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LEARNING TO SEE: THE VIEWPOINT OF THE BLIND

Lourdes Figueiras
Universitat Autònoma de Barcelona
Lourdes.figueiras@uab.cat
Abraham Arcavi
Weizmann Institute of Science, Israel

Visualization goes beyond "seeing". On the one hand, it includes other sensorial perceptions, relationships with previous experiences and knowledge, verbalization and more. On the other hand, visualization can develop also in the absence of vision. On the basis of these premises, we attempt to revise the processes of visualization in mathematics education by a) analyzing learning and teaching of mathematics by blind students with an expert blind mathematics teacher, and b) simulating blindness with mathematics teachers with normal vision.

Key words: visualization, blind

INTRODUCTION

The general goal of this work is to study how certain processes of mathematical knowledge construction rely on the use of sensorial/perceptual resources available to human beings. One of these resources which has gained increased attention in the mathematics education community in the last two decades is visualization.

Attending to the blind in order to study visualization is a relatively new avenue of research. In his book on visualization, Rivera (2011) devotes a chapter to the blind and draws implications for mathematical reasoning. Similarly, Lulu Healy and her research team in Brazil (see, for example, the regular lecture in this Conference) are studying the blind. One main and general conclusion seems to arise from these and other studies, as well as from self-reports by blind mathematicians, (see, for example, Jackson 2002): there is much more to the visualization than the sense of vision, and impaired vision does not necessarily preclude our faculties to visualize.

Our aim is to reflect upon the characteristics of visualization in connection with the spectrum of sensorial and other resources we employ, how we employ them and what are those that we could have employed but we don't.

For this purpose, we have analyzed some regular upper secondary mathematical classes conducted by a blind teacher at a special school for the blind. This data was not collected as part of an experimental program, but rather as documentation of the practices of an expert teacher, specialized in mathematics education for the blind. The aim of our analysis is neither to further studying the processes of learning mathematics by the blind per se nor to develop didactical aids for them. Rather, we aim at reflecting on the processes of visualization in the construction of mathematical knowledge within different mathematical topics: geometry (the cone and its sections) and quadratic functions and their graphs. Inspired by those two lessons, we designed two mathematical tasks and tried them with pre- and in-service teachers and with researchers in mathematics education (all with normal sight) in which they covered their eyes while performing.

In this presentation, we do not pursue theory building, neither have we described a methodologically sound empirical study. Rather, we report on our observations and reflections from both settings which helped us to pinpoint interesting phenomena (some of which we were not aware of) and which inspired the raising and formulation of research questions to explore further.

THE EPISODE OF THE CONE

The excerpt transcribed below is taken from a video-taped math class in which there are three students (two totally blind and one with residual vision), and the totally blind teacher mentioned above. They manipulate a wood model of a cone made of five pieces (see Figure 1) in order to learn to recognize the conic sections.

Figure 1: A wooden model for conic sections

The section obtained by cutting the cone with a plane parallel to its base allows the teacher to refer to the truncated cone. His aim is to guide the students to visualize the truncated cone as a solid of revolution generated by a trapezium (i.e. a trapezoid). As a preparation step, the teacher appeals to the students' prior knowledge and asks one of them -totally blind- to build a right-angled triangle and to convince his mates that its rotation around one of its legs generates a cone (as a solid of revolution).

Teacher: So, you say that with a right-angled triangle you get a cone?

Johnny: Yes

Teacher: Let's see if you convince Mercedes, because I think that Mercedes is not at all convinced

Johnny: I stand up

Teacher: ... and I doubt that Manolo is convinced

Johnny: [In front of Manolo, who has residual vision] with a right-angled triangle, when rotating it, its trail reproduces a cone. One is convinced…

Johnny: [In front of Mercedes, holding one of her hands – see Figure 2] Look Mercedes, imagine that here we have the triangle, upright, and here we have a rotation axis. By rotating it, the trail it leaves to us, if we fill it up with matter, it would reproduce a cone, when the round is completed. Another one is convinced.

Figure 2: Johnny and Mercedes rotating a paper right-angled triangle

Let us compare how Johnny speaks to his two mates, keeping in mind that Manolo has residual eyesight and that both, Johnny and Mercedes, are totally blind In both instances, his goal is to facilitate to his listener the visualization of a cone as a revolution solid generated by a rotating right-angled triangle, and he uses the metaphor for the cone as the trail left by the triangle. In the demonstration for Manolo, the explanation is brief and several elements remain implicit, for example, the position of the triangle and its rotation axis. We infer that Johnny assumes that Manolo does not need explicit references to locate the figure in space or its rotation axis. This is not the case when he speaks to Mercedes. In the absence of vision much less can be assumed or remain implicit, and thus he makes many elements explicit by means of a verbal description. Moreover, when speaking to Manolo, the metaphor of a trail is not explicitly related to the mathematical content of the conversation, whereas in the case of Mercedes, there is a clear reference to the cone as a three dimensional figure - not just a surface - by stating that matter could be used to fill the space in which the triangle completed a round around its axis.

Moreover, we note that the explanation to Manolo is impersonal and lacks argumentative structure. In contrast, in the explanation to Mercedes, a perlocutionary effect is at play from the beginning (when she is asked to evoke previous images) to the end (when the physical experience of filling the space with matter is proposed to be done in collaboration by holding hands and talking about it – see Figure 2). The validity of the conclusion that the generated volume is a cone is related to enacting that experience (“the trail it leaves to us, if we fill it up with matter, it would reproduce a cone”) and it has an argumentative structure.

It would seem that when speaking to someone who can see -with the eyes-, the speaker takes for granted that vision makes obvious many things and thus making the reference to them superfluous. The implicit assumption that develops is that what it may be obvious for oneself, it will be for others as well. There is an important possible conclusion for the mathematics education of students with normal eyesight: by making fully explicit those issues considered to be "obvious" and thus not mentioned, one could help the building of sound knowledge. Moreover, such explicitness may enable to pinpoint possible points of miscommunication between two people (teacher and student, or student and student) who are looking at the same object and yet possibly seeing different things while being completely unaware of that. This may carry important morals for the integration of visualization into the practices of the mathematics classroom. First of all, it seems to be clear that, poor references to position and spatial relationships between people who are not blind should be enhanced by verbal descriptions as if it were assumed (contrary to what would seem natural) that our interlocutor is not seeing them.

In order to further analyze this type of situations, we conducted the following experience with two blindfolded teachers: We gave them a wooden stick with an attached right-angled triangle through one of its legs. The task had two parts. Firstly, both of them had to describe the object they received, and secondly, they had to describe and explain which solid would be obtained by rotating the stick, and thus rotating the attached figure. However, in the second part of the assignment the first teacher was just requested to explain, whereas the second was requested to explain it to a blind person.

We observed similar differences between the two explanations of these teachers to those given by Johnny, first to Manolo and then to Mercedes, respectively. The first explanation was considerably briefer and concluded with the teacher raising his hand and rotating the stick in order to "show" to some colleagues who observed the experience.

However, in the explanation of the second teacher he did several other things, especially in trying to provide an accurate characterization of the geometrical object attached to the wooden stick: a) he leaned the triangle on the table in order to emphasize the existence of a right angle; b) he made an explicit attempt to compare the lengths of the sides of the triangle by sliding two fingers along the sides at the same ‘sliding speed’ and then comparing sliding times; c) he verbally described the position of the rotation axis; d) he produced the metaphor of the cone as a funnel when the description of the movement came to a end. During these descriptions, there were a lot of redundancies.

Despite their differences, there are interesting parallels between Johnny’s two explanations and the explanations of the two blindfolded teachers regarding the implicitness of the characteristics of what we see when we see it, and on the need to get a hold of many other resources to make the implicit explicit.

THE EPISODE OF THE PARABOLA

Let us go back again to the same mathematics classroom of blind students in order to analyse this time some excerpts from a lesson on the quadratic function. In the conversation transcribed below, the same teacher (who is totally blind) guides the students to draw the graph of the parabola $y = 4x^2 + 4x$. As the extract is longer than the previous one, we have divided it into three units according to the intentions and the actions carried out by the teacher. In the first unit, the teacher elicits students' previous knowledge for plotting the graph. In the second unit, he guides them towards the conclusion that this is a parabola with two roots by reasoning by contradiction. In the third unit, he induces students to manipulate the algebraic equation of the parabola to determine the coordinates of the intersection points with the axes and of the vertex in order to sketch the graph.

Unit one: elements for sketching the graph

Teacher: Ah! A parabola. And what shall we do to draw a parabola?

Manolo: We will draw the roots, the vertex

Teacher: First of all. This one. Goes up or down?

Manolo: It goes up?

Teacher: It goes up. OK. And why?

Manolo: Because... The coefficient of *x* squared is positive.

Teacher: The coefficient of *x* squared is positive. You say the roots, do you? Then let's go for it, let's see the roots.

Teacher: zero...

Manolo: and zero

Teacher: Really? Does it have only one root? That is amazing. Sure?

Manolo: And minus two.

Teacher: And minus two?

Manolo: It makes zero and four

Teacher: Zero and four. Zero is clear. If I substitute zero for *x* we get zero. And if I substitute four, we don’t

Manolo: minus four.

Teacher: Minus four. Let's see. Two multiplied by minus four squared. How much is it?

[the students whisper]

Unit two: reasoning by contradiction

Teacher: How do we calculate the roots? It has only one, therefore it meets *x*-axis, or does it?

Johnny: Yes

Teacher: And in how many points?

Manolo: zero, zero

Teacher: Ah, Only in one point. And this point, what could it be then?

Manolo: The vertex

Teacher: It would be the vertex, and if the vertex is in zero, zero, the graphic, how could it be?

Johnny: the top part....

Teacher: the graphic would be symmetric, perhaps?

Students: Yes

Teacher: Yes? Therefore the right....

Johnny: side...

Teacher: ...would be equal to the...

Johnny: ...left side

Teacher: and what's the name of those functions having the right side equal to their left side?

Students: even

Teacher: What?

Students: even

Teacher: And what happens with the formulae of even functions?

Johnny: That we get the same if we substitute *x* or minus *x*

Teacher: Let's see if this is true. Wherever there is an *x*, I put a minus *x* to see that whenever there is a minus *x*, the left side equals the right side whenever there is an *x*.

Manolo: ...

Johnny: four *x*

Teacher: Then this is not even, therefore I am afraid that this is neither...

Johnny: symmetric

Teacher: and as a consequence the vertex cannot be at zero zero. One root is certainly zero. If I substitute *x* for zero, *y* is zero. But, which is the other root?

Unit three: algebraic solution of the equation

Teacher: Either we apply the formula [of the quadratic equation] or we decompose that.

Johnny: minus two

Manolo: One point would be zero and the second one...

Johnny: Look, Manolo, take out *x* as a common factor.

Teacher: And even something more can be taken out.

Manolo: Minus two.

Teacher: Then let's go for it, we write *y* equals two *x*... do you follow me?

Manolo: *x* plus two

Teacher: x plus two. For which values of x does it become zero?

Johnny: for zero and minus two

Teacher: zero and minus two. Therefore the roots are...

Johnny: zero and minus two

Teacher: And, what are the roots of a polynomial, geometrically, in the graph?

Manolo: The places where it meets the x-axis.

Teacher: The places where it meets the x-axis. Because for that value of x, the y, the ordinate, is zero. It remains at that point. Zero and minus two. We know already that this is a parabola, it is second degree, that it is positive, it goes up, the roots are zero and minus two. Well. Anything else to get a full sketch?

Johnny: That the vertex will be in minus one because...

Teacher: That the vertex will be in minus one because...

Johnny: ...because it is the middle point between minus two and zero

Teacher: The middle point between the two roots. Because there we will see the axis of symmetry of the parabola. Very well. Minus one, what?

Johnny: Minus two

Teacher: Minus one, minus two. Let's see. Two multiplied by minus one squared equals two. Plus 4 minus one minus four equals minus two. How do we denote the vertex? Capital V, do we? Minus one, two.

We first note that the conversation does not lose consistency or mathematical meaning if the second unit is suppressed. Moreover, the first and third units concatenated could be considered paradigmatic of many math classes. However, the second unit is precisely what captured our attention. The teacher decided not to correct the student's calculation mistake (see the end of the first unit) instead he pursues the reasoning that, if this parabola has a root at (0, 0), then it must have another one. Probably, most of us would have solved the quadratic equation directly in order to deduce the existence of a second root, and, at the same time, find the coordinates of the point of intersection with the x-axis explicitly. This is indeed what this teacher will do later, as it is shown in the third unit of analysis. So, why does the teacher invest time in pursuing the argument shown in the second unit? Which may his intention be?

All blind mathematicians and mathematics teachers agree that without vision it is extremely easy to make mistakes when recording information in writing. This is due to three essential limitations of the Braille system: a) the use of various signs to construct a single mathematical symbol induces frequent errors and there may be confusion about the meanings of blank spaces, b) the "linearity" of writing makes the creation of notations like exponents or subscripts a complicated endeavour, and c) the lack of resources such as deleting, annotating an equation, and so on, imply much work load.

Studies on the learning of algebra have shown that translating word problems into equations constrains in many cases the use of useful representations and the creativity of the solution (e.g. Friedlander & Tabach, 2010). Indeed, it is recommended as good practice to ask students to solve such problems without using equations in order to emphasize the importance of nurturing a certain "algebraic sense". In many cases, students are reluctant to set equations aside, especially when they feel comfortable enough with solution techniques. Instead, the blind resort to algebra only when it is strictly necessary. Using other forms of reasoning, and developing some creative imagination is a must if they do not want to be immersed in cumbersome calculations performed with the aid of the Braille system. Table 1 compares the most common way of representing a parabola with what happened in the episode above, considering the mathematical content involved.

Some common steps in order to begin to sketch the graph of the parabola $y=2x^2+4x$	**Steps followed by the blind in order to begin to sketch the graph of the parabola $y=2x^2+4x$**
	Detect the intersection point at (0,0)
	Reason by contradiction regarding the existence of a second root: If there is only one root, it must be the vertex (relies on previous knowledge) If the vertex is at (0,0), the graph is symmetric with respect to the y-axis Check if $f(x)=f(-x)$ If the function is symmetric, then it is even The function is not even, therefore this root is not unique.
Solve the equation $y=2x(x+2)$	Solve the equation $y=2x(x+2)$
Find the two roots $x=0$ and $x=-2$	Find the two roots $x=0$ and $x=-2$
Conclude that there are intersection points at (0,0) and (-2,0)	Conclude that there are intersection points at (0,0) and (-2,0)

Table 1: Comparison of steps followed to sketch $y=2x^2+4x$

Further analysis of this excerpt leads us to highlight another essential point of our discussion. In the absence of vision, the teacher chooses to guide the students to use logical reasoning that compels them to invoke images generated beforehand (a parabola that has a unique intersection point with the x-axis at (0,0) has its vertex necessarily at that point). Probably this image is related to others, like parabolas with the vertex on the y-axis and no points of intersection with the x-axis. In addition, students are requested to connect some concepts and definitions (symmetry, even functions) with images, and follow an argument by contradiction. The teacher's decision to delay to solve the equation in order to obtain the roots of the equation has redirected students' mathematical reasoning to rely on their images of a parabola as the graph of a quadratic equation.

As we did in the episode of the cone, we also tried a task related to the graph of a quadratic function in the context of professional development of teachers. This time, we provided the blindfolded participants with a relief plot of the parabola $y=2x^2+4x$. They were told that they are given the graph of a function in the Cartesian plane in the upright position and their task was to establish what graph is it and then to attempt to find its equation. The plot was printed on a special material, the thickness of the axes differed considerably from the thickness of the graph of the parabola, and the lattice of integer values was also marked. The aim of our experiment was to detect which properties were detected and invoked by touching, as well as to use the experience to reflect with the teachers on their own practices. Findings in repeated experiments with different groups of teachers allowed us to establish three main types of reactions:

Difficulties to get started: Some teachers with a good knowledge of mathematics had difficulties to make sense of the graph only by touching and expressed their difficulties to distinguish the lattices from the axes, and the axes from the graph. Some of them attributed this to the lack of a fine tuned sense of touch. However, if they overheard others talking and exchanging information of what they were finding, they were able to redirect themselves and started to make better sense of what they experienced.

Attention to local properties: A common and rather unsurprising outcome of the experience was that the amount of time taken to identify the curve to be a parabola was much longer than it would have been by using eyesight. If one knows what a quadratic function is and how the parabola represents it in the Cartesian plane, only a quick glimpse suffices to identify it globally as such. By means of some mental or written work, one would be able to establish its symbolic representation using clues from the graph (position, location of the roots and vertex, intersection with the y-axis). The most interesting observations for us refer to the first stage when that global glimpse that is enabled by our eyesight is not possible. In this case, we saw that there are still some features of globality that people attempted to grasp (running their fingers at once from end to end of the curve) in order to get an overall feeling of the shape and to try to fit it with mental images. However this was interspersed with many attempts to detect local features (turning points, intersections, "sense of curvature"). This interplay between the local and the global features of a graph in order to identify it was interesting and productive. For example, one of the teachers explained that sometimes, while she was sliding down her fingers along the descending branch of the parabola, she felt the horizontal axis and related these two objects (x-axis and curve). Then she realized that the amplitude of the angle with which the curve leaves the axis was an important clue to imagine the equation of the graph, something she did not realized when watching parabolas. This and similar instances regarding the need to engage all local and global information available were an eye opener for us: the gift of vision by virtue of its immediacy saves us from the task to note how local and global features interlock in order to conform a unit which we "see" at once.

Attention to the analytical steps needed for the plot: A subgroup of the teachers dedicated enough time to find out the coordinates of x-intercepts by estimating distances with their fingers, and some of them did the same to find out the vertex of the parabola. One of the teachers concluded that he could not be sure that the represented graph was not the graph of a polynomial of fourth degree or even higher (actually this distinction is equally undetectable by seeing the graphs). Many of them concluded that the graph represented the curve $y=2x^2+4x$, because they had only taken into consideration the points where the curve met the axes. All of them made explicit that they knew that they need the vertex to obtain the equation, but trying to memorize simple calculations and coordinates was extremely difficult; moreover, in their own view, this was probably the reason why they had not checked the coordinates of the vertex. At this stage, when the main task was to produce the symbolic equation of the function, it became apparent how importantly related are the sense of vision and short term memory. When one derives a symbolic formula on paper, one relies heavily on the sense of vision and its power to unload memory from abundant information, a quick glance at previous steps suffices when one is engaged in a symbolic calculation. The appreciation of the effortless way in which vision unloads our working memory becomes very apparent when one has to perform the same calculation on one's head only due to absence of the sense of vision.

CONCLUSIONS

We observed that, in the absence of vision, people cope with mathematical tasks by involving a rich variety of resources and mathematical reasoning processes. In contrast, when facing similar tasks, people with normal eyesight make use of only a small subset of the resources they have available. In the presence of vision, the presence of images may make verbal descriptions superfluous: one sees something and thus there is no need to describe it in words because the image speaks for itself. Note the widespread expressions in the mathematics education literature related to visualization such as "proofs without words" or "one picture is worth a thousand words", which implicitly confirm that one can do without verbal descriptions. However, the experience of understanding some very clever visual proofs show that whereas the images may lack words and symbols, there may be many words and sentences needed to unpack the proof and to make sense of it. In our observations, we noted the richness of the oral descriptions and the resourcefulness in the use of mathematical knowledge, when vision is not available. Such richness can be a source of inspiration for mathematics education of people with normal eyesight. This can be of particular importance because verbal communication regarding mathematical concepts usually involves implicit elements of which one does not talk about. Following our observations, we would like to suggest that such implicitness is due to our reliance on the sense of vision. It is assumed that the visual image communicates information that is not necessary to describe. Moreover, sometimes we also generalize our experience and assume that people who see are able to detect in the image some properties that are not explicit. We suggest that the explicitness in the verbal descriptions by the blind or for the blind can be adopted for all, even if in some cases it may sound redundant. This practice not only may help in avoiding misunderstandings but it may also support communication skills and clarity of formulations.

Regarding verbal communication, it t is also noteworthy that blind people pay careful attention to what is said by others. This was also very apparent with blindfolded teachers who were at lost when they had to identify the parabola and the talk by others in the same situation were a powerful resource to make progress. The lack of vision seems to promote attentive listening as yet another resource for knowledge construction. In addition to verbalization, knowledge construction and mathematical reasoning of blind people is supported by other resources, such as haptic perception, which could easily be incorporated as resources for general eyesight settings (for example, touch and speed of movement could be a way of apprehending length of segments). The perception by touch provides access to spatial details which may not be easily perceived otherwise and its use in the construction of mathematical knowledge by people with normal eyesight has not been explored. The global immediacy of vision may prevent us from interweaving local and global characteristics of approaching mathematical objects such as graphs of functions. Our experiences with the blindfolded teachers showed how the sense of touch allows them to integrate local and global perception in order to create a mental image of the graph they were trying to identify.

There are several difficulties for the blind people in creating and following chains of symbolic developments, like solving an equation. On the one hand, the Braille system may not support well the editing and correction of errors in symbolic derivations, and, on the other hand, these derivations which may be highly demanding to perform, even partially, in our head. Thus the blind may tend to develop alternative ways to eschew these difficulties, as it was the case of the blind teacher when he tried to foster, by means of a logical argument, the rejection of the the initial temptation of assuming that (0,0) was the vertex, instead of going right away to a symbolic calculation for that purpose.

Thus, another important realization, especially for the participant teachers, was the appreciation of the enormous power which we are granted by being able to unload our working memories into written displays which can be quickly scanned by our sense of vision, which plays a key role as a memory enhancer. These also led to the realization of a somehow paradoxical situation: those mathematical areas less associated with images (algebra, analysis) rely more on vision in order to unload memory than those which are richer in images. This is probably the reason why the areas in which most blind mathematicians work is geometry, or geometry related. This leads to propose that more research and development efforts in visualization in mathematics education should be devoted to its role in such areas as algebra.

Finally, we would like to suggest that the possible differences in the construction of mathematical knowledge that exist between the blind and the non blind people should be regarded as an issue of deployment and implementation of resources for learning and not merely in terms of the presence of the absence of the sense of vision. And thus, there is much to learn from the blind in this respect. We, the non blind, are rarely encouraged to perform the spectrum of potentially rich actions of a blind. Thus it makes all the sense to explore further how the development of reasoning and knowledge construction under visual impairment can broaden our own learning processes in mathematics, and that is what we attempted to do in this presentation.

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