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FROM PRACTICAL GEOMETRY TO THE LABORATORY METHOD: THE SEARCH FOR AN ALTERNATIVE TO EUCLID IN THE HISTORY OF TEACHING GEOMETRY[1]

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The teaching of Geometry has been greatly influenced by one work: The Elements of Euclid. But, along with the logical/deductive aspect, another main aspect of Geometry was present from the very beginning of its history: Different populations or authors in different times developed practical geometrical skills according to their needs.

Texts devoted to practical geometry were developed starting from the work of Fibonacci in 1220. Slowly this practical geometry, born to give a concrete help to persons involved in the trade or even in astronomy, underwent a transformation that underlined its didactical value and turned first into a teaching via problems and then into a pseudo-practical/intuitive geometry that could be opposed, in teaching, to the deductive/rational one. The question is not only that of a practical or intuitive introduction to rational geometry, but also that of a presentation of the latter in a way different from the one derived from Euclid, in which even proofs can be substituted by different forms of argumentation. The development of Mathematics Education brought to analyze the links between the different aspects when teaching and learning geometry and to develop further teaching methods in the direction of experimental geometry.

This evolution will be highlighted on the hand of textbooks that proposed themselves as alternative presentations of geometry.

Key words: practical geometry, history of mathematics education, textbooks.

INTRODUCTION

In the course of history of teaching many attempts were made to teach geometry following a path different from that proposed in the Elements of Euclid. The most diffused tendency was to change the order of the theorems and of the problems, also accepting hypothetical constructions and a limited use of arithmetic. In this lecture we will not discuss all attempts, nor the most important, but only those based on the development of the practical aspect of geometry.

Geometry, as mathematics and other sciences, arose out of man's contact with nature. Men first discovered intuitive rules, they then recognized relations between space and its

[1] Reduced draft version for ICME 12 pre-proceedings

measure, classifications, and patterns. In this way the geometrical work of the Egyptians, the Babylonians, the Chinese, and the Indians arose.

The Ancient Greeks, basing their work on the practical geometry of the Egyptians, developed a system of logic that culminated with the great work of Euclid (Walker Stamper, 1909). Practical geometry was nevertheless present also in the Greek world, as shown by the work of Hero.

The Romans were mainly interested in practical geometry (land-surveying and engineering of warfare), also for the teaching in schools; they were influenced by Hero, more than by Euclid. We find among the Romans an interesting didactical appreciation of geometry: Indeed *Quintilianus*, though referring to practical geometry, considered the logical aspect of geometry useful as an aid in oratory, "this science, differently from others, is not useful when it is part of the knowledge, but *in the moment in which it is learned*" (Quintilianus, 35/40-96 a.C., *Istitutio Oratoria*, I, 34 ff.).

From the 10th century a teaching of practical geometry based on the texts of the Roman "agrimensores" was present, joint with the Arab rules based on the use of the astrolabe. But, when speaking of *textbooks* of practical geometry, we mainly refer to that stream which characterised geometry of the Late Middle Ages and started with the work of Leonardo of Pisa, better known as Fibonacci.

So we will begin our excursus with Fibonacci's book "De practica geometriae" (1223). Although this kind of texts can hardly been regarded as schoolbooks, Fibonacci's was largely used in the last grades of the Italian *Scuole d'abaco*, which were parish-schools for pupils who wanted to learn a trade. In these schools pupils were supposed to learn by heart and to memorize rules, but in the text of Fibonacci there is something that goes beyond these assumptions.

PRACTICAL GEOMETRY. MEASURES, NUMBERS, AND OBSERVATIONS.

Fibonacci, "Practica geometriae" (1223) [2].

Fibonacci doesn't write his book in contrast to the Euclidean one. On the contrary, he refers very often to Euclid's propositions. His geometry is simply a different thing, it has the scope of measuring areas and to make partitions of fields. Fibonacci never speaks of axioms or theorems. His proofs are verifications with numbers. Only in the case of the theorems of the 2nd book of Euclid, those concerning geometric algebra, he reports the proofs "translated" in an algebraic language.

Most texts of practical geometry of that time are "really" practical. Rules are given without any explanation. For instance, to measure the volume of a heap of wheat that is in a corner of a room, you have first to render it smooth, then you must insert along the wall, on the two

[2] The English version of the problems is taken from Hughes B., ed., (2008). Fibonacci's de practica geometrie, New York: Springer. Hughes' translation is based on the transcription of the Latin Manuscript by Boncompagni.

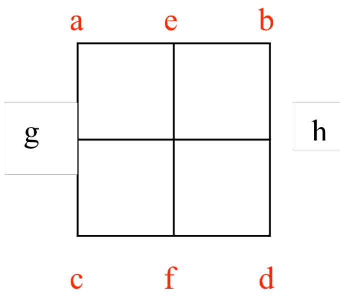
sides of the heap, a wooden meter, keeping it possibly parallel to the ground, then you have to read and multiply the two measures and divide by two (Bassi, 1666).

The book of Fibonacci is different. At the beginning he lists definitions and "principles", which broadly correspond to axioms or to constructions that can be performed (this means, in the sense of Euclid, to existing elements). Then he goes on to the calculation of areas, to the extraction of the square root (which is the opposite operation), to the Pythagoras theorem, to the partition of fields according to given rules, to the determination of heights and distances using similar triangles, and then to other "curious" problems.

These last two topics will characterize the geometry of the late Middle Ages, with contents between "practical" problems and "challenging" problems (Alberti, 1450).

But let's look at one of the first properties shown by Fibonacci, the calculation of the area of a square. As he always does, the rule is given with an example:

> Given a quadrilateral, equilateral and equiangular field having 2 rods on each side, I say that its area is to be found by multiplying side *ac* by its adjacent side *ab*, namely two rods by two rods.
>
> Let lines *ab* and *cd* be divided into two equal parts [this is one of the allowed constructions listed at the beginning] at points *e* and *f*, and draw the line *gh*. Thus quadrilateral *abcd* has been divided into four perpendicular squares, each of which is measured by one rod on a side. Thus there are 4 plane rods in the entire square quadrilateral *abcd*.



This explanation will be sufficient in all later texts of practical geometry. But Fibonacci wants to show that the four quadrilaterals are squares. The proof is based on the fact that, being *ef* equal to and equidistant from *ac* and *bd*, and being *gh* equal to and equidistant from *ab* and *cd*, also *ae* is equidistant and equal to *cf*. Therefore the angle *aef* is equal to the right angle *acf*, a.s.o

[...]

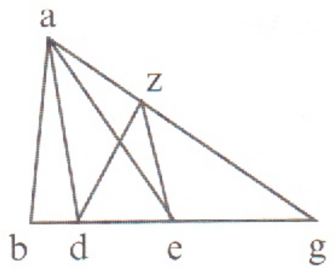
Then Fibonacci gives the rule for the area of a triangle. He doesn't show that a triangle can always be seen as half a rectangle. He shows this only in a case of a rectangular triangle. For the other triangles, he tries to come back to rectangular triangles by dividing the figure. For instance he tries to prove, by absurdum, that

> If a cathete is drawn in a triangle *abg* from angle *bag* that is not less than either angle *abg* or *bga* to side *bg*, then if it is draw from vertex *a*, I say that it will fall within triangle *abg*.
>
> [...]

Also the theorem of Pythagoras is used to measure areas. Fibonacci proves this theorem referring to theorems concerning the similarity of the involved triangles already proved by Euclid.

Problems of "dividing fields" are quite abstract, as the following, which shows how

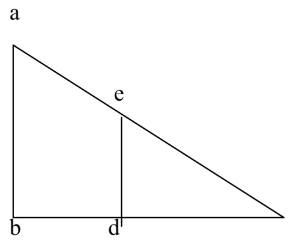
> To divide a triangle in two equal parts by a line drawn from a given point on a side.



Fibonacci first shows the particular case in which the given point is the middle point of the side, and gives later the general case:

> In the triangle *abg*, consider the point *d*. I will divide side *bg* in two equal parts at point *e*, and I will join lines *ad* and *ae*.
>
> [...]

In the chapter about measuring heights, depths, and longitudes of planets, we find practical problems as:



> If you wish to measure a height [*ab*], fix a staff [*ed*] perpendicular to the ground. Step away from the staff and the object you wish to measure. Stoop down the ground level from where you can see the top of the object across the top of the staff, and mark the place from where you looked [*c*].

Considering similar triangles, Fibonacci explains how to find the height. He makes different examples, repeating the argumentations.

Books as that of Fibonacci, may be with a little more Euclidean proofs or at least argumentations, and a minor discussion about the different existing measures will last for about 300 years and will influence teaching in the old universities as well as in the first secondary schools of the 16th century. Examples are the texts of the Italian Luca Pacioli in 1494 and of the French Orontius Finaeus in 1556. In all these texts the tendency is to hold to the special, i.e. to justify a rule by means of a particular case. Rarely results from previous exercises are used, instead most processes are explained again.

Petrus Ramus, "Geometriae libri" (1569)[3]

In the Renaissance Italy loses its mathematical "supremacy" in favour particularly of France. In the 16th century we find another interesting work, that of Pierre de la Ramée (Petrus Ramus) (1515-1572). Also the text by Petrus Ramus should be of practical geometry. Ramus presents more drawing and work instruments then Fibonacci, but he is also a philosopher and an educator, and he explicitly criticizes the presentation of Euclid.

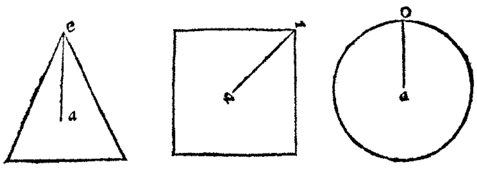
Ramus writes that "geometry is the art of measuring well" (*geometria est ars bene metiendi*). To measure well it is necessary to consider the nature and affections of every thing that is to be measured: to compare such things one with another, to understand their reason and proportion and similitude.... This end of Geometry will appear much more beautiful when you observe "astronomers, geographers, land-meaters, sea-men, enginers,

[3] For the English text we used the translation by William Bedwell**:** *The way to Geometry - Being necessary and usefull for Astronomers. Engineres. Geographers. Architecks. Land-meaters. Carpenters. Sea-men. Paynters. Carvers, &c.,* London: Printed by Thomas Cotes, 1636.

architects, carpenters, painters, and carvers" in the description and measuring of the "starres, countries, lands, engins, seas, buildings, pictures, and statues or images".

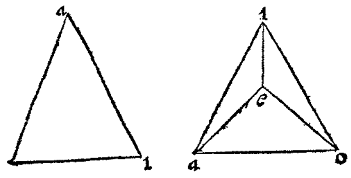
Ramus starts with an extended "presentation" of geometrical entities and figures by means of drawings, of the first measures, of simple constructions. We could call this an *observational* geometry, with which the reader becomes acquainted with the figures and their properties. Some examples:

> 6. [The Diameter is a right line inscribed within the figure by its center].
>
> - The diameters in the same figure are infinite.
>
> - The centre is in the meeting of the diameters.



The latter assertion is, for instance, explained on the basis of the drawn examples and of the definition of the diameter. Ramus makes various observations about the concepts that he introduces, for instance that if a figure has all equal diameters it is a circle.

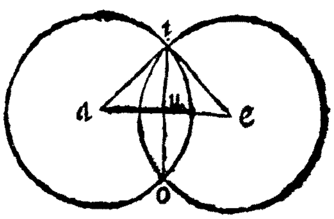
> 11. A prime or first figure, is a figure which cannot be divided into any other figures more simple then it selfe.



[...]

Ramus introduces *drawing instruments* (ruler and compasses), then, referring to previously mentioned properties ha states:

> 12. If two equall peripheries from the ends of a right line given, doe meete on each side of the same, a right line drawne from those meetings, shall divide the right line given into two equall parts.
>
> Let the right line given bee ae. And let two equall peripheries from the ends *a* and *e* meete in *i* and *o*. Then from those meetings let the right line *io*. be drawne. I say, that *ae* is divided into two equall parts, by the said line thus drawne. For by drawing the raies of the equall peripheries ia and ie, the said *io* doth cut the angle *aie* into two equall parts. Therefore the angles *aiu* and *uie* being equall and equicrurall (seeing the shankes are the raies of equall peripheries, by the grant) have equall bases *au* and *ue*. Wherefore seeing the parts *au* and *ue* are equall, *ae* the assigned right line is divided into two equall portions.



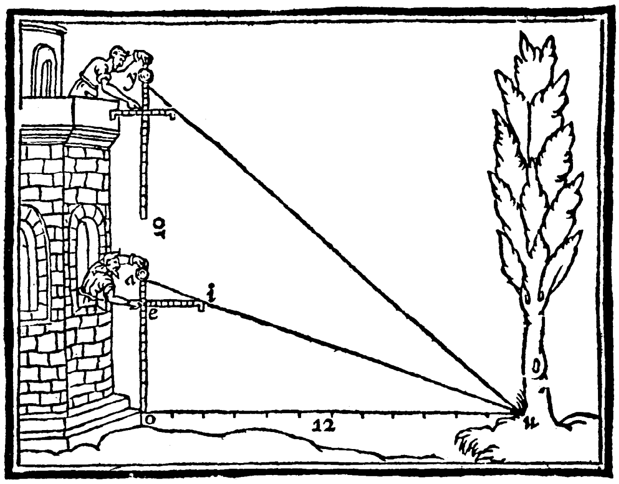
So we can see that Ramus explains his constructions, he proofs that they work.

[...]

The text of Ramus is full of drawings, describing the properties of the figures and the considered relations. Many of them cannot found normally in textbooks. Ramus brings the reader to observe, but in an active way. And he also presents proofs. For instance he proves the construction (that we have seen also in Fibonacci) that leads to the division of a triangle in two parts. [...]

Then Ramus comes to the "classical" problems of practical geometry.

> 7. If the sight be from the beginning of the Index right or plumbe unto the length, and unto the farther end of the same, as the segment of the Index is, unto the segment of the transome, so is the heighth of the measurer unto the length.

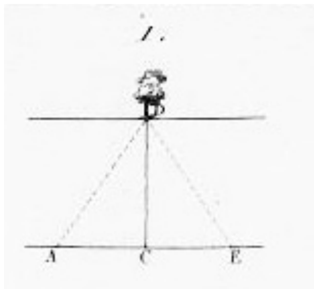


Let therefore the segment of the Index, from the toppe, I meane, unto the transome be 6. parts. The segment of the transome, to wit, from the Index unto the opticke line be 18. The Index, which here is the heighth of the measurer, 4. foote: The length, by the rule of three, shall be 12. foote. The figure is thus, for as ae, is to ei, so is ao, unto ou, by the 12. e vij. For they are like triangles. For *aei*, and *aou*, are right angles: And that which is at *a* is common to them both: Wherefore the remainder is equall to the remainder.

GEOMETRICAL CONSTRUCTIONS

Alexis Clairaut, Elements de Géometrie (1741)[4]

In 1741, again in France, Alexis Clairaut writes his *Elements de Géométrie*. Clairaut is *not* interested in teaching a practical geometry, but he uses it as a didactic tool, to teach via problems. We can say that here we have a real shift from practical geometry as a goal to practical geometry as a means to teach geometry.



The aim of Clairaut is to "construct" the elements he needs to solve a problem.

> A person placed on D, on the bank of a river, wishes to ascertain how far it is from the other bank AB. It is clear, in this case, that to have this measure, it is necessary to take the shortest of all the right lines *DA*, *DB*, etc., which may be drawn from the point *D* to the right line *AB*. [...].

Clairaut at first shows other cases in which a perpendicular is needed, for instance to draw a rectangle, and then goes on with the construction:

> Suppose that on the point C, in the line AB, we wish to raise the line CD, perpendicular to AB. Supposing then that C is at the same distance from A and B, and that the right line CD does not lean either way, it is clear that every point of it will be at an equal distance from A and B. All therefore we have to do is to find any one point D, whose distance from A and B shall be equal;

[4] For the English text we used: Clairaut, Elements of geometry: for the use of the Irish national schools M. Goodwin & Co., 1836

> for in that case if we draw a right line CD, through this point and C, this line will be the perpendicular required.
>
> The point D might be found by repeated trials, but this method would leave the mind unsatisfied...
>
> Take a common measure, or a pair of compasses with a certain opening, according as you may going to operate either on the ground or on paper....

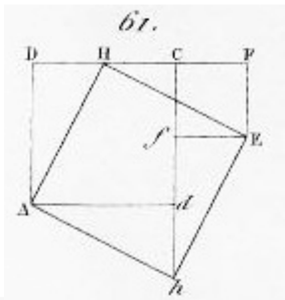
We note the precision of language. Clairaut's text doesn't contain proofs, but constructions and argumentations.

After having shown that the area of a rectangle is given by the product of its base and height (making an analogy with arithmetic and counting the number of unit squares in a rectangle) he shows that the area of a triangle in half the product of basis and height, because "we may easily perceive that this figure (the rectangle) transversely divided by the line AC, which is called a diagonal, resolves itself into two equal triangles". He doesn't prove that the triangles are equal, but he shows that a triangle is completely determined (that means it can be constructed) if: three sides are given, or two sides and the angle between them, ...

The proportionality of the corresponding sides of similar triangles is shown with an example.

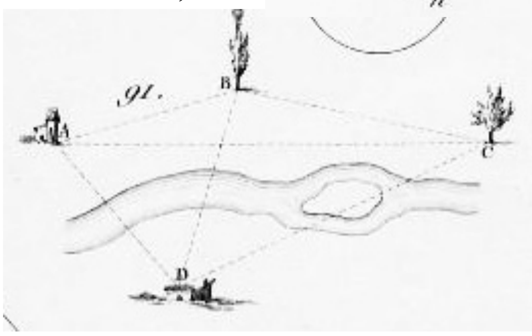
The second part of the text is more rigorous, as Clairaut affirms, not because there are proofs, but because the allowed instruments are only ruler and compasses. But he always proceeds via problem solving, as, for instance, to transform a rectangle into another rectangle with equal area. To sum two squares "*we can transform the smallest square into a rectangle whose height is the side of the other square, so obtaining a new rectangle*".

> [...]
>
> To sum two squares in order to obtain another square, we can work on the decomposition of the two squares, as in following figure, where *H* is taken so that *DH* = *CF* (and consequently *HF* = *fh* = *DC*).



From this figure the theorem of Pythagoras can be deduced (but Clairaut doesn't mention Pythagoras).

Clairaut doesn't make use of the classical theorems of Euclid, but he proves the theorems concerning the angles in a circumference and then goes back to problems of land-surveying; this has the scope of using in a proper way the protractor so as to construct a similar figure on the paper "Knowing the distances between the given points *ABC*, we want to find their distances to a fourth point *D* from which the three are seen".



Clairaut's success came about one century after his text was written. In 1836 it was translated for the use of the Irish national schools and was reprinted in French in 1852 and officially adopted. Also in Italian technical schools (which correspond to the middle schools for the technical instruction) it was used till to the beginning of the 20th century.

At this point we briefly mention the work of Adrien Marie Legendre, whose *Eléments de Géométrie* were published for the first time in Paris in 1794. Legendre's work is well known and his text doesn't concern practical geometry. Legendre displays his topics of geometry by following the classic axiomatic method. The axioms are listed in the initial part of book I. But we can observe some differences with respect to the traditional treatise:

- The number of Legendre's axioms correspond to the number of Euclid's postulates, but their contents are different. His five axioms are not sufficient to infer all his theorems of elementary geometry.

- Legendre first exhibits proofs of his theorems, and later makes use of the obtained results in order to resolve the various problems (that for the most part are constructions). This fact sets up another difference with Euclid's text; in fact, as Euclid argues about figures of known construction, he mixes theorems and problems.

- Legendre uses arithmetic *notations* and elementary algebraic rules.

- Legendre's principal point of view shifts a bit from a logical perspective to an intuitive one. Even within a deduction, he often uses intuitive reasoning.

Therefore there are some elements (the use of arithmetic, a major use of intuition, ...) that also characterize practical geometry and that will slowly find their way in the teaching of geometry.

The text of Legendre was a very successful alternative to Euclid, may be the first one since the existence of stable school systems.

DRAWING, CONSTRUCTIONS, PROOFS

Franz Ritter von Mocnik, Anfangsgründe der Geometrie (≈1860).

The text of Mocnik is addressed to Unterreal- and Bürgerschulen, which correspond to technical and professional schools. The text had many reprints and was used in the Austro-Hungarian territories, including the northern of Italy before the Italian unification; it was also adapted to different schools and school-levels by Spielman, and the reprints will last till into the 20th century.

Mocnik divides his text into a first part of *Formenlehre*, which includes hand-free drawing, some hints to projective geometry, and applications to practical measurements, and a second part of *Grundlehre*, which includes constructions with ruler and compasses (as Clairaut) and also proofs.

Formenlehre reminds to the educator Pestalozzi, who had a particular success in Mittel- and North-European countries, and who supported an initial teaching based on intuition. Pestalozzi is mentioned also by Wilhelm Fiedler, who holds that, starting from central projection, which corresponds to the process of viewing, we can develop the fundamental part of projective geometry in a natural and complete way (Fiedler, 1878, p.248). He feels supported by Pestalozzi and sees these strategies as the best method for the reform of geometry teaching at all levels.

Here we see that a practical geometry can suggest methods of introduction of geometry.

[...]

Mocnik starts with practical indications on how to represent points, on how to represent a line by free hand or with a ruler. He gives many exercises that require free hand drawing.

> If we want to plant a pole, on a field, that has to be aligned with two other poles A and B, then we need to stay behind the pole B while an assistant goes with the pole approximately where the pole should be planted. Then we sign him with the hand how to move till we see the pole C aligned with A and B

Fig. 8.

The concept of distance is primitive; an exercise asks, for instance, to draw five horizontal lines at the same distance; then it is suggested how to compare the length of two segments and how to draw their sum or difference (there is no reference to numbers). Also multiples or submultiples of a segment have to be found approximately by drawing. Then the unit measure is introduced and some information is given about new and old measures, and about measuring instruments.

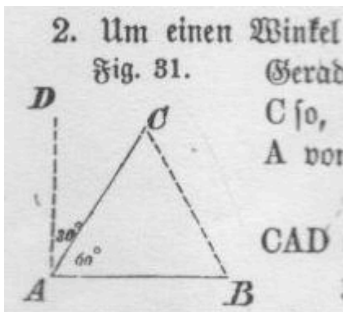
The angle is introduced as the difference of the orientation of two straight lines, and the perpendicular [to a line *AB*] is defined as in Clairaut, as a line *which must not lean to the line AB more in the direction of the right hand than of the left.* But he also adds that it must form two equal angles with the line *AB*.

This and other considerations (as the diagonals of a square) help to draw perpendiculars with the free hand. On a field a perpendicular has to be drawn using a special cross.

Then the protractor is introduced. We note that measuring instruments are introduced only after that the concept has been explored by free hand drawing and other reflections. But the use of instruments is not an aim, their use is not often suggested.

For instance, after the introduction of the protractor, Mocnik shows how to draw an angle of 60°:

> We take a segment *AB*, and a point *C* whose distance from *A* and *B* should be as the distance from *A* to *B*. Then we trace *AC*. *BAC* = 60°.
>
> If we then trace the perpendicular *AD* to *AB*, then *CAD* = 30°

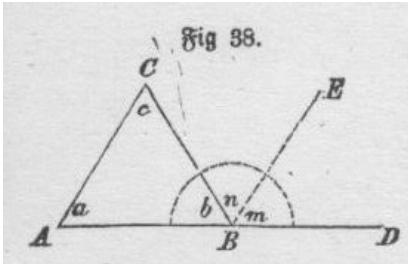


[...]

Mocnik doesn't explain all the rules and methods that he gives, nevertheless we cannot say that his is really a practical geometry. The suggested work is not for professional workers, but aims at giving the pupil a familiarity with the considered topic. In a certain sense it has something in common with Ramus.

And in fact he slowly increases the level of rigor.

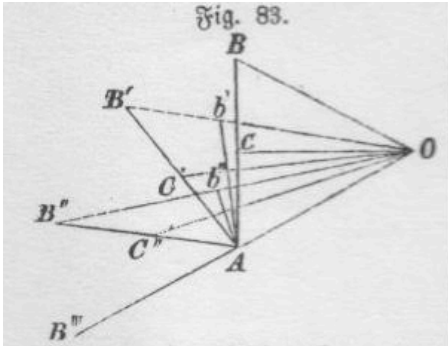
The first proof (which in anyway still in the part of "Formenlehre") concerns the exterior angle of a triangle. It is presented after having simply stated the equalities between angels formed by a transversal to two parallel lines:



Congruent triangles are, in this first part, simply triangles which have three equal sides and three equal angles. Mocnik suggests to draw congruent triangles using a translation. Angles at the centre and at the circumference are at first defined without stating properties.

Then he passes to describe solids in the space and gives the first elements of perspective, for instance:

> If a line [a segment] is rotated from the position *AB* ... around *A* to the positions *AB*', *AB*'', *AB*''', it is seen under different angles and with different lengths. The apparent length of the segments is obtained when, from point *A*, we draw – within the corresponding angle of view – a perpendicular to the central ray

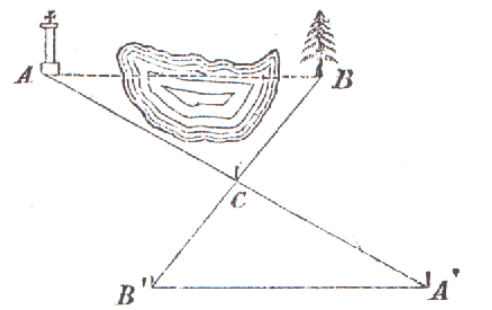


In the part concerning *Grundlehre*, the first proof concerns the equality of two triangles with three equal sides. The proof is made showing how to construct the triangle with the compasses, and then superposing the corresponding parts. [...]

The theorems are always alternated with exercises that require constructions. For instance, after having proved properties of quadrilaterals, Mocnik shows the construction of a perpendicular using ruler and compasses. Then he comes back to practical applications of the congruence of triangles:

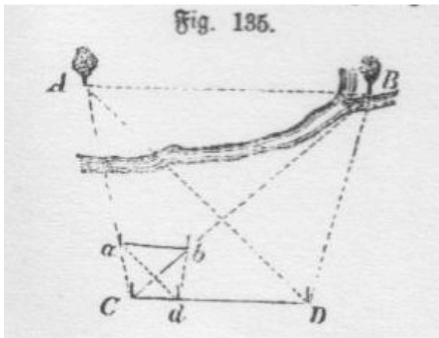
> To determine the distance of two points on a field, if this cannot be measured directly due to an obstacle between them, but if it is possible to measure from a third point to both.

Fig. 119.



... and to practical applications of the similarity:

"To determine the distance between two un-reachable points" you have to fix two points C and D, and to measure the angles CDB...



Only at this point Mocnik comes to the measurement of areas. The area of a square is determined at first with an example, dividing each side in three parts. Exactly as Clairaut, he at first sees the figure as decomposed in three rectangles of area 3 units, and then the whole area of 9 units.

[...]

The theorem of Pythagoras is proved exactly in the way of Clairaut, but Mocnik also proposes to draw the figure on paper and to cut the various parts in order to obtain the considered squares.

Again he comes back to practical problems, as to copy parts of the land on paper. Here he also describes instruments used by surveyors. [...]

He also comes back to theorems, proving those concerning the circumference, and also conic sections and geometry in space.

GEOMETRICAL TRANSFORMATIONS

In the second half of the 19th century many texts modified the introduction to geometry trying also to include "new geometries". We find the first approaches that use geometrical transformations, in particular translations, to introduce the concepts of straight line and parallelism; and rotations to introduce the concept of angle and perpendicularity (see Meray, 1874). In this period also new topics as conic sections, a larger part of trigonometry, and analytical geometry make their first appearance in the curriculum.

Julius Henrici & Peter Treutlein, Lehrbuch der Elementar Geometrie (3 v., 1881-1883)

This is a text for the Gymnasium, more precisely for pupils in the Tertia, which means in the age of about fourteen years. It is a text of synthetic geometry, with axioms and theorems. But it presents relevant differences from the Euclidean text. The authors state that they pursue "a logical order of the concepts rather then of the theorems", as in many recent developments of geometry. In particular the matter is developed according [...] to the necessary geometrical transformations. The authors add also that it is important that the teacher concretely performs the movements in the classroom using models.

The use of geometric transformation is here more relevant then in Méray. At first figures with a centre are introduced, and the central symmetry is use to define parallel lines.

To draw parallel lines the use of a set-square is suggested.

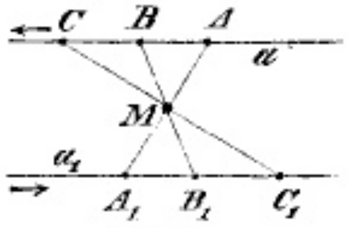


Fig. 43.

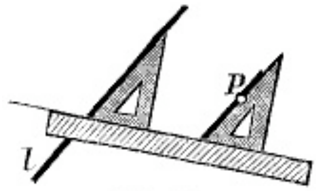


Fig. 39.

Then figures with an axis of symmetry are introduced, and the axial symmetry is used to prove theorems related to the isosceles triangle.

Let's see how the central symmetry is used to prove that the medians of a triangle meet in one point.

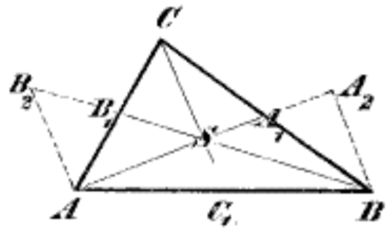


Fig. 77.

> If two of them, AA_1 and BB_1, meet in S, we rotate the segment CS around B_1 in AB_2 and around A_1 in BA_2, so CS is parallel and equal to AB_2 and BA_2. So also AB_2 and BA_2 can be brought to coincide and the centre of the rotation must be S. The three parallel CS, AB_2, BA_2 must cut equal segments on AB as on AA_2 and BB_2 [so also CS is a median].

But, also the kind of theorems change. After the introduction of translations and rotations, we find proofs of "new theorems" concerning, for instance, how to find the transformations that bring two equal triangles to coincide.

[...]

After the circle had been introduced, the drawing with ruler and compasses can be performed, and it is a relevant part. A method is given on how to solve problems of constructions. The first are the usual problems to construct a perpendicular or a parallel line, as we can find in Clairaut, but we arrive soon to more demanding problems, as to draw tangents to circles with certain conditions, or, for instance, to draw the common tangent to two given circles.

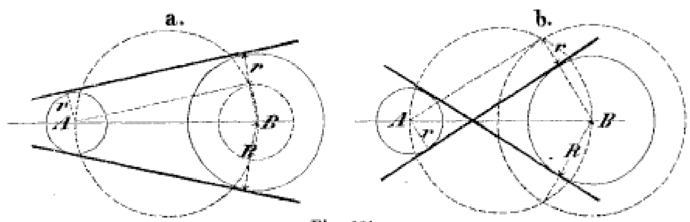
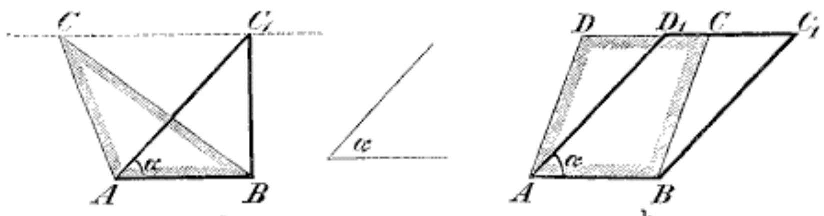


Fig. 121.

[...]

Areas are introduced with reference to the II book of Euclid (geometric algebra), to the proportions for the area of rectangles of equal bases, to the equi-decomposition (which leads also to the "Parallelverschiebung").



a. Fig. 178. b.

The theorems of Euclid and Pythagoras are thus proved with reference to equivalent transformations and not to proportions.

This chapter is about the "transformations of figures", and contains exercises similar to those of Clairaut in an analogous chapter, but also problems of "divisions of figures" that are exactly those of Fibonacci about the "division of fields".

[...]

In the second volume similarity is introduced, and also elements of projective geometry. This latter is more extended with respect to the part on perspective of Mocnik, who rather refers to the geometric optic of Euclid. We find topics as harmonic points, pole and polar, projective correspondences; [...]

Then the text presents problems of *practical geometry*, mainly as an application of trigonometry. For instance a problem that Mocnik solves with the help of similar representation and concrete measurement is solved using Carnot's theorem. [...]

Èmile Borel, Geometrie: premier et second cycles (1905).

Èmile Borel writes in his introduction that geometry is the study of the groups of movements and that the tendency is to substitute a dynamic study to a static one. He uses a mainly synthetic geometry, numbers appear only in the 3rd part, with proportions and geometric algebra.

Borel starts describing drawing instruments, as ruler, compasses, set square, giving only a hint about their use. Perpendiculars are introduced by successive folding of a sheet of paper.

He defines triangles and quadrilaterals and goes on to the measures, which are "the scope of geometry". The rules for finding the measure of plane and solid figures are given without an explanation, Borel doesn't even explain why the area of a triangle is half the basis times the height. He also "gives" the theorem of Pythagoras.

Geometrical transformations are used to *define* particular relations or figures. For instance:

Two figures are equal if they come to coincide "moving them" or folding the paper. In this latter case the axial symmetry is introduced and perpendicular lines are (again) defined, as the lines joining two corresponding points of an axial symmetry.

In all his text Borel comes often back to an argument, giving new definitions or properties and increasing the level of rigor. The first proof is the following:

The geometric locus of the points which are at the same distance from two point A and B is the perpendicular to AB through its middle point.

[...]

After the introduction of the circumferences with their secants and tangents the construction with ruler and compass of the perpendicular is introduced. But then again the construction of the perpendicular with the set square is given.

Parallels lines are again defined by means of a translation, so parallelograms can be introduced. Then Borel introduces the rotation, the angles and their measures, the protractor, the central symmetry.

As in Clairaut, the equality of triangles is seen as the possibility of constructing them.

There are proofs in Borel's text, even if we don't find explicitly axioms. Geometric transformations are not used to prove theorems, except the idea of symmetry. Even if he speaks at the beginning of groups of transformations, ha never makes compositions of transformations.

[...]

NEW TRENDS IN GEOMETRY TEACHING

Intuitive geometry and concrete materials

In 1901 texts of *intuitive geometry* comes to life in Italy, addressed to pupils in the first three years of the Gymnasium (the "lower Gymnasium" corresponding to the present middle school for pupils in the age 11-14). An earlier intuitive experimental approach is considered a good help for students to overcome the difficulties caused by the logical deduction of Euclid's textbook. Geometrical drawing should also contribute to overcome these difficulties.

These textbooks share ideas with many foreign textbooks of the time looking for a new methodology in geometry teaching. [...] Geometric transformations too are considered suitable for an intuitive introduction to geometry: as a *tool*. Motions can in fact be carried out experimentally and can be used to introduce simple concepts and to compare segments and figures. Venonese (1901) introduces parallel lines with a central symmetry, as Henrici & Treutlein, in order to avoid the concept of infinity. Frattini (1901) uses paper folding to introduce the concept of perpendicularity, as later Borel, and adds in his text some practical proofs. For instance, to prove that *the diagonals of a parallelogram bisect each other,* he proposes to cut out the parallelogram from a piece of paper, and to fill the empty space placing the parallelogram back after a rotation of 180 degrees. In a certain sense, we find here the first reference to the use of concrete materials, that will develop with the work of Amaldi (1941) and then of Emma Castelnuovo in 1946 (Menghini, 2010), who was also largely inspired by the approach via problems of Clairaut.

[...]

Experimental geometry

At the beginning of the 20th century an international reform movement starts, which on the one hand aims to introduce analytical geometry and calculus in secondary school, on the other hand proposes new methodologies in mathematics teaching. In many countries new syllabi are established. As to geometry a practical/intuitive approach is suggested. The text of Borel, already mentioned, is in fact written to fulfil the new French sillabi of 1905 (see Nabonnand, 2007). The various reprints of Henrici & Treutlein seem to fulfil the German sillabi of 1907 (see Becker, 1994).

The "experimental geometry" of John Perry will have a major influence at an international level. Perry (1901) emphasizes the educational value of experimental procedures in the first approach to Euclidean geometry. He retaines that a larger part of elementary geometry be assumed as axiomatic and that the subject matter be taught with reference to its utility and to the interest of the child (see Barbin & Menghini, to appear).

Joseph Harrison (1903) Practical plane and solid geometry.

This text, which had many reprints, was written according to the proposals of Perry, and even presents an appreciation of Perry himself in the preface.

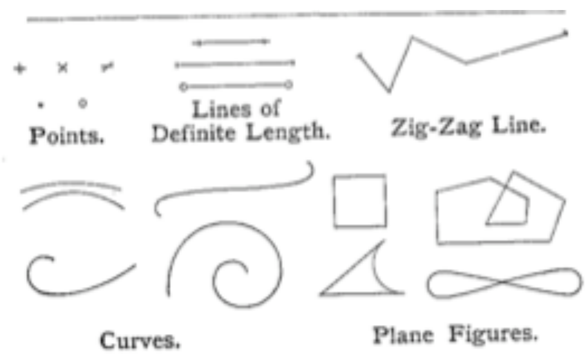
Harrison states in the preface that many of the British schools and colleges are equipped with laboratories in which experimental work involving quantitative measurements can be carried on as part of the ordinary school course, and it is coming to be recognized that elementary mathematics should be taught in relation to such work.

The syllabus that the author reports at the end of the book (Science Subject I, Board of Education, South Kensington) states that "This subject [practical plane and solid geometry] comprises the graphical representation of position and form and the graphical solution of problems. [...] in the elementary stage the main object of the instruction will be to familiarize the student with the fundamental properties of geometrical figures and their applications [...] it is not intended that the student shall follow the Euclid's sequence [...]. In the Advanced stage and in Honours the subject will be developed in its applications [Engineering and physical group..].

According to Harrison, in geometry the principal aim is that the reader shall become acquainted with the representation and the measurement of lengths and angles in space.

Harrison starts with a detailed description of drawing instruments and their use, and gives a sort of definition of points, lines surfaces which is in between Euclid's definitions, Ramus' drawings and Mocnik's representation of points.

The aim of the drawing of points is to explain how a point is visible in a drawing.

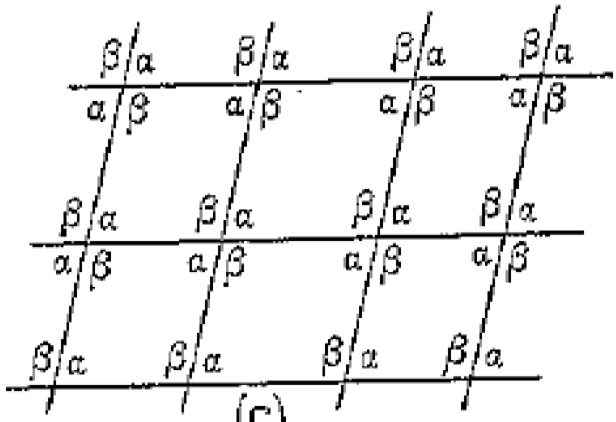


Harrison gives a long explanation of the measures of the angles, he gives also the definition of the basic trigonometric functions and the explanation of trigonometric tables.

Then he explains how to draw parallel and perpendicular lines using various kinds of instruments (ruler and set square, tee square and clinograph). He also presents the construction with ruler and compass, without explaining it.

Defining a parallelogram he states:

> Any figure with four straight sides is called a quadrilateral. A parallelogram is a quadrilateral in which opposite sides are parallel. Thus in fig. c the two systems of parallel sides give rise to a series of parallelograms. From the figure it is evident that opposite angles of a parallelogram are equal. And by measurement the student will easily satisfy himself that opposite sides are also equal.



(C)

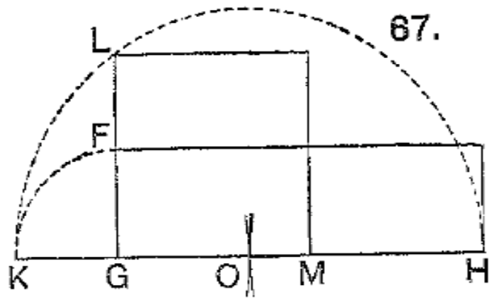
No other explanation is given.

Again by measuring or by paper cutting the student can verify the Pythagoras theorem

[...]

The problem: "To reduce a given rectangle *FGH* to a square of equal area" is solved:

> Produce one side *HG* to *K*, making *GK* equal to the other side *GF*. Bisect *HK* in *O*, and with centre *O* describe the semicircle on *HK*. Produce (if necessary) *GF* to *L*. On the side *GL* construct the required square.



Harrison writes that "the reason will appear in the next chapter". In fact, were find the *theorem*:

> If lines passing through a fixed point cut a circle, the product of the segments is constant.
>
> Verification. Draw any circle, centre C. Mark any inside point O. Join CO. Draw MOM perpendicular to CO. Observe that MM is bisected at O. Now confirm, by careful measurement and arithmetic that:
>
> OA x OA' = OB x OB' = OM x OM = OM^2 = constant.

[...]

In the same way the theorem is "proved" that:

> The angle in a semicircle is a right angle.
>
> Verification: Draw any semicircle, diameter AA. Set out several angles ABA in the semicircle. Verify that in all cases the angle ABA is 90°.

[...]

Practical geometry in space is on methods on representing solids, mainly by projections on the coordinate planes. These exercises are exercises of technical drawing, which are not very elementary. They require, for instance, the use of trigonometry to calculate angles and lengths (of planes, lines, segments).

Charles Davison, Plane Geometry for secondary schools. Cambridge: At the University Press (1907).

We mention this textbook only as an answer to the text of Harrison. [...] According to the author:

> Most recent textbooks of geometry contain an introduction on practical geometry. While presupposing a short preliminary course of this nature, we have preferred to leave it to the teacher to devise himself. In this direction we think that the recent reforms have gone too far, and we feel sure that, as regards secondary schools, it will be necessary to retrace our steps. Too much time spent on experimental and graphical work is wearisome and of little value to intelligent pupils. They can appreciate the logical training of theoretical geometry, while experiments of far greater interest can be made in the physical and chemical laboratories.

George Birkhoff & Ralpf Beatley, *Basic Geometry*, 1941

The authors start with many exercises that require the use of drawing instruments, measuring of lengths and angles, and verifying rules about their addition and subtraction. But measure is not an aim. The aim of the practical work is to bring pupils to accept the axioms, in particular those about

> Line measure: the points of any straight line can be numbered so that number differences measure distances.

and

> Angle measure: All half lines having the same end point can be numbered so that number differences measure angles.

[...]

School Mathematics Project, 1970

In the 60s, in the period of the New Math reforms, the British School Mathematics Project still presents elements of practical geometry. A chapter is devoted to the measuring of an angle, to the use of the protractor, and to applications as “To measure the height of a hill”. Pupils are supposed to find the answer by drawing a similar situation on paper and measuring.

A chapter is explicitly devoted to land-surveying. The aim is to apply what has been seen in the previous chapters, but also to introduce some new concept. The exercises mainly require to copy on paper a piece of land by measuring certain distances and angles and using similarity (which hasn’t yet had a proper chapter), as in some examples of Mocnik.

Geometrical transformations are also introduced, with a new kind of theorems regarding their composition.

[...].

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[...]