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THE ROLES OF MATHEMATICS COMPETITION IN SINGAPORE MATHEMATICS EDUCATION

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The roles of mathematics competitions in Singapore mathematics education have expanded beyond helping the country in identifying and supporting of mathematical talents. In this note, test items from the past years mathematics competition were examined. It was proposed that mathematics competitions can potentially play three important roles in Singapore mathematics education: to (1) stretch students to explore mathematics beyond the usual school curriculum; (2) set direction in higher order thinking skills could be infused into the usual classroom teaching; and (3) preserve the "elementary mathematics" within the constantly evolving national mathematics curriculum. This note further presents some episodes of students' responses to some competition questions from previous years. It was found that some students developed incomplete or incorrect mathematical reasoning but gave the correct answers to these questions, which is contradictory to the intention of setters of the questions. Readers are cautioned to the existence of a mismatch between the intentions of these competition questions and the actual format and structure of the competitions.

Keywords: Mathematics competition, problem solving, higher order thinking

INTRODUCTION

Mathematics competitions have always played the important role of identifying and supporting mathematical talents for a country, building the national pool of mathematically gifted students and preparing the best among them for the most prestigious International Mathematical Olympiad.

More Singapore schools are beginning to organize mathematics competitions for students beyond their schools. Anecdotal evidence shows that many schools are not only sending their highest achievers to participate in these competitions, but are also exposing other students to these competitions. This provides opportunity for more students to participate in such competitive events. Furthermore, some Singapore schools are incorporating their students' performance in the various mathematics competitions as one of their key performance indicators.

According to the data from the Singapore Mathematical Society (Figure 1), there has been a steady increase in the total number of participants in the Singapore Mathematical Olympiad (SMO), the most prestigious national level mathematics competition organized annually by the Singapore Mathematical Society.

Table 1: Number of participants in the SMO for the past three years

Year	Junior	Senior	Open	Total
2011	5048	3592	2028	10666
2010	5082	3452	1992	10526
2009	4933	3354	1877	10 164

The influence of the mathematical competitions has expanded beyond the few mathematically gifted students to include all the students who participate in these mathematics competitions. Students' broadened exposure to mathematics through the competitions could also influence teachers' pedagogical practice and mode of lesson delivery in the usual mathematics classrooms. For example, the traditional drill-and-practice approach of teaching mathematics might not satisfy the curiosity of these students.

The objective of this note is to identify the key roles (and contributions) that mathematics competitions can potentially play in Singapore mathematics education, through an examination of the test items from the mathematics competitions in the previous years.

ORIGIN AND PURPOSE OF MATHEMATICAL COMPETITIONS

It is not easy to trace the origin of mathematics competitions for school students. As early as 1885, it was reported that a primary school mathematics competition was held in Romania (Berinde, 2004). Within a few decades after the First International Congress of Mathematicians in 1897, several countries began organizing their own mathematical olympiads (Kenderov, 2006).

The most prestigious International Mathematical Olympiad (IMO) was first held in 1959 in Romania. Ever since, this competition has taken place annually (except 1980). It first emerged as a small-scale mathematics contest with participants from several European countries and gradually transformed to a large-scale contest with more than 80 participating countries. According to Reiman (2005), "wherever mathematics education has reached a moderate level, sooner or later the country has turned up at the IMO" (p 1).

To better identify talents and to prepare these talents more proficiently to participate in the IMO, mathematical societies were set up in various countries, and mathematical competitions at the national level were organized. Such has obviously impacted these countries in several ways (Reiman, 2005):

> ... has enriched the publishing activity in several countries. Math-clubs have been formed on a large scale and periodicals have started ... if the educators regard the competitions not as ultimate aims, but as ways to introduce and endear pupils to mathematics, then their pedagogical benefit is undeniable. (p1).

Literature abounds with research on the use of mathematics competitions in development of the gifted students (for example, see Bicknell, 2008; Campbell & Walberg, 2010; Kalman, 2002). Expanding beyond the role of mathematics competition on gifted students, some researchers propose the use of competition as an enriching problem solving experience for the general student population (Grugnetti & Jaquet, 2005). There are organizers of mathematics competitions which have transcended the objective of talent identification; they are making the competitions more inviting to the general student population by designing their competition around the content of common school mathematics (for example, Swetz, 1983).

MATHEMATICS COMPETITIONS IN SINGAPORE

In Singapore, many mathematics competitions have been organized by various schools and professional bodies for students. In this note, our discussion focuses primarily on the Singapore Mathematical Olympiad (SMO), the most prestigious national mathematics competition organized by the Singapore Mathematical Society, for two main reasons: (1) this competition attracts the largest number of participants annually in Singapore, and (2) information about the SMO is available from the publications of the Singapore Mathematical Society.

The aims of the SMO as stated in its official document are: (1) to test the ingenuity and mathematical problem-solving ability of participants; and (2) to discover and encourage mathematical talents in Singapore schools (Chua, Hang, Tay & Teo, 2007). It is clear that objective (1) is aligned to the Singapore mathematics curriculum, which has placed mathematical problem solving as the heart of the curriculum. Objective (2) can be seen as consisting of two sub-objectives: to *discover* and *encourage* mathematical talents. To *discover* mathematical talents matches the role of identifying mathematically gifted students for Singapore, which acts in the country’s national interest. To *encourage* talents identifies the competition’s more active role in developing talents in mathematics. Milestones along the development of mathematical talents include the processes of building up students’ interest, confidence and competency in the subject.

The other mathematics competitions in Singapore generally do not serve to discover national talents, but aim to provide opportunities for more students for wider exposure to mathematics beyond the usual school curriculum by participating in some forms of mathematics competitions. Other than the highest achieving students, many students can also potentially benefit from mathematics competition.

The publication of the SMO solution books for the yearly competitions since 1993 has enabled the SMO competition to reach out to a wider student population and to “enhance students’ problem-solving skills they learn in school” (Tay, To, Toh & Wang, 2011) as one of the threefold aims of the publications. The other mathematics competitions also have their channel of disseminating information to the general student population either formally or informally. The test items from the mathematics competition of the previous years could be used for preparing students for future competitions or could even be adapted for usual classroom lessons.

In this note, an examination of the SMO test items and the published solutions of the questions was carried out. The published solutions of the questions provide one with a clear picture of the underlying intention of the competition questions and the overarching objective of the entire competition. It is proposed that mathematics competitions could play three key roles in the Singapore mathematics education scene: to (1) stretch students' learning beyond the school mathematics curriculum; (2) engage students in developing higher order thinking skills in mathematics; and (3) preserve "elementary mathematics" content within the mathematics curriculum.

LEARNING BEYOND THE MATHEMATICS CURRICULUM

Students' involvement in the various mathematics competitions has resulted in the emergence of preparatory lessons organized by the various Singapore schools to prepare their students for the mathematics competitions. Many schools either involve their own mathematics teachers or engage external coaches (usually university mathematics lecturers or retired mathematics teachers) in conducting these preparatory lessons. Anecdotal evidence has shown that, in some schools, these preparatory lessons have evolved into *mathematics enrichment lessons* in which more students are allowed to attend for mathematics enrichment, even if not to prepare for the competitions.

There is generally no fixed "syllabus" for these competitions. Questions from the past year papers of the competitions show that the topics that are tested in these competitions generally include those which are beyond the usual school curriculum. Challenging students to extend their learning beyond the limits of the school curriculum is an important step to mathematical problem solving. This could arouse students' interest and confidence in mathematics (Toh, 2011). However, mathematics teachers have often neglected this aspect in their usual mathematics classrooms because of the heavy mathematics content and limited curriculum time. The mathematics enrichment lessons could serve to fill this gap by providing students the opportunity to learn mathematics content which is beyond the curriculum.

Extension of learning beyond the curriculum should not be solely aimed at purely building up additional "cognitive resources" (Schoenfeld, 1985) for the students. It should be seen as a motivation for the students to appreciate mathematics and a means to introduce them into advanced mathematical thinking which is not emphasized in the school mathematics.

An examination of the past years' test items from the SMO competition reveals that there are significantly many items which require the contestants to be proficient in mathematics beyond the usual school mathematics curriculum. These items can be classified under three broad categories discussed below.

Category 1: Problems solvable without sophisticated content knowledge

These problems are contextualized in topics which are beyond the usual mathematics curriculum (e.g. Number Theory, Combinatorics). Students need to be familiar with the basic terminologies peculiar to these topics before they are able to solve these problems. However, solving these questions does not require sophisticated content knowledge; students can generally solve these problems based on their learning experience in the usual mathematics classrooms, once they have understood the basic terminologies related to these topics. Figure 1 shows an example of two such questions in the Junior Category of SMO (for students aged 13 and 14). These two questions are broadly classified under Number Theory.

Q1. The last two digits of 9^{2004} is

(A) 21 (B) 81 (C) 09 (D) 61 (E) 01

(SMO 2004)

Q2. How many zeros does the number 50x 49 x 48 x … x 3 x 2 x 1 end with?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

(SMO 2005)

Figure 1. Two SMO questions of Category 1 type

To solve Q1 and Q2 in Figure 1, one does not need to know modulo arithmetic, Fermat's Little theorems or other advanced knowledge in Number Theory. The solutions of these problems are accessible to the students even without any additional content "topping-up", if they are familiar with the terms *last two digits* in Q1 and the significance of *how many zeros* in Q2. The solutions of these problems require proficient use of problem solving strategies (which is an important emphasis in the Singapore mathematics curriculum): understand the problems (for example, what is meant by "the last two digits" in Q1; "number of zeros a number ends with" in Q2), use appropriate heuristics to tackle the problem (for instance, replace large numbers by smaller manageable numbers, and to look for suitable patterns).

In fact, without knowing too much sophisticated knowledge about modulo arithmetic for Q1 and the special formula $\sum_{k=1}^{\infty}\left\lfloor \frac{n}{5^k} \right\rfloor$ for the "number of zeros in n!" as in Q2, students can be more actively engaged in the problem solving processes and gain a more enriching experience with mathematical problem solving. This enriching learning experience is crucial for lower level students that is aligned to the Singapore syllabus requirement.

This category of problems requires students to know basic facts and terminologies outside the usual mathematics curriculum. Once students have an understanding of the contexts, the problems are solvable by the various problem solving strategies. These questions, when used appropriately, could provide students with enriching problem solving experiences. They could even be used in the usual mathematics classrooms.

Category 2: Problems that deepen understanding and appreciation of school mathematics.

These problems stress on skills or mathematical concepts which are usually not emphasized in the usual school mathematics curriculum, although the contexts of the questions are within the school curriculum. Mastering these skills or concepts would necessarily deepen students' understanding and appreciation of school mathematics. Figure 2 shows a sample of three SMO questions of this category.

Q3. In a triangle ABC, it is given that AB = 1cm, BC = 2007 cm and AC = a cm, where a is an integer. Determine the value of a. (SMO 2007)

Q4. Let a, b, c, d be integers and $(a^2+b^2)(c^2+d^2)=29$. Then the value of $a^2+b^2+c^2+d^2$ is ________. (SMO 1996)

Q5. Find the value of $\frac{2007^2+2008^2-1993^2-1992^2}{4}$. (SMO 2007)

Figure 2. Three SMO questions of Category 2 type

Q3, which is based on elementary geometry, examines the relation between the three sides of a triangle. In school mathematics, students learn about angle properties of triangles, properties of triangles involving the lengths, length and angle properties of similar and congruent triangles. It is not addressed in the school curriculum (at least explicitly) on the constructability of a triangle given three lengths – the condition of the triangle inequality. This question addresses a "gap" in geometry concepts taught within the school mathematics curriculum. The frequent occurrence of questions related to the triangle inevitably lead to this knowledge and application as a compulsory "syllabus" for the mathematics enrichment lesson.

Students usually solve or manipulate an algebraic equation procedurally. The focus of teaching algebra in the usual mathematics classroom is frequently procedural manipulation at the expense of students understanding what they are doing (Lee, 2006, p 44). It is thus not uncommon that many students may not have acquired an appreciation or deep understanding of school algebra.

Q4 introduces an alternative skill of dealing with mathematical equations – that of recognizing what each of the algebraic terms (including both symbol and operation) in an equation represents. This question compels the learners to re-examine the procedural algebra they have learnt and hence deepens their understanding of school algebra. Q5 introduces to students an appreciation of the various algebraic expressions (factorization and expansion) through experiencing its application in evaluating numerical expressions without the use of calculating devices.

There are also questions which challenge students to re-examine their current knowledge of concepts and skills in school mathematics. Two sample questions of this type are shown in Figure 3.

Q6. Let $a < 0$. Find $\sqrt{a^2} + \sqrt{(1-a^2)}$ in terms of a.

(A) 1 (B) -1 (C) $2a - 1$ (D) $1 - 2a$ (E) None

(SMO 1996)

Q7. How many real numbers x satisfy the equation $\frac{x^2 - x - 6}{x^2 - 7x - 1} = \frac{x^2 - x - 6}{2x^2 + x + 15}$?

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

(SMO 2004)

Figure 3. Two more SMO questions of Category 2 type

That the identity $\sqrt{a^2} = a$ holds for all real numbers a is a common misconception held by many students (Toh, 2006, p 67). Also, students commonly apply the simple method of simplification in solving an algebraic equation by "cancelling" off like terms on both side of the equation without examining the implication of such a cancellation process (Toh, 2006, pp. 72 - 74). The questions in Figure 3 serve to invite the students to re-examine their existing knowledge and skill in school mathematics. These questions can also be incorporated into the usual mathematics classrooms to deepen students' understanding of mathematics.

Category 3: Problems that require new knowledge and skills

These are the "out of syllabus" problems that require students to be proficient in mathematical knowledge and skills beyond the school curriculum in order to solve them. Usually, the "new" content knowledge forms the foundation of advanced mathematics which students would encounter at higher levels. Challenging higher achieving students to higher level mathematics content would expose them to more advanced mathematical thinking earlier. Examples of this category of problems are shown in Figure 3.

Q8. Let $f(x) = \frac{x^{2010}}{x^{2010} + (1-x)^{2010}}$. Find $f\left(\frac{1}{2011}\right) + f\left(\frac{2}{2011}\right) + f\left(\frac{3}{2011}\right) + \ldots\ldots + f\left(\frac{2010}{2011}\right)$.

(SMO 2010)

Q9. Find $\sqrt{14^3 + 15^3 + 16^3 + \ldots\ldots + 24^3 + 25^3}$. (SMO 2010)

Figure 3. Sample of SMO questions on summation of series.

Notice that Q8 requires the special techniques of finding the sum of the series (in the case of Q8, appropriate "pairing", which is motivated by Gauss' method to find the sum of an arithmetic progression). This problem demonstrates that Gauss' method can be extended to finding sums of series which are arithmetic progression. On the other hand, Q9 requires new formulae (in this case, the sum of cubes of the first n positive integers) as part of the solution of the problem.

The Three Categories of Problems and Implications for Mathematics Teachers

Category 1 problems expose students to problem solving strategies and other classroom learning experiences in "new" contexts provided by the problems. Having rich learning experiences (e.g. meaningful experience in problem solving processes) in the usual mathematics classrooms is an important part of students' learning, other than acquiring mathematical content knowledge. Rich learning experience in classrooms enables students to solve non-routine problems in "new" contexts.

Category 2 problems introduce students to situations which are not usually encountered in the usual mathematics classrooms; nevertheless, they facilitate to deepen students' understanding and appreciation of school mathematics. Category 3 problems are more appropriate for higher achieving students, since they require students to acquire new mathematical content knowledge beyond the usual school curriculum. This category of problems provides a useful platform for mathematics teachers to reflect on their own classroom teaching of mathematics, and the aspects of mathematical understanding that could be infused into their classroom teaching to deepen students' understanding of school mathematics.

Category 3 problems generally are more suitable for the higher achieving mathematics students. Appreciation of these problems will put the teacher in a better position to appreciate the position of the school mathematics curriculum within the broader context of mathematics education.

ENGAGING STUDENTS IN HIGHER ORDER THINKING (HOT)

The traditional attitude towards mathematics is usually one that emphasizes extensive drill-and-practice within a formulaic environment (Crawford & Brown, 2002). As students' learning environment and the world at large are changing rapidly, educational expectations are also shifting. As such, emphasis on higher order thinking skills should be of primary concern within mathematics. According to Thomas, Thorne & Small (2001),

> Higher Order Thinking, or HOT for short, takes thinking to higher levels than just restating the facts. HOT requires that we do something with the facts. We must understand them, connect them to each other, categorize them, manipulate them, put them together in new or novel ways, and apply them as we seek new solutions to new problems (quoted by Crawford & Brown, 2002).

Based on the above criteria, it is obvious that most of the problems discussed in the three categories in the preceding sections are questions that require HOT skills. In this section we present another example that might be pedagogically important.

Getting students to examine their misconceptions and the implications of their misconceptions are ways to incorporate higher order thinking skills into teaching mathematics. Learning mathematics instrumentally without much relational understanding has led students to develop many misconceptions. Two identities that are common among secondary school students are shown below.

$$(a+b)^2 = a^2 + b^2 *$$

$$(a+b)^3 = a^3 + b^3 *$$

It is not difficult to observe that the students' errors in algebraic manipulation lie the erroneous generalization of the distributive law:

$$2(a+b) = 2a + 2b \quad \rightarrow \quad (a+b)^2 = a^2 + b^{2\,*}$$

$$3(a+b) = 3a + 3b \quad \rightarrow \quad (a+b)^3 = a^3 + b^{3\,*}$$

There are many good sample activities for students to understand their misconceptions in the above rules using numbers or pictorial approach (see for example, Lee, 2006, p 46).

An example of turning the understanding of this erroneous generalization into a higher order investigative activity for students to examine the implications of their misconceptions in a creative way is shown in Figure 4.

<u>Student Activity</u>

"If $(a+b)^2 = a^2 + b^2$ is true for some real numbers *a* and *b*, what can you say about the numbers *a* and *b*?"

"If $(a+b)^3 = a^3 + b^3$ is true for some real numbers *a* and *b*, what can you say about the two numbers *a* and *b*?"

Figure 4. Sample of a HOT activity for students

An examination of many of the problems in the past year SMO competition papers shows that there are questions that attempt to develop the higher order thinking of students, by developing from these "wrong formulae" in algebraic identities. Two examples are presented in Figure 5.

Q10. Suppose x_1, x_2 and x_3 are the roots of $(11-x)^3 + (13-x)^3 = (24-2x)^3$. What is the value of $x_1 + x_2 + x_3$?

(A) 30 (B) 36 (C) 40 (D) 42 (E) 44

(SMO 2007)

Q11. Find the sum of all the real numbers x that satisfy the equation

$$(3^x - 27)^2 + (5^x - 625)^2 = (3^x + 5^x - 652)^2$$

(SMO 2005)

Figure 5. Sample of two SMO questions emphasizing on HOT.

Observe that Q10 and Q11 compels students to manipulate the equations further (for example, replacing each of the "cumbersome" expression in the algebraic equations by single letters) in order to present the equation in a manageable form, which will be the two "wrong formulae" above. This type of questions eventually leads students to connect different algebraic entities and invite them to apply their algebraic manipulation skills creatively.

Implications for Teachers on HOT tasks

The competition questions have provided a glimpse into the culmination of HOT skills. A deeper examination of these questions provides ideas for mathematics teachers on how HOT skills can be incorporated into the usual classroom teaching of mathematics.

PRESEVING ELEMENTARY MATHEMATICS IN CURRICULUM

The national mathematics curriculum is evolving to meet the needs of the modern world. In particular, the mathematics content is changing. Technologies and other new initiatives are introduced into the mathematics curriculum in order to enable students to cope with the new demand in the real world. With the introduction of the new initiatives into mathematics curriculum, it is natural that some mathematics content has to give way. It is lamented that "the official curriculum in mathematics today is far from the level of the 1980s" (France & Andzans, 2008).

Mathematics competitions such as the SMO play a crucial role as a very strong consolidating factor. For example, the competition "syllabus" (although there has never been an official one) and the structure of the competitions (paper-and-pencil test in which scientific calculators or other calculating devices are not allowed) remain unchanged. These competitions usually cover a broad spectrum of topics in elementary mathematics. In this way, mathematics competitions help to preserve some "elementary mathematics" that mathematicians believe are important for students as part of the advanced mathematics education, and which have to be removed from the usual mathematics curriculum.

In the Singapore mathematics curriculum, the use of technology forms an indispensable part of learning mathematics. According to the syllabus document (Ministry of Education, 2006), it was stated that all secondary school students must be able to "[m]ake effective use of a variety of mathematical tools (including information and communication technology tools) in the learning and application of mathematics." (p 1). As such, technology (in this case, electronic calculators) becomes an indispensable part in classroom teaching, the school examinations and even the high-stake national examinations. Some topics or subtopics within the usual mathematics curriculum are rendered obsolete under the new educational setting with the use of technologies. For example consider the questions in Figure 6.

Q12. Suppose $a = \sqrt{6} - 2$ and $b = 2\sqrt{2} - \sqrt{6}$. Then
(A) $a > b$ (B) $a = b$ (C) $a < b$
(D) $b = \sqrt{2}\,a$ (E) $a = \sqrt{2}\,b$
(SMO 1999)

Q13. Simplify $144\left(\sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}\right)$. (SMO 2004)

Q14. Find the smallest integer greater than $\left(1+\sqrt{2}\right)^3$. (SMO2006)

Q15. Given that $2\sqrt{x} - \sqrt{4x-11} = 1$, find the value of $x^2 + x + 1$.
(A) 71 (B) 81 (C) 91 (D) 47 (E) 63
(SMO 2004)

Figure 6. Sample of four questions “preserved” in SMO

The techniques needed to solve the four questions in Figure 6 include skilful manipulation of surdic expressions (e.g. irrationalizing the denominator of a surdic expression in Q12, squaring surdic expressions and skilful application of the rules of surds in Q13, Q14 and Q15). Q12, Q13 and Q14 are no longer emphasized in the new mathematics curriculum, since the answers to these questions are easily obtained using a scientific calculator. Solving surdic equations is considered an obsolete topic not included within the school curriculum. Nevertheless, the skills involved in such algebraic manipulation are considered important as part of elementary mathematics.

Besides preserving those sections of the mathematics curriculum which are rendered obsolete by the new initiatives in the curriculum, mathematics competitions also serve to preserve some mathematical topics which never have a rightful place in the mathematics curriculum. An example: theory of polynomial equations, with particular emphasis on the properties between the roots and coefficients of a polynomial equations (degree higher than 2) (see Figure 7).

Q16. Find the sum of the squares of all real roots of the equation
$x^4 + 4 + 11x^2 = 8(x^3 + 2x)$.

Figure 7. Another sample of two questions “preserved” in SMO

Use of Competition Questions to Generate “New” Questions

The above discussion demonstrates the richness of the test items from past years mathematics competitions. Problems from past years mathematics competition provide good resource for mathematics teachers to generate “new” problems by innovating on existing problems. Some mathematics educators have advocated this technique of problem generation. In particular, Vistro-Yu (2008) discusses in great length on this technique; some of the problems that she used as a basis for her discussion were mathematics competition questions from past years.

STUDENTS' RESPONSES TO THE OLYMPIAD QUESTIONS

While it is the intention of the competition questions to encourage mathematical reasoning and advanced mathematical thinking in the questions, the structure of the competition might limit the full development of the students' arguments. Most mathematics competitions only require students to give the "correct" answers without full solution. Thus, students' incorrect or incomplete reasoning (and which yields correct answers to the questions) may remain undetected.

We consider the SMO as an example. The competition consists of two papers: the first paper (which all candidates shortlisted by the schools will participate) consists of some short questions to be answered by candidates within 2.5 hours. Candidates are expected to provide the answers, and not required to show their working. The second paper, which only candidates who are shortlisted based on outstanding performance in the first paper will sit, requires full solution of five questions within 2.5 hours. As such, the second paper is limited to the selected few who have done exceptionally well in the first paper. Our discussion is restricted to sample students' responses to questions selected from first paper only.

This section reports some episodes of students' responses to selected past SMO questions. These episodes were collected over past several years in which the author came into contact with several potential students identified by their schools to participate in the various mathematics competitions.

In this section, some of the "incorrect" (or incomplete) student responses to selected competition questions which yield correct answers are presented and discussed.

Response 1: "Heuristic" approach to solving algebraic problems

The following three problems in Figure 8 on elementary algebra presented to several lower secondary students (ages 13 and 14) which the author has contact with.

A. If x and y are real numbers such that $xy = 24$ and $x + y = 11$. Find the value of x^2+y^2. (SMO 2000)
B. Suppose x and y are two real numbers such that $x - y = 8$ and $x^2 + y^2 = 194$. Find the value of xy. (SMO 2004)
C. Suppose $x - y = 1$. Find the value of $x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4$. (SMO 2005)
D. If $x + \sqrt{xy} + y = 9$ and $x^2 + xy + y^2 = 27$, find the value of $x - \sqrt{xy} + y$. (SMO 2007 Junior Q24)

Figure 8. Four problems on elementary algebra problems in SMO

A brief description of the expected response in the published SMO solution booklet yields is as follows:

- Expected response to A: Use the identity $(x+y)^2 = x^2 + y^2 + 2xy$.
- Expected response to B: Use the identity $(x-y)^2 = x^2 + y^2 - 2xy$.
- Expected response to C: Factorize the expression $x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4$ in terms of expressions in $x - y$.
- Expected response to D: Use the identity $(a+b)(a-b) = a^2 - b^2$.

Students are expected to be proficient in the use of algebraic identities and factorization. However, a comparison with some students' responses shows a stark contrast in the approach: many lower secondary students (ages 13 and 14) used the heuristic of "guess-and-check" in obtaining the correct answer.

- Response to A: It can be seen that $x = 3$ and $y = 8$ by guess-and-check. Therefore, the correct answer is $3^2 + 8^2 = 73$.
- Response to B: By guess-and-check, $x = 13$ and $y = 5$. Therefore, the answer is 13 x 5 = 65.
- Response to C: It is clear that $x = 1$ and $y = 0$ (obviously, many students did not consider there are many other possible pairs of x and y). Direct substitution into the expression yields the answer 1.
- Response to D: By guess-and-check, $x = 3$ and $y = 3$ (some students were unsure that the letters x and y may represent the same number). Direct substitution into the expression yields the answer 3.

The following observations made were of great concern to the author:

- Many past year questions involving algebraic manipulation were constructed in a way that the answers, being small integer values, could easily be obtained by guess-and-check.

- Students (especially students from lower secondary levels) are resistant to apply algebraic identities to solve the algebra problems in competition problems. This is partly due to their experience of success in obtaining the correct answers for many past year problems by guess-and-check. In other words, while the questions attempt to encourage students to use algebra, it further reinforces students' reluctance to use algebra.

Response 2: Solving a problem by considering a "special" case.

If geometry proof or deductive reasoning is not required as part of a submitted solution in mathematics competition, geometry problems are most notorious in enabling one to obtain the correct answer by considering a special case instead of the general case specified by the problems. Consider the following question:

In the following figure, ABCD and PQRS are two identical squares with side 20cm. The point P is placed at the centre of the square ABCD. What is the area of the quadrilateral PXCY in cm^2? (SMO 2004)

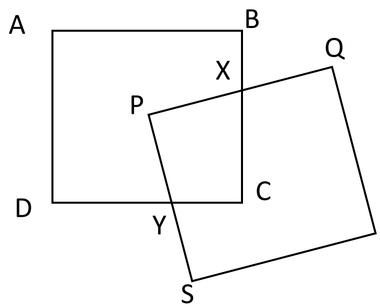

Figure 9. A geometry problem in SMO

The official solution as published in the SMO solution book shows that the following justifications are required for a complete solution of the problem in Figure 9:

- Consider the special case when PQ is parallel to AB.
- In that special case, the area of PXCY equals 100 cm^2.
- Justify, by using congruent triangles, that the area PXCY in the general position in Figure 9 also have an area of 100 cm^2.

Students must be able to be able to begin solving the problem by consider a special case (one of the problem-solving heuristics emphasized in the primary school mathematics curriculum), and progress to generalize the solution for the general case by using properties of congruent triangles.

A number of student responses to this geometry problem yielded the correct answer 100cm^2 and stopped at considering the special case when PQ is parallel to AB. No further justification was provided by the students.

Using a special case to obtain the answer for a general problem is also observed in the students' responses to an algebra problem on logarithms (Figure 10).

Suppose that *a, b* and *c* are real numbers greater than 1. Find the value of

$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}.$$

(SMO 2009)

Figure 10. A problem on logarithms in SMO

According to the official solution published in the SMO solution book, students are expected to combine the individual terms in the denominator of each of the fractions into logarithmic expressions and then apply the change of base rule in logarithm to combine the three terms together.

Students are generally not well versed in a combination of several rules of logarithms within an algebraic expression. Some interesting responses obtained by students are their substitution of suitable numbers, with $a = b = c$ (e.g. $a = 2$, $b = 2$, $c = 2$), in which case the answer of the question is easily seen to be 3.

From the sample students' responses above, they might not have acquired the mathematical skills or reasoning which the questions aim to test. The irony is that while the setters of the SMO expects students to acquire sophisticated skills, concepts and reasoning behind each problem (as indicated by the publication in the official solution book), students are *not* required to write down the full solution of each problem. This mismatch could lead to students giving the correct answers without fully able to perform the full mathematical reasoning required by the problems.

If the structure and format of the mathematics competition were to stay, it would be necessary that the competition question setters be "stickler for detail and can anticipate errors and misconceptions" (Wright, 1993), and anticipate the students' incorrect responses which yield correct answers of the questions. The competition questions must be tweaked to ensure that only sound rigorous mathematical reasoning could lead to correct answers.

CONCLUSION

The role of mathematics competitions in Singapore have expanded beyond the objective of identifying mathematically talented students in Singapore and reached out to a much wider Singapore student population. In this note, through an examination of the test items of the competitions in the past years, the roles that mathematics competitions can play in mathematics education are discussed. The pedagogical values of mathematics competition cannot be ignored. Sample of students' responses to some of these problems based on incomplete mathematical reasoning which yield correct answers are presented, showing the mismatch between the good intention of the competition and the actual outcomes due to the limitation of the phrasing of the questions and the format of the competition.

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