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# UNDERSTANDING THE NATURE OF THE GEOMETRIC WORK THROUGH ITS DEVELOPMENT AND ITS TRANSFORMATIONS

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*The question of the teaching and learning of geometry has been profoundly renewed by the appearance of Dynamic Geometry Software (DGS). These new artefacts and tools have modified the nature of geometry by changing the methods of construction and validation. They also have profoundly altered the cognitive nature of student work, giving new meaning to visualisation and experimentation. In our presentation, we show how the study of some geneses (figural, instrumental and discursive) could clarify the transformation of geometric knowledge in school context. The argumentation is supported on the framework of Geometrical paradigms and Geometric Work Space that articulates two basic views on a geometer's work: cognitive and epistemological. A possible extension to all the mathematical work is explored in the concluding section.*

*Geometric work, figural and instrumental genesis, geometrical paradigm*

## INTRODUCTION

The influence of tools, especially drawing tools, on Geometry development at school has recently improved greatly due to the appearance of DGS. As Straesser (2001) suggested, we need to think more about the nature of Geometry embedded in tools, and the traditional opposition between practical and theoretical aspects of geometry has to be rethought. It's well known that we can approach Geometry through two main routes:

1. A concrete approach which tends to reduce geometry to a set of spatial and practical knowledge based on material world.
2. An abstract approach oriented towards well organized discursive reasoning and logical thinking.

With the social cynicism of the Bourgeoisie in the mid-nineteenth century, the first approach was for a long time reserved to children coming from the lower class and the second was introduced to train the elite who needed to think and manage society.

Today, in France, with the "college unique", this conflict between both approaches stays more hidden in Mathematics Education but such discussions have reappeared with the social expectation supported by the OECD (Organisation for Economic Co-operation and Development) and its "bras armé" PISA (Programme for International Student Assessment) with the opposition between "Mathematical literacy" and "Advanced Mathematics".

In the present paper, I will leave aside sociological and ideological aspects and focus on what could be a didactical approach, keeping in mind a possible scientific approach to a more practical geometry referring to approximation and measure, in the sense Klein used when he suggested a kind of approximated Pascal's theorem on conics:

> Let six points be roughly located on a conic: if we draw the lines roughly joining points and they intersect at a, b and c, then these points are roughly aligned. (Klein, 1903)

The present presentation will be supported by a first example showing what kind of contradiction exists in French Education where no specific work on approximation exists during compulsory school. This contradiction appears as a source of confusion and misunderstandings between teachers and students. We were lead to introduce some theoretical perspectives aiming at understanding and solving this trouble. In the following, our theoretical framework for studies in Geometry will be introduced and used to launch some perspectives.

## COMPLEXITY OF THE GEOMETRIC WORK

Mathematical domains are constituted by the aggregation and organization of knowledge. As Brousseau (2002) emphasized, this organization will not inevitably be the same as the actual implementation in a classroom. A mathematical domain is the object of various interpretations when it is transformed to be taught. These interpretations will also depend on school institutions. The case of geometry is especially complex at the end of compulsory school, as we will show in the following.
The following problem was given for the French examination at Grade 9 in 1991 and was used in a study we conducted (Houdement and Kuzniak, 2003a).

Construct a square ABCD with side 5 cm.

1) Compute BD.

2) Draw the point I on [BD] such that BI=2,8 cm, and then the point J on [BC] such that JC=3 cm.
Is the line (IJ) parallel to the line (DC)?

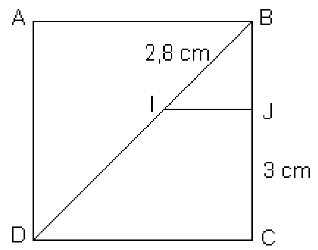


Table 1. A geometric problem

The intuitive evidence (the lines are parallel) contradicts the conclusion expected from a reasoning based on properties (the lines are not parallel). Students are faced with a variety of tasks referring to different, somewhat contradictory conceptions and the whole forms a fuzzy landscape:

1. In the first question, a real drawing is requested. Students need to use some drawing and measure tools to build the square and control and validate the construction.
2. Students then have to compute a length BD using the Pythagorean theorem and not measure it with drawing tools. But which is the nature of the numbers students have to use

to give the result: An exact value with square roots, or an approximate one with decimal numbers which is well adapted to using constructions and that allows students to check the result on the drawing?

3. In the third question – are the lines parallel? – students work again with constructions and have to place two points (I and J) by measuring lengths. Moreover, giving the value 2.8 can suggest that the length is known up to one digit and could encourage students to use approximated numbers rounded to one digit. In that case  is equal to 1.4 and both ratios are equal, which implies the parallelism by the Thales' Theorem related to similarity. If students keep exact values and know that  is irrational, the same Theorem implies that the lines are not parallel.

**With Grade 9 students**

The problem was given in a Grade 9 class (22 students), one week after a lecture on exact value with square roots and its relationships to length measurement. After they had spent 30 minutes working on the problem, half of the students answered that the lines were parallel and the other half answered that they were not. On the teacher's request, they used the problem of approximated values to explain the differences among them. At the teacher's invitation, they started again to think about their solutions. At the end, 12 concluded the lines were not parallel, 8 that they were and 2 hesitated.

Indeed, after studying their solutions and their comments on the problem, we can conclude that students' difficulties did not generally relate to a lack of knowledge on geometric properties, but to their interpretations of the results. They had trouble with the conclusions to be drawn from Thales' Theorem. Even after discussion, students expressed their perplexity about the result and its fluctuation. One student said "I don't know if they are parallel for when I round off, the ratios are equal and so the lines are parallel, but they are not parallel when I take the exact values". For students, one answer is not more adequate than another. This gives birth to a geometric conception where some properties could be sometimes true or false. How to make students overcome the contradiction? A first possibility is to force the entrance in the didactical contract expected by the class's teacher, who explained us that at this moment in Grade 9, it must be clear that "a figure is not a proof".

Working on approximation and thinking about the nature of geometry taught during compulsory school open a second way we will explore with geometrical paradigms in the following.

## GEOMETRICAL PARADIGMS AND THREE ELEMENTARY GEOMETRIES

The previous example and numerous others of the same kind show that a single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend both on the evolution of mathematics and school institutions. At the same time, we saw that students are strongly disturbed by this diversity of approaches. Geometrical

paradigms were introduced into the field of didactics of geometry to take into account the diversity of points of view (Kuzniak and Houdement, 1999, 2003b).
The idea of geometrical paradigms was inspired by the notion of paradigm introduced by Kuhn (1962, 1966) in his work on the structure of the scientific revolutions. In a global view, one paradigm consists off all the beliefs, techniques and values shared by a scientific group. It indicates the correct way for putting and starting the resolution of a problem. Within the restricted frame of the teaching and learning of geometry, our study is limited to elementary geometry, and the notion of paradigm is used to pinpoint the relationships between geometry and belief or mathematical theories.
With the notion of paradigms, Kuhn has enlarged the idea of a theory to include the members of a community who share a common theory.

> A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm (Kuhn, 1966, p. 180).

When people share the same paradigm, they can communicate very easily and in an unambiguous way. By contrast, when they stay in different paradigms, misunderstandings are frequent and can lead, in certain cases, to a total lack of comprehension. For instance, the use and meaning of figures in geometry depend on the paradigm. Sometimes it's forbidden to use the drawing to prove a property by measuring and only heuristic uses of figures are allowed.
To bring out geometrical paradigms, we used three viewpoints: epistemological, historical and didactical. That led us to consider the three following paradigms described below.

**Geometry I: Natural Geometry**

Natural Geometry has the real and sensible world as a source of validation. In this Geometry, an assertion is supported using arguments based upon experiment and deduction. Little distinction is made between model and reality and all arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments. Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Fold or cutting the drawing to obtain visual proofs is also allowed. The development of this geometry was historically motivated by practical problems.
The perspective of Geometry I is of a technological nature.

**Geometry II: Natural Axiomatic Geometry**

Geometry II, whose archetype is classic Euclidean Geometry, is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms could be incomplete and partial: The axiomatic process is a work in progress with modelling as its perspective. In this geometry, objects such as figures exist only by their definition even if this definition is often based on some characteristics of real and existing objects.
Both Geometries have a close link to the real world even if it is in different ways.

**Geometry III: Formal Axiomatic Geometry**

To these two approaches, it is necessary to add a third Geometry (Formal Axiomatic Geometry) which is little present in compulsory schooling but which is the implicit reference of teachers' trainers when they have studied mathematics in university, which is very influenced by this formal and logical approach.

In Geometry III, the system of axioms itself, disconnected from reality, is central. The system of axioms is complete and unconcerned with any possible applications in the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call in" to perceptive evidence such as convexity or betweenness. Moreover, axioms are organized in families which structure geometrical properties: affine, euclidean, projective, etc.

These three approaches (and this is one original aspect of our viewpoint) are not ranked: Their perspective are different and so the nature and the handling of problems change from one to the next. More than the name, what is important here is the idea of three different approaches of geometry: Geometry I, II and III.

**Back to the example**

If we look again at our example, students – and teachers – are not explicitly aware of the existence of two geometrical approaches to the problem, each coherent and possible. And students generally think within the paradigm which seems them natural and close to perception and instrumentation – Geometry I. But in this geometry, measurement is approximated and known only over an interval. Parallelism of lines depends on the degree of approximation. Teachers insist on a logical approach – Geometry II – which leads the students to conclude blindly that the lines are not parallel, against what they see.

It could be interesting to follow Klein's ideas and introduce a kind of "approximated" theorems, more specifically here an "approximated" Thales' Theorem: If the ratios are "approximately" equal then the lines are "almost" parallel. In that case, it would be possible to reconcile what is seen on the drawing and what is deduced based on properties.

Developing thinking on approximation in Geometry can be supported by DGS which favour a geometric work into Geometry I but with a better control of the degree of approximation. It is the case, for instance, with the CABRI version we used during the session with students. In this version, an "oracle" is available which can confirm or not the validity of a property seen on the drawing. Here, the parallelism of both lines was confirmed by the "oracle" according to the approach with approximation of the problem.

Many problems allow discussion of the validity of a theorem or property in relationship to numerical fields. For instance, CABRI oracle asserts that (EF) and (BC) are parallels in a triangle ABC when E and F are respectively defined as the middles of [AB] and [AC]. But, if E is defined as the middle of [AB], when we drag a point F on [AC] it is possible that CABRI oracle never concludes that (EF) and (BC) are parallel for any position of F. These variations in the conclusion need an explanation and provoke a discussion among students

which can be enriched by the different perspectives on Geometry introduced by geometrical paradigms.
To discuss the question in-depth and think about new routes in they teaching and learning of geometry, we will introduce some details about the notion of Geometric Work Space (Kuzniak 2010 & 2011).

## THE NOTION OF GEOMETRIC WORK SPACE WITHIN THE FRAMEWORK OF DIDACTICS OF GEOMETRY.

At school, Geometry is not a disembodied set of properties and objects reduced to signs manipulated by formal systems: It is at first and mainly a human activity. Considering mathematics as a social activity that depends on the human brain leads to understanding how a community of people and individuals use geometrical paradigms in everyday practice of the discipline. When specialists are trying to solve geometric problems, they go back and forth between the paradigms and they use figures in various ways, sometimes as a source of knowledge and, at least for a while, as a source of validation of some properties. However, they always know the exact status of their hypotheses and the confidence they can give to each one of these conclusions.
When students do the same task, we are not sure about their ability to use knowledge and techniques related to Geometry. That requires an observation of geometric practices set up in a school frame, and, more generally, in professional and everyday contexts, if we aim to know common uses of mathematics tools. The whole work will be summarized under the notion of Geometric Work Space (GWS), a place organized to enable the work of people solving geometric problems. Individuals can be experts (the mathematician) or students or senior students in mathematics. Problems are not a part of the Work Space but they justify and motivate it.
Architects define Work Spaces as places built to ensure the best practice of a specific work (Lautier, 1999). To conceive a Work Space, Lautier suggests thinking of it according to three main issues: a material device, an organization left at the designers' responsibility and finally a representation which takes into account the way the users integrate this space. We do not intend to take up this structure oriented to the productive work without any modifications, but it seems to us necessary to keep in mind these various dimensions, some more material and the others intellectual.

### The epistemological level

To define the Geometric Work Space, we introduced three characteristic components of the geometrical activity into its purely mathematical dimension. These three interacting components are the following ones:

- A real and local space as material support with a set of concrete and tangible objects.
- A set of artefacts such as drawing instruments or software.
- A theoretical frame of reference based on definitions and properties.

These components are not simply juxtaposed but must be organized with a precise goal depending on the mathematical domain in its epistemological dimension. This justifies the name of epistemological plane given to this first level. In our theoretical frame, the notion of paradigms brings together the components of this epistemological plane. The components are interpreted through the reference paradigm and in return, through their different functions, the components specify each paradigm. When a community can agree on one paradigm, they can then formulate problems and organize their solutions by favouring tools or thought styles described in what we name the reference GWS. To know this GWS, it will be necessary to bring these styles out by describing the geometrical work with rhetoric rules of discourse, treatment and presentation.

## The cognitive level

We introduced a second level, centred on the cognitive articulation of the GWS components, to understand how groups, and also particular individuals, use and appropriate the geometrical knowledge in their practice of the domain. From Duval (2005), we adapted the idea of three cognitive processes involved in geometrical activity.

- A visualization process connected to the representation of space and material support;
- A construction process determined by instruments (rules, compass, etc.) and geometrical configurations;
- A discursive process which conveys argumentation and proofs.

From Gonseth (1945-1952), we retained the idea of conceiving geometry as the synthesis between different modes of knowledge: intuition, experiment and deduction (Houdement and Kuzniak, 1999). The real space will be connected to visualization by intuition, artefacts to construction by experiment and the reference model to the notion of proof by deduction. This can be summarized in the following diagram:

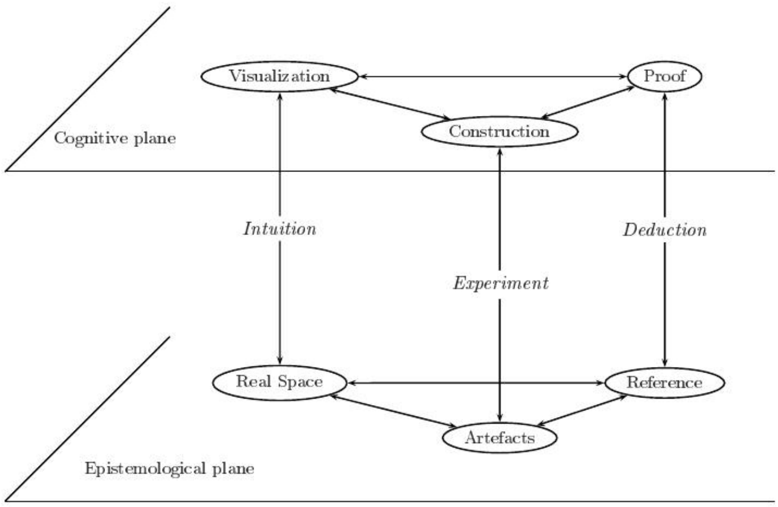


Figure 1. The Geometric Work Space

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# BUILDING A GEOMETRIC WORK SPACE: A TRANSFORMATION PROCESS

## On the meaning of genesis

In the following, we will consider the formation of GWS by teachers and students. Our approach intends to better understand the creation and development of all components and levels existing in the diagram above. The geometric work will be considered as a process involving creation, development and transformation. The whole process will be studied through the notion of genesis, used in a general meaning which is not only focused on origin but also on development and transformation of interactions. The transformation process takes place and, finally, forms a structured space, the Geometric Work Space.

### Various GWS levels

In a particular school institution, the resolution of geometric tasks implies that one specific GWS has been developed and well organized to allow students to enter into the problem solving process. This GWS has been named appropriate and the appropriate GWS needs to meet two conditions: it enables the user to solve the problem within the right geometrical paradigm, and it is well built, in the sense in which its various components are organized in a valid way. The designers play here a role similar to architects conceiving a working place for prospective users. When the problem is put to an actual individual (young student, student or teacher), the problem will be treated in what we have named a personal GWS. The geometric work at school can be described thanks to three GWS levels: Geometry intended by the institution is described in the reference GWS, which must be fitted out in an appropriate GWS, enabling an actual implementation in a classroom where every student works within his or her personal GWS.

## Various geneses of the Geometric Work Space

As we have seen, geometrical work is framed through the progressive implementation of various GWS. Each GWS, and specifically the personal GWS, requires a general genesis which will lean on particular geneses connecting the components and cognitive processes essential to the functioning of the whole Geometric Working Space. The GWS epistemological plane needs to be structured and organized through a process oriented by geometrical paradigms and mathematical considerations. This process has been named "epistemological genesis". In the same way, the cognitive plane needs a cognitive genesis when it is used by a generic or particular individual. Specific attention is due for some cognitive processes such as visualization, construction and discursive reasoning.

Both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process supposes some transformations that it is possible to pinpoint through three fundamental geneses strictly related to our first diagram:

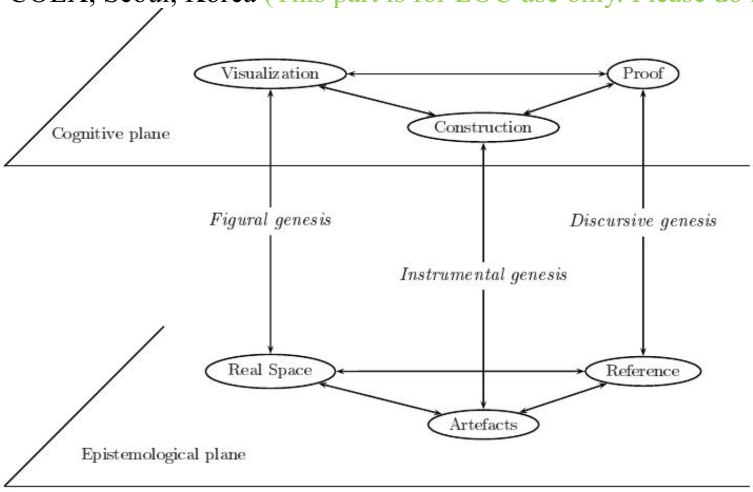



Figure 2. Geneses into the geometrical Work Space

An instrumental genesis which transforms artefacts in tools within the construction process.

A figural and semiotic genesis which provides the tangible objects their status of operating mathematical objects.

A discursive genesis of proof which gives a meaning to properties used within mathematical reasoning.

We will examine how it comes into geometrical work by clarifying each genesis involved into the process.

## On figural genesis

The visualization question came back recently to the foreground of concerns in mathematics and didactics after a long period of ostracism and exclusion for suspicion.
In geometry, figures are the visual supports favoured by geometrical work. This led us, in a slightly restrictive way, to introduce a figural genesis within the GWS framework to describe the semiotic process associated with visual thinking and involved in geometry. This process has been especially studied by Duval(2005) and Richard (2004). Duval has given some perspectives to describe the transition from a drawing seen as a tangible object to the figure conceived as a generic and abstract object. For instance, he spoke of a biologist viewpoint when it is enough to recognize and classify geometric objects such as triangle or Thales' configurations often drawn in a prototypical way. He also introduced the idea of dimensional deconstruction to explain the visual work required on a figure to guide the perceptive process. In that case, a figure need to be seen as a 2D-object (a square as an area), a set of 1D-objects (sides) or 0D-objects (vertices). Conversely, Richard insists on the coming down1 process from the abstract and general object to a particular drawing.

## On instrumental genesis

A viewpoint on traditional drawing and measuring instruments depends on geometrical paradigms. These instruments are usually used for verifying or illustrating some properties of the studied objects. The appearance of computers has completely renewed the question of the role of instruments in mathematics by facilitating their use and offering the possibility of dynamic proofs. This aspect is related to the question of proof mentioned in the preceding paragraph, but the ability to drag elements adds a procedural dimension which further increases the strength of proof in contrast to static perception engaged in paper and pencil environments. But the ability with the use of artefacts is not easy to reach by the students. At the same time, teachers need to develop specific knowledge for implementing software in a classroom. Based on Rabardel's works on ergonomic, Artigue (2002) stressed the necessity of an instrumental genesis with two main phases that we can insert in our frame. The coming up transition, from the artefacts to the construction of geometric configurations, is called instrumentation and gives information on how users manipulate and master the drawing tools. The coming down process, from the configuration to the adequate choice and the correct use of one instrument, related to geometric construction procedures, is called instrumentalisation. In this second process, geometric knowledge are engaged and developed.

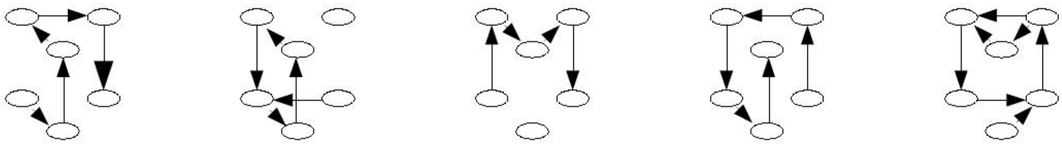
**On discursive genesis of reasoning**

The geometrization process, which combines geometric shapes and mathematical concepts, is central to mathematical understanding. We saw the strength of images or experiments in developing or reinforcing certainty in the validity of an announced result. However, how can we make sure that students understand the logic of proof when they do not express their argumentation in words, but instead base it on visual reconstructions that can create illusions? A discursive explanation with words is necessary to argue and to convince others.
The nature and importance of written formulations differ from one paradigm to another. In most axiomatic approaches, it is possible to say that mathematical objects exist only in and by their definition. This is obviously not the case in the empiricist approach, where mathematical objects are formed from a direct access to more or less prototypical concrete objects. Such as for artefacts, we can pinpoint two geneses. The coming up sense relates to a proof process based on initial properties (Balacheff, 1982) and the other sense could be seen such as a defining process (Ouvrier-Buffet, 2007) and relates to institutionalization for Coutat and Richard (2011).

## TOWARDS A COHERENT GEOMETRIC WORK AT THE END OF COMPULSORY SCHOOL

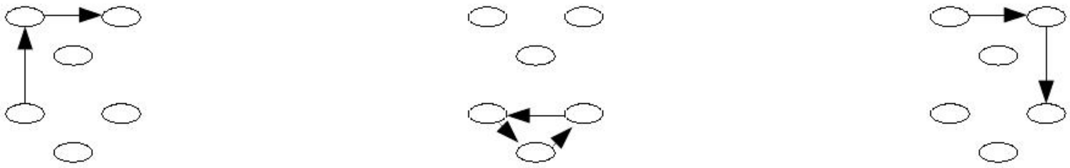
In practice, the personal and adequate Geometric Work Spaces do not lean on a sole paradigm but rather on articulated and connected paradigms. When users know how to recognize and connect them effectively, we will speak about a game between paradigms otherwise we speak of a shift to emphasize that the user does not control these relationships. Using the theoretical framework introduced above, we will insist here on some contradictory ways we encountered in French geometry education and highlight what could be a coherent approach using both geometric paradigms. For that, we draw some conclusions from a work of Lebot (2011) who has studied different teaching organization for introducing the notion of angles at Grades 6 to 8. Using the GWS diagram, it is possible to describe possible routes students may take when they use software or drawing tools to construct figures and solve problems. Lebot has observed interesting differences visible on the following diagrams and we will discuss some among them.

(1) (2) (3) (4)



(5) (6) (7) (8) (9)

Besides this complex routes, he had observed some very incomplete schemas like this



(10) (11) (12)

*A coherent GI Work Space*

Generally, a geometric task begins with a construction performed using either traditional drawing tools or digital geometric software. Each time, the construction is adjusted and controlled by the gesture and vision.

In this approach to geometry, the trail into the GWS diagram is like the one of Diagram 5 and done in a first sense (Instrumental - Figural and then Discursive) which characterizes an empirical view on geometric concepts.

A coherent way to work theoretically in Geometry I would be to use “approximated” theorems in the sense we introduced (sec. 2) where the numerical domain is based on decimal numbers rather than real numbers. Theoretical discourse must justify what we see and not contradict it. This approach has been developed by Hjelmsev (1939) among others.

*A coherent GII Work Space*

In the Geometry II conception, the focus is first on the discourse that structures the figure and controls its construction. This time, the route is trailed (Diagram 8) in an opposite sense (Discursive - Figural – Instrumental) and figure rests on its definition: All properties could be derived from the definition without surprises.

In the traditional teaching and learning of geometry, students are frequently asked to start geometric problems with the construction of real objects. This leads them to work in the sense (I-F-D) of the Diagram 5. But for the teacher, the actual construction of an object is not really important. The discursive approach is preferred and expected, as in the Diagram 8 covered in sense (D-F-I) : what I know is stronger than what I see and measure. In this pedagogical approach, elements coming from Geometry I support students' intuition for working in Geometry II, leading the formation of a (GII /gI)1 Work Space. But at the same time, students may believe that they work in a (GI /gII) Work Space where the objective is to think about real objects using some properties coming from Geometry II (Thales and Pythagorean Theorems) to avoid direct measurement on the drawing.The geometric work made by students could be incomplete as in Diagram 6 where students stay in an experimental approach without any discursive conclusion. They have paid attention to the construction task which requires time and care, but this work is neglected in the proof process expected by the teacher, where figures play only a heuristic supporting role. That can lead to another form of incomplete work but this time favoured by teachers as in Diagram 4 where there exists only interaction between proof and figure.
The inverse circulation of the geometric work in Geometry I and Geometry II can lead to a break in the geometric work that forms, when only one approach is explicitly privileged. We support the idea that both geometric paradigms must be included in geometry learning to develop a coherent (G|GII) Work Space where both paradigms have the same importance. Only when this condition is met can an approximation could have both a numerical and geometrical meaning, and can a work space be created suitable for introducing "almost parallel" lines in relationship to decimal numbers and where "strictly parallel" relate to real numbers. That would help resolve problems of mathematical coherency such as those experienced by students who asserted that they did not know if the lines were parallels because "the lines (IJ) and (DC) are parallel if we round off, but they are not if we take the exact value".

## BEYOND THE GEOMETRIC WORK SPACE

How the notion of GWS could be extended beyond the Geometry? First, we can take into account the context within the geometric work is developed. This context can be of social and cultural nature as in a socio-epistemological approach (Cantoral and ali, 2006). Another extension could deal with the cognitive dimension in the teaching and learning processes as Arzarello (2008) did by introducing the "Space of Action, Production and Communication" viewed as metaphorical space where the student's cognitive processes mature through a variety of social interactions. Within these frameworks, it is clear that the notion of GWS can operate and pinpoint on what, at the end, is the goal of an educational approach in mathematics: to make an adequate mathematical work. This assertion leads us to another kind of generalization related to what is mathematical work. In this direction, we have started some investigations with researchers interested in Calculus, Probability or Algebra. A third symposium on this topic will be held at Montreal in 2012 and some elements on this approach are given in Kuzniak (2011). The generalization supposes an epistemological study in-depth of the specific mathematical domain and of its relationships to other domains. Indeed, each domain relates to a particular class of problems and the crucial question is to find an equivalent to the role that space has in geometry. Variations and functions for calculus, chance and data for probability and statistics, can play the same role as space and figures in geometry. If, it seems that the two planes, epistemological and cognitive, keep the same importance as in geometric work, figural genesis and visualisation should be changed and reinterpreted through semiotic and representation processes in relationship to the mathematical domain concerned.

**Epistemological and cognitive levels: a double articulation**

From our study of geometric work, we retain the idea of articulating two levels in the Spaces of the Mathematical Work, one of epistemological nature related to mathematics content and the other to cognitive aspects. The mathematical work is the result of a gradual and evolutionary process that will allow an inner joint at each of the epistemological and cognitive levels and the articulation of these two levels.

If artefacts and theoretical frame of reference are two basic components of any epistemological level associated with a particular mathematical field, the component related to space and geometric configurations must be changed. In the case of GWS, this component is closely related to the visible and tangible form of geometric objects. To extend it to other areas of mathematics and in accordance with a conception of mathematics based on semiotic representations, we think relevant to introduce the notion of sign or *representamen*, according Peirce. The *representamen* or sign is a thing that represents something else: its object. The interest of the idea of *representamen* is that it can be connected to objects in more or less abstract forms: icons, indexes and symbols.

A sign refers to its object in a iconic way when it evokes its object in a similarity connection. It refers to its object in an indexed way when it is really affected by this object, for example, a knock at the door is the sign of a visit or a symptom of a disease is the index of this disease. A sign is a symbol when referring to its objects under rules. Rules may have been made a priori, by agreement, or have been made a posteriori, by attendance and cultural habit. Mathematics is generally concerned with the symbolic level but in learning and in an empirical conception of mathematics, some signs may have an iconic or indexed meaning. This is, for example, the case with figures in geometry or dice in probabilities. On the other hand, these signs will be organized in registers of semiotic representation to enable a mathematical work.

The cognitive level is influenced by the importance we grant to signs and representation in the formation of mathematical work. Indeed, processes at work in the cognitive level have to allow the development of some components by using specific modes of knowledge. Notions of proof and construction could be kept without any change but visualization process requires a fundamental adaptation to find its place in the Space of Mathematical

Work.

We propose the following diagram to describe our first approach on Spaces of Mathematical Work.

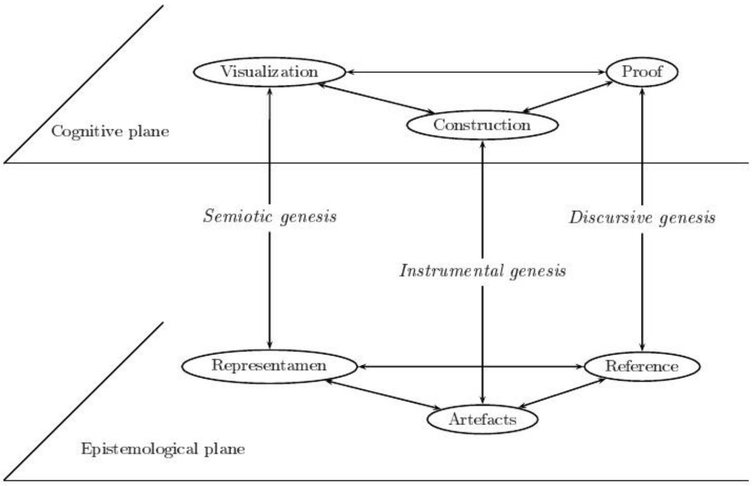



Figure 3: Space of Mathematical Work and its Geneses

We introduced the idea of a semiotic genesis associated to representations of mathematical objects. We keep the term of visualization which should be associated strictly to the operating schemes and to intuition.

**Tools for a study of Spaces of Mathematical Work**

The study of the mathematical workspace seems to rely on various tools that were used in the study of GWS, these tools should allow including:

- Description and differentiation of various GWS such reference, appropriate or personal GWS.
- Description of the specific epistemological and didactic issues in connection to mathematics domains and paradigms.
- Studies of the development of the various geneses at work in formation of mathematical work.

Some of these tools were evoked in the text as the instrumental, semiotic, ergonomic or praxeologic approach. This list is not restrictive and have to be completed with precise studies on specific subjects.

**CONCLUSION**

The presentation of the notion of Geometric Work Spaces and of its extension to all the mathematical work was based on different geneses. Our objective was to define and also to clarify how these notions could impact on didactics of mathematics and relate to the teaching and learning of mathematics.

A first genesis, internal to our works in didactics, is concerned by the passage of the initial notion of GWS limited to geometry to the more general notion of Spaces for Mathematical Work. Around the central notion of GWS, the notions of geometric work and spaces of work are linked and organised. The first one relates to the impact of specific contents on the evolution and transformation of the work and the second defines the structure which allows this particular work. Thanks to the Spaces of Work, it is possible to introduce three views on mathematical work concerned by the material device with its components, the organization of this space by its designers and the mental representations of its users.

Space of Mathematical Work is structured in two levels by epistemological and cognitive geneses to better understand the interactions that exist within the teaching and learning of mathematics.

> The epistemological genesis structures the mathematical organization of MWS by giving it a meaning which geometrical paradigms help to define in the case of geometry.
>
> The cognitive genesis structures the working space when a generic or particular individual use it. Even there, the example of geometry attract attention on some cognitive processes as visualization, construction and the discursive reasoning already important within the framework of GWS.

How then to articulate both levels in order to realize the expected mathematical work? It seems possible to introduce three fundamental geneses strictly related to the theoretical frame developed:

> An instrumental genesis makes artefacts operative in the construction process.
>
> A semiotic genesis based on semiotic registers of representation which provides the status of mathematical objects to the tangible objects.
>
> A discursive genesis of proof which gives a meaning to properties used within mathematical reasoning.

To answer to theses questions, we need further studies on specific contents similar at that we made in Geometry. In this direction, some investigations are done by researchers interested in Calculus, Probability or Algebra and will be presented during the third ETM symposium at Montreal in October 2012.

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