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VISUALIZING MATHEMATICS AT UNIVERSITY? EXAMPLES FROM THEORY AND PRACTICE OF A LINEAR ALGEBRA COURSE

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With this communication, I will try to promote a discussion on visualization adapted to university level: how I understand it, why may it be important to understand advanced mathematics and, mainly, how it is currently taught. With this aim, five examples –obtained by the observation and my reflective practice in a Linear Algebra course– will be presented. The analysis of these episodes will enable to go deeper into some issues of visualization, relevant in this particular context: characteristics of visualization, some obstacles and opportunities of teaching visualization and some actions needed to improve the teaching of visualization at university level.

Key-words: *Visualization, Linear Algebra, Teaching at University Level, Participant Observation.*

INTRODUCTION AND MOTIVATION

There are several reasons to believe that visualization is helpful to understand advanced mathematics. In fact, so many authors have supported its use at university:

> "Mathematical concepts, ideas, methods, have a great richness of visual relationships that are intuitively representable in a variety of ways. The use of them is clearly very beneficial from the point of view of their presentation to others, their manipulation when solving problems and doing research." (De Guzmán, 2002, p. 2)

> "A more cognitively appropriate approaches are postulated, some with empirical evidence of success. These include: […] the use of visualization […] to give the student an overall view of concepts and enabling more versatile methods of handling the information." (Tall, 1991, p. xiv)

In my opinion, two words need of a better specification in order to argue the importance of visualization at university level: understanding and visualization. Inspired by the notion of concept image (see Vinner's chapter in Tall, 1991) and Duval (1999), to *understand a concept* implies the construction of a network –a cognitive structure– in which the concept occupies a new node. This new node should be connected to previous nodes, which could be other elements such as: mathematical knowledge on this concept (definition, properties, theorems, proofs, etc.), mathematical experiences with this concept (examples, problems, conceptions, etc), different representations of the concept (in the table, symbolic or geometric registers, in natural language, etc.), experiences with representations (creation, transformation (treatment and conversion), coordination with other representations, etc.), previous knowledge, intuitions, other concepts, etc. (Figure 13 could illustrate this idea). In order to achieve understanding, this network should be complete and adequately articulated in such a way that a rich mental image of the concept is formed on individual's mind. This image allows the learner to make sense of the concept and look at it from different points of view. Moreover, the construction of this rich mental image provides a new starting point, cognitively higher, that enables *Advanced Mathematical Thinking* (AMT) (Tall, 1991); being this one of the main objectives of the teaching and learning at university.

On the other hand, I agree with Arcavi's definition of *visualization*:

> "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings." (Arcavi, 2003, p.217)

Eventually, the importance of visualization for the understanding in mathematics, underlined in the quotations above, could be illustrated by the following example.

> Imagine that I am in a foreign country where people speak an unknown language to me. I would like to walk from my hotel to another place. The only things I know about this place are its name and that it is close. Since I do not understand the person in the reception, I cannot ask. I decide to try by myself with Internet, as in similar situations in my country. First, I find out the address in the website of the destination. It is a sequence of words and numbers that do not make any sense. Then, I look for it in Google Maps. I introduce the addresses of the hotel and the destination, and voilà! I obtain my route's description in two different and complementary ways: one in written language (which I do not understand) and a map. Finally, I know that I can have the "street view" too. It is more accurate and could be helpful in order to determine the route in the real street. With all this information, I feel ready to go outside and walk to my destination by following the map.

When students arrive to some new subjects at university –such as Linear Algebra (LA)– and try to understand a concept, they may feel likewise in a foreign country, with an unknown language, in where they need to handle different representations (Duval, 1999; Pavlopoulov's chapter in Dorier, 2000), languages (Hillel's chapter in Dorier, 2000), modes of thinking (Sierpinska's chapter in Dorier, 2000) and points of view (Alves-Dias' chapter in Dorier, 2000). In LA context, this is called "cognitive flexibility" and has been pointed out as a source of difficulties for students (Dorier, 2000). I believe that, like in the example, visualization could be a useful tool to overcome such difficulties. Moreover, in this example, visualization has been essential in order to connect the two different points. Similarly, visualization is essential to achieve deep understanding, that is, to obtain a rich mental image of concepts that enables AMT. However, is visualization available to students? In the example, I was supposed to have this ability at my disposal from previous experience. Likewise, I defend that teaching should facilitate it to students.

In order to find out more about how the teaching of visualization is at university courses, I participated as a teacher and observer in a LA course during the academic year 2010/2011. This course was taught at the School of Mathematics of the Universidad Complutense de

Madrid (UCM), which is one of the most important universities in Spain. In this communication, I will describe five episodes that took place during this period. With these episodes, I do not try to show an exhaustive analysis of the results of this participant observation, but to promote a discussion about some issues I found relevant in relation to the teaching of visualization at university level.

EPISODES OF VISUALIZATION IN A LINEAR ALGEBRA COURSE

About the context

The LA course is part of the first year of a four year-long Degree of Mathematics. LA was one of the four annual subjects in this first year. It was divided into two semesters (November-February; February-June), each ended with a period of *partial exams*. There were two additional examination periods for *final exams* (one at the end of June and another in September). There were six hours per week for teaching LA: four hours of *lectures* dedicated to the theoretical contents of the subject and two hours of *seminars* dedicated to work on sheets of problems. There were fourteen sheets –one for each content unit– with 15 problems on average. Students were divided into three groups for seminars. I taught one of these seminar groups during the second semester. I also observed and videotaped lectures from January until the end of the course. In relation to the *materials* used, lectures followed a textbook (Fernando, Gamboa, & Ruiz, 2010). For the seminars, in addition to the sheets of problems, other worksheets were given to students as voluntary homework. It counted for the continuous evaluation. Despite this, exams consist of the main method of *assessment*.

Episode 1: Spontaneous use of a diagram as an aid for solving a problem in class

This first episode is about the use of a diagram in a problem solving situation. It took place in a lecture as a consequence of a students' intervention. This episode reminded me an anecdote told by Miguel de Guzmán in his works on visualization: Noerbert was giving a lecture and suddenly got stuck; he only was able to continue thanks to a figure that he drew (and erased quickly) in a corner of the blackboard (De Guzmán, 2002). In the next episode, something similar happened. Fortunately, in this case, the diagram was loudly commented.

Obs. 1: If A is antisymmetric and n is odd $\Rightarrow$ $det\,A=0$, since

$$det(A) = det\,(-A^t) = (-1)^n\,det\,(A^t) = -\,det(A)$$

Obs. 2: Both, the rank and the antisymmetric character of a matrix, are preserved if I do this: $f_i \rightarrow f_j$ and $c_i \rightarrow c_j$

Obs. 3: Let r be the rank of A. Let B the matrix obtained by interchanging rows and columns like in Obs. 2, so that the minor in B formed by the first r rows and columns is no null.

Obs. 4: $B = \left(\begin{array}{c|c} * & \\ \hline & \end{array}\right)$ antisymmetric $\Rightarrow$(*) antisymmetric

Conclusion: $\left.\begin{array}{r}\det(*) \neq 0\\ (*)\text{ antisymmetric}\end{array}\right\} \overset{Obs.1}{\Longrightarrow} r \text{ is even}$

Figure 1: Lecturer's solution to the problem as it was written on the blackboard.

February's examination period was close and the lecturer left one day for students' queries. The day before, one student had asked the lecturer about one of the tasks in the voluntary worksheets: *"Let A be an antisymmetric matrix. Does A with odd rank exist?"* The lecturer began the class by explaining, his solution to this problem, on the blackboard (Figure 1). Another student raised his hand and showed his disagreement with the "Observation 3". He said: "The third… How do you know that you can do this? […] If you have a 4x4 matrix and a 2x2 minor no null on the upper right corner, you cannot move it to the upper left corner by doing this" (20110202_23:02)[1]. The lecturer wrote an example of a 4x4 antisymmetric matrix on the blackboard (Figure 2, on the left). In this manner, a long discussion started, which lasted almost the whole class. The lecturer got stuck, trying to find out why his solution did not work, and thought aloud for the class. Then, the following diagram appeared (Figure 2, on the right). The lecturer used it to think, with generality, on the transformations that keep the antisymmetric character of any 4x4 matrix. The diagram was also helpful to communicate these thoughts to the class.

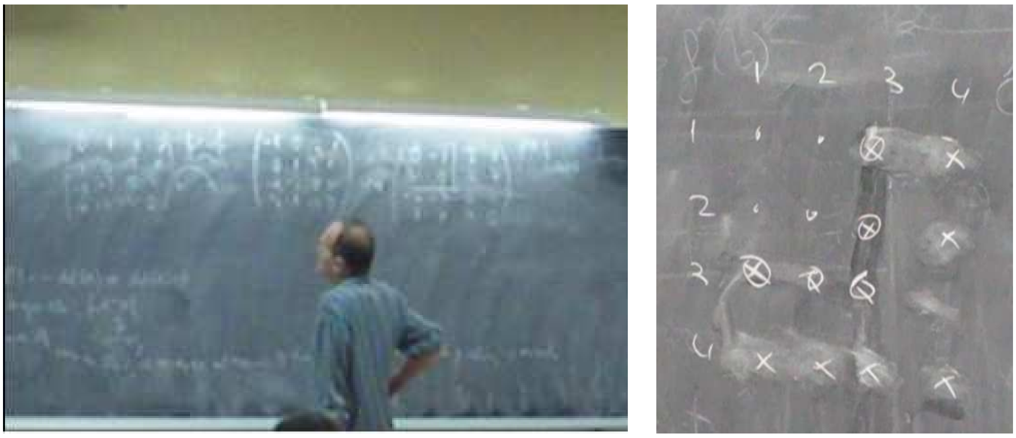


Figure 2: The lecturer thinking about the problem with a concrete example (on the left) and with a diagrammatic representation (on the right)

Episode 2: The paradox of making representations explicit, the transformation "coordinates"

Students often confuse the vector and its representation in coordinates with respect to a basis; mostly if the vector space is IK^n. In this case, the original vector and its coordinates are both an *n*-tuple, looking like the same object (see Hillel's chapter in Dorier, 2000, p.201). This second episode concerns the transformation "coordinates", which I found useful to help students to overcome this difficulty. Moreover, this transformation helped me to realize about the existence of a paradox in relation to make explicit issues about representations.

The lecturer introduced the transformation "coordinates" ($coord_B$) during the explanation about the coordinates of a vector with respect to a basis *B*. This transformation maps an abstract vector space *E* onto IK^n and assigns each vector *u* to its coordinates in terms of *B*, that is, a *n*-tuple. At this time, it was described in symbolic language, though I found the same transformation represented in other registers (like the geometric) in other LA textbooks (Figure 3). This transformation makes the change of representation explicit and allows the vector and its coordinates to be distinguished, which I believe facilitates the reasoning.

[1] I refer the quotations from videos as follows: Date(YYYYMMDD)_Time(minute:second)

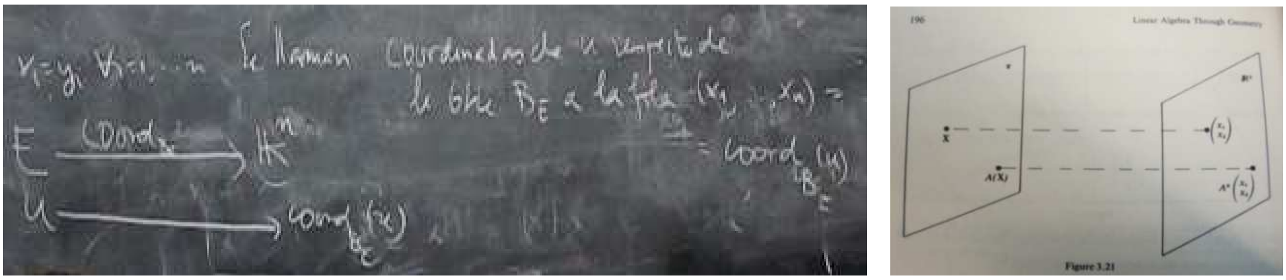


Figure 3: Representations in symbolic register used on the blackboard (on the left) and geometric register found in other textbooks (on the right[2])

Nevertheless, the textbook followed a different approach. This transformation was not introduced at this point (unit 7 in the textbook). It appeared two units after, just to note that it was an isomorphism. Contrary to what happened in class, the isomorphism was not used to make any reasoning. It was just said: "The reader could recognize this isomorphism behind many arguments used in unit 7" (Fernando et al., 2010, p. 191). Why do authors in the textbook decided to hide this isomorphism while the lecturer decided to include it? I asked the lecturer about his decision to include this isomorphism in class:

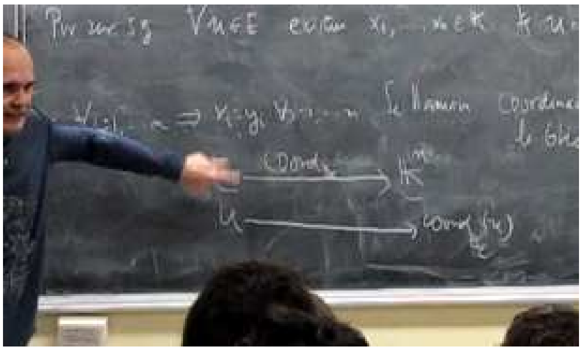
> L: It is useful and we constantly use it. […] Thus, it is advisable to shape it, to give it a notation. I realized that notation in mathematics helps a lot. […] To transform things in equalities of symbols helps enormously to think. […] This is contrary to what many people have defended, is it not? Formulas: the less the better. I do not agree with this at all. Formulas: the more the better. Thus, the "general non sense" of mathematics can lead you.

Therefore, there is a difficult decision to be taken in relation to the communication of issues about representations such as: how to distinguish one representation from another or from the represented object; how a transformation is made; which representation is better for a particular aim; etc. The lack of clarity with such issues could be a source of students' difficulties, as it was pointed out above. In order to help students to avoid this kind of difficulties, a specific language –that allows the different kinds of representations and their transformations to be referred– is needed. In this case, mathematics provided this language[3] (the transformation "coordinates"). In other cases, different languages should be created and research, in Mathematics Education, could be a good source of inspiration (see macro/ micro language explained in the Episode 3). However, the use of this language (initially introduced to help students) could be a new source of difficulties for students too, as textbook's authors might have thought. Thus, the decision is: is it worthy, from students' point of view, to make explicit this kind of information about representations despite introducing a new language? This is what I have called the paradox of making representations explicit.

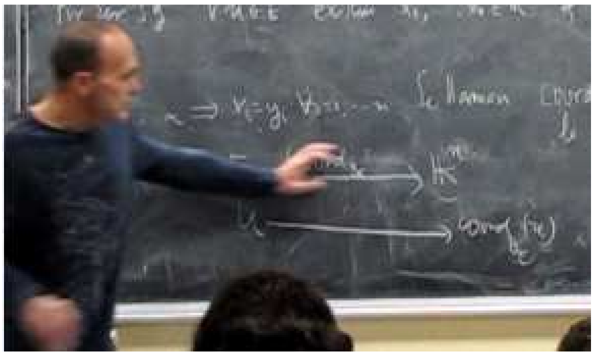
[2] The source of this image is: Banchoff, T., & Wermer, J. (1992). *Linear Algebra Through Geometry*. Undergraduate texts in mathematics. (2nd. Ed.). New York [etc.]: Springer.

[3] In fact, there are many theorems and results in LA about changes of representations: matrix of a change of basis, matrix of a linear transformation, subspaces as intersection of hyperplanes, etc. This gives an idea of the degree of difficulty that this kind of processes could reach in advanced mathematics.

In this case, the lecturer had a clear answer to this question: YES, it was worthy since this transformation exhibits a common way of thinking in mathematics, in general, and in LA, in particular. It allows the work done with rows in the first part of the course to be translated to abstract vector spaces. This way of reasoning is well explained, in the corresponding extract of the class (Figure 4, 20110124_41:42), through the previous symbolic representation of the transformation, gestures and natural language:

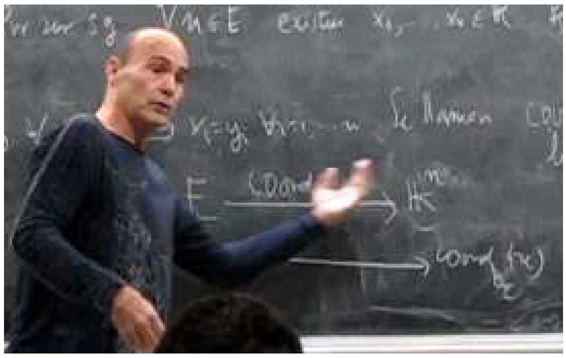


1. We will study this with more detail. But the idea is the following. Always that I am asked to do something in here,



2. I travel, via "coord", pshium (he makes a sound while he passes the hand on the arrow) to IK^n.

3. I do there whatever, in IK^n [he shakes his hands], where I know well how to move –it is a space that I master, its elements are rows–



4. and I come back afterwards.

Figure 4: Lecture's explanation of a way of thinking via the transformation "coordinates"

Episode 3: Making sense of difficult concepts for students through several visualizations, the concept of Quotient Vector Spaces

Most of the explanations in lectures used to be in the table and symbolic register and were exposed in a logical sequence. However, the next episode provides evidences of certain variety in this routine. Explanations about quotient vector spaces (QVS) attracted my attention because different kind of representations, examples or metaphors –from now on, I will say "visualizations"– were used to introduce, motivate, favour intuition or, more generally, to make sense of the concept. Likewise, QVS were one of the few concepts geometrically represented in the textbook (Fernando et al., 2010, p. 176). In an informal conversation, the lecturer told me that QVS were one of the most difficult concepts for students to grasp in this LA course. This could have motivated the amount of visualizations (bigger than with other concepts). As a result, this episode serves to shed some light on how visualization could be used in order to help students making sense of difficult concepts, particularly on QVS: which different kinds of visualizations can be used, how the communication about them can take place, what are they useful for. Below, I will expose some of these episodes and visualizations that QVS gave rise. I describe before the contents, that concern QVS, in the chronological order in which they were explained during the course:

- The **formal definition** of *E/V* as quotient set –built from a vector space *E* and an equivalence relation dependent on a subspace *V*– provided with a vector space structure inherited from *E*.
- The **geometric representation** of *E/V* as a family of subspaces parallel to *V*. In lectures, it took place in between the formal definition: after the definition of the quotient set and before providing it of vector space structure.
- The **dimension and basis** of *E/V* and the calculation of the **coordinates** of a given equivalence class.
- The properties of the **canonical projection**, $\pi: E \rightarrow E/V$: it is linear, surjective (since $im\ \pi = E/V$) and no injective (since $ker\ \pi = V$)
- The Canonical Factorization (better-known as the **First Theorem of Isomorphism**) and its **application** (**factorization** of transformations via a QVS).

Z_n example

This example concerns the group of integers modulo *n*. The lecturer used it to introduce the formal definition of QVS as an attempt to help students to connect it with previous knowledge acquired in other subjects of the Degree. Moreover, this example served to motivate the usefulness of quotients: "What for? Why? Why quotients were created? Well, you may have realized that the use of these Z_n is really useful to think about divisibility with simplicity." (20110217_13:43). Finally, Z_5 was used to introduce the notation of QVS as something familiar (Figure 5) and Z_{17} was used to show why a transformation starting in a quotient has to be well-defined.

Figure 5: Notations on the blackboard

The parameterization of the circumference

This second example was introduced, with the following sentence, in the same lecture about the formal definition of QVS: "I think that quotients were created because of situations like this. Would you know how to parameterize the circumference S^1? (20110217_15:59)". He asked students to build such a parameterization. First, students proposed the parameterization obtained by isolating one of the variables in the equation of S^1. The lecturer started by considering this proposal just to show that it is not a good one, since two branches are needed to cover the whole circumference. He uses a geometric representation to support this reasoning (Figure 6). Second, students proposed a unique parameterization, with sine and cosine, that covers the whole circumference. The lecturer argued that it is not an injective map. At this point, he said: "Whenever you have a map that is surjective but not injective, the reasonable thing to do in mathematics is what follows (20110217_21:45)". Thus, he introduced a relation of equivalence in *IR* and therefore, a quotient space. Eventually, this teaching sequence, provided by the problem of circumference's parameterization, served to motivate the introduction of the concept of QVS. Moreover, it was recalled –only with a few words this time –during the explanation of the First Isomorphism Theorem (the problem of having a non-injective map appears there again). This is an example of how can be reified a whole teaching sequence into a new unit of visualization, that could be easily referred to afterwards.

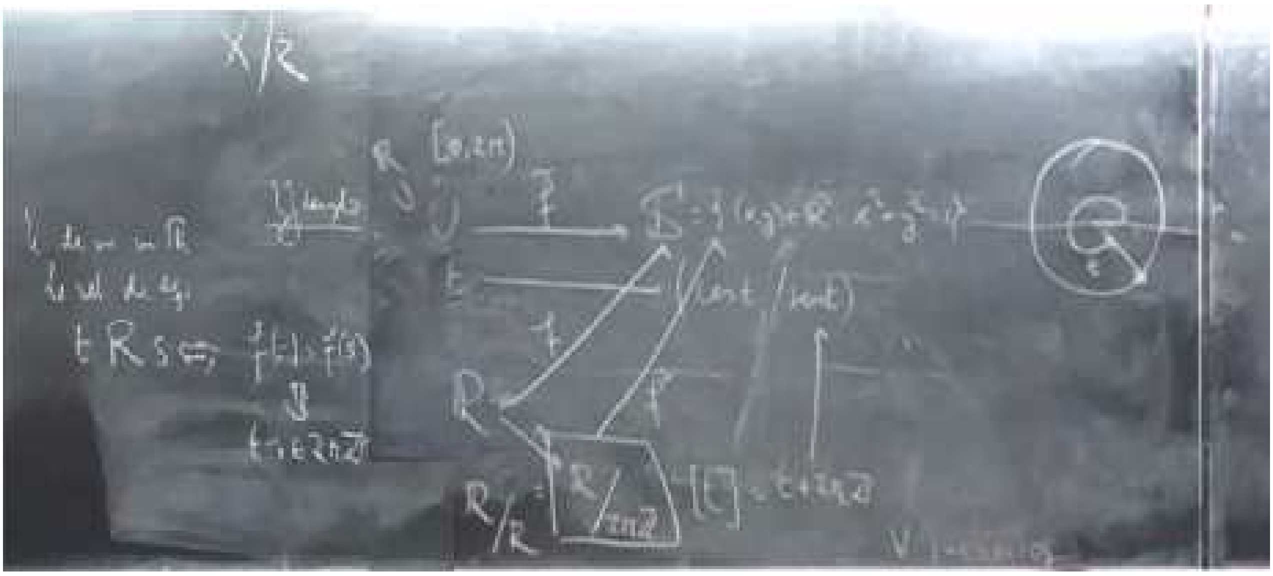


Figure 6: Blackboard afterwards the explanation of the parameterization of the circumference

Geometric representations in IR^2 (micro and macro) [4]

Two different episodes, concerning the geometric representation of QVS, will be described in this section. The first episode took place in relation to the formal definition too. The lecturer defined the relation of equivalence, proved that reflexive, symmetric and transitive properties held and described algebraically the classes of equivalence through several treatments of symbolic representations (Figure 7, on the left). At this point, the geometric representation of QVS was presented as an example of how to visualize the equivalence classes. It was introduced with the following message:

> L: During my studies, some lecturers showed examples, though few. The examples they did were not useful, they were all trivial. And the examples I wanted them to do, lecturers did not. Thus… I have a bad experience with that. Of course, the thing I am going to do now [the example], it is not useful at all for a person who is in her right mind. But OK, there is always someone who is not in her right mind. (20110227_36:38)

Additionally, the lecturer described the example as "antipedagogic". In his opinion, the main point about quotients was that their elements are subsets from the original set. "But even God cannot represent that! Thus, what am I going to do? To represent these subsets"

[4] While the kind of representations and transformations referred in this section are possible in IR^3 too (in fact, we also used them in seminars), I am going to refer here only representations in IR^2 because they were more common.

(20110227_37:22). He turned back to the blackboard, wrote the equation of *V* (a subspace from IR^2) and represented geometrically *V* and a vector, $u = (2,1)$. Step by step, he added *u* to vectors in *V* by the parallelogram rule and asked students: "Conclusion, what is $u+V$?" Some answered that it was a parallel line (Figure 7, on the right). At this point, he insisted on the problem of the geometric representation of QVS:

> L: I have just represented, as a subset of *E*, this parallel line (he follows it with the hand). OK? Like a subset of E. Like a point of the quotient, there is no way! I do not know how to draw it. I cannot draw it as an element of the quotient. (20110227_36:48)

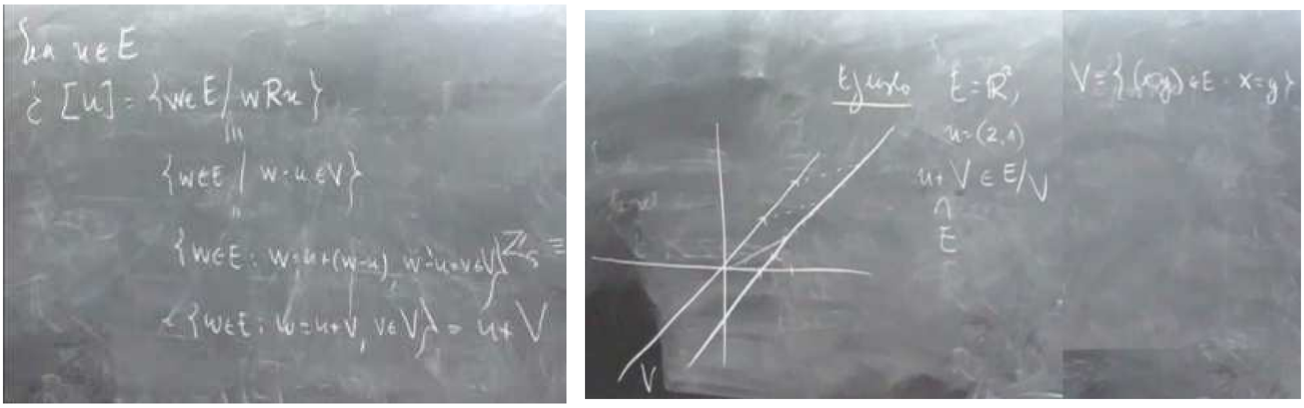


Figure 7: Symbolic representations (on the left) and geometric representations (on the right) for classes of equivalence of QVS

Finally, he recognized that the geometric representation was useful to see the properties of partitions: each element from the original space belongs to one and only one equivalence class; two any classes are either disjointed or the same; and the union of all the classes is the original space. He added a message to remark the particularity of the image: "It would be an unimaginable surprise that the opposite happened. This is a particular case (20110227_41:40)". I found this kind of message relevant to students.

The second episode, involving a geometric representation in IR^2, took place in one of my seminars. Several students came to me and claimed that they did not understand QVS. Despite I knew it could take some time, I decided to explain them again trying to use some different visualizations. I started the class with the *nails metaphor*, which will be explained below. It was followed by the *geometric representation in* IR^2. I asked one student to come to the blackboard and to find vectors related to a given *u*, which was represented geometrically. The idea was to use the definition of the relation of equivalence in order to find the class (instead of the $u+V$ description used in the lecture). However, I realized this approach may bring more difficulties to students. It involves a subtraction, and the geometric interpretation of subtraction seemed to be less familiar to students. Thus, I felt the need of explaining it (Figure 8, on the left part). Moreover, depending on how this subtraction is done, the result can be a vector which origin is not the point *(0,0)*. Therefore, this task was cognitively demanding and the degree of difficulty varied depending on the starting point chosen for the conversion from symbolic to geometric register.

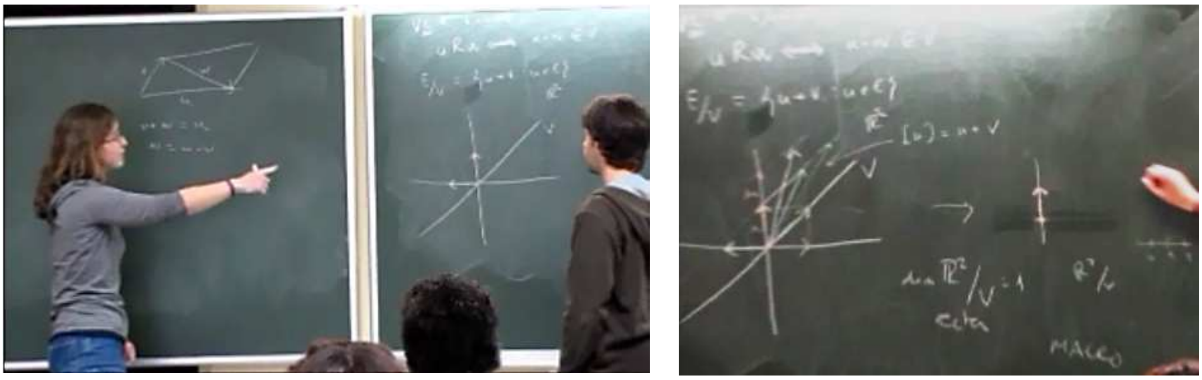


Figure 8: Geometric representations for subtraction (on the left) and micro and macro representations of QVS (on the right)

Afterwards, a new geometric representation was introduced. I called it "macro", inspired by a previous work about the dual space (De Vleeschouwer & Gueudet, 2011). The motivation was to solve the problem of representation for quotients pointed out by the lecturer. The name "macro" refers to the fact that this representation could be thought as the result of making "zoom out" from the previous geometric representation. Thus, in the example above, each parallel line to V becomes a vector and the quotient space obtained can be represented as a vector line. This representation enables to see QVS aspects' such as their vector space structure, a basis[5], the dependence of classes, etc. However, it hides the subspace V, the properties of partitions and the fact that each element is actually formed by many vectors from the original vector space. I tried to solve this last issue by saying that the vectors in the macro representations were thick and I reproduced this idea in the geometric representation (Figure 8, on the right). If all these properties of QVS want to be seen again, it is enough to undo the transformation by doing "zoom in". This recuperates the initial geometric representation that thus, I will call "micro". This micro/ macro transformation of the geometric register enables to exhibit and to communicate a change in the point of view from which we look at QVS. Similarly to Episode 2, it involves the use of a new language and provides, as compensation, a good tool that could offer a deeper understanding of the concept.

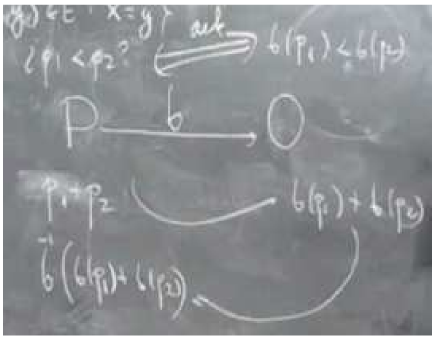


Figure 9: Diagram to "transport the structure": (the bijection b allows traveling from P ("poor" and unknown space) to O ("organized" and well-known space))

Diagrammatic representation

Once the lecturer had defined QVS and had represented them geometrically, he said: "You did not finish here when you were explained Group Theory. Likewise, we have not finished yet. The next step is to provide the quotient set with a vector space structure" (20110227_41:43). He continued saying: "And this is also done with the "general non sense" of Mathematics I told you about before. The

[5] In fact, the transformation to the macro representation needs of the choice of a representative for each equivalence class, and therefore a basis of the quotient.

hand writes by itself. You do not have to think anything" (20110217_41:43). He called this process "to transport the structure" and is very similar to the process described above, in Episode 2, for the "coordinates" transformation (Figure 4). He started to explain this process by speaking but when he asked the students if they had understood, there was no answer. Thus, he decided to explain it with a diagram (Figure 9).

Bags and Nails metaphors

Metaphors were introduced in the course mainly to highlight that QVS are partitions, that is, QVS are the result of a process of ordination or classification into subsets. Moreover, these metaphors enable to talk about issues related to the choice of a representative of each class of equivalence; in particular, to the property of being something well-defined.

The *bags metaphor* consists of thinking that "the quotient space is a space that has bags as elements" and the "canonical projection is to move from one space to this space of bags". This metaphor was introduced during the explanation of the First Isomorphism Theorem, just after recalling the example of the *parameterization of the circumference*. It was also used to explain how quotients serve to make injective a non-injective transformation. The lecturer used a diagrammatic representation to support his explanation (Figure 10, right part).

> L: How do I convert this [the transformation defined in the upper part of the diagram in Figure 10] in injective? I pass from this space (he writes an arrow going down) to the space that has, as points, the bags. Let us see if I can do it… (He draws a square with three sets. They are similar to the upper representation but coloured instead of dotted). One, two, three. And I say: "look if this is not bijective!" (He draws three arrows, one from each subset). Of course it is bijective, because this set has three elements: three bags. The first bag goes to the 1, the second to the 2 and the third to the 3. This is what it does, ok? It is obvious; I even do not know what to say! (20110228_26:00)

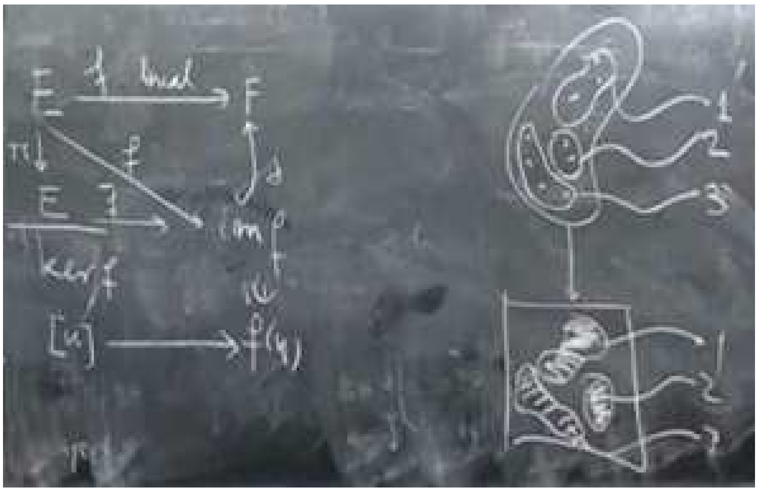


Figure 10: Lecture's explanation of the 1st Isomorphism Theorem with the diagrammatic (on the right part) and symbolic representations (on the left part)

Afterwards, this process was "translated" to the symbolic register, which the lecture qualified as "less visual" (Figure 10, right part). To have both representations, one next to the other, made this process easier. Additionally, the lecturer used this metaphor in order to justify why it was necessary to prove that the built transformation was well-defined. In this case, gestures served to communicate it (Figure 11). Finally, the lecturer used both, the diagram and gestures, to answer a student's question about the injectivity of the original transformation.

> L: All the transformations in the world that begin in a quotient except these here, with drawings, all use the following resource. We take the class of someone, the class of *u*. We introduce the hand into the bag. We take out someone, *u.* (He makes the gesture of introducing and taking out something from an imaginary bag). And we use it to define the transformed of the bag. Worry! What if other person introduces his hand, takes a different thing and, when applying *f* to it, the result is different? (He repeats the previous gestures). Thus, which is the image of the bag? I do not know! Let us prove that it does not matter who introduces the hand and what is taken from the bag. That is it, let us do it! This is called to be well-defined. (S6200110228_28:55)

"We introduce the hand into the bag"

"We take out someone, *u*"

Figure 11: Lecturer explanations with gestures

The *nails metaphor* was introduced in the seminar I explained QVS again as response to students' difficulties. This metaphor consists of thinking that "quotient is the desk of a hardware store", "classes are drawers of things such as nails" and "representatives are the labels on the drawers". I used this metaphor in order to highlight that the quotient is not a subspace of the original space, since the elements in each space have different nature: nails in the original space and drawers in the quotient. Moreover, I used it to show the importance of notation for classes. I defined two different equivalence relations: to have the same colour and to have the same dimensions. Each relation led to different quotient sets (they had even different number of drawers). However, some drawers from different quotients can have the same label, the same representative, and it is not possible to distinguish them when using the bracket notation (*[u]*). In this metaphor's language: "brackets (*[u]*) do not allow us to see what is inside the drawers ($u+V$)". I supported the explanation with a graphic representation on the blackboard made with colour chalks (Figure 12)

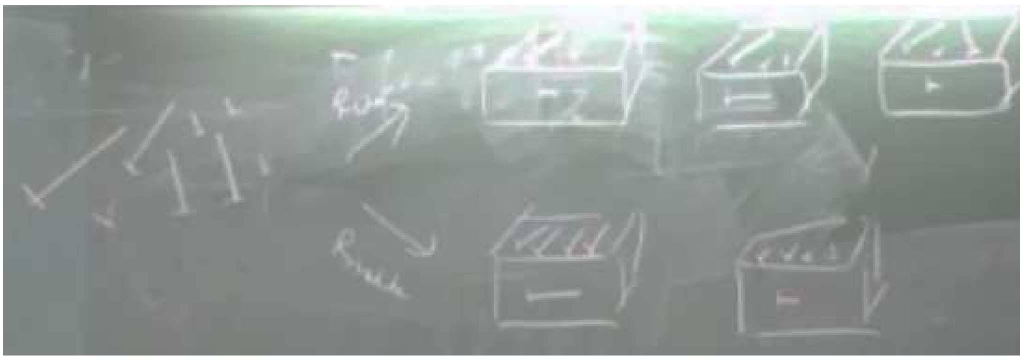


Figure 12: Graphic representation used to support the explanation of the nails metaphor

To sum up, in the following schema (Figure 13) are represented the contents involving QVS explained in the course (light blue boxes in the centre), the different visualizations used to explain these contents (dark blue boxes with bold letters), the kind of language or representation used to communicate these visualizations (green boxes) and the aim for their introduction in the different episodes described above. The more visualizations to explain a content are used, the bigger the box for this content is.

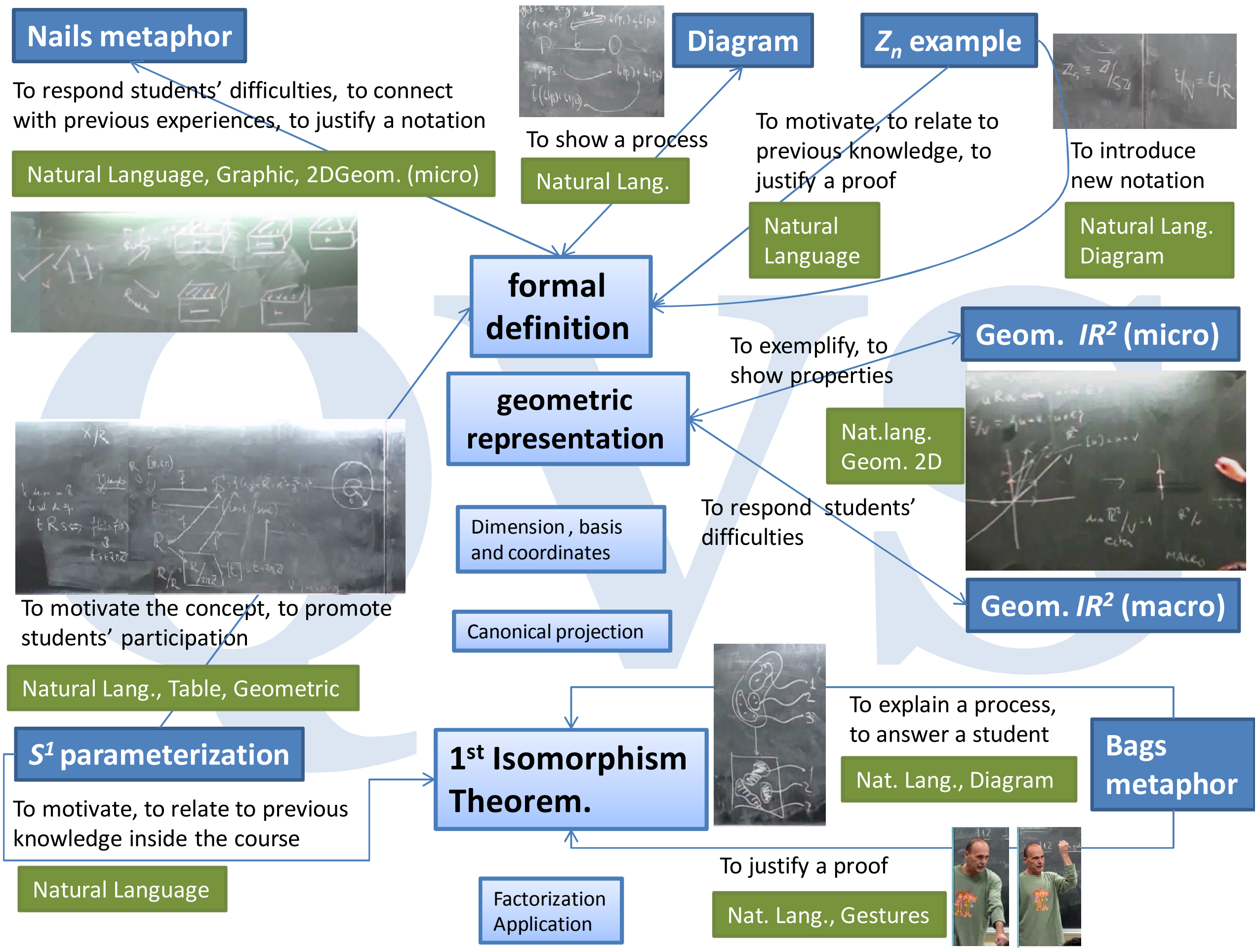


Figure 13: Schema of the visualizations used to make sense of QVS

Episode 4: Persistence of students' difficulties, the gap between theory and practice

In the previous episode, I pointed out how students' difficulties with a concept motivated different kind of visualizations. Now, I would like to reflect on the effect that these visualizations have in students' understanding of the concept. According to my view, explained in the introduction, about how learning is produced, the presence of more representations, examples and different points of view should favour a better understanding. In a conversation with a student repeating this year, he recognized that QVS "were easier to understand in the way they had been explained this year". However, the fact is that, despite all these explanations, most of the students still have difficulties with the concept and they did not demonstrate a good understanding of it when solving problems.

The persistence of these difficulties in the understanding of QVS could be explained by saying that this is a difficult concept from both, the cognitive and epistemological viewpoints (see Souto-Rubio & Gómez-Chacón (in press) for more details). Nevertheless, the effort made in the teaching of the concept was very valuable and it should have smoothed the path towards its understanding. At this point, I remembered a Chinese proverb: "*I hear and I forget; I see and I remember; I do and I understand*". Thus, I had a look at the fourteen problem sheets and at the two problems in each exam. In relation to QVS, I found nine problems in the sheets and one section in the first problem of each final exam (June and September). The schema in Figure 14 summarizes the following information about these problems: the contents about QVS related to each problem (light blue boxes in the centre); the kind of the representation used (green boxes) and the aim of the problem. Problems are denoted with two numbers (dark blue boxes) –the first number refers to the content unit and the second to the position in the sheet– and some key words briefly describe them.

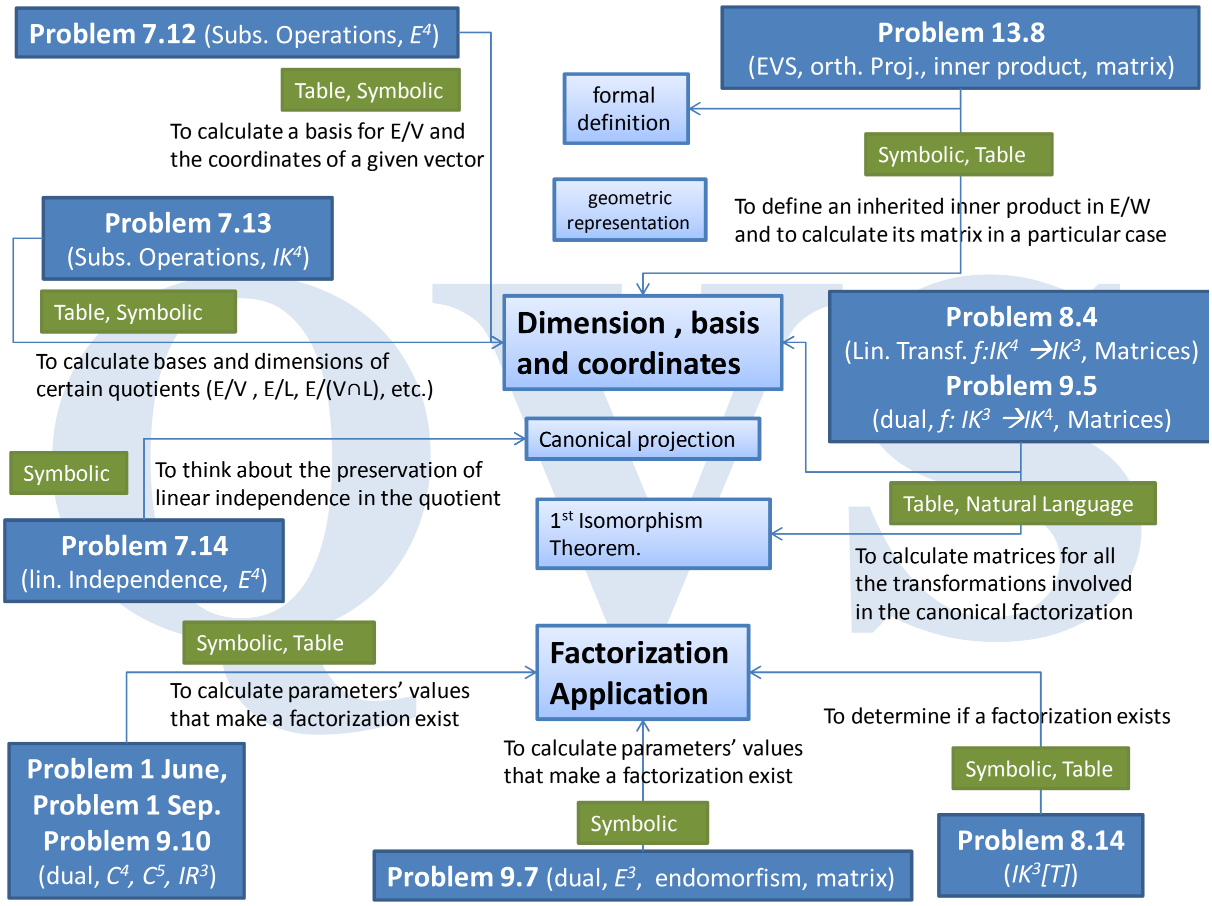


Figure 14: Schema of the problems found in worksheets and exams that concerns QVS

The contents most referred in the problems are represented in bigger boxes: Dimension, basis and coordinates in QVS and the factorization of functions as an application of the 1st Isomorphism Theorem. This set of contents hardly intersects the set of contents explained in class with more visualizations (see Figure 13). Moreover, all the problems are posed in a table or symbolic registers and most of them are routine and mechanic. I know from previous research (Souto-Rubio, 2009) that problems with these characteristics normally need only instrumental understanding and table or symbolic registers to be solved. Thus, there is a gap between theory and practice: when solving these problems, students can leave aside all the images, metaphors and different kind of representations used in class. In other words and recalling the Chinese proverb, what students have heard and seen in explanations does not correspond to that they have to do. Therefore, the teaching effort did, in order to make sense of QVS, may actually not help students to understand this concept.

In order to avoid this gap, I think that tasks of different nature should be included in the course: tasks that develop the instrumental character of visualization (it also exists); more constructive tasks which lead students to build rich mental images; more conceptual tasks, which need of these mental images, to be solved. How should these tasks be designed in order to help students to understand? How could these tasks be introduced in the LA course?

Episode 5: Visualization and assessment, a difficult couple.

This last question leads to reflect on the institutional point of view of visualization. We have already witnessed –in some of the episodes above– facts, attitudes and messages that give an idea of the status of visualization in this LA course. In this last episode, I will narrate two experiences, around assessment, that make me specially reflect about this issue.

Last year's class, the big failure

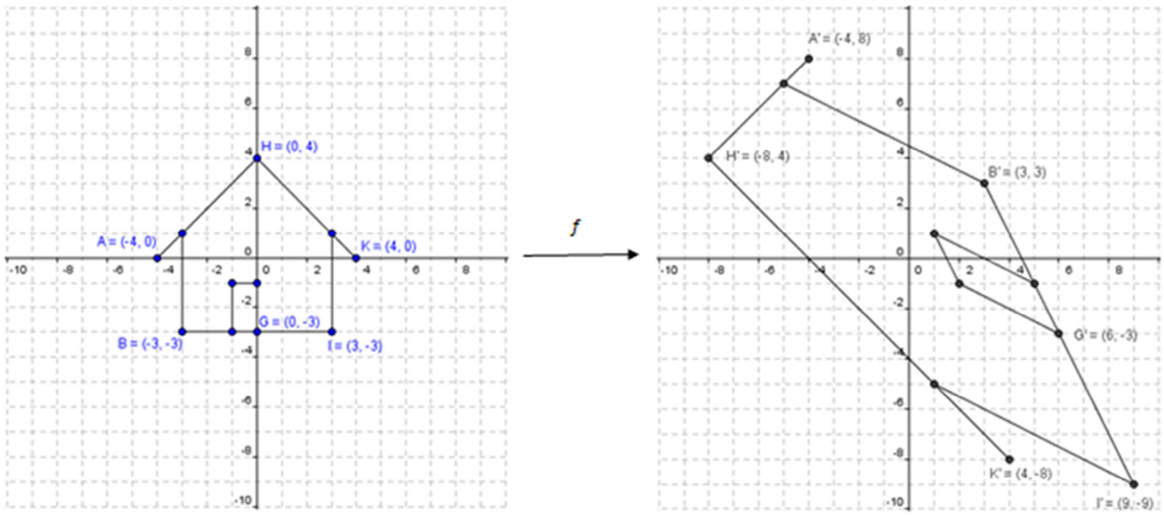
The end of the course was coming. Only one week left for the second partial examination. I decided to give a special last class with the aim to review the most important contents seen in the course and to answer students' queries before the exam. This class was open to students from other seminars groups and some came. I had noticed that the closer the exam was, the more time I spent at the blackboard and the less visualization I included in my seminars. Thus, I decided to design a special material for this last class. The material started with a task of representing six endomorphisms in IR^2either graphically (with a house, like in Figure 15) or algebraically. Next, there were some questions to reflect on these endomorphisms and their representations. Finally, a conceptual map (about different points of view for a symmetrical matrix) was included to be filled. I had planned to spend the first hour with this activity, since it could serve to review the main concepts in the course and to give a geometric interpretation of them. The second hour would be for students' queries.

However, this plan did not work. I was feeling nervous about the activity. I thought students may be only interested in solving their queries. Just one week left to the exam and such things were not going to be asked. Moreover, students were performing the activity much slower than I thought. At this point, I started to solve the problem on the blackboard. I found this harder than in the paper: I needed to place the exact points without a grid; I was too close to the blackboard to see the global picture, etc. The communication with students was also difficult, mainly with those who were not from my seminar group (fact that led me to think, once more, about the importance of the development of a language that enables to talk about visualization). Thus, after having made just four examples and motivated by a students' question, I decided to skip the activity and make a more traditional review. At the end, I recuperated the conceptual map's question from my activity. I thought it could be useful for summarizing what has been said. The result was that most of the students felt confused. They probably had never made such an activity and they did not know exactly what to do. After a brief comment about it, I felt defeated and left students to start asking their queries.

Attempting to change, visualization in the final exam

With this experience, I realized that assessment was a key factor to take into account in order to make changes in relation to the use of visualization in such LA course. It had strongly affected my way of teaching in seminars and my attitude to my own materials. Thus, when the lecturer wrote an email, asking for opinion about the second partial exam, I answered back encouraging him to introduce a more visual question. My proposal is shown in Figure 15. It was based on one of the questions I skipped from the activity described in the section above:

(6) **(0.6 points)** Let IR^2 be an euclidean space with the standard inner product and the following endomorphism $f: IR^2 \rightarrow IR^2$, which sends the first set of vectors into the second:



a) Diagonalize graphically the endomorphism, that is, find and draw a basis $B = \{u_1, u_2\}$ such as $M_f(B)$ is diagonal and write this matrix. Explain the procedure followed in order to find both, the basis and the matrix.

b) Write the matrix of f with respect to the standard basis $\mathcal{E}$ of IR^2 and find a matrix P invertible such as $P^{-1} \cdot M_f(\mathcal{E}) \cdot P$ is diagonal.

c) Is f a selfadjoint endomorphism? Why?

Figure 15: Question included in the final exam of June

The lecturer accepted to introduce it as an optative question, that allowed students to obtain 0.6 extra points over 10, and it would be as a part of the final exam (instead of the second partial). Twenty seven students took the final exam and only twelve tried to do something in this question. I note that some students, who passed the subject with partial exams, did not take the final exam and therefore they are not included in the following results (Figure 16). The first thing to be noticed, in these results, is that students who tried this activity did not perform it very successfully. Thus, if it would have been higher considered in the final mark it would be detrimental for students. Second, the section with worse results is the more visual one, which is the section a) about what I have called "graphic diagonalization" (see Figure 15). The section with better results is the more algebraic one, the section b), and it is remarkable that most of the students started with it instead of the a). Moreover, since both sections were asking the same in two different ways, this result is also saying that, in general, students are not able to connect the algebraic and the visual modes of thinking about diagonalization. From my point of view, this is worrying because it means that they do not completely understand this notion. Finally, the third question admitted both a visual and an algebraic answer. Only three students answered it. All of them did it correctly by giving the algebraic one. This result can be interpreted in two very different ways: either students naturally use the algebraic mode of thinking (they may be more used to it); or students do not use intentionally the visual mode of thinking because either they think they are not expected to do it or they may even have been penalized for using it before (Souto-Rubio, 2009).

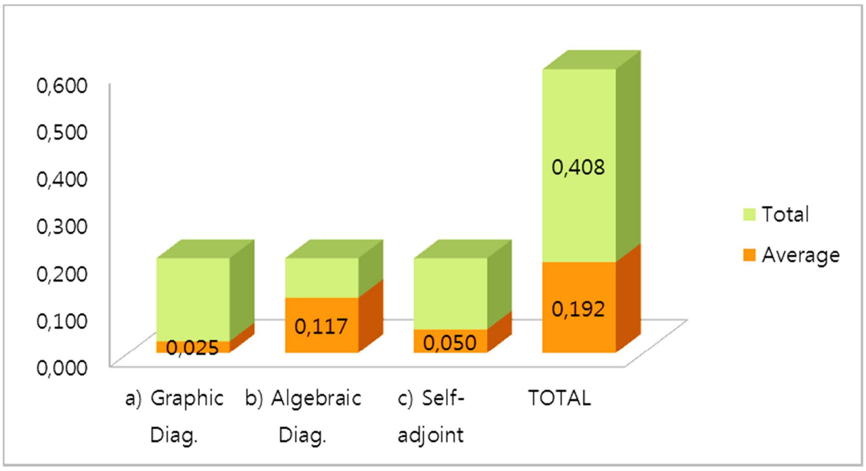


Figure 16: Results of the question included in June's final exam

DISCUSSION AND CONCLUSIONS

The episodes presented in this communication come from a concrete case of a LA course, teachers and students. Many of the aspects of the teaching of visualization may be particular of this context. However, as I pointed out in the introduction, I think these examples can serve to highlight important issues of visualization at university, or at least, at a LA course.

There is visualization at university level

Lectures with a proof- oriented focus, such as those observed in this LA course, are commonly criticized by its traditional style described as "definition- theorem- proof" (DTP) (Weber, 2004). Nonetheless, I agree with Weber (2004) when he claims that the teaching styles included "under the umbrella of traditional DTP instruction may vary widely" and that more empirical studies on what happen during lectures are needed (and I would add seminars too). Episodes 1 and 3 give evidences of such variety. Moreover, such kind of episodes helps to give a better characterization of visualization in LA. This characterization is represented in Figure 17, taking into account the three dimensions involved in Arcavi's definition: visualization as product (green and blue boxes), visualization as a process (orange boxes with capital letters); and visualization as an ability useful for several aims (white box).

However, these episodes were exceptional, since most of the explanations and problems used table and symbolic registers and followed a logical sequence. What was special in these cases? In the first one, the lecturer got stuck and needed to think aloud in class. In the second one, students' difficulties with a concept could have motivated a considerable amount of visualizations. Therefore, visualization seems to be an aid in situations of cognitive difficulty. If this is the case, why is there no more visualization in the rest of the course?

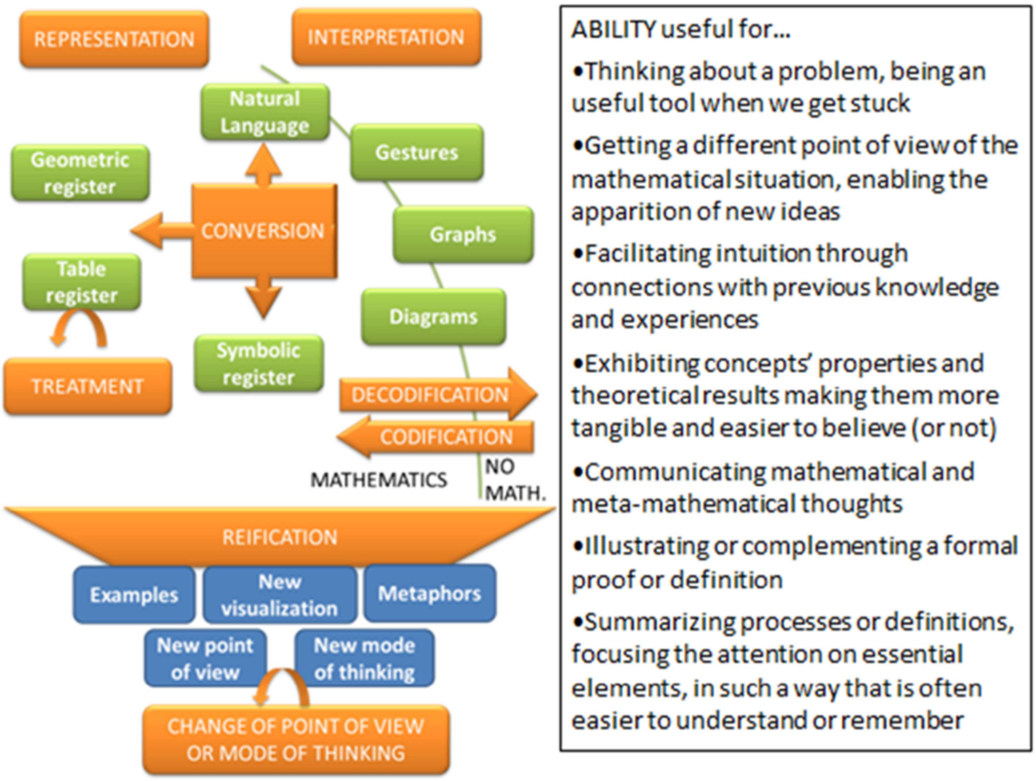


Figure 17: Schema of the characterization of visualization in LA

There are obstacles in the teaching of visualization

Authors like Eisenberg and Dreyfus (1991) and De Guzmán (2002) have pointed out some obstacles that make difficult the use of visualization in class. Some of these difficulties have been revisited through the episodes presented. I reformulate them as follows: visualization can lead to misinterpretations, visualization can be a limitation, visualization is not serious mathematics, visualization is not part of the curriculum, visualization is cognitively demanding, visualization needs of more knowledge and visualization is hard to communicate. I am going to comment the last two, since they may be less clear and I found them important.

<u>Visualization needs of more knowledge</u>- Visualizations in Episode 3 require more teacher knowledge (Ball, Thames, & Phelps, 2010) than a traditional class. The lecturer needed to know more than the common definitions, properties and proofs around QVS (Common Content Knowledge). First, he also needed to know different kind of representations, to handle the rules for their transformations, etc. (Specialized Content Knowledge). Second, he needed to know which visualizations are better in order to explain a particular property, or which example better motivates a concept, etc. (Knowledge for Content and Teaching). Similarly, students also need part of this knowledge in order to understand what the teacher is doing. For example, they cannot understand the micro geometric representation of QVS if they do not know how to add or rest two vectors geometrically.

Visualization is hard to communicate- Different kinds of knowledge need of different levels of communication. First, it is difficult to convert in a teaching sequence to motivate QVS a non-linear and maybe personal relation between the parameterization of the circumference and this concept (Episode 3). Second, as happened in Episode 2, it could be important to make explicit some issues about representations. This needs of a specific language, which leads to a paradox in relation to students' difficulties. Regardless, the experience in the last year's class highlighted the importance of the development of such a language to talk about visualization.

At this point, it might be argued: if it implies so many difficulties, why to insist on teaching visualization?

In defense of visualization

First, as I explained in the introduction, visualization is essential to reach AMT. Second, as some of the episodes evidenced, visualization offers interesting opportunities for teaching and learning that should be exploited (see Figure 13 and Figure 17). However, the main reason to pay attention to visualization is that it is unavoidable, even in a subject such as LA. As the episodes showed, there is visualization, whether we want to or not. Geometry is one of the bases for the historical development of this subject (Dorier, 2000) but it is also in the core of other subjects. Diagrams appear to represent transformations and relations among them and help us to think more abstractly. Metaphors or graphs could be a good aid to answer students –some could be visualizers (Presmeg, 2006)– who ask for different explanations, etc. Thus, for me, the question is not if teaching or not teaching visualization, but how to handle visualization when teaching? And the answer is clear to me: it is important not to leave the whole responsibility only to students; on the contrary, the course should pay explicit attention to it. This is the only way to break the vicious circle that exists around visualization. I believe that the more you practice something, the easier it results. If students were used to the languages and characteristics of visualization before, it would take less time to explain a new one, its understanding and communication would be easier, possible misinterpretations of images would be more likely to be noticed by students, etc. Thus, visualization would become the useful and helpful tool for understanding that it could be.

In order to achieve this goal, I think three actions are needed. First, it is important to improve the teacher knowledge about visualization, thus more visualization and more conversations on visualizations could emerge in class. A systematization of this knowledge, either from a general approach (Figure 17) or from a concept's perspective (Figure 13), could benefit this improvement. Subsequently, the challenge will be how to transfer this knowledge to university teachers. Second, it is important, not only to expose students to visualization, but also to make them to practice and reflect on it. How to design such kind of tasks is still an open question. Third, none of the both previous actions will be effective if they are not accompanied by the legitimization of the visualization. This involves the institutional dimension, introducing changes in the curriculum and assessment, but also the personal dimension, promoting the individual reflection on this issue. For this reason, I consider very important to continue promoting this kind of debates among the community of mathematics educators. With this aim, and to show that there is still much to be done, I have presented here these episodes.

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References

Arcavi, A. (2003). The Role of Visual Representations in the Learning of Mathematics, *Educational Studies in Mathematics, 52*(3), 215-24.

Ball, D. L., Thames, M. H., & Phelps, G. (2010). Content Knowledge for Teaching: What Makes it Special? *Journal of Teacher Education*, *59*(5), 389 –407.

De Guzmán, M. (2002). The Role of Visualization in the Teaching and Learning of Mathematical Analysis. *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*. Univ. of Crete, Greece.

De Vleeschouwer, M. & Gueudet, G. (2011). Secondary- Tertiary transition and evolutions of didactic contract: the example of duality in Linear Algebra. In M. Pytlak, E. Swoboda & T. Rowland (eds.) *Proceedings of CERME 7,* 1359-1368,Univ. of Rzeszów, Poland.

Dorier, J.-L (2000). *On the teaching of linear algebra*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt y M. Santos (Eds.), *Proceedings of the 21st North American PME Conference, 1*, 3-26.

Eisenberg, T. & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmemann, W. & Cunningham (Eds.) *Visualization in Teaching and Learning Mathematics, MAA Notes, 19*, 25-37. Washington, D.C

Fernando, J. F., Gamboa, J. M., & Ruiz, J. M. (2010). *Álgebra Lineal* (Vols. 1-2). Madrid: Sanz y Torres.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics, *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future. PME 1976-2006*. Sense Publishers, 205-235

Souto-Rubio, B. (2009). *Visualización en matemáticas. Un estudio exploratorio con estudiantes del primer curso de Matemáticas*. (Master Dissertation), UCM. Retrieved from www.mat.ucm.es/vdrmat/TI-08-09/trabajo-master-curso-2008-09-blanca-souto.pdf

Souto-Rubio, B. & Gómez-Chacón, I. (in press). "Ways of looking" at quotient spaces in Linear Algebra. How to go beyond the modern definition? *Proceedings of the 36th PME,* Taipei, Taiwan: PME

Tall, D. (1991). *Advanced mathematical thinking*. Dordrecht : Kluwer Academic Publishers.

Weber, K. (2004). Traditional Instruction in Advanced Mathematics Courses: a Case Study of One Professor's Lectures and Proofs in an Introductory Real Analysis Course. *The Journal of Mathematical Behavior*, *23*(2), 115–133.