12th International Congress on Mathematical Education

Program Name XX-YY-zz (pp. abcde-fghij)

8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

# DOING RESEARCH WITHIN THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC: THE CASE OF SCHOOL ALGEBRA

Marianna Bosch

IQS School of Management, Ramon Llull University

marianna.bosch@iqs.edu

*Since its emergence in the early 80s with the study of didactic transposition processes, the Anthropological Theory of the Didactic maintains a privileged relationship with school algebra and its diffusion in school and outside school. I have chosen this case study to introduce the main "gestures of research" this framework promotes and, more particularly, the tools used to help researchers detach from the dominant viewpoints of the institutions where teaching and learning processes take place or which affect these processes in the distance. The construction of alternative reference models concerning school algebra and teaching and learning processes leads to some recent teaching experiences that break down the established didactic contracts, raising new research questions that need more in-depth analysis in the way opened by the "procognitive paradigm".*

*School algebra, anthropologic theory of the didactic, didactic transposition, arithmetic calculation programme, algebraization process*

## RESEARCH, THEORY AND THE "DETACHMENT PRINCIPLE"

My aim at this lecture is to introduce the Anthropological Theory of the Didactic (ATD), a research framework where I have been working for more than twenty years now, growing in it as a researcher and having the chance of participating in its development. At the beginning we were a small group of French and Spanish people collaborating with Yves Chevallard in Marseilles, a group that has now become a community of about one hundred researchers mainly from Europe, Canada and Latin American countries. A good outline of the problems approached and the results obtained by this community can be found in the proceedings of the three International ATD Conferences held since 2005 in Spain and France (Estepa, García & Ruiz Higueras 2007, Bronner et al 2010, Bosch et al 2011).

In spite of the word "theory", the ATD is, as the Hans Freudenthal Award recognizes it, "a major cumulative programme of research" in mathematics education. Like in many other cases, "theory" is used here as a synecdoche to refer to a whole research activity naming only one of its elements: the organization of the concepts, assumptions, relationships and other notional tools used to problematize reality. However, "theory" is not always the best way to access a research approach and the body of knowledge organised by it, even if we are used to it, since many centuries of diffusion strategies seem to overvalue "theories" dimension as the main entrance to knowledge organisations. We will choose another entry: the one of the research problems raised by the ATD and the main methodological "gestures" used to approach them, which also includes the kind of empirical evidence considered as experimental basis. We will restrict the entry even more and focus on a single case study, the problem of school algebra, which has been at the core of the ATD development since its very beginning and thus provides a good illustration of the different treatments this research framework proposes.

Even if the ATD is much more than a theory, it is also true that the role played by its theoretical constructions is essential in a very specific sense, especially when we compare it to other approaches in mathematics education, which can be subsumed in a basic principle that permeates all its methodological gestures. I will call it the "detachment" principle, after the work of the German sociologist Norbert Elias (1987). Because researchers in didactics deal with a reality that takes place in social institutions, and because they often participate at these institutions (as researchers, teachers, students, or in several positions at the same time), we need to protect ourselves—to emancipate—from the institutional points of view about this reality, that is, from the common-sense models used to understand it. This effort of detachment is a basic gesture in sociological and anthropological research (see, for instance, Bourdieu & Passeron 1962; Berger & Luckmann 1966; Elias 1984; among many others). It is also coherent with the double assumption made by the ATD that persons are the *subject* of the set of *institutions* they enter during their lives and that what they think or do (their knowledge and know-how) derives in a personalised way from *institutional* knowledge and know-how. In this context, human practices and human knowledge are entities arising in institutional settings. Thus a person acquires knowledge and practice by entering the institutions where this knowledge and practice exist. As institutions are made of people, institutional praxeologies evolve because of the changes introduced by their subjects.

The word "institution" has to be taken here in a non-bureaucratic sense, as it is used by the anthropologist Mary Douglas in her work *How institutions think* (Douglas 1986). As Y. Chevallard presents it (Chevallard 2005, our translation):

> "An institution lives through its actors, that is, the persons that are subjected to it—its subjects—and serve it, consciously or unconsciously. […] *Freedom* of people results from the power conferred by their institutional subjections, together with the capacity of choosing to play such or such subjection against a given institutional yoke."

This dialectic between the personal and the institutional perspective is at the core of the ATD. It is important to say that the personal *subjection* to institutions must be understood as a productive subjection instead of as a loss of freedom. We do not act nor know as individualities, but as part of some collective constructions we participate in, assuming their rules and contributing to making them evolve. The idea of being empowered (both cognitively and practically) through the subjection to institutions can be illustrated by the metaphor of the bicycle: when the wheels are free, the bicycle does not move; movement is possible through the subjection of the wheels. The principle of “detachment” has to be understood in this context, since researchers’ institutional subjections affect the way of conceiving and understanding reality. It could be very difficult to understand the mechanism of the subjection of the chain while riding the bicycle.

When trying to adopt an external point of view from the reality we want to study, we often need to question the institutional dominant point of view, which initially appears as “transparent” or natural to the subjects of the institution. It is here where theoretical constructions acquire their functionality, by providing alternative perspectives about the reality we want to study. However, except if we adopt a hyper-empiricist perspective—which we will not—, the way of delimiting and even defining this reality also depends on the perspective adopted. As we will see, the “detachment” required by the ATD methodology also implies an important enlargement of the empirical unit of analysis considered.

But let me first introduce a theoretical notion that we will need to use soon and which is also part of the effort of detachment we are considering here. In didactics research, almost all problems deal with teaching and learning processes where “something” is learnt or taught. This “something” is usually a particular “piece of knowledge” that can be of a different size: the whole “mathematics”, the practice of “mathematical modelling”, a whole domain like when we talk about “algebra” or “geometry”, a sector of this domain like “first degree equations” or “similar triangles”, or even a smaller piece like “the concept of variable” or “transposing and cancelling”. The ATD proposes to talk about “praxelogies” to refer to any human practice and, in particular, to mathematical (and also teaching and learning) activities (Chevallard 1997, 1999, 2006; see also Barbé et al 2005). The term “praxeology”, made of the Greek words *praxis* and *logos*, enables us to consider two terms that are often opposed within the same entity: the “practical block” or know-how and the “theoretical block” or knowledge (in its narrow sense) made of the discursive elements (*logos*) used to describe and justify the practice. A praxeology is made of four components: type of tasks, techniques, technologies and theories (sometimes called the “four Ts”). The *praxis* or “practical block” contains a set of *types of tasks* to be carried out and a set of *techniques* to do so, “technique” being considered here in a very general sense of “ways of doing”. The *logos* or “theoretical block” is made of a double-levelled discourse. A *technology* or “discourse on the technique” to explain what is done, to let others interpret it and to provide a first justification or control of it. The general models, notions and basic assumptions that validate the technological discourse and organise the praxeological elements as a whole, form what we call the “theory”. As well as tasks and techniques, technological and theoretical discourses can be elaborated and well-grounded, or just incipient, routine-based and naturalised.

Scientific praxeologies try to make their technologies and theories explicit, so as to control the assumptions made, to formulate the problems and phenomena approached and, as Allan Schoenfeld outlines it in his “Reflections of an Accidental Theorist”, “to elaborate clearly for yourself ‘what counts’ and how things supposedly fit together” […] as well as to “hold yourself accountable to data" (Schoenfeld 2011, p. 220). The synecdoche I mentioned before about referring to a whole research praxeology by naming only its theoretical component is a classic one when dealing with scholar knowledge. Praxeologies culturally considered of a "lower level" are also usually designated through an opposite synecdoche, naming the practical component as if there were no theoretical block associated to it, that is, as if there was "nothing to say" about the practice or, at least, as if there was not a strong enough institutional theoretical construction around it. The use of the term "praxeology" enables us to escape from these institutional evaluations and view through the same prism the different mathematical, teaching and learning praxeologies that form our object of study. It is meaningful, for instance, that educational institutions can easily talk about "teaching theories and practices", but tends to refer to "teachers' practices" much more than to "teachers' theories"…

## WHAT IS "SCHOOL ALGEBRA"? DIDACTIC TRANSPOSITION PROCESSES

The first "detachment gesture" proposed by the anthropological approach has to be found in the first formulations of the didactic transposition process (Chevallard 1985). It consist in questioning the nature and origin of the mathematical knowledge that is taught at school, looking at the work done by different institutions during different periods of time to select, reorganize, adapt and develop the mathematical praxeologies from their first appearances in the "scholar" institution (main responsible for knowledge production) to their designation as "knowledge to be taught" and their implementation at school as "taught knowledge". A lot of decisions are made during this transposition process that should be taken into consideration to better understand what conditions (in terms of mathematical, didactic and other kinds of praxeologies) are made available to teachers and students and what constraints hinder or even impede the development of many others.

The notion of didactic transposition appeared as a powerful theoretical tool to break with the dominant viewpoints with regard to the "disciplinary knowledge" didactics research has to deal with. Before focusing on how children learn and how we can teach them—the viewpoint of the teachers' institution—the attention is first put on what is learnt and taught, its "nature" (what it is made of), "origin" (where it comes from) and "function" (what it is for). In spite of the dominant viewpoint on mathematics brought about by the scholar and the school institution, leading to the impression that there is only *one* school algebra and that the problem is how to teach or learn *it*—as if the decisions were always beyond the epistemological dimension of teaching and learning processes—, the ATD starts questioning "what is being taught" and showing its undefined nature. What is this thing called "school algebra"? What kind of praxeologies is it made of? What could it be made of under other institutional constraints? How does it vary from one school institution to another, both in time (from one historical period to another) and in the institutional space (from one country or educational system to another)? Where does it come from? What legitimates its teaching?

It seems obvious that to answer those questions, the kind of empirical evidence necessary may not be reduced to observing the didactic processes as they are currently taking place in the classroom. We need to look into the different institutions (present and also past ones) that influence the transposition processes, amongst them “scholar mathematics” and, more particularly, the “noosphere”, that is, the sphere of people who “think” and make decisions about educational processes, such as curriculum developers, policy makers, mathematics advisors, associations of teachers, educational researchers, etc.

Research about the teaching of elementary algebra in France (Chevallard 1985, 1989a, 1989b, 1989c; Assude 1993; Grugeon 1995; Coulange 2001a, 2001b; Artigue et al 2001) and their contrast with the Spanish case (Gascón 1993, 1999; Bolea 2004; García et al 2006; Ruiz-Munzón 2011) have all shown a similar evolution of the didactic transposition processes that has led to a dispersion of the contents traditionally assigned to “elementary algebra” in secondary school curricula, splitting up the classic triad of arithmetic-algebra-geometry that used to structure school mathematical curricula before the New Mathematics reform. With slight variations depending on the historical periods and regions, we can observe that the existence of algebra as a school mathematical domain (or “block of content”) is at the most fluctuating. It has disappeared from the French and Catalan official curricula, and has only recently been reintroduced in those of some other Spanish regions. For instance, in Catalonia, the present curriculum (2007) proposes five blocks of contents, *Numeration and calculation; Relations and change; Space and shape; Measure; Statistics and randomness*, which appear to be very similar to the “overarching ideas” proposed by the OECD/PISA commission: *Quantity; Space and shape; Change and relationship; Uncertainty* (OECD 2009). It is interesting to notice that, in this new organisation of mathematics proposed by the PISA evaluators, the correspondence with what they called the “traditional topics classification” confine algebra to the “Change and relationship” strand, as if apparently there were no need for algebraic techniques in the other domains (OECD 2009, p. 28). It could be interesting to study how transpositive processes are currently influenced by this type of international evaluations, a phenomenon that seems to affect the different societies that take part in these processes in the same way, although the effects appearing in each educational system may be considerably different from each other.

Apart from the loss of visibility of the mathematical organisation of school algebra, which may vary significantly from one country to another, what is much more common and what has been commented in numerous research projects, is the establishment, since the reform reaction that follows the New Math, of a *formal* approach of the algebraic tool, to the detriment of a *functional* approximation in which algebra would appear as a way of modelling other kinds of systems or mathematical realities (Chevallard 1989a). In traditional teaching preceding Modern Mathematics, the introduction of the algebraic tool and the use of letters to name both known and unknown quantities or numbers allowed to systematically solve (and often, though not always, more efficiently) the corpus of problems of elementary and mercantile arithmetic that represent most of the mathematical work done in primary school. Arithmetic calculations and the structured corpus of arithmetic problems act as the reference of the new algebraic construction, which in turn marked the entrance to a higher level of education. In this function of “generalized arithmetic” attributed to algebra, the interplay between parameters and unknowns was necessary to cope with the richness of the discursive models that support the arithmetical techniques (Chevallard 1989; Bosch 1994). It was also necessary to later give room to using formulas and connecting algebra to functional calculus and analytic geometry.

Nowadays, however, the reference to traditional arithmetic and its important corpus of problems that used to give the teaching of algebra its raison d’être has disappeared. The opposition—which was also an affiliation—between “arithmetic” and “algebraic” techniques to solve problems, which during long time marked the entrance to the algebraic work, does no longer make sense. At current secondary schools, elementary algebra is largely identified with solving equations, and mainly first and second degree equations, with some subsequent “applications” to a set of “word problems” coming out of nowhere. This limited domain is often preceded by a short “introduction to the language of algebra” used to introduce the specific terminology required (algebraic expression; evaluation; terms, members and coefficients; similar terms; equations, equalities and identities; etc.), a formal frame where students learn how to “develop”, “factorise” and “simplify” expressions as a goal in itself, where algebraic expression and equalities between expressions are no longer presented by what they designate but only by their formal structure and by the mathematical objects they are made of. This formal learning is unable to recreate the big variety of manipulations that are needed to use algebra in a functional way, which will be required when students arrive at higher secondary education and find “completely algebraized” mathematics.

It may seem easy to criticize—by showing its limitations and deficits—the praxeological entities school algebra is made of. Knowing how algebra is understood in mathematics classrooms, at school and even in our societies, as well as the kind of praxeological elements that are not (but could be) conceived as part of it, is, however, an essential questioning to investigate the conditions of possibility for educational changes not being reduced to mere local innovations. It is important to understand the transpositive constraints that have shaped algebra as “taught knowledge”, especially when some of the detected traits (that have only been succinctly described here) seem robust and stable enough to remain in most current educational systems.

## THE DIDACTIC ECOLOGY OF SCHOOL ALGEBRA

The study of the didactic transposition processes points out the existence of constraints of different natures influencing the teaching of algebra at secondary school. The study of these constraints is part of the “ecology” of the praxeologies (Chevallard 2002b), that is, of all the conditions necessary for a specific praxeological organisation to exist and evolve in given institutional settings. These constraints can be of a very specific nature, related to the way the different domains, sectors, themes and issues are organised in a given mathematical curriculum, or they can be more generic, that is, not directly related to mathematics but affecting the school teaching and learning of any discipline at school, or at any educational institution, or even affecting the dissemination of any kind of knowledge or praxeological organisation in the society at large. Y. Chevallard (2002a, 2002b, 2007) introduced a hierarchy of "levels of didactic codetermination" to clarify the scope of the considered constraints and also to uphold that the study of phenomena arising at very general levels of determination should be taken into account by research in didactics, since they can strongly affect the conditions of possibility and evolution of teaching and learning processes. The scale consists in the following sequence:

Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline ↔ Domain ↔ Sector ↔ Theme ↔ Issue

Figure 1. Scale of levels of didactic codetermination

Research on school algebra leads to identify important constraints in almost all levels of codetermination. It provides a good illustration of how the most generic levels, especially phenomena arising at the level of our Western civilisation and of our societies can influence mathematical praxeologies, at the lowest levels of specification.

## The Western relationship to orality and literacy

According to the work of the classical and humanistic scholars, Eric A. Havelock (1963) and Walter J. Ong (1982), in traditional Western cultures, oral formulations are regarded as the direct expression of thought, and writing is viewed as the mere written transposition of oral discourse. The French philosopher Jacques Derrida (1967) describes this metaphysical position as *logocentrism.* It is assumed that thought is something residing in our head that first comes out through the discourse before being transcribed to writings. Thus, (verbal) "reasoning" is often opposed to (written) "calculations", as illustrates the current recommendation "First say it with words, then write it down". This assumption permeates our teaching practices and can explicitly be found in several teaching documents about the "danger" of introducing writing manipulation too early, before the "meaning" is constructed. See, for instance, the following suggestion about the construction of "number sense" in early arithmetic by Julia Anghileri (2006, p. 45) quoting the British Department of Education and Employment:

> Current recommendations propose that "oral and mental competence" is established "before written calculation methods are introduced" […]. This does not mean that there will be no written recording but that children will learn to record their thinking with progressive formalization, learning first to use words to record results they can already talk about.

A comment that is preceded by a synthetic indication about how "Progression in learning may be summarized" (*Ibid.*, p. 44):

DOING … TALKING ABOUT … WRITING ABOUT … SYMBOLIZING

It is important to notice that in the algebraic symbol manipulations, this relationship between oral and written work is reversed: writing comes first and orality is just a "secondary" accompaniment of the written algebraic formulations, which are furthermore not always easy to “oralise”. Contrary to our mental habits, written algebraic symbolism is not a derivation of oral language: it is the source, the manifestation and the touchstone of algebraic “thinking”.

The school ecology of algebra has always been hindered by what we can call a cultural incomprehension of its written nature. In fact, the relationship to symbolism is still an important barrier to the acceptance of scientific work in the realm of highly valued cultural practices. A small sample of this situation can be the number of books in different languages pretending to popularize scientific fields using none or very little symbolism: ‘Spaceflight without formulae’, ‘Special relativity without formulae’, ‘Quantum mechanics without formulae’, ‘Statistics without formulae’, and even the Russian ‘Mathematics without formulae’ in two volumes! (Pujnachov & Popov 2008). In the introduction to the book, we can read the following statement that the authors attribute to the famous mathematician Sofia Kolvalésvkaya and that reintroduces the common idea that formulae are something secondary in the production of knowledge (our translation):

> In mathematical works, the most important is the content, ideas, concepts, and only afterwards, to express all this, mathematicians have their language: formulae.

The lack of meaning assigned to written formulae by our Western culture has its effects in the school introduction of algebra. As we have shown in our research on the *ostensive tools* used in mathematics (Bosch & Chevallard 1999), the “rupture” between arithmetic and algebra is also a cultural break from an essentially discursive world, based on oral techniques scanned by simple operations—the “reasoning” realm—to a mostly written world, where techniques are difficult to “oralise”, where a specific descriptive discourse (a *technology* of the written calculations) has to be explicitly constructed and that has always been harder to accept by the cultural environment. Algebra thus appears as a kit of tools that allows doing things more quickly in detriment of the meaning or reasoning, written mechanics against verbal thought. A quotation of an old French textbook of elementary algebra would give an idea of this dominant viewpoint that has still not completely disappeared (Blanc & Soler 1933, p. 12, our translation):

> If the algebraic solution is quicker than the arithmetic solution, we do not have to forget than it is the latter which mainly contributes to develop reasoning. Thus with problems the solution of which includes reasoning, it is necessary to find both solutions: the arithmetic and the algebraic one.

**The cultural pejoration of algebra**

The first investigations on school algebra carried out within the framework of the ATD (Chevallard 1985 and 1994) immediately highlighted a fact of society closely related to the primarily written nature of algebra and that can be designed as *the cultural pejoration of algebra*. As we showed in (Chevallard and Bosch, *to appear*), a research carried out at the beginning of the 1980s using a *semantic differentiator* technique, it could grasp what we postulate to be an almost invariable trait in secondary school students: while to them geometry would be pretty, warm deep and feminine, algebra turned out to be ugly, cold, superficial and masculine. Again, we can find several pieces of evidence of such relationship our society maintains with algebra. A quite surprising one comes from a voluntarily provocative comment from the great mathematician Sir Michael Atiyah that clearly illustrates this cultural pejoration of algebra (Atiyah 2001, p. 659):

> Algebra is the offer made by the devil to the mathematician. The devil says: ‘I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine. […] the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning. I am a bit hard on the algebraists here, but fundamentally the purpose of algebra always was to produce a formula that one could put into a machine, turn a handle and get the answer. You took something that had a meaning; you converted it into a formula; and you got out the answer.

It is not strange that, in this state of mind, introducing students to “proof”, “demonstration” or “deductive reasoning”, the mathematical domain *par excellence* is usually geometry, and rarely algebra. It is difficult to accept algebra as a domain of proof, because algebraic work seems to consist mainly in calculations, which is supposed to imply little “reasoning”.

Let me finish this illustration of constraints coming from the generic levels of the scale of didactic codetermination with a final example of a fact that can be located at the Society level, even if in this case the society is one that I, as a citizen, am not so familiar with. Some years ago, to introduce the proposal of “algebrafying” into an elementary mathematics experience, James Kaput depicted the result of an evolution of the American didactic transposition process that led to what the author names “Algebra the Institution” and that shows a very different picture of the one described above by the French and Spanish cases (Kaput 1998, p. 25):

> ‘Algebra the Institution’ is a peculiarly American enterprise embodying the standard courses, textbooks, tests, remediation industry, and their associated economic arrangements, as well as the supporting intellectual and social infrastructure of course and workplace prerequisites, cultural expectations relating success in algebra to intellectual ability and academic promise, special interests, relations between levels of schooling, and so on. Exhortation for and legislation of Algebra For All tacitly assume the viability and legitimacy of this Institution. But this algebra is the disease for which it purports to be the cure! It alienates even nominally successful students from genuine mathematical experience, prevents real reform, and acts as an engine of inequity for egregiously many students, especially those who are the least advantaged of our society.

The scale of levels of didactic codetermination is a productive methodological tool for the detachment principle I mentioned at the beginning of this paper, to be aware of the factors that influence what can be done—and what cannot be done—at school related to the teaching and learning of algebra, and avoid taking for granted the current assumptions about the nature of algebra and its relationships with the other mathematical domains (the specific mathematical levels of the scale) nor the implicit evaluations and judgement that we, as members of a particular society and civilisation, attribute to the phenomena we observed. However, as we said before, the best way to free research from all these implicit institutional assumptions that always impregnate teaching and learning processes, is to build an alternative reference model from which to look at the phenomena from another point of view and, of course, with other assumptions that research theory should try to make as explicit as possible. This is especially important when dealing with the specific levels of codetermination, that is, when we are considering school algebra as a (more or less well-defined) mathematical domain.

## WHAT COULD SCHOOL ALGEBRA BE? A REFERENCE EPISTEMOLOGICAL MODEL

When analysing any teaching or learning process of mathematical contents, questions arise related to the interpretation of the mathematics involved in it. The different institutions interfering in the didactic processes propose more or less explicit answers to said questions. If researchers assume those answers uncritically, they run the risk of not dealing with the empirical facts observed in a sufficiently unbiased way. Therefore the ATD proposes to elaborate what are called *reference epistemological models* for the different mathematical sectors or domains involved in teaching and learning processes (Bosch & Gascón 2003). This explanation of the specific epistemological viewpoint adopted—which is always an a priori assumption constantly evolving and continuously questioned—determines, amongst other things, the amplitude of the mathematical field in regard to which research problems are set out; the didactic phenomena which will be "visible" to researchers and the attempted explanations and actions that are considered "acceptable" in a given domain of research.

In the ATD, those reference epistemological models are formulated in terms of local and regional praxeologies and of sequences of linked praxeologies. With respect to school algebra, our proposal is to interpret it as a *process of algebraization* of already existing mathematical praxeologies, considering it as a *tool* to carry out a modelling activity that ends up affecting all sectors of mathematics. Therefore, algebra does not appear as "one more content" of compulsory mathematics, at the same level as the other mathematical praxeologies learnt as school (like arithmetic, statistics or geometry) but as a general modelling tool of *any* school mathematical praxeology, that is, as a tool to model previously mathematized systems (Bolea, Bosch & Gascón 1998, 2001, 2004; Ruiz-Munzón 2011; Ruiz-Munzón, Bosch & Gascón 2011).

This vision of algebra can provide an answer to the problem of the status and rationale of school algebra in current secondary education. On the one hand, algebra appears as a privileged tool to approach theoretical questions arising in different domains of school mathematics (especially arithmetic and geometry) that cannot be solved within these domains. A well-known example is the work with patterns or sequences where a building principle is given and one needs to make a prediction and, thus, find the rule or general law that characterises it. This feature highlights another differential feature of algebra that is usually referred to as "universal arithmetics": the possibility of using it to study relationships independently of the nature of the related objects, leading to "generalised" solutions of a whole type of problems, instead of a single answer to isolated problems, as is the case in arithmetic. Another essential aspect of the rationale of algebra is the need to organise mathematical tasks in *types* of problems and to introduce the idea of "generalisation" in the resolution process.

In this perspective, the introduction of the algebraic tool at school needs to previously dispose of a system to model, that is, a well-known praxeology that could act as a *milieu* (in the sense given to this term in the Theory of Didactic Situations) and that is rich enough to generate, through its modelling, the different entities (algebraic expressions, equations, inequalities, formulae, etc.) essential to the subsequent functioning of the algebraic tool. In the model we propose, we will take as initial system the set of *arithmetic calculation programmes*, a "calculation programme" (CP) being a sequence of arithmetic operations applied to an initial set of numbers or quantities that can be effectuated "step by step" and provides a final number of quantity as a result. The corpus of problems of classic elementary arithmetic (and also some geometrical ones) can all be solved through the verbal (or graphical) description of a CP and its execution. The starting point of the epistemological reference model for elementary algebra is therefore a compound of elementary arithmetical praxeologies with techniques based on the verbal description of CP and their effectuation "step by step". Working with this praxeology soon presents some technical limitations (for instance when trying to solve complex arithmetic problems of sharing quantities in given ratios) and also raises theoretical questions about, on the one hand, the reasons for obtaining a given result, justifying and interpreting it and, on the other hand, the possible connections between different kind of problems and techniques. All these questions lead to an enlargement of the initial system through successive modelling processes giving rise to different stages of the algebraization process that we will briefly summarize hereafter. A more detailed description can be found in (Ruiz-Munzón 2011; Ruiz-Munzón, Bosch & Gascón 2011).

The first stage of the algebraization process starts when it is necessary to consider a CP not only as a process but as a whole, representing it in a "sufficiently material" way—for instance written or graphically—to manipulate it. This does not necessarily mean the use of letters to indicate the different numbers or quantities intervening in a CP (that we can call the "variables" or "arguments" of a CP). However, it requires making the global structure of the PC explicit and taking into account the hierarchy of arithmetic operations and their hierarchy (the "bracket rules"). This new practice generates the need of new techniques to create and simplify algebraic expressions and a new theoretical environment to justify these techniques. It is here where the notion of "algebraic expression" takes its sense as the symbolic model of a CP, as well as the "equivalence" between two CP. In what concerns written manipulation of CP, and following the classic terminology about equations, we can say that this stage requires the operation of "simplifying" and "transposing" equivalent terms but not the operation of "cancelling".

The passage to the second stage of algebraization occurs when simplification techniques and equivalence between PC are not enough to solve the problems because the initial data or the problem stated are given as a relationship between variables of the CP. The structure of the types of problems typical of this stage can be represented by an identity between CP containing two non-numerical arguments. In this stage, algebraic techniques of CP manipulation become more complex and include considering equations as new mathematical objects, as well as the technical transformations needed to solve them. In the specific case where the unknown is one of the non-numerical arguments of the PC, the problem is reduced to solving a one-variable equation. Nowadays, school algebra mainly remains in this last case (without necessarily having passed through the first one): solving one variable equations of first and second degree and the word problems that can be modelled with these equations However, the work done at school does not achieve this second stage of the algebraization process, which also comprises the consideration of equations with parameters.

The third stage of the algebraization process appears when the number of arguments of the PC is not limited and the distinction between unknowns and parameters is eliminated. The new praxeology obtained contains the work of production, transformation and interpretation of *formulae*. It is not much present at current secondary schools even if it appears under a weak form in other disciplines (like physics or chemistry). Nowadays, at least in Spain, the use of algebraic techniques to deal with formulae is hardly disseminated outside the study of the general “linear” and “quadratic” cases. However, they play an essential role in the transition from elementary algebra to functions and differential calculus, a transition which is nowadays quite weakened in school mathematics. On the one hand, algebraic techniques are centred on the problem of solving equations (and some simple inequalities) of first and second degrees and on the formal manipulation of more complex algebraic expressions, to “factor”, “develop” or “simplify” polynomials, rational fractions or expressions with radicals. On the other hand, functions emerge in a numerical environment, to deal with problems that are rarely motivated by the difficulties found during the algebraic work done previously (for instance, in the solving of inequalities).

Furthermore, secondary school mathematics does not usually include the systematic manipulation of the global structure of the problems approached, which can be reflected in the fact that letters used in algebraic expressions only play the role of unknowns (in equations) or variables (in functions), while parameters are rarely present. However it can be argued (Chevallard & Bosch, to appear) in which sense the omission of parameters—that is, the use of letter to designate “known” as well as “unknown” quantities—can limit the development of efficient modelling algebraic tools and constitutes a clear denaturalisation of the “algebraic art”.

The effort to explicitly state an epistemological reference model for elementary algebra has different purposes. It can first be used as a descriptive tool to analyse the kind of algebraic praxeologies that exist at school and to study the ecological effects (conditions provided and constraints imposed) of these praxeologies in other mathematical contents. It is also a productive tool when trying to connect investigations about school algebra carried out from different theoretical perspectives as it helps specify the reference epistemological model of algebra more or less explicitly assumed by each research and compare the results provided by each one. An example would be looking at the similarities and differences between the “structural approach” of the research strand on *Early algebra* (Carraher et al. 2000, 2006; Malara 2003), Subramaniam & Banerjee 2004; Warren 2004) or the “algebrafying” paradigm promoted by J. J. Kaput (2000) and the first stage of the algebraization process and its possible implementation in the classroom. Another interesting exploitation consists in considering what aspects of elementary algebra are not taught at school and inquire about the possible reasons of their absence, as well as the ‘nature’ and ‘origin’ of these reasons. This kind of study, which in the ATD is called the “possibilistic problem” (Chevallard & Bosch, to appear), would help us progress in our knowledge of the conditions needed to modify a given “institutional ecology” in a given way. As we will see in the next section, the epistemological reference model also provides a way to experiment new teaching processes that are supposed to bring a new insight on this institutional ecology from the response obtained to the changes operated in it. A clinical analysis of the teaching interventions can really reinforce the approach of the possibilistic problem, as it usually highlights restrictions that are normally hidden or “silent”. Finally, we will just mention a last important use of reference epistemological models in the research cooperative work with teachers or directly in teachers’ training programmes (Sierra, Bosch & Gascón 2011).

## HOW TO TEACH ALGEBRA AT SCHOOL? STUDY AND RESEARCH PATHS

Given the results obtained by the ecological analysis of school systems related to the teaching and learning of algebra, it could seem that the only possible way to integrate algebra as a modelling tool in compulsory education is to operate effective changes in both the pedagogical and epistemological models prevailing in these institutions. However, the final aim of the ecological analysis cannot be reduced to the description of how things are and why they seem to be as they are, but to enquire into the possible ways of making them evolve. Of course not much can change without understanding the constraints or barriers of any kind (material as well as ideological or conceptual) that hinder the set of praxeologies that can be brought into play in the classroom, at school as well as outside school. The phenomenon of *logocentrism* and the written symbolism pejoration, the cultural supremacy of discourse and geometrical work in front of algebraic calculations, or the disappearing of formulae from the school mathematical work are part of these constraints and are affecting any local attempt of modification. As a consequence, it could seem that any attempt to renew the teaching of school algebra requires significant changes that are out of the scope of didactics research.

The way chosen by the anthropological approach to face this situation is to carry out *clinical analyses* of current teaching processes (Chevallard 2010), proposing local modifications, studying the conditions of possibility of such modifications and exploring the answers or reactions to them. To progress in this way, and in the frame of investigations focused on the new problematic opened by the *paradigm of questioning the world* that Y. Chevallard is presenting in this congress, most of our investigations since 2005 have been centred on the implementation of new teaching proposals based on *research and study paths* (RSP), working in close collaboration with secondary school teachers from the metropolitan area of Barcelona. In the case of school algebra, these RSP have been designed so that the initial questions that are at the starting point of the process would promote the transition through the different stages of the process of algebraization.

The first type of RSP are built around the well-known "Think of a number" games, which are used as a *milieu* to informally introduce the students to simple arithmetical calculation programmes. Carrying out these games can soon highlight the limitations of arithmetical techniques (based on "step by step" calculations) to work with CP and raises new theoretical questions about how to justify the "magic" of the games, for instance that the result of a given CP is always 75 or that, independently of the initial number taken, the final result of two different CP is always the same, etc. The work carried out during this study generates the need to progress through the first and second stages of the algebraization process.

In close relation to this RSP, and once the students can work at the first level of algebraization with the writing and simplification of algebraic expressions (without solving equations yet), a second kind of didactic process is introduced, more tightly led by the teacher, with the aim of introducing negative numbers in the context of the algebraic work (Cid & Bolea 2010; Cid & Ruiz-Munzón 2011). In this proposal, instead of putting the "conceptual construction" of negative numbers as vectorial quantities before their formal manipulation, the chosen option is to propose situations where negative numbers appear as natural needs of the algebraic work (for instance to simplify expressions obtained by a modelling process, such as $(3x + 2) – (x + 8)$) and afterwards deduce the kind of theoretical construction that can give coherence to the manipulations carried out.

The second type of RSP has been carried out with school students in the transition from lower to upper secondary level. They are based on initial questions of different natures, related to economics and financial issues ("Selling T-shirts", "Financing a students' trip") so that their study and resolution need the transition from the second to the third stage of the algebraization process and the connection with functional modelling, which is usually absent from Spanish secondary school curricula (Ruiz-Munzón 2011, Ruiz-Munzón et al to appear).

These investigations have shown different gaps to make the ecology of algebraic teaching practices evolve. We can mention, for instance, the possibility to introduce algebraic techniques of the different stages of the algebraization process from the study of questions related to the technical and theoretical limitations of the previous stage, that is, arising from situations where algebra appears as a tool to progress in the modelling of both mathematical and extra-mathematical issues. We have also confirmed the possibility for the students to work, from the first stages of the algebraization process, with expressions involving several variables, exchanging the role of letters as unknowns and as parameters. However, a lot of constraints have appeared, some of which can be located at the levels of didactic codetermination linked to the curricular organisation of contents (sublevels of the discipline) and to the discipline and pedagogy levels, especially related to the change of the didactic and pedagogical contract that hinder the passage from the paradigm of "visiting the works" to the one of "questioning the world" (Bosch 2010).

We are currently studying the new needs in mathematical and didactic *infrastructures* required by the implementation of SRP at secondary and tertiary level, and beginning to analyse the possible use of SRP, together with the reference epistemological and pedagogical models that support them, in pre and in-service teachers' training. This work participates of the latest developments of the ATD which promote to focus research efforts on the study and development of a new school ecology based on the "questioning the world" paradigm. It opens new and complex problems the scope of which seem to go beyond the research work done in "classroom laboratories" and even beyond the collaborative research work with pre and in-service teachers. However, the small progress already made in these contexts seems to open a fruitful line of research. It also shows that the "detachment gestures" we mentioned at the beginning of this paper are completely useless if we are not able to get efficiently involved in the social problems that we should face as mathematics educators.

## References


Anghileri, J. (2006). *Teaching Number Sense* (2d edition). London: Continuum International Publishing Group.

Artigue, M., Assude, T., Grugeon, B., & Lenfant, A. (2001). Teaching and learning algebra: approaching complexity through complementary perspectives. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra Proceedings of the 12th ICMI Study Conference*, Vol. 1 (pp. 21-32), Melbourne, Australia: The University of Melbourne.

Assude, T. (1993). Écologie de l'objet « racine carrée » et analyse du curriculum. *Petit x, 35*, 43-58

Atiyah, M. (2001). Mathematics in the 20th Century, *American Mathematical Monthly*, 108(7), 654-666.

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice. The case of limits of functions in Spanish High Schools. *Educational Studies in Mathematics*, *59*, 235-268.

Berger, P. L., & Luckmann, T. (1966). The social construction of reality: A treatise in the sociology of knowledge. Garden City, NY: Doubleday.

Blanc, J.-F., & Soler, S. (1933). *L'algèbre à l'école primaire*. Marseilles, France: Ferran.

Bolea, P. (2003). *El proceso de algebrización de organizaciones matemáticas escolares.* Monografía del Seminario Matemático García de Galdeano, 29. Zaragoza, Spain: Departamento de Matemáticas, Universidad de Zaragoza.

Bolea, P., Bosch, M., & Gascón, J. (2001a). La transposición didáctica de organizaciones matemáticas en proceso de algebrización. *Recherches en Didactique des Mathématiques, 21*(3), 247-304.

Bolea, P., Bosch, M., & Gascón, J. (2001b). ¿Cómo se construyen los problemas en Didáctica de las Matemáticas? *Educación Matemática*, *13*(3), 22-63.

Bolea, P., Bosch, M., & Gascón, J. (2004). Why is modelling not included in the teaching of algebra at secondary school? *Quaderni di Ricerca in Didattica, 14,* 125-133.

Bosch, M., & Chevallard, Y. (1999). La sensibilité de l'activité mathématique aux ostensifs. *Recherches en didactique des mathématiques, 19*(1), 77-124.

Bosch, M., & Gascón, J. (2005). La praxéologie comme unité d'analyse des processus didactiques. In Mercier, A. et Margolinas, C. (Coord.) *Balises en Didactique des Mathématiques* (pp. 107-122), Grenoble, France: La Pensée sauvage.

Bosch, M. (1994). *La dimensión ostensiva en la actividad matemática. El caso de la proporcionalidad* (Doctoral dissertation). Universitat Autònoma de Barcelona, Spain.

Bosch, M. (2010). L'écologie des parcours d'étude et de recherche au secondaire. In G. Gueudet, G. Aldon, J. Douaire, & J. Traglova (Eds.) *Apprendre, enseigner, se former en mathématiques: quels effets des ressources? Actes des Journées mathématiques de l'INRP*. Lyon, France: Éditions de l'INRP.

Bosch, M., Gascón, J., Ruiz Olarría, A., Artaud, M., Bronner, A., Chevallard, Y., Cirade, G., Ladage, C., & Larguier M. (2011). *Un panorama de la TAD. An overview on ATD* (pp. 579-604). CRM Documents, vol. 10. Bellaterra (Barcelona, Spain): Centre de Recerca Matemàtica.

Bourdieu, P., Chamboredon, J. C., & Passeron, J. C. (1968). *Le métier de sociologue: Préalables épistémologiques*. Paris: Mouton de Gruyter.

Bronner, A., Larguier, M., Artaud, M., Bosch, M., Chevallard, Y., Cirade, G., & Ladage, C. (2010). *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action.* Montpellier, France: IUFM.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. M. (2000). Early algebra, early arithmetic: Treating operations as functions. *The Twenty- second Annual Meeting of the North American Chapter of the International Group for th Psychology of Mathematics Education,* Tucson, AZ.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Aritmetic an Algebra in Early Mathematics Education. *Journal for Research in Mathematics Education, 37*(2), 87-115.

Chevallard, Y. (1984). Le passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collège – Première partie. L'évolution de la transposition didactique, *Petit x, 5,* 51-94.

Chevallard, Y. (1985). *La Transposition Didactique. Du savoir savant au savoir enseigné*. Grenoble, France: La Pensée Sauvage (2[d] edition 1991).

Chevallard, Y. (1989a). Le passage de l'arithmétique à l'algébrique dans l'enseignement des mathématiques au collège - Deuxième partie: Perspectives curriculaires : la notion de modelisation. *Petit x, 19,* 45-75.

Chevallard, Y. (1989b). Le passage de l'arithmétique à l'algébrique dans l'enseignement des mathématiques au collège – Troisième partie: Perspectives curriculaires: voies d'attaque et problèmes didactiques. *Petit x, 25*, 5-38.

Chevallard, Y. (1990). Le passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collège – Troisième partie. Perspectives curriculaires : vois d'attaque et problèmes didactiques, *Petit x*, 23, 5-38.

Chevallard, Y. (1994). Enseignement de l'algèbre et transposition didactique. *Rendiconti del seminario matematico Università e Politecnico Torino,52*(2), 175-234.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, *19*(2), 221-266.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie & regulation. In J.-L. Dorier, Artaud, M., Artigue, M., Berthelot, R. & Floris, R. (Eds) *Actes de la 11e École d'Été de didactique des mathématiques* (pp. 41-56). Grenoble, France: La Pensée sauvage.

Chevallard, Y. (2005). Didactique et formation des enseignants. In B. David (Ed.), *Impulsions 4* (pp. 215-231), Lyon, France: INRP.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In Bosch, M. (Ed.) *Proceedings of the IV Congress of the European Society for Research in Mathematics Education* (p. 21-30). Barcelona, Spain: FUNDEMI-IQS.

Chevallard, Y. (2007). Passé et present de la Théorie Anthropologique de Didactique. In A. Estepa, L. Ruiz, & F. J. García (Eds.), *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD)* (pp. 705-746), Jaén, Spain: Publicaciones de la Universidad de Jaén.

Chevallard, Y. (2010). La notion d'ingénierie didactique, un concept à refonder. Questionnement et éléments de réponse à partir de la TAD. In C. Margolinas, M. Abboud-Blanchard, L. Bueno-Ravel, N. Douek, A. Fluckiger, P. Gibel, F. Vanderbrouck, F. Wozniak (Coord.), *En amont et en aval des ingénieries didactiques* (pp. 705-746), Grenoble, France: La Pensée sauvage.

Chevallard, Y., & Bosch, M. (to appear). L'algèbre entre effacement et réaffirmation. Aspects critiques de l'offre scolaire d'algèbre. *Recherches en Didactique des Mathématiques*. [Special issue on algebra]

Cid, E., & Bolea, P. (2010). Diseño de un modelo epistemológico de referencia para introducir los números negativos en un entorno algebraico. In A. Bronner, M. Larguier, M. Artaud, M. Bosch, Y. Chevallard, G. Cirade & C. Ladage (Eds.), *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action* (pp. 575 - 594). Montpellier, France: Université de Montpellier.

Cid, E., & Ruiz Munzón, N. (2011). Actividades de studio e investigación para introducer los numeros negativos en un entorno algebraico. In M. Bosch, J. Gascón, A. Ruiz Olarría, M. Artaud, A. Bronner, Y. Chevallard, G. Cirade, C. Ladage, & M. Larguier (Eds.), *Un panorama de la TAD. An overview on ATD* (pp. 579-604). CRM Documents, vol. 10. Bellaterra (Barcelona, Spain): Centre de Recerca Matemàtica.

Coulange, L. (2001a). Enseigner les systèmes d'équations en Troisième. Une étude économique et écologique. *Recherches en Didactique des Mathématiques, 21*(3), 305-353.

Coulange, L. (2001b). Evolutions du passage arithmétique-algèbre dans les manuels et les programmes du 20ème siècle. Contraintes et espaces de liberté pour le professeur. *Petit x*. *57*, 61-78.

Derrida, J. (1967). *De la grammatologie.* Paris: Les Editions de Minuit.

Douglas, M. (1986). *How institutions think.* Syracuse, NY: Syracuse University Press.

Elias, N. (1978). *What is Sociology?* London: Hutchinson.

Elias, N. (1987). *Involvement and Detachment. Contributions to the Sociology of Knowledge*. Oxford, United Kingdom: Blackwell.

Estepa, A., García, F. J., & Ruiz-Higueras, L. (2006). *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD)*. Jaén, Spain: Publicaciones de la Universidad de Jaén.

García, F. J., Gascón, J., Ruiz Higueras, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics, *ZDM International Journal on Mathematics Education, 38*(3), 226-246.

García, F. J. (2005). *La modelización como herramienta de articulación de la matemática escolar. De la proporcionalidad a las relaciones funcionales* (Doctoral dissertation). Universidad de Jaén, Spain.

Gascón, J. (1993). Desarrollo del conocimiento matemático y análisis didáctico: Del patrón análisis-síntesis a la génesis del lenguaje algebraico. *Recherches en didactique des mathématiques, 13*(3), 295-332.

Gascón, J. (1993-1994). Un nouveau modèle de l'algèbre élémentaire comme alternative à l'« arithmétique généralisée ». *Petit x, 37*, 43-63.

Gascón, J. (1999). La naturaleza prealgebraica de la matemática escolar. *Educación matemática. 11*(1), 77-88.

Gascón, J. (2011). Las tres dimensiones fundamentales de un problema didáctico. El caso del álgebra elemental. *Revista Latinoamericana de Investigación en Matemática Educativa, RELIME*, 14 (2), 203-231.

Grugeon, B. (1995). *Étude des rapports institutionnels et des rapports personnels des élèves à l'algèbre élémentaire dans la transition entre deux cycles d'enseignement: BEP et Première G.* Doctoral dissertation. Université de Paris 7, Paris.

Havelock, E. A. (1963). *Preface to Plato*. Cambridge, MA: Harvard University Press.

Kaput, J. J. (1983). Errors in translations to algebraic equations: Roots and implications, *Focus on Learning Problems in Mathematics*, 5, 63-78.

Kaput, J. J. (2000). *Transforming algebra from an engine of inequality to an engine of mathematical power by "algebrafying" the K-12 curriculum.* National Center for

Improving Student Learning and Achievement in Mathematics and Science, Dartmouth, MA.

Kaput, J. J. (1998). Transforming Algebra from and Engine of Inequity to an Engine of Mathematical Power by "Algebrafying" the K-12 Curriculum. In *The Nature and Role of Algebra in the K-14 Curriculum* (pp. 25-26). Washington, D.C.: National Council of Teachers of Mathematics and the Mathematical Sciences Education Board, National Research Council.

Malara, N. A. (2003). Dialectics between theory and practice: theoretical issues and aspects of practice from an Early Algebra Project. In N. Pateman, G. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education North America*, Vol. 1 (pp. 33-48).

OECD (2009). *Learning Mathematics for Life: A Perspective from PISA. Programme for International Student Assessment.* http://www.oecd.org/dataoecd/53/32/44203966.pdf

Ong, W. J. (1982). Orality and Litteracy. The technologizing of the world. New York: Routledge.

Pujnachov, I. V., & Popov, I. P. (2008). *Matemáticas sin formulas: Libro 1 y 2.* Moscow: URSS.

Ruiz Munzón, N. (2010). *La introducción del álgebra elemental y su desarrollo hacia la modelización functional* (Doctoral dissertation). Universitat Autònoma de Barcelona, Spain.

Ruiz Munzón, N., Bosch, M., & Gascón, J. (2007). The functional algebraic modelling at Secondary level. In D. Pitta-Panzati & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 2170-2179). Nicosia: University of Cyprus.

Ruiz Munzón, N., Bosch, M., & Gascón, J. (2011). Un modelo epistemológico de referencia del álgebra como instrumento de modelización. In M. Bosch, J. Gascón, A. Ruiz Olarría, M. Artaud, A. Bronner, Y. Chevallard, G. Cirade, C. Ladage, & M. Larguier (Eds.), *Un panorama de la TAD. An overview on ATD* (pp. 743-765). CRM Documents, vol. 10. Bellaterra (Barcelona, Spain): Centre de Recerca Matemàtica.

Ruiz-Munzón, N., Matheron, Y., Bosch, M., & Gascón, J. (*to appear*). Autour de l'algèbre : les entiers relatifs et la modélisation algébrico-fonctionnelle. *Recherches en Didactique des Mathématiques.* [Special issue on algebra]

Schoenfeld, A. H. (2011). Reflections of an Accidental Theorist. In M. Pitici (Ed.), *The Best Writing on Mathematics* (pp. 219-235). Princeton, NJ: Princeton University Press.

Sierra, T., Bosch, M., & Gascón, J. (2011). La formación matemático-didáctica del maestro de Educación Infantil: el caso de «cómo enseñar a contar». *Revista de Educación, 357,* 231-256.

Subramaniam, K., & Banerjee, R. (2004). Teaching arithmetic and algebraic expressions. In M. Johnsen, & A. Berit (Eds.), *Proceedings of the 28th International Group for the*

*Psychology of Mathematics Education*, Vol. 3 (pp. 121-128). Bergen, Norway: Bergen University College.

Warren, E. (2001). Algebraic understanding and the importance of operation sense. In M. Heuvel–Penhuizen (Ed.), *Proceedings of the 25th International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 399-406). Utrecht, The Neetherlands: Freudenthal Institute.

Warren, E. (2004). Generalizing arithmetic: supporting the process in the early years. In M. Johnsen y A. Berit (Eds.), *Proceedings of the 28th International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 417-424). Bergen, Norway: Bergen University College.