

SUBTLETY AND COMPLEXITY OF MATHEMATICS TEACHERS' DISCIPLINARY KNOWLEDGE

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I discuss teachers' disciplinary knowledge of mathematics, developing three main points as I report on a two-year study involving 22 practicing teachers. First I argue that teachers' knowledge of mathematics might be productively construed as a complex evolving form, a significant dimension of which is tacit knowledge. Second, based on the first point, I suggest that this knowledge is better understood as a learnable disposition within a participatory frame than a body of knowledge within a mastery frame. Finally, I address the question of how teachers might learn such knowledge by describing an design experiment that unfolded in collaboration with multiple groups of practicing teachers, oriented to structuring settings to support teachers disciplinary knowledge of mathematics.

Keywords: mathematics knowledge for teaching; concept study; teacher education

INTRODUCTION

Since the 1970s, investigations of the impact of teachers' formal mathematical preparation on their effectiveness in the classroom have reaffirmed an early finding by Begle (1979): there is little correlation between the number of mathematics courses that teachers took in college and the performance of their students on standardized tests. These findings have troubled many within the field of mathematics education for decades. How could it be that deeper disciplinary backgrounds are not more strongly linked to teacher effectiveness?

Widely endorsed answers to this question began to emerge in the 1990s. Several researchers noted that stock courses in mathematics are focused on completed results that often hide the messiness and complications that led to their production. In contrast, teaching in primary school entails struggling with the messiness and working through the complications as students produce novel mathematical insights. Since teachers must regularly employ associative reasoning as they grapple with the images, analogies, logical connections, and other associations that tend to be buried inside finalized formulations, their knowledge is necessarily different from mathematicians' knowledge. As Ball and Bass (2003) characterized this difference, whereas it is the research mathematician's task to *pack* insights into tight formulations (theorems, formulae, etc.), it is the teacher's task to *unpack*. Aligned with this insight, Ma (1999) elaborated Shulman's (1986) notion of "pedagogical content knowledge" to put forward the construct of "profound understanding of fundamental mathematics." The name describes the particular character of teachers' disciplinary knowledge of mathematics as *fundamental* – that is, "foundational, primary,

and elementary” (p. 116) – and as *profound* – that is referring to a “deep, vast, and thorough” knowledge of concepts and their interconnections (p. 120).

The suggestion that a specific type of pedagogical content knowledge is associated with the teaching of mathematics has reached near-consensus among researchers. In a recent review of studies of teachers’ disciplinary knowledge of mathematics, Baumert and colleagues (2010) summed it up as follows:

Findings show that [teachers’ content knowledge of mathematics] remains *inert* in the classroom unless accompanied by a rich repertoire of mathematical knowledge and skills relating directly to the curriculum, instruction, and student learning. (p. 139, emphasis added)

Despite widespread agreement on this point, there is a decided lack of consensus around the nature of teachers’ “inert” mathematical content knowledge, and how it might be accessed or activated. A large number of formal investigations have focused on explicit manifestations of disciplinary knowledge – that is, the sorts of insights that can be assessed directly through observation, interviews, or written tests. Such explicit knowledge is typically deemed to be teachable, as reflected in studies of the disciplinary content of teacher preparation programs (e.g., Schmidt, Houang, & Cogan, 2011).

Other researchers (e.g. Adler & Davis, 2006; Davis, 2011; Davis & Renert, 2009), influenced largely by the work of Polanyi (1966), have suggested that the most important knowledge for teaching tends to be enacted and tacit, and so neither easily identified nor readily measured. While in no way denying the importance of teachers’ ability to demonstrate competence in formal mathematics, some of these investigations have emphasized the networks of association that underlie the conceptual fluency of expert teachers. As has been established in other domains of expertise, expert performance is enabled by well-established and automatized (i.e., not necessarily accessible to consciousness) webs of association, from which the expert effortlessly “selects” an interpretation, scenario, or cluster of actions that is likely to best fit unfamiliar situations (see Ericsson et al., 2006). Experts are typically unable to explain or justify their choices when asked about them; they simply recognize their interpretations or actions as appropriate in the circumstances encountered. I believe that teachers’ content knowledge manifests largely in this way. As I elaborate, viewing teachers’ knowledge in this way carries significant implications for how it is studied, assessed, and developed.

This discussion is framed by a critique of Ma’s notion of *profound understanding of fundamental mathematics*. While I resonate with her interpretation of the word *profound*, particularly as illustrated through her highly networked diagrams, I am less comfortable with the adjective *fundamental*. Specifically, I propose that characterizing teachers’ disciplinary expertise as “foundational, primary, and elementary” – terms which suggest a closed set of insights and understanding that might be catalogued and assessed — may be antithetical to the project of researching the complexity of teachers’ knowledge. As an alternative to Ma’s construct, I develop the notion of “profound understanding of *emergent* mathematics.” In brief, I argue that the knowledge needed by teachers is not simply a clear-cut and well-connected set of basics, but a sophisticated and largely enactive mix of

familiarity with various realizations of mathematical concepts and awareness of the complex processes through which mathematics is produced. (Note: the term *realizations* borrows from Sfard, 2008, and is used to refer to associations that a learner might use to make sense of a mathematical construct.) In anchoring my usage of the term “*emergent*” to the adaptive, evolutionary dynamics described by complexity researchers, I intend to flag the coherent-but-never-fixed character of the complex form of teachers’ knowledge.

The distinction between fundamental and emergent is not a subtle one. As I argue, teachers’ disciplinary knowledge of mathematics is vast, intricate, and evolving. Moreover, no individual could possibly be aware of the whole range of interpretations that might be invoked in primary school mathematics. Rather than think of this knowledge as a discrete body of foundational knowledge held by individuals, then, it may be more productive to view it as a flexible, vibrant category of knowing that is distributed across a body of professionals. With regard to individual teachers, then, I frame teachers’ knowledge of mathematics as a learnable participatory disposition within an evolving knowledge domain.

The notion of “learnable participatory disposition” is central to my research, and it is one that I attempt to explicate as this report unfolds. The participatory “concept study” research project that I report on in this article was undertaken *with* teachers rather than *on* teachers with the intention of affecting the ways they think about, feel about, and engage with mathematical concepts – as individuals, with colleagues, and with their students. The deconstructive and constructive activities in which the teachers engaged included: identifying extant interpretations, investigating entailments, blending them into new meta-interpretations, and using resultant insights to inform teaching. Although still in its early stages, my research indicates that teachers’ involvement in such shared activities can have immediate and significant impact on their knowledge of mathematics and on their teaching practices. In particular, these activities appear to support a shift in teachers’ perceptions of the nature of mathematics, away from pre-given and unchanging facts, toward open and evolving human understandings that can be discussed and debated.

The bulk of this article describes my collaborative work with teachers in the framework of a concept study of multiplication. I begin with a brief introduction to the notion of *emergence*. I then move on to recount a series of meetings in which practicing teachers took part in critical examinations of some of their mathematical understandings. In the process, I explain my conception of teachers’ knowledge of mathematics as a learnable disposition, and of structures that might be used to support the development of disciplinary knowledge.

EMERGENCE

Capra (2005) offered a systemic analysis of the building blocks of life in diverse phenomena: biological cells, consciousness, and social reality. Capra’s synthesis is based on the assumption that there is a fundamental unity to life and that different living systems exhibit similar patterns of organization. This assumption is supported by the major findings of complexity science, systems theory, and cognitive sciences of the past three decades. Accordingly, the pattern of organization of living systems is the self-generating network. Living systems are cognitive learning systems, where cognition is closely related to the

process of *autopoiesis* (self-organization, self-generation) and can be observed as *evolution* on multiple scales. Within the complexity sciences, the word *emergence* has been adopted to collect these notions.

As Goldstein (1999) explained, emergence is “the arising of novel and coherent structures, patterns and properties during the process of self-organization in complex systems” (p. 51). The notion of emergence can be applied in a number of ways to mathematics for teaching. Specifically, it can be used to understand and characterize the dynamics and structures of this branch of knowledge. It highlights not only situation-specific selections and adaptations but the systemic, distributed, and self-organizing character of teachers’ content knowledge.

It is not uncommon to encounter suggestions that mathematics “is a living, breathing, changing organism ...” (Burger & Starbird, 2005, p. xi) or that it “emerges as an *autopoietic* [i.e., self-creating and self-maintaining] system” (Sfard, 2008, p. 129). As an example, Mazur (2003) offered some historical case studies of the emergence of mathematical concepts. Multiplication, for example, was likely originally conceived as repeated addition and/or some sort of a grouping process. However, this conception breaks down when one attempts to add $\frac{5}{8}$ to itself $\frac{3}{4}$ times, π to itself d times, or to assemble -2 groups of -3 . Mathematicians struggled to elaborate multiplication as new number systems unfolded. Their responses were never purely logical and, in most cases, new realizations – that is, associations that might be used to interpret a concept, including definitions, rules, analogies, applications, and so on – arose out of emergent arguments and were incorporated into always evolving conceptual constructs.

It is this sense of emergence I intend to invoke in the phrase “profound understanding of *emergent* mathematics.” This construct gestures toward an understanding of teachers’ disciplinary knowledge as a responsive and evolving autopoietic system of realizations that is distributed across a body of educators. Staying with the example of multiplication, and on the basis of my analyses of North American textbooks, I feel justified in asserting that most learners will have encountered at least a dozen distinct realizations of multiplication by the 8th grade. They include: skip counting, hopping along a number line, stretching and compressing a number line, grid making, area generation, dimension jumping, scaling, a linear function, and a process that might be illustrated by repeated folding, splitting, or layering. Such diversity contributes to robust and flexible understandings (see, e.g., English, 2003; Harel & Confrey, 1994; Lakoff & Núñez, 2000).

The subtleties of the capacity to move among interpretations tend to be difficult for expert practitioners to appreciate. In particular, teachers and other expert knowers are often unable to differentiate among realizations which novices are unable to reconcile (Sfard, 2008). To illustrate, figure 1 shows four different interpretations of multiplication along with entailments for the meanings of multipliers, multiplicands, and products. While some obvious overlaps exist among these instances, shifting from one instance to the next involves significant conceptual leaps. It is not simply that the underlying images are different; the actions that are mapped onto the concept of multiplying (i.e., clustering *vs.* rectangulating *vs.* compressing *vs.* sloping) are lodged in very different webs of association. Each mapping opens up and shuts down different interpretive possibilities.

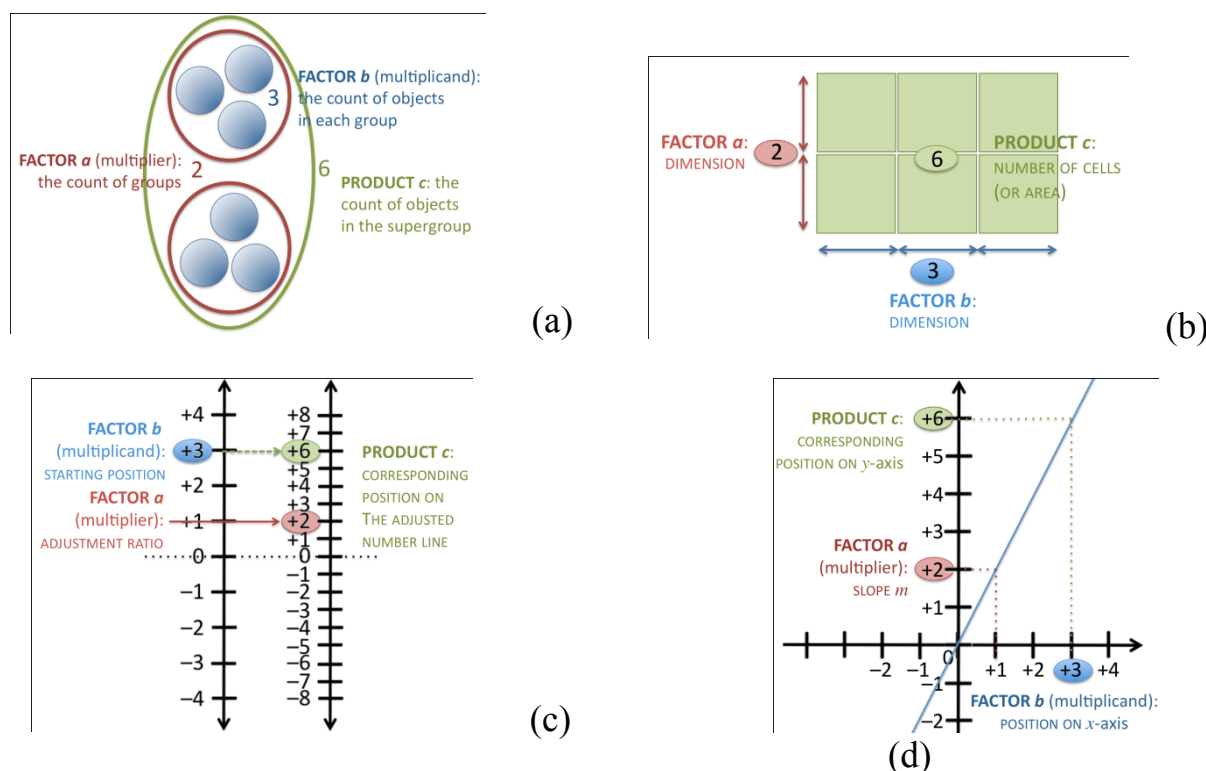


Figure 1. “ 2×3 ” as viewed through four interpretations of multiplication: Multiplication as (a) repeated grouping, (b) grid (or area) making, (c) number-line stretching or compressing, and (d) a linear function ($y = mx$).

These webs of association, I contend, are not merely accumulative, but aspects of emergent forms. That is, as they interact, they cohere into grander constructs that can open up surprising new dimensions. In this sense, a concept such as multiplication is one element in an ever-evolving ecosystem of elements, which constitutes the dynamic system of Western mathematics. A profound understanding of emergent mathematics, then, entails both the complex dynamics at work in the development of mathematical knowledge and the specific realizations of elementary concepts that might be relevant and meaningful to learners.

In critiquing Ma’s construct I do not intend to dispose of it or minimize its importance. On the contrary, *emergent* mathematics does not preclude what has already been embodied in a cultural system. With regard to teachers’ knowledge, profound understanding of emergent mathematics includes but broadens profound understanding of fundamental mathematics. In particular, a profound understanding of emergent mathematics foregrounds and embraces the necessary tensions of stability-and-novelty and coherence-and-decentralization, affording the latter elements of these dyads the same status as the former.

CONCEPT STUDY

This report is organized around what I call “concept study”, a collaborative structure to engage with teachers in the examination and elaboration of mathematical understandings. The phrase “concept study” combines elements of two prominent notions in contemporary mathematics education research: *concept* analysis and *lesson study*. Concept analysis, which was well represented in mathematics education research from the 1960s to the 1980s,

focuses on explicating logical structures and associations that inhere in mathematical concepts. As Usiskin et al. (2003) described it, concept analysis

involves tracing the origins and applications of a concept, looking at the different ways in which it appears both within and outside mathematics, and examining the various representations and definitions used to describe it and their consequences. (p. 1)

Usiskin et al. extended their description to include ways of representing ideas to learners, alternative definitions and their implications, histories and evolutions of concepts, applications, and learners' interpretations of what they are learning.

I blend this emphasis with the collaborative structures of lesson study, through which "teachers engage in to improve the quality of their teaching and enrich students' learning experiences" (Fernandez & Yoshida, 2004, p. 2). Lesson studies are oriented towards new pedagogical possibilities through participatory, collective, and ongoing engagements. Concept study is thus guided by these assumptions:

- At the individual level, understandings of mathematical concepts and conceptions of mathematics are emergent.
- At the cultural level, teachers are vital participants in the creation of mathematics, principally through the selection of and preferential emphasis given to particular interpretations over others.
- As the level of social collectives, teachers' knowledge of mathematics is largely tacit but critical elements of it can be made available to interrogation in group settings.
- Individual and collective knowing cannot be dichotomized; collective possibilities are enfolded in and unfold individual understandings.

The last two points are of particular relevance. A common observation across concept studies has been that individual teachers can rarely identify more than a handful of interpretations of a given concept when presented with a direct question (e.g., "What is multiplication?"). In contrast, in my experience, cross-grade groups of teachers consistently generate rich lists of metaphors, analogies, and images when invited to situate the concept in the context of their teaching experiences. These lists then act as gateways into the excavation of embodied and figurative dimensions, a process that can quickly result in a reworking of the concept itself. I refer to this process as *substructuring*.

Substructuring is derived from the Latin *sub-*, "under, from below" and *struere*, "pile, assemble" (and the root of *strew* and *construe*, in addition to *structure* and *construct*). To substruct is to build beneath something. In industry, *substruct* refers to reconstructing a building without demolishing it – and, ideally, without interrupting its use. Likewise, in concept studies, teachers rework mathematical concepts, sometimes radically, while using them almost without interruption in their teaching.

While the term "substructuring" may connote a descending motion, substructuring is both reductive and productive. It is reductive in that it starts by re-collecting and re-memorizing experiential, linguistic, and other elements that infuse the meaning of a concept. It is productive in a complexivist sense, in which acts of re-presentation often compel new integrative structures and novel interpretations. These constructs may become substructures of

subsequent knowing. This recursive process corresponds with the understanding of emergent knowledge as both dynamic and stable/coherent: always deepening, crystallizing, and becoming, while always embodying the memory of its evolution in its structures. To this end, the process of substructuring operationalizes the structures of emergent mathematical knowledge. Substructuring might be contrasted with the process of “unpacking” (Ball & Bass, 2003), which corresponds with a view of mathematical knowledge as manifesting in compressed and relatively static forms. In any case, the purpose of concept study and substructuring activities is not to immerse teachers in ongoing debate on the ontological status of multiplication, specifically, or mathematics, more generally. Rather, the focus is on how humans come to understand mathematical concepts.

This report focuses on concept study work carried out over a period of two years with a cohort of 22 teachers enrolled in a Master’s program focused on the theme of Mathematics for Teaching. The teachers in the group represented almost all grade levels. Mathematics backgrounds varied from no formal coursework beyond high school to graduate level study.

The cohort met twice a month in daylong sessions. About two hours of each session were devoted to concept study work. I led some of the concept studies; the teachers led others. Each of the participating teachers was required to design and lead a concept study, either individually or as part of a group, and to report on the findings in a concluding paper for the program. The participants’ motivations to engage in concept studies were therefore very different from those we encountered in previous concept study sessions, which tended to be less formal and involved in-service teachers seeking to improve their practice.

The structures of concept study described in this report have a 10-year history, and arose initially out of casual encounters of groups of teachers who shared an interest in better understanding specific concepts and topics in mathematics (cf. Davis & Simmt, 2006). Based on that work, four emphases were identified that had proven productive for the ongoing collective elaboration of mathematical concepts within various groups and across various mathematical topics (Davis & Renert, 2009): *realizations*, *landscapes*, *entailments*, and *blends*. These emphases served as starting points for organizing the cohort’s concept study work, and first introduced them, in the context of a course on research methods, as strategies that had previously proven useful for substructuring mathematical concepts.

I proceed with brief descriptions of the four emphases and end by describing a more recent fifth emphasis, *pedagogical problem solving*, which unfolded through work with the cohort. It is important to note that I do not offer the emphases as “steps” or “levels.” I see them as aspects that are always already present (alongside other interpretive strategies that we either have not noticed or have not made explicit). The word *emphasis* was chosen to signal the simultaneity of such elements. That said, the strategies were not implemented rigidly. Indeed, as events unfolded, the members of the cohort revised and refined the strategies in a recursively elaborative manner.

EMPHASIS 1: REALIZATIONS

As noted above, the term *realizations* is used to collect all manner of associations that a learner might draw on and connect in efforts to make sense of a mathematical construct

(Sfard, 2008). More precisely, a realization of a signifier S refers to “a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S” (p. 154). The distinction between a signifier and a realization is often blurred, as mathematical realizations can often be used as signifiers and realized further. Among many possible elements, realizations might draw on:

- formal definitions (e.g., multiplication is repeated grouping)
- algorithms (e.g., perform multiplication by adding repeatedly)
- metaphors (e.g., multiplication as scaling)
- images (e.g., multiplication illustrated as hopping along a number line)
- applications (e.g., multiplication used to calculate area)
- gestures (e.g., multiplication gestured in a step-wise upward motion).

To be clear, the assertion and assumption here is not that any particular realization is right, wrong, adequate, or insufficient. It is that personal understanding of a mathematical concept is an emergent form, arising in the weaves of such experiential and conceptual elements.

The process of collectively identifying realizations, and their related signifiers, is neither linear nor obvious. Each knower holds and utilizes a personal set of realizations. Some of these are common to all participants, while others are idiosyncratic or shared by only a few. Moreover, realizations are not fixed. They evolve through the process of learning. Not only they become more numerous, some earlier ones are discarded or expanded when new applications arise. Well-rehearsed realizations (e.g., “multiplication is repeated addition”) can be so well rehearsed that they may eclipse other interpretive possibilities.

To circumvent the tendency to go directly to well-rehearsed definitions, I framed the first concept study session as an invitation to explore how the concept of multiplication is introduced, taken up, applied, and/or elaborated at different grade levels. The collected response, assembled after small groups worked for 30 minutes, is presented in figure 2.

Some Realizations of Multiplication

- grouping process
- repeated addition
- times-ing
- expanding (i.e., distributing across factors; e.g., how can you write $(x + 3)$, $(x + 2)$ times?)
- scaling
- repeated measures
- making areas (continuous)
- making arrays (discrete)
- proportional/steady increase/slope/rise
- splitting, folding, branching, sharing, and other ‘dividings’ (i.e., multiplyings)
- skip counting (jumping along a number line)
- transformations
- stretching/compressing a number line

Lingering worries:

- none of these on their own seem to do much to illuminate -2×-3
- what’s going on when units are introduced – e.g., $3 \text{ g/L} \times 5 \text{ L}$?
- most of the entries rely on/suggest a linear or rectilinear basis/image of multiplication, which might work through middle school, but won’t stretch across some things encountered in and beyond high school

Figure 2: Teacher-generated realizations of multiplication, and related concerns

It bears noting that there are some striking similarities and differences between this list and those generated by other groups (e.g., Davis & Renert, 2009; Davis & Simmt, 2006). I have observed remarkable stability across groups of teachers – pre-service and practicing, novice and experienced, grade-specific and cross-grade – in the range and quantity of interpretations generated. In every context so far, participants have generated about a dozen interpretations, of which **GROUPING** and **REPEATED ADDITION** are always the first. A distinction is almost always made between realizations appropriate to continuous applications and those appropriate to discrete applications. In contrast, the list in figure 2 lacks some distinctions made by other groups – e.g., grouping realizations that use number lines and reordering according to grade level. I suspect this lack of consolidation is due to the rather short timeframe allocated to this concept study.

Returning to the construct of profound understanding of emergent mathematics, I see realizations as interacting elements within the evolving system of teachers' mathematics knowledge. Since realizations evolve through learning, part of mathematics learning might be productively understood as the evolution of networks of realizations.

EMPHASIS 2: LANDSCAPES

There are dramatic differences of conceptual worth among realizations. Some can reach across most contexts in which a learner might encounter a concept; others are situation-specific or perhaps learner-specific. This realization compelled a strategy to organize and contrast assembled lists of realizations. Briefly, a landscape is a macro-level view, whereas a realization is a micro-level view, of a concept. The strategy was to invite participants to identify moments in the K–12-mathematics curriculum that compel significant shifts in understandings of multiplication.

Drawing on Greer's (1994) observation that many different criteria could be used to organize this information, the group explored distinctions that might be used as axes in a mapping. The following dimensions were proposed: grade level, grounding metaphor, underlying image (e.g., number line; grid; area, graph), dimension(s) of underlying image (0, 1, 2, 2+), types of factors (i.e., discrete/continuous), curriculum topics, processes versus objects, keywords (e.g., "by," "of," "times"), applications, factors with/without units, and errors/misconceptions. The collective settled on grade level and grounding metaphor as most useful. The mapping presented in figure 3, in which Lakoff and Núñez's (2000) notion of "grounding metaphor" is used to organize constructs along the horizontal axis, is one of more than a dozen distinct landscapes created by the group.

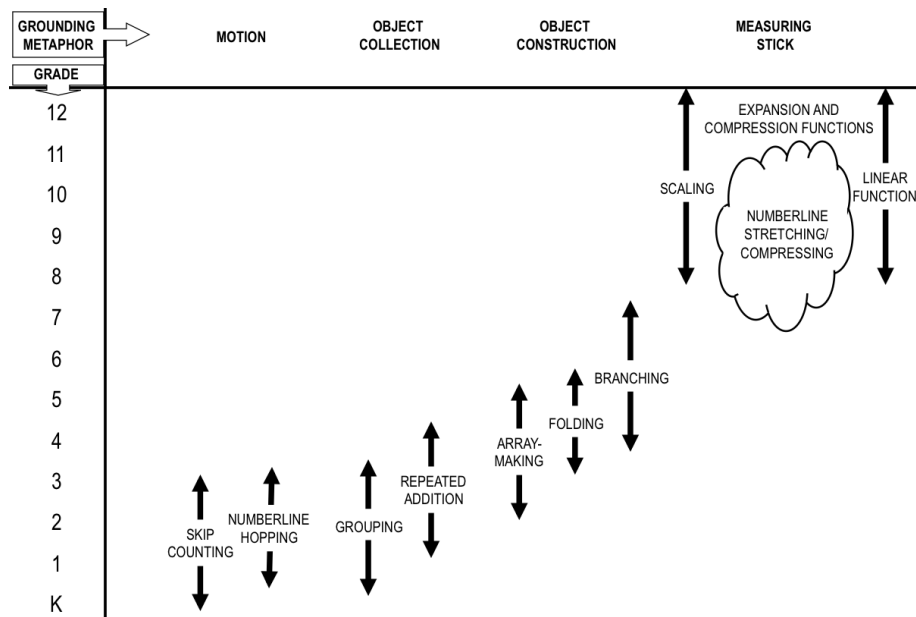


Figure 3: A collective mapping of some of the realizations

It is important to emphasize that this exercise is not merely representative. In fact, in this case it proved to be quite productive. Some observations that arose in the discussion were:

- Multiplication is introduced informally (but not explicitly) in terms of motion as early-grade teachers skip count (in particular, by 2, 3, 5, and 10). Frequently, skip counting is accompanied by tracing out the movement on number lines and in number charts.
- The movement across categories of interpretation in different grade levels seems to be fluid and deeply connected; yet it is rarely made explicit. For example, the sequence of “skip-counting → number-line hopping → repeated addition” had not been previously noticed by any of the participants.
- There are also pedagogical moments in which elaborations are important yet non-intuitive. In particular, there appears to be a major conceptual leap in the shift from “object collection” interpretations (suited to whole numbers applications) to “object construction” interpretations (useful for continuous applications).

EMPHASIS 3: ENTAILMENTS

As mentioned, each realization of a concept carries a set of implications. The intention of this emphasis is to examine the entailments of different realizations to related concepts (e.g. how does understanding multiplication as number-line hopping impact our understanding of the commutative property of multiplication). In the process of exploring entailments, participants are forced to consider the concept afresh and not only in well-rehearsed ways.

The third session was thus organized around “entailments charts” (fig. 4), a strategy developed by a different group of teachers in an earlier concept study (Davis, 2008) to explore connections and differences. A chart comprises a list of realizations in the first column, with as many additional columns as desired to record unpackings of the usually uninterrogated implications of those realizations for related concepts and topics.

If multiplying is a factor is a product is commutativity is a prime is ... (necessary conditions, but not sufficient)
REPEATED ADDITION	ADDEND or NUMBER OF ADDENDS (2×3 : 2 added to itself 3 times or vice versa)	a SUM	$2 + 2 + 2 = 3 + 3$	sum of ones
REPEATED GROUPING	NUMBER OF GROUPS or NUMBER OF ELEMENTS IN EACH GROUP	a SUM: total number of all the elements in the groups (cardinality of the set)	2 groups of 3 = 3 groups of 2	one group, or one element in each group
MAKING A GRID OR RECTANGULAR ARRAY	DIMENSION: number of rows (number in each column) and number of columns (number in each row)	NUMBER of cells	90° -DEGREE ROTATION (a 2-by-3 grid has the same number of cells as a 3-by-2)	one of the dimensions has to be 1
SKIP COUNTING	SIZE OF THE JUMP and NUMBER OF JUMPS	END DESTINATION (the last number you land on)	a jumps of distance b lands you in the same place as b jumps of length a	must make only a single jump or jump one space at a time
SCALING	SCALE FACTOR and ORIGINAL MEASURE	MEASURE OF THE FINAL MAGNIFICATION/ REDUCTION	size a scaled by a factor of b gives the same result as size b scaled by a factor of size a	when a magnification/ reduction can only be reached in unit increments or directly
AREA GENERATION	DIMENSIONS (lengths and widths)	AREA	90° -ROTATION: $lw = wl$	one dimension must be 1
NUMBER-LINE STRETCHING AND COMPRESSING	SCALE FACTOR and STARTING POSITION ON UNALTERED NUMBER LINE	CORRESPONDING POSITION ON STRETCHED/ COMPRESSED NUMBER LINE	If c corresponds to <i>point a</i> when line is scaled by b , it will correspond to <i>point b</i> when scalar is a	to get to <i>point c</i> , you must EITHER start at 1 with a scalar of c OR vice versa
FOLDING	NUMBER OF HORIZONTAL AND VERTICAL DIVISIONS (made by the folds)	NUMBER OF LAYERS	folding into a layers, then into b layers gives the same number of layers as b first, then a	can only be folded directly using $a - 1$ folds
BRANCHING	NUMBER OF STEMS and NUMBER OF BRANCHES PER STEM	TOTAL NUMBER OF BRANCHES AT THE LAST LEVEL	a branches of b stems has the same product as b branches of a stems	must either have 1 stem or 1 branch/stem
LINEAR FUNCTION $y = mx$	SLOPE and X-COORDINATE	Y-COORDINATE	If $y = c$ when $m = a$ and $x = b$, then $y = c$ when $m = b$ and $x = a$.	To get to the desired y-coordinate c , either $m = c$ and $x = 1$ or $m = 1$ and $x = c$.

Figure 4: Some analogical implications of different realizations of multiplication (Note: UPPERCASE is used to flag realizations and their analogical entailments)

The teachers worked on aspects the entailment chart in small groups. Figure 4 represents a subsequent consolidation of their work into a single chart. Just as had been encountered in a previous concept study with a different group of teachers (Davis, 2008), the resulting chart was met with expressions of surprise and confusion. For example, upon comparing different ‘readings’ of commutativity, one participant commented, “I get the idea of the communicative property – flipping the factors doesn’t change the product – but I’m realizing I don’t get what’s under it.” This comment was in specific reference to the

“obviousness” of commutativity when multiplication is interpreted in array or area terms, and the obscurity of the concept for most other interpretations of multiplication.

Another topic of engagement was the implicit criteria that teachers might be using to select realizations. Possibilities included mathematical appropriateness, sufficiency, and familiarity. Agreement was quickly reached that, while not often considered, other vital criteria might be conceptual reach, potential for elaboration, and transparency. That was coupled to the qualification that the teacher must also be thinking systemically while choosing. As the example of commutativity illustrates, some concepts become incoherent when associated with particular realizations.

The discussion closed with some reflection on the likelihood that the ability to move fluidly among realizations was not so much a matter of “having a firm handle” on those realizations; it was, rather, more about having lost the ability to differentiate among them – reminiscent of Sfard’s (2008) reminder that, as with many expert knowers, teachers are often unable to distinguish among elements that their students and novices cannot reconcile.

EMPHASIS 4: BLENDS

The three emphases described so far are focused mainly on making fine-grained distinctions among realizations and their entailments. Not surprisingly, many of the participating teachers voiced some frustrations as the shared work unfolded. Multiplication is, after all, a mathematically coherent concept, not an assemblage of images and implications. The blending emphasis is about seeking out meta-level coherences by exploring the deep connections among identified realizations and/or assembling those realizations into a more encompassing interpretation – which, of course, might introduce emergent possibilities.

For this emphasis, I drew on research into conceptual blends (Fauconnier & Turner, 1998). As diSessa described, metarepresentational skills comprise “modifying and combining representations, and ... selecting appropriate representations” (p. 296), subcomponents of which include inventing and designing new representations, comparing and critiquing them, applying and explaining them, and learning new representations quickly. In previous concept studies, a few blends/metarepresentations arose spontaneously (Davis, 2008; Davis & Renert, 2009; Davis & Simmt, 2006). However, in this group, I tried to condition the emergence of blends in a more deliberate manner. I found out that *causing* emergent insight was impossible. This is not to say that the time set aside for this layer was unproductive. I had decided in advance that, should explorations stall, I would introduce metarepresentations from other concept studies, which sponsored some rich discussion.

The first blend to be introduced was from a concept study reported by Davis & Simmt (2006), in which a grid-based algorithm was used to highlight some connections among standard algorithms (fig. 5). The second was from a study reported by Davis & Renert (2009), through which participants developed an interpretation that brought together multiplication as NUMBER-LINE STRETCHING/COMPRESSING, as SCALING, as a LINEAR FUNCTION (fig. 6). The emergent product had several ‘bonus’ features, among which was a new interpretation of integer multiplication in terms of a number line that inverts as it compresses through zero. (The explanation is not a brief one, and so is omitted here.)

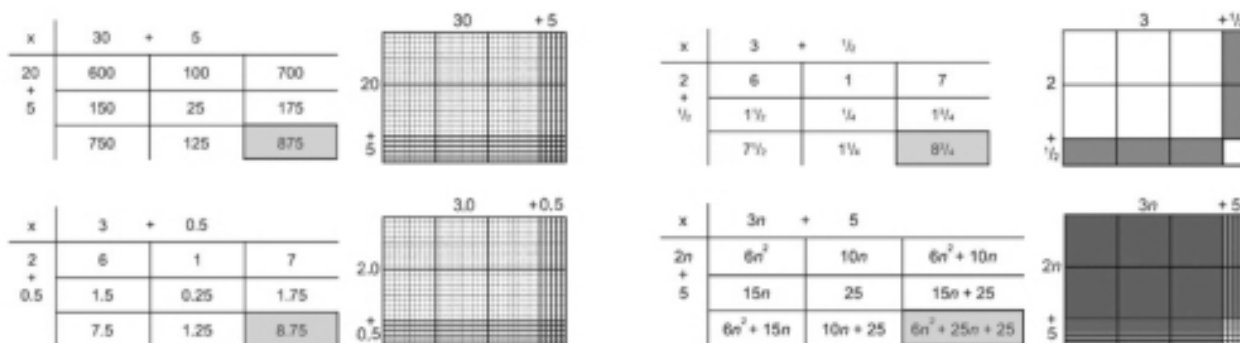


Figure 5: A grid-based blend that highlights the similarities of multiplicative processes involving additive multiplicands

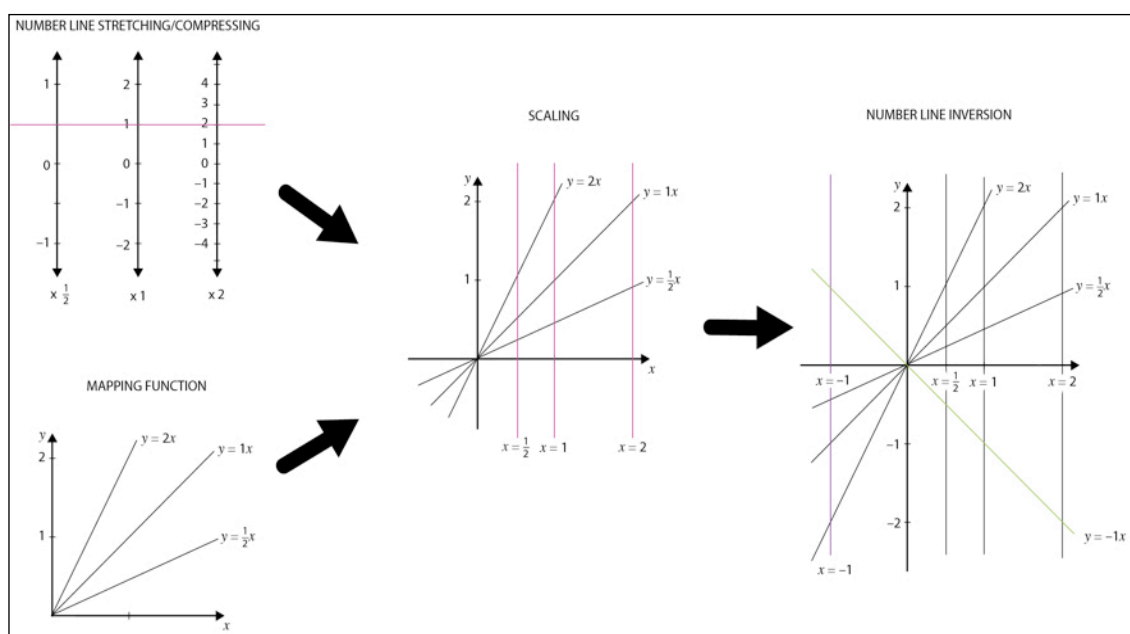


Figure 6: A graph-based meta-realization that blends linear models of multiplication

The surrounding conversation focused mainly on the manner in which some of the previously identified realizations did or did not fit within these metarepresentations. Another prominent theme was the extent to which the two metarepresentations might be reconciled into a unifying blend. More generally, discussions revolved around the conceptual novelty that can emerge in metarepresentations – and how in turn, such metarepresentations might be blended into even more encompassing interpretations.

And so, although there were no new constructs, outcomes of expanded awarenesses of the web of associations and the relevance of mathematics education research were met.

EMPHASIS 5: PEDAGOGICAL PROBLEM SOLVING

By the completion of the fourth session together, participants appeared to be comfortable with the construct of “emergent mathematics” insofar as it related to learning and teaching – that is, with the suggestion that mathematics understandings and insights arise in the interplay of realizations and other experiences.

The encounter described in this section took place seven months after the fourth session. During the intervening period, participants were involved in small-group concept studies. Clusters organized in self-selected groups, and each group took on an investigation of a specific mathematical topic of their choice, employing and adapting the emphases described above. Groups varied in size from two to five, with most involving three participants.

By the time that I introduced this emphasis, then, participants were comfortable with “emergent mathematics.” The hope at this point was to focus their explorations of concepts onto persistent conundrums of mathematics teaching. I thus introduced an emphasis on “pedagogical problem solving,” defined in terms of grappling with questions that are commonly encountered in schools and that may be deceptively easy to pose but very difficult to answer in a manner that satisfies many students. Participants were asked to identify examples in advance of the concept study meeting, and their suggestions included:

- Is ∞ a number?
- What does it mean to divide by zero?
- What’s the difference between *undefined*, *indeterminate*, and *infinite*?

To the experienced mathematics knower, these questions might seem trivial, with established and unambiguous (or, at least, context-specific) responses. Such is not usually the case for novices, as most experienced teachers will attest.

The unexpressed (to the group) intentions around the notion of profound understanding of emergent mathematics were very specific here. I wondered if challenging participants to bring their diverse areas of conceptual expertise to bear on a common question might occasion a sophisticated and productive exploratory conversation that would be useful to them in their teaching.

In this specific concept study meeting, I posed the question, “What’s $5/0$?”, with the intention of addressing the last three of the five questions listed in the previous paragraph. Before delving into the discussion, I offer three preliminary definitions, taken from Wolfram MathWorld:

Indeterminate: A mathematical expression can ... be said to be indeterminate if it is not definitively or precisely determined.

Infinite: Greater than any assignable quantity of the sort in question.

Undefined: An expression in mathematics which does not have meaning and so which is not assigned an interpretation. (<http://mathworld.wolfram.com/>)

It bears noting that, despite the very different definitions, these three words have similar original meanings. According to the *Oxford English Dictionary*, *indeterminate* is drawn from the Latin *in-* (“not”) + *de-* (“off”) + *terminare* (“to mark the end or boundary”); *infinite* derives from the Latin *in-* (“not”) + *finis* (“end”); *undefined* combines *un-* (“not”) + *definire* (“to limit, determine” – which, in turn derives from the root *finis*). In other words, all three terms carry senses of openness, unending, and unspecifiability, which likely contribute to confusions and conflations. To complicate matters, it turns out that all three words can be applied to $5/0$, depending on the perspective taken. I offer three such perspectives from among the many that arose during the concept study.

The complexity of the situation arises in the fact that there are a great many interpretations of '5' (as a discrete count, a position on a continuous number line, comparison value, etc.), the slash '/' (as divided by, inverse of, out of, per, in, to, etc.), and 0 (nothing, starting position, center point, stationary, etc.). Participants were invited to bring their particular expertise to bear on the discussion. For the sake of brevity, I focus only on those from the subgroups that had concentrated on division and factoring.

Among the interpretations of division that the division group brought to bear on the construct, $5/0$ were sharing, inverse of multiplication, and repeated subtraction. If the division is interpreted as sharing and the 5 and 0 as quantities, $5/0$ can be understood either quotitively (i.e., how many groups of 0 are in 5?) or partitively (i.e., what is the size of each group when 5 is partitioned into 0 groups?). The group consensus was that the former "makes more sense" than the latter, which seems to contain a contradiction (i.e., if there are no groups, it makes no sense to ask about the size of each group). Following the definitions presented above, they argued, it seems that multiple answers are defensible. That is, $5/0$ is

infinite, in the quotitive interpretation, because there are clearly a boundless number of size-0 groups can be drawn from a group of 5 – i.e., the quotient is greater than any assignable quantity of the sort in question.

undefined, in the partitive interpretation, because the expression is meaningless and cannot be assigned a sensible interpretation.

Similar responses were derived by interpreting division as repeated subtraction. The situation was not much clarified when subgroup that had been focusing on factoring offered their response. As they saw things, the quotient $5/0$ is equivalent to the missing factor, n , in $0 \times n = 5$ (or $n \times 0 = 5$). There is no such factor, and so $5/0$ must be

indeterminate, because although the expression can be assigned a sensible interpretation, it cannot be definitively or precisely determined.

Once again, this reporting is only a thin slice of the discussions in the concept study. Another subgroup, for example, approached the question by focusing on the meaning of the zero and found distinct rationales for all three responses.

SUBSTRUCTING TEACHERS' MATHEMATICAL KNOWLEDGE

The intention in identifying and describing the five emphases of concept study above is to better understand aspects of the complexity of teachers' profound understanding of emergent mathematics. The first four emphases are focused on realizations of mathematical concepts: their multiplicity, their roles in teaching and learning, their shifting significances, their nuanced interconnections and varied entailments, and their blending potentials. Whereas the first four emphases are focused on mathematical concepts, the fifth emphasis – pedagogical problem solving – represents a deliberate movement into processes entailed in pedagogical practice. Pedagogical problem solving aims to capitalize on the interpretive potentials that arise on the collective level when individual expertise is drawn together around perplexing problems of shared interest. It resituates the enterprise of concept study in the everyday complexities of mathematics teaching and problem solving, where multiple concepts are more typically at play.

It is important to emphasize that the purpose of engaging with the question of the meaning of $5/0$ was not to home in on a correct response. The core concern is not with the nature of mathematics but with the complexities of learning and teaching mathematics. Enhanced understandings of established mathematics was important but not the primary consideration. The intention was to attune awareness to the complexity of such constructs as $5/0$, and in the process to grapple with the confusions and frustrations that young learners might experience. All of the teachers in the group confessed to having brushed aside “division by 0” in their teaching with the dismissive comment “it can’t be done”. In contrast, subsequent to this concept study, several of the teachers undertook studies of division by 0 with some of their classes, adapting discussions as appropriate to Grades 5, 7, 10, and 11. These teachers reported back that not only were their students able to engage in nuanced discussions when prompted to consider diverse interpretations of number, division, and zero, they did so with an enthusiasm that was unusual for mathematics classes.

Much of the literature on teachers’ disciplinary knowledge to date has sought to identify clear connections between mathematics for teaching and teacher effectiveness. The belief that such clear connections do exist appears to be founded on a prevailing view of teachers’ mathematical knowledge as concerned with a relatively static collection of established results. Since I view teachers’ disciplinary knowledge as an emergent phenomenon, I tend to look for more fulsome descriptions of classroom impact. These might, for example, include teacher narratives. I offer one illustrative account written by a Grade 6/7 teacher:

“Thou Shalt Not”

In 2007, the mathematics curriculum was changed. While reviewing the new document, I noticed an addition to the standard learning outcome that covers the divisibility rules for 2, 3, 4, 5, 6, 8, 9, and 10: explain “... why a number cannot be divided by zero.”

My first response was, “That’s new.” My second response was, “Wait, I don’t have *any* idea why you can’t divide by zero! How can I teach that?”

In my first try teaching the concept I relied on the grade-level resources, which recommended an “exploratory” lesson. I did this after a warm up intended to highlight the relationship between division and the other operations (particularly subtraction and multiplication). Reading student work afterwards, I realized that the lesson had failed in a fundamental way: students didn’t differentiate between the questions $0 \div 8$ and $8 \div 0$. Even though they gave an answer of “zero” for the first and an answer of “no answer”, or “undefined”, for the second, students concluded that both of these responses meant “nothing”, and therefore the two questions were the same. This led to re-teaching with explicit instruction: $0 \div 8 = 0$, which is an *answer*. $8 \div 0$ is undefined, which means it has *no* answer, and is *not* possible. It is *not* the same question as $0 \div 8$.

For my second try the next year, I went back to the teacher’s guide. I found more explicit background information using models of repeated subtraction and inverse multiplication. After the failure of the exploratory lesson, I tried a more structured approach, where I introduced the models to the students first with possible divisions, then with division by zero. This time, students were able to differentiate the different questions, but their explanations tended to ignore the models I had provided for their own responses based on

grouping. These explanations were fuzzy, vague, and often incorrect; students were able to state that a number could not be divided by zero, but could not explain why.

I sought out several grade-level colleagues for advice on how they taught the lesson. The responses? They didn't – they skipped the concept entirely. I asked some high school teachers how they respond to student questions on division by zero. The responses: don't. Just don't.

My third try teaching the concept was the first after participation in the M₄T cohort,¹ and in a concept study of multiplication. At the beginning of the year, I had engaged my students in an exploration of the models of multiplication, with some of the results from our concept study that the students would have encountered so far (multiplication is repeated addition, grouping, array making, and area). Two months later, my introduction for the division-by-zero lesson was the question: what is division? Students worked in groups to brainstorm models of division, which we brought together on chart paper as a class. Collectively, students had developed models based on grouping, repeated subtraction, number line hopping (including the goal being zero), fractions and the inverse of multiplication. We then used the models to specifically model the equation $6 \div 2 = 3$. Next, the students were asked to use the models to represent $6 \div 1 = 6$, and $0 \div 6 = 0$. I then changed the question to $6 \div 0 = ?$, and asked students to work with the models. Students began to respond with phrases like “it doesn't make sense” and “it's all wonky”, which lead to introduction of the term “undefined”. In follow-up work, the success was clear: students could answer the questions about division by zero correctly, and also clearly explain why, using the meaning of division. The concept study of multiplication seems to have led directly to students being able to grasp this difficult mathematical concept.

The above narrative is one of many that I have collected through the course of working with the teachers. Rather than thinking of it as evidence of increased teaching effectiveness, I view it as an indication of the profound conceptual shifts that can occur through subtle pedagogical adjustments. It provides one answer to the question: *why does concept study matter?* The third lesson described above demonstrates the emergent interpretive possibilities that can arise when conceptual diversity and learner collectivity are engaged.

I continue to receive regular and consistent reports (some of which are supported by objective in-class observations) that this freely interpretive manner of engagement is readily generalized to many topics in mathematics class, not just those into which the teachers delved in concept study. The key benefit of concept studies is not so much that they foster familiarity with specific substructured mathematical concepts, but that they impart strategies for substructuring any and all mathematical concepts. Participating teachers regularly report that their amount of class time that they spend on active and explicit discussion of interpretations has increased dramatically. In some cases it has become the principal mode of engagement. In other words, casting teachers' disciplinary knowledge of teaching as an emergent, participatory domain – at least on the level of the group at the centre of this report

¹ The “M₄T [mathematics-for-teaching] cohort” is the title of the Master's of Education program that served as the context of the events described in this paper.

– appears to contribute to immediate, generalizable, and sustained transformations in pedagogy.

The manner in which specialized knowledge of specific mathematical topics (e.g., division, zero) can be excavated from a cluster of teachers who engage in a sustained dialogue about them offers a cogent illustration of how expert knowledge can be distributed across a system. The distributed nature of knowledge provides all participants in the system with enhanced access to deeper insights. From this vantage point, mathematics for teaching appears to be more about conscious and deliberate participation in a professional community than it is about a specifiable and testable body of knowledge.

One of the more interesting “accidental” areas of investigation that has arisen in ongoing work with teachers revolves around *which* aspects of their formal mathematics knowledge are invoked as they substruct concepts and engage in pedagogical problem solving. For example, the construct $0/0$ arose regularly in investigation of division by 0. This construct is often analyzed using the tools of differential calculus. Participants in the group who were familiar with differential calculus were able to drill deeper into the construct than others. It bears mention that early decades of research into teachers’ knowledge of mathematics did not draw distinctions among areas of advanced disciplinary competence. As concept studies expand to include more concepts within the school curriculum, it would be interesting to keep track of other areas of advanced study, such as number theory and topology, that might engender deeper analysis these concepts.

Generally, we found that many of the participants with extensive mathematical preparation were reluctant at first to depart from their well-rehearsed interpretations of concepts. However, once they began to do so, they were able to go much further than their colleagues in identifying emergent interpretations, articulating connections, and crafting extensions. My observations are consistent with the long-standing commonsense suspicion that advanced mathematical study *must* matter in effective teaching. It appears that preparation in advanced mathematics both constrains and enables. The concept study environment can serve to remove some of the constraints of formal mathematical study and to bring about an openness to novel interpretations and elaborations.

LOOKING AHEAD

Returning to the Baumert et al.’s statement about the inert nature of teachers’ mathematical knowledge, we assert that concept studies point to a critical element of pedagogical content knowledge needed to activate otherwise “inert” formal mathematics – emergence. I also believe much could be gained if math teaching included concept-study emphases.

Teachers’ disciplinary knowledge of mathematics might be productively framed in terms of their *profound understanding of emergent mathematics*. Given the extensive, evolving, and distributed character of emergent mathematics, teachers’ disciplinary knowledge is then better understood as a learnable disposition than as a specific body of knowledge. I end by clarifying that how I understand the relationship between *mathematics* and *mathematics for teaching*. The principal focus in concept study is on the creation of new possibilities for mathematics teaching that are rooted in more nuanced understandings and elaborations of

extant mathematics; the goal is not to create new formal mathematics – a task that would require very different validation criteria. The essential questions for us do not revolve around the ontological status of mathematical concepts or around teachers’ production of new mathematics. Rather, I seek to study new emergent possibilities for understanding mathematics. This framing is consistent with and elaborates the now-common observation that effective teaching is never simple a matter of transmission. Teaching always entails transformation, but that transformation is typically understood to happen to the learners. Following others, I include the body of mathematical knowledge within the space of teachers’ transformative influence. After all, teachers have the most direct and pervasive influence in defining what is mathematically interpretable for most of the population.

As a research community, mathematics educators are still far from making definitive claims about the relationships between teachers’ profound understandings of mathematics and their students’ mathematical understandings. My suspicion is that efforts to address this vexing quandary will require more fine-grained analyses than large-scale assessments, in large part because many of the most important aspects of teachers’ knowledge are simply unavailable for explicit and immediate assessment. They are tacit and can only emerge through participation in collective explorations, such as concept studies.

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