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EXPLORING THE NATURE OF THE TRANSITION TO GEOMETRIC PROOF THROUGH DESIGN EXPERIMENTS FROM THE HOLISTIC PERSPECTIVE

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The gulf between empirical and deductive reasoning is a global problem that has produced many students who have extreme difficulties learning proofs. In this paper, we explore the conditions that aid students in entering into proof learning and how they can increase their ability before learning proofs through design experiments. First we discuss the theoretical backgrounds of the holistic perspective and didactical situation theory, and set our research framework as the transition from empirical to theoretical recognition consisting of the three aspects of inference, figure, and social influence. Next, we report our design experiments in plane geometry redesigned for the seventh grade, and examine how students may enter the world of proof by learning geometric transformation and construction as summarized in the three aspects of the framework. Finally, we suggest key ideas for designing lessons that promote transition.

Key words: Transition to geometric proof, holistic perspective, empirical and deductive reasoning

PROBLEMS WITH PREREQUISITES AND THE NECESSITY OF LEARNING PROOF

Proving is an essential activity in mathematics that has occupied an important content area in school mathematics. However, extreme difficulties with proof learning have continued to be a global problem. Research has indicated that the gaps between empirical and deductive reasoning cause a large number of secondary students to fail to learn proof (Hirabayashi, 1986; Harel and Sowder, 2007). Nevertheless, it remains unclear how students can increase their abilities to bridge this gap, which is the issue we explore in this paper.

We consider the gap between empirical and deductive reasoning as being identified by the van Hiele model (1986) as a difference between the second and third levels of thinking. In the second level of thinking, students “explicitly attend to, conceptualize, and specify shapes by describing their parts and spatial relationships between the parts” while in the third level they “explicitly interrelate and make inferences about geometric properties of shapes” and “logically organize sets of properties (Battista, 2007). Several researchers have indicated that even after students learned formal proof many of them continue to use empirical arguments from the second level of thinking (Chazan, 1993; Koseki, 1984), such as basing their responses on appearances in drawings or proving statements by providing specific examples, and are not able to distinguish between inductive and deductive arguments (Harel and Sowder, 2007). However, our intention is not to compare between the empirical-inductive and deductive forms of argument, but rather to reexamine them from the perspective of transition from the former to the latter.

Moreover, several Japanese researchers with teaching experience have indicated a problematic situation in proof teaching whereby students do not feel the necessity for learning proofs (Ohta, 1998; Souma, 1998; Kunimune, 2003). As reasons for this they suggested that students already know the properties and thus few new properties need to be explored, so it is a determined geometric system that is given to students. For example, the equality of base angles in an isosceles triangle has already been confirmed by folding a piece of paper in elementary school, and therefore the students find it hard to understand the significance of proving the properties of isosceles triangles and parallelograms. Instead they proposed the idea of "proving as inquiry" (Ohta, 1998; Sekiguchi, 2002). For proof learning to be accompanied by necessity, Ohta insists that we should "situate proofs in the learning tasks to solve problems from a real life or to explore the interesting properties of geometric figure, and to encourage students to see the significance and structure of proof through gradually systemizing them". Stylianides (2007) also indicated that the sudden introduction of proof causes difficulties. We think that these factors cause students to miss the point of why they are learning proof.

We thus consider it important to emphasize the functions of proof (de Villiers, 1990), in particular the function of explanation because formal proof that only focuses on verification causes students to remove much of the significance (Hanna and Jahnke, 1996). Furthermore, as Hanna (1991) shows, mathematicians regard the appeal of the theorem to be more important than that of rigorous proof, and it may be necessary to consider what interesting situations the theorem can treat and solve. Tall (2008) also indicated that "written formal proof is the final stage of mathematical thinking; it builds on experiences of what theorems might be worth proving and how the proof might be carried out, often building implicitly on embodied and symbolic experience". Without considering this, students may not direct their attention towards formal proof.

We examine in this paper what conditions enable students to learn proof and how they can develop their reasoning abilities before learning proof so that they overcome the gap between empirical and deductive reasoning. We think that this is consistent with recent attempts that continually foster proving of statements from the elementary stage under a wider concept of proof (NCTM, 2000; Stylianides, 2007). At the same time, we think that we cannot avoid the problem of necessity. If students begin learning the significance of proof, they will have a good chance of learning successfully. Thus, our second focus is how we can create a situation in which students are able to prove with necessity. For these purposes, a large number of practical studies will be necessary. However, very little research has so far looked at proving practices in the classroom (Knipping, 2008). We later examine how students get their ability for proof learning through our classroom design experiments.

Below, we begin by considering a perspective that permits proving as inquiry.

THE HOLISTIC PERSPECTIVE AND DIDACTICAL SITUATION THEORY

We now introduce the philosophical stances of the holistic and systemic perspectives (Miller, 2007; Wittmann, 2001), not distinguishing between the two terms but showing the central ideas.

The holistic perspective is in a sense an antithesis of the mechanistic and atomistic view that has dominated our way of thinking and living in various respects since the industrial revolution (Hirabayashi, 1987; Sato, 1996; Wittmann, 1995, 2001). In the atomistic view, the whole consists of the separate parts and so the whole is acquired by learning the parts separately. This view may induce teachers to break up the content, arrange it sequentially and teach it piece by piece. Harel and Sowder (2007) reported that the stereotype of a US class is "students check homework, teacher illustrates something new, students then do seat-work or homework to practice the new material," which we suggest to be an influence of the atomistic and mechanistic view.

Meanwhile, holism is the view that the whole is not the sum of its parts but is the whole itself in principle. When students, even those who have good mathematical competence, often say, "I don't know what we are learning, where we are going, what mathematics is good for", these comments can be interpreted as anxiety at not seeing the whole. In this regard, we may note that holism does not approve of totalism which consolidates all things into the fixed total, but emphasizes the relations between the whole and the parts, or that the whole can come to be a part of the larger whole.

Holistic education is also a result of reflection on the history of education in which policy has gone back and forth between emphases on the individual child' knowledge and skills and on the application to real life contexts, and tries to synthesize both to open up new horizons for education (Yoshida, 1999). We think that the definition of holistic education given by Miller (2007) is important in mathematics education:

> The focus of holistic education is on relationships: the relationship between linear thinking and intuition, the relationship between mind and body, the relationships among various domains of knowledge, the relationship between the individual and community, the relationship to the earth, and our relationship to our souls. In the holistic curriculum the student examines these relationships so that he or she gains both an awareness of them and the skills necessary to transform the relationships where it is appropriate.

Holistic education has several keywords such as "relationships", "balance", "inclusion", and "connection". We may consider at least three aspects of this even if we limit our scope to mathematics. One aspect is knowledge formation where emphasis is placed on connections between intuition, logic, prior or everyday knowledge, and the ideas of others. The second aspect is human relationships, which may include fellowships, norms, or attitudes in the classroom. The third aspect is the innermost self. Education aims to give students encounters of experience so that the self is moved, where there is a connection between the external ego and the internal self. Here "the student is not reduced to a set of learning competencies or thinking skills but is seen as a whole being" that includes aesthetic, ethical, physical and spiritual aspects.

Holistic education integrates two strands. One is humanistic education that concerns the growth of humanity in each student, and the other concerns social change towards an equal and cooperative society. In mathematics education, the former has been integrated into mathematics as a human activity by several founders, such as Gattegno, Wheeler, Brown and Hirabayashi (Hirabayashi, 1987, 2001; Koyama, 2007). The latter strand has been recognized by teachers in Japan. Here, some excellent teachers share their thoughts: "Only after the feelings of 'everybody is different and everybody is nice' come into being in child's mind the rich mathematics lesson in creativity is possible" (Tsubota, 2001), and "I specialize elementary mathematics teaching. Through teaching mathematics, I am guiding students' life, deepening child's compassion, and fostering classroom camaraderie" (Takii, 2001). We think that the holistic view may reflect more or less an East Asian way of thinking. The theme of PME 31 in Korea was "School Mathematics for Humanity Education". The holistic view sympathizes with the Chinese philosophy of Laozi and Zhuangzi (Hirabayashi, 2001; Wittmann, 2001). Wittmann expresses it as "leaders should not interface with the natural powers and inclinations of their clients, but should instead build upon self-organization and offer help for self-help". Here, Wittmann considers mathematics education as design science, which sympathizes with our research stance.

We may then consider how we can realize the holistic view in the classroom. We think that it may be a basic didactical device for constructing rich situations in which students can identify various relationships (Hirabayashi, 2001; Wittmann, 2001; Yoshida, 1999). In this sense, we turn out attention to Brousseau's (1997) didactical situation theory for that realization. Brousseau conceives of knowledge (knowing) as characterized by a (or some) "adidactical situation" and describes the learning process by which knowledge and situation develop reciprocally. He distinguishes three statuses of knowledge in the history of mathematics: protomathematical, paramathematical, and mathematical. An example he gives of the protomathematical stage is that al-Khowarizmi had constructed many ideas with rational numbers but not really with real numbers. An example of the paramathematical stage is function in the nineteenth century, where "in the absence of recognized mathematical status, their terms used are tools which respond to the needs of identification, formulation and communication and that their use is based on a semantic control". In the final mathematical stage, a concept is put "under the control of a mathematical theory" and has "its exact definition in terms of structures in which it intervenes and of the properties that it satisfies".

The three types of knowledge are sustained by situations for action, formulation, and validation, respectively. Students first construct their (informal) ideas from their interaction with a fundamental situation like a game or a problem that is the situation for action. In situations for formulation they use their ideas practically or explain them to others, so the ideas may take on a character of usability (tool) and the ideas of others enter the situation. Furthermore, in situations for validation, students objectify the messages that have been exchanged with others and develop their knowledge of mathematical and theoretical ideas. Okazaki (2003) schematized the process by which the three situations develop as follows (Fig. 1).

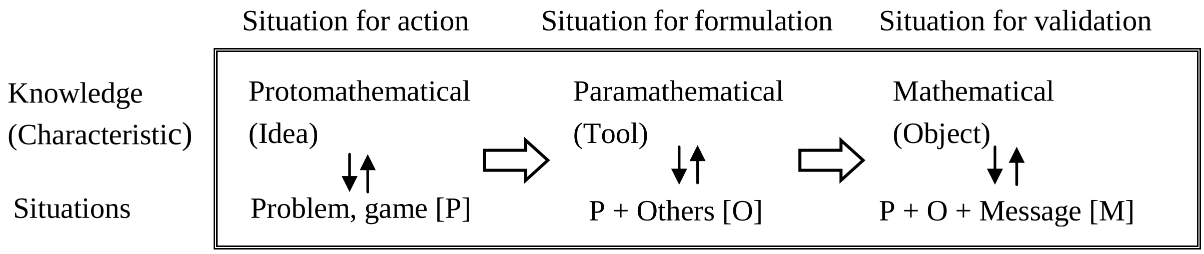

Figure 1. Learning process in didactical situation theory

There are several holistic characteristics within didactical situation theory. The process shows that students develop their ideas among various relationships through interacting with the situations and with other people. It also attempts to design mathematics classes that cover the improvement of human relationships by didactical engineering (Douady, 1997). It seems that didactical situation theory suggests a way of realizing holistic views.

Finally, if we see the theory in terms of proving activity, it places us in situations for validation. The theory suggests that the preceding stages are necessary to reach proof learning. Brousseau (1997) also indicates that the paramathematical stage has continued for a long period in the history of mathematics. If we wish to invite all students into a world of proving, we should clarify what activities are needed in situations for action and formulation to prepare for effective proof learning, what opportunities create the feeling of the necessity to prove something and how these proving activities can emerge and develop.

TRANSITION FROM EMPIRICAL TO DEDUCTIVE RECOGNITION

First let us briefly introduce the teaching of proof in Japan. The importance of proof has been recognized since the end of the nineteenth century when Japan imported western mathematics. Formal proof is now introduced in the eighth grade where propositions about isosceles triangles, right triangles, and parallelograms are proved mainly using conditions for congruent triangles. In the seventh grade, a year before introducing proof, students learn geometric constructions and transformations, not in a rigorous way, but rather intuitively as a prerequisite for learning proof. However, the connection between constructions, transformations and proof has not succeeded so far, and proof has remained one of most difficult concepts for Japanese students to learn.

Empirical and deductive proof schema

Harel and Sowder (2007) indicate several typical views of proof that students have: (A) external conviction proof schemes (authoritarian, ritual, and non-referential symbolic), (B) empirical proof schemes (inductive and perceptual), and (C) deductive proof schemes (transformational and axiomatic). (A) is a heteronomous case that depends on the authority of the teacher and textbook, or the superficial appearance of a written proof. (B) is a conception of proof by giving examples or experimenting and measuring. Kunimune (2003) reported that 92% of Japanese eight graders and 77% of ninth graders thought it acceptable to use experimentation and measurement. Chazan (1993) also reports that students have the belief that evidence is proof and that proof is just evidence. (C) is divided into deductive reasoning and the understanding of the meaning and roles of axioms.

(B) and (C) may be historically likened to pre-Greek and Greek mathematics, respectively. Harel and Sowder see not just that the Greeks pushed mathematics from a practical tool to the study of abstract entities and produced a proof method, but also that the consideration of the nature of existence and inference applied to existence progressed in parallel, namely that the object and the method are epistemologically dependent. If so, even if we teach students just the method of proof without changing the existence of a geometric figure, an inconsistency between the method and the object may arise. Moreover, they indicated that Greeks wished to create a consistent system that avoided paradoxes. The study of consistency presupposes the existence of a substantial number of laws, as was the actual situation in Greece. However, it does not seem that current geometry students have as many laws as they feel necessary for organization before learning proof. Hirabayashi (1991) indicated that "an introduction of proof may be impossible until students come to view geometric figure as a set of properties and relations, but not as the intuitive shape".

They also point to the continuity between the empirical and the deductive proof schemes. They state that "the construction of new knowledge does not take place in a vacuum but is shaped by existing knowledge," and "the empirical proof schemes are inevitable because natural, everyday thinking utilized examples so much. Moreover, these schemes have value in the doing and the creating of mathematics... The question is how to help students utilize their existing proof schemes, largely empirical and external, to help develop deductive proof schemes?" Our research question addresses this very point. We consider the teaching and curriculum for developing empirical proof schemes towards deductive schemes below.

THE RESEARCH FRAMEWORK FOR ANALYZING THE TRANSITION FROM EMPIRICAL TO DEDUCTIVE RECOGNITIONS

We consider next the research framework for analyzing the transition process students follow towards geometric proof, which consists of inference, figure, and social influence.

Learning geometric construction is crucial for extending students' empirical recognition to deductive recognition. Mariotti (2000) states that "geometrical constructions have a theoretical meaning. The tools and rules of their use have a counterpart in the axioms and theorems of a theoretical system, so that any construction corresponds to a specific theorem," where she emphasized a need to shift from the construction procedure to a justification of the procedure itself. Tall (2008) also refers to the shift of the focus of attention from the steps of a procedure to the effect of the procedure in the compression from procedure to process.

Okazaki and Iwasaki (2003) identified several functions of geometric constructions in the teaching experiments: evoking shapes and putting their properties into play, constructing propositions, facilitating the recognition of hypothesis-conclusion, and enhancing the recognition of definition. However, we do not think this is possible with the current teaching where the procedures of drawing perpendicular bisector, perpendicular, and angle bisector are taught in a manner such as: 'Draw a circle with center O and any radius and let the intersections with side OA and OB be C and D respectively. Draw circles with centers C and D of equal radius and let the intersection be P. Draw the ray OP'. Such knowledge may not provide a good basis for students in their proof learning.

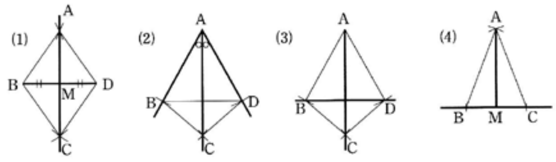

Figure 2.

Instead, we should consider that perpendicular bisector, perpendicular, and angle bisector are integrated in a kite or a rhombus (van Hiele, 1986). Thus, if we imagine a kite, then we can construct each of these (Figure 2). Because of the definition for constructing a kite ("two pairs of adjacent sides are equal"), perpendicular and angle bisector are then deduced as properties of the diagonals. If we symbolize this as if AB = AD and BC = DC then $AC \perp BD$, then it may be changed into a proposition. Thus, geometric construction is not just the procedure but also the tool for exploring the relationships between the properties of geometric figure. Therefore, it is important to adequately situate geometric construction in the transition stage.

Next, we must also consider that proving can be essentially characterized as the interactions between an individual's discursive inferences and visualizations (Koseki, 1987; Duval, 2002; Battista, 2007). Murakami (1994) states that the figures in proof have characteristics of the proof model which shows structure, tool and variability. Also, Duval (1998) sees the figure in proof as configurations of several constituent gestalts in 1D, 2D, and 3D, and states that the relationships between several constituents need to be recognized by discursive language for each constituent figure. Thus it is necessary to recognize that the figure can represent geometric relations and contain the data, and if we can clarify the nature of reasoning using shape then proof teaching will be improved. However, the opportunities for reasoning by seeing a figure as a set of constituents are rarely encountered before the introduction of proof in the current curriculum. Later, we consider this in the learning of geometric transformation.

Moreover, social influence is another factor that we regard as essential in the proof learning. Fawcett's (1938) study remains fresh today. He emphasizes the importance of re-examining the hypothesis of orienting human behavior behind belief or of clarifying the significance of definition and premise, with the purpose of cultivating critical and reflective thinking in accepting or rejecting conclusions. We think it necessary to clarify the characteristics of the social influences when students construct proof in classroom discussions.

We determine our research framework by considering the three aspects of inference, figure and social influence so as to clarify students' development towards proof learning, and we clarify the substance in a bottom up manner through the design experiments below. We also regard the three aspects not as independent but as interrelated with each other.

Elementary stage: To comprehend the properties of figure inductively and empirically

↓

Transitional stage:

- Aspects of inference
- Aspects of figure
- Aspects of social influence

Secondary stage: To prove the statements deductively and using symbols and languages

Figure 3. Triad of aspects for analyzing the transition towards geometric proof.

DESIGN EXPERIMENTS ON THE TRANSITION TOWARDS PROOF

Participants, the design of teaching units, and methods

We conducted our design experiments around the seventh grade unit on plane geometry in two classrooms at a public junior high school in Japan. We had 21 lessons of 40 minutes each for each classroom. The classrooms were typical in Japan in which the students' abilities were judged as average for public schools from the usual tests. They had already learned the properties of geometric figures empirically in elementary school.

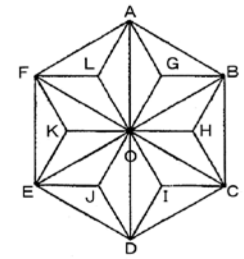

Figure 4

We intended to redesign the lessons on geometric transformations and constructions to bridge the gap to formal proof in the eighth grade. We chose the hemp leaf figure (Fig. 4) from traditional Japanese design as a fundamental reference figure for the whole unit, and constructed the teaching based on figure. The teaching unit was divided into four subunits: (1) the discovery game for figures, (2) the jintori game (transformations), (3) the discovery game for constructions, and (4) the construction and proof of the center of rotation. In designing the teaching sequence for each subunit, we referred to the ideas of situations for action, formulation and validation from Brousseau's (1997) didactical situation theory. Moreover, for teaching and learning, the classroom was often divided into two groups where the students were encouraged to propose their discoveries and refute them with each other, each proposal being given a point and each counter-example two points. Brief lesson overviews are given in the following table.

Subunit	Nth lesson	Overview of the lesson
1	1	Making the hemp leaf using Origami
	2	The construction of a hexagon using compass and ruler
	3-7	The discovery game for figures. The students found the figures included in the hemp leaf (e.g. rhombus, kite, cube) and the properties (e.g. parallel, perpendicular, symmetry), and justified their findings with each other.
	8-9	The properties of the figures of line symmetry and point symmetry
2	1	Introduction of three transformations and jintori game (described below)
	2	The games within small groups and the reflections
	3	Multiple transformations and how to transform to the empty places
	4	The games in the whole class session and summary
3	1	Introduction and discovery games for geometric constructions
	2-3	The construction of a kite and its relationship with the constructions of perpendicular, perpendicular bisector, and angle bisector
	4	Symbolizing the procedures of the three constructions
4	1-4	The construction and proof of the center of rotation (described below)

Our design experiment was conducted according to the methodologies of Cobb et al. (2003). All lessons were recorded on three video cameras and using field notes. We then made transcripts of the video data and conducted two types of data analysis. First was an ongoing analysis after each lesson. Here we analyzed what happened in the classroom in terms of the students' activities and utterances. Thus, the original lesson plans would often be modified as a result of the analysis. Second was a retrospective analysis after all the classroom activities had finished. We used the grounded theory approach (Glaser and Strauss, 1967) to encode and conceptualize the students' views by analyzing the transcripts and video data. Finally, we summarized those views in the theoretical framework.

Jintori (position taking) game by transformations in the hemp leaf

The second subunit dealt with geometric transformations in the Jintori (position taking) game. Our aim was to clarify how students may enhance their views of geometric figures and reasoning towards geometric proof. The rules of the game are that a pair of students first each decide upon a base position from the 18 isosceles triangles in the hemp leaf, then obtain the remaining positions alternately by translation, symmetry, and rotation from the base position. The person with the most positions wins. We also added the rule that they obtain a position if their explanation of how to transform from the base to the target position is accepted by the partner, since the intention is to improve recognition of relationships between figures and to have rich experiences of justification. We also prepared a tool for checking the transformations of each student.

(1) Introduction of game and imagining transformations

The teacher explains translation, symmetry, and rotation with demonstration, and then introduces the game with a help of a student (Noza) seated in the front row. Noza (base position ΔJDE) and the teacher (ΔHOC) alternately got the positions as shown in Fig. 5 (No.: turn, triangle: teacher, star: Noza, T, S, and R: translation, symmetry, and rotation). After the explanation, the students played the game in pairs without being confused by the rules. The teacher instructed them to articulate the direction of translation, the axis of symmetry, and the center and the angle of rotation.

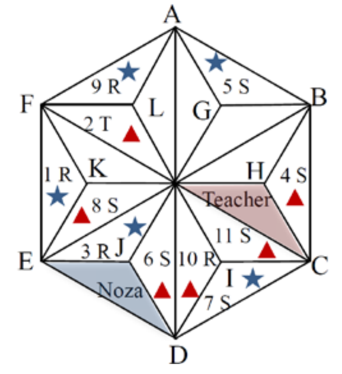

Figure 5 Noza vs teacher

We noted several characteristics among the students. First, they often envisioned their partner's move by pointing to the target place just after his or her initial utterance like "I chose *BE* as the axis". Second, they tried to get to positions that the partner may target. That is, they imagined how their partner may act. We also observed that they discussed with each other how the remaining positions could be reached beyond winning or losing.

(2) Utilizing transformations as tools

In the second lesson, we found that they used several strategies such as trying the same moves with their partner and changing the home position. Here, it seems that their interests were shifting from just enjoying the game to exploring the strategies and structure of the game. When the teacher asked what strategies they used, Noza stated that "I summarized all the positions I can get" (Fig. 6: base position: ΔAGO). We think that Noza's memo is essentially his strategy for winning the game and more importantly that it played a role in orienting the learning of the structure of the hemp leaf by transformations. The

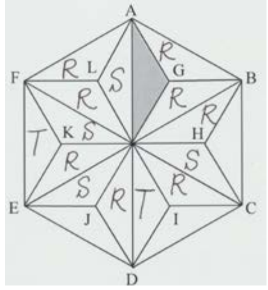
Figure 6 Noza's memo

teacher asked whether there are other transformations different from Noza, and many students showed their interest by stating that there were the multiple transformations. Moreover, although he also asked whether moves to the three empty places in Noza's memo were possible, nobody was able to respond.

The teacher proposed that all students create Noza's memo from another base place, ΔKEF , after confirming that there were two kinds of the base place, outer and inner. We found that they came to fully comprehend what places can be reached. We consider that the transformations gradually became their tools for exploring the structure of the hemp leaf.

(3) Objectifying transformations

In the third lesson, the teacher again asked them whether it is possible to move to the empty places. After some small group discussion, Miya explained "We use two moves. We first rotate it 60 degrees clockwise with the center at *O* and next flip it with *BH* as a crease. Then we can move it to the empty ΔBCH ". However, the other students argued against this because the rules did not allow two consecutive transformations. The teacher then negotiated with the students and they agreed on the new rule that two moves are permitted. Under this rule they were able to fill in all the empty places.

The teacher again asked if these transformations were possible by a single move. Ura stated "if we use the midpoint of *BO* and rotate 180 degrees, we can move to ΔHCB" and demonstrated it on the blackboard. The teacher gave each student a checking sheet and asked them to check it (Fig. 7). As many students had thought that one move was impossible, they were surprised and became inquisitive. Next, Noza proposed "we can maybe move to ΔEJD if we put *C* as the center and rotate 120 degrees anticlockwise", though the angle was wrong. The students checked it using a sheet and marveled at being able to move to a place they had thought impossible. When the teacher asked Noza how he found it, Noza stated "there is a rhombus *A*, *B*, *C*, and *O*, and *C*, *O*, *E*, and *D* also form a rhombus. Then, I displaced them." We found that he not only considered the move from ΔAGO to ΔEJD , but the correspondence in terms of the rhombuses *ABCO* and *EOCD*, which include the object figures (Fig. 8). This idea helped the students recognize the rotation more clearly.

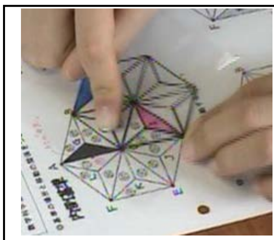

Figure 7 checking

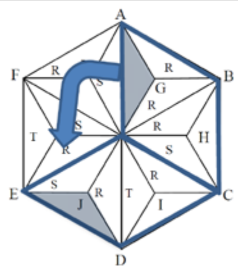

Figure 8

At this point, the only remaining target was ΔICD . The students were convinced that some move existed, saying "we can absolutely move it". A little later, Hato discovered and stated that "point H is center and we rotate 120 degrees" and all the students checked it using a sheet. They were astonished about this, too. When we examined Noza's notebook after the lesson, there was the picture of a kite in it (Fig. 9). While we assumed that the discoveries made by Ura, Noza, and Hato initially were exclusively for them, all students were able confirm them, with feelings of surprise, imaginarily or using the checking sheet.

Figure 9

(4) Discussion

We next discuss the three aspects of the transition in the framework.

From the aspect of inference, we think that the idea of 2-fold correspondences as seen above shows the characteristic of deductive reasoning. We can describe it as (a) the transformation from rhombuses *ABCO* to *EOCD* is possible (the major premise), (b) the triangle *AGO* in rhombus *ABCO* is same position as triangle *EJD* in *EOCD* (the minor premise), and therefore (c) the transformation from triangles *AGO* to *EJD* is possible (the conclusion). We think that this reasoning echoes Duval's (1998) view of proof in which the relationships among the subfigures in the given figure are recognized through discursive language. It is noteworthy that this emerged from the students' own explanations.

We think that this is rooted in several conditions. First, the students used their abilities to imagine the transformations in their minds. They seemed to progress from predicting their partner's move from his or her words to trying to take positions that the partner would not wish them to take. Next, the students were able to examine the possibilities and limits of transformations. Here, Noza's memo that organized the transformations beforehand was a new horizon. It led to examining different moves and pursuing whether all positions can be reached. We think that the transformations became a tool for exploring the hemp leaf situation in the students' recognition.

Next, we think of the aspects of figures with reference to the above ideas as seeing various correspondences or relationships among the figures and the umbrella figures. As a premise for this, it may be necessary to be able to variously combine the figures.

We should also pay attention to the above factors emerging with some social influences. It was by imagining the partner's moves and pursuing the others' thinking that the initial activity proceeded. The influence of Noza's memo was crucial as stated above. Moreover, the idea of 2-fold correspondences emerged as a way of explaining concepts to others. We may thus think of the progress of learning as a process that internalizes and objectifies others peoples' actions and thinking.

Geometric construction of the center of rotation and its proof

The task for the fourth unit was the construction of the center of rotation and the corresponding proof, since the students had already realized that rotation played a big role in the Jintori game in the second subunit and had just learned geometric constructions in the third subunit. As the students had not been taught what formal proof was, we regarded it as important that their arguments were based on some given premises and that they had feelings of conviction and persuasion.

In the first lesson the teacher explained that in the jintori game, rotation made moves from one place to many other places possible. Then, after putting two triangles from the hemp leaf in arbitrary places on the blackboard, he asked "We want to superpose triangle ABC on triangle DEF. Where is the center of rotation?"

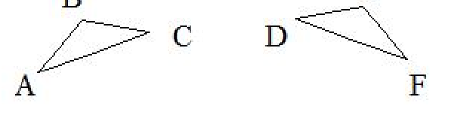

Figure 10

(1) Investigation by trial and error

The students first formed small groups to explore the problem. At the end of the lesson, each group presented its investigation to the whole class (Fig. 11).

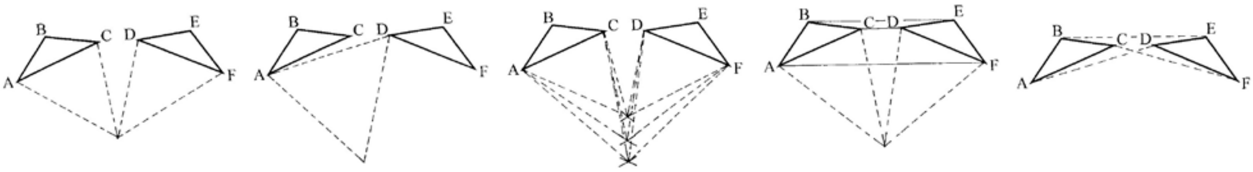

Figure 11. Students' constructions ((a), (b), (c), (d) and (e) from the left)

Construction (a) is an exploration based on the figures done by connecting an intuitively assumed point with the vertices of the triangles. Construction (b) shows an equilateral triangle that has edges of the length of AD. In (c), the students find the points equidistant from the vertices B and E and connect them with the other vertices, where they try to find the center by trial and error. In (d), we found from the remarks that they considered the perpendicular bisectors of AF, CD and BE. In (e), they connect the corresponding points to each other. We think that all ideas are effective in solving the problem, since if these are integrated it would lead to the discovery of how to construction the center of rotation.

(2) Construction using the perpendicular bisector

In the second lesson, a student, Simi, connected the pairs of corresponding points A and D, B and E, and C and F, and drew the perpendicular bisectors of the three segments. Although the third perpendicular bisector deviated a little from the intersection of first two bisectors, he displayed these as a point by making a small circle (Fig. 12). He seemed to be conscious of concurrency. Next, Miya drew three circles centered at the point Simi found that passed through the points A, B and C, and stated that the points D, E and F were on each circle (Fig. 13). The classmates agreed. By the end of the second lesson, Simi's construction was accepted as a way of finding the center.

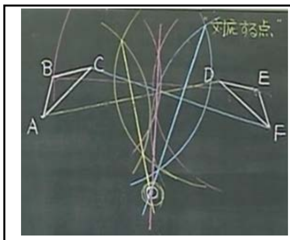

Figure 12.

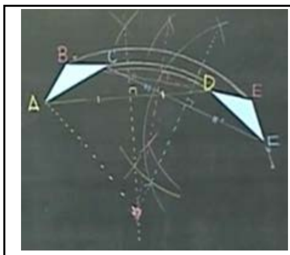

Figure 13.

(3) Justification of the construction

In the third lesson, the teacher asked the students to justify why Simi's construction was correct. After the individual activities, Seki stated, "I used the hemp leaf" (Fig. 14) and explained:

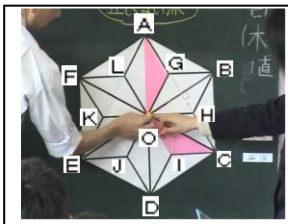

Figure 14.

Seki: The transformation from triangle AGO to CIO is possible by a rotation with the center at O. And we connect the corresponding points of triangles AGO and CIO. Then the intersection of the perpendicular bisectors of segments AC and GI is point O. So it is correct.

They became more convinced by her idea that used an example from the hemp leaf and thus extended the application field inductively. We found that the activities involving the hemp leaf in the former subunits formed a foundation. However, Oda then tried to refute the idea.

Oda: When they are lined up like triangles FLO and OHC, the perpendicular bisectors of the segments connecting the corresponding points do not intersect. So the explanation does not apply to the parallel case. Therefore, it can't be valid (Fig. 15).

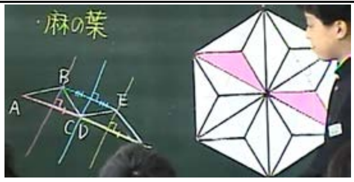

Figure 15.

Many students agreed with the counterexample Oda presented. However, at the beginning of the fourth lesson Oda himself stated his opinion that the explanation is valid if we exclude the parallel cases, and the classmates again agreed with it. Here, while the teacher reformulated it as a proposition, his argument seemed to be acknowledged as natural by the others, as they were able to immediately begin the proof construction activities in small groups. After the group session, some students explained:

Matu: The center of A and D is this line, the center of C and F is that line. Also, the center of the circle of B and E is the perpendicular bisector. So, the intersection O of these lines is the center of all circles (Fig. 16).

Iori: O is a point that satisfies all things

Noza: They are equal anywhere on perpendicular bisector CF.

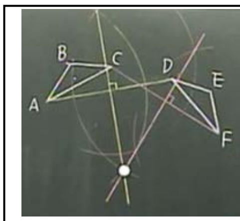

Figure 16.

We think that Matu was explaining that if O is the point of intersection of the perpendicular bisectors of segments AD and CF, then O is also at the center of a circle through B and E, and is the center of rotation of the figure. The teacher then augmented the sketch (Fig. 17), which was followed by Noza's statement "Because O is equidistant from A and D as well as from C and F, it is a center". The other students agreed with this statement. We observe here that the teacher's indication of the relationship using the segments in Fig. 17 guided Noza's second statement.

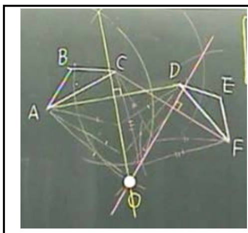

Figure 17.

To complete the proof, the students had only to show the congruence between △OAC and △DOF by adding the statement of AC = DF. However, they did not reach this stage within the given hours. We note further that in several experiments conducted later, students were deeply influenced by the argumentation. We think that this is due to continuity through the idea of 2-fold correspondences in the hemp leaf in the second unit.

(4) Discussion

We now discuss the three aspects of transition in the framework.

We consider there to be two critical points in inference: examining the validity and extent of the proposition in inductive and empirical ways and by presenting a counterexample, and objectifying the construction procedure and using it for justification.

There are several important reasons for examining the validity and extent of the proposition. First, the students' justification began with a demonstration to confirm whether the corresponding points were on the same circles, and they checked inductively to see if the construction method remained valid in the hemp leaf situation. We found that they were able to enhance their conviction of the proposition through these approaches. Second, they reconceived the extent to which the construction method was sound after Oda proposed the counterexample of the parallel case. Moreover, we think that the teacher reformulating the method as a proposition in the if-then form played a role in shifting their empirical reasoning towards the deductive scheme. We note that the proposition here became meaningful for them through their explorations, and we think that they took a vital step towards proving it through these processes.

We found that their proof proceeded by objectifying the construction procedure and then using it to support the justification. The first half of the process was as follows. First, Matsu reflected on the construction procedure and stated that the distances from two corresponding points to any point on the perpendicular bisector were equal to each other. Next, the teacher visualized these as segments. Finally, Noza made the logical step 'any point on the perpendicular bisector of segment AD is equidistant from A and D, and likewise any point on the bisector of segment CF is equidistant from C and F. The intersection O of the perpendicular bisectors satisfy OA = OD and OC = OF. Therefore, O is a center of rotation'. We remark that they reinterpreted the construction procedures as the conditions for justification and drew a conclusion based on them.

Next, we examine the aspects of figure so as to understand the development of their reasoning in terms of recognition of the figures. First, we focus on the students drawing the equilateral triangles in exploring the construction and trying to find the center of rotation by changing the radius. The figures here were not static for them but entailed images of equilateral relations and the continuous movement of points. We think these were the resources that they used to discover the construction method. Second, we think it important that, when in the justification stage a student found that the distances from two corresponding points to any point on the perpendicular bisector were equal to each other, the teacher visualized these as segments, because through that visualization the later reasoning proceeded to a successful conclusion. The segments here are variable and with equality relations, and thus consist of a part of the reasoning. We think that the figures implying movement, variability or relationship may be used to produce the construction and deductive reasoning.

Finally, regarding the aspect of social influences, we first found that the counterexample led to exploring the extent to which the construction method was valid and to making sense of the proposition. Also, through the proving process, the efforts to reformulate and improve each others' explanations helped to develop their reasoning. We can list the social influences in this subunit as the classroom lessons in the form of conjectures, refutations and agreements, absorbing criticism from others, giving the counterexample and reflecting on its meaning, and stating other people's ideas more clearly.

DESIGNING AND CLARIFYING THE TRANSITION FROM EMPIRICAL TO DEDUCTIVE RECOGNITION

We here clarify our framework for transition by incorporating the findings of our analysis in design experiments and give some suggestions on the design of lessons and units for the transition.

On the theoretical framework for analyzing the transition from empirical to deductive recognition

What we identified from our experiments as the aspect of inference in the second unit (U2) was the importance of finding the transformation by using and combining figures and properties, including examination of different or multiple transformations and working with the idea of 2-fold correspondences. In particular, we note that the idea of 2-fold correspondence takes on an aspect of deductive reasoning, though it may not be developed using only language. In the fourth unit (U4), we identified two main things. One is enhancing recognition of the proposition through examining the validity and extent of the construction method, in the same way that students checked validity empirically in the special situation of hemp leaf and shifted their focus on the limit to which the construction method worked out. The other is that it was necessary to objectify the construction procedure and using it for justification. We regard this as deductive, and also as the reflective abstraction that extracts ideas from actions of the construction and reorganizes them mathematically (Piaget, 2000).

We may regard the aspects of figure as emerge from inference as the flip side of the same coin. For example, the idea of 2-fold correspondences in U2 was dependent on seeing the configurations and relations in the figure. Also, the reasoning in U4 was based on seeing the figure dynamically to actualize the meaning of construction. We think that it is crucial to be able to see the figure as implying variables and relations for the transition to proof.

Last, when we examine the aspects of social influence, we may see whether the first two aspects go hand in hand with this one. In U2, we observe the students' attitudes towards assuming or pursuing others' actions and thoughts, trying transformations different from others, and revising explanations in their justification. Further, in U4 we may consider the attitude of giving a counterexample and efforts to reformulate and revise the explanations of others as solid background factors for development from the game to proving.

We summarize these three aspects of the framework in Fig. 18. We think that this framework will play a useful role in designing classroom lessons for transition and in evaluating students' activities. Moreover, we think that with refinement and further design experiments, this framework can be applied more widely, in particular to the upper grades in elementary school. It is our intention to further investigate this.

Elementary stage: To comprehend the properties of figure inductively and empirically

Transitional stage:

A. Aspects of inference

(1) to find methods of construction and transformation by using and combining figures and their properties.

(2) to confirm whether a proposition works empirically and the extent to which it works inductively and with counterexamples, and to enhance the understanding of the proposition.

(3) to reinterpret and use the construction procedures as conditions for proving.

B. Aspects of figure

(1) to select suitable figures and to use them in combination for constructions and inference.

(2) to see figures as variables which can be changed by dynamic transformations.

(3) to see figures as relations among the whole and partial figures through reasoning diagrammatically.

C. Aspects of social influence

(1) to base the learning environment on students' conjectures, refutations and consensus.

(2) to make and develop conjectures while accepting criticism from others.

(3) to interpret others' explanations and express them more precisely.

Secondary stage: To prove the statements deductively and using symbols and languages

Figure 18. Revised triad of aspects for analyzing the transition towards geometric proof.

We note that our discussion does not suggest that these three aspects work separately, but rather that they have an integrated and interdependent nature.

Moreover, if we think that a certain type of learning transformations and constructions produced the above aspects in students' recognition, it may be reasonable to see these aspects as in Fig. 19. It is useful to note that the students' reuse of the construction procedures in their justifications and seeing the figure as implying variables and relations, both of which aided the proving process, are reflectively abstracted from the dynamic actions when learning constructions and transformations. We may thus proceed to study the curriculum for how teaching the content of geometric construction, transformation and proof can be reorganized as teaching units for introducing secondary geometry to students.

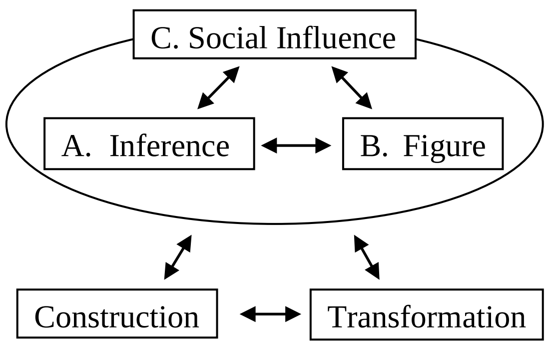

Figure 19. A modality of three aspects

Some suggestions for designing lessons for the transition to geometric proof

We consider that, to accomplish the transition to geometric proof, it is essential not only to propose the theoretical framework but also to develop and show the classroom practices and situations for the transition, which was one of our efforts in this paper. We finish by commenting on the design of the lessons and units in terms of the continuity of context and necessity, the recursive growth of explanation, and the humanistic aspects.

We first consider that continuity of context through the whole unit was crucial for the students' construction of the proof. This should be made clear by conceiving of the fourth unit as the situation for validation in terms of didactical situation theory. The first three units can then be regarded as situations for action and formulation, where the fundamental ideas and concepts like rotation and perpendicular bisector, the ways of reasoning using the figure, and the learning style of conjecture and refutation were fostered. Such stages seem to be inevitable when we consider how long the stage of formulation has lasted in the history of mathematics. Moreover, as the students explored proving by returning to the hemp leaf situation, we conclude that the Jintori game was a solid foundation for them as the situation for action. If this situation had not existed, they would not have explored the meaning of the proposition and the extent to which the construction method works. In other words, we consider that learning the three aspects of our transition framework in the same situation facilitated the students' development of proving.

In association with the above points, our experiments show that the students' activities in understanding the theorem itself occupied more time than of the actual proof. In the Jintori game, it was important to find whether a single rotation enabled the move from the base triangle to the separate target triangle. Also, in constructing the center of rotation the exploration of whether the three perpendicular bisectors intersect or not was important. If we regard the activity of understanding the theorem as a situation for formulation in didactical situation theory, we think that more focus should be given to understanding theorems and how this understanding is intertwined with the proving process. This is important for solving the problem of necessity.

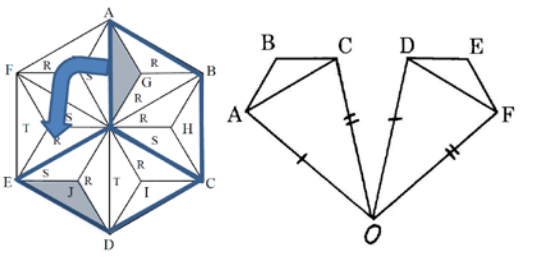

Figure 20.

Second, we think that it is important to design the lessons for transition as a generalization process. In our experiments, the idea of 2-fold correspondences in the Jintori game has the same structure as the proof of the center of rotation. They are both deductive, while the former is action based and the latter is a language-symbol based exploration of the general case (Fig. 20). Thus the students can reciprocate between the particular and the general cases. We use the terminology of Pirie and Kieren (1994), concluding that students develop their understanding of the latter situation through folding back to the former situation. This implies that the students' proving ideas may grow transcendent-recursively. This process will become one of our important tasks.

Finally, we think it important to consider the aspects of humanity when introducing proof to students. That is, students' justification and proving are conducted by exploring and explaining curious phenomenon, and it is necessary that they include human relationships such as collaboration with each other, accepting criticism from others, and refining others people's explanations, as our design experiment demonstrated. More importantly, we emphasize that students construct knowledge of proof with surprise or good emotion. In this we find the relationship between humanity and mathematics.

FINAL COMMENTS

In view of the increasing gulf between empirical and deductive recognition of geometry, we have explored in this paper what connections exist between these recognitions and how they can be developed. We think that clarifying students' recognition of the transitional state is paramount in the global research that falls under a wider concept of proof and proving.

We challenged our exploration through the design experiments from a holistic perspective. One reason why we adopt the holistic perspective is that we think students can better embody mathematical concepts with affirmative emotions such as interestingness, surprise and beauty, whereas many students experience negative affects alongside their difficulty in understanding proof. Therefore, we have just not investigated the proving process, but how the students feel the necessity of proving from the games in the hemp leaf situation and how they enhance their power of reasoning from the three aspects of inference, figure, and social influence.

In research into learning algebra we often see discussion of what prerequisites are necessary for understanding symbolic expressions, and it is indicated that seeing the special case of numerical expressions as a general case from the perspective of algebra is important in this respect. For geometry, we consider the ability to see transformations and constructions from the perspective of deductive geometry. Thus, we hope that the theoretical framework discussed in this paper contributes to clarifying the geometrical perspective in the transitional stage. More importantly, we hope that our efforts make a substantial contribution to practical requirements for teachers and students in the mathematics classroom.

References

Battista, M. (2007). The development of geometric and spatial thinking. F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 843-908). National Council of Teachers of Mathematics.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359-387.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Chino, K. et al. (2009). Tasks and perspectives of proof studies in Japan. *The Booklet for Working Groups*, 64-69. Japan Society of Mathematical Education. [in Japanese]

Douady, R. (1997). Didactic engineering. T. Nunes and P. Bryant (eds.). *Learning and Teaching Mathematics: An International Perspective* (pp.373-401). Psychology Press.

Duval, R. (1998). Geometry from a cognitive point of view. C. Mammana *et al*. (eds), *Perspectives on the Teaching of Geometry for the 21st Century* (pp.37- 52). Kluwer.

Duval, R. (2002). Proof understanding in mathematics: What ways for students? *Proc. of 2002 Int. Conf. on Mathematics: Understanding Proving and Proving to Understand*, 23-44.

Hanna, G. (1991). Mathematical proof. D. Tall (ed.), *Advanced Mathematical Thinking* (pp. 54-61). Kluwer Academic Publishers.

Hanna, G. and Jahnke, H. (1996). Proof and proving. A. Bishop *et al*. (eds.), *International Handbook of Mathematics Education* (pp. 877-908). Kluwer.

Harel, G. and Sowder, L.(2007). Toward comprehensive perspectives on the learning and teaching of proof. F. Lester (ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp.805-842). Information Age.

Hirabayashi, I. (1986). For the effective mathematics education. *Bulletin of Nara University of Education*, *36* (2), 13-29. [in Japanese]

Hirabayashi, I. (1987). *Studies on the development of activism in mathematics education*. Toyokan. [in Japanese]

Hirabayashi, I. (1991). Overviews of elementary geometry and considerations of the points. *New Courses of Teaching Elementary Mathematics: Early and Middle Elementary Grades* (pp.39-72). Kaneko Shobo. [in Japanese]

Hirabayashi, I. (2001). Recent perspectives of research in mathematics education: "Culture" and "Ecology". *Research in Mathematics Education*, *7*, 1-6. [in Japanese]

Knipping, C. (2008). A method of revealing structures of argumentations in classroom proving processes. *ZDM Mathematics Education*, *40*, 427-441.

Koseki, K. (1984). *The teaching of proof in geometry*. Meiji Tosho. [in Japanese]

Koyama, M. (2007). Need for humanizing mathematics education. In J. H. Woo *et al*. (Eds.), *Proc. 31st Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 115-118). Seoul, Korea: PME.

Kunimune, S. (2003). Teaching and learning with a necessity of doing proof. *Booklet for the Working Sessions*, 220-223. Japan Society of Mathematical Education. [in Japanese]

Mariotti, M. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics*, *44*, 25-53.

Miller, J. (2007). *The holistic curriculum*. University of Toronto Press.

Murakami, I. (1994). The developmental process of the recognition of figure in the teaching of geometry. *Proc. 27th Conf. of Japan Society of Mathematical Education*, 125-130. [in Japanese]

NCTM. (2000). *Principles and standards for school mathematics*. NCTM.

Okazaki, M. and Iwasaki, H. (2003). Geometric construction as an educational material mediating between elementary and secondary mathematics. *Research in Mathematical Education*, *80*, 3-27. [in Japanese]

Okazaki, M. and Komoto, S. (2009a). The recognition of geometric figures cultivated through geometric transformations: The design experiment aimed at the transition towards proof. *Journal of Japan Society of Mathematical Education*, *91* (7), 2-11. [in Japanese]

Okazaki, M. and Komoto, S. (2009b). The transition process towards the logical proof through synthesizing the geometric transformation and the geometrical construction: A design experiment of seventh grade unit "plane geometry" (3). *Research in Mathematics Education, 15* (2), 67-79. [in Japanese]

Ohta, S. (1998). The place of proof in geometry teaching. *Booklet for the Working Sessions*, 103-108. Japan Society of Mathematical Education. [in Japanese]

Pirie, S. and Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165-190.

Sato, M. (1996). *Educational methods*. (pp.7-32). Iwanami. [in Japanese]

Souma, K. (1998). Teaching of geometric proof with an emphasis on the necessity. *Booklet for the Working Sessions*, 97-102. Japan Society of Mathematical Education. [in Japanese]

Sekiguchi, Y. (2002). Proof for the inquiry activities. *Booklet for the Working Sessions*, 181-184. Japan Society of Mathematical Education. [in Japanese]

Stylearides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38* (3), 289-321.

Takii, A. (2001). *Elementary mathematics lessons for fostering classroom cultures*. Toyokan [in Japanese].

Tall, D. (2008). The transition to formal mathematical thinking in mathematics. *Mathematics Education Research Journal*, *20* (2), 5-24.

Tsubota, K. (2001). *Creating elementary mathematics lessons: Towards symbiotic and mutually creative learning*. Toyokan. [in Japanese]

van Hiele, P. (1986). *Structure and insight: A theory of mathematics education*. Academic Press.

Wittmann, E. (1995). Mathematics education as a design science. *Educational Studies in Mathematics*, *29*, 355-374.

Wittmann, E. (2001). Developing mathematics education as a systemic process. *Educational Studies in Mathematics*, *48*, 1-20.

Yoshida, A. (1999). *The holistic education: Philosophy and movement*. Nippon Hyoron-sha. [in Japanese]