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MAPPING MATHEMATICAL LEAPS OF INSIGHT

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Mathematical leaps of insight – those Aha! moments that seem so unpredictable, magical even – are often the result of a change in perception. A stubborn problem can yield a surprisingly simple solution when one changes the way one looks at it. In mathematics, these changes in perception are usually structural – new insights develop as one notices new mathematical objects, properties and relationships in the underlying mathematical structure. This paper describes a methodological approach for studying these insights visually. The approach uses diagrams to display evidence of students' mathematical knowledge structures as they evolve over time. Significant reorganisations in these structures correspond to mathematical leaps of insight, and the diagrams are used to compare the strength, depth and robustness of students' resulting mathematical structures.
Mathematical insight; calculus; diagrammatic analysis

INTRODUCTION

Mathematical leaps of insight are often described on affective and aesthetic levels—the excitement of understanding something that was previously incomprehensible (Liljedahl, 2005), the pleasure of a beautiful, elegant solution (Hadamard, 1954; Poincaré, 1956). On a mathematical and conceptual level, leaps of insight involve shifts in perceived structure, whereby students notice and create deeper, stronger and more robust mathematical structures to replace older ones. In this paper, I propose a way of visualising these leaps of insight through network diagrams that display evidence of the mathematical structures students perceive as they work on challenging mathematical tasks. This visual approach is inspired by the success of another visual technique in a cognate field: functional magnetic resonance imaging (fMRI). Just as fMRI techniques enable neuroscientists to document and measure neural activity visually, so the diagrams described in this paper are intended to provide a visual way of studying shifts in students' perceived mathematical structure.

Not all shifts in perceived mathematical structure qualify as a *leap* of insight, however. Sometimes, students will elaborate on an existing mathematical structure by adding new mathematical objects, properties, operations and relationships, like a variation on a theme. By contrast, a leap of insight involves a significant reorganisation of mathematical structure that goes beyond incremental additions to an extant structure. This paper uses network diagrams to compare the size and quality of the shifts in mathematical structure that two participants perceived while working on an antiderivative task.

PERCEIVED MATHEMATICAL STRUCTURE

The discipline of mathematics can be characterised as consisting of different types of structure: "Pure mathematics is the study of structures" (Shapiro, 1997, p. 75). These structures are made up of mathematical objects (such as counts, measures, sets), properties (e.g. few, large, open), operations (e.g. combine, enlarge, invert), and relationships (e.g. greater than, equivalent to, isomorphic), and different mathematical structures have different mathematical components. Applied mathematics also involves creating and manipulating mathematical models to describe structures in the world: "Mathematical models are distinct from other categories of models mainly because they focus on structural characteristics (rather than, for example, physical, biological, or artistic characteristics) of systems they describe" (Lesh & Harel, 2003, p. 159).

The term "structure" is also used in theories of learning to describe how mathematical *knowledge* is represented in the mind. For example, Skemp (1987) describes understanding as the development of schemas, which are interconnected networks, and Piaget (1970) describes the growth of children's cognitive structures. In this instance, the term "structure" is used to characterise *mathematical knowledge*, rather than simply *mathematics*, although the two characterisations are closely linked. I adopt a structural view of *mathematics* in my analysis of students' mathematical insights. Accordingly, I seek to identify the mathematical structures (and consequently the mathematical objects, properties, operations and relationships) that students perceive and work with. I am more interested in characterising the content of students' mathematical interpretations (i.e., what mathematical structures have they constructed and perceived?) than theorising how those interpretations are represented in their mind. Therefore, I use the term "mathematical structure" in this paper to refer to the mathematical objects, properties, operations and relationships that students perceive, rather than terms such as "construct", "conceptual system" or "schema", which are often associated with assumptions about the representation of knowledge more generally.

I acknowledge that a person's perceived mathematical structures can only be identified through the observable signs the person produces and works with. According to Arzarello, Paola, Robutti and Sabena (2009), these signs or semiotic systems may involve spoken language, gestures, written text, symbols, diagrams, graphs and physical artefacts. Consequently, I analyse the semiotic systems that students produce while working on a task in order to identify the mathematical structures they perceive. The use of the term "perceive" is not meant to suggest that the mathematical structures are "in" the problem, hidden for students to find. Instead, I use the term "perceived mathematical structure" to emphasise that I am trying to identify the mathematical structures that students themselves construct, recall and manipulate in order to solve the problem.

METHOD

The data used in this paper were collected as part of a project that investigated students' construction of calculus concepts (Yoon, Dreyfus & Thomas, 2010). Twelve participants worked in pairs on four calculus tasks of one-hour duration each. Ten participants were undergraduate students, two were secondary school mathematics teachers, and all had basic school-level knowledge of calculus at the time. The participant pairs worked in the presence of a researcher who clarified task instructions but refrained from giving mathematical direction. The pairs were videotaped and audiotaped on each of the four tasks, which led to 24 verbal transcripts. These transcripts were annotated to include the gestures, inscriptions and nonverbal cues that participants performed or created.

This paper reports on the shifts in perceived structure identified from participants working on the first task—the tramping activity (described below). In order to identify these shifts in structure, I developed a coding scheme using design research principles (Kelly, Baek, & Lesh, 2008). I began by developing an initial list of codes to identify the mathematical objects, properties, operations and relationships that could be encountered during the activity. Then, two research assistants and I entered into design cycles of implementing, testing and revising the coding scheme on three transcripts (which were analysed in conjunction with the associated video and images of student work). We coded portions of transcripts independently then met to compare and identify coding discrepancies. I used these discrepancies to inform revisions in the coding scheme, which we again tested through independent coding, comparisons, and further revisions. Following this, a third research assistant who had not been involved in the process joined us in further design cycles. This measure was taken to enhance the reusability of the coding scheme.

Table 1: A selection of objects and properties from the coding scheme

	OBJECTS	**PROPERTIES**
Related to the gradient graph	*y*-value	Large, small, zero, bigger than, etc.
	Change in *y*-value	Increasing, decreasing, constant
	Sign of *y*-value	Positive, negative
	Maxima/minima	Number, order, location
	x-axis intercepts	Number, order, location
	Size of bumps/dips on g-graph	Big, small, relative size
Related to the tramping track (antiderivative)	Steepness of slope	Steep, gentle, flat, steeper than, etc.
	Change in steepness of slope	Getting steeper, getting gentler, no change
	Direction of slope	Up, down, flat
	Absolute height	High, low, at or below sea level, etc.
	Points of inflection	Location, number, order
	Horizontal distance of track	Long, short, longer than, shorter than, etc

The design process lasted six months, during which the design team met more than 20 times to compare and discuss our coding. The process was repeated to develop a coding scheme for the fourth task (for a description of the fourth task, see Yoon, Thomas & Dreyfus, 2011a), which informed further revisions of the coding scheme for the first task. All of the six transcripts of the first task were then recoded using the final coding scheme (which consists of over 100 codes), and the mathematical structures were represented in network diagrams. Table 1 shows a small selection of the codes from the final scheme that were used to identify mathematical objects and properties that students perceived during the first task. Antidifferentiation was the main mathematical operation in this task, and the list of mathematical relationships included equivalences, comparisons, and logical relationships.

Description of the tramping activity

The activity begins with a warmup, where students are given a distance-height graph of a tramping (hiking) track (Figure 1a), and instructed to calculate gradients of the track at certain points, and sketch the gradient graph (derivative) of the track (Figure 1c). Students are then shown a graph of a similar, flatter tramping track (Figure 1b), and are asked to sketch the gradient graph of this track (Figure 1d) without calculating any actual gradients.

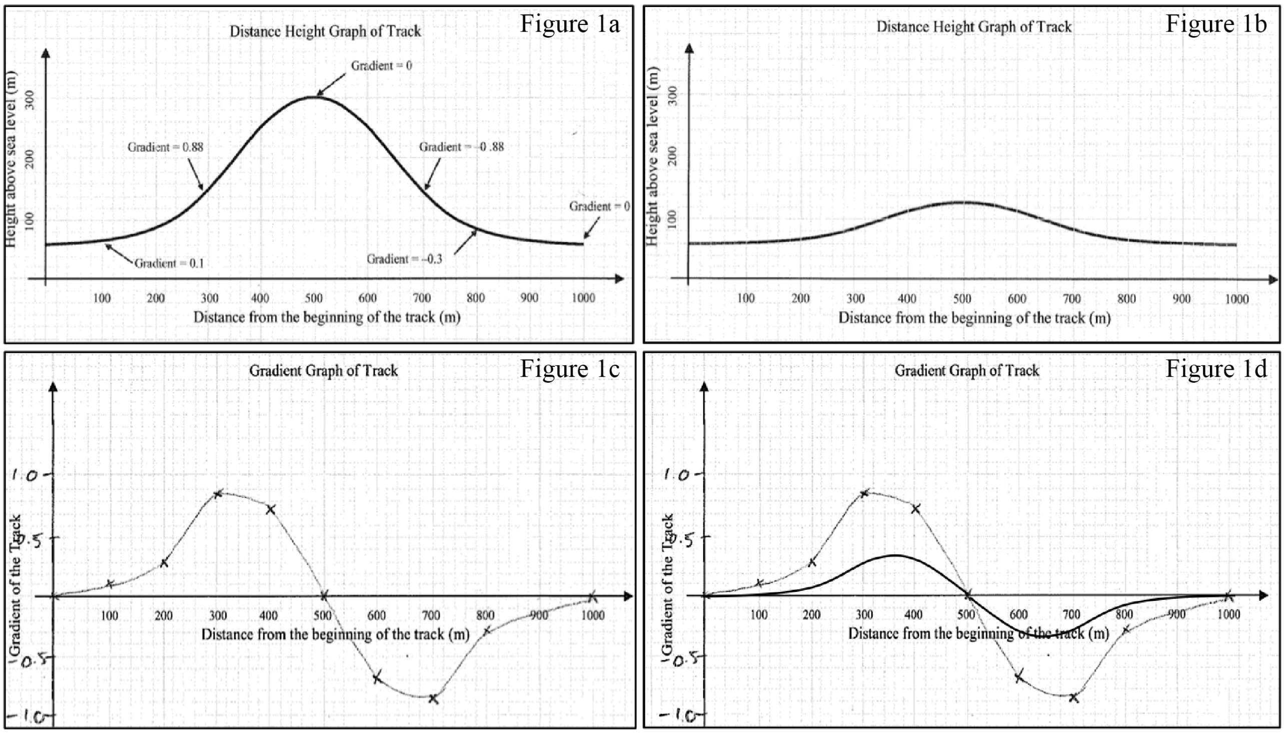
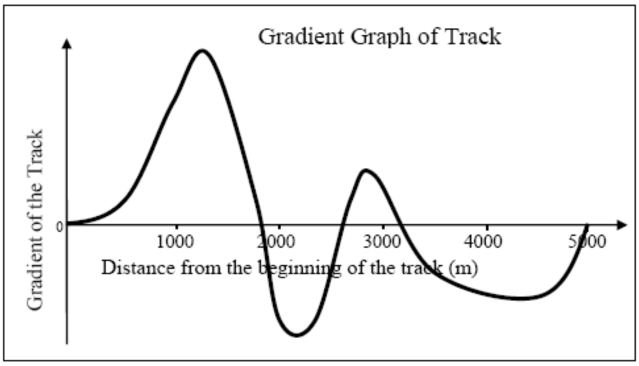


Figure 1: Graphs of two tramping tracks (1a & 1b) and their gradient graphs (1c & 1d)

Next, students are given the problem statement (see Figure 2), which asks them to design a method for drawing the distance-height graph of any tramping track from its gradient graph. This task is mathematically equivalent to finding the graphical antiderivative of a function presented graphically. Students are asked to explain their method in the form of a letter addressed to hypothetical clients (the O'Neills) and to use their method to find features of a specific track whose gradient graph is given in Figure 2.



The O'Neills need your help!

Design a method that the O'Neills can use to sketch a distance height graph of the original track (like the one given in the warm up question). You can assume that the track begins at sea level.

Write a letter to the O'Neills explaining your method, and use your method to describe what the tramping track will be like on the day. In particular, you must clearly show any summits and valleys in the track, uphill and downhill portions of the track, and the parts of the track where the slopes are steepest and easiest.

Most importantly, your method needs to work not only for this tramping track, but also for any other tramping track the O'Neills might consider.

Figure 2: Graph of a tramping track's gradient and the problem statement

RESULTS

The diagrams presented in this section describe the evolution of two kinds of mathematical structure that Ava and Noa (the pair of teachers in the study) perceived during the tramping activity (for more detail about these participants, see Yoon, Thomas & Dreyfus, 2011b). The first type of perceived structure involved determining whether *x*-axis intercepts in the gradient graph correspond to maxima or minima in the tramping track. The second dealt with interpreting how the "bump size" in the gradient graph relates to the height (*y*-value) of summits and valleys in the tramping track. Ava and Noa also perceived many other types of mathematical structures, but these two were chosen because they illustrate different kinds of shifts in perceived structure. Ava and Noa frequently referred to features of the gradient graph and the tramping track in their work. These features are shown in Figures 3a and 3b: the latter diagram shows the distance-height graph of the tramping track that Ava and Noa drew, which is a reasonably accurate depiction of the main features of the track.

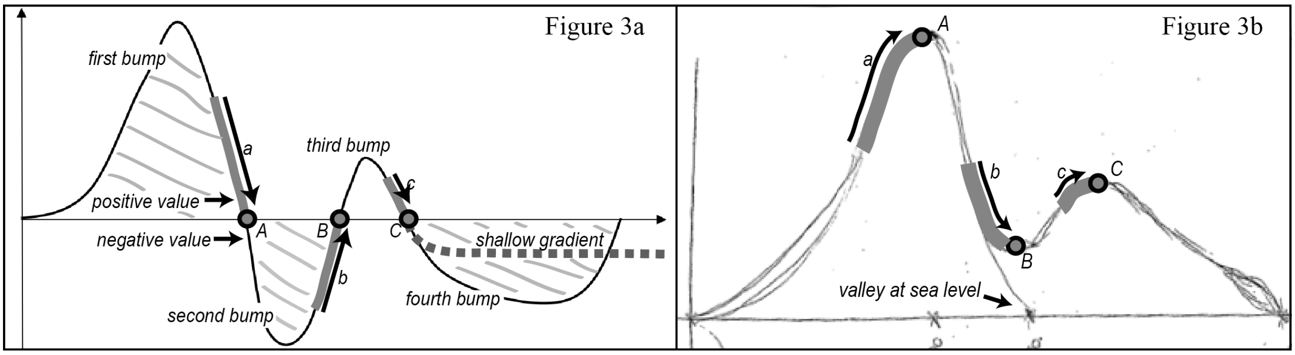


Figure 3. Some features of the gradient graph (a) and the tramping track (b)

1. Perceived mathematical structure regarding *x*-axis intercepts on the gradient graph

During the warmup to the activity, Ava and Noa notice that the gradient is zero at the summit of the track (Figure 1a), and they plot this in the corresponding gradient graph (Figure 1c). When they begin the actual antiderivative problem (Figure 2), they reason that the first *x*-axis intercept in the gradient graph (point *A* in Figure 3a) corresponds to a flat gradient of the track, but are unsure whether this is due to a summit or a valley. They notice that the shape of the warmup gradient graph (Figure 1a) to the left of the *x*-axis intercept is similar to the shape of the analogous part of the given gradient graph (i.e., the curve to the left of point *A* on Figure 3a). They reason that the first *x*-axis intercept in the given gradient graph corresponds to a summit since the *x*-axis intercept in the warmup gradient graph does.

The diagrams in Figure 4a and 4b portray the relevant mathematical structures described so far. Notably, Ava and Noa do not interpret the shape of the graph to the left of the *x*-axis intercept in terms of the gradient of the track (e.g., positive gradient, with increasing then decreasing steepness), but rather in terms of its geometry, and its similarity to that of the warmup gradient graph was used to argue the presence of a summit. This changes when Ava and Noa trace along the gradient graph (from left to right) with one hand, and visualise the track using gestures in their other hand, then draw the graph of the track (see Yoon et al., 2011b for more detail on these gestures). They attend to the sign and changing size of the *y*-value of the gradient graph before each *x*-axis intercept, interpret the corresponding direction and changing steepness of the gradient, and thereby determine whether the intercepts corresponded to peaks or valleys in the track. For example, they notice that the *y*-value is positive but decreasing over section *a* in Figure 3a, they reason that this corresponds to the track being uphill but getting gentler over section *a* in Figure 3b, and that point *A* in Figure 3a corresponds to a summit. They reason similarly for sections *b* and *c* and points *B* and *C* in Figures 3a and 3b. The elaborations to the previous mathematical structures are shown in Figure 4c.

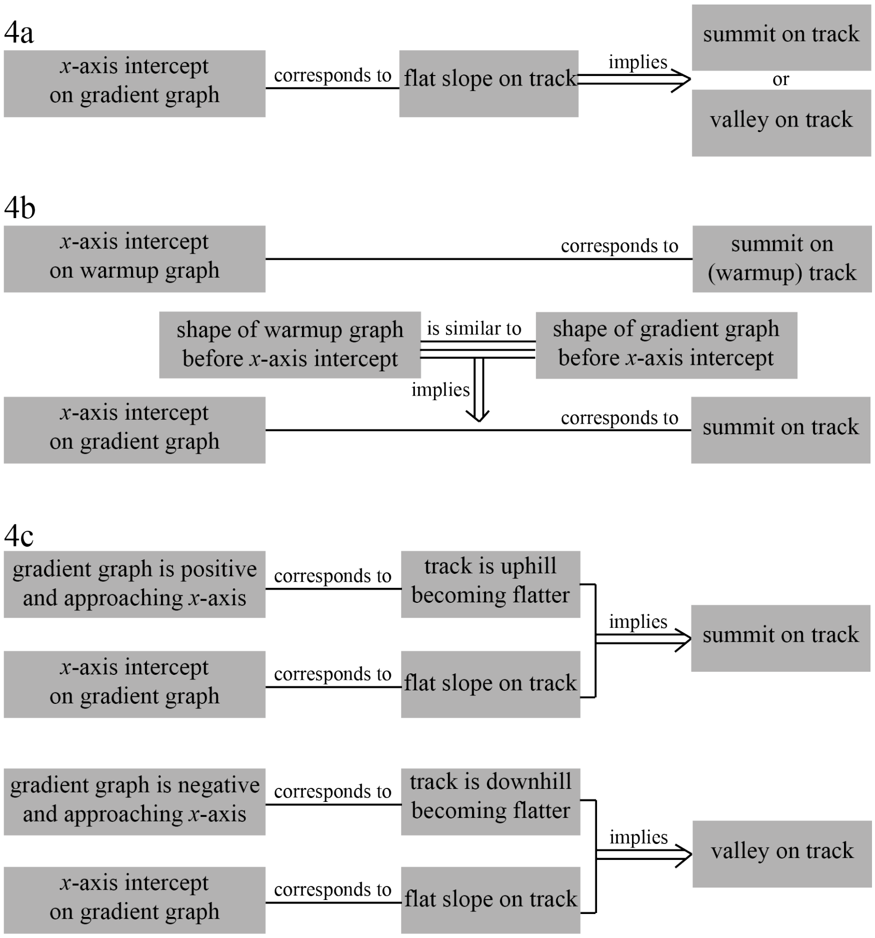


Figure 4(a-c): Diagrams showing the construction of concepts related to *x*-axis intercepts

The next shift involves a more significant reorganisation in Ava and Noa's perceived mathematical structure. They generalise that when a gradient graph crosses from positive values to negative values, the *x*-axis intercept corresponds to a peak (maximum) in the track and vice versa for valleys (minima). In this new perceived structure, Ava and Noa widen their focus to consider the gradient after the *x*-axis intercept as well as before. At the same time, they narrow their focus to consider only the discrete change in the sign of the gradient graph (positive to negative) rather than the shape of the graph, or the continuous variation in its *y*-value. The revised structure is simpler and more elegant as it uses few elements to synthesise many of the previous objects and relationships: the change in sign eliminates the need to consider the continuous change in *y*-value before each *x*-axis intercept.

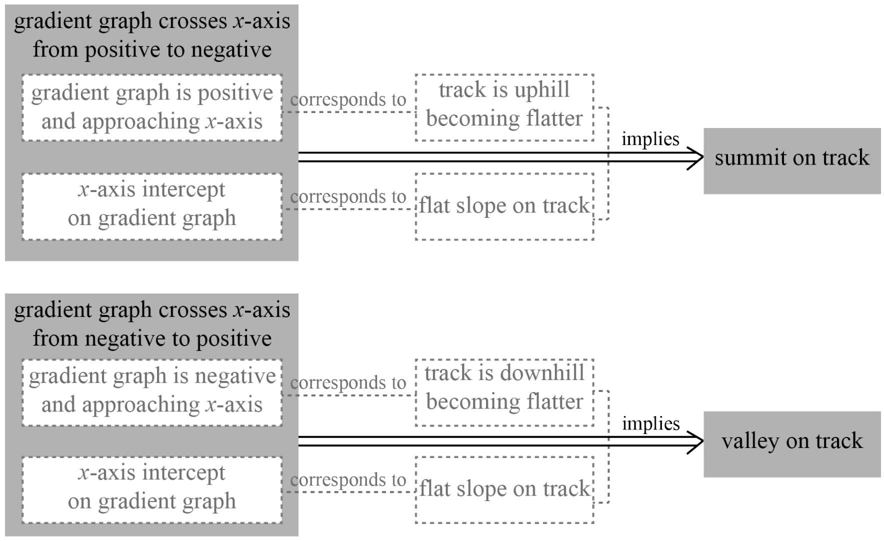


Figure 5: The final mathematical structure regarding *x*-axis intercepts on the gradient graph

2. Perceived mathematical structure regarding the height of the tramping track

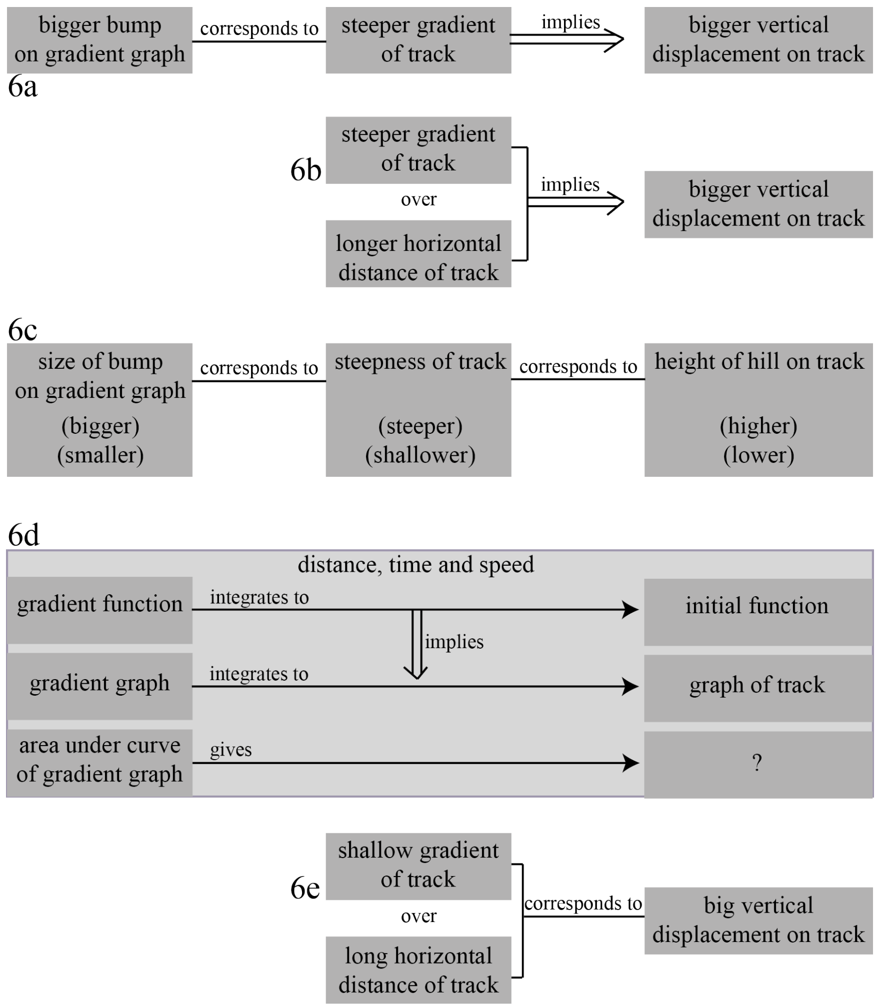


Figure 6(a-e): Diagrams showing the construction of concepts related to height

Ava and Noa first consider the height (*y*-value) of features of the tramping track when Ava draws the graph of the track. Initially, she draws the valley at sea level (this is visible in

Figure 3b), but Noa points out that the valley may be above sea level. Ava revises her drawing, reasoning that as the first bump is bigger than the second bump on the gradient graph (see Figure 3a), the ascent to the first summit is steeper than the descent to the valley and therefore, the valley must be above sea level. Noa agrees, saying, “Steeper for longer must be higher”. Figure 6b shows that this comment suggests the structure Noa perceives is slightly different to Ava’s (Figure 6a): Noa’s structure introduces a new mathematical object and relationship, the horizontal distance during which steepness is measured. Next, they notice that the larger bumps in Figure 1c correspond to a higher summit in Figure 1a, and the smaller bumps in Figure 1d correspond to a lower summit in Figure 1b, and reason that these relationships hold generally. Figure 6c shows that their perceived mathematical structure is an elaboration of the same basic structure perceived in Figure 6a.

Then, Ava and Noa try to consider how integration and the area under the curve could help them establish the height of the track’s features more accurately. Ava says, “If you’ve got the gradient function how do you get back to the initial function? You need to integrate. Right, so we could integrate this function to provide a value? Yes, we could, couldn’t we?” and shades in the area under the bumps in Figure 3a. Noa agrees and they try to apply their knowledge of integration and area under the curve to the problem. However, their discussion of integration is firmly rooted in the context of distance, time, and speed, rather than the context of slope and height, and they abandon this pursuit after failing to link their knowledge of integration to the mathematical structures they have created.

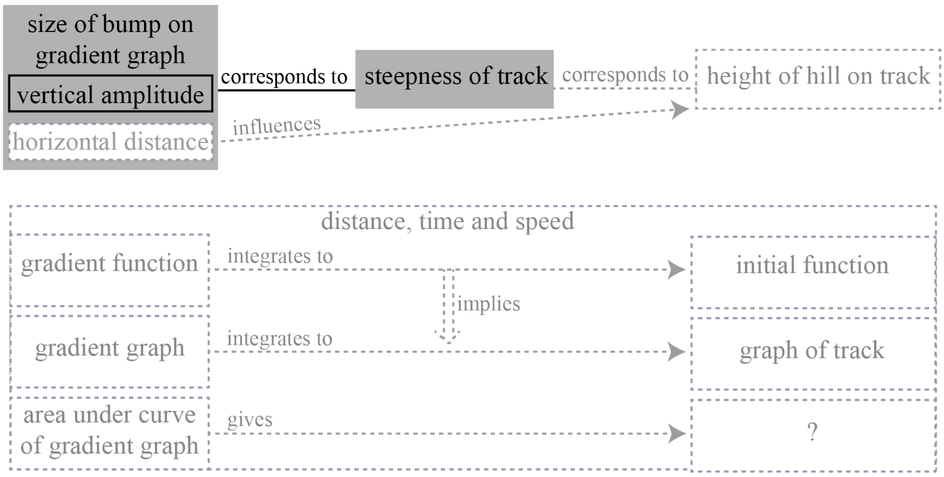


Figure 7: The final mathematical structure regarding the height of the tramping track

They return to their previous assertion that the size of bumps on the gradient graph corresponds to the steepness of the track, and is consequently an indicator of the vertical displacement in the track. Noa then rejects this, saying, “You could have [a bump] that’s just got a shallow gradient but just goes on for ages (*traces shallow gradient from point C in Figure 3a*), so you could end up being really high, but not really steep to get there.” This demonstrates a more sophisticated understanding of the interaction between steepness and track height, as it incorporates (like Figure 6b) the additional dimension of horizontal distance (see Figure 6e). However, Noa uses this as a counterexample of their previous rule, rather than an opportunity to expand the mathematical structure. In the end, they adopt a more conservative rule—that the vertical amplitude of the bumps on the gradient graph indicates the steepness, but not the height of the track.

The diagram in Figure 7 displays the final mathematical structure perceived by Ava and Noa regarding the question of the height of the antiderivative. Their final structure fails to incorporate many of the objects and relationships that were considered throughout Figures 6a-e, and which could be linked to explain how the area under the curve of the bumps (approximated by their vertical amplitude multiplied by their horizontal length) informs the height of the track at various points. Instead, the structure shrinks to a more conservative set of objects and relationships, and abandons the question of the height of the track altogether.

DISCUSSION AND FUTURE DIRECTIONS

The results show considerable variation in the size of shifts in perceived mathematical structure. Incremental changes are characterised by the objects being refined, and varied, without changing the structure significantly: such changes are visible between Figures 4b to 4c, and 6a to 6c. More significant changes occur when new objects, properties and relationships are incorporated into the structures. For example, Figure 6b shows a new object, the horizontal distance, being connected to the previous structure from Figure 6a, and Figure 6d shows the introduction of a completely new structure.

A leap of insight can be identified in Figure 5, where previous objects describing the continuous variation in size of the *y*-value from Figure 4c are incorporated into a mathematical structure that is *stronger*, *deeper* and more *robust*. I will clarify what I mean by these terms by comparing the structures depicted in Figures 5 and 7. First, the strength of a structure refers to its internal connectivity. Whereas the structure shown in Figure 5 is well connected, with previous objects being incorporated into new objects, the structures in Figure 7 are poorly connected, with the structure regarding integration disconnected altogether to the size of the bump on the gradient graph. The depth of a structure can be visualised as layers in a structure. The structure in Figure 5 consists of two layers: the objects and relationships with dotted lines can be visualised as sitting below the rest of the structure, much like a secondary layer of explanation that props up the first. In comparison, the structures in Figure 7 are all of the same depth as there has been no incorporation of one layer into another. Finally, the robustness of a structure describes its explanatory power. The structure in Figure 5 can be expanded easily to identify points of inflection from *x*-axis intercepts in the gradient graph, not just turning points. Thus, it is more robust than the structure in Figure 4c, which would need to be altered significantly to include objects and relationships that describe the behaviour of the gradient graph after an *x*-axis intercept in order to identify points of inflection. Similarly, the structure in Figure 7 is less robust than that in Figure 6a, as it cannot explain the height of the antiderivative track.

The results shown in this paper are a small part of a larger research programme dedicated to mapping students' mathematical conceptual shifts in order to understand how such shifts come about, and how to facilitate their occurrence in practice. Similar analyses have been carried out to identify the shifts experienced by different participants working on different tasks. My immediate goal is to display the network diagrams in a dynamic format, so that researchers can observe the mathematical structures evolve over time. In the long term, I plan to create a software programme that automates the construction of these dynamic network diagrams, so that other researchers can use similar procedures to study mathematical leaps of insight in any mathematical domain.

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