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TEACHING MATHEMATICAL MODELING IN SCHOOL MATHEMATICS

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Modeling is a cyclical process of creating and modifying models of empirical situations to understand them better and improve decisions. The role of modeling and teaching mathematical modeling in school mathematics has received increasing attention as generating authentic learning and revealing the ways of thinking that produced it. In this paper and interactive lecture session, we will review a subset of the related literature, discuss benefits and challenges in teaching and learning mathematical modeling, and share our attempts to improve traditional textbook problems so that they can become more authentic modeling activities and implications for instruction and assessment as well as for research.

Models, representations, modeling activities, teaching modeling

MODELS AND MATHEMATICAL MODELING

The terms *models* and *modeling* are used variously in the literature. Lesh and Doerr (2003) view models as conceptual systems that are expressed for some specific purpose using some (and usually several) representational media and modeling as a process of developing representational descriptions for specific purposes in specific situations. That is, models are purposeful interpretations, descriptions, explanations or symbols that are used to construct, manipulate, or predict the systems that are being modeled. Mathematical models are used to interpret real-world situations or non-mathematical situations in mathematical formats (English, Fox, & Watters, 2005). For example, graphs, tables and equations are used to model and make intelligible interpretations of complex relationships among various phenomena. Because the need to develop models (or other conceptual tools) seldom arises unless part of the goal includes shareability (with others) and reusability (in other situations), Greer (1997) sees that modeling is inherently a social enterprise and that mathematical modeling is thus seen as building a way of making sense of our physical and social world and mathematics as a set of abstract, formal structure by negotiating the interchange between the world and its mathematical counterparts. Lesh and Lehrer (2003) assure that the distinction between the model and the world is not merely a matter of identifying the right symbol-referent matches; rather it depends intimately on the accumulation of experience and its symbolic representations over time. Models bootstrap the world and the world pushes back toward revision of one's models. This suggests that models are inherently provisional and modeling usually involves a series iterative testing and revision cycles– even though they may endure for longer periods of time, and even though they generally are intended to be shareable and reusable in a variety of structurally similar situations. In a modeling cycle, competing interpretations are gradually sorted out or integrated or both– and in which promising trial descriptions and explanations are gradually revised, refined, or rejected.

Information processing in problem solving and modeling

Traditionally problem solving is understood as the search for a powerful procedure that links well-specified givens to well-specified goals. In this view, applied problem solving is treated as a special case of traditional problem solving. Lesh and his collegues (2000) offer a modeling perspective of problem-solving in that a modeling perspective views the interpretation of the givens and goals as the major challenge, making selection and application of procedures, a cyclical process integrated into the interpretation phases of problem solving. Rather than using a fixed interpretation or procedure to process data, students are operating primarily on their own interpretations of both the goal and the given information. In this alternative view, illustrated in Figure 1, traditional problem solving is treated as a special case of modeling activities (p. 603).

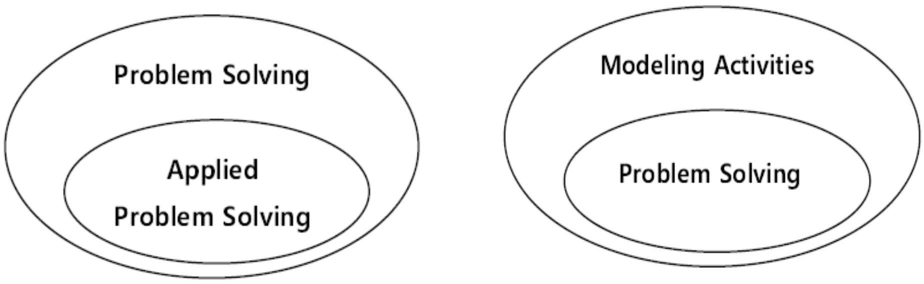


Figure 1. A modeling perspective of problem solving

PROCESS OF MODELING

From a review of the literature (e.g., Abrams, 2001; Dossey, McCrone, Giordano, & Weir, 2002; Kang, 2010; Meyer, 1984; NCTM, 1989; Swetz & Hartzler, 1991), aspects of the modeling process can be characterized as (1) examining the situation and setting up the goals to be accomplished, (2) identifying variables in the situation and selecting those that represent essential features, (3) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (4) analyzing and performing operations on these relationships to draw conclusions; if the implementation of the performed operations cannot be complete, then revise the selection of the variables used to formulate the model, (5) interpreting the results of the mathematics in terms of the original situation, (6) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, and (7) applying the model to similar situations for evaluation and refinement. The process of developing sufficiently useful models for a specific purpose usually involves a series of iterative testing and revision cycles. Also, choices, assumptions, and approximations are present throughout the modeling cycle. This characterization of the modeling process is illustrated in Figure 2.

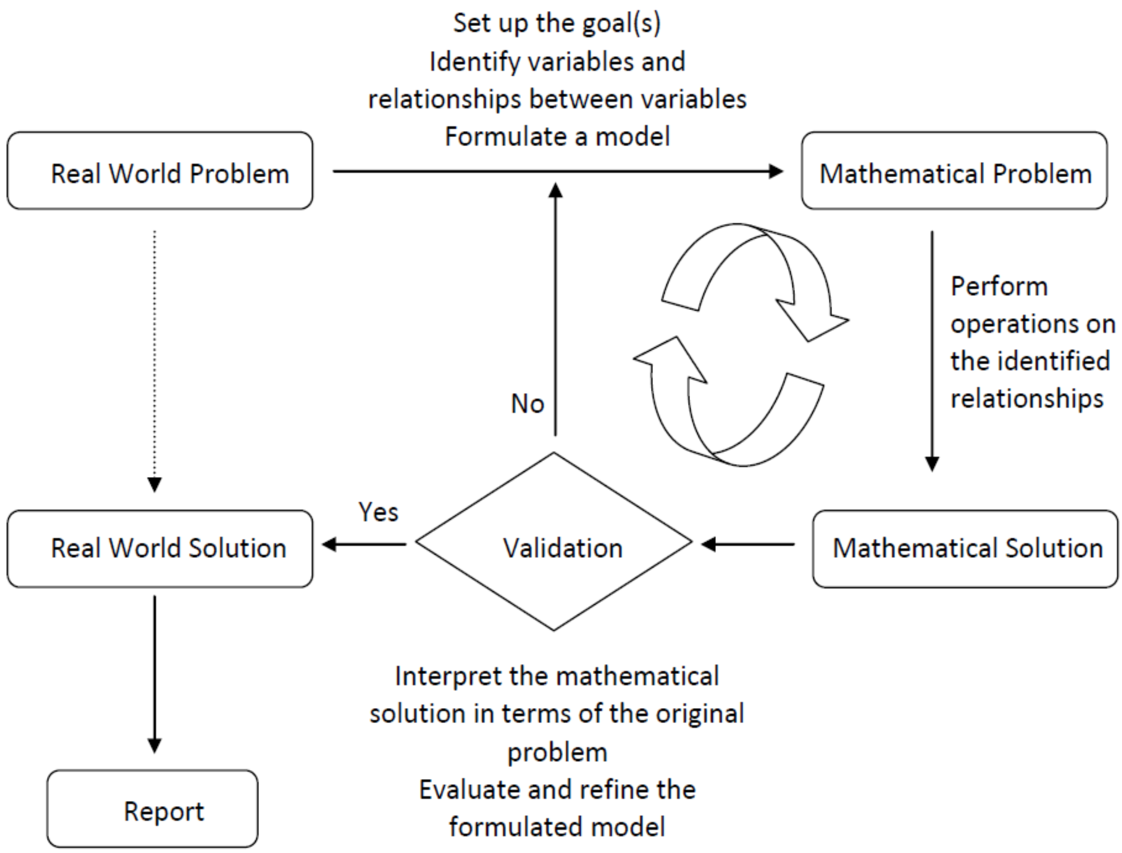


Figure 2. A synthesized view of the modeling process from existing literature

EVALUATION OF MATHEMATICAL MODELS

As much as all steps that are taken during a modeling cycle are important, it makes an important difference whether it is a good model or a bad one if a mathematical model is used to improve decisions. The model one has available may not be good enough to use. Or there may be more efficient models available for use in a given situation. Meyer (1984) views evaluation to be in the forefront of the thinking about mathematical modeling and suggests six principles to go by in taking the measure of a model: Accuracy, descriptive realism, precision, robustness, generality, and fruitfulness.

Definitions of the six principles

A model is said to be:

- *Accurate* if the output of the model (the answers it gives) is correct or very near to correct.
- *Descriptively realistic* if it is based on assumptions which are correct.
- *Precise* if its predictions are definite numbers (or other definite kinds of mathematical entities: functions, geometric figures, etc.). By contrast, if a model's predictions is a range of numbers (or a set of functions, a set of figures, etc.), the model is imprecise.
- *Robust* if it is relatively immune to errors in the input data.
- *General* if it applies to a wide variety of situations.
- *Fruitful* if its conclusions are useful or it inspires or points the way to other models.

To illustrate the characteristics of some of these principles, Meyer (1984) provides examples using a college enrollment trajectory problem as follows:

<u>Model 1</u>
Suppose this year there are 10 million people in the entire population who are in the age category from 18 to 22, which forms the bulk of the college students population. Also suppose the number of college students is 5 million this year. From these figures the following equation can be obtained:

$$E = 0.5P$$

, where E = number of students enrolled and P = population aged 18 through 22 years. This equation is based upon the following two assumptions:

A1. Each college student is in the age category 18-22.
A2. In the 18-22 age group, one out of every two is enrolled in college.

If the Bureau of the Census determines that next year there will be 11 million people in the age category 18-22, then:

$$\begin{aligned} E &= (0.5)(11{,}000{,}000) \\ &= 5{,}500{,}000 \end{aligned}$$

If it turns out next year that there are really are 5,500,000 students enrolled in college (or pretty near that number), this model has the characteristics of accuracy. Model 1 has been tailor-made to be correct for this year, but we won't know until next year whether 5,500,000 students it predicts for next year is correct or nearly correct. This is a common stumbling block of nearly all models. A genuine evaluation of the accuracy of this model requires observation over a number of years, which is not realistic in this given situation looking for a model to be used for next year.

<u>Model 2</u>
Assumption A1, made in formulating Model 1, is not correct. Although the bulk of college students come from the age category 18-22, there are many older college students and some

younger ones. With available statistics for this year, the following assumptions might be more reasonable:

A3. Students in college can be divided into three age categories.
   (a) Aged 18-22
   (b) Aged 23 and over
   (c) Aged under 18

A4. In each age category of the population, a certain percent is enrolled in college.
   (a) In the 18-22 age category, 30 percent
   (b) In the 23-and over category, 3 percent
   (c) In the under- 18 category, 1 percent

If P_a, P_b, and P_c are used to denote the sizes of these age categories respectively, then this assumption produces:

$$E = 0.30P_a + 0.03P_b + 0.01P_c$$

The accuracy of Model 2 cannot be determined, in regard to the prediction for next year, any better than that of Model 1. But because Model 2 is more descriptively realistic than model 1, one should be inclined to trust it more. In this case realism serves as a sort of a “stand-in” for accuracy.

<u>Model 3</u>

For the precision principle, Model 3 is developed from Model 1 by making assumption A5 instead of A2.

A1. Each college student is in the age category 18-22.

A5.The fraction of the 18-22 age group which is enrolled in college in any particular year is always between 0.46 and 0.5 (in the past six years supposedly).

Using assumptions A1 and A5, the following inequality is obtained:

$$(0.46)(11{,}000{,}000) \le E \le (0.5)(11{,}000{,}000)$$
$$5{,}060{,}000 \le E \le 5{,}500{,}000$$

When Model 1 and Model 3 are compared, it appears that, if Model 1 is chosen for precision, it is done so at the expense of descriptive realism.

<u>Model 4</u>

To illustrate the concept of generality, Model 4 is based upon the following two assumptions:

A1. All college students are in the age group 18 through 22.

A2′. Each individual college will have its enrollment expand or decline by the same ratio, that ratio being the ratio of next year’s to this year‘s population in the 18-22 age category.

With assumption A2′, any individual college can use the same model. In this reason Model 4 appears to be more general than Model 1.

The fruitfulness principle can often be evaluated often before any of the details of the model. For example, the discussed college enrollment models could be useful planning tools for a governmental agency administrating a student loan program.

MODELING IN SCHOOL MATHEMATICS

A sense of connection among representations for the concept of a mathematical idea is enhanced through a variety of experiences with applied problem settings that allow students to describe, explain, manipulate, and predict a wide range of problem situations. Many current reform recommendations value the mathematical modeling of phenomena as one of the most powerful uses of mathematics, and emphasize modeling and contextualized problem solving across the mathematics curriculum (NCTM, 1989, 2000). In *Principles and Standards for School Mathematics* (2000) it is recommended that high school students should be able to develop, identify and find the best fitting model for real-world data by drawing on their own knowledge of ideas and methods that they have developed. They should also be able to explain why that model seems reasonable. Because a teacher makes choices of which problems to engage students in, a teacher's capacity to use and appreciate the importance of the concept in varying contexts is critical.

In the US' recently developed mathematics curriculum standards, the Common Core State Standards (2010), modeling is not only in the standards for Mathematical Practice, which describe habits of mind, or productive ways of thinking that support the learning and application of formal mathematics, we want to develop in our students, but also is a content standard. In the Mathematical Content Standards, modeling is defined as: "Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (p. 72).

However, research indicates that most school problems being posed to students do not involve the students in creating, modifying or extending systems of representations for meaningful problem situations (e.g., Doerr, 1995). Even in solving typical textbook "word problems," students generally try to make meaning out of questions that are often simply a thin layer of words disguising an already carefully quantified situation. Most textbook modeling problems are that students must make symbolic descriptions of the situations being modeled (Lesh & Lehrer, 2003).

An important aspect of modeling problems is diagnosing the given situation. In a modeling activity, students are required to develop, extend, and/or revise a model that is useful for accomplishing some specific purpose. What needs to be produced is a model to make sense of the situation for which students' currently available interpretations of the givens and goals lack enough detail, elaboration, precision, and development (Zawojewski & Lesh, 2003). In contrast with many classroom mathematics problems, modeling activities promote problem posing as well as problems solving primarily because they evoke repeated asking of questions and posing of conjectures (Brown & Walter, 2005). As in real-life situations, modeling activities often comprise information that might be incomplete, ambiguous, or undefined, with too much or too little data. Numerous questions and problems arise as students try to make sense of this information, elicit and work with the embedded mathematical ideas, and modify and refine their models.

Levels of modeling problems

Based on the completeness and ambiguity of the information composing a problem, modeling problems can be categorized into three levels with Level 3 being the most authentic type, modified from the work of Galbraith and Clatworthy (1990), as follows:

- Level 1: Problems at this level are already carefully defined so there is little ambiguity about what needs to be done and how to do it. They contain all the information necessary to formulate a model. They either specifically call for a certain procedure to be used or its use is evident on prior instruction or placement of the task. Students are expected to search for the needed information that is hidden in the problem, recall the (implicitly or explicitly) called for procedure, and carry it out correctly. There is no need to collect additional data to formulate a model.
- Level 2: Problems at this level still have a little ambiguity about what needs to be done and how to do it. However, they do not provide all the information needed to successfully complete the task. Although students may be given a direction of what data is needed, they need to devise a meaningful way to gather the needed data and test if the gathered data would produce a reasonable answers.
- Level 3: Problems at this level are comprised of information that is open-ended, incomplete and/or redundant. There is not a well-rehearsed approach or pathway explicitly suggested by the task. Students are expected to analyze the task to find what needs to be done and actively examine tasks constraints that may limit or suggest possible solution strategies and solutions.

TEACHING MODELING IN SCHOOL MATHEMATICS

There is a distinct difference between teaching mathematical *models* and teaching mathematical *modeling*. Whereas in the former the emphasis is on the product (the models), in mathematical modeling, the focus is on the process of arriving at a suitable representation of the physical, real world situation. One begins with a real problem and progresses step by step towards possible solutions. Related literature (e.g., Lesh & Lehrer, 2003; Lesh, Zawojewski, & Carmona, 2003; Schorr & Lesh, 2003) shows that modeling activities often lead to remarkable mathematical achievements by students formerly judged to be too young or lacking in ability for such sophisticated and powerful forms of mathematical thinking. They often create productive interdisciplinary niches for mathematical thinking, learning and problem solving that involve simulations of similar situations that occur when mathematics is useful beyond school. A selection of the research studies on modeling instruction is followed.

In an exploratory teaching experiment study conducted by two Belgian researchers, Verschaffel and De Corte (1997), they tested and provided support for the hypothesis that it is feasible to develop in 10- and 11-year-old students a disposition toward (more) realistic mathematical modeling by immersing them in a classroom culture in which work problems are conceived as exercises in mathematical modeling, with a focus on the assumptions and the appropriateness of the model underlying any proposed solution. Cheng (2001) presented examples of how the process of mathematical modeling could be introduced in the Singaporean secondary classroom using basic mathematical ideas and concepts and found many challenging and exciting skills emerging in developing models which have often been ignored in traditional school mathematics. Kim and her colleagues (2010) viewed a mathematical modeling problem as a non-routine problem that involves real-world applications and mathematical concepts that lead to the creation of a mathematical model. In their study involving Korean sixth graders, modeling problems involving a variety of mathematical ideas such as fractions, shapes, measurement, probability and statistics were created and used in instruction multiple times throughout one entire semester. These problems focused on searching for patterns and developing problem solving skills. On an assessment item sketching the planets' orbits around the sun, students were able to meaningfully relate concepts such as rate, ratio and proportion and make a relevant use of them to the problem. Other studies conducted at the elementary level include third graders' learning of changes and rates of changes (English et al, 2005), fifth graders' learning of data analysis and interpretation (English, 2003), and third and sixth graders' information processing and decision making skills (Doerr & English, 2003). Lee and Kim (2004) investigated eighth grade students' strategies used in modeling activities involving linear relationships and found the level of sophistication in the students' solution strategies was improved. Shin and Kim (2011) examined the effect of modeling activities on a group of middle school students' uses and conceptions of graphical and symbolic representations of the absolute value. In Lee's study (2006) involving high school students, modeling activities appeared to be helpful in improving problem solving skills.

In summary, it is suggested that models and modeling practices can be introduced to schools and accomplished meaningfully. In addition, when assessment recognizes the importance of a broader range of mathematical understandings, more students tend to emerge as having exceptional potential. However, this review of the literature has revealed an interesting observation, which is that the modeling problems and activities used in those studies were tasks that were carefully developed for research. Those problems were not presented in their mathematics textbooks. This creates a complex situation in which classroom teachers need to search for or develop such activities themselves if they want to use them in their instruction. Although a teacher's role as a researcher should not be overlooked, such a demerit discourages or creates resistance towards using modeling problems in their classrooms. To address this, we present our attempts to transform traditional textbook problems into more authentic modeling problems in two examples: one from a sixth grade textbook and the other from an eighth grade textbook. In each example the original textbook problem is anlayzed and is presented with a possible improvement and a description of the classroom implementation of the modeling version of the textbook problem.

A sixth grade example

<u>Textbook problem</u>

The following table presents the suggested traveling time between Pusan and Seoul, Korea for each of the three transportations. There are some blanks in the table. Complete the table using the clues provided below.

Transportation	Distance it travels in one minute	Traveling time	Traveling distance
High-Speed Train	3.2km	( ) hours ( ) minutes	416km
Train	( ) km	( ) hours ( ) minutes	432km
Car	( ) km	5 hours	( )km

Clues:

- If you travel by high-speed train, it takes you 20 minutes less than half of the time needed by car.
- A train travels 50% of the distance a high-speed train travels in one minute.
- The traveling distance ratio of train and car is 36 to 35.

(Source: Mathematics 6-2 (2011) p.129)

Analysis

It is easy to think of this problem as a real-life problem in the sense that students use mathematics to deal with a situation outside the classroom. But, in a real situation, it would not be sensible for someone to want to know the answer to the questions as they are stated. In fact, computing the traveling time using the intended rule depends on ignoring common sense and/or practical experience. In reality, the traveling time depends on road conditions or the weather, on how many stops you (or the train) make and how long, on what route you take and how fast you drive when traveling by car, and so on.

In this problem, the problem (not students) determines what to investigate and identifies the variables that need to be used to formulate a mathematical model. Although it does not specifically call for a certain procedure to be used, the kind of operations that need to be performed is evident based on the set of the identified variables in the problem and the given clues. In this problem, the "math answer" obtained by performing the operations is an end in itself. In an authentic modeling problem, the math answer is a means to an end, that is, a tool for informing actions, decisions, and judgments. However, little decision-making is required of students working on this problem. In addition, the criteria for judging the quality of the answers are not implicit to the situation, and the solutions are judged according to whether students conform to the calculation that is expected (students will have used similar methods), rather than according to whether they succeed in any practical or meaningful sense of the world.

A possible improvement

You have relatives who live in Pusan and your family is planning to visit them for a vacation, and you are taking a car. What time should you and your family leave home to make it in time to the dinner that is planned at 6pm at their house in Pusan? Explain your reasoning.

<u>Classroom implementation</u>

When this open-ended, unstructured problem was presented to a group of nine sixth grade students, the students' immediate reaction was that this problem wouldn't be solvable because there wasn't sufficient information in the problem to begin with. The teacher then encouraged students to recall a car trip their family made recently and talk about many things about their trip such as what time they left home and arrived at the designation, whether they stopped in to rest, who was driving and how fast, and how the traffic was. The teacher did not give any plan or procedure for the students to use, but was hoping for them to see the need to determine important factors in which the situation might depend upon and come up with a reasonable plan to gather the information they need. Students were encouraged to talk with their classmates to formulate and solidify their ideas and solutions. However, they were expected to come up with a solution on their own, partly because of the small number of students participating in this activity. Two class periods, 50 minutes each, were spent on this activity with outside classroom work time required. Each student wrote a report of and presented their results.

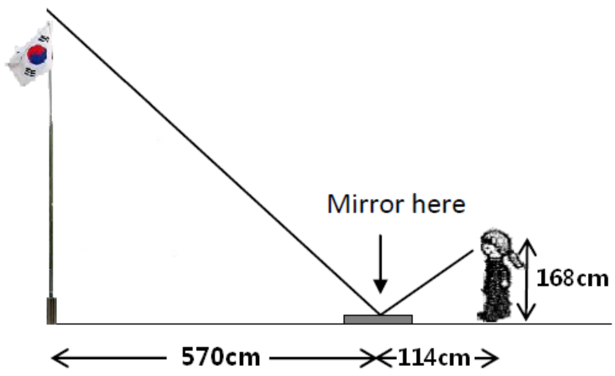
When the students' modeling activity process and product were assessed, based on our (modified) level analysis guide, three out of the nine participating students appeared to have the ability to complete Level 3 modeling tasks. These students were able to specify the modified problem clearly and formulate a meaningful model (a reasonable plan about traveling time in this case) by choosing important variables and finding their relative importance and relations with little or no guidance from the teacher. Another four students exhibited some troubles with identifying important factors– some of the factors they considered were irrelevant, incomplete and/or redundant. The teacher had to make the problem more transparent to students regarding factors and suggest possible ways to gather data for the chosen factors. In doing so, the modified problem was lowered to a Level 2 task. These students however could put the information together to come up with a reasonable plan in the end. For the remaining two students, the modified problem was too unstructured for them to think about the situation reasonably. They were searching for a certain procedure to be used. They were unsuccessful in analyzing the situation constraints that may limit or suggest possible solutions. After their initial attempts failed, they refused to engage in the task. They could complete the problem only after they were given, by the teacher, all the information about what needs to be done and how it can be done. It is not confident to say that these students would be able to do problems beyond Level 1.

An eighth grade example

<u>Textbook problem</u>

> The following picture shows a method for measuring the height of a flagstaff using a flat mirror. Measurements such as the distances from the flagstaff and the mirror, from the mirror and the person who's measuring, and from the person's eye level to the ground are provided. Determine the height of the flagstaff using similar triangles. (We will assume that the angle of incidence is equal to the angle of

reflection at the point where the mirror is located.)



(Source: Middle School Mathematics 2 (2011) p. 204)

Analysis

When working with traditional curricular activities and assessments like this problem, teachers frequently find themselves trying to figure out what a particular solution tells the student did or did not do the problem correctly. In most cases, they can say nothing more than anything about what the student does know or what they can do. When working through a modeling activity, students are asked to produce descriptions, explanations, procedures, and constructions. In the original textbook problem use of a certain procedure (that is, similar triangles) is specifically called for and the information needed to carry out the procedure is already provided. So, the focus is on recalling a previously learned procedure to obtain an expected answer, not their thought process. When examining solutions to this problem, it would not be easy to observe how students are thinking about a mathematical situation and to find out what they can do. In addition, this problem does not request students to assess the quality of the model being used. That is, students do not need to determine whether the mirror method is an effective way to measure the height of a flagstaff.

A possible improvement

Our school needs to replace the two flagstaffs in the playground because they are aging. We need your help to measure the length of each flagstaff (They are different in length) so that we can give the measurements to the company that makes flagstaffs. Please provide a manual that tells us how to measure the length of any flagstaff.

Classroom implementation

Analysis of students' manuals and their discussions while writing the manuals revealed new information about students' proportional reasoning and their abilities to use similar triangles in various situations, as well as valuable insights regarding the effectiveness and usefulness of this activity. Select strategies employed by students are presented, in Figure 3, to illustrate the ways in which this activity revealed their thought processes. Consider the following strategies (in the form of rough sketches) that were produced by 5 groups (Groups 2, 4, 5, 6 and 7) for the modified problem. Originally students wrote their manuals on the chalkboard, with sketches of the methods they used to develop the manuals. However, only students' sketches are displayed in Figure 3. Their manuals were removed from the pictures because they were written in Korean.

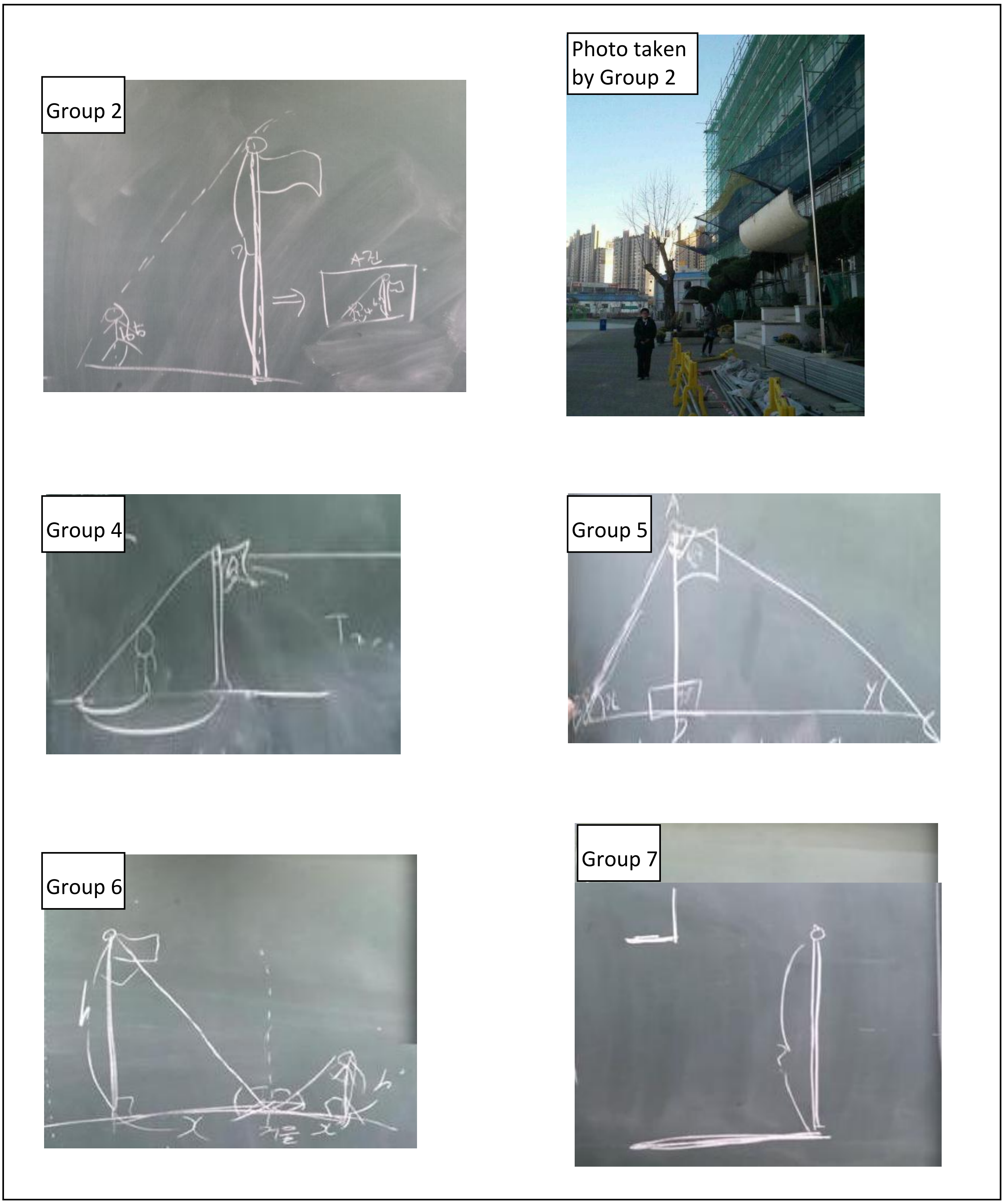


Figure 3. Select groups' strategies

These sketches provide interesting insights into how these groups thought about the situation. Group 2 took a photograph of the flagstaff with a person standing before the flagstaff and their plan was to use a proportion relating the measurements from the photo and the actual measurements. Group 4 had a similar start as Group 2 by showing a person at some distance from the flagstaff. This group then constructed two similar triangles with: a straight line that connects the tip of the flagstaff, the tip of the head of the person and the ground, and; the distances from the point on the ground to the person and to the flagstaff. Group 5 also used the concept of similar triangles but in a different way. Their idea was to construct two right triangles that would be similar. This group suggested using a laser pointer to help determine the two end points of the largest triangle. Group 6, another group that used the similar triangles idea, came up with the method illustrated in the original textbook problem. Group 7 suggested using the shadows of the flagstaff and a stick and setting up a proportion relating the lengths of their shadows and their actual measures. While the methods used by Groups 2 and 7 could incorporate the idea of similar triangles, it was not mentioned in their sketches nor manuals.

After sharing their methods, they started talking about the logistics of actually carrying out their methods such as measuring the length of the flagstaff's shadow and locating a point at an angle from the ground. Students needed to think about what can be done and might not with the resource they had or had access to. This process of negotiating one final product encouraged students to justify and verbalize their responses (e.g., the reason for their sketch) and to compare their methods with those of other groups. These negotiations provided additional insights into each student's concept development and reasoning patterns.

CONCLUDING REMARKS

In this paper a selected literature on modeling and teaching modeling in school mathematics was reviewed, our modified guide that can be used for analyzing levels of modeling problems was discussed, and lastly attempts to improve modeling problems was presented. Our case study examples are meant to illustrate the process of using textbook problems to create a more authentic situation where students are engaged in an iterative practice of identifying variables, formulating a model, interpreting the result, and validating the model. The teacher's role is critical to help students express-test-revise their thinking in productive direction. Therefore, the results observed in our case study may have been influenced by the teachers' conceptions of modeling activities and abilities to guide their students' thinking towards the goals of the activities. This necessitates further studies focusing on the nature of teachers' developing knowledge, abilities, and on-the-job classroom-based professional development of teachers where modeling activities for students provide contexts in which teachers' teaching experiences become productive learning experiences to support teacher development. Because modeling practices are thought-revealing activities, they are useful for instruction and assessment as well as for research. When teachers observe their students working on such tasks and when they examine the results that their students produce, they are able to gather useful information about their students' conceptual strengths and weaknesses and become more familiar with their students' ways of thinking, so that teaching can be more effective. With this reason, it seems sensible to encourage the future teachers to use this type of activities to gain access to the developing understandings and reasoning patterns of their students. The implications for future teachers' teaching practices may be significant, because they are being introduced to an efficient and powerful means of gaining regular access to their students' thinking.

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