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MATHEMATICAL PROBLEM SOLVING BEYOND SCHOOL: DIGITAL TOOLS AND STUDENTS MATHEMATICAL REPRESENTATIONS

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By looking at the global context of two inclusive mathematical problem solving competitions, the Problem@Web Project intends to study young students' beyond-school problem solving activity. The theoretical framework is aiming to integrate a perspective on problem solving that emphasises understanding and expressing thinking with a view on the representational practices connected to students' digital mathematical performance. Two contextual problems involving time-variation are the basis for the analysis of students' digital answers and an opportunity to look at the ways in which their conceptualisations emerge from a blend of pictorial and schematic digital representations.
Keywords: Problem solving; Expressing thinking; Digital mathematical performance; Competitions.

INTRODUCTION

For many years, mathematical problem solving has been positioned as a central research theme in mathematics education even though with fluctuations in intensity level and obvious nuances in trends across countries and research groups (Törner, Schoenfeld & Reiss, 2007). Alongside mathematics curricula and educational orientations tend to renew the attention devoted to problem solving skills among the range of mathematical abilities that students are expected to develop in general and vocational studies throughout their school trajectories.

Not less significant than mathematical problem solving is the mobilization of efforts in the research on technology use in mathematics teaching and learning. So far most of the research has put its gravitational centre in particular resources (software packages, calculators, spreadsheets, applets, or interactive whiteboards) that continue to be developed and implemented in straight connection with the teaching and learning of particular mathematical topics and mathematical methods. Moreover the studies on the use of digital technology focusing on classroom tasks are in obvious dominance. This body of research has already shown that a crucial factor in the success of ICT integration lies in the teacher and students' roles and in the ways in which the content material is approached, i.e., in the type of learning environment that is generated through the use of technology.

It is now naturally expected that research into problem solving seek for new advancements by taking into account the solver's use of technological devices. Although not many studies consider these two aspects in a clear and neat way it may be argued that research involving the integration of technology in the classroom generally reflects a problem solving approach or an investigative perspective into mathematical topics. In general there is a growing awareness of the ways in which the use of digital tools, namely computer technology, largely due to their multi-representational and dynamical nature, changes and reshapes mathematical problems.

Furthermore we are now educating a generation of students who more often than not use digital technologies out of school, regardless the serious efforts that many countries are endorsing to commit schools and educators to the technological uptake. Surely many of those students are regularly surfing the Internet or communicating on Facebook and doing a lot of things far from mathematics and mathematical problem solving. But in some cases they may actually be at their personal computers, possibly at home, and engage in mathematical problem solving while participating in web-based mathematical competitions.

One of such particular contexts, which is presented in detail in the following section, combines explicit mathematical activity in problem solving with the use of digital tools, in particular home computer technology, in a web-based environment that extends beyond school. It is therefore a propitious context to study specific aspects of students' mathematical problem solving in view of their use of digital tools. It renders the opportunity to discover how students engage in problem solving and what kind of technology use they reveal in their solutions to mathematical problems beyond their school learning.

As part of the ongoing research work within the Problem@Web project, I will address here examples of the *forms of expressing mathematical thinking* in problems involving time-variation from participants in digital mathematical competitions. Such problems may of course evoke kinesthetic images on the problem solver and our purpose is to identify aspects of their use of digital representations that reflect their understanding of the time-variation situations and represent powerful conceptual models beyond typical school-like solutions.

THE CONTEXT OF WEB-BASED MATHEMATICAL COMPETITIONS

One of the conclusions drawn by large scale international studies on students' mathematical performance is the fact that students really learn mathematics outside school. On the other hand learning mathematics beyond the classroom is particularly supported today (sometimes encouraged by peers, teachers and parents) considering the availability of versatile technological environments (Freiman, Kadijevich, Kuntz, Pozdnyakov, & Stedøy, 2009; Haapasalo, 2007).

> The fact that students seem to learn as well mathematical as technical skills effectively outside the classroom, forces us to ask if there is something wrong inside school as far as the question "how to learn" is concerned (Haapasalo, 2007, p. 9).

The numerous and diverse mathematical competitions and enrichment activities taking place regionally, nationally or internationally are a way of extending mathematics learning beyond the classroom. As the recent 16th ICMI Study has pointed out learning mathematics beyond the classroom may be based upon multi-day mathematical competitions for students of a wide range of mathematical abilities (Kenderov, Rejali, Bussi, Pandelieva, Richter, Maschietto, Kadijevich, & Tayor, 2009). Also, as remarked by authors from countries with a long tradition in mathematical competitions and highly competitive-driven systems, the uprising of a large number of new competitions “reflects a possible shift in the focus and purpose of competitions away from a strictly talent-search model to a more inclusive ‘enrichment’ approach” (Stockton, 2012, p. 37).

Despite the many variants in content, duration and participants in today’s widespread competitions, it has been shown that they all can have high motivational effects, especially for younger students (Kenderov et al, 2009; Freiman, Vézina & Gandaho, 2005; Freiman & Vézina, 2006). For example, in what concerns the benefits perceived by the students, the CAMI project highlights very positive and largely distributed advantages for different ability-level students: “the children that might benefit most from the project are mathematically able students (44 of 66 answers), next come children having difficulties (31 of 66), and slow-working kids (25 of 66)” (Freiman & Vézina, 2006, p. 7).

The overall aims of mathematical competitions are therefore reaching much further than the identification and selection of mathematically gifted students. They are becoming one of the many places of mathematics education where mathematics is presented as challenging, exciting, accessible to average students, socially and emotionally engaging – since many of them set up team work or inter-schools tournaments or mathematics dissemination to the public – and closer to the daily aspects of students’ lives. As Freiman & Vézina (2006) state, these new forms of mathematical competitions, including virtual and on-line contests and the surrounding attractive materials created, become good examples of partnership between schools, universities and families.

While these new formats and purposes of mathematical competitions are being established, the type of mathematical knowledge targeted and the questions aiming for higher problem solving skills start to give way to the idea of challenges and challenging situations to the average people. “While challenges have always been part of mathematical exposition in some small way, they have now come to the forefront in our conception of classroom practice and public exposition” (Barbeau, 2009, p. 9). Thus, according to Kenderov et al (2009), mathematics education “beyond the classroom” has attained an important and irreplaceable role: to challenge the minds, skills and talent of youngsters. This explains why mathematical competitions are being conceived as challenging environments where young people can expose and develop their skills in the field of mathematics. In the words of the authors, there is now a “world of mathematics competitions” involving millions of young people, teachers, mathematicians, educators and schools, sponsors and parents.

As a result, two different kinds of competition are presently coexisting: inclusive, for students with a wide variety of capacities, and exclusive for especially talented students generally requiring filtering and selection of the participants. Web-based mathematical competitions that run online and involve substantial electronic communication are typically among those of the inclusive genre, characterised by being open to a large number of participants, closer to schools and teachers, and surrounded by a certain sense of community development.

Two of such competitions are promoted by the Mathematics Department of the Faculty of Sciences and Technology of the University of Algarve, in the southern region of Portugal – Sub12® and Sub14®.

A description of Sub12 and Sub14

The two competitions have been running since 2005 and are aimed at two school levels: the Sub12 addresses students in 5th and 6th grades (10-12 year-olds) and the Sub14 addresses students in 7th and 8th grades (12-14 year-olds). The two competitions are web-based, located in two web-pages in the same web-site (http://fctec.ualg.pt/matematica/5estrelas/), have similar rules and operate in parallel. They involve two distinct phases: the *Qualifying* and the *Final*. The Qualifying phase develops entirely at distance through the website and consists of a set of ten problems each posted every two weeks. Students are invited to engage with the competition by means of informative flyers distributed to all the schools of the region covered by the competition. There is no formal enrolment process since the way to step in is just answering to the first problem posted. Failure in one of the problems will be signalled with a yellow card, and those participants who exceed two yellow cards get a red card and are eliminated from the competition. The failure cases fall into three modalities: no answer given to the problem, a wrong answer, or critical omissions in the solution, especially the lack of an explanation of the problem solving process.

During the Qualifying participants may participate individually or in small teams of two or three elements. They send their answers to the problems by e-mail or through the electronic message editor available at the website, having the choice of writing their solutions completely on the e-mail window or attaching files to their messages. The problem statement is displayed on the web-page and it can be downloaded as a pdf file. Students' answers are received in e-mail accounts specifically devoted to that purpose and a team of senior mathematics teachers reply to every participant, by giving a formative and encouraging feedback, suggesting reformulations when needed and offering clues to help overcoming obstacles or just praising good answers and cheering the progresses made. Students are allowed to submit revised solutions as many times as needed within the respective deadline.

The web-page also includes a specific area for news where the organising team places relevant information not only to the participants but also to teachers and parents, concerning the rules of the competition or announcements, or even an incentive for teachers and parents to encourage their youngsters. A table with the participants' results is periodically posted and 10-20 selected answers, seeking to illustrate the diversity of solutions produced and showing good clear explanations of the mathematical processes, are also made available.

The number of students participating in the two competitions has been growing over the years and it now reaches about 2000 participants in Sub12 and 800 in Sub14. The development of the Qualifying is characterised by an asymptotic decrease in the number of participants, mostly caused by abandoning, which is more pronounced around the middle of the competition. Usually 10% to 15% of the total initial participants reach the Final phase.

The Final is a half-day on-site contest held at the university campus with the presence of the finalists, their families and also teachers. At the Final students are given a set of five problems to be solved in one hour and a half. Everyone is competing individually and there is no technology available. The students' written answers to the problems are corrected anonymously by a jury. In the meantime parents, teachers and other accompanying guests have a program devoted to them, including a workshop, an exhibition, a seminar or other forms of interactive activities related to mathematics and mathematical problem solving. The Final includes a coffee-break, sometimes music and folk dance, and culminates with the awarding ceremony of the three winners who receive prizes and honour diplomas.

Throughout the history of this competition a number of distinctive characteristics have been standing out: i) it is based on mathematical challenges that can be labelled as *contextual word problems* usually allowing several ways to be solved; ii) it values communication competences and explicitly requires exposing the strategy and the procedures followed to find a solution; iii) it ensures that students have the mathematical knowledge required do deal with the problems proposed; iv) it is curriculum-detached, meaning that problems are not chosen to fit any particular school curricular topic (yet they may involve solid and plane geometry, algebra, counting, logic, numbers, variables and change); v) it is close to teachers and families in the sense that it encourages their support to the young participants; vi) it is formative and friendly assessing by offering opportunities for problem solving improvement and giving hints if necessary; vii) it welcomes all types of media to attain and to deliver solutions (either the use of digital tools or image-scanned paper and pencil work); viii) it favours persistence and commitment often related to the involvement of parents (although e-mail communication always addresses the participants themselves); ix) it gives public recognition to the more precise, creative, aesthetic, interesting solutions by publishing them on the website; x) it concentrates the competitive component in the Final, promoting collaboration and sharing during the Qualifying (for instance some teachers discuss the problems with their students in mathematics classes or in monitored study and help them on technological matters).

THE PROBLEM@WEB PROJECT: RESEARCHING WEB-BASED MATHEMATICAL PROBLEM SOLVING

In Portugal, despite curricular orientations, problem solving based practices have been proved deficient over the years in school mathematics (Matos, 2008). Official reports have highlighted important weaknesses regarding students' mathematical competence at the end of compulsory school: low in problem solving across curricular topics; low in communication, namely in the interpretation and use of diverse mathematical representations; fair in reasoning about simple situations but very low in deductive thinking. One of the specified reasons to explain these facts is that students are not frequently exposed, in their regular schooling, to problem solving where analysis and interpretation are required.

In 2006, the Ministry of Education has launched a Plan of Action for Mathematics, aimed at raising students' attainment in mathematics (Ministério da Educação, 2006). As a result, the development of three transversal capacities is now considered mandatory in K-9 mathematics: (a) problem solving, (b) mathematical reasoning and (c) communication (Ministério da Educação, 2007). Therefore, in conjunction with the growing interest of teachers and mathematics educators on mathematical competitions, mathematical problem solving has been repositioned on the agenda of mathematics education in Portugal.

On the other hand the massive research produced for more than five decades on developing students' abilities in problem solving is judged to having little to offer to school practice because it misses to explain average students' difficulties related to devising a model of the situation presented in a problem (Lester & Kehle, 2003; Lesh & Zawojewski, 2007).

Also Francisco & Maher (2005) reflect on the excessive focusing of research on generating and describing taxonomies of students' problem-solving heuristics which results in an inability to perceive what is actually the mathematical reasoning of students when they solve problems. In their study, they view "mathematical learning and reasoning as integral parts of the process of problem solving" (p. 362). They have concluded that engaging students in strands of complex tasks promotes meaningful and thoughtful mathematical activity and showed that students were able to overcome cognitive obstacles sometimes by reference to some prototypical problems. Moreover, their research sparked the idea of the ownership of students' activity in problem solving, accounting for the fact that they were able to come up with different and interesting ways of thinking about the problems.

It is now becoming consensual that research on problem solving needs to find new directions and new empirical fields to understand the nature of humans' approaches to *mathematizable* situations. English, Lesh & Fennewald (2008) have identified some of the important drawbacks in the existing knowledge about mathematical problem solving, including the following: limited research on concept development and problem solving, and limited knowledge of students' problem solving beyond the classroom.

This also takes us to the recognition of a new generation of students – youngsters who are developing, mostly out of school, a large number of competences, which grant them the skills and sophistication required to learn beyond the school barriers. Often described as "digital natives" (Prensky, 2001, 2006), they access information very fast, are able to process several tasks simultaneously, prefer working when connected to the Web and their achievement increases by frequent and immediate rewards. In the specific context of digital mathematical competitions participants can communicate their reasoning on the problems in an inventive way and can resort to any type of technological tools. Home digital technologies play a role in tackling the problems and in communicating about them, thus adding competences that sometimes school neglects or forgets (Jacinto, Amado & Carreira, 2009).

In light of the above directions and trends, the Problem@Web project was launched to seize the opportunity of studying students' mathematical problem solving beyond the mathematics classroom. By looking at the global context of the competitions Sub12 and Sub14 as a rich multi-faceted environment, the project intends to explore, in an integrated way, issues that combine cognitive, affective and social aspects of the problem solving activity of 21st century young students. The research field is clearly based on inclusive mathematical competitions, mainly taking place through the Internet, inducing strong digital communicative activity and having resonance with students' homes and lives.

The research focuses of the project concern:

(a) Ways of thinking and strategies in mathematical problem solving, forms of representation and expression of mathematical thinking and technology-supported problem solving approaches;

(b) Creativity as manifested in the expression of mathematical solutions to problems and its relation to the use of digital home technologies;

(c) Attitudes and affect in mathematics and mathematical problem solving, considering students, parents and teachers;

(d) The role of feedback, communication and participation in mathematical activity within a digital and virtual environment.

The empirical work integrates two main modes of data collection (extensive and detailed) and the data analysis will combine quantitative and qualitative methods.

The extensive data include:

- Records of the exchanged e-mail messages in each “15 days stage of the competition”.
- Records of selected participants’ solutions to all the problems in the course of the competition.
- Online questionnaires to the participants (in the middle and in the end of the competition).
- Collection of all the finalists’ papers in the Final.

The detailed data include:

- Interviews with a small number of participants who get to the Final.
- Interviews with a small number of participants who drop out of the competition.
- Interviews with parents (or family members) of participants.
- Interviews with teachers who have students participating.
- Video recordings of school classes or other school sites where teachers work on the problems with their students.
- Video recordings of the on-site Final.

The project also aims to develop a coherent theoretical framework to investigate problem solving within the context of participation in virtual mathematical competitions. The theoretical developments are being undertaken in three directions: i) Problem solving and creativity; ii) School and beyond school mathematics; and iii) Communication and feedback in virtual environments.

THEORETICAL FRAMEWORK

Concerning the overarching theoretical perspectives from the ongoing research project, the emphasis is being placed on problem solving as part of understanding mathematics and being able to engage in mathematizable situations. At the same time, theoretical concepts regarding the use of digital technologies in students’ mathematical activity are focusing on mathematical representation and on the role of imagery for explaining thinking.

Problem solving from the point of view of expressing thinking

One of the characteristics of the data gathered from students’ answers to the problems proposed in the competitions Sub 12 and Sub14 is the fact that they have a digital format, totally framed by electronic communication. Participants are required to give a clear explanation of the problem solving process to some virtual people on the other side of the e-mail connection. This well marked aspect of our data is leading us to pursue the notion of *explaining*, *exposing*, or *expressing* mathematical problem solving as a fundamental aspect of the problem solving process as a whole. In this sense, we share the kind of questioning formulated by Lesh & English (2005, p. 193):

> In what ways is "mathematical thinking" becoming more multi-media – and more contextualized (in the sense that knowledge and abilities are organized around experience as much as around abstractions, and in the sense that relevant ways of thinking usually need to draw on ways for thinking that seldom fall within the scope of a single discipline or textbook topic area)?

One of the fundamental ideas we endorse is that mathematical problem solving means *ways of thinking about challenging situations where a mathematical approach is appropriate*, even if the problem solver may not recognise such thinking as being a typical mathematical activity or may not draw on school mathematics knowledge, as it is often the case of our participants.

In this sense, particular concepts and notions derived from the models-and-modelling perspective (MMP) (Lesh & Doerr, 2003a) reveal promising directions to understand mathematical problem solving as an activity mainly organized around experience. One of such notions refers to productive ways of thinking. Lesh & Zawojewski (2007) put it clearly by stating that a problem may be any situation or task where the problem solver feels the need to find a productive way of thinking about it. Productive ways of thinking do not mean direct paths between the givens and the goals of the situation; on the contrary they are the result of *seeing* the situation in effective ways that may involve several iterations of *interpreting*, *describing* and *explaining*. The proponents of MMP have provided evidences that students are able to create conceptual tools while looking for ways of thinking about a situation. A result of productive thinking is a *conceptual model* of the situation.

> Students produce conceptual tools that include explicit descriptive or explanatory systems that function as models which reveal important aspects about how students are interpreting the problem-solving situations (Lesh & Doerr, 2003b, p. 9).

The authors further explain that such conceptual models are expressed in several different ways that students resort to, like images, diagrams, symbols, and representational materials, all explicit elements that give visibility to their understandings, as for example, the quantities they think about, the rules they consider, the relationships established, etc.

For our present research it is also important to replace the notion of "getting an answer to the problem" with the idea of "creating an explanation" – a more useful construct that encapsulates the answer and the process.

> …descriptions, explanations, and constructions are not simply processes students use on the way to "producing the answer", and, they are not simply postscripts that students give after the "answer" has been produced. They ARE the most important components of the responses that are needed (Lesh & Doerr, 2003b, p. 3).

As argued by Reeuwijk & Wijers (2003), getting students to show their reasoning, thinking or strategy may be a question of introducing mathematical norms or of including prompts to incite students to give justifications and to show their work. Part of such mathematical norms should be the practice of equally valuing all strategies although it may also depend on the way the tasks are defined.

In Sub12 and Sub14, all problems include a prompt of the kind: “Do not forget to explain your problem-solving process”. This is one of the rules of the competition that is announced from the beginning in the news board of the competition web-page. Additionally, as students get regular feedback on the solutions they send by e-mail, they are impelled to present their ways of thinking about the problem, in their own words and with whatever means they decide to use, or else to elaborate more on their responses. Omissions in problem-solving explanation become a penalty to the participant who fails to offer a sufficiently understandable and convincing picture of the reasoning developed. Considering problem solving performance in those terms is also a consequence of the underlying assumption that it necessarily involves mathematization and mathematical communication – or rather, achieving a model and its presentation (Reeuwijk & Wijers, 2003). Thus a mathematical representation, such as for instance an equation or a tree diagram, should not be taken as “the reasoning” even if it is a key part of the solution process. Instead it has to be placed within a descriptive story that contains both the original context of the problem and the mathematical representation, in a way that echoes the following conception of mathematical understanding:

> …a blurring of task, person, mathematical activity, nonmathematical activity, learning, applying what has been learned, and other features of problem solving (Lester & Kehle, 2003, p. 516).

Rather than having problem solving subsuming mathematical understanding, it is proposed that mathematical understanding subsumes problem solving and posing. Thus when looking at problem solving we should be looking primarily at mathematical understandings or, more precisely, mathematical ways of understanding situations.

Problem solving as expository narrative

The number of research studies addressing students’ problem solving in virtual and beyond-school empirical fields is still very small especially when compared with the considerable corpus of research carried out in classroom settings. Nonetheless the study developed by Stahl and his collaborators on virtual math teams (Stahl, 2009a) is particularly helpful in offering clues to our present research.

The Virtual Math Teams project (VMT) consists of one of the many online services offered by the quite well known Math Forum website, currently accessed by millions of visitors a month. The VMT service has grown out of another service in the Forum, the Problem of the Week (PoW), where challenging mathematical problems are posted and students can send their solutions and receive feedback for improvement. The VMT is another way of working on more open-ended problems, in a collaborative mode, with students interacting in groups of peers in mathematical discussion chat rooms. Specific software tools available in the VMT environment allow for maintaining group coordination and mathematical problem solving, such as the case of the whiteboard for graphical representations or the tools to edit mathematical symbols.

The fact that the data stored in the Problem@Web project consist mainly of digital solutions that can not provide any information regarding immediate or face-to-face interactions, our logs are clearly different from the ones captured during chats running in group problem solving sessions. Nevertheless, following the research of Stahl and his team, the concept of *expository discourse* (which they distinguish from exploratory discourse and see as complementary in their data analysis) is an important tool for an analysis of problem solving as *expressing thinking*. In fact, the aforementioned view on problem solving highlights representation, communication and explanation of thinking. From this perspective a large number of signs, considerably propelled by the use of digital tools, become significant as part of an expository discourse: the use of colour, natural language, mathematical language, drawings, pictures, photos, icons, diagrams, arrows, labels, notations, pre-symbols, symbols, outputs of specific software (spreadsheets, dynamic geometry systems, graphing tools), tables, letters, numbers, characters, and so on. As Stahl (2009b) describes it, expository discourse is the telling of a story about how the problem was solved, usually providing a sequential account of the essential elements that constitute the problem solving process. Medina, Suthers & Vatrapu (2009) also reporting on the study of VMT, move into the question of *representational practices* and describe how inscriptions become representations in students' problem solving attempts. In describing a group of students engaged in finding a formula to translate a geometrical pattern, the authors highlight how students' inscriptions in the whiteboard guided the group's activity and turned into representational resources with the attribute of working as indexical signs to the problem solvers and the solution readers. This is also the case of many of the stories told by participants in Sub12 and Sub14 in their answers to the problems. Their expository narrative is often quite rich in inscriptions with strong indexical value: "this is how the water level rose in the tank", or "the arrows indicate opposite directions of walkers". Most of these pieces of information are meaningless without the original context of the problem and outside of the complete story of the problem solving. But they actually have a profound role not just as a post-script of the problem solving but as part of the representational practices students engage in.

Another idea that may deepen the concept of expository narratives, as thought-revealing activities or as ways of communicating math experiences, is the notion of *performance*. Gadanidis, Borba, Hughes & Scucuglia (2010) used the performance lens to analyse students who wrote scripts about their mathematical experiences, and afterwards performed, recorded and shared the videos online to a wider community. For the authors those are also "digital stories" that reveal a relationship between the performance and the audience. And the new digital media are obviously offering new possibilities for storytelling and for communication to be displayed. Thus technology itself becomes an actor in the *digital mathematical performance* of the students. Referring to the work of Hughes (2008, cited by Gadanidis et al), the authors also mention the idea of authorship as being akin to the new digital power of publishing your own stories, contents, emotional states. Students' digital mathematical performance seems to embody this sense of authorship, sometimes described as being the intersection of personalization, participation and productivity representative of the networked society (McLoughlin & Lee, 2008). It is also convergent with the students' sense of ownership reported by Francisco & Maher (2005) that emerges when mathematical reasoning becomes the fundamental aspect of problem solving.

Humans-with-media **solving mathematical problems**

At the same time we are emphasising digital tools as powerful media to *express* mathematical thinking in problem solving processes, the educational community is calling attention to this new power of digital technologies already impacting on mathematical practices outside school, as stated in the report of the Joint Mathematical Council of the United Kingdom:

> School and college mathematics should acknowledge the significant use of digital technologies for *expressive and analytic purposes* both in mathematical practice outside the school and college and in the everyday lives of young people (Clark-Wilson, Oldknow & Sutherland, 2011, p. 18) (my emphasis).

"Because technology has the potential for broadening the representational horizon" (Zbiek, Heid, Blume & Dick, 2007), representational fluency is acquiring obvious importance, including knowing how to use particular representations to describe, illustrate or justify assertions and ideas. Representational fluency can also be considered as a lens to examine mathematical activity on technology-based representational media (Zbiek et al, 2007; Johnson & Lesh, 2003).

Results from our research have already pointed out that representational fluency flows from the expository narratives of the participants in the competition and such fluency is strongly interlinked with their use of digital media. For example, Jacinto, Amado & Carreira (2009) looked at how participants in Sub14 perceived the role of the technological tools they used during the competition. The participants valued the opportunity of *communicating their reasoning in an inventive way*, since they could resort to any type of attachments, in particular those they felt more comfortable with or found adequate to the problem itself. They resorted mainly to the text editor MSWord, but also to MSPaint and MSExcel, all examples of home digital technology. The use of images is often a result of their efforts on expressing their reasoning in the best possible way. Moreover, we noticed their awareness of the different representations that could materialize their reasoning and even some decision ability when selecting among the options they had at hand. In another study, Nobre, Amado & Carreira (2012) reported on how students dealt with one of the competition problems with the use of a spreadsheet. It was clear that students interpreted the problem in light of their mathematical knowledge and of their knowledge of the digital tool. When the problem was later explored in the classroom with their mathematics teacher, the relationship between the symbolic language of the spreadsheet and the algebraic language was clear to the students.

These results converge with conclusions drawn by Johnson & Lesh (2003), according to whom "important functions of technology-based representational media (eMedia) are: (a) to describe or explain complex systems, and (b) to express complex constructs by providing new ways for people to communicate with both others and with themselves" (p. 273).

Finally, we endorse the theoretical stand that rejects a separation between the user and the mediational means, as it is elaborated by Borba & Villareal (2004) through the notion of humans-with-media, and also by Moreno-Armella & Hegedus (2009) through the idea of co-action. In the former perspective, we want to highlight one particular point that resonates with our data and knowledge on students' online participation in the competitions: visualization and humans-with-media. Computers change the status of visualization in mathematical activity and bring in new tools to express ideas through visual forms. It has been a constant trace in many of the students' answers that the media used to develop visual representations goes much further than just embellishment. The ways in which students *see* the problem solution and *express* it with digital media supports the statement: "what we see is always shaped by the technologies of intelligence that form part of a given collective of humans-with-media, and what is seen shapes our cognition" (Borba & Villareal, 2004, p. 99).

Expression of mathematical problem solving mediated by home computer technologies has been pushing our research into the question: how do young 21st century problem solvers expose their problem solving processes and how do visual aspects of home digital technologies emerge as part of their digital-mathematical- performance?

PROBLEM SOLVING RELATING TO TIME-VARIATION

To get some insight into the question above, I will be focusing on two of the problems proposed in the competitions, one from Sub12 and the other from Sub14, both involving imaginary situations that contain movement and time as a variable.

The purpose is to look at how students deal with time-variation in their ways of representing the situation and to identify features of their mathematical representations within the media used to express their thinking.

A problem from the Sub12 competition

The following is Problem #8 proposed in the 2010/11 of Sub12 (Figure 1). This was a problem that came out near the end of the Qualifying and therefore participants were quite familiarised with the rules and with the operational aspects of the competition. In particular, they had already solved seven problems and got a fairly good experience on sending their answers by e-mail. At this stage, many of the students were choosing to send attached files rather than just typing their answers in the e-mail window. The majority were Word documents, but a few answers came also with Excel, PowerPoint, and Publisher files.

On some of the week days Paulo gets a ride from a schoolmate to go from home to school but at the end of classes he walks back home. On such days he takes a total of 40 minutes to go and return. On the other week days he gets a ride to go to school and also to come back home. On such days he takes half of the total time to go and return.
Unfortunately last week, as his mate's mum was ill and could not drive, he had to walk to school and to walk back home after classes. How long did it take him to walk to school and back home?

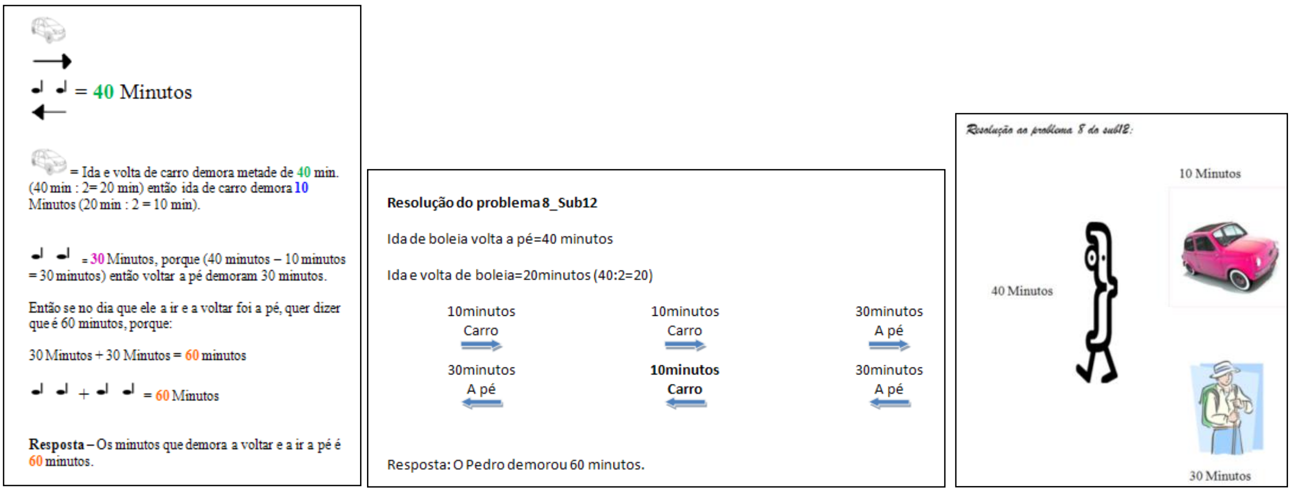
Figure 1. Problem #8 of Sub12, edition 2010/11

Very few (only two in total) of the 5th and 6th graders exposed their mathematical thinking through symbolic equations. It is worth mentioning that students in this grade levels have a limited knowledge on equations and variables although they may come across introductory algebraic language in pattern description and generalization.

Most of the participants took different approaches to the problem. It became evident from their answers that their understanding of the problem stood on realising that it involved the identification of three distinct possibilities of the journey from home to school and back.

Those students who managed to understand this aspect were able to solve the problem. Therefore, it becomes relevant to consider in what ways the students manifested this kind of thinking and how this was expressed in their digital-mathematical-performance.

Three excerpts of students' answers featuring their approaches to the problem are given in Figure 2 (snapshots (a), (b), and (c)). All of them depict the situations described in the problem and a common trace is the use of iconic signs to represent them. In the case of excerpt (c), the sent file included a second page with the explanation of the reasoning developed to get the solution, thus indicating that the first page was actually a way of showing the three distinct ways mentioned in the problem of travelling from home to school and back.



(a) (b) (c)

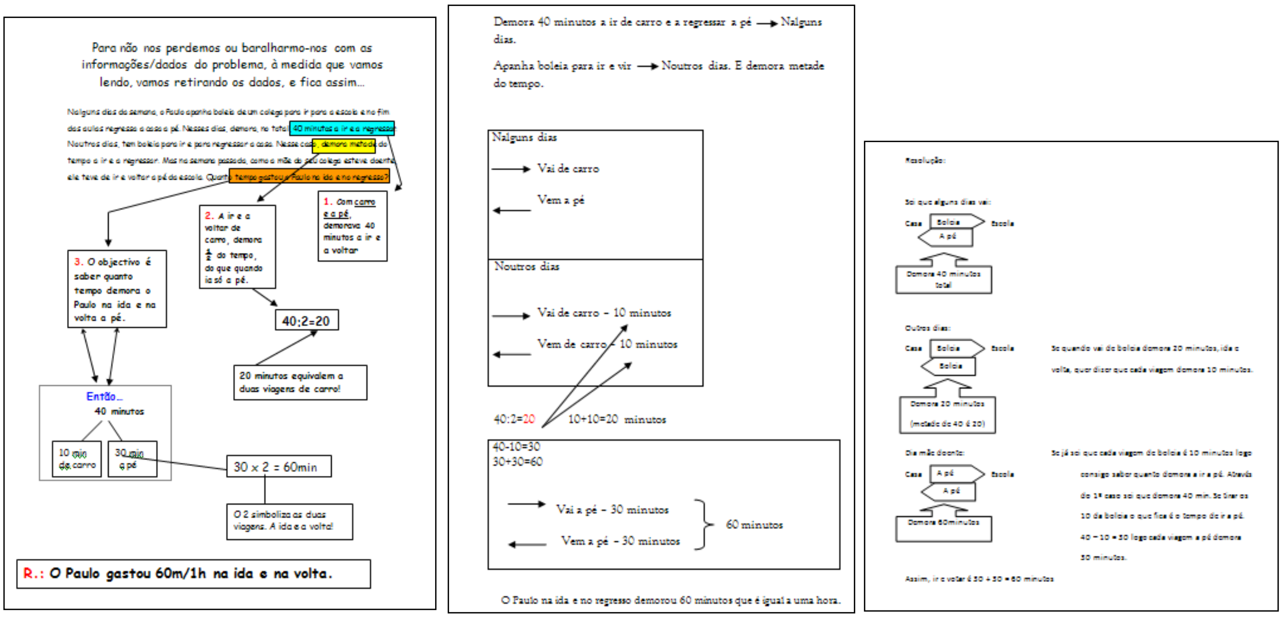
Figure 2. Print-screens of excerpts of students' answers

In the first solution (a), the student starts to present a picture of a car, with an arrow beneath pointing to the right, and an iconic version of two feet, with an arrow beneath pointing the left. One colour (green) and a large letter size are used to write '40 Minutes' on the side. It indicates a fundamental piece of information extracted from the problem. Both the two arrows pointing to opposite directions and the two images used reveal an understanding of the information: going to school by car and returning home on foot takes 40 minutes. The next piece of the answer (second paragraph) only uses the image of the car. Natural language and mathematical symbolism (indicating elementary computations) are introduced to state that going and returning by car takes half of 40 minutes, and thus dividing 20 by 2 gives the time of '10 Minutes' for a one-way trip by car. In the sentence, the number 10 is written in blue, signalling a new relevant piece of information and one that refers to a different aspect of the problem. The third paragraph starts with the icon for the walking trip immediately followed by the words '30 Minutes', with the number 30 coloured in pink. It goes on explaining how this result is obtained from the difference between 40 and 10. Afterwards, the answer states that walking to school and back takes 30 plus 30 minutes, in a total of ‘60 Minutes’. The number 60 is written in a new colour (orange), again showing that it refers to another of the cases described in the problem. Another line is finally included, where the student rephrases the previous conclusion using the iconic language and showing the sum of two walking icons being equal to 60 minutes (maintaining the orange colour).

In general, both the iconic elements and the use of four different colours to display different numbers were relevant inscriptions embodying the reasoning: 40 (the time for going by car and returning on foot); 10 (the time of a one-way trip by car); 30 (the time for a one-way walking trip); and 60 (the time for going and returning on foot). These pictorial-visual meanings keep the thinking directed to the several cases of the trip and allow having them differentiated while integrating the data and intermediate results to get the answer.

The second solution (b) also describes the three situations with the use of arrows and labels. The labels indicate ‘by car’ and ‘on foot’. And in each part, the two arrows beneath each case point to opposite directions. The reasoning developed is not so much detailed as in solution (a). The answer starts with the calculation of 40 divided by 2 for knowing the time of the round trip by car. The central iconic display indicates the two one-way trips by car and the respective time, suggesting that the 20 minutes were divided in 10 minutes for either way. The label concerning the car trip from school to home is highlighted in bold, also suggesting the relevance of this data. It suggests that it was the key to find the time for the same trip on foot (on the left side), which lead to the time spent in each of the one-way walking trips (on the right) and thus to the total time.

Further examples of students’ graphical arrangements of information, explanations and representational forms (Figure 3) also highlight the pictorial use of arrows and colours to dissect the situation in three clear cases. The reasoning flows out of the well distinguished cases, digitally described through the use of diagrammatic elements.



(a) (b) (c)

Figure 3. Print-screens of excerpts of students’ answers

A problem from the Sub14 competition

The Problem #1 from the 2011/12 edition of Sub14 (for 7th and 8th graders) also involved variation with time (Figure 4). This problem mentions two individuals walking towards each other with different velocities and different departure times. Although this was the first problem of the season, many of the participants were already used to send attached files.

Alexander and Bernard live at a distance of 22 km from one another and they want to meet but the only way to do it is… walking! On a holiday morning they decided to walk towards one another to get together. Alexander left his home at 8 a.m. and went walking at a speed of 4 km per hour. Bernard left his home an hour later and walked at a speed of 5 km per hour. Neither of the two friends took his watch but we can know at what time they met. What time was it?

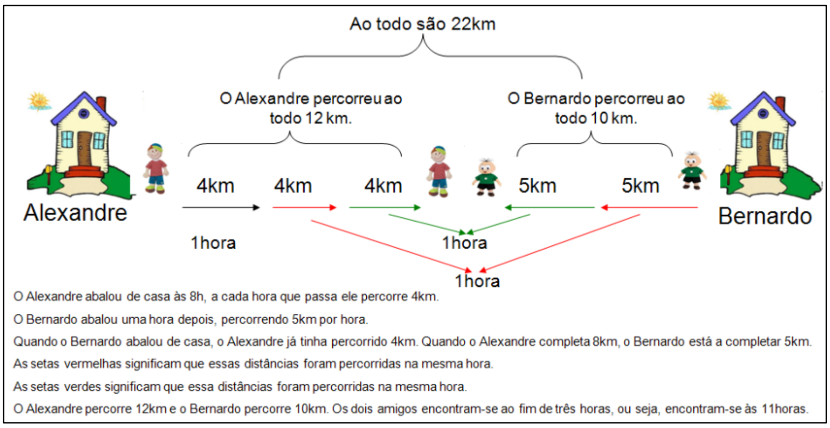
Figure 4. Problem #1 of Sub14, edition 2011/12

Most of the participants who got the problem wrong on their first attempts solved it as if the two friends were walking in the same direction; they started to analyse the distance walked by each of the friends with time, and saw that both linear patterns reached the value of 20: after 5 hours Alexander had walked 20 km and after 4 hours Bernard had also walked 20 km. As the first left one hour earlier, the answer given was that the two friends met each other at 1 p.m. There were also a number of seventh graders who invoked the notion of the least common multiple of 4 and 5 to address the problem. Such answers showed a weak understanding of the problem conditions and apparently a tendency to apply school knowledge to a problem that may have looked like a standard situation for using the least common multiple.

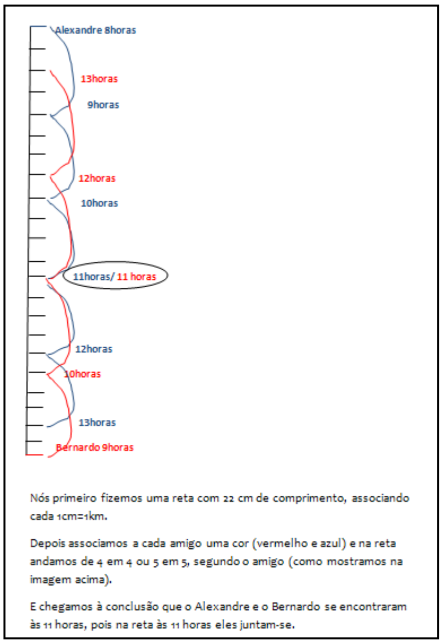
On the contrary, students who got correct answers payed attention to the fact that the two friends walked towards each other. Many of the solutions were presented with a table describing the positions of each of the friends at every hour, from 8 a.m. until 11 a.m., the time of the meeting.

There were also many others that involved graphical and pictorial representations, consistently highlighting the distance walked by each of the two friends and their opposite directions (Figure 5). The use of colours and indicators of constant steps (curved lines and/or a line scale marked) were quite frequent. Usually, the drawings are followed by natural language explanations, as in the case of Figure 5 (b), where the students state: "We first draw a line with 22 cm and considered 1 cm to be equivalent to 1 km. Then we assigned a colour to each of the friends (blue and red) and on the line we followed in spaces of 4 marks or 5 marks, according to the friend (as shown in the picture). And we concluded that Alexander and Bernard met at 11 a.m. because it's where they joined on the line". Also it is very clear, in both the answers of Figure 5, the use of labels that seek to explain the movement developing: the time changing and the position of both friends getting closer to each other.

Two additional examples (Figure 6) illustrate the movements of the two friends and highlight the fact that the distances travelled by each of them must add up to the distance of 22 km between the houses. This explanation was given emphasis with colour (b) or it was included in the text to underline the match between what the picture showed and what the problem meant (a).

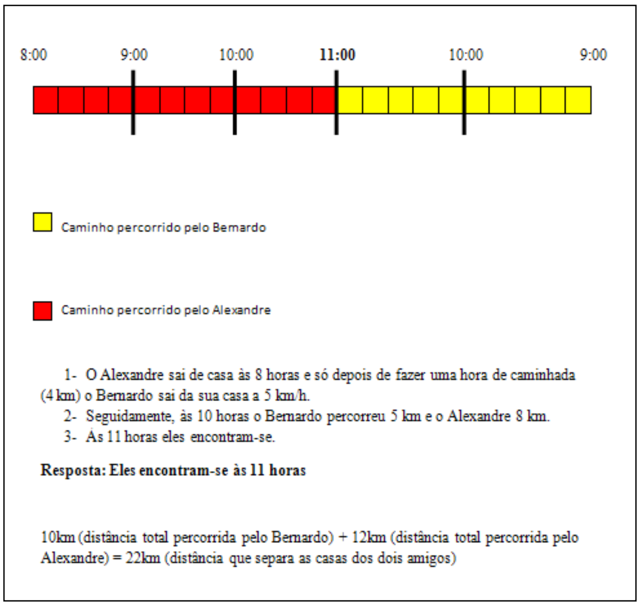


(a)

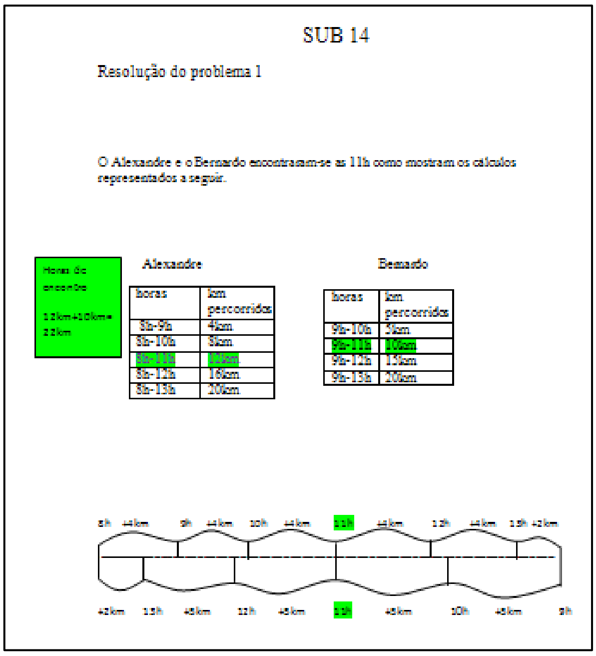


(b)

Figure 5. Print-screens of excerpts of students' answers



(a)



(b)

Figure 6. Print-screens of excerpts of students' answers

The above set of answers is rich in details, showing that different students presented several important aspects of the situation: the time-variation, the changing position relative to the origin, the distance travelled by each individual and the position relative to the other individual. Thus good models of the situation or productive ways of thinking about it are clearly expressed in students' story telling mediated by their effective use of digital representations.

CONCLUSIONS AND FURTHER DEVELOPMENTS

The solutions to the problems concerning time-variation provide some important insights into the nature of representational practices in students' problem solving processes when digital tools become natural tools to express thinking.

Online mathematical competitions, where communicating and expressing reasoning through electronic media is a central feature of the mathematical activity, can reveal the forms of expository narratives that youngsters produce (Stahl, 2009b). Such expression of mathematical thinking becomes an integral part of the problem solving process and seems to be sustained and reinforced by the use of digital tools beyond the direct prompts that may be offered by the competition itself. It may be described as a digital-mathematical-performance, in the sense suggested by Gadanidis et al (2010), where graphical, iconic, pictorial, indexical, and schematic means are smoothly intertwined with mathematical thinking and become inherent to the thinking.

Most of students' answers as the ones considered in the previous section are not sophisticated solutions inasmuch as they primarily intend to create and present a clear picture of the problem. Neither their use of digital tools can actually be seen as sophisticated. Yet both the solution approach and the representations afforded by the use of the tools look as friendly and clear-cut ways of creating mathematical models and performing mathematically.

Creativity is also an important aspect of many of students' visual ways of expressing thinking. It indicates how representational fluency is clearly tied to the problem solving environment and it suggests that co-action between the human agent and the digital medium (as described by Moreno-Armella & Hegedus, 2009) is actually a source of creative activity. As the proponents of the concept argue, mathematical objects are refracted in the digital medium and as a result new ways of justifying and presenting mathematical ideas come to the surface. This stands out from the solutions produced by the young participants – mathematical objects, ideas, and models are being refracted in the digital media they use to think and express their thinking. Models are therefore more than mathematical expressions, algorithms or symbols. Models are essentially forms of understanding and they lead much of the successful problem solving processes of the participants in the two competitions. They reveal how situations are conceptualised and how such conceptualisations develop from inscriptions: pictures, schematic representations, language, letters, and iconic elements easily available and displayable through digital tools.

Research has provided evidence of the differences between visualisers and verbalisers in problem solving. Moreover, Kozhevnikov, Hegarty & Mayer (2002) described two types of visualisers, the iconic type and the spatial type: those whose imagery is primarily pictorial and those whose imagery is primarily spatial, abstract and schematic. They also found that the first group had more difficulties in kinematics problems, especially in understanding graphs of motions. Although the problems here discussed were not real kinematics problems they involved motion and time-variation. The data presented reflect a relevant type of answers given in the competitions that may relate to a sense of "performing" or "expressing" thinking when developing a solution. On the other hand, many of the digitally mediated solutions not only exhibit pictorial representations but a combination of those with spatial schematic representations, thus suggesting a blend of iconic and spatial characteristics. This brings the question of whether digital mathematical representations with which young students are fluently expressing mathematical activity influence their representational preferences.

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