THE SOCIAL DIMENSION OF ARGUMENTATION AND PROOF IN MATHEMATICS CLASSROOMS

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Argumentation and proof have received increasing attention in mathematics education in recent years. However, social dimensions of proof and argumentation have not been emphasised. These include the social, argumentational dimension of proving in academic mathematical practice, the social process that transforms mathematical proving and argumentation in the context of school mathematics classrooms, and the interplay between the socio-cultural backgrounds of students and the social expectations around proving and argumentation in schools. Without adequate attention to these dimensions there is a danger that classroom argumentation could become a social filter, emphasising students' preexisting advantages and disadvantages. Attention to the structures of argumentations in mathematics classrooms combined with research on social dimensions can provide a better understanding of the filtering effect of argumentation in classrooms. This could provide a basis for minimising unexpected and undesirable consequences of a greater focus on argumentation and proof.

Argumentation, proof, mathematics classrooms, equity, socio-cultural background

INTRODUCTION

Argumentation and proof have received increasing attention in mathematics education research in recent years (Hanna & de Villiers 2012, Mariotti 2006). There have also been calls for a new focus on argumentation and proof in mathematics classrooms from the beginning of schooling on (e.g., NCTM 2000). In research a range of conclusions have been reached on the role and relationship of argumentation and proof for the learning of mathematics. But the nature of the argumentation, the nature of proof and the nature of the relationship between them is far from clear. Beginning with an examination of mathematical practice, including social dimensions, is helpful in bringing some clarity to this situation. In this paper I will discuss the 'social dimension of argumentation and proof' in three different ways. First, the social, argumentational dimension of proving in academic mathematical practice. Second, the social process that transforms mathematical proving and argumentation in the context of school mathematics classrooms. And third, the interplay between the socio-cultural backgrounds of students and the social expectations around proving and argumentation in schools.

MATHEMATICAL PROOFS AS ARGUMENTS

Reuben Hersh is one of those who bases his reflections on the investigation of mathematical practice. He characterizes mathematical proofs as follows:

> Mathematical discovery rests on a validation called 'proof', the analogue of experiment in physical science. A proof is a conclusive argument that a proposed result follows from

accepted theory. ‘Follows’ means the argument convinces qualified, sceptical mathematicians. Here I am giving an overtly social definition of ‘proof’ (1997, p. 6)

In his definition Hersh emphasises the ‘social’ dimension of proof. A proposed result is validated not through introspection and not by formal derivation, but through the formulation of an argument that convinces other colleagues in the field. In mathematics education Balacheff (1988) was one of the first who pointed out the significance of this social process for proofs in mathematics and in the context of learning about mathematical proof. Unlike Hersh, Balacheff makes a distinction between proof (‘preuve’ in French, Balacheff, 1991, p. 109, Note 2; 1987) and mathematical proof (‘démonstration’ in French) He describes ‘preuve’ as an explanation accepted by a given community at a given moment and ‘demonstration’ as an explanation of a specific form, organized as a succession of statements following specified rules (Balacheff, 1987, p. 148). In his early writings Balacheff states that only explanations of this form are accepted as proof within the mathematical community.

Aberdein (2012) offers a model of proof that is based on mathematical practice. He characterises mathematical proof as an argument with a parallel structure, comprised of *argumentational* and *inferential* structures (see Figure 1).

Argumentational Structure:	**Inferential Structure:**
Mathematical Proof, P_k	Mathematical Inference, I_k
Endoxa: Data accepted by mathematical community	Premisses: Axioms or statements formally derived from axioms
argument ↓	*derivation* ↓
Claim: I_k is sound: that is, an informal counterpart of S_k should be accepted too	Conclusion: An additional formal statement, S_k

Figure : The parallel structure of mathematical reasoning (Aberdein, 2012, p. 353)

The argumentational structure is composed of arguments by means of which mathematicians seek to persuade each other of their results or, more generally, to achieve goals appropriate for whatever dialogue they are having. The inferential structure is composed of derivations which offer a formal counterpart to these arguments (p. 352).

Analysing actual mathematical proofs Aberdein comments that proofs generally comprise not single arguments, but structures of arguments “by which mathematicians attempt to convince each other of the soundness of the inferential structure” (p. 352-353). According to Aberdein:

> This account both conserves and transcends the conventional view of mathematical proof. The inferential structure is held to strict standards of formal rigour, without which the proof would not qualify as mathematical. However, the step-by-step compliance of the proof with these standards is itself a matter of argument, and susceptible to challenge. Hence much actual mathematical practice takes place in the argumentational structure. (p. 353).

He concludes that an adequate characterisation of mathematical reasoning requires an account of both structures.

Aberdein describes different *argumentation schemes* according to how "their instantiations are related to the corresponding steps (if any) of the inferential structure" (p. 356). Arguments that "correspond directly to a derivation of a rule of the inferential structure" (p. 356), he calls A-schemes. Mathematical proofs of this form can be described as rigorous and are considered as the only form of acceptable proofs in a foundational, formalist view of mathematics.

B-schemes, according to Aberdein, are "less directly tied to the inferential structure" (p. 356). They collapse many inferential steps into a single argument, and may refer to propositions proved elsewhere. But they could be, in principle, "formalized as multiple inferential steps" (p. 357). The use of B-schemes in proofs is "intrinsic to established mathematics, and fluency in their use is a prerequisite for participation in mathematical practice" (p. 360).

Mathematical practice is not limited to established mathematics and methods. It also includes argumentations involving innovative and informal mathematical practices, both deductive and non-deductive. Aberdein describes these as C-schemes. "C-schemes are even looser in their relationship to the inferential structure, since the link between their grounds and claim need not be deductive" (p. 357). C-schemes allow one to describe arguments in informal mathematical practice, for example plausible mathematical reasoning, analogies or visual arguments.

Which argumentational schemes are acceptable in a mathematical proof and the relationship between the argumentational and the inferential structure, depends on the context of the mathematical practice. In a proof in formal logic, only A-schemes are used and the argument is simply a representation of the inference. Typical proofs published in mathematics journals make use primarily of B-schemes, referring to results proved elsewhere, skipping over steps that the reader can easily provide, and leaving implicit definitions that the reader is expected to know.

> Any steps which are purely mechanical may be omitted from an ordinary mathematical text. It is sufficient to give the starting point and the final result. The steps that are included in such a text are those that are *not* purely mechanical — that involve some constructive idea, the introduction of some new element into the calculation. (Davis & Hersh, 1981, p. 139)

C-schemes are used in informal discussion of proofs, at conferences and in classrooms, to help an audience unfamiliar with the context of the proof understand its topic and basic structure. "There is often a long distance between the original proof and textbook or oral classroom versions, full of hints aimed at making it accessible to a wider audience." (Dufour, 2012, p. 177). C-schemes are also used in research when a new conjecture or a new proof is sought.

It is important to note that unlike mathematical inference, mathematical argumentation does not follow explicit rules. Instead what is acceptable argumentation is negotiated in a social community. In the community of academic mathematicians implicit criteria for argumentation in proofs are established, except in unusual cases like the computer assisted proof of the four colour theorem. When proof becomes a topic in school mathematics, however, there are no previously negotiated criteria for argumentation. In addition, academic mathematicians recognise the mathematical inferences that are referenced by A-schemes and B-schemes, but the existence of these inferences is unknown to school students. This is a serious issue for the teaching of proof. How can one make the nature of mathematical inferences known to students while negotiating the criteria for mathematical argumentation at the same time?

Next I will describe some attempts to solve this problem. The first attempted to establish explicit criteria for argumentation by introducing a specific format, the two-column proof, into school mathematics. This format uses A-scheme argumentations to make the structure of the mathematical inference more visible. The others embed the process of proving in a larger process of conjecturing and proving, but in very different ways.

"DOING PROOFS" – A HISTORICAL PERSPECTIVE

Patricio Herbst (2002) analyses the history of two-column proofs in the US. At the beginning of the twentieth century the traditional high school course in Euclidean geometry underwent important modifications, intended to make it a good context for learning proof, while learning the content of geometry was largely shifted to the junior high school and earlier. In the high school course 'proof' was defined, and textbook definitions included a requirement that every step be stated and justified, as in a mathematical inference. A proof that followed this definition was considered a 'formal' proof.

> In line with the aims of the Committee of Ten, a premium was put on students learning to write formal proofs, even if some propositions in a text were only 'informally' proved. The fact that the two-column format emphasized in so evident ways the formal aspects of proving, enforcing the notion that a proof consisted of steps of statements and reasons, made it useful at the time. (p. 298)

In the textbooks of the time a distinction was made between two kinds of statements to be proven. 'Fundamental propositions' were significant theorems in geometry that students were expected to know, but not to prove. The teacher proved the fundamentals and in so doing provided the students with a model of how they should prove the 'exercises'. These exercises were intended to give the students many easy opportunities to prove, and they could be proven directly from the fundamentals. This meant that the subject of geometry was presented as divided into two sorts of propositions. There were mathematically significant theorems with lengthy or otherwise challenging proofs, the 'fundamentals', and trivial theorems with simple proofs, the 'exercises'. What unified the subject was no longer the importance of the theorems studied (as it had been in the nineteenth century) but instead the way the theorems were proven. The teachers' proofs of the fundamentals showed the students what a proof should be, and that meant it had to follow the same definition and format as the proofs the students were intended to prove. The two-column proof format was invented to make this clear.

The invention of the two-column proof served an additional purpose. In the beginning of the twentieth century high school enrollments increased “at a striking rate” (Stanic, 1986 p. 194). The percentage of 14 to 17 year olds in school doubled between 1890 and 1910 (p. 194). “It was claimed during this period … that with the increase in the school population came a decrease in the overall quality or intellectual capacity of that population” (p. 194). So at the same time the curriculum of the high school geometry course shifted from learning significant theorems of geometry to learning to produce proofs, the audience for that curriculum was becoming more diverse.

> Teachers had to take proactive steps to ensure that the course served its purpose. The argument was common at the time and addressed what educators thought of students as learners. D.E. Smith (1911, p. 70) suggested that to give the opportunity to prove might not be enough because the diversity of students in geometry classes made it unrealistic for teachers to expect all students in their classes to be enthusiastic “over a logical sequence of proved propositions.” But whereas it was not reasonable to expect that all students would “discover new truths,” proving truths stated by somebody else was something that all students should be able to do (ibid., p. 160). The task of ensuring that all students would do proofs was one that the teaching profession had to take on. (Herbst, 2002, pp. 299-300)

The two-column proof format supported teachers as they attempted to ensure that “all students would do proofs”.

In terms of the distinctions Aberdein (2012) makes between different types of argumentation schemes, two-column proofs are supposed to employ only A-schemes, so that the inference structure is more evident. This comes at some cost, however.

For one thing, it means that the proofs students see in high school do not resemble the proofs of academic mathematicians, including the proofs they will see if they study mathematics at a university. B-scheme argumentations that were part of high school geometry proofs in the nineteenth century (e.g., “Similarly it can be proved that the angles BEC and AED are also equal.” Euclid’s Elements, Book I, Prop. 15) were replaced with lengthier sequences of A-scheme argumentations.

A second cost is the shift in attention from the construction of geometrical knowledge to proofs. “The two-column proving custom … brought to the fore the logical aspects of a proof at the expense of the substantive role of proof in knowledge construction” (Herbst, 2002, p. 307). Because the focus was on revealing the inferential structure, the meaning of the theorem, its role in geometry and its place in relationship to other theorems was obscured.

> To make student proving possible, a system of resources had to be developed and coordinated with a norm for accomplished proofs. The integration of all those elements produced a stable geometry course oriented toward students’ learning the art of proving embodied in the two-column format. However, that stability came with a price – that of dissociating the doing of proofs from the construction of knowledge. (p. 307)

Senk (1985) and others have studied empirically the extent to which the high school geometry course as it evolved actually succeeded in making student proving possible, and she concludes that writing proofs remains difficult for most students at the end of the geometry course. Herbst's (2002) analysis includes some indications of why students do not learn to prove well in spite of the support of the two-column format. The form of the two-column proof and the teaching practices associated with it serve to make the task of writing a proof easier for the students to do and for the teacher to teach. However, they also lead to the teacher providing significant guidance to the students resulting in a division of labour that could be described by saying "the teacher proves and the students write down the proof" (Reid & Knipping, 2010, p. 217). This mostly tacit division of labour in classroom undermines the goal of engagement in mathematical reasoning by students. This historic lesson also shows the didactic complexity of approaching proof, due to the complexity of proof itself, to external social factors, and to the constraints on teaching in schools.

The approach of using two-column proofs attempted to address this complexity by focussing on the inference structure, at the cost of trivialising proofs. Next, I will describe some approaches that attempt to address this complexity by relating the process of proving to the process of conjecturing, in various ways.

CONJECTURING AND PROVING – RECENT PERSPECTIVES

In this section three teaching approaches will be described that rely on conjectures generated by students. These approaches differ in the way they conceptualise argumentation, proof and the relationship between them.

Serious attention to the relationship between conjecturing and proving could be said to have begun with the work of Lakatos (1976). His starting point was an examination of mathematical practice and a critique of formalist descriptions of mathematics. Rather than beginning with axioms and definitions, Lakatos says, mathematicians begin with conjectures. After a conjecture comes a proof, but proving is a means of analysing the conjecture, not establishing its truth. It is "a rough thought-experiment or argument, decomposing the primitive conjecture into subconjectures or lemmas" (p. 127). He emphasises the argumentational structure of the proof, which he sees as a means to improve, but never finalise, the proof as a whole.

For Lakatos, proof and conjecture are intertwined, a hypothesis that has been adopted by many, but not all, mathematics educators interested in proof teaching. In the following sections I will describe the teaching approach of Duval and Egret, who reject Lakatos' hypothesis while seeing conjecturing as important, the debate approach which is strongly influenced by Lakatos but which empirically turned out to be problematic, and an approach based on the idea of cognitive unity between conjecturing and proving.

Duval and Egret

Duval and Egret (1989, 1993) advocate teaching students to produce proof-texts involving only deductive reasoning. They propose having students come to a conjecture in a 'heuristic phase' which also includes identifying the key ideas in their conjectures. This is followed by a distinct phase of 'deductive organisation' in which the students develop a graphical representation of their conjecture, breaking apart its antecedents and consequents and citing theorems that connect them. Once this graphical representation is complete the students use it to produce a proof-text.

The use of graphical representations addresses the problem of students' lack of awareness of inference structures. They are supposed to make the logical structure of the proof visible, prior to the construction of the proof-text. Hence, they can be considered A-scheme argumentations. One would expect this focus on an argument closely tied to the inference structure to suffer the same drawbacks as two-column proofs. However, because no analyses from empirical research on the use of this approach in classrooms have been reported, we can only speculate.

Duval and Egret would reject the description of the graphical representations and the proof-text as argumentations, as they see argumentation in a more restricted way. They see no link between the argumentation that occurs in the heuristic phase and the activity of deductive organisation. On the contrary, they see proving as opposed to argumentation. They adopt the position of Perelman and Olbrechts-Tyteca, expressed in their famous *New Rhetoric: A Treatise on Argumentation* (1958), that arguments and mathematical proofs (demonstrations) are by their nature distinct. Dufour (2012) presents this approach as one of the two "mid-twentieth century renewal of academic reflection on argumentation, Toulmin being the other" (p. 166) and characterizes its rationale as follows:

> Perelman's starting point was his dissatisfaction with the principle held by some philosophers … that (formal) logic could provide a general theory of human reasoning and a suitable tool to analyse human inferences. He did share the positivist idea that logic is convenient for science, but denied that beyond the area of logic, mathematics and empirical sciences, human thinking is fuzzy and even irrational since it does not lend itself to logical analysis. As a jurist, Perelman could not discard value or moral judgments as irrational since they are an essential part of legal argumentation (p. 166).

In the *Treatise* and other writings of Perelman (1977) it is obvious that scientific, logical and mathematical reasoning is not distinguished, but construed as opposed to dialectical and rhetorical arguments. Dufour (2012) questions this dichotomy for different reasons. First of all he criticizes "the Perelmanian confusion between logic and mathematics" (p. 167). He then problematizes the "purely semantic" view of mathematics, only concerned with the truth of propositions, as "it lacks any pragmatic dimension brought about by human interactions" (p. 167). This becomes problematic when it comes to the evaluation of an argument or a proof. "For Perelman, as far as argumentation is concerned, it is up to the audience to decide" (p. 168). At the same time, "Perelman seems to believe that scientific proofs cannot be controversial since they are 'logical' … his conception of scientific reasoning precludes the possibility of an argument among scientists, especially in mathematics." (p. 169).

But as the quote from Hersh (1997) above indicates, proofs are not simply logical, they are convincing not in themselves but by convincing someone, "qualified, sceptical mathematicians" (p. 6). And in fact, proofs can be controversial.

> Even to the "qualified reader," there are normally differences of opinion as to whether a real proof (i.e., one that is actually spoken or written down) is complete and correct. These doubts are resolved by communication and explanation, never by transcribing the proof into first-order predicate calculus. Once a proof is "accepted," the results of the proof are regarded as true (with very high probability). (Davis & Hersh, 1981, p. 354)

In other words, the argumentational structure of the proof, that part "that is actually spoken or written down" can itself be the subject of argumentation, as to whether there in fact exists an inference structure to which it corresponds.

I have described two approaches (two-column proofs and Duval and Egret's) that focus on the inferential structure. I will now describe a third approach that focuses on the argumentational structure, and is strongly influenced by Lakatos' ideas about the connection between conjecturing and proving.

The debate approach

Colleagues working in Grenoble and Lyon in the late 1980s explored the potential of a teaching approach centring around 'scientific debates' (Arsac, Chapiron, Colonna, Germain, Guichard & Mante, 1992, Arsac, Balacheff & Mante, 1992). The teaching is based on students' exploring mathematical tasks chosen so as to be within the students' capabilities to explore and likely to provoke disagreement and debate. In the 'research' phase the students explored the tasks in groups and presented their solutions on a poster. Then, in the 'debate' phase, they critique each other's solutions with the teacher intervening to manage the debate. In the debate evidence for a conjecture is offered to the community and others comment on it and point out flaws. These are then corrected and perhaps new arguments are brought forth until finally the proposition is accepted into the body of mathematical knowledge. The influence of Lakatos (1976) on this teaching model is clear. It is through a social process that conjectures are refined, with proofs being argumentations that support that process. But Lakatos was describing the practice of mathematicians who have some idea of the inferential structures of their proofs to guide their argumentations. School students lack this knowledge of inferential structures.

One role of the teacher in this approach was to compensate for the students' lack of knowledge of inferential structures. She was expected not to interfere at a mathematical level during the so called research period, but had the responsibility of "to institutionalize the debate's outcome at the very end" (Arsac, Balacheff & Mante, 1992, p. 9). So in the final synthesis of the debate it was her role to draw attention to the "mathematical rules" that have been used in the debate and note the insufficiency of "pragmatic proofs" as insufficient. In other words, the teacher formulates B-schemes used in the debate and endorses their use, while rejecting C-schemes, using her awareness of the inferential structure to do so.

However, in practice this approach was not entirely successful. Arsac, Balacheff and Mante report that the arguments offered were often not founded on mathematical bases, but include appeals to social and personal factors. Students relied on their personal authority as members of the social structure of the class to verify their statements by reference to their own authority. They used C-schemes instead of B-schemes. As the teacher's role was limited to endorsing some arguments and rejecting others this presented her with a dilemma. What should she do if the only arguments available were C-schemes?

When examined in terms of the argumentational and inferential structures of a proof, a further shortcoming of this method becomes evident. Even if B-schemes are offered and endorsed by the teacher, such schemes give little access to the inferential structure. As the students cannot be expected to have such a structure in mind (they were about 11 years old) they experience only the argumentational structure. Without access to the inferential structure they have no independent way to evaluate arguments in the argumentation structure. They must rely on the teacher to indicate what kinds of arguments are acceptable.

Cognitive unity

Lakatos' (1976) vision of mathematics has also inspired colleagues from Italy who have developed a teaching approach and studied it in different learning and school contexts (see e.g., Boero, Garuti, Lemut, & Mariotti, 1996, Pedemonte, 2007). Some teaching experiments took place over an extended period of time, including individual and group work as well as whole class discussions; other experiments were conducted in a much shorter period of time without teacher interventions. Common to all these experiments is the design of a phase of conjecturing and a phase of proving, as the aim was to investigate empirically the 'cognitive unity' between the 'logic of discovery', where argumentation plays a major role, and the 'logic of justification', which leads to proof.

> During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingling with the justification of the plausibility of his/ her choices.
>
> During the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain. (Boero et al. 1996, p. 113)

Sun shadows provided one basis for conjecturing and proving in grade 8 classrooms, which allowed explorations of the mathematics of triangles and parallel lines supported by everyday experiences, experiments, and drawings. In this context it was observed that:

> When the phase of producing a conjecture had shown a rich production of arguments that aimed to support or reject a specific statement, it was possible to recognise an essential continuity between these arguments and the final proof; such continuity was referred to as Cognitive Unity. (Mariotti, 2006, p. 183)

In other words, in this context students were able to recast their argumentation in the conjecture phase, which may have included C-schemes, into B and A-schemes in the proving phase. Boero and his colleagues did their work in schools with many students from low socio-economic environments, who would be expected to find engaging in mathematical argumentation difficult (see below), but given "very strong teacher mediation" they were able to engage in conjecturing and proving similar in some ways to that of academic mathematicians' practice. (Boero, 1999). "The teacher must necessarily play the role of a committed 'dissenter' opposing the naive or non-'scientific' ways of thinking of the students and, often, of the same environment they come from.” (Boero, Dapueto, Ferrari, Ferrero, Garuti, Lemut, Parenti, & Scali, 1995, p. 164)

Pedemonte (2007) uses the Toulmin (1958) layout of arguments to describe and compare the production of a mathematical conjecture and the construction of a proof-text by pairs of students solving a problem with no teacher intervention. She reports that also in this context “argumentation activity might favour the construction of a proof” (Pedemonte, 2007, p. 25) and that “the idea of cognitive unity can be used to foresee and analyse some difficulties that students might have in the construction of proof” (p. 25).

It is significant that Pedemonte builds on Toulmin’s work. Toulmin set out to develop a model that could be used to represent both arguments in everyday discourse and proofs in mathematics, a strong contrast to Perelman who opposed argumentation and proof. However, Pedemonte goes beyond Toulmin who used his layout to model finished proofs (1958, Toulmin, Rieke & Janik, 1979), but not to show connections between the argumentation in conjecturing and the argumentation in proving.

The researchers and teachers who work with the idea of cognitive unity address the social dimension of proof at several levels. They see proof as a social practice of mathematicians, and also look at the actual practices of students, social practice in classrooms and what this means for learning mathematics and teaching. Boero and his colleagues are also concerned with the socio-cultural backgrounds of students and the role they play in the classroom (Boero, Dapueto, Ferrari, Ferrero, Garuti, Lemut, Parenti, & Scali, 1995). However, they don’t specifically focus on the heterogeneity of students and how students’ different approaches to learning mathematics could relate socio-cultural differences of students’ backgrounds. I will take up this issue in the next section.

ARGUMENTION AND STUDENTS’ BACKGROUNDS

While the increased emphasis on argumentation in the mathematical classroom is a welcome development, we need to be careful that classroom argumentation does not become a social filter, emphasizing students’ preexisting advantages and disadvantages.

Lubienski (2000) describes how efforts to improve the teaching of problem solving, in ways that were expected to help especially students with lower socio-economic status, had the opposite effect. She notes that “instruction centered around open, contextualized problems might seem particularly promising for lower SES students” (p. 456) because research has shown that such students have less exposure to open problems, and their “families tend to be more oriented towards contextualized language” (p. 456). However, in her research she found that a focus on teaching through open, contextualised problems “could improve both lower SES and higher SES students’ understanding of mathematics while also increasing the gap in their mathematics performance” (p. 478). In giving a central role to argument in mathematics teaching we risk similar unintended consequences, unless attention is paid to the structures of argumentations in mathematics classrooms combined with research on the mechanisms of stratification in them. This could provide a better understanding of the filtering effect of argumentation in classrooms and in turn provide a basis for minimising undesirable consequences of a greater focus on argumentation and proof.

Next I will introduce a sociological framework that allows one to describe different forms of discourse that are essential to mathematical argumentation, and students differential access to them.

FORMULATION AND DECONTEXTUALISATION

Basil Bernstein has theorised language and knowledge acquisition from a sociological perspective (Bernstein, 1971, 1996, 1999). His concepts of a ‘horizontal’ and a ‘vertical discourse’ allow one to not only describe different forms of discourse, specifically features of school discourse, but also to account for the differential access of students to knowledge in school.

Bernstein (1999) says horizontal discourse:

> is likely to be oral, local, context dependent and specific, tacit, multi-layered, and contradictory across but not within contexts. However […] the crucial feature is that it is segmentally organized. (p. 159)

Speech acts and the immanent knowledge of horizontal discourse are focused on concrete people and situations, a direct correspondence is characteristic for this type of discourse. Formulation might be incomplete, but coherent within a given situation. But as the discourse is sequentially organised it tends to be inconsistent and contradictory between different situations. Much of everyday language, focussed as it is on the here and now, is horizontal.

Vertical discourse on the other hand aims for coherence across situations, independent of specific contexts. Vertical discourse:

> takes the form of a coherent, explicit, and systematically principled structure, hierarchically organized as in the sciences, or it takes the form of a series of specialised languages with specialised modes of interrogation and specialised criteria for the production and circulation of texts, as in the social sciences and humanities. (p. 159)

Mathematics, like other academic disciplines, uses vertical discourse, and axiomatic inference structures could be considered the most vertical kind of discourse possible. Mathematical argumentation is unlike everyday argumentation, because it parallels an inferential structure. This is especially true of argumentation composed of B and A-schemes.

Gellert (2011) points to work of Martin (1993, 2007) and Halliday (1985), who have analysed characteristics of vertical discourse from a systemic functional linguistic perspective. Martin summarises the role of abstract metaphorical text and grammatical metaphors for the construction of vertical discourse in schooling as follows:

> It appears that the institutional boundary between primary and secondary school symbolizes the ontogenesis of grammatical metaphor in students’ language development; and discipline-specific secondary school discourses depend on abstract metaphorical text to construe specialized knowledge. Such a transition makes apprenticeship into *written abstraction* a fundamental rite of passage in secondary school. (Martin, 1993, p. 152, cited by Gellert, 2011, p. 103, my emphasis)

As Epstein (2012) notes, mathematics is based on abstraction, "much more than any science" and so engaging in mathematical argumentation requires that students recognise the need to use abstractions and develop ways of doing so.

As Gellert (2011) notes the transition into 'written abstraction' is not limited to writing, but is also characteristic for the oral instructional discourse in school. Writing can actually support students in engaging in learning mathematics and this form of abstraction, as Morgan (1998) describes, but is not typical for all mathematics classrooms. In German classrooms that I have studied proof-texts were scarce, even when considerable time was spent on proving activities (Knipping, 2003b). For participation in horizontal discourse contextualised language is characteristic and sufficient, but for vertical discourse, which is structured by an internal 'logic' of a specialised practice, a decontextualised language is needed (Gellert, 2011, p. 103).

Hasan (2001) clarifies the nature of decontextualised language:

> A discourse is decontextualised/disembedded, not because what it refers to is not physically present to the senses here and now, but because it refers to something that is by its very nature incapable of being present in any spatio-temporal location whatever. (pp. 53-54)

Of course, how one comes to know of "something that is by its very nature incapable of being present in any spatio-temporal location whatever" is through a process of abstraction. And the objects of proof, of mathematical argumentation in parallel with inference, are all abstractions. Proving requires recognising and engaging in a decontextualised discourse.

Hasan has studied the opportunities children have to engage in this kind of discourse in their homes before they begin schooling. In some homes the talk of adults to small children remains tied to the context, related directly to the activities and objects that are present to the senses. In others there is already at this early age a fluid shifting between decontextualised and contextualised language. Hasan says that in these homes there are 'con/textual shifts', by this she means "that on the one hand the speakers are shifting from an ongoing context, they are *reclassifying* the discursive situation, and on the other hand the new context to which the shift has been made is still being integrated into the discursive context from which the shift is being indicated" (p. 63). For example children and their mothers shift from talking about the immediate context, a cat walking by, to talking about abstract notions such as death, and then back again to use the fruit in the garden as a specific example of something that dies.

There is reason to believe that the division Hasan found, between home situations in which con/textual shifts occur often and those where they occur rarely, is not random. Rather, social class influences the use of decontextualised and contextualised language and the shifting between them.

> Bernstein (1975) argued that linguistic codes (or the underlying principles of speech) are affected by the class system. … Lower status families use *restricted codes,* language with implicit and context-dependent meanings that make sense in contexts in which emphasis is placed on the community and common knowledge and values are assumed to be

> shared. Additionally, Holland (1981) found that middle-class children tended to categorize pictures in terms of transituational properties (e.g., grouping foods together that were made from milk or came from the sea), whereas working-class children tended to categorize pictures in terms of more personalized, context-dependent (e.g., grouping foods they eat at Grandma's house). Holland concluded not that children could not thinking differently but that they had been raised with a particular orientation. (Lubienski, 2000, p. 456-457)

In other words, the middle-class children used abstract categories and vertical discourse to classify the pictures, while the working-class children use contextualised, horizontal discourse. When encountering a mathematical argumentation pointing to an abstract, hierarchical, and invisible inference structure it is understandable that children from different classes might perceive and engage with the argumentation differently.

Not only in mathematics, but in schooling in general decontextualised language is expected and privileged. Hasan (2001) has observed that the kinds of environments that she believes are "the best … for learning to use disembedded language" (p. 74) allow for contextual shifts. However, such environments are rare in schools. This means that "it seems very unlikely that schools provide the best environment for learning how to use such language for those children who do not already possess this expertise to some extent before they enter the school" (p. 74).

If some children are not exposed to environments in which they can learn to use decontextualised language before they start school, and schools do not provide such environments, then it is not surprising that these children have difficulty recognising decontextualised language and 'written abstraction' in mathematics in general and in proofs in particular.

When, for example, a proof in geometry makes reference to a triangle ABC, and a triangle with that labelling is present on the page or chalkboard, it is being used as a generic example, to stand for all triangles, so that the proof applies generally. But to see that triangle as a generic example requires abstraction, requires that it be recognised as a generic example, rather than a specific triangle, and that the argument be understood as decontextualised, independent of the specific triangle depicted. If a student cannot recognise that the argumentation is meant to be general, or cannot abstract the generic example from the specific context, or cannot engage with the decontextualised language of the argumentation, then that student will be at a disadvantage in learning what mathematical argumentations and proofs are.

On the other hand, a student who can engage in con/textual shifts is at an advantage in the mathematics classroom. For example, in Knipping (in press) I describe how a student, Max, shifts from filling in a table describing specific information about two children, to describing in contextual language an abstract relation he sees, to describing the same relation symbolically. Such a student can both recognise that making such shifts is expected in mathematics, and can make them when needed. For others, it is difficult to recognise the implicit expectation to make these shifts and to recognise these shifts when Max makes them in his discourse. This makes it unlikely that they will be able to learn to make such shifts in this context.

But this does not mean that pre-school experiences with decontextualised language determines success in school mathematics and specifically success in learning what proofs are and how to produce them. It is admittedly challenging to see, given the diversity of the students, how we can make the inferential structure visible and the argumentational structure accessible. One step towards this is the already common practice of developing a classroom version of a proof "full of hints aimed at making it accessible to a wider audience" (Dufour, 2012, p. 177).

> We do not hesitate to qualify seemingly different proofs as versions of the same one. This allows modification or rephrasing of the initial version to make it more explicit or to make its necessity more salient for people who are not experts. In my opinion, these strategic reworkings belong to the field of mathematical argumentation. (p. 177)

Such argumentation that makes a proof more accessible has analogues in conjecturing and proving that offer some hope for students who begin school at a disadvantage. Recall that Boero and his colleagues found that students from low socio-economic environments, who would be expected to find engaging in mathematical argumentation difficult, were able to engage in conjecturing and proving given "very strong teacher mediation" (Boero, 1999). Further research on and description of such mediation would be helpful.

Another way in which proof might be made more accessible focuses on gradually making the inference structure more visible, by beginning with local deductions. I will consider this idea next.

LOCAL ORGANISATION AND ARGUMENTATION

As I noted above, it is not evident how a teacher could make the nature of mathematical inferences known to students. In reaction to the emphasis on axiomatic approaches in vogue at the time, Freudenthal (1971, 1973) offered an alternative.

Freudenthal (1971) claims that proving must begin with what he calls "local organization" as opposed to the "global organization" of an axiomatic system. In a globally organized system the definition of parallelogram would either be explicitly taught or would become known through its use in proofs. Freudenthal describes another approach: A discussion, for example, of the properties of parallelograms can begin by simply listing all those that are apparent to the students. Similar lists might be made for rectangles and rhombuses. In examining such lists, Freudenthal claims, "There are a host of visual properties which ask for organization. Here starts deductivity; rather than being imposed it unfolds from local germs. The properties of the parallelogramme become deductively interrelated" (p. 424). Finally, one property emerges as a definition from which the others can be deduced. This is local organisation. It can be extended as the properties of parallelograms are related to the properties of rectangles, rhombuses and squares.

At first C-schemes justify the properties that are listed. They are evident from drawings, or recalled from authoritative texts or people. But to organise those properties simple links are made between them. These links might also be justified using C-schemes if the simple coincidence of a parallelogram having two properties used to justify that one property implies the other. However, the situation is one in which there are a host of simple deductions that relate the properties, simple enough that they can be pointed to by A-schemes. As the A-schemes developed in this local context become interrelated, a deductive structure grows, one that follows closely the applicable part of the inference structure. This could make that part of the inference structure visible to students, giving them some access to its nature. Later, perhaps much later, these glimpses into parts of inference structures will provide the basis for global organisation.

Freudenthal's implications for teaching are clear:

> In general, what we do if we create and if we apply mathematics, is an activity of local organization. Beginners in mathematics cannot do even more than that. Every teacher knows that most students can produce and understand only short deduction chains. They cannot grasp long proofs as a whole, and still less can they view substantial part of mathematics as a deductive system. (p. 431)

If mathematical knowledge is not globally organised, but rather left as a set of islands of local organization, this raises the question for teaching of what islands to develop. What parts of the inference structure should be made visible through A-scheme arguments, and what parts can be left implicit? Freudenthal's examples (in *Mathematics as an Educational Task*, 1973) offer a resolution to this problem. The decision as to what to leave implicit and what to make explicit must be made in order to develop students' understanding of mathematical concepts and ability to apply them. In cases where the A-scheme proof brings a new understanding of the concepts involved, the proof is useful.

CONCLUDING REMARKS

Here I have discussed the teaching of proof while taking into consideration the social dimension of argumentation and proof. This includes the social process of argumentation that is a part of a proof, parallel to its inference structure. As I have described, if one pays attention to the actual practices of mathematics, this element of social argumentation is inescapable. Another social dimension occurs in classrooms, where teaching proof is tied to argumentation. There mathematical practices are transposed into the context of schooling, and argumentation and proof take place in a new community with different needs and goals. Finally, I have discussed the social backgrounds of students, which must be taken into consideration in mathematics education whenever argumentation is involved.

There is more to be said about teaching proofs than I could here, and much more to be said about social aspects of mathematics education. For example I have not even mentioned cultural differences in approaches to argumentation, or issues related to argumentation in a second or further language. Both of these are important, and either could have been the basis for an equally long discussion.

I have argued that there must be a focus on argumentation in mathematics teaching. Will this shift make the teaching and learning proof easier? No. As I have discussed above, it presents many obstacles. But I agree with Boero (1999) who says that the relationship between argumentation and proof is complex, but also unavoidable. If we want to teach proof we have to include argumentation. And if we want to acknowledge the social dimension of argumentation we must understand better the argumentation that takes place in the practice of teaching mathematics. My work on argumentation structures (e.g., Knipping, 2003a,b, 2008, Knipping, C. & Reid, D., in press) is a step in this direction. The next step is to integrate this work with research on the relationships between social background and argumentation. Much work remains to be done in this area.

Faced with these challenges in teaching and research, some might propose giving up on teaching proof. They could ask the same question Schoenfeld (1994) raises, "Do we need proof in school mathematics?" (p. 75). And I would echo his answer:

> Absolutely. Need I say more? Absolutely. Proof is one of the most misunderstood notions of the mathematics curriculum, and we really need to sort it out. What is it, what roles does it play in mathematics and mathematical thinking, and how and when can students learn to deal with it? (p. 75)

We need to sort out especially the social dimension of proof and argumentation, and not only because this is important for the learning of mathematics. As we become aware of the ways that sociological factors related to the use of decontextualized language in proof and argumentation, and as we explore ways of taking these factors into account, we have to opportunity to change mathematics education so that it allows all students to engage in mathematical argumentation, and in so doing we can help students to be successful not only in learning about mathematical proof, but also in negotiating the language of academic discourse more generally, opening up opportunities for students that would otherwise be closed.

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