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LEARNING MATHEMATICS BY CREATIVE OR IMITATIVE REASONING

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This paper presents 1a) a research framework for analysing learning difficulties related to rote learning and imitative reasoning, 1b) research insights based on that framework, 2a) a framework for research and design of more efficient learning opportunities through creative reasoning and 2b) some related ongoing research.

Key words: Learning difficulties, rote learning, creative reasoning, problem solving.

INTRODUCTION

A central problem in mathematics education is that we want students to understand mathematics and to become efficient problem solvers, but even after 30 years of research and reform many students still do inefficient rote thinking (Hiebert, 2003; Lithner 2008). This is one of the main reasons behind learning difficulties in mathematics. Even the learning of routine procedures does not function well by rote learning, since students largely following the rules "like robots with poor memories" (Hiebert, 2003, p. 12).

There are probably several reasons behind this problem such as social, cultural, political, etc. This paper will focus on some of the reasons that are directly related to how the subject mathematics is handled in teaching and learning situations. Even with respect to this aspect, the reason that the rote learning problem is (largely) unsolved in many countries is probably a combination of several factors related to the immense complexity of mathematics learning (Niss, 1999) and to the lack of research insights concerning the effectiveness of different teaching designs (Niss, 2007). In addition, there seems to be many choices made in ordinary teaching that lead to rote learning as an unintended by-product, mainly connected to attempts to help students by reducing the cognitive complexity (Doyle, 1988; Schoenfeld, 1985; 1991).

The purpose of this paper is to present research frameworks for 1) analysing existing learning difficulties related to rote learning and for 2) a design research approach to constructing more efficient learning opportunities for mathematics students. A research framework is in this paper seen as “a basic structure of the ideas (i.e., abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated” (Lester, 2005, p. 458). These two parts respond to the questions "what are the causes and characteristics of mathematical learning difficulties?" and "what can be done to improve the situation?" respectively. Of course, this paper does not fully respond to these questions but aim to treat some central sub aspects of them.

PART ONE: INSIGHTS IN SHORTCOMINGS OF IMITATIVE LEARNING

Part one of the framework contains the following components:

- Goals for mathematics teaching and learning, as a basis for analysing what to aim for and where we fail. Problem solving, reasoning and conceptual understanding will be in focus.
- Rote learning, the unintended but common way to try to learn mathematics through superficial imitation, which is a main cause behind learning difficulties since the goals mentioned above are not attained.
- Imitative and creative mathematical reasoning, a characterisation of the thinking processes that students activate in learning situations. A basic assumption in this paper is that students opportunities to learn mathematics is largely determined by the thinking processes they activate in learning situations, and their explicit reasoning is seen as traces of their thinking processes. The empirically based reasoning framework will be used to specify how students reason by imitation in rote learning and what is missing with respect to more efficient creative reasoning.

Learning goals in mathematics: Problem solving, reasoning and conceptual understanding

In order to understand the learning difficulties that students encounter and in order to suggest measures to take, it is not sufficient to describe learning goals merely in terms of mathematical content since such descriptions do not capture what students are supposed to be able to do with the content. The NCTM Principles and Standards (NCTM 2000) complements its five content standards (number and operations, algebra, geometry, measurement, and data analysis and probability) with five process standards (problem solving, reasoning and proof, connections, communication, and representation). Judging by the impact and large number of references to the NCTM Principles and Standards (and its earlier versions) it seems fair to say that this type of learning goal description is largely accepted by the mathematics education research community. A framework presented by Niss & Jensen (2002) and Niss (2003) contains factors similar to the NCTM process standards but denotes them competencies. Competence is the ability to understand, judge, do, and use mathematics in a variety of mathematical contexts and situations. Three competencies are particularly relevant for this framework: problem solving ability, reasoning ability and conceptual understanding.

In this paper *problem solving* is defined as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 51). This definition implies that in this perspective there are only two types of tasks: problems and non-problems (often denoted 'routine tasks'). Note that some aspects often included in similar definitions of problem solving are not included in the definition above, for example that the task is necessarily a challenge (Schoenfeld, 1985) or that the task requires exploration (Niss & Jensen, 2002). The main difference between solving a problem and a routine task is that in the former the solver has to at least partially construct the solution method by herself, while in a routine task the method is already known by the solver or provided by an external source such as the book or the teacher. One may also note that in determining if a task is a problem or not it is insufficient to consider properties of the task alone, instead the relation between the task and the solver has to be considered (Schoenfeld, 1985). For example, to find the number of combinations when having 3 pants and 4 shirts available to choose from may be a problem for a grade 3 student but a routine task for a grade 7 student. And if the grade 3 student (that master elementary addition and/or multiplication) has solved several similar problems and realised that the number of combinations can be found by multiplication or repeated addition, or if the teacher describes this method to the student, then every task of this particular type becomes a routine task to this student. The problem solving competency includes identifying, posing, and specifying different kinds of problems and solving them, if appropriate, in different ways (Niss 2003). Schoenfeld (1985) formed through a series of empirical studies the probably most cited problem solving framework based on four key competencies (p. 15): Resources (basic knowledge), Heuristics (rules of thumb for non-standard problems), Control (metacognition: monitoring and decision-making), and Belief Systems (one’s mathematical world view). He found that novices often had sufficient resources but were lacking in the other three competencies.

The NCTM Principles and Standards (2000) recognize *reasoning and proof* as fundamental aspects of mathematics. “People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove” (p. 56). The reasoning competency goes beyond constructing reasoning, and includes abilities like following and assessing chains of arguments, knowing what a proof is and how it differs from other kinds of reasoning, uncovering the basic ideas in a given line of argument, and devising formal and informal arguments (Niss 2003). Creative and imitative reasoning will be described more in detail below.

The concept of *understanding* is very complex (Sierpinska 1996), and will not be pursued here beyond noting that several of the theoretical constructs concern relations between rote learning and deeper understanding. Skemp (1978) distinguishes between ‘instrumental understanding’ and ‘relational understanding’ of mathematical procedures. The former can be apprehended as ‘true’ (relational) understanding, but is only the mastering of a rule or procedure without any insight in the reasons that make it work. A similar distinction, between ‘action’ and ‘process’ is made by Asiala et al. (1996), and Hiebert & Lefevre (1986) distinguish between conceptual and procedural understanding. It seems difficult to find a precise definition of mathematical understanding, but the NCTM connection and representation standards can be seen as more well-defined sub-components of understanding. The notion ‘conceptual understanding’ (or just ‘understanding’) will here be used in a relatively intuitive way, referring to insights in the origin, motivation, meaning and use (Brousseau 1997) of a mathematical fact, method or other idea.

Seeing problem solving ability, reasoning ability and conceptual understanding as key learning goals, the next section will describe how learning environments that promote rote learning deny students of proper opportunities to reach such goals.

Rote learning

Rote learning is "the process of learning something by repeating it until you remember it rather than by understanding the meaning of it" (Oxford Advanced Learner's Dictionary). Rote learning in mathematics mainly includes facts and procedures, and can vary from simple, e.g. the fact that a+b=b+a or the procedure of one-digit addition by finger-counting, to complex such as a long proof or a set of techniques for integration of composite functions. The characteristics, causes, and consequences of rote learning in mathematics can to a large extent be connected to an unwarranted and far-reaching reduction of complexity in terms of an algorithmic focus (Skemp, 1978; Hiebert & Carpenter, 1992; Tall, 1996; Vinner, 1997; Hiebert, 2003; Lithner, 2008).

> "After several years of executing procedures they do not understand, students' behaviour is so rule-governed and so little affected by conceptual understanding that one can model their behaviour and predict the errors they will make by looking only at the symbol manipulation rules they have been taught and pretending that they are following these rules like robots with poor memories" (Hiebert, 2003, p. 12).

Rote learning alone is not likely to be a goal of any mathematics curricula. Schoenfeld (1985) argues that many of the counterproductive behaviours that we see in students are unintended by-products of their mathematics instruction that result from a strong classroom emphasis on performance, memorising, and practicing, which ultimately causes students to lose sight of rational reasons. Referring to "massive amounts of converging data" in studies from USA, Hiebert suggests that the baseline conclusion is that students are learning best the kinds of mathematics that they are having the most opportunities to learn, which is simple calculation procedures, terms and definitions through memorization (Hiebert, 2003). Similar opportunities to learn mainly how to handle procedures were found in a Swedish large-scale study including observations of 200 mathematics classrooms (Boesen et al., 2012).

Rote learning is in itself not problematic. On the contrary, memorising facts and procedures, even without understanding, is a central aspect of mathematics learning. It is not reasonable to expect that students should be able to understand or (re)construct every mathematical idea. At least some of these ideas are too difficult to be fully understood (for a specific educational level) and may have to be learnt by rote or with limited understanding. The problem is when rote learning becomes dominating since it is not possible to develop other central competencies like problem solving ability and conceptual understanding by rote learning alone. For example, it is well known from the extensive research on problem solving from the eighties (e.g. Schoenfeld, 1985) that there is no transfer from rote learning of basic facts and procedures to the ability to solve non-routine mathematical problems. From literature reviews (e.g. Hiebert, 2003) and from the empirical studies exemplified below, it is reasonable to draw the conclusion that rote learning is one of the main causes behind the learning difficulties that large groups of mathematics students of all age levels encounter. As will be discussed below, the avoidance of meaning is a key to clarifying both advantages and disadvantages of rote learning.

So far, the main ideas in the presentation of rote learning and problem solving has been quite general. However, one reason that impact of rote learning may be particularly strong in mathematics is that the historical progress of the subject itself is to such a large extent based on the inventions of powerful concepts (such as the zero and the decimal position system) and powerful procedures (such as algorithms for arithmetic calculations), of which many can be learnt by rote. As an extreme example, it is fairly easy to teach seven-year old kids to differentiate simple polynomials, which would yield them some points on an upper secondary mathematics exam. These kids have of course no insight whatsoever into the underlying concepts of polynomials, functions or differentiation and have learnt the procedure only by rote. At the same time, due to the clear and simple structure of mathematics and the possibility to (partially) detach mathematical tasks from the complexity of the real world and hereby choose a suitable task difficulty level, mathematics is particularly suited to learn problem solving and mathematical reasoning (Pólya, 1954).

Creative reasoning

This and the next section contain a modified summary of selected parts of a research framework (Lithner, 2008) that is based on the outcomes of a series of empirical studies on the relationship between reasoning and learning difficulties in mathematics.

"Mathematical reasoning is no less than a basic skill" (Ball & Bass, 2003, p. 28). Despite this pronouncement, the term 'reasoning' is often used by mathematics educators without being defined under the implicit assumption that there is universal agreement on its meaning (Yackel & Hanna, 2003). The purpose of this section is to provide three things: 1) a broad definition of reasoning that allows the inclusion (and comparison) of both low- and high-quality arguments; 2) the underlying notions that make it possible to define creative, mathematically founded reasoning; 3) a characterization of imitative reasoning as the opposite of creative reasoning.

Reasoning is defined in this paper as the line of thought that is adopted to produce assertions and reach conclusions when solving tasks. Reasoning is not necessarily based on formal logic and is therefore not restricted to proof; it may even be incorrect as long as there are some sensible (to the reasoner) reasons supporting it. This example illustrates that "reasoning" is used in a broad sense in this framework to denote both high- and low-quality argumentation; the quality of the argument is characterized separately. Reasoning can be seen as thinking processes, as the product of these processes, or as both. The data for the investigations discussed here are behavioural; thus, we can only speculate about the underlying thought processes (Vinner, 1997). Because one purpose of this framework is to characterize data, the choice is to see reasoning as a product that (primarily) appears in the form of written and oral data as a sequence of reasoning that starts in a task and ends in an answer.

In a task-solving situation (including sub-tasks) two types of argumentation are central. 1) *Predictive argumentation* (why will the strategy solve the task?) can support the strategy choice. The 'strategy' can vary from local procedures to general approaches, and 'choice' is defined in a broad sense (choose, recall, construct, discover, guess, and so forth). 2) *Verificative argumentation* (why did the strategy solve the task?) can support the strategy implementation.

School tasks normally differ from the tasks addressed by professionals such as mathematicians, engineers and economists. Within the didactic contract (Brousseau, 1997) in the school context, it is allowed, and sometimes encouraged, to guess, to take chances, and to use reasoning without any strict requirements on the logical value of the reasoning. Even in examinations, it can be acceptable, as in Sweden for example, to have only 50% of the answers correct, while it would be absurd if mathematicians, engineers, or economists were satisfied in being correct in only 50% of their conclusions. This framework proposes a wider conception of logical value that is inspired by Pólya (1954): "In strict reasoning the principal thing is to distinguish a proof from a guess, [...] In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess." Thus, a *plausible argument* can be constructive without being logically valid (in contrast to, for example, a proof which must be logically true).

What does it mean for an argument to be based on mathematics? Schoenfeld (1985) found that novices used naive empiricism and judged that geometrical constructions were correct if they 'looked good,' whereas experts used more relevant properties (for example, congruence). Thus, the reference to the mathematical content is important: what are the arguments about? To address this question, the notion of *anchoring* is introduced (Lithner, 2008). Anchoring does not refer to the logical value of the argument but refers to its fastening the relevant mathematical properties of the components one is reasoning about—objects, transformations, and concepts—to data. The object is the fundamental entity; it is the 'thing' that one is doing something with, for example, numbers, variables, functions, and diagrams. A transformation is what is being done to the object, and the outcome of the transformation is another object. A sequence of transformations, finding polynomial maxima for example, is a procedure. A concept is a central mathematical idea built on a set of objects, transformations, and their properties, such as the concept of a function or of infinity. The status of a component depends on the situation. $f(x)=x^3$ can be seen as a transformation of the input object 2 into the output object 8. If f is differentiated, then the differentiation is the transformation; $f(x)$ is encapsulated (Tall, 1991) into an input object, and $f'(x)$ is the output object.

Arguments can be anchored in either *surface* or *intrinsic properties*, and the relevance of a mathematical property can depend on context. In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a surface property that is insufficient to resolve the problem (a conclusion based on this property alone is that 9/15>2/3 since 9 and 15 are larger than 2 and 3), while the quotient captures the intrinsic property. The intrinsic/surface distinction was introduced because one of the reasons behind students' difficulties was found to be the anchoring of arguments in surface properties (Lithner, 2003).

Because this framework addresses ordinary students' thinking, imputing creativity only to experts is not sufficient.

> "Although creativity is often viewed as being associated with the notion of 'genius' or exceptional ability, it can be productive for mathematics educators to view creativity instead as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (Silver, 1997, p. 75).

The aspect of creativity that is emphasized in this framework is not 'genius' or 'exceptional novelty,' but the creation of mathematical task solutions that can be modest but that are original to the individual who creates them. Thus, creative is the opposite of imitative.

The discussion above leads to a definition of *Creative Mathematically Founded Reasoning* (CMR) that fulfils all of the following criteria.
i) Creativity. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created, in a way that is sufficiently fluent and flexible to avoid restraining fixations.
ii) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation explaining why the conclusions are true or plausible.
iii) Anchoring. The arguments are anchored in the intrinsic mathematical properties of the components that are involved in the reasoning.

Imitative reasoning

The empirical studies behind this framework have identified two main types of imitative reasoning: memorized and algorithmic. In *Memorized Reasoning* (MR), the strategy choice is founded on recalling an answer by memory, and the strategy implementation only consists of writing it down. This type of reasoning is useful as a complete solution method in only a relatively small proportion of tasks (Lithner, 2008), such as recalling every step of a proof or the fact that one litre equals 1000 cm^3. When school tasks ask for calculations, it is normally more appropriate to use *Algorithmic Reasoning* (AR) (Lithner, 2008), where the strategy choice is to recall an algorithm and the strategy implementation is to apply the algorithm to the task data.

The term 'algorithm' includes all pre-specified procedures (not only calculations), such as finding the zeros of a function by zooming in on its intersections with the *x*-axis with a graphing calculator. "An algorithm is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems" (Brousseau, 1997, p. 129). The importance of an algorithm is that it can be determined in advance. The nth transition does not depend on any circumstance that was unforeseen in the (n-1)st transition - not on finding new information, any new decision, any interpretation, or thus on any meaning that one could attribute to the transitions. Therefore, the execution of an algorithm has high reliability and speed (Brousseau, 1997), which is the strength of using an algorithm when the purpose is only to solve a task.

However, if the purpose is to learn something from solving the task, the fact that an algorithm is independent of new decisions, interpretations or meaning implies that all of the conceptually difficult parts are taken care of by the algorithm, and thus only the easy parts are left to the student. This segmentation may lead to rote learning. In particular, the resultant argumentation is normally superficial and very limited, as seen in the main AR types that are found in studies. *Familiar AR/MR* includes a strategy choice that can be characterized by (perhaps superficial) attempts to identify a task as being of a familiar type with a corresponding known solution algorithm or a complete answer. Justifying a successful solution by simply describing the algorithm is an accepted sociomathematical norm (Yackel & Cobb, 1996) in most practice and test situations studied (Lithner, 2008). In *Delimiting AR*, the algorithm is chosen from a set of algorithms that are available to the reasoner, and the set is delimited by the reasoner through the included algorithms' surface property relationships with the task. For example, if the task contains a second-degree polynomial $p(x)$, the reasoner can choose to solve the corresponding equation as $p(x)=0$ even if the task asks for the maximum of the polynomial (Bergqvist et al., 2008). In *Guided AR*, the reasoning is mainly guided by two types of sources that are external to the task. In person-guided AR, a teacher or a peer pilots the student's solution (see Section 4 for examples). In text-guided AR, the strategy choice is founded on identifying, in the task to be solved, similar surface properties to those in a text source (e.g., a textbook). Argumentation may be present, but it is not necessary because the authority of the guide ensures that the strategy choice and the implementation are correct.

In students' attempts to resolve problematic task solving situations, the CMR criteria i-iii (see Section 2.1.5) were found to capture the main differences seen in reasoning characteristics between MR/AR (where i-iii are absent) and constructive CMR (Lithner, 2008). A task solution in MR is immediate through recollection, in AR, it follows a known algorithm and in CMR, it is created (although CMR normally includes elements of MR/AR). Furthermore, in CMR the epistemic value (degree of trust, see Duval, 2002) lies in the plausibility and in the logical value of the reasoning. In MR and AR, it is determined by the authority of the source of the imitated information.

Students often use superficial imitative reasoning of the types presented above in laboratory tests and when working with tasks (e.g. textbooks or assessment) in regular classroom contexts, which is a major hurdle both when it comes to learn and to use mathematics (e.g. Lithner, 2000; 2003: 2008; 2011, Bergqvist, Lithner & Sumpter, 2008; Boesen, Lithner & Palm, 2010). In addition, teaching, textbooks and assessments mainly promote rote learning in the sense that Guided AR is provided by teachers and textbooks, and that most practice and test tasks can be solved by AR (e.g. Bergqvist 2007; Palm, Boesen & Lithner, 2011; Bergqvist & Lithner, 2012; Boesen et al., 2012). Judging from the quote by Hiebert in the introduction this may be the case also outside Sweden, for example as found in common American calculus textbooks (Lithner, 2004).

Summary Part one

Rote learning is sometimes necessary in mathematics and in general efficient in a limited and short-sighted perspective, but when dominating it does not provide students with opportunities to develop central mathematical competencies such as problem solving ability, reasoning ability and conceptual understanding. The message from large parts of the mathematics education research community (e.g. NCTM, 2000) is quite clear: students need to engage in activities including problem solving and reasoning in order to develop mathematical competence. Under the assumption that learning mathematics is affected by the reasoning that the student actually activates when solving practice tasks, it is hypothesised that in order to learn mathematics better students need to engage in problem solving and CMR to a larger extent than what seems to be the case in for example in USA and Sweden. The next section will present ongoing research on the design of such learning opportunities.

PART TWO: DESIGNING TEACHING THROUGH CREATIVE REASONING

Part two of the framework contains the following components:
- A summary of the principles of design research, the approach chosen to study alternatives to rote learning.
- Brousseau's Theory of Didactical Situations, that clarifies in what ways and why learning through problem solving can be more efficient than rote learning.
- A ongoing teaching experiment comparing learning by Algorithmic and Creative reasoning.

Design research

Concerning constructive teaching there are some insights that certain approaches can better enhance learning, but in general we lack deeper knowledge regarding *how* and *why* different teaching approaches affect different aspects of learning (Niss, 2007). The ongoing research described below can be characterised as *design research* which in this paper refers to the use of scientific methods to develop theories, frameworks and principles of existing or envisioned educational designs. An *educational design* can be seen as a plan produced to show the function (including purpose and means) of an educational artefact or practice. Such plans can be of different grain size and character, for example from local informal to global formal. The plan can refer to various components of the educational system, e.g. the classroom, teacher education, textbook production and large scale assessment.

The meaning of design experiments have not been settled in the literature (Schoenfeld, 2007). Plomp (2009) argue that authors may vary in the details of how they picture design research, but they all agree that design research comprises of a number of stages or phases:
- preliminary research: needs and content analysis, review of literature, development of a conceptual or theoretical framework for the study
- prototyping phase: iterative design phase consisting of iterations, each being a micro-cycle of research with formative evaluation as the most important research activity aimed at

improving and refining the intervention
- assessment phase: (semi-) summative evaluation to conclude whether the solution or intervention meets the pre-determined specifications. As also this phase often results in recommendations for improvement of the intervention, we call this phase semi- summative.

A key characteristic of design research is thus that it is strongly aligned with effective models linking research and practice, which, according to Burkhardt and Schoenfeld (2003), "the traditions of educational research are not". This is also emphasised by Cobb et al. (2003): "Design experiments have both a pragmatic bent – 'engineering' particular forms of learning - and a theoretical orientation - developing domain-specific theories by systematically studying those forms of learning and the means of supporting them." This is also in line with (Gravemeijer & Cobb, 2006): "the purpose of the design experiment is both to test and improve the conjectured local instruction theory that was developed in the preliminary phase, and to develop an understanding of how it works."

All empirical design research do not have to take place in classrooms, but may for example be in the format of laboratory pre-stages to classroom design in form of the clinical trials described by Schoenfeld (2007). Research questions in design research are typically in the form "what are the characteristics of an intervention X for the purpose/outcome Y in context Z?" and the research results in interventions (programs, products, processes), design principles or intervention theory and professional development of the participants involved in the research (Plomp, 2007). In contrast to most research methodologies, the theoretical products of design experiments have the potential for rapid pay-off because they are filtered in advance for instrumental effect. They also speak directly to the types of problems that practitioners address in the course of their work (Cobb et al., 2003).

The Theory of Didactical Situations

The theoretical foundation for the attempts presented in this paper to design better learning opportunities for mathematics students is Brousseau's Theory of Didactical Situations (1997), which is a theory of how mathematics can be learnt through non-routine problem solving. It emphasises "the social and cultural activities which condition the creation, the practice and the communication of knowledge" (p. 23). In the theory, the milieu is "everything that acts on the student or that she acts on" in a learning situation (p. 9). Didactique studies the communication of knowledge and one central aspect of Brousseau's didactical situations is the devolution of problems. The student has to take responsibility for a part of the problem solving process, but she cannot in general learn in isolation. The teacher's task is to arrange a suitable didactic situation in the form of a problem. Between when the student accepts the problem as her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. This part of the didactic situation is called an adidactical situation. The student must construct the piece of new knowledge and the teacher must therefore arrange not the communication of knowledge, but the devolution of a good problem. If the student avoids or does not solve the problem, the teacher has the obligation to help. Then a relationship is formed that (mainly implicitly) determines what each party will be responsible for: the didactic contract that ensures the functioning of the process. Wedege and Skott (2006) note that the term 'didactic contract' is used outside France as a metaphor for the set of implicit and explicit rules of social and mathematical interaction in a particular classroom, which is an extension outside Brousseau's didactical situations and more in line with a definition by Balacheff (1990).

Temporarily incomplete or faulty conceptions in the form of obstacles are in Brousseau's theory not in general seen as failures but are often inevitable and constitutive of knowledge. An obstacle produces correct responses within a particular, frequently experienced context but not outside it and may withstand both occasional contradictions and the establishment of a better piece of knowledge. Clarifying obstacles helps the student see the necessity for learning, not by explaining what the obstacle is but to help her discover it. Good problems will permit her to overcome the obstacles. The teacher may (e.g. to reduce complexity) try to overcome the obstacle and force learning by devolving less of the problem to the student. Brousseau exemplifies this by the Topaze effect (p. 25) when the teacher lets the teaching act collapse by taking responsibility for the student's work and letting the target knowledge disappear (as in Guided AR). Telling the student that an automatic method exists relieves her of the responsibility for her intellectual work, thus blocking the devolution of a problem. If this is the normal didactic situation the student meets then the didactical contract is formed accordingly, which may not be the teacher's intention. The teacher expects the student to learn problem solving reasoning, while the student expects that an algorithm should be provided that relieves her of the responsibility of engaging in the adidactical situation. This avoids dealing with the obstacle that can therefore become insurmountable.

So the key issue with respect to this paper is to find a suitable devolution of problem, with the aim of providing learning opportunities through CMR instead of AR. It is in general easy to design imitative (MR or AR) tasks, since the structure of the task is based on repeating the fact or algorithmic procedure and follow therefore directly from the fact or procedure. For example, after the procedure to solve linear equations ($ax+b=cx+d$) is described then a large number of AR tasks are obtained trivially by just formulating different equations. If the purpose is just to design any mathematical problem (recall that a problem is a non-routine task) suitable for a particular student group, then the situation is a bit trickier but the literature and the internet is full of good mathematics problems. However, if the purpose is to design a problem that can help the student to construct (by devolution) a particular target knowledge then the design becomes much more complicated. In addition, the central target knowledge within mathematics curricula is often such that a set of problems (and adidactical situations) rather than a singular task is required. For example, if the goal is that the student shall herself construct a general method for solving linear equations it is unrealistic that this can be done in a single adidactical situation. It probably requires the solution of a series of equations of increasing complexity, and unpublished pilot studies indicate that students can get quite far this way.

A teaching experiment comparing learning by Algorithmic and Creative reasoning

This pre-clinical design experiment is a part of a larger project that studies teaching designs that give students different opportunities to learn with respect to imitation or creative construction of knowledge. In this experiment two ways of teaching are compared:

I) An algorithmic method for solving a type of tasks is presented, and students apply this method on a set of practice tasks. The structure is founded in the framework for AR and in the empirical studies of common teaching mentioned above.

II) Guiding the individual into by herself constructing a solution method for the same type of tasks as in I. This structure is founded in the Theory of Didactical Situations, in research on mathematical problem solving and in the framework for CMR.

In order to be able to compare these two ways of teaching, it is prioritised a) that similar target knowledge of the teaching experiments can be reached by both ways and b) that the target knowledge may be learnt both by rote and by other types of learning leading to higher understanding. A suitable form of target knowledge is task solving methods that can be economised as algorithmic mathematical procedures. This is a central aspect of mathematical knowledge (Kilpatrick, Swafford & Findell, 2001) and since the teaching of such procedures seems to constitute some 50-100% of mathematics teaching (Lithner, 2008; Boesen et al., 2012) at least in Sweden but maybe also in other countries (Hiebert, 2003). Other types of knowledge, e.g. understanding of concepts, the heuristic strategies or metacognitive control ability, is impossible (or at least unlikely) to be learnt by imitative reasoning and therefore not suitable as target knowledge when comparing these two different teaching modes.

One consequence of this choice of target knowledge is that this study does not primarily address the question of how to better learn non-routine problem solving, which is another central aspect of the rote learning problem. However, this has been extensively researched with uniform (at a general level) results: In order to become proficient solvers of non-routine problems students must practice non-routine problem solving, there is no automatic transfer from extensive drills of routine algorithms alone to this competence (Schoenfeld, 1985). Thus the overall background question posed is: "how to best learn mathematical task solving methods that can be formulated as algorithms"? Is it to practice standard algorithms by large amounts of drill exercises, or by the students' own construction of the algorithms? Concerning this issue the discrepancies between research and practice, and between different research perspectives seem large (Arbaugh et al., 2010). In addition, there seem to be little empirical evidence backing the rather few theoretical claims made.

In the imitative teaching mode I a set of algorithms (e.g. rules for solving equations or rules for two-digit multiplication) is described and a set of practice tasks (e.g. equations that can be solved by the given rule) is given to the subject. This teaching mode is hypothesised to lead the subject into rote learning of algorithms by AR without understanding the foundations of the algorithm. It is relatively easy (Lithner, 2008) in mathematics, which is essential for the experiments in this project, to design teaching situations where students are likely to learn a task solving algorithm without understanding it.

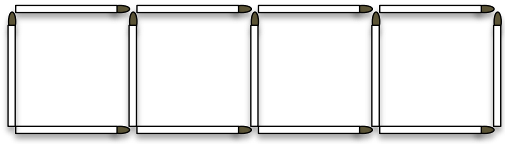
In the creative teaching mode II the subject is not given a method that can be directly applied to solve a set of practice tasks. Instead, a sequence of exploratory tasks is given. The tasks are denoted problems, meaning that the solver does not from the start have access to a complete solution scheme and CMR is required if the tasks are to be solved successfully. This devolution of problem is intended to make pure rote learning impossible and the subject has to understand the method in order to solve the task.

Compared to rote learning, the individual's construction of knowledge is to a larger extent the ideal among educational researchers and in particular within the constructivist paradigm. However, as far as I can see it is still not sufficiently clarified empirically if, why and in what sense this approach should be better. For example, Brousseau's motivation why devolution of problem is necessary is somewhat vague: "The student must construct the piece of new knowledge since she can only truly acquire this knowledge when she is able to put it to use by herself in situations outside the teaching context." (Brousseau, p. 30). One argument behind the hypothesis that task that require CMR will lead to a constructive adidactical situation with a real devolution of problem is related to the three defining criteria of CMR: i) Novelty, that the task cannot be solved by familiar imitative reasoning, ensures the devolution of some kind of reasoning that the student has to be responsible for. ii) The presence of arguments, supporting the plausibility of the conclusions, is necessary to guide and verify the construction of new insights. iii) The necessity to anchor the reasoning ensures that the mathematical obstacles are addressed and that the resolutions are based on properties of relevant mathematical facts and concepts (in contrast to, for example only superficial clues and imitative reasoning).

The research question of this experiment is: What are the characteristics of an adidactical situation that leads to a devolution of problem where learning through CMR is more efficient than learning through AR (in the format common in school)? The present pre-clinical experiment is carried out in a laboratory context with no peer-peer or peer-teacher interaction, and serves to clarify basic phenomena as a preparation to pose the same question in a real classroom context.

Several iterations and revisions of task designs have been carried out. In one of the designs two groups of students, matched by basic cognitive tests, learn task solving methods in the form of algebraic formulas by AR and CMR respectively. An example of an AR practice task is given in Figure 1 and a corresponding CMR practice task is given in Figure 2. One week after the practice session both groups take the same post-test, and among the data registered are the number of correct responses and corresponding response times.

When squares are put in a row it looks like the figure to the right. 13 matches are needed for four squares:



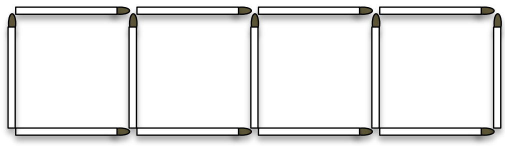
If x is the number of squares then the number of matches y can be calculated by the function $y=3x+1$

Example: If 4 squares are put in a row then $y=3x+1=3\cdot 4+1=13$ matches are needed.

How many matches are needed to get 6 squares in a row?

Figure 1, example of an AR practice task.

When squares are put in a row it looks like the figure to the right. 13 matches are needed for four squares:



How many matches are needed to get 6 squares in a row?

Figure 2, example of a CMR practice task.

One may note that compared to the CMR group, the AR group has an advantage since they are provided with more information. This implies that the AR group could solve the task in exactly the same way or a better way than the CMR group. However, the empirical studies mentioned above show that if students are given an algorithmic solution method to a task, they will probably mainly apply AR to solve the task without considering the underlying meaning of the concepts, representations or connections. Thus they will probably not even try to understand meaning of the algebraic formula, which in this example is the relation between the figure of matches and the formula $y=3x+1$. If this actually is the case in the experiment and if the devolution of problem is successful, then the Theory of Didactical Situations implies that the CMR group may learn better in some ways. The preliminary analyses of data indicate that this is the case, in the sense that the CMR group on average has more correct test responses and shorter response times.

Parallel to the experiment above, other complementary studies are carried out within the research project. One example is an ongoing study using functional Magnetic Resonance Imaging (fMRI) to compare brain activity for students from AR and CMR training groups. This study is exploratory with the aim to analyse non-behavioural information about students' thinking processes. One question asked is if students from the two groups activate different neural networks, and how this relates to earlier research findings about the brain and mathematics. Another question is if students from one group show higher brain activity (in some regions), and what the cause may be. For example, brain activity in the CMR group could be higher if they have created some kind of richer neural networks or lower if they have developed more rational solution methods. Another example of ongoing research uses eye-tracking methods to compare the strategies used by the AR and CMR student groups.

Further research

Schoenfeld (2007) provides a structure that is useful in design research, with four phases of evidence-based educational research and development: Pre-clinical studies, design experiments, contextual studies and large scale validation studies. The ongoing research presented above reside in the pre-clinical stage and concerns the design of mathematical tasks that are suitable for devolution of problems where students may solve the tasks by CMR. One aim is to form a basis for clinical (classroom) studies in phase two. However, it is not just to take tasks designed and evaluated in the pre-clinical phase into the classroom. Stein, Engle, Smith & Huges (2008) argue that teachers who attempt to use inquiry-based, student-centred instructional tasks face challenges that go beyond identifying well-designed tasks and setting them up appropriately in the classroom:

> "Because solution paths are usually not specified for these kinds of tasks, students tend to approach them in unique and sometimes unanticipated ways. Teachers must not only strive to understand how students are making sense of the task but also begin to align students' disparate ideas and approaches with canonical understandings about the nature of mathematics" (p. 314).

Thus one major challenge for the further research is how the design research can incorporate peer – peer, teacher – peer and teacher - class interaction that enhances suitable devolutions of problems through CMR. A second challenge is to design tasks that are more open to the students' own initiatives, and a third to design adidactical situations that encompass wider and deeper target knowledge than the algebraic formulas in the design experiment above.

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