteer's voice recorded earlier. The result is shown to be similar to that produced by the third experiment. But the result contains a surprise: The human voice always sings many frequencies at the same time, as would any string instrument. This means that if the singer wants to sing an A at 440 Hz, she or he does simultaneously sings tones at 880, 1320, 1760 Hz, ..., on top of the desired one. These tones have frequencies that are integer multiples of the base note. This is a well-known fact among musicians, but a surprising one for many young scientists. Each experiment teaches a particular observation about the vibration of a medium and its mathematical description. The relationship between the three experiments is stated, but the participants do not need to understand them all to have a clear idea of the whole argument.

The last experiment aims at obtaining the natural frequencies of the vibrating string and of the square drum. Even though the students know the concept of derivative (the only one really necessary), this is a more difficult exercise. I usually invite one of the professors in the audience. The result shows that the natural frequencies of a string and those of the volunteer's vocal cords are integer multiples of the lowest frequency, the *fundamental one*. For the square drum, the frequencies do not follow an easily recognizable pattern.

Finally the last part reveals the possible discovery by the Indian artisans of the best tabla possible. It follows easily from the previous experiments and a simple drawing giving a quantitative measurement of the consonance of tablas as a function of their density and radius.

Two scripts, common characteristics

Clearly the topic of the last lecture is more modest than Villani's. After all, the clash between Darwin and Kelvin is a determining moment in the history of science as a whole, a statement that can be hardly made of the history of the tabla. The lecture on the tabla has the advantage of calling for students' participation, an opportunity that is offered by the homogeneity of the audience. This participation slows down the flow – this is a good thing – and, each time a volunteer is in front of the hall, the attention is increased dramatically.

Despite their differences, the two scripts have many things in common. They both start by an introduction that wants to be engaging and bring confidence among the audience. The scientific explanations of each part are based on simple elements, but the understanding of one part is not crucial for the following. And more importantly, the whole lecture is constructed as to answer *one* question, *one* mystery. The solution, revealed at the end, gives a feeling of completeness and satisfaction, as does the script of a well-plotted thriller.

EXAMPLES OF "EXPERIENCING MATHEMATICS"

The previous section was devoted to examples of the overall structure of public talks. The present one focuses on parts of public talks where a single mathematical idea is captured with minimal notations and definitions. They deliver to the public the "experience of mathematics" at its best.

Ziegler's checkboard

Günther Ziegler of the Freie Universität Berlin was invited to deliver one of the public talks at the last International Congress of Mathematicians (ICM), held in 2010 in Hyderabad, India. His talk was entitled *Proofs from THE BOOK*. Proofs are at the heart of mathematics and it seems natural to make an effort to describe to the public what they are and what they achieve. In fact, Ziegler wrote a book, with Martin Aigner (2003), about some proofs that are so clear, direct and insightful, that they seem to have been written by God's hand. His book and lecture share the same title. The "public" for the public lectures at the ICM of Hyderabad comprised a large part of high-school students, brought in by the bus load (Casselman, 2010). It is clear that this fact is in Ziegler's mind.

The first few slides of his presentation are devoted to explaining the role of proofs in mathematics and for the mathematical community. This is done by formulating some simple questions that are very difficult to prove and also reproducing some quotes from famous mathematicians about proofs. Some are deep, others simply witty. Then Ziegler introduces THE BOOK, the one containing "definitive" proofs. And then he gives an example. It is his first real mathematical moment and it is likely to have been magical for many of the high-school students. Here it is.

A slide appears, totally blank, but for one sentence and one drawing. The sentence is: "Theorem. The 'chessboard without corners' cannot be covered by dominos." And the drawing is the one on the left in Figure 1. One can guess that Ziegler explains the statement that, even though it is possible to cover a whole chessboard with dominos, each covering two neighboring boxes, once two opposite corners of the chessboard are deleted, this task of filling the new chessboard by dominos becomes impossible. A student who has not encountered this problem before might feel the urge to play with dominos to see why this is impossible.

But a very simple argument proves the Theorem, without any such attempt! It is given in Ziegler's next slide that contains only the right drawing of Figure 1 where the usual pattern of alternating black and white boxes has been added. There, in this single drawing, one is reminded that each domino will always cover precisely one black and one white boxes, wherever they are placed. But one can also see at once that the removal of opposite corners of the chessboard has deleted two boxes of the same color. The number of black boxes is now two less than the number of white ones and it is therefore impossible to accomplish the covering by dominos.

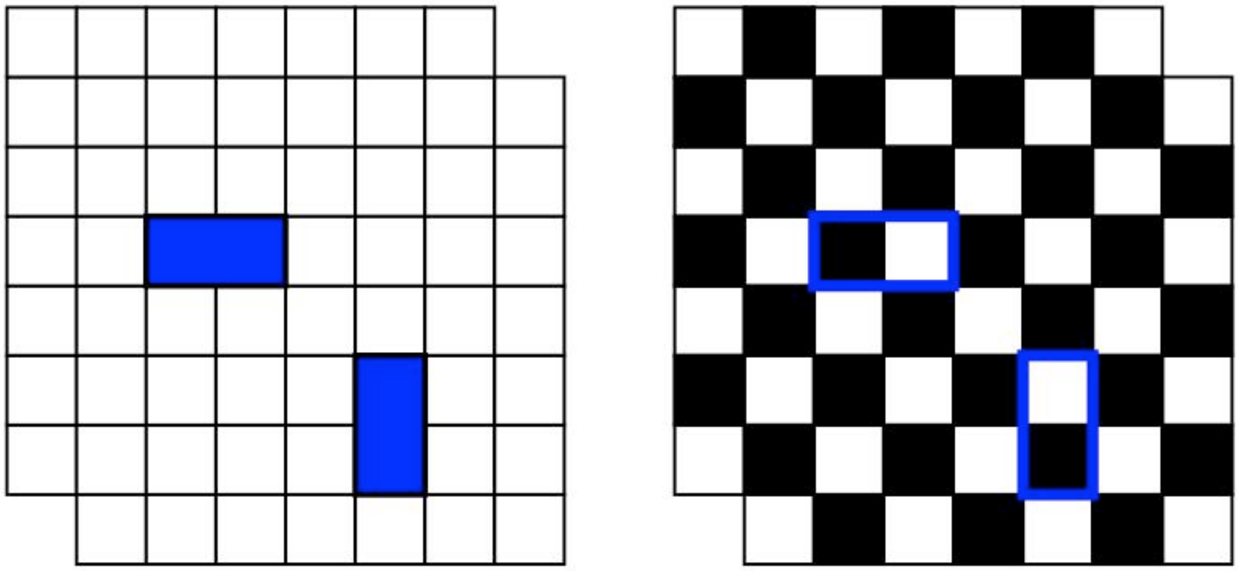

Figure 1. A chessboard without corners cannot be covered by dominos.

The argument, by its simplicity and elegance, can be understood by most people and creates one of these magical mathematical moments in the mind of any scientifically inclined person. It is beautiful and, being placed at the beginning of the talk, captures all listeners in the experience. And it certainly belongs to THE BOOK!

The original algorithm that launched Google

Whoever is old enough to have discovered the beginning of the world wide web will remember how Google imposed itself, over all other search engines, by the efficiency of its algorithm. In fact many of us wondered how a computer algorithm could possibly guess what was important for each of us!

One of my public lectures is aimed at explaining to a large audience how this algorithm works. The lecture supposes little mathematical knowledge and has been given to high-school graduates. It covers first the basic working of classical search engines, like library catalogues or police registries, and second the new problems raised by cataloguing the pages of the world wide web: The lack of uniformity among webpages, the wide range of information quality of webpages, the exponential increase of their number with time, the lack of uniformity of users, and the absence of consensus about what should be searchable.

After, the real mathematical idea behind Google's algorithm is explained. Figure 2 depicts a very small "world wide web". Each of the five circles marked A, B, C, D and E represents a webpage. For example C could be the home page of the ICME-12, A its scientific program, E the home page of EXPO 2012 Yeosu Korea, and so on. The arrows between the circles represent links between the pages. The arrow starting from C and leading to A means that the webpage C (the page of ICME-12) has a link which, once clicked, will bring the browser to webpage A (the scientific programme of the Congress). Of course, the world wide web has more pages. (On July 25 of 2008, Google passed the threshold of 10^{12} indexed pages.)

But the small graph allows us to understand how an order of importance among these five pages can be computed.

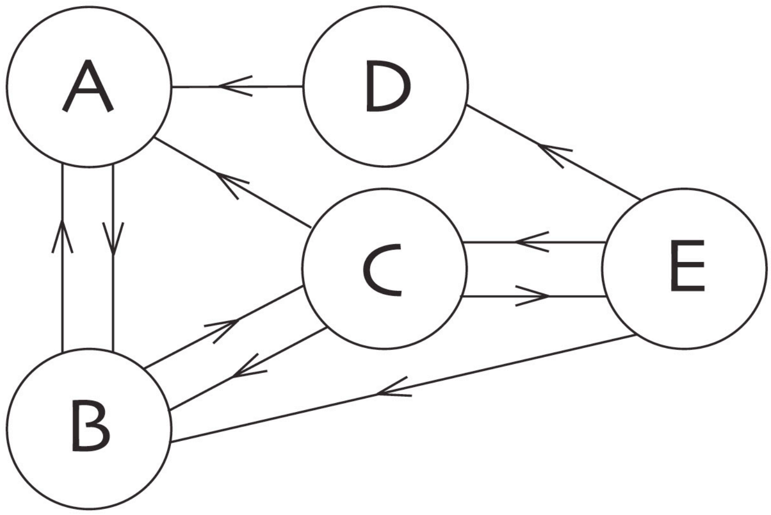

Figure 2. A small graph representing the world wide web.

Suppose that an impartial web surfer is put at time $t = 0$ on page C. After each period of time, say of one second, the impartial surfer is asked to click on one of the arrows that leave the page where it stands. If there are more than one, it will choose with equal probability between all the arrows available. It is hard to determine where this surfer will be after a minute or an hour. Or more precisely, it is hard to determine what is the probability that it will stand, say, at page A. But, before explaining more in details the mathematics, I ask the audience on which page the impartial surfer is most likely to be after many clicks. A vote among the participants leads usually to the right answer. I guess most think as follows: The surfer can get to pages D and E only through a single arrow. It is therefore less likely to be there than on pages A, B and C. These three pages received exactly three arrows from other pages. But B is favoured as each time the surfer visits A (and D and E), the surfer will visit B one step after (and D after two and E after three). Since there are other ways to get to B, it is more likely to be on B than on A. For that reason, it is not surprising that the audience votes mostly for B. And the participants are right. The mathematical treatment developed after explains why, for most graphs like the one in Figure 2, the behaviour of the surfer after a long period (the asymptotic behaviour) exists and is unique and it can be calculated using methods of linear algebra. (The asymptotic behaviour is the eigenvector associated to the largest eigenvalue of the matrix representing the Markov chain of the surfer. See for example Rousseau and Saint-Aubin (2008).) In the case of the example, page B will be visited 16/41 ~ 39% of the time, more than any other.

The mathematics is not too difficult for the aimed audience. What is great with this simple example is that the audience can understand why the original algorithm used by Google is so remarkable. Once the probabilities of the visit by the impartial surfer are known, they are used to order the pages found after a user's request. The first pages presented by Google in reply to the request are those that would have gotten visited the largest number of times by the surfer. Each time a webpage points to another page, the latter gets a "vote" and the algorithm is therefore poling the users by studying the arrows (the links) between the pages of the world wide web. And it is not the creator of a webpage who decides if this page is important. It is the other web users that do by linking to it.

I ask questions to the audience once the algorithm has been digested. For example, how should it be changed to avoid dead ends (a page that does not have any exiting arrow)? How would you do to get a better ranking of your personal webpage? Etc. It is clear that understanding the basics of Google algorithm has given the participants an awareness of the web community and its hierarchy.

The basic theorem of Fourier analysis

Many technologies rely crucially on the mathematical theory founded by Fourier, especially those around signal transmission and analysis. Kelvin's estimate of the age of the earth was based on this theory, so was the analysis of the sound of the tabla. It is tempting to convey, in such talks, the basic ideas behind this theory. Unfortunately it requires fairly advanced mathematical concepts and, in an undergraduate program, it is usually introduced in the last two years. Describing even the simplest idea of this theory is a challenge.

The simplest (and central) result of Fourier analysis is somewhat intimidating when one sees it for the first time: Any periodic function of period L can be written as a series of sine and cosine functions whose periods are all of the form L/n for integer n. There exists a simple formula for the amplitudes of these functions and these amplitudes are uniquely determined by the function. (There are some minor hypotheses of smoothness on the function, but they can be ignored for the present discussion.) Another way to state it is the following: Draw the graph of a function over the interval $[0, 2\pi]$. Then one can always reproduce this graph by adding functions $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, $\sin 3x$, $\cos 3x$, ..., and maybe a constant. The coefficients in front of these functions are determined by Fourier analysis. In other words, such a series always exist and it is unique. Can one convince a non-mathematical audience of this fact or even give it an intuitive idea of what it means?

I chose to do it using a small interactive program. I invite a member of the audience to come play with it. The participant's experimentation will proceed in two steps. For the first, this participant is presented with the left image of Figure 3, and then asked to move the cursor using the mouse so that the two curves meet. (When the cursor moves, the amplitude of the pale curve (green) is changed.) The participant rapidly finds the correct position of the cursor. I then ask her or him whether any other position of the cursor would lead to the same match between the curves. The answer is obviously "no" and the answer comes without hesitation. (Note that the dark graph (in red) is one period of the sine function and the pale graph (in green) is $a \sin x$ where the amplitude a is controlled by the cursor.)

The second step is the real challenge. There are now three cursors (middle image of Figure 3) and the graph to be reproduced by moving these cursors is the dark (red) curve. This graph was chosen so that only three functions are necessary, namely $\sin x$, $\sin 2x$, and sin 8*x*. For Figure 3, the cursors have been put away from their "correct" positions, those that would reproduce correctly the graph. The participant finds these correct positions with some trial and error. I then ask the same question: Are there any other positions of the cursors that would match the two curves? Usually, but not always, the participant claim, somewhat prudently, that no, there can be no others. This is the uniqueness stated above.

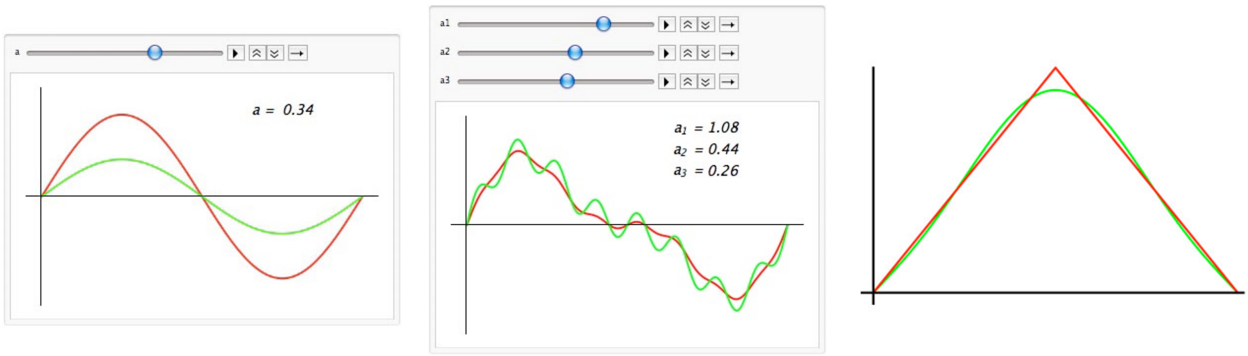

Figure 3: Approximating a graph with trigonometric functions.

I keep the participant with me on the stage for one more question. I show her or him an animation where the triangle graph is approximated by the addition of recursive addition of sine functions (right image of Figure 3). The match is still not perfect at the end of the animation as I programmed it to stop after about ten non-zero terms and an infinity would be necessary. Still the approximation is very good. (On the Figure, only two non-zero terms are used.) I finally ask the participant whether a perfect match seems possible by adding more terms and whether such approximations could be done for any functions I could come with. Most think that it is plausible that, by adding more terms, the match will become as good as desired but, for the second part of the question, most hesitate. Some are bold enough to risk a positive answer (which is Fourier's result), others refuse to take position. I then thank the participant who, often, gets cheered by the rest of the audience. I conclude this experiment by saying that, yes, it is possible to approximate graphs by adding sine and cosine functions and that there is a unique way to do it.

The time spent for the whole experiment is between five and ten minutes. I believe that the small interactive program gives a concrete visual picture of what an approximation of a graph through trigonometric functions is and that such approximation might exist for a large class of functions. These might seem modest achievements, but they are a reasonable success considering the time invested. The main disadvantage of the experiment is that, contrarily to other graphical arguments, this one does not give any idea how to prove the statement. But that might be beyond the goal of a public talk.

CONCLUDING REMARKS

As an organizer of a public lecture series, I seek leaders in their mathematical research field that are superb communicators. This is a very small group. However, when I visit Quebecer cegeps, the institutions whose level is just before the one where I teach, I consider that I am giving indeed a public talk. At least, I follow the templates put forward in the previous sections. In that extended sense, every mathematics educator could give a "public talk", that is, a talk to a group of interested students in a free setting. In fact, every mathematics educator should be given the opportunity to live such an experience. For once, no syllabus to abide by, no rush to make sure the material is covered, no need to manage students' stress, no exams to grade. Only the pleasure of communicating something one likes. This is a pregnant experience.

Since my first public talks years ago, I have come to distinguish between elements of my teaching experience that are applicable to public talks and those that are actually in conflict with them. Over the years, this analysis has hopefully improved the way I conceive a public talk. But this *is not* the most important point.

Public talks are only a small part of my professional life. I earn my life doing research and teaching in a university. And the conception of public talks had a definite influence on my teaching, particularly at the undergraduate level. Graphical arguments or even metaphors are useful to capture the essence of a mathematical concept. Most, I included, would agree that this is not how a mathematical proof is done. But, when there are a useful step toward a mathematical formalization, I am definitely willing to use them.

Unfortunately, historical evolution, cultural impact, and real life applications are never part of syllabi of courses like differential geometry, group theory, real analysis or even linear algebra. Now, in all my classes, I try to keep fifteen, twenty minutes every two or three weeks to tell a story, as I would do in a public talk. These stories are intimately related to the course material, but definitely not in the syllabus. They try to paint mathematics as tightly woven into the fabric of human history. For example, in my linear algebra class, I tell the story of cuneiform tablet VAR 8389 of the Vorderasiatisches Museum in Berlin. This tablet, dating from 2000 BC, asks a simple question about the areas of two fields, given the rent the owner must pay on them (Grcar (2011)). The problem leads to a system of two equations in two variables. This tablet is a clear indication that recipes to solve such systems were taught four thousand years ago. Clearly this linear algebra course has a fairly long history. Another example tied this time to my differential geometry class has, as main actor, the great architect Frank Gehry. One of his most famous buildings, the Guggenheim Museum in Bilbao, Spain, is made of bent and curved surfaces. These surfaces are covered with small tiles of metal. Gehry realized that, depending on the shape of the curved surfaces, the bending could be done manually or needed to be molded in a workshop. The first ones did not cost more than if the surface had been a plane, the second needed to be limited as they increased the budget. The difference between the two surfaces is that the Gaussian curvature, a central concept in the course, is zero in the first case and non-zero in the second.

Hence the important point *is* that public talks have had a direct influence on my classroom teaching. This is why I wish every teacher should get an opportunity to give a public talk.

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