THE 26TH ICMI STUDY: ADVANCES IN GEOMETRY EDUCATION

DISCUSSION DOCUMENT

INTRODUCTION AND BACKGROUND

This document announces a new *ICMI Study* being conducted by the International Commission on Mathematical Instruction (ICMI). The Study, the 26th led by ICMI, addresses the advancements and current challenges in the teaching and learning of geometry, with particular attention to how the landscape has changed in the past three decades since the ICMI Study 9 on "Perspectives on the Teaching of Geometry for the 21st Century" (Mammana & Villani, 1998). Fundamentally, the International Program Committee (IPC) considers that geometry is positioned as a powerful pathway to support the understanding of complex ideas within school mathematics and beyond. In fact, we boldly suggest that geometry education is at its most exciting and quickly advancing period of the past 100 years. This ICMI Study examines the nature and role of geometry in mathematics education and related fields with an eye to the increasing depth and range of geometry education research, knowledge, pedagogies, and approaches on an international scale.

AIMS AND RATIONALE

The primary aims of the Study are first to report the state of the field of mathematics education in the area of geometry education with respect to theory, research, practice, and policy; and, second, to suggest new directions of research that consider contextual, cultural, national, and political dimensions of practice.

Since there are different ways of understanding geometry education, the Study aims to include multiple theoretical perspectives and methodological approaches. Thus, the IPC encourages contributions that use a variety of methodological strategies including, among others, large-scale experimental and descriptive studies, case studies, and iterative or cyclical processes such as design research and action research. The IPC welcomes contributions from mathematics education researchers and from mathematics teachers, to ensure that teachers' voices are given prominence in accounts of their learning and teaching. Moreover, in this new Study, the IPC invites contributions from those working in the wider geometry space (such as cognitive scientists and educational psychologists). This interdisciplinary approach aims to provide innovative and collaborative opportunities not afforded by the current state of play that can leave work siloed within individual disciplines.

This Study aims to produce and share new knowledge about promoting effective pedagogy in the teaching and learning of geometry for all. It takes into account the large body of existing theory and research, socio-cultural diversity, cultural differences in curricular and institutional constraints, as well as progress the field. In particular, the IPC acknowledges the following specific goals for geometry education and anticipates that these will be developed by participants through the course of the ICMI Study:

- Bring together an expert reference group to analyze the state-of-the-art of research and practice in geometry education in order to contribute to better understanding of the challenges that geometry education faces in diverse contexts.
- Bring together communities of international scholars, with diverse representation within the ICMI community, and across regions and nationalities, to address research and practices in geometry education, ultimately resulting in the production of an ICMI Study volume.
- Emphasize the role of geometry as a facilitator of advances in logical reasoning, strategic thinking, and understanding of hierarchical relationships between mathematical (geometrical) objects.
- Analyze the influence of affective components of teaching and learning such as confidence, motivation, anxiety, efficacy, and others, on teachers' decisions and methodological choices and on students' behavior and related learning results.
- Facilitate advances in multi- and inter-disciplinary approaches (including cooperation with other bodies and scientific communities) to research and development in geometry education.
- Explore the role of Geometry as an arena or lever for developing children's and adolescents' mathematical creativity and flexible thinking by means of conjecturing, problem posing, and proving theorems.
- Disseminate scholarship in mathematics education —research, practices, methodologies, theories, findings and results, curricula design, etc.— in geometry teaching and learning.
- Identify and anticipate new research and development possibilities, challenges, and questions related to geometry education and research in geometry education.
- Produce sets of recommendations on effective resources for researchers, teachers, teacher educators, policy makers, curriculum developers, analysts, and the broad range of practitioners in mathematics and mathematics education.
- Act as an instigator of new research and innovation to be produced in the future.
- Promote and assist in the discussion of geometry education and related research in action at local and international levels.

STRUCTURE: TOPICS, SUB-TOPICS, AND QUESTIONS

The activity of the Study is organized based on four focused topics, aimed to provide complementary perspectives and approaches to the teaching and learning of geometry.

Contributions to the topics will be organized around sets of related sub-topics, each sub-topic focusing on a specific issue and stating a set of questions aimed to lead discussions. The four topics are:

A. Theoretical perspectives
B. Curricular and methodological approaches
C. Resources for teaching and learning geometry
D. Multidisciplinary perspectives

Topic A: Theoretical perspectives

Several theoretical and conceptual perspectives have been used to study geometry education —some have emerged within the field itself while others have been drawn from the education disciplines or applied from psychology or sociology. Theories specifically about geometry education that continue to be evident in research include the Van Hiele model (Van Hiele, 1986), the theory of figural concepts (Fischbein, 1993; Mariotti & Fischbein, 1997), and the theory of figural apprehension (Duval, 1998). Researchers are continuing to develop, refine and apply these theories. More recently, theories emerging in geometry education include embodied (Ng & Sinclair, 2015) and ecocultural perspectives (Owens, 2014), prototype phenomenon (Gal & Linchevski, 2010), and semiotic mediation (Bartolini & Mariotti, 2008).

In the recent ICMI Survey of geometry education, Sinclair et al. (2016) maintained that over the past decade, there has been an increased focus on theories related to visuospatial reasoning, the role of gestures and diagrams, and the use of digital technologies. As Sinclair and Bruce (2015) reinforce, post-Euclidian developments have moved beyond the emphasis of naming shapes and their properties to "a more active meaning-making orientation to geometry (including composing/decomposing, classifying, mapping and orienting, comparing and mentally manipulating two- and three-dimensional figures)" (p. 320).

Rather than revisiting the well-documented theoretical perspectives, this topic aims to dive more deeply into several emerging areas of research activity in geometry teaching and learning. Such emerging areas include new conceptions of the teaching and learning of proofs in geometry contexts, the role of spatial reasoning and the implications for geometry teaching and learning, and, in a related area, the role of visualization in geometry reasoning.

Sub-topic A1: Teaching and learning of proof in geometrical contexts

In the study of geometry, the concept of proof appears as a key element to support and validate mathematical statements. The IPC conceives of the proving process as a way to convince oneself or another person, through the presentation of different types of arguments in favor of the pertinence and validity of mathematical statements. In Euclidean geometry contexts, it is important to recognize that, depending on the education level of the students, the type of proofs to support conjectures vary. For instance, at elementary education, concrete examples could be sufficient to present and support mathematical relationships, but when students' progress or advance in their studies, they should refine their ideas and transit from empirical examples to present informal deductive proofs and, finally, formal proofs. In this context, teachers' educational programs should explicitly include the development of knowledge and experience in problem posing, formulating conjectures, and validating them. These are creative processes as much as they are logical thought processes.

It is also important to acknowledge that the consistent and systematic use of digital technologies opens different routes for teachers and students to represent and explore geometrical properties and also to support the truth of mathematical conjectures or statements.

Some overarching questions for contributions on the teaching and learning of mathematical proof in geometry include:

- How can the concept of mathematical proof in geometry be characterized at different educational levels? What types of geometry proofs should students learn during the different school levels? Are there paths or trajectories during the process of learning to prove geometric statements?
- What types of teaching sequences are helpful to promote the learning of geometric proofs?
- How might teacher and/or student uses of digital technologies provide a set of affordances to think of different types of arguments (visual, empirical, formal)?
- How do undergraduate students value the correctness of geometric proofs? Is this in line with their teachers' values?
- How do dynamic geometry environments affect conceptions of what is meant by geometric proof or what a geometric proof is?
- What affective components (e.g., anxiety, confidence, and efficacy) influence the learning of geometric proof? Are there gender differences in approaches to geometric proofs?
- In which ways can students be engaged in formulating conjectures and doing geometric proofs?

Sub-topic A2: The role of external visual inputs in geometric reasoning

Visualization is a broad construct that is important in mathematics. Simply put, it involves "the mind seeing things." Use of imagery is known to support geometric thinking and is instrumental in working in novel or complex situations. Along with many others, Mason's (2004) work on the "structure of attention" intersects with these notions of visualization. However, challenges prevail in mathematics education where canonical/prototypical and static inputs tend to dominate students' encoding of visual inputs. Consequently, it is important to consider how students can be offered a much wider variety of experiences (and inputs) as early as possible.

Dynamic visualizations, where there is movement, such as in visualizations of compressions and stretches are one example of expanding these cases. Building on the work of Duval (1998), what is seen is quickly classified in the minds' eye as "triangle", for example; this is the iconic vision, and these canonical/prototypical constructs can be deconstructed. For instance, a square would need to be seen not only as a shape of square in an iconic way, but also as four lines intersecting orthogonally at equal distance and ultimately four points. New technologies offer interesting new affordances worthy of exploration. A burgeoning area of research related to visualization focuses on objects-to-think-with or implicit tools for use in a broad range of contexts. There is much to explore at this time in the arena of external visual inputs.

The IPC invites contributions that explore these key questions:

- In what ways are visualization and geometry linked, and how does visualization support geometric thinking?
- What affordances and challenges are engendered through technology uses for visualization?
- How might learners break away from canonical visual depictions of geometric figures? Why is this important?
- Are visual and analytic geometric processing distinct? How do they overlap and when should they overlap?
- How do objects-to-think-with, implicit structures, and/or forms of visualization operate as a tool for geometric thinking and supporting analytic reasoning?

Sub-topic A3: Spatial reasoning interventions and transfer to geometry

Spatial reasoning is integral to activities in daily life, such as packing a car for a trip or reading maps. At a broad level, spatial reasoning describes the ability to mentally represent, analyze, and transform objects and their relations, and is comprised of distinct, yet related, skills with strong ties to mathematics achievement. Spatial reasoning involves the location and movement of objects and ourselves, either mentally or physically, in space. It is not a single ability or process but actually refers to a considerable number of concepts, tools, and processes (National Research Council, 2006). The role of spatial reasoning within mathematics problem solving and performance have been noted for decades (Clements & Battista, 1992). In fact, a recent meta-analysis (Hawes et al., 2022) confirms that improving spatial reasoning skills causally supports improvements in mathematics performance.

Spatial reasoning is malleable, with improvements observed even in short periods of explicit instruction or exposure to spatial tools and playful experiences. Spatial reasoning is especially relevant to geometry education, with numerous (recent) studies demonstrating transfer to geometric understanding (Lowrie et al., 2019). What are less known are the mechanisms that support the transfer from spatial development to geometric understanding. This topic seeks advances in the field that describe interventions and learning programs that spatialize the geometry curriculum as well as spatial learning activities that support geometric understanding. Topics include the relationship between spatial reasoning and geometry in 2D and 3D transformations, composing and decomposing shape, perspective taking and activities involving symmetry and rotations of 2D shapes and 3D objects.

The IPC invites proposals that address some of the following questions:

- What are the mechanisms that support transfer from spatial reasoning to geometric understanding?
- What constructs or types of spatial reasoning should be included in school (Pre-K to 12) mathematics?
- Given the strong relationship between spatial reasoning and geometric understanding, what are promising ways of doing this?
- How is spatial reasoning positioned in school curricula and how does this vary from nation to nation? Is this serving students well? What is the impact?

Topic B: Curricular and methodological approaches

A core pillar of the 26th ICMI Study relates to geometry curricula and methodological approaches to teaching geometry, from the early years through to higher education. There are novel developments that are changing the landscape of geometry education with intensity of research activity in the early years and in higher education. When developments in curriculum for school geometry are considered, increased attention to spatial reasoning is apparent, alongside and/or within the geometry curriculum. In many nations, curriculum constitutes policy, which carries weight and encourages greater classroom time devoted to geometry learning. Recent research on professional learning focused on geometry teaching and learning has demonstrated that longer and more intensive professional training programs, which are situated in classrooms directly involving students, are proving to have lasting impact (Clements & Sarama, 2011). This includes attending to individual differences of learners and cultural and political contexts of learning.

Sub-topic B1: Geometric thinking across age levels (from preschool to higher education)

Despite decades of robust research in geometry education, there continues to be gaps in students' geometry experiences and learning across age levels from preschool to higher education. The gaps include limited opportunities to engage with real-world and mathematical objects that foster geometric modeling and reasoning, uneven quality of curriculum and teaching that students experience, limited connections made between spatial, geometric, and other mathematical concepts, and what students are asked to "do" (e.g., doing, describing, analyzing, proving, etc.). There is an urgent need to develop and/or refine ways of analyzing these gaps. In the last 30 years, there has been a flurry of research activity in the early years and in higher education that focuses on improving spatial sense and geometric reasoning. The literature on building spatial reasoning for young learners and for those pursuing careers in STEM education is developing at an accelerated pace.

Some of the important issues on which researchers and teachers alike could contribute are:

- What is the current status of geometry curricula across age levels and how do such curricula support student's conceptual understanding and development?
- What curricular and classroom-based innovations could help to understand, analyze, and fill the gaps among early, primary, secondary, and higher education? How can recent research efforts in the early years and in post-secondary education be built on?
- How do informal and formal geometry curricula seek to include the use of language, gestures, drawing, modelling, or representations to support student thinking? How can the effectiveness of these aspects of conveying meaning be analyzed?
- What are effective ways to investigate children's developing understanding of 2D and 3D shapes in early years? How are early years spatial-related experiences linked to later formal learning of geometry?
- What efforts have been made to ensure that formal geometry curricula build on earlier understandings?
- How are post-secondary programs in STEM areas supporting the geometric thinking required for professions such as engineering, medicine, chemistry (and beyond)?

Sub-topic B2: Individual differences, including students with learning difficulties or mathematical giftedness

The ability to study geometry successfully can be a consequence of learning opportunities provided to students or their innate capabilities, that may impede or advance the development of their geometry knowledge, skills, and interest in learning geometry.

Difficulties in processing geometric concepts can depend on a range of factors. For example, dyslexia or difficulties related to reading comprehension may affect students' capabilities to formulate, describe, and characterize geometrical object and their properties and read geometrical text, such as definitions and theorems. Students with learning difficulties, for example, may find visual-spatial supports that do not require language and number to provide new pathways into complex geometry. Dyscalculia can influence solving calculation problems in geometry whereas dysgraphia can be an obstacle for drawing geometric figures. Visual impairments also impact students' geometrical learning and processing. Students with learning disabilities need special programs and approaches to teaching geometry that allow them to study fluently and experience success in geometry lessons.

Moreover, students with high mathematical abilities who enjoy solving challenging geometry problems also need special attention. As Krutetskii (1976) demonstrated, mathematically gifted students are able to memorize mathematical objects, schemes, principles, and relationships and often reason both logically and spatially. Such mathematical processing at these high levels across different branches of mathematics, including geometry, provide opportunities for creativity-directed activities such as conjecturing, proving, or refuting geometry statements using different theorems and with different auxiliary contractions. *Mathematical cast of mind* (Krutetskii, 1976) reflects students' special capability to see mathematical objects within the real world as well as to be creative in mathematics. Mathematically gifted students can be exceptionally creative, and their creativity needs to be encouraged and promoted.

The IPC invites contributions that explore teaching and learning geometry as related to a continuum of mathematical capabilities. Submissions are encouraged of studies that include, but are not limited to, the following questions linked to (a) various student learning disabilities and (b) students with high mathematical abilities:

- How do student capabilities affect studying geometry?
- Are there types of geometry tasks that map to different capabilities?
- What teaching approaches are most effective for students with different characteristics?
- How can mathematics teachers be supported to work with students of varying capabilities?
- What type of assessment tasks are effective when teaching students with special educational needs?
- How can technological tools be used to support and challenge the teaching and learning of geometry by students with special learning needs, including gifted students?
- How can creativity-directed activities or creativity-promoting activities support learning geometry for students with learning disabilities or mathematically gifted students?

Sub-topic B3: Influence on the curriculum of different cultural and political traditions and contexts around the world

Curriculum is regarded as the knowledge, skills, values, and attitudes presented to the learners in order to change their behaviors to become active members of society. A school mathematics curriculum should reflect the political and cultural contexts of the students for which it is planned. Political context refers to the way power is achieved or used in a society. Cultural context refers to the shared values, attitudes, beliefs, and practices of a community. Geometry should not be regarded as a value-free curricular content area. In fact, some argue that Mathematics, including geometry, should equip students to deal with political and cultural issues; for example, closing the learning gap between informal and formal geometric knowledge.

The IPC invites proposals that address some of the following questions:

- What is the influence on the curriculum of different cultural traditions and contexts around the world regarding geometric concepts?
- What is the influence on the curriculum of different political traditions and contexts around the world regarding geometric concepts?
- How can political and cultural contexts of mathematics curricula help to recognize and respect the history, tradition, and mathematical thinking developed by members of distinct cultural groups through the development of geometric thinking?
- What is the significance of the influence on the curriculum of different political and cultural contexts for researchers, policy makers, and school teachers?

Sub-topic B4: Professional training and development in geometry education (pre-service and in-service)

There are a wide range of professional learning models related to geometry education both for pre-service and in-service educators. Early years teachers often receive mathematics professional training that focuses on playful approaches to exploring patterns (repeating patterns), informal volume and capacity activities (emptying and filling containers), size comparisons (ordering objects from smallest to largest), and in the area of geometry, naming and recognizing 2D shapes. Recently, programs that focus on mathematics for young children have intensified the attention on geometry and spatial reasoning (Moss, Bruce & Bobis, 2015).

In terms of educator challenges, some primary teachers have limited knowledge about geometry and do not necessarily gain further understanding in the pre-service years due to a typical focus on general mathematics that affects the amount of attention on geometry in the classroom. For example, there is evidence that teachers concentrate on the precision of geometrical drawings rather than on the validity of the geometrical constructions. Moreover, the integration of the spatial reasoning components in geometry learning are yet to be fully realized in elementary curricula. By contrast, some secondary teachers might find it challenging to present pedagogical practices that are engaging, especially if they lack an appreciation of the nature of geometry, the foundations of geometry and how to teach these principles and baseline concepts. Educators also face challenges when presented with professional training programs. These include a lack of time to engage in professional learning, a lack of resources to implement new strategies (to release teachers and provide support materials), fear of mathematics and a lack of geometry knowledge, and/or a devaluing of geometry as an important strand/area of mathematics.

Despite these challenges, some professional learning opportunities are showing promise in supporting geometry educators. For both pre-service and in-serving teaching, the opportunity to explore models of active teaching and use and design rich geometry tasks can help support the development of effective pedagogical practices. In-service training in geometry can be deeply supported by more intensive forms of professional learning such as lesson study and collaborative action research. Working directly and consistently with students and teachers in the classroom has also been shown to be effective in building effective practices.

Methodologically, many research-based professional learning programs are guided by a design research approach. This involves the collaborative and continual refinement of materials, tools and meaningful tasks that address learning needs which have been identified in the analysis of student thinking. Using this type of approach, classroom trials ultimately engender refinements that are increasingly scalable.

The IPC invites proposals that address some of the following questions:

- What models or approaches to professional learning with a focus on geometry are proving to be effective in building geometric reasoning, and helping teachers and students to develop understanding of fundamental ideas in geometry?
- What are some effective approaches to designing meaningful tasks (including tasks that involve technology use) that support and reveal geometric reasoning and/or proof building from intuitive ideas, and theoretical understanding of geometric principles? How can these tasks be leveraged to support pre-service and in-service teacher training? How can teachers themselves be supported in task design that promotes geometry reasoning?
- How can pre-service and in-service teachers be supported in understanding the importance of explicit and implicit use of instruments in the learning of geometry? How does this translate to classroom practice?
- To what extent do pre-service and in-service teachers value geometry and how can their epistemology of the subject be reinforced?

Topic C: Resources for teaching and learning geometry

To teach and promote geometry understanding, teachers need to access resources and promote their use with students. These resources may be technological or non-technological in nature. Depending on the teaching context, these resources may operate as constraints or affordances, and be imposed, overabundant or limited. This topic explores questions about the impacts of various resources in supporting the teaching and learning of geometry. The inherent limitations of resources due to external factors are also considered.

Sub-topic C1: Digital technologies in learning and teaching geometry

Seven lenses through which to examine technological tools in teaching and learning of geometry, the development of spatial capabilities, and advancement of geometrical reasoning and creativity are as follows: (1) the types of technology: software vs. hardware; (2) the goals of uses of technology: teaching and learning vs. evaluation and analysis; (3) the level of openness of the tools: open vs. half-baked vs. closed; (4) the educational functions: exploration vs. verification vs. proving vs. demonstration; (5) the context: pure mathematics vs. real/practical situation vs. Virtual Reality; (6) the nature of space and shape —2D vs. 3D vs. … n-dimensional— and how it is represented; and (7) the types of geometry: Euclidean, Projective, Affine, Non-Euclidean geometries.

The tools under consideration could be, for example, dynamic geometry software, 3D spaces, virtual reality, intelligent tutors using AI, learning analytics, videos, programming tools and more. The hardware used in the studies can include, among others, computers, tablets, interactive whiteboards, 3D printers and robots. Information and communication technologies have the power of enhancing visual representations of shapes and space, readily conveying geometrical relations, and connecting multiple representations of concepts which are not possible by the tool of paper-and-pencil; mediating individual (teacher and student) and classroom discourse; broadening the scope of information provided by teachers and textbooks; and promoting students' multimodal, action-based, independent, and self-regulated learning.

Contributions can be about completed or in-progress research or innovation projects. Research can be empirical, experimental, or theoretical. The IPC invites proposals that address some of the following questions:

- What is the impact of using different types of technological tools in relation to one or more lenses proposed above?
- How does the impact of technological tools on geometry education depend on the types of tools, their openness, or their educational functions?
- How do the uses of different kinds of technologies by students, teachers, or teacher educators impact on their practices and thinking in geometry?
- What are some of the opportunities and challenges that are presented by the use of technology in geometry education?

Sub-topic C2: Manipulatives and visual tools in teaching and learning geometry

Visual tools and concrete manipulatives are often seen as important resources in promoting geometrical thinking. A review of the state of the art of research on the contribution of manipulatives to learning and teaching geometry is needed. Are manipulations and visualization sufficient to allow effective learning processes of geometry, including deductive reasoning, proving, and constructing in geometry? How is the use of manipulatives linked to modelling, abstracting, using language to describe geometrical objects, and proving?

Manipulatives and visual tools can be examined from different theoretical perspectives and regarding different goals of the use of manipulatives and visualization: creating experience, identifying critical properties, formulating conjectures, and proving or refuting the conjectures.

The questions posed can be about different types of visualization and objects such as material tools, photographs, video, and drawings. The contributions may address different uses of visualization and objects, depending on the ages of students and the specific difficulties they might have (e.g., blindness, developmental coordination disorder, etc.).

The IPC invites proposals that address some of the following questions:

- What are the links between the use of manipulatives, visual tools, and modelling activity?
- What is the difference between using the manipulatives or dynamic diagrams constructed by educational designers and constructing manipulatives or working in an open dynamic environment (e.g., GeoGebra)?
- What is the role of visual tools and manipulatives on geometrical proof processes?
- How can bridges be created between manipulation and visualization and mathematical thinking in geometry?
- How can students be helped to abstract from perceptive and concrete properties of physical objects?
- What is the place of language, signs, gestures in activities based on manipulatives?

Sub-topic C3: Learning geometry with resource constraints

The scope and focus of this sub-topic address issues relating to the teaching and learning of geometry in resource-constrained classrooms and under-resourced school communities. Resources, in this context, include physical resources (e.g., digital tools and non-digital manipulatives), non-tangible resources (e.g., the language of instruction), and human resources (e.g., specific languages used by visual or hearing-impaired students). Constraints refer to limitations in the use of resources. Three sources of resource constraints can be identified: (1) the resource itself: any physical resource has limitations in the scope of its use, and it is necessary for teachers to be aware of them, so they do not pose tasks to their pupils where the resources are an obstacle more than a help; (2) the users: some students may have difficulties in handling digital or manipulative resources due to some disability, and some teachers may not have adequate training in using resources, with consequences for their pupils' learning; (3) the socio-economic conditions: many schools do not have access to resources for teaching mathematics, so students cannot benefit from their use to help them develop their geometric conceptions.

The IPC aims at composing an overview of the current state of the difficulties encountered by students in resource-constrained classrooms in the learning of geometry by (1) documenting measures and attempts to address the constraints; (2) addressing theoretical issues relating to the teaching and learning of geometry in resource-constrained classrooms; and (3) suggesting ways forward in meeting the needs of students in learning geometry in resource-constrained classroom.

We invite proposals that address some of the following questions:

- What theoretical lenses on geometry learning in resources-constrained classrooms and communities are emerging, and how do these inform the understanding of students' learning of geometry in such classrooms?

- What practices have proved to be effective in resource-constrained geometry classrooms and under-resourced communities?
- How should instructional activities be designed in resource-constrained geometry lessons, in particular for students with disabilities?
- How can the teaching and learning of geometry be improved in resource-constrained contexts where use of software (e.g., dynamic geometry) and manipulatives is limited?
- How are local languages (e.g., indigenous verified mathematics) used as a resource for supporting mathematical access in teaching and learning geometry in multilingual countries?

Topic D: Multidisciplinary perspectives

The multidisciplinary perspectives can be seen as a polysemic interaction between different knowledge fields such as ethnomathematics. One of the perspectives can be related to how teaching and learning of geometry can contribute to the development of other disciplines regarding professional, social, and cultural activities and practices. Another perspective addresses the possible contributions of other sciences such as neuroscience and psychology disciplines, and professional needs, in the process of teaching and learning geometry. The third perspective relates to the roles that geometry may play within different disciplines and knowledge fields. The main goal is to further advance research and understanding in relation to geometric knowledge and its polysemic multidisciplinary connection with other knowledge fields and sociocultural contexts.

Sub-topic D1: Connections between geometry and professional contexts

Geometry has inherent importance. There are also practical implications in professional contexts where practitioners are using geometry consistently. Efforts have been made to examine the role of mathematics in Science, Technology, Engineering and Mathematics (STEM) professions (Van der Wal, Bakker & Drijvers, 2017). With respect to geometry, it is the case that compulsory schooling, initial professional training and workplace training differ in the nature and the emphasis on geometric learning. Some of these connections are obvious while others not. Interest in the way school geometry is taught in schools and colleges and the mathematics needed in a variety of work-related activities has been evident in the literature for some time (see Roth, 2014, for a description of bridging between formal mathematics and enacted practical geometric experiences). Carpenters for example, rely on Pythagoras' theorem (especially the 3-4-5 triangle) in the form of a set square for much of their daily activities. Construction professionals such as engineers, architects and builders use trigonometry to calculate angles and distances routinely in their drawings and interpretation of plans. Creative professionals including fashion designers use symmetry and tessellations to create patterns while town planners utilize Thales' theorem to level ground accurately over long distances. In a recent example, technological developments allow sonographers to measure the head circumference of an unborn baby in various static and dynamic images of 2D-3D space.

This topic seeks research links between school mathematics and applications for on-the-job training and applications in the real world. Such links raise several questions to be researched:

- Where might school geometry be needed in emerging professions and changing workplaces?
- How can school-based geometric tasks be designed to be meaningful for out-of-school professional applications?
- How can professional activities inform the design and development of innovative geometric experiences in the classroom?
- How does geometry needed in initial professional and on-the-job training differ from geometry taught in schools?

Sub-topic D2: Ethnomathematics and indigenous ways of understanding geometry

Cultural artifacts are objects created by members of distinct cultural groups that convey cultural meanings and information about their creators and users. Geometric ideas, procedures, techniques, and practices are related to the development of cultural artifacts that are socio-culturally situated as well as distributed among these members from generation to generation. This approach also includes embodied cognition in which cognitive activities use symbols as external resources that assist these members to develop their mental representations and manipulation of objects. Geometric representations exist in all cultural groups and their meaning may be different from one specific culture to another. It is important to emphasize that ethnomathematics is also a program that studies geometric concepts and theories developed locally as well as it is related to dynamic pedagogical actions that respond to the environmental, social, cultural, political, and economic needs of the students, which enables them to develop their imagination and creativity. Thus, there is a necessity to understand indigenous ways of understanding geometry through ethnomathematics.

The IPC invites proposals that address some of the following questions:

- What role does ethnomathematics play in the understanding of diverse indigenous ways of doing geometry? What are the sociocultural influences of the use of this knowledge in distinct cultural groups? What investigations have been done in this area to date? What has been learnt from these investigations?
- What cultural traits contribute to the construction of local geometric knowledge? Does this lead to the development of embodied cognition and situated learning regarding geometry? How might mainstream/traditional mathematics learn from these constructions of local geometric knowledge?

- In what ways does ethnomathematics contribute to and improve current formal geometry content and teaching in schools?
- How do members of distinct cultural groups extend their cognitive processes beyond the brain, the body, and their immediate environments in order to develop diverse ways of understanding geometry? Does this mean that cognition is not only embodied and situated, but also distributed as evidenced in their ways of creating, processing, accumulating, and diffusing information conjointly, including geometric knowledge?

Sub-topic D3: Contributions of psychology and neuroscience to research in mathematics education focusing on geometry

Cognitive scientists posit that student mathematical processing in general and geometrical processing in particular are determined by general cognitive traits and socio-emotional characteristics. Students' cognitive characteristics such as memory, attention, pattern recognition, speed, and types of information processing are proven to affect student mathematics learning and processing. In this sense, cognition research may inform mathematics education research on the role of general cognition and the role of affective factors in studying geometry.

In the field of Neuroscience, results from research using fMRI, EEG, and eye-tracking studies (as examples) suggest that solving mathematical problems, including geometry problems, are connected to student characteristics. The research also suggests that students with varied levels of mathematical performance exhibit different patterns of brain activity (Ansari, 2016; De Smedt & Grabner, 2016; Grabner, 2022).

Teacher efficacy, attitudes, beliefs, and knowledge in geometry is another related area of robust research activity. For example, in parallel work in Psychology and Mathematics Education, research on student self-efficacy in geometry is aiming to explicitly link confidence, motivation, persistence, anxiety, and comfort with the teaching and learning of geometry.

Beyond discrete disciplinary research, it is in the intersection of mind, brain, and education research where novel contributions are increasingly expanding conceptions of geometry and spatial reasoning learning. For example, the work on mental transformations of 2D and 3D figures has been jointly propelled by research activity in clinical and classroom settings and shows significant promise as a predictor of mathematics achievement but is particularly related to geometric reasoning (see Mix et al., 2016, Uttal et al., 2013). This includes research on transformations, mental mapping, coding, perspective taking, and first-person insertion technologies, for example, that may be subsequently linked to improving the teaching and learning of geometry.

The IPC invites proposals that address some of the following questions:

- What insights from the recent research on spatial reasoning (such as mental rotations) can be integrated into research on geometry thinking? What are the implications for teaching and the resultant learning?
- What affective constructs impact on geometry teaching and learning?
- How can students' cognitive characteristics be taken into account and what are the implications for geometry education?
- How can neuro-cognitive research inform geometry education about individual differences in geometric processing and in learning geometry?
- Is there room for transdisciplinary research and what should it focus on as it relates to geometry teaching and learning?

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THE 26TH ICMI STUDY PRODUCTS

The ICMI Studies are a major activity of ICMI. Their global aims are to contribute to a better understanding of the challenges faced by mathematics education in our multidisciplinary and culturally diverse world and to collaborate in advancing to their resolution. The first ICMI Study was launched in 1980. More detailed information can be found in the ICMI web page (https://www.mathunion.org/icmi/activities/icmi-studies-activities).

The main products of an ICMI Study are a conference and a volume of the ICMI Studies Series. The conference addresses a topic or issue of particular significance in contemporary mathematics education. It is aimed to gather together leading scholars and practitioners specialized in the field of inquiry of the Study, for them to engage in a productive interaction and collaboration to advance in the knowledge of the topics of the Study.

Based on the activity during and results from the conference, the final and most relevant product of an ICMI Study is a volume offering a coherent state-of-the-art view of the topic addressed by the Study. This volume will be part of the New ICMI Study Series.

The ICMI Study Conference

The 26th ICMI Study focuses on *Advances in Geometry Education* and is planned to provide a platform for teachers, teacher educators, researchers, policy makers, and other stakeholders around the world to share research, practices, projects, and reports that advance the understanding of geometry. As in every ICMI Study, the 26th ICMI Study is built around an International Study Conference that is directed towards the preparation of a volume published in the New ICMI Studies Series. In the conference, substantial time will be allocated for collective work and discussion on significant problems within the four topics and the related sub-topics. Outcomes from the conference activity provide the foundations for the Study volume.

The Study Conference will be organized around working groups on the sub-topics presented above. The working groups will meet in parallel during the conference. It is the work of these groups that is captured as chapters in the ICMI Study volume.

Authors should nominate the topic (one of four) and sub-topic in which they would like their paper considered, by examining the content and questions outlined in the topic sections presented above. Contributions are encouraged that are analytical and innovative rather than solely descriptive in nature. It may be the case that interconnections between sub-topics emerge and warrant attention; consequently, papers may be re-allocated by the IPC if beneficial.

Participation

As is the usual practice for ICMI studies, participation in the Study Conference is by invitation only. Proposed papers will be reviewed, and a selection made according to the quality of the contribution, the potential to contribute to the advancement of the Study, with explicit links to the sub-topics proposed in this Discussion Document, and the need to ensure diversity among the perspectives and representation. The number of invited participants is limited to approximately 100 delegates. As part of the invitation process, potential participants will receive a registration code, which will be required to complete registration details. Unfortunately, an invitation to participate in the conference does not imply financial support from the organizers, and participants should be prepared to finance their own attendance at the conference.

The ICMI Study Volume

The first product of the 26th ICMI Study is an electronic volume of the Conference proceedings, to be made available prior to the conference on the conference website and later the ICMI website. The proceedings will contain all the accepted papers as reviewed papers, so they can be cited as a refereed (peer-reviewed) publication with an ISBN number. They are published online only.

The second and main product is an edited volume published by Springer as part of the New ICMI Studies Series. The editing process and content of the volume will be the subject of discussion among the International Program Committee (IPC). It is expected that the structure of the volume will follow the organization and topics set out in this Discussion Document, although some changes might be introduced as a consequence of the discussions raised during the conference. The chapters in the volume will collectively and consensually integrate the outcomes of the discussions of the parallel working groups at the conference, informed by the papers presented. It must be appreciated that there is no guarantee that any of the papers accepted in the Study conference proceedings will appear in the volume. Furthermore, chapters in the volume may be an amalgamation of several presented papers.

A progress report on the Study will be presented at the ICME15 (Sydney, 7-14 July, 2024).

CALL FOR CONTRIBUTIONS

The IPC for the 26th ICMI Study invites submissions of contributions of several kinds, which include research, theoretical, and innovation papers on the four study topics, all of them pertaining to geometry education, namely: (a) theoretical perspectives; (b) curricular and methodological approaches; (c) resources for teaching and learning geometry; and (d) multidisciplinary perspectives.

Authors must select one topic, plus one of the sub-topics listed below each topic —see the *Structure: Topics, Sub-topics, and Questions* section of this Discussion Document. To ensure a rich and varied discussion, participation from countries with different economic capacity and different social, political, or cultural heritage and practices is encouraged.

The IPC encourages early career researchers to submit their paper drafts at least one month in advance of the submission deadline (that is, by August 15, 2023) and request assistance for finalizing their contribution. This assistance concerns its structure or the choice of the paper topic and sub-topic, not the writing of the text or the editing of English language. In this way, the IPC continues a tradition of helping newcomers to the international mathematics education community.

An invitation to the conference does not imply that a formal presentation of the submitted contribution will be made during the conference.

Location and dates

The Study Conference will take place at the University of Reims Champagne Ardenne - INSPE - Reims, France (http://www.univ-reims.fr/inspe/)

Dates[1]: Tuesday 23rd April – Friday 26th April, 2024.

Submission

A template for the submission of papers is available on the 26th ICMI Study Conference website (https://icmistudy26.sciencesconf.org/). Papers have to be written in English (the language of the Study Conference) according to the template instructions, with a maximum of 8 pages.

Deadlines

Submissions must be made online no later than *October 6, 2023*, but earlier if possible. Papers will be reviewed and decisions made about invitations to the conference. Notifications of decisions will be sent to the corresponding/main author from *November 30, 2023.*

Information about the venue, registration, visa application, costs, travel, and accommodation is available on the 26th ICMI Study Conference website (https://icmistudy26.sciencesconf.org/).

Summary of dates:

a. Call for proposals (paper submission): from April 1, 2023
b. Deadline for proposals (paper submission): October 6, 2023
c. Invitations to participate mailed: from November 30, 2023
d. Registration: opens on November 30, 2023, and closes on March 1, 2024

[1] Note: For our Jewish community members, we understand that the timing of the conference over Passover brings about some challenges. Please be assured that the organizing committee will make appropriate arrangements for those who require Kosher for Passover food. Furthermore, we will work with the local Jewish Cultural and Social Association of Reims to ensure all those who need a place for Seder will have somewhere to go.

e. Proceedings published online: March 31, 2024
f. Conference Opening: Tuesday April 23, 2024

Registration and fees

For information about registration fees, procedure of registration, and deadlines, please consult the 26th ICMI Study Conference website (https://icmistudy26.sciencesconf.org/).

MEMBERS OF THE INTERNATIONAL PROGRAM COMMITTEE

Co-chairs

Angel Gutiérrez (Spain, angel.gutierrez@uv.es)

Thomas Lowrie (Australia, thomas.lowrie@canberra.edu.au)

IPC Members

Cathy Bruce (Canada, cathybruce@trentu.ca)

Fabien Emprin (France, fabien.emprin@univ-reims.fr)

Keith Jones (United Kingdom, jones.keith2013@gmail.com)

Roza Leikin (Israel, rozal@edu.haifa.ac.il)

Lisnet Mwadzaangati (Malawi, lmwadzaangati@unima.ac.mw)

Oi-Lam Ng (Hong Kong SAR, oilamn@cuhk.edu.hk)

Yukari Okamoto (United States, yukariokamoto@ucsb.edu)

Milton Rosa (Brazil, milton.rosa@ufop.edu.br)

Manuel Santos-Trigo (Mexico, msantos@cinvestav.mx)

Ex-officio members

Frederick K.S. Leung (ICMI President, Hong Kong SAR, icmi.president@mathunion.org)

Jean-Luc Dorier (ICMI General-Secretary, Switzerland, icmi.secretary.general@mathunion.org)