Chapter 0
Introduction

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0.1 Challenging: a human activity

In modern society, mathematics is a prominent part of the school syllabus. It is praised for its utility and regarded as a foundation of our modern technological society. Yet school mathematics is the locus of much concern and criticism. Many leave school uncomfortable with, if not disdainful of, the subject.

Even those who get good grades may lack fluency and appreciation of its structure and significance. Mathematicians themselves may see little that reflects the character of mathematics as they experience it.

The blame for this situation is often laid at the door of the demands of a rigid syllabus and the imperatives of assessment.

We need to analyze these complaints. Mathematics is a highly structured subject. It is hard to see how one can proceed very far without orchestrating topics and assessing the mastery of its students from time to time to make sure they are prepared to make further progress. But does this mean that mathematical instruction should embrace so much mechanical learning and rely on recall and stock situations?

The issue is really one of ownership—who owns the mathematics?

Too often, the answer is "the system" or "the teacher". From the pupils' perspective, it seems imposed from without to achieve extrinsic goals. For many, it makes little sense. To be sure, mathematics can be difficult, but is it a difficulty one would want to surmount?

In the third book of the "Divine Comedy", Dante's pilgrim is advised by Beatrice that

> convienti ancor sedere un poco a mensa
> però che 'l cibo rigido c'hai preso,
> richiedi ancora aiuto a tua dispensa.

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Apri la merte a quel ch'io ti paleso
a fermalvi entro; ché non fa scienza
sanza lo ritenere, avere inteso.

Or, in English

you must stay longer at the table
because the food you have eaten is tough
and needs time for digestion.

Open your mind to what I shall reveal
and keep it there; whoever has heard
and not retained, knows nothing.

Paradise, Canto V, 37–42

This could be applied to the student of mathematics. She must learn to be patient, open the mind, and seek resilient rather than transient understanding.

These are qualities that require the learner to draw on internal resources. If the student is permitted to be passive, she may become alienated and resist accepting this responsibility.

The task of the teacher and expositor is to present the subject in such a way as to awaken these resources.

Danesi (2000) suggests the key to this conundrum. He notes that "puzzles have been around since the dawn of history" and that people have "been so fascinated by seemingly trivial posers, which nonetheless require substantial time and mental effort to solve, for no apparent reward other than the simple satisfaction of solving them". He asks,

Is there a *puzzle instinct* in the human species,
developed and refined by the forces of natural selection
for some survival function? Or is this instinctual love
of puzzles the product of some metaphysical force buried
deep within the psyche, impelling people to behave in ways
that defy rational explanation?

Danesi sees this propensity for puzzles as tightly bound up in human culture. There is a natural curiosity to be found in every human civilization; the environment poses wonders, threats and opportunities that must be understood and exploited by beings lacking many of the physical advantages of the beasts but endowed with a powerful mind. However, it is not just necessity but choice that leads people to accept challenges, as is borne out by the popularity of such recreations as Sudoku, topological puzzles and games of all sorts.

We should not overlook the social aspects of challenge that involve both cooperation and competition. In sports and the arts, the participant is motivated by a goal—a game or concert—and collaborates with others, a coach or conductor for expert guidance and colleagues for support and inspiration for the achievement of excellence.

The pursuit of the goal leads to the acquisition of new skills.

However, the goal must be commensurate with the abilities and characteristics of the group; one plays with teams of comparable skill or performs musical pieces whose technical requirements can be mastered with reasonable effort and expenditure of time.

Thus, growth is promoted through the presence of an appropriate challenge.

On the one hand, there is an aversion to the aridity of much school mathematics, and on the other a natural attraction to challenge. This Study is predicated on the possibility that the first can be counteracted by an exploitation of the second.

First, we discuss the fundamental question, "what is challenge?"

We shall examine sources of challenge and identify contexts in which expositors can introduce challenge. We will address the fundamental question as to whether challenges enhance learning.

This Study places the issue on the international agenda and has as its purpose to press for an answer to this question and to develop our understanding of the role of challenge.

0.2 Challenges and education

The tendency to see education in terms of formal institutions designed to meet societal goals has become very pronounced in many countries. We are told that children must be educated in order to fit them for various careers, in order that the nation becomes competitive and in order that economic success is achieved.

This is too narrow an outlook. Every society has provided some kind of initiation to prepare its young to accept the demands, responsibilities and privileges of adulthood.

In less sophisticated societies, this has tended also to provide children with a succession of socially useful tasks, so that they maintain a feeling of cohesion with the larger community. This is described in a very graphic way in the early chapters of Haley's novel *Roots* (1976) which recounts in considerable detail the first seventeen years of the life of Kunta Kinte as he is progressively integrated into the Madinka tribe in Juffure (now in Gambia) prior to his enslavement in the eighteenth century.

However, modern education often can separate children from the larger society and insulate children in their own world. So we should think of education in terms of three time frames.

Modern citizens may as a matter of course spend a third of their lives in some kind of school. Accordingly, we have to think of education for the present, a schooling that is immediately rewarding for the student in and of itself, a schooling in which the pupil experiences the joy of learning and growth as well as integration into the adult world through a broadening of interests and points of contact.

Properly handled, mathematics can be part of this. In many mathematical situations, children with their curiosity and mental agility are in a position of equality with adults. In particular, mathematical challenges become not only a way in which they can feel intellectually alive and productive, but also something that can be shared outside of their own age group.

To be sure, schooling must allow students to function as citizens and employees, and to provide opportunities for work that is appropriate and productive (in whatever sense we want to use this word) for the individual. This is education for the near future that will open doors and prepare for the next stage of life. It can be claimed that exposure to mathematical challenges can support the resourceful flexibility of thought and deeper mathematical understanding that will make the preparation of students more successful.

But there is more to life than a job; many seek fulfillment through other activities and jobs. This is often movingly portrayed in descriptions in the press of rich lives of some quite ordinary people who have suffered some disaster.

In particular, through schooling we produce new generations whose interest and curiosity will see that the arts and sciences and all such tokens of human excellence do not vanish, but transcend our individual mortality. This is education for the far future.

This dichotomy between the transitory nature of human existence and the nobility of human achievement is strikingly expressed in the two answers that the Old Testament provides for the question, "What is Man?"

> Man is like to vanity: his days are as a shadow that passeth away.
>
> Ps. 144:4

> Thou has made him a little lower than the angels, and has crowned him with glory and honor. Thou madest him to have dominion over the works of Thy hands; Thou has put all things under his feet.
>
> Ps. 8:5–6

The lesson of the first is to redeem each moment in education, and of the second, to plunge students into the broad stream of civilized achievement and development.

As we shall see in this volume, mathematical challenges have a place in all aspects of education and intellectual growth, whether in a formal school setting or through informal means such as clubs, museums, books, magazines, games and puzzles, or the Internet.

The history of mathematics is a long saga of great thinkers pushing the bounds of their knowledge by formulating and solving problems. It seems clear that they exulted in their growing mastery of ideas as step by step they progressed from the most basic ideas in number and geometry to the magnificent edifice that we know today as mathematics.

This ICMI Study is predicated on the premise that we can duplicate some of this sense of entitlement and mastery among the public, both within and without school, by the use of mathematical challenges.

0.3 Debilitating and enabling challenges

Already, mathematics often challenges both school children and the general population, but for reasons that often discourage and alienate.

Difficulties may result from poor curricular and pedagogical design that shroud in mystery what should be clear and overwhelms with tedious detail what should be surveyable.

Schools may be afflicted with teachers who themselves are uncertain of their mathematics, or whose mathematical training ill equips them to anticipate possible stumbling blocks to learning even basic mathematics.

Sometimes, students get locked into a conceptualization that is not at all productive, so that they cannot move beyond their frustration.

Is it possible for students to learn mathematics and the general public to understand it if they must accept someone else's formulation of it? Or should we bring in lay people as participants, providing a direct opportunity to grapple with the ideas and devise their own patterns of thought?

If the latter, is it through the posing of challenges, problems and investigations, that the learner is brought into a collaboration with the teacher, each alive to his own particular responsibilities?

0.4 What is a challenge?

For the purpose of the Study, we will regard a challenge as a question posed deliberately to entice its recipient to attempt a resolution, while at the same time stretching their understanding and knowledge of some topic. Whether the question *is* a challenge depends on the background of the recipient; what may be a genuine puzzle for one person may be a mundane exercise or a matter of recall for another with more experience.

Furthermore, a challenge may or may not be appropriate. An inappropriate challenge is one for which the background of the recipient is so weak that she may not understand what is at stake or does not possess or is unable to create the tools needed to engage with it. A good challenge is one for which the person possesses the necessary mathematical apparatus or logical skill, but needs to use them in a nonstandard or innovative way.

A good challenge will often involve explanation, questioning and conjecturing, multiple approaches, evaluation of solutions for effectiveness and elegance, and construction and evaluation of examples.

Following Danesi (2002), we can expect a latent willingness for people to accept challenges provided a suitable stimulus can be devised. This is an optimistic message which, as we shall see in the following chapters, has already been validated in many individual situations across the globe. Indeed, we have seen enough instances that we can analyze what is likely to be effective and what is likely to be counterproductive.

We see in *mathematical challenge* an idea that will revitalize discourse about the role of mathematics in the educational culture. In schools, it may help to equip students to face future challenges in life by fostering desirable attributes such as patience, persistence and flexibility, to learn content more richly and exploit connections, to identify and develop their mathematical capabilities, to become self-actualized and confident, to experience the pleasure of engagement and the joy of success and to participate in a community of learning.

We can see through challenge a kind of relationship between a learner and a learning opportunity, mediated by the engagement of the individual. In the diagram below, we list the ingredients in the sphere of the learner alongside activities that could be in the learning environment that affect them.

LEARNER		LEARNING OPPORTUNITIES
development cognition knowledge metacognition beliefs motivation/beliefs values desire to learn and succeed	E N G A G E M E N T	exploring inquiring problem solving developing, valuing, comparing, evaluating multiple approaches explaining generating ideas and questions conjecturing constructing forms judging effectiveness and elegance

The opening chapter provides many examples of challenges. We look at where they come from, what they are about and what makes them work. It is hoped through this chapter to indicate to the educator or teacher in mathematics who may not have an advanced background in the subject the scope of challenges, and to encourage them to think through some of these themselves as a way of honing their ideas about their possible use with students and the public.

We emphasize that good challenges are like good musical compositions or poems; they infiltrate into the broader culture and are passed down from one generation to another.

Some will have broad appeal, while others will be treasured only by the cognoscenti.

The second chapter studies challenges beyond the classroom. Over the last century, particularly in the last fifty years, many different occasions for being challenged mathematically have been introduced, from newspapers and magazines, and an ever-increasing variety of competitions, to the work of artisans in the creation of topological and other puzzles (the Rubik's cube being a notorious example), as well as to mathematics “museums”, clubs and web sites. The scope of these is examined and several examples are studied in detail.

Of course, no examination of this topic would begin to cover it without acknowledging the intensive intervention of technology. Technology not only serves to augment the effectiveness of traditional resources, such as books and journals, lectures and schools, but it also provides a handy and extensive library of information and problems; it provides electronic tools to aid learning, experimentation and understanding on a scale impossible to conceive of until now.

Its novelty and power has made possible brand new interactive programs and the capacity for in-depth investigations in number theory, combinatorics, probability and geometry by students. With the modern computer, new areas of mathematics have been opened out, such as dynamical systems and stochastic processes, and some of this is accessible to students. Moreover, the Internet has made it convenient for mathematicians and students to collaborate easily.

Chapter Three takes stock of these developments, sorting out what is available, assessing how they relate to existing methods of pedagogy and dissemination, analyzing the issues at stake, and describing ways in which they can support the use of challenge.

In Chapter Four, the promotion of mathematical learning through the use of challenges is examined through several case studies. The role of the teacher is critical, for she must seek out appropriate material, calibrate and orchestrate the challenges given, ensure that students are properly prepared to meet them, analyze what makes them successful and see that the classroom environment is conducive to an effective experience.

Chapter Four touches briefly on an area that this author feels has not been given due attention—the use of mathematical fallacies. Two decades ago, he persuaded the editors of the *College Mathematics Journal* to initiate a department, “Fallacies, Flaws and Flimflam”, devoted to the collection of flawed mathematical proofs and solutions in the hope that this might be of use to teachers. His own experience suggests that it can be a serious problem for students to troubleshoot a flawed argument, that the attempt to do so may lead to an appreciation of some rather subtle mathematical points, and that students come to appreciate the need for care. Indeed, the reader may have been in the position of marking a student solution, “smelling a rat”, but being hard pressed to winkle out the error, for good students can make interesting mistakes. Even “howlers”—manifestly incorrect or inappropriate techniques that lead to a correct answer—can lead to a fruitful investigation into the situations for which they may actually “work” and the reasons for this. One source of such material is Movshovitz-Hadar and Webb (1998).

In Chapter Five, the emphasis is on the cooperative facing of a challenge by some community of students and their teachers or mentors, whether it be in a mathematics laboratory, special schools, a school assembly, a classroom or a jamboree where teams compete in the consideration of an experiment or research problem.

The success of any educational regime in the schools depends on a well-prepared corps of teachers. If the use of challenges in schools is to be successful, then the formation and professional development of teachers needs to be suitably reformed. Teachers need to be convinced that what they are required to do authentically represents mathematics and is of lasting benefit to the student. They must be prepared to reassess the way in which they interact with the students.

Most importantly, they must come to see themselves as practitioners of mathematics, sharing their own experience and joy of mathematics. Thus, in their training, teachers need to grapple with mathematical challenges themselves, so that they know how to support their students and can model good mathematical behavior. Such considerations are the burden of Chapter Six, which opens with a discussion of the nature of challenges in school and why they are important. Some examples of challenges, their design, and the responses they elicit are given.

Psychological considerations are important. What is it that might prevent teachers from using challenging problems? What effect does their knowledge and beliefs have on their willingness and ability to handle challenging problems? What can be said about the development of the brain? The chapter next deals with the factors that lead to effective pedagogy and the role of professional development.

Finally the chapter provides a description of some pre-service and in-service programs. In China, the new curriculum that promotes the use of challenging problems has resulted in a Shanghai study, "Teacher Action Education", to promote the effectiveness of teachers in adopting the spirit of the reforms and achieving student learning goals.

In Münster, Germany, cadet teachers directly experience mathematical challenges and have to reflect on their own mental processes; their formation involves teamwork, critical discussion of videotaped lessons and research, resulting in the production of educational materials. Similarly in New Zealand, a numeracy development project brought about extensive professional development.

Likewise, at Northern Kentucky University in the USA, pre-service teachers take a course that embraces work on challenging problems alone and in groups, with sharing of solutions and preparation of examples to be used in schools.

Chapter Seven follows this up with a look inside the classroom. The issue is one of priorities, as teachers have to make sure that they cover the syllabus and prepare students for various tests. Nevertheless, it is argued that challenges should and can be part of the classroom experience. However, in many situations this goes against the grain of the existing culture. Teachers need to examine their own attitudes and be prepared to interact with the pupils in a less authoritarian way.

Those responsible for curriculum design and assessment need to question whether their policies inhibit or promote an authentic and productive mathematical experience in the classroom. This is the theme of Chapter Eight in which assessment issues in the framework of challenge are considered in four countries, and assessment is evaluated as to its role in promoting learning and its implications for curriculum and assessment. The chapter concludes with some research questions needing further investigation.

While challenges have always been part of mathematical exposition in some small way, they have now come to the forefront in our conception of classroom practice and public exposition. This Study has intervened at a time when there has been a lot of activity and experience that can be assessed. The time has come for a gathering of the available materials and the formulation of research and field trials involving the use of challenges that will allow us to move forward in a sound and measured way.

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Chapter 1
Challenging Problems: Mathematical Contents and Sources

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This chapter gives many examples of challenging problems, categorized according to the settings in which they occur, both in and beyond the classroom. Some challenges arise as extensions of normal classroom practice; other challenges, some of very recent origin, become popular among the general population, while still others are created especially for contests for different populations of students. In some cases, a detailed discussion of their origins and uses is provided. It should become clear to the reader the extent to which they betoken the vitality and the creativity of the mathematical enterprise and show how much of a "human endeavor" mathematics is. The final section provides a summary of the place of context and content in the creation and use of challenges. This chapter focuses on challenges given in the form of individual problems; challenges that occur in extended investigations are mentioned briefly in the final section and will be considered in later chapters.

1.1 Introduction

Challenges in mathematics are not new. Once people began to observe numerical and geometric patterns and sought to account for them, difficult problems emerged naturally to challenge their wits and to force them to organize their knowledge and explain the underlying concepts more precisely. During the Renaissance, mathematicians who had discovered a new technique might show off their knowledge by posing a challenge problem that others, not privy to their strategy, could not solve. In the Enlightenment, prizes were offered for those who could make some progress on some pressing mathematical problem of the day.

In modern times, the first challenges aimed specifically at students were posed in magazines and in competitions, with Hungary being an early leader. These

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challenges were created especially for this purpose, so that it became a battle of wits between the mathematicians who set them to confound the students, and the students who might arrive at a solution, often in an unanticipated way.

However, as the popularity of games like bridge, topological puzzles, and Sudoku and other puzzles in the popular press indicate, there is a taste for challenges of a mathematical type among non-mathematicians as well. Indeed, in almost every culture there is an ancient tradition of posing mathematical problems.

It is therefore plausible that both tradition and culture support the introduction of challenges both into the classroom and into public events to enhance the appreciation of and facility in mathematics among both students and the general population.

However, the mathematician or teacher who introduces challenges into an environment must be aware of the particular circumstances. In the classroom, it is important to be as inclusive as possible, while for extracurricular activities, where participation is voluntary, the educator cannot force anyone to take part and must select material carefully to ensure success.

As we shall see in the final section of this chapter, the research literature has devoted insufficient attention to the issue of the appropriate selection of challenges. The aim of the next four sections is to encourage educators to engage and contribute to the discussion of challenges and their use. We provide a set of problems used in a variety of situations and earnestly encourage the reader first to try them, then reflect on the thought processes that they evoke, what they convey about the nature of mathematics and how different types of recipients might respond.

A good challenge can be thought of as a work of art, similar to a poem, a musical composition or a painting. All of these are of no account without an appreciative audience. Accordingly, the creativity and elegance of a challenge should be matched by the discernment to calibrate it so that the challenge entices but does not overwhelm those with whom the challenger wishes to make communion.

1.2 Challenges within the regular classroom regime

Quite a bit of time in the normal classroom is spent in teaching standard results and techniques, and applying them to stock situations. Understandably, pupils may not appreciate the significance of the work nor retain and use it effectively later.

Accordingly, it may be useful to adapt a mundane question to make pupils think more deeply about structure of this mathematics and shine a light on its salient features. Here are some examples:

Challenge 1.2.1 (Ages 6 to 8): Six-year-old Danny is discussing even and odd integers with his father, a research mathematician and a talented teacher. Danny has learnt what even and odd integers are: “It’s an even number of people when they can split up into pairs and no one will be left out, and an odd number if one person doesn't have a partner to hold hands with."

Danny is asked to prove that the sum of two odd numbers is an even number. After some hesitation and mumbling, his face suddenly lights up and he cries out: "Of course, of course, the two people who didn't have a partner, they find each other, they start holding hands, so the number is even!"

Discussion: If it can be said that one purpose of education in general is to encourage the child to be aware of and discriminating towards the world around her, this is particularly true in mathematics. Necessarily, the mathematics syllabus covers a lot of concepts and procedures; it is a hazard that pupils can fall into a mechanical mode, reciting definitions and performing operations while unaware of the significance and utility of what they are doing.

We need to insinuate into the situation something to increase awareness and fluency. The posing of questions designed to make a pupil pause and take stock is one way to do this.

The reader is invited to formulate her own response as to why the sum of two odds is even, and then think about situations in which this can be asked and how one might expect pupils of various ages to respond.

Two aspects stand out. First, in order to answer the question, the responder needs a workable definition of "even" and "odd". By workable, we mean something that can be appealed to. What would be the understanding of a young child? Perhaps it might be that odd and even numbers alternate in counting, putting 1 in the odd pile, 2 in the even, and so on. How would a child with this viewpoint tackle the question? Perhaps the criterion is the remainder upon division by 2. Would an eight-year-old work with this definition? A secondary student with some knowledge of arithmetic modulo 2 might use this approach. Or would children be likely to follow Danny in pairing off?

The second aspect is that the pupil is being asked to prove a general proposition with infinitely many instances. Normally, one thinks of proofs as occurring much later in the school career, perhaps not until high school geometry. Is a child of Danny's age likely to be "up to" handling such a proof?

That the sum of two odds is even might be accepted through empirical observation—every example shows that it is true. In the same way, every robin has a red breast. Some children might not progress much further than this. But this particular challenge puts on the table a new and different aspect of mathematics. Accompanied by similar future challenges, a child's perception of mathematical truth and the power of reasoning to illuminate what can never be checked directly will be deepened.

Thus, we see how a challenge can originate from the desire of a teacher to induce her charges to probe more deeply into the mathematics.

It is this feeling of elation (Danny's response) that one should try to induce in setting challenges before classes.

Challenge 1.2.2 (Ages 9 to 11): Take the digits 1, 2, 3, 4, 5 and 6. Using each exactly once, form two numbers for which the difference is as small as possible.

Discussion: Generally, children master skills by working examples provided by the teacher. But one might better create a sense of ownership by asking children to create examples of their own. To begin with, a teacher might simply ask pupils to form two numbers from the six digits and find their difference. However, by introducing the optimization question, she provides a goal that induces the pupils to look more carefully into the ingredients of the situation and pay attention to what factors govern the size of the difference.

There are a number of realizations that students will reach, probably implicitly. The first is that to make the difference small, the numbers chosen should both have three digits. The teacher may wish to draw out explicitly why this is so.

The second is that the leading digit of the larger number (minuend) exceeds the leading digit of the smaller (subtrahend) by 1. The third is that the digits after the leading one of the subtrahend should form as large a number as possible while the digits after the leading one of the minuend should form as small a number as possible. The answer is given by $412 - 365 = 47$.

This is an example of what a teacher can do with a simple computation. The set of digits could be varied and both the largest and smallest sums, differences, products, quotients and exponentials formed by two numbers using those digits can be found. Another nice problem that lends itself to group work is to take the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 and use them to construct three numbers, the largest of which is the sum of the other two. What are the smallest and largest such sums?

1.2.1 Challenge from observation

Sometimes a challenge can be built on an interesting observation. There are likely many pupils who have noted with interest that the sum and product of two 2s are the same.

The teacher can exploit this by asking pupils to find other examples of pairs of numbers that have the same sum and product. The numbers involved can be just positive integers, or rationals.

Here is another way to generalize the property:

Challenge 1.2.3 (Ages 10 to 15): Determine all possible ways of finding two pairs of positive integers such that the sum of each of the pairs is equal to the product of the other.

Discussion: Take a few minutes to think about this "two pairs" problem. Do we need to know something about the relative sizes of the sum and the product of two positive whole numbers? How easy is it to generate an example? How would we know that we have a complete set of examples?

In tackling this challenge, one makes the key observation that for one of the pairs, the sum must be at least as great as the product. It may be that some children have never considered this possibility, having dealt only with situations for which the opposite is true. The number 1 might not have been used as a multiplier. We can see the possibility of counteracting a bias that multiplication always makes things larger, a prejudice that can later confound a child's handling of products of numbers other than integers.

Pre-algebra students may solve this problem informally and intuitively using trial and error. With algebra, a more systematic attack is possible. The condition that $a + b \geq ab$ can be converted to $(a - 1)(b - 1) \leq 1$, which for unequal positive integers a and b implies that one of the variables is equal to 1. Or one can draw from the conditions $a + b = cd$ and $ab = c + d$, the equation $(a - 1)(b - 1) + (c - 1)(d - 1) = 2$. The only possibilities are $\{(2, 2), (2, 2)\}$ and $\{(2, 3), (1, 5)\}$.

The challenge can be extended. What happens if we allow negative integers? Or we could ask that the product of each pair is twice (or some other multiple of) the sum of the other.

1.2.2 Challenge from a textbook problem

Sometimes a challenge can be made from a standard textbook problem that will encourage students to look beyond a merely algorithmic approach to a more holistic stance towards a problem.

Challenge 1.2.4 (Ages 11 to 14): (This problem appeared in a first algebra text.) A man is standing in a theatre line. 5/6 of the line is in front of him and 1/7 of the line is behind him. How many people are in the line altogether? Without setting up an equation, argue that the answer must be 42.

More generally, we can pose this situation. A man is standing in a theatre line with the fraction x of the line in front of him and the fraction y behind, where x and y are fractions written in lowest terms. If x and y are such that the problem makes sense, then that answer must be the least common multiple of the denominators of x and y.

Discussion: This problem was originally posed in a graduate course on problem solving for practicing secondary teachers, as part of a discussion on the creative use of textbook exercises. Is it clear that the answer must be 42? Would the reader have realized this without being prompted? Certainly, one can set up an equation and solve it; the exercise came from a text chapter on this very topic. But does the algebraic formalism reveal aspects of the situation that are worth noting?

The key observation is that, in the original problem, the number of people in the line must be divisible by 6 and 7, so that the numbers before and behind are integers. So the total number of people is a multiple of 42. Why must it be 42 itself, rather than a larger multiple? Expressing the reason clearly and completely is an expository challenge for even the brightest students.

1.2.3 Increasing fluency with fractions

While we are on the topic of fractions, there are some challenges that can be used to help students become more fluent with them and gain some understanding of inequality relations among them.

A good area for this Study is that of Egyptian fractions, those whose numerators are 1.

Challenge 1.2.5 (Ages 11 to 15): Solve the following equation for the natural numbers x, y, z:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

Discussion: There is an obvious solution for this: $(x, y, z) = (3, 3, 3)$. Some pupils may also know about $(x, y, z) = (2, 3, 6)$. The challenge here is to get an exhaustive set of solutions.

The first task in meeting this challenge is to reduce the level of complication by taking advantage of symmetry. Without loss of generality, we can assume that $x \leq y \leq z$. Making this additional assumption is a mathematical gambit that would probably not be taught as a regular part of the syllabus, but one does not require much experience in problem solving to see how such assumptions become standard.

Ordinary syllabus problems also tend to be directive in the sense that the student follows a standard procedure and goes directly after the solution. This particular challenge illustrates that it is often a good idea to get an overview of the situation before beginning the grind of hunting for solutions. We note that the integers required cannot all be very large. In particular, if all the integers exceed 3, the left side must be less than 1. Also it is clear than $x > 1$.

So we have to consider two cases: $x = 3$ and $x = 2$. The first yields only one possibility. The second requires that $y = 3$ or $y = 4$, so we can now list all the solutions, the two mentioned above and $(x, y, z) = (2, 4, 4)$.

If students become interested in this challenge, it can be generalized to having any number, say n, of unit fractions on the left. If there are n fractions on the left, we might ask how large the largest denominator can be.

Challenge 1.2.6 (Ages 11 to 18): Determine all the integers n exceeding 2 for which there exist distinct positive integers x, y, z such that

$$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

Discussion: It is a conjecture of Paul Erdös that $4/n$ admits such a representation for all $n \geq 3$. This problem works well with secondary students in particular because most cases can be handled within a short space of time.

Like many challenges, progress depends on making a key observation, in this case that

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}.$$

Many students discover this after playing around a bit. This allows us to dispose immediately of the case that n is an even number: i.e. we can write $n = 2m$. Thus,

$$\frac{4}{n} = \frac{2}{m} = \frac{1}{m} + \frac{1}{m} = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m(m+1)}.$$

It turns out that generally within an hour a class can arrive at a set of cases that cover every number n that does not leave a remainder 1 when divided by 24. This is a nice example that illustrates how one can make some progress on a natural mathematical question, yet find that some parts of the question remain intractable.

1.2.4 Engaging with algebra

For many secondary students, the study of algebra is quite tedious. The fact that nowadays a large percentage of the population studies high school mathematics and many of these find frustration with algebra probably accounts for its downgrading in the syllabus.

However, the result is that many students do not engage in algebra to a degree that permits fluency and ability to make actual use of it. Without this ability, it is easy for students to see their studies as pointless.

Some students have no desire to study technical mathematics, and, if their ambitions lie in another direction, they should not be penalized for avoiding it. The experience of the remaining students can be enriched if we restore to the standard syllabus more challenging material.

A standard complaint about many students of mathematics is that they tend to be driven by formulae and do not appreciate the structure behind a formula. Factoring polynomials is one way to ameliorate this deficiency.

There is a "trapdoor" aspect to factoring. It is a mundane task to expand the product of two polynomials; to factor this product into its irreducible components requires a range of skills and sensitivity to structure that will help students mature as mathematicians. Here are a few examples:

Challenge 1.2.7 (Ages 13 to 17): Factor

(a) $4(a^2+b^2) + 21b^2 - 20ab - 36$;
(b) $6x^2y - 15y - 5x + 18xy^2$.

Discussion: While there is no grand scheme that allows us to factor any polynomial, nevertheless there are some rules of thumb that students will pick up from experience and allow them to factor with increasing success. Success will also depend on recognizing certain forms and analyzing the structure of the polynomial to be factored. For example, in (a), the student might combine the terms in b^2 and note the difference of two squares, of a linear polynomial and of 6. In (b), one might isolate the terms of like degree and pull out common factors.

We can observe an analogous situation in elementary calculus, where differentiation of functions can be handled by a set of easily learnt rules, but integration requires more skill and judgment.

Challenge 1.2.8 (Ages 13 to 17): Factor

(a) $a^{10} + a^5 + 1$;
(b) $2(x^5 + y^5 + 1) - 5xy(x^2 + y^2 + 1)$.

Discussion: Part (a) is a nice challenge in which the "breaking down" approach is unlikely to lead to success. A good start is the observation that the polynomial has the form $x^2 + x + 1$, and recognizing that this in turn is a factor of $x^3 - 1$.

So we build up to factor $a^{15} - 1$, which has the given polynomial as a factor. However, this binomial is not only a difference of cubes but also a difference of fifth powers, so that it can be factored according to two different strategies:

$$a^{15} - 1 = (a^5 - 1)(a^{10} + a^5 + 1) = (a^3 - 1)(a^{12} + a^9 + a^6 + a^3 + 1).$$

At this stage, the students need to understand the significance of the result that every polynomial is uniquely given as a product of irreducibles. In particular, $a^2 + a + 1$ is an irreducible factor of $a^3 - 1$, and so must be a factor of $a^{10} + a^5 + 1$.

Indeed, the required factorization is

$$(a^2 + a + 1)(a^8 - a^7 + a^5 - a^4 + a^3 - a + 1).$$

An alternative approach recognizes that all the roots of the given polynomial are those 15th roots of unity that are not 5th roots of unity and that two of its roots are imaginary cube roots of unity.

Part (b) provides a challenge of a different sort. In this case, one can work from the symmetry in x and y. Making the substitution $s = x + y$ and $p = xy$, we find that the polynomial can be rendered as

$$\begin{aligned}2[s(s^4 - 5s^2p + 5p^2) + 1] - 5p(s^2 - 2p + 1) = 10p^2(s + 1) &- 5p(2s^3 + s^2 + 1)\\ &+ 2(s^5 + 1),\end{aligned}$$

which has $s + 1$ as a factor. The desired factorization is

$(x+y+1)(2x^4-2x^3y+2x^2y^2-2xy^3+2y^4-2x^3-x^2y-xy^2-2y^3+2x^2-xy+2y^2-2x-2y+2)$.

Challenge 1.2.9 (Ages 15 to 19): Find the smallest possible value of

$$f(x) = \cos 2x - x\cos x + x^2/8 \text{ for all real } x.$$

Discussion: Here is a challenge designed to help rid students of the habit of blindly following an algorithm without paying attention to any special characteristics of the situation. One can see how the problem was created using a square involving cos x and then made more mysterious by converting $\cos^2 x$ to its equivalent involving cos $2x$.

This is a "wolf in sheep's clothing" sort of challenge. The student tries a standard derivative approach and gets into a mess. How can this be avoided? Noting the middle term, a mixture of x and cos x, one might recall that cos $2x$ can be written in terms of $\cos^2 x$ and see if we can get a perfect square somewhere. Indeed

$$f(x) = 2(\cos x - x/4)^2 - 1.$$

1.2.5 Pedagogies to help development

The foregoing classroom challenges indicate how one can start with straightforward material, and by providing either a twist or asking a natural question, can help authenticate the mathematical experience. In addition, for this to be effective, we need a pedagogy that does not force students to engage with these on their own unsupported, as has so often happened in the past.

Students can often be asked to work in groups so that they can share ideas and can be allowed time to reflect on the problems, knowing that they cannot always be expected to answer questions immediately. Any decision to introduce challenges in the classroom also requires that the whole system of teaching and assessment be reviewed so that different aspects of the classroom experience are not working at cross purposes.

1.2.6 Combinatorics

In addition, there are problems not immediately connected with the syllabus that can be used in the classroom to good effect.

Combinatorics is an area in which little specific background is needed for some situations and children can be expected to be as successful in meeting a challenge as an adult. We discuss a few possibilities.

Challenge 1.2.10 (Ages 9 to 15): In a supermarket there are different goods that weigh from 1 kg to 40 kg (all the weights are integers). The boss who likes mathematical problems decides to buy only four different counterweights for weighing any of the goods. Will he succeed? Which counterweights should he buy?

Alternatively: What is the minimum number of counterweights sufficient to weigh all goods with integer weights between 1 kg and 40 kg inclusive, with a conventional balance and two pans?

Discussion: This has broad appeal. One approach is to simplify the situation, either by reducing the number of weights available or by trying to weigh goods from 1 kg up to a smaller number of kilograms. The key to a successful prosecution of the problem is to realize that counterweights can be used in the same pan as the goods to be weighed. Thus, students might note that two counterweights of 1 kg and 3 kg will weigh goods from 1 kg to 4 kg inclusive. Extending to three and then to four weights, students might come up with the table:

Number of counterweights	Counterweight	Weight of goods
1	1	1
2	1, 3	1, 2, 3, 4
3	1, 3, 9	1–13
4	1, 3, 9, 27	1–40

Now the students can come to a conclusion. In the elementary grades, students might not have the tools to justify their answers rigorously, but they will be able to understand that their answer is correct. This particular challenge invites a generalization to n weights; by looking at the data, some students will be able to conjecture what the situation will be.

1.2.7 Geometry

Geometry is another area that lends itself to challenges that can be approached by the general school population.

Challenge 1.2.11 (Ages 11 to 17): Given four distinct points in the plane, there are six distinct pairs of them and each pair determines a distance between the two points. In general, one would expect the six distances so obtained to be all different, but sometimes some of them are equal. Find all the configurations for which the six distances involve at most two different numbers.

Discussion: One issue that arises with this challenge is for students to realize what is being asked—that is, possibilities that are *essentially* different. Most children realize intuitively that two configurations are not to be distinguished if one can be obtained from the other by an isometry or similarity, but some do have a difficulty with this point.

This challenge has the agreeable characteristic that most classes can find a possibility quite readily: the four points are at the vertices of a square. In all there are six possibilities, and pupils experience more or less difficulty in finding them all. One is a 60-120 rhombus and there are three more for which three of the points are at the vertices of an equilateral triangle. The one that is generally missed is 72-108 isosceles trapezoid. However, in the case of one class, the *first* example given was the trapezoid. When asked how this was arrived at, a student responded that he took a regular pentagon and deleted one of the points.

Challenge 1.2.12 (Ages 11 to 15): (a) A unit square $ABCD$ is partitioned into four regions by means of line segments AE and BF where E is the midpoint of BC and F is the midpoint of CD. Suppose that AE and BF intersect at P. Determine the areas of the four regions, BPE, $PECF$, $PFDA$ and ABP.

(b) A unit square $ABCD$ is partitioned into nine regions by means of segments AE, BF, CG and DH where E, F, G and H are the respective midpoints of BC, CD, DA and AB. Determine the area of the middle region.

Discussion: Here are challenges designed to break a particular mindset among pupils, to wit the automatic application of formulae and making of calculations. Teachers who find their pupils calculating distances and trying to use the standard area formulae should challenge them to see if the structure of the situation can be exploited.

Once students take this perspective, they progress quite rapidly. In the case of (a), they do not arrive at sufficiently many relationships to complete the solution:

$$[APB] + [BPE] = 1/4$$
$$[PECF] + [BPE] = 1/4$$
$$[APB] + [APFD] = 3/4.$$

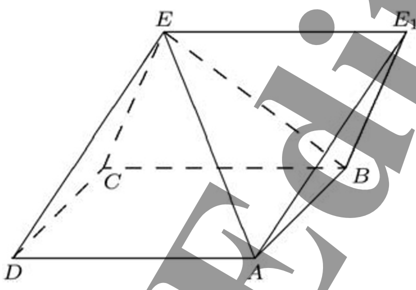
The additional bit of information that is needed is to observe that triangles APB and BPE are similar with factor 2, so that $[APB] = 4[BPE]$.

Problem (b) can be solved by folding out small corner right triangles onto the adjacent trapezoids and converting the figure to a cross of five squares. However, in one class, an interesting issue was raised by a student, that is, whether the middle region was indeed a square. This was a question that needed an answer, and it turned out to be a significant challenge for the class to find one. When they were persuaded that the figure was most probably a square based on the symmetry, it became a non-trivial task to actually describe that symmetry.

Challenge 1.2.13 (Ages 12 to 17): A regular tetrahedron $ABEF$ has a common face ABE with a square-based regular pyramid $ABCDE$ with apex E, where all edges of both figures have the same length. This conjunction of the two solids yields a new polyhedron. How many faces does it have?

Discussion: This problem is a misfire on a multiple-choice examination in 1980 from the US Educational Testing Service (#44 on 3CPT1). It was evidently thought that the correct answer was 7 (the conjunction of the two solids obliterates one face of each, so that the resulting solid inherits three faces from the tetrahedron and four from the pyramid); some takers of the test objected that the answer should be 5. Indeed, after the conjunction, two faces of the pyramid are coplanar with two respective faces of the tetrahedron.

That this is true is not obvious, especially if one conceives of the tetrahedron sitting obliquely like a carbuncle on a face of the pyramid. However, a change in perspective makes the matter clearer. Translate the pyramid along the edge DA so that we get a second pyramid $ABB_1A_1E_1$ that abuts the first along the edge AB.



All edges have unit length, and we note that EE_1 has unit length (being the distance of translation). Thus, $ABEE_1$ is a tetrahedron all of whose sides have length 1, and so must be regular and congruent to $ABEF$.

This example shows that a problem can be made challenging if a straightforward way of solving it does not work; students are encouraged to look for a different approach. See Barbeau (2000, pp. 61–62), Brown and Walter (1990) for further information.

1.2.8 Other settings for school challenges

Some challenges for pupils can be suggested from the events of everyday life, applications and puzzles.

Challenge 1.2.14 (Ages 11 to 17): On Tuesdays after school, Ivan has tennis practice near the Taganskaya metro station, whereas his school is near Kievskaia. Since both metro stations are on the circular line and are diametrically opposite, Ivan got into the habit of taking the first train arriving at the Kievskaia station independent of its direction (clockwise or anticlockwise).

After a while, he noticed that he would ride in the clockwise direction on average about three times as often as in the anticlockwise direction. He could not understand why this was occurring, as he knew that the interval between trains in either direction was always two minutes and the train schedule on Tuesday afternoons was always the same. Why did this occur?

Discussion: This problem was given in a mathematics circle for 15- and16-year-olds whose participants were learning elements of probability theory. At first, the students could not understand why this was possible. They insisted that the events of going in either direction were equally probable. Finally, one of them figured it out. The anticlockwise train apparently always arrives at Kievskaia 30 seconds after the clockwise train, he noticed, and then the next clockwise train comes 90 seconds later.

This problem (in company with Challenge 1.2.12) illustrates how students can be stymied by jumping to a conclusion while neglecting some possibilities. In this case, implicit in the problem, but not spelled out, is the possibility that two recurrent processes with the same period can have different phases.

In mathematical research, as well as in real life, in order to achieve something, care must be taken to analyze and understand the situation, place it in an appropriate context, make sure that possibilities and assumptions are clear, and carefully formulate it. It is rather rare that problems are perfectly stated. Challenges with ambiguities, vague wording and gaps generate the need to work with students so that they can appreciate what the challenge really involves. This is an important facet of mathematics education, both for those students who will become professional mathematicians and for those who will use their mathematically trained minds to overcome real-life problems.

One practical problem is that of college admissions. Students apply to a list of colleges, ranked according to preference; colleges admit students ranked according to some criterion. The colleges hope to do this with a minimum number of reversions wherein the students who are accepted pass up the invitation in favor of a superior offer. A simplified version of this is the Marriage Problem (Gale and Shapley 1962):

Challenge 1.2.15 (Ages 15 to 18): There are an equal number of men and women. A matchmaker has the task of pairing them off into married couples, but he must do this in such a way that there is no incentive for a mixed pair to cheat. In other words, he must avoid a situation in which some man prefers someone else's wife to his own and at the same time that other wife prefers him to her own husband. Can this always be done? If so, describe a procedure that will achieve this.

Discussion: This is a fine problem for secondary students, as it involves no technical mathematics whatsoever, just imagination and basic reasoning. Students can be walked through some specific examples involving small numbers of men and women to get a feel for the situation. However, the perhaps surprising result is that an algorithm to find an appropriate pairing always exists and is straightforward to describe.

Briefly, the process involves several rounds. Each man and each woman ranks the individuals of the opposite sex in order of preference. In the first round, each man proposes to the top woman on his list; and each woman provisionally accepts the proposal of the most preferred suitor. If each woman gets a proposal, then the process terminates and the pairing has the desired property. Otherwise, there is at least one woman without a proposal and at least one man who has been refused.

At the beginning of subsequent rounds, each man crosses from his list all those women who have refused him, and proposes to the top woman remaining. Any woman receiving a proposal provisionally accepts the most preferred among any individual provisionally accepted in previous rounds and her suitors on this round. This continues until the process terminates with no more refusals (as it must do); the provisional acceptances become permanent and a pairing is obtained that satisfies the condition.

Of course, there is another, possibly different, solution where the women do the proposing.

Challenge 1.2.16 (Ages 12 to 18): A magician has 100 cards numbered from 1 to 100 inclusive. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers of the chosen cards. Given this sum, the magician identifies the box from which no card has been taken.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put in a different box.)

Discussion: Even though this problem was posed on an IMO examination (2000), like the $4/n$ problem of Challenge 1.2.6 it has a simpler component that makes it suitable for an ordinary class. While it is no easy task to determine all possible allocations of cards to boxes, it is possible with a little imagination to envisage such possibilities.

One can only speculate on how the discovery of the result of the problem was made, or whether this problem was constructed especially for competition or arose out of some research result. In any case, it is attractive because of its natural setting and the ease with which it is posed.

Some students may realize that there is nothing special about the number 100 and perhaps experiment with fewer cards to get a feel for the situation. So this challenge has the advantage that the student can get into it right away. Doubtless, some students relying on their experience would realize that one could sort the cards according to residues modulo 3, putting all cards with numbers congruent to 0 in the red box, congruent to 1 in the white and congruent to 2 in the blue. Some might arrive upon the solution of putting card 1 alone in the red box, card 100 alone in the white box and the rest in the blue.

Challenge 1.2.17 (Ages 15 to 18): Determine a number which at once (a) is divisible by 2007, (b) has 2007 digits, (c) ends with 2007 and (d) has 2007 as the sum of its digits.

Discussion: The observation that $2007 = 9 \times 223$ suggests a simple strategy: take 2007 occurring 223 times in the digits (making sure that the number starts and ends with 2007) and then fill up with 0s. For example, 20070...[1115 times]...02007...[222 times]...2007.

One can get more elaborate examples. Since $2007 = 9 \times 223$, we just need to arrange that the sum of the digits is a multiple of 9, the number is clearly divisible by 223 and has the requisite number of digits. We include the digits in order 223 $9x$ times and 2007 y times so that $7(9x) + 9y = 2007$, or $7x + y = 223$, and then fill in with 0s. We can take $x = 31$ and $y = 6$ and have the number 223...[279 times] ...2230...[1146 times]...02007...[6 times]...2007.

This is a nice challenge of a somewhat different character than the previous ones, as it is usable with a wider audience. Its attractiveness lies in its rather symmetric statement with respect to 2007. Students could be challenged to find more "interesting" examples.

One way in which pupils may experience the "thrill of the chase" is in the detection of patterns. While formally, mathematics is all about establishing results, some of the hardest work is done in isolating what it is that one really wants to prove.

This depends on vigilance, experimentation and sensitivity to patterns. For example, a pupil seeing the numerical equations $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$ may be encouraged to wonder whether these are but two of a whole flock of similar or analogous relations. What "similar" and "analogous" means is a matter of judgment. Sometimes, it is not always the case that one's expectations of a generalization are satisfied, or perhaps satisfied in the way one may expect. Consider the pair of equations: $3^2 + 4^2 = 5^2$ and $3^3 + 4^3 + 5^3 = 6^3$. Another example of a pattern that does not continue is $1! = 1!$, $1!3! = 3!$, $1!3!5! = 6!$, $1!3!5!7! = 10!$.

There is another way to proceed from the basic 3-4-5 Pythagorean relationship.

Challenge 1.2.18 (Ages 11 to 17): Verify that the following numerical equations are true:

$$3^2 + 4^2 = 5^2$$

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2.$$

These are the first three equations in a sequence of equations. What do you think the next two equations should be?

Discussion: This has been tried a number of times quite successfully. There are many ways that the students can try to extend the pattern. It is an early observation that the number of terms on each side of the equation increases by one with each equation and that there is one more term on the left side than on the right. Usually, students focus on the leading term of the left side, often by checking first differences. However, some notice that $3 = 1 \times 3$, $10 = 2 \times 5$, $21 = 3 \times 7$, which leads to a conjecture for the leading term of the fourth equation. Occasionally, students can extend the pattern knowing only the first two equations.

While a formal proof of the general equation requires a bit of skill in setting it up, verifying a particular case in an insightful way gives pretty convincing evidence that the general case is likely to be true. The only background needed is the result that the sum of the first n odd positive integers is n^2.

Thus, for example,

$$\begin{aligned}
&(25^2 + 26^2 + 27^2) - (22^2 + 23^2 + 24^2) \\
&= (25^2 - 24^2) + (26^2 - 23^2) + (27^2 - 22^2) \\
&= 49(1 + 3 + 5) = 7^2 \times 3^2 = 21^2,
\end{aligned}$$

where a difference of squares factorization is used on the second line.

To find the 57th (say) term in the sequence, note that it will have 58 terms on the left side, 57 on the right side. Since $(57 + 58)^2 = 115^2 = 13225 = 6612 + 6613$, and $57 \times 115 = 6555$, we get the equation

$$6555^2 + \ldots + 6612^2 = 6613^2 + \ldots + 6669^2,$$

whose truth can be checked by imitating the verification of the third equation given above.

Pattern challenges of this type not only enhance the propensity of students to be observant and resourceful, they tend to work well with students of varying abilities since there are often several ways of describing the pattern. For more advanced classes, they pose the additional challenge of providing a proof of conjectures made.

1.3 Challenges in popular culture

In the introduction, reference was made to popular culture as the source of challenges. Some problems go back hundreds of years. For example, many cultures have a version of the following boat crossing problems:

(a) A man has a wolf, a goat and a cabbage, and wishes to cross a river. However, the size of the boat requires that he can take at most two of his possessions with him. If he cannot leave the wolf alone with the goat, or the goat alone with the cabbage, explain how he can make the crossing.

(b) Three couples (husbands and wives) come to a river and wish to cross. The only boat available can hold at most two people. If no woman can be left in the presence of a man unless her own husband is also present, explain how the couples can cross the river.

Other popular types of problems involve having a number of vessels of varying capacities, some full of liquid, with the requirement of obtaining a fixed volume of liquid. For example, three ungraduated beakers can hold 19, 13 and 7 liters of fluid; the 13-liter and 7-liter beakers are full, the 19-liter beaker is empty. By pouring liquid from one beaker to another, obtain exactly 10 liters.

However, new challenging problems are appearing all of the time. The most obvious recent manifestation is the popularity of Sudoku puzzles, which are carried in many newspapers and magazines around the world. Another problem that has attracted a great deal of public attention is the Monty Hall (Car-and-goats) problem:

Challenge 1.3.1 (Ages 12 to 110): A game show host tells a contestant that he wins the prize behind one of the doors A, B, C that the contestant selects. The contestant is told that behind two of the doors there is a goat and behind the remaining one there is a car. The contestant naturally would prefer the car. The contestant points to door A. However, before revealing what is behind door A, the host opens door C to reveal a goat. He then invites the contestant to consider if he wishes to stick with door A or switch his choice to door B.

Since now there is a goat behind one of the doors A or B, and a car behind the other, it appears that it does not matter whether the contestant switches. What would you advise?

Discussion: While some form of this problem has been in existence for more than a century, it came into prominence about two decades ago after appearing in a syndicated column, "Ask Marilyn", published in weekly supplements of US papers. Readers sent in their questions to Marilyn Savant. This was one, and Savant's answer was hotly contested by many of her readers, including several mathematicians. A history of this problem and the controversy is described in the references below.

The problem has become part of the educational regime and has appeared in many professional development sessions and textbooks. Those that immerse themselves in the problem often resolutely hold fast to their opinion, so the teacher needs to be careful to let the discussion evolve so that students are not squelched by the dead weight of authority (which was attempted by several mathematicians who responded to the original column of Marilyn Savant). For further references see Barbeau (2000, pp. 86–90) and Berlov et al. (1998).

Challenge 1.3.2 (Ages 10 to 110): You are given twelve billiard balls and an equal-arms balance. The twelve balls look identical and eleven of them have exactly the same weight. The twelfth has a different weight. Using the balance a minimum number of times, determine which ball has the different weight from the rest and whether it is heavier or lighter.

Discussion: This problem is at least sixty-years-old, and was passed around Toronto schoolyards in the 1940s. It is a sort of problem that children can get into as it admits a lot of trial and error. Indeed, it is likely that some people will have a good instinct as to how to select the balls at each weighing.

However, it does yield to a more systematic and reasoned approach. First, one can argue that at least three applications of the balance are required. There are 24 possible “states of the world”, as there are 12 possibilities for the odd ball and two possibilities as to its weight relative to the others.

Each application of the balance has three possible outcomes according to whether the pans balance or the left or right pan goes down. Thus, each outcome of the balance splits the number of outstanding possibilities into three categories.

After the first balancing, one result will leave at least eight outstanding possibilities to be tested. After the second, there may be at least three. If it can be arranged that there are no more than three, then the third balancing might be contrived to distinguish them.

This realization allows one to devise a strategy for solving the problem. Thus, the first balancing should compare one set of four balls with another set of four. It can be seen that whatever happens will be consistent with exactly eight of the possibilities. The second weighing should narrow the number of cases to at most three.

1.3.1 Another schoolyard problem

Another “schoolyard problem” from the 1940s has recently made a reappearance.

Challenge 1.3.3 (Ages 10 to 110): Three men go into a hotel and pay \$10 each for a room. After the men go upstairs, the desk clerk realizes that he made an error and should have charged only \$25 for the three of them. Accordingly, he calls over a bellhop and asks him to return five dollars to the men. As he climbs the stairs, the bellhop figures, “What the heck? Those three guys won’t know the difference. I’ll give them one dollar apiece and keep the other two.” That is what he does. So in the end, each man pays \$9 for the room—that’s \$27, and the bellhop gets \$2—that’s \$29. What happened to the other dollar?

Discussion: This problem is a trap for the facile thinker, and requires of the solver a willingness to carefully analyze the status of the various disbursements. Of course, the two dollars kept by the bellhop along with the twenty-five kept by the desk clerk made up the twenty-seven dollars ultimately spent by the three individuals.

1.3.2 A Russian problem

Of more recent vintage is the following problem, purported to be from Russia as a test for talented students. However, this exquisitely contrived problem deserves broader public exposure.

Challenge 1.3.4 (Ages 12 to 110): A table that is free to rotate has four deep wells embedded in its surface, symmetrically placed at the vertices of a square.

Inside each well is a drinking glass, either upright or inverted, but not all oriented the same way.

The wells are deep enough that you cannot see their contents. The table rotates and stops at random. You are permitted to place your hands in up to two of the wells, determine the state—upright or inverted—of each glass, and change the state of none, one or both of them. This is repeated. Your task is to ensure that the glasses are eventually all in the same state—all upright or all inverted. A bell sounds at the moment the task is successfully completed.

Is it possible to ensure success? If so, how can it be achieved?

Discussion: The implicit answer is that the task can be completed, although it is far from obvious to many that this is so. After all, one might reason, there may be a well that one never has the opportunity to visit.

The apparent randomness of the outcome of each rotation of the table seems to confound an orderly approach. However, in order to make progress, the solver needs to realize three things. First, it is only required that the glasses end up the same way, not that they end up in a particular state. So if one well is never visited, one should try to arrange things so that all the other glasses have the same state as that in the unvisited well.

Secondly, one in fact does have distinguishable choices at each move—either select adjacent wells or diagonally opposite wells.

Thirdly, the sounding or non-sounding of the bell actually does give information upon which inferences can be made.

Now the problem can be solved in a straightforward way, as there are only two ways to make each move. After the first two moves, if the bell does not sound, one can always arrange that three glasses are upright and one is inverted. It turns out that at most three more moves are needed for success.

Astonishingly, if one tightens the rules to deny to solver the ability to determine the state of the glasses, but only to specify the two wells to be visited and the final state of the glasses in the well, the problem can be solved with a maximum of seven moves.

1.3.3 The Microsoft problem

The next problem is known as the Microsoft problem as it is said to be used by Microsoft interviewers to determine suitable candidates for employment.

Challenge 1.3.5 (Ages 12 to 110): Four men and a flashlight (with a weak battery) are out for a walk on a very dark night. They come to a bridge, which is so rickety that at most two can cross at a time and for which one needs to carry the flashlight to see one's way. One man can cross the bridge in one minute, the second takes a minimum of two minutes, the third a minimum of five minutes and the fourth a minimum of ten minutes. What is the minimum amount of time required to get all four men from one bank to the other?

Discussion: It is clear that at least three crossings of the bridge are necessary, with one person bringing the flashlight back after the first two crossings. This problem has the treacherous feature that most people accept without question that the fastest man returns with the flashlight so the others can cross. It is thus easy to lock into the answer of 19 minutes for three crossings, where the lengths are determined by the slower three men in turn (accompanied by the quickest) and two returns by the quickest.

Salvation comes only with the realization that the one returning with the flashlight after the second crossing need not be one of the two who has just crossed the bridge. Thus, we can have the two slowest cross together (to save time in the forward direction), provided we have arranged that the two quickest are already on the other side to cover the two return trips with the flashlight.

For school use, the problem can be generalized to the following: Replace the numbers 10, 5, 2 and 1 by a, b, c and d respectively, where $a > b > c > d > 0$. What is the condition on the variables that the obvious solution of sending the quickest back with the flashlight each time is in fact the optimum solution?

1.3.4 A problem from children's literature

A recent trend in problems is the occurrence of logical problems where people guess the color of a hat that is placed on their heads when they can see the hats on all heads but their own. One such problem has in fact made it into the children's literature (Nozaki and Anno 1985).

Challenge 1.3.6 (Ages 10 to 110): There are three people and five hats, two of which are green and three are red. The people are seated in a ring, so that each can see the heads of the other two, but not her own head. Three of the five hats are placed at random on their heads. Every ten seconds a gong sounds. At the sound of the gong, all those who are certain of the color of their hats raise their hands (but do not speak).

Is it possible for each person eventually to determine the color of her own hat?

Discussion: At first blush, it seems that it is not possible for anyone to be sure of her own color when both green hats are not placed. But it is important to realize that, as in the rotating table problem with the bell, the lack of an action may give useful information. If two green hats are placed, the remaining person immediately infers her own hat is red and raises her hand, the others then realizing that this sudden decision is possible only when both wear green. If no hand is raised after the first gong, then there are either three red hats placed or two red and one green hat.

Since, at this stage, everyone knows that there is at most one green hat placed, anyone seeing such a hat will raise her hand. And each person can use logical reasoning to identify their hat.

1.3.5 A probabilistic element

An interesting twist on the hat problem introduces a probabilistic element.

Challenge 1.3.7 (Ages 12 to 110): Three knights have rescued the beautiful daughter of the king from a fierce dragon. In return, the king has offered the three a chance to win a vast fortune. He tells the knights that he will place them in a circle and place on the head of each, either a green or a red hat. He tells them that, while each can see the hats of the others, no one sees his own hat. Furthermore, there must be absolutely no communication among them.

When he claps his hands, at least one person must speak and all speakers must identify the color of their own hats. If so, they will win the fortune. If anyone is mistaken, the fortune is lost.

At this stage, it seems that the best that can be done is for one knight to speak and he will have an even chance of being correct. However, the king gives them a chance to improve the odds in their favor. He tells them that they can confer ahead of time and agree on a strategy for speaking. Is it indeed possible to devise a strategy that will allow the knights a better than even chance of winning the fortune?

Discussion: It seems counterintuitive that the odds can be improved. However, the knights can arrange that their probability of bagging the fortune is 3/4. Simply have any knight who sees the same color on the heads of his two companions guess the opposite color for his own hat. This strategy loses only if all three have the same color hat.

1.3.6 Concluding comments

All of these modern problems have the potential to become classics.

They are constructed with a minimum of complication, but often have a paradoxical character to them. It may be that what is required seems impossible or counterintuitive, or that there is insufficient information. Sometimes it is easy to overlook a possibility or to jump to a conclusion, based on an assumption that is not supported by the problem.

In each case, the problem deserves respect for its elegance and the creativity of its (often unknown) author. But what is most attractive is a kind of communion with the poser whose wits are to be matched. The gauntlet is thrown down; is the solver equal to the challenge?

1.4 Challenges from inclusive and other teacher-supported contests

As will be discussed in Chapter 2, competitions can be classified either as inclusive or exclusive. Inclusive competitions are sourced from outside the classroom and written at possibly many schools and by up to hundreds of thousands or even millions of students. Examples of these are the AMC competitions held by the Mathematical Association of America, the Canadian Mathematics Competition held by the University of Waterloo, the Australian Mathematics Competition, the European Kangaroo, and the Challenges run by the UK Mathematics Trust.

There are also other competitions for the broader, but still talented, student population, which are again conducted by an organization outside the classroom. These are written within schools under the supervision of teachers, who receive support materials to help them deal with anticipated questions. One example of this type is the Mathematics Challenge for Young Australians, conducted by the Australian Mathematics Trust, and in which students have several weeks to explore problems under teacher supervision.

In these competitions, the assumed syllabus does not go beyond what is taught in the classroom. However unlike TIMSS and PISA, which are designed to test the classroom knowledge of students with "curriculum-bound" problems not normally regarded as challenging, these competitions will contain problems in which students use their knowledge to explore problems to which they can relate.

This can lead to the students bridging their knowledge with reasonably accessible tools such as the pigeonhole principle, proof via methods such as contradiction, cases or invariance, Diophantine equations, enumeration techniques, graph theory and discrete optimization.

The more talented among these students tend to become independent learners, though many continue to be mentored by teachers or professors from outside the classroom. Some progress to Olympiad competitions, which will be discussed in Section 1.5.

In this section we will discuss and illustrate the skills developed with examples drawn from such competitions as those listed above and the International Mathematics Tournament of Towns, an international competition based in Russia, which is also discussed in Chapter 2.

1.4.1 Diophantine equations

Diophantine equations, which are linear equations with integer solutions, provide an excellent extension path for secondary students. The following problem is taken from an inclusive competition and had a good response.

Challenge 1.4.1 (Ages 13 to 15): Red rose plants are for sale at \$3 each and yellow ones for \$5 each. A gardener wants to buy a mixture of both types (at least one of each) and decides to buy 13 in total, with more yellow ones than red ones. The number of dollars he spent could be

(A) 51 (B) 67 (C) 65 (D) 58 (E) 57.

Discussion: Because of the finite nature of the problem, the student could canvass all the possibilities, working out the amounts when the number of yellow flowers varies from 7 to 12 inclusive (yielding all odd numbers between 53 and 63 inclusive). However, the problem could be done algebraically. Here, the students are challenged to define suitable variables and construct the necessary functions. Logical thinking, along with strict attention to the conditions of the problem, will lead students of average ability to the solution (which is borne out by the statistics of the competition).

1.4.2 Pigeonhole principle

This elementary idea (also discussed in Chapters 6 and 7), thought to have been first articulated as such by Dirichlet (see Chapter 7) and often known as Dirichlet's principle, is simply a statement that if there are pigeons to be placed into pigeonholes, and there are more pigeons than pigeonholes, then some pigeonhole will contain more than one pigeon. The statement can be extended to cover cases where the number of pigeons is more than double, treble, and so on, the number of pigeonholes, requiring the existence of pigeonholes with at least three, four, and so on, pigeons inside. The following is an example of an accessible problem whose solution is best wrapped up using this idea.

Challenge 1.4.2 (Ages 13 to16): Ten friends send greeting cards to each other, each sending 5 cards to different people. Prove that at least two of them sent cards to each other.

Discussion: The words "at least" are the ones which give the experienced student the clue that the pigeonhole principle will be useful here. However the student lacking such experience might ask how many routes from sender to recipient are possible. Since each of the ten friends can send to nine others, there are 90 available routes. However, each pair of friends is involved in 2 routes, so that there are 45 pairs. If more that 45 cards are sent, then by the pigeonhole principle, two of the transmissions must be on the same route in opposite directions. In this case since each student actually sends 5 cards, there are 50 transmissions altogether and thus two friends do send cards to each other.

Such challenges can generate discussion as to other situations where the pigeonhole principle is applicable, such as in combinatorics, number theory and geometry. However, while a useful tool, it does require special circumstances for its application. Two problems can look quite similar. One can be handily dispatched by the principle and the other can be very difficult indeed. It requires judgment and insight to detect when the principle can be used and to identify the "pigeons" and the "pigeonholes".

1.4.3 Discrete optimization and graph theory

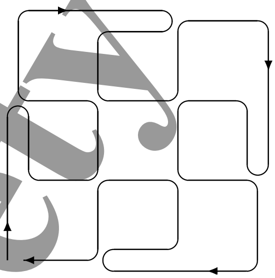
Discrete optimization is quite a different skill than that found in calculus. The standard method, which should be applied in an optimization problem with integer variables, involves two steps, one showing existence, and the other showing optimality, that is, giving an argument to show that the proposed solution cannot be exceeded. The following example, from the International Mathematics Tournament of Towns, is one in which there is a nice use of Eulerian graph theory, which is also a useful tool in networking problems.

Challenge 1.4.3 (Ages 15 to 18): A village is constructed in the form of a square, consisting of 9 blocks, each of side length l, in a 3 by 3 formation. Each block is bounded by a bitumen road.

If we commence at a corner of the village, what is the smallest distance we must travel along bitumen roads, if we are to pass along each section of bitumen road at least once and finish at the same corner?

Discussion: This problem is also an excellent interactive classroom problem. Students can try for some time to improve their first results until everyone is convinced they have a result which cannot be beaten.

The shortest route does turn out to be of length $28l$, with existence shown by the diagram.



The optimality part of the proof is a little more difficult, requiring the graph theory reminiscent of the Königsberg bridges. It is noted that there are three types of node, those with 2, 3 and 4 joining lines. In the case of the even-numbered ones a shortest path can go through them optimally, with an inward route matched by an outward route. However there are eight nodes with three joining lines, which means they have to be visited twice, involving an apparent wasted visit. Assuming they are in four pairs and that an extra route can be shared between a pair, there are at least four extra routes required. Since there were 24 routes in any case, the shown route travelling along 28 links is optimal.

Diagrams: Composers of problems such as this often face the issue of whether or not to include a diagram in the problem statement. The level of challenge can be quite different for such problems dependent on whether or not a diagram is provided. In inclusive competitions and in classroom use diagrams are more common. In advanced competitions such as the International Mathematical Olympiad, constructing the diagram is usually part of the challenge and diagrams are rarely if ever provided in the description of the problem.

1.4.4 Cases

Quite often experimentation with a situation leads to a conclusion that a result can only be established after an exhaustive consideration of a mutually exclusive set of cases. This method is usually known as proving by cases. The challenge is two-fold. First, one needs to identify the cases that might apply and to describe them in a way that is clear, efficient and preferably non-overlapping. Secondly, one needs to ensure that the cases are exhaustive, that nothing is left out. This can be illustrated by the following problem, one of the more challenging problems from the Australian Mathematics Competition.

Challenge 1.4.4 (Ages 15 to 18): The sum of n positive integers is 19. What is the maximum possible product of these n numbers?

Discussion: This problem is also excellent for classroom interaction. Students can try to obtain maximum products with various selections but soon discover that high numbers adding to 19 don't seem to help, while at the other extreme the number one is also useless. Students will eventually see that a summand m bigger than 4 can always be replaced to good effect by 2 and $m - 2$. They will also see that 4 can always be replaced by 2 and 2. They will also be able to formally dispose of the case of an integer being 1. So they are left to consider only sums that use the numbers 2 and 3. The situation is finally resolved by noting that replacing any three 2s in the sum by two 3s will increase the product.

In this problem, students might ask whether the number 19 plays an essential role, or whether the same argument is available when this number is replaced by some other. A common practice in competition problems is to ask students to look at a particular instance of a general result; a student who is aware of this may often establish the general result, often by a more effective argument as the solver is not distracted by irrelevant factors particular to the situation.

A secondary challenge for a more senior student studying calculus is to decide whether there is a continuous version of the challenge and formulate it exactly.

1.4.5 Proof by contradiction

One of the most famous proofs in mathematics, that the square root of 2 is irrational, is made by contradiction and is accessible from school mathematics. It is often seen that a direct proof promises to look very complicated and these are the occasions to try contradiction. The following problem, taken from the

International Mathematics Tournament of Towns, is most easily solved by contradiction.

Challenge 1.4.5 (Ages 15 to 18): There are 2000 apples, contained in several baskets. One can remove baskets and/or remove any number of apples from any number of baskets. Prove that it is possible to have an equal number of apples in each of the remaining baskets, with the total number of apples being at least 100.

Discussion: This is hardly in the form that a student might encounter at school, and the initial challenge is to figure out what is being asked. The indeterminacy of the situation and the variety of possibilities for removal of apples and baskets boggles the mind. An efficient way to control the situation is to suppose that the result is false. As the reader will see in the solution, within this large contradiction are a number of smaller ones to be negotiated.

Suppose that we have a configuration of apples and baskets for which the result fails to hold. It does not change the problem if we assume that all empty baskets have initially been removed. Then the number of remaining baskets does not exceed 99; otherwise, we could leave an apple in each basket and get a contradiction.

In a similar way, we see that number of baskets with at least two apples cannot exceed 49; the number of baskets with at least three apples cannot exceed 33, and so on. We now estimate the number of apples in the baskets. This cannot exceed

$$
\begin{aligned}
99 + 49 + 33 + 24 + \ldots &< 100(1 + 1/2 + 1/3 + 1/4 + \ldots + 1/99) \\
&< 100(1 + 1/2 + 1/2 + 1/4 + \ldots + 1/64) < 100(7) = 700.
\end{aligned}
$$

However, this contradicts the assumption that there are at least 2000 apples.

1.4.6 Enumeration

Combinatorial problems are popular in challenges because they can be less dependent on classroom knowledge and therefore be fair ways of identifying potential problem solvers. Enumeration is a popular source of such problems. Enumeration problems, properly set, can be solved in the time allocated and they have the advantage of challenging the student later to try to generalize, to enable similar problems to be solved from an algorithm.

A good example is the derangement class of problems, of which the following problems from the Australian Mathematics Competition are good accessible examples.

Challenge 1.4.6 (Ages 13 to 15): In how many ways can a careless office boy place four letters in four envelopes so that no one gets the right letter?

Discussion: It is possible for a junior high school student to list and count all the cases. Then the student might find the answer if there had been five letters instead of four. If she looks at higher cases too difficult to count, she might find the famous derangement formula.

Challenge 1.4.7 (Ages 13 to 16): In the school band, five children each own their own trumpet. In how many ways can exactly three of the children take home the wrong trumpet, while the other two take home the right trumpet?

Discussion: This is a variation of the derangement problem in which some matchings are correct and others incorrect. It is possible again for students to count the cases, as the statistics from this inclusive competition showed, and to look for generalizations.

1.4.7 Invariance

Discovering an invariant in a problem can lead to a simple resolution of an otherwise intractable problem. The method of invariance applies in a situation where a system changes from state to state according to various rules, and some property which is important to the statement of the problem remains unchanged in each transition.

This method is very well illustrated by the following famous problem from the International Mathematics Tournament of Towns, not just for its mathematical properties, but for other various associated aesthetic features.

Challenge 1.4.8 (Ages 15 to 17): On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet, they both simultaneously change color to the third color (e.g. if a grey and brown chameleon meet they both become crimson). Is it possible they will all eventually be the same color?

Discussion: If the original number of chameleons had been 15 of each color, it is clear that pair-wise choices of chameleons of the same color pairs would lead to 45 chameleons all of the third color. However with this starting configuration all attempts to obtain the same result fail. The student needs to find a basic property of the starting numbers 13, 15, 17 which remains unchanged during every meeting of two chameleons of different colors. In fact, no two of the three numbers of colored chameleons leave the same remainder upon division by 3.

1.4.8 Inverse thinking

Sometimes there can be useful challenges involved by thinking in the inverse direction. Here is a problem from the Mathematics Challenge for Young Australians.

Challenge 1.4.9 (Ages 14 to 16): A Fibonacci sequence is one in which each term is the sum of the two preceding terms. The first two terms can be any positive integers. An example of a Fibonacci sequence is 15, 11, 26, 37, 63, 100, 163, ...

- Find a Fibonacci sequence which has 2000 as its fifth term.
- Find a Fibonacci sequence which has 2000 as its eighth term.
- Find the greatest value of *n* such that 2000 is the nth term of a Fibonacci sequence.

Discussion: Generally one thinks of a Fibonacci sequence in the forward direction. Here, as is common in an inverse thinking scenario, instead of being given the data and then finding the results, we are given the results and are asked to find the data. It is a challenge for students to think this way.

The student can do this by searching through various second-last terms and working back. In doing so, depending on which term they choose, they can work back uniquely but some choices will not go back far. If the second last term is less than 1000, the third last term is greater than 1000 and that is as far as we can go, as the next term would be negative. We do not do much better if the second last term is too high.

The student can eventually focus in on a small range of values for which the sequence can be traced back a few terms, and then finally the one which goes back optimally. The Golden Ratio can be discovered in extended thinking of this problem, which makes a nice surprise.

1.4.9 Coloring problems

There is a famous problem in which an 8 by 8 checkerboard has it top-left and bottom-right squares removed. One is then asked whether 31 dominoes (1 by 2) can be placed over the remaining 62 squares. At first sight this can seem a tantalizing problem with the student trying for some time to show an arrangement. However, as with the chameleon problem, a solution is not reached and the student is left wondering why not. In the end, the reason is obvious. Each domino necessarily covers one square of each color in the normal checkerboard coloring scheme. However the two squares removed are of the same color, leaving an imbalance.

The following problem, taken from the International Mathematics Tournament of Towns, is an extension of this idea.

Challenge 1.4.10 (Ages 14 to 18): A 7 by 7 square is made up of sixteen 1 by 3 tiles and one 1 by 1 tile. Prove that the 1 by 1 tile lies either at the centre of the square or adjoins one of its boundaries.

Discussion: This problem has a rather surprising result and at first sight, with all the combinations possible, seems almost impossible to prove. But an extension of the domino question above, coloring with 3 colors instead of 2 and looking at the resultant way in which a 3 by 1 domino might cover squares of the board, makes the problem accessible.

1.4.10 Concluding comments

The common feature of all these problems is that no difficult calculations are needed anywhere, which helps to ensure that students in a normal class can reach out to the problem from their normal experience. All of the problems require the discipline of clear thinking, which will enhance the general problem-solving capacity of the student.

Major advances in our society often emerge from the disciplined solution of apparently simple problems. For example, Euler's analysis of the bridge problem led to a higher level of knowledge which is central to modern technology today. For him to have been able to develop an idea beyond what was known is similar to the challenges which people continually face on a day-to-day basis.

1.5 Challenges from Olympiad contests: Students independent of classroom teacher

There are students who wish to go beyond mass contests. They have to become more independent, as their teachers often cannot support them adequately. Usually they participate in national Olympiads and a fortunate few participate in the International Mathematical Olympiad. Calculus is not involved in these contests.

However, students are expected to be fluent in concepts of proof, and to learn in detail advanced topics in geometry, algebra, number theory and combinatorics. A complete list of IMO problems from 1959 to 2003 can be found at www.kalva.demon.co.uk/imo.html.

The teachers who organize and coach students for contests of this kind are highly dedicated and proficient mathematicians, who are themselves expert creators and solvers of problems and skilled at leading students forward.

The posing of challenges is most refined when the audience is specialized. Many appear in problems sections of mathematical journals, for which the solvers must have significant skill and experience.

While problems have to be relatively elementary for mass contests (and there is a particular talent in setting problems that are at once interesting and accessible to an ordinary school student), it is at the level of national and international contests that the setting of challenges is most demanding and rewarding.

On one side, problems committees often consist of mathematicians and teachers of some eminence, and on the other, the solvers are students with a great deal of talent and natural intuition augmented by a technical sure-footedness and breadth of knowledge in the traditional contest areas.

Since both coaches and students can find a wealth of material on the Internet, there is a premium placed on problems that are new, interesting and yet not so abstruse as to bar almost everyone from attempting them.

Challenge 1.5.1 (Ages 13 to 20): Let $A = 4444^{4444}$, B be the sum of the digits of A, C the sum of the digits of B and D the sum of the digits of C. What is D? (IMO 1975).

Discussion: In principle, the problem is trivial. Simply work out what A is, sum the digits and continue until you are finished. In fact, in one class where this was tried, one student used the mathematical software Maple to compute A; the result ran to several pages of digits. However, this is not only impractical but highly prone to error.

If this problem is to be solved within the time allotted for a contest without electronic aids, the question has to be reworked. Can we find out bits of information about D that will allow us to deduce its value, even without knowing exactly the value of A, B and C? An experienced student will recall that one invariant under the summing of digits of a number is the remainder modulo 9 ("casting out nines"), so once this remainder is known for A, we know it also for D.

It is clear that $A > B > C > D$; the question is how fast is the decrease. Since $A < 10{,}000^{4444} = 10^{17776}$, we see that A has fewer than 20,000 digits so that the value of B is less than $9 \times 20{,}000 = 180{,}000$, and the value of C is less than 46. This makes D no bigger than 12.

Even though this problem is of Olympiad caliber, it also works well in a class where the teacher is in a position to prompt the students towards the winning strategy. The value of problems of this type is that students learn that when one cannot enter by the front door, there is often a back entrance that can be used.

Challenge 1.5.2 (Ages 14 to 20): We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating. (IMO 2004)

Discussion: This problem is inviting in that one can begin by looking at particular examples. One can eliminate multiples of 20. However, as expected for a problem in the IMO, the problem was difficult.

One solution was provided by an Australian student who proceeded by a number of steps, showing in turn that powers of 2, powers of 5, twice powers of 5 and numbers not divisible by either 2 or 5 have alternating multiples. Any number not divisible by 20 is equal to a multiple of one of these. The crucial lemma is that, if a has an alternating multiple and b is divisible by neither 2 nor 5, then ab has an alternating multiple.

Challenge 1.5.3 (Ages 14 to 20): Twenty-one girls and twenty-one boys took part in the mathematics competition. Each contestant solved at most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove that there was a problem that was solved by at least three girls and at least three boys. (IMO 2001)

Discussion: Combinatorial problems of this type are standard fare on high-level competitions. While the easier ones can be handled with a few basic principles, such as the pigeonhole principle, more advanced ones require a lot of combinatorial experience on the part of both the setter and the solver. The initial solution of this one was quite complex, requiring at least two pages and double summations, but in the end a simpler solution was found.

Challenge 1.5.4 (Ages 14 to 20): Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P. [IMO 2006]

Discussion: This result is easy when the polygon is a quadrilateral or regular with any number of sides. It thus becomes a natural conjecture for convex polygons in general. It is attractive, simply stated and quite believable.

In fact, this is a really tough problem; it was the final problem on the IMO. It is further discussed at imo2006/dmfa/si/problems.html . The solution can be found at imo2006.dmfa.si/imo2006-solutions.pdf .

Challenge 1.5.5 (Ages 12 to 20): A young man walks into a 7-Eleven store and asks for four items. The assistant tells him that his bill is 7.11 euros, since the product of the four prices is exactly 7.11. The young man explains indignantly that one is supposed to add together the four prices, not to multiply them. “Oh, dear!” exclaims the shop assistant, who then sums the four numbers. But, can you imagine, the right sum turns out to be 7.11 too. How much did each item cost? [Swedish, Challenging problems, 2003/2004]

Discussion: This is a particularly difficult example of a type of problem that can be given in a public venue. In principle, the solution is straightforward, a matter of “search and destroy”. The trick is to set up the structure so that this can be done efficiently and quickly. If there are too many cases, then one runs a high risk of either making a mistake or leaving something out.

The use of decimal fractions may be for some a confounding feature, so one can convert the problem to one of cents rather than euros. This leads to the pair of equations for the prices:

$$a + b + c + d = 711 = 3^2 79$$
$$abcd = 711(10^6) = 2^6 3^2 5^6 79.$$

Now it is a matter of gathering evidence. Exactly one price is a multiple of 79 (one of 79, 158, 257, 316, 395, 514, 593, 632) and at most three prices are multiples of 5.

It is not possible for three prices to be a multiple of 25, so one of them must be a multiple of 125 and, clearly, not at the same time a multiple of 79 (giving the possibilities 125, 250, 375, 500). This reduces the field of possibilities, but the analysis still requires some care and proficiency.

The answer is (1.20, 1.25, 1.50, 3.16) in euros.

Challenge 1.5.6 (Ages 15 to 20): Solve the equation nontrivially

$$8\cos x\cos 4x\cos 5x = 1.$$

(Barabonov et al. 2002)

Discussion: Trigonometric problems such as this one are always enticing for the connoisseur, as they are particularly suited to challenge a student who must select from a wealth of identities those that will suit the purpose. An obvious trial solution by inspection is to let x be a multiple of $\pi/6$, but attempts along these lines lead to -1 rather than $+1$ as a value of the left side. This "red herring" makes the problem more delicious.

However, a stroke of inspiration is still possible. If one hits upon the idea of making $\cos 4x = 1$, then we should try $x = \pi/12$. The left side becomes

$$4\cos\pi/12\sin\pi/12 = 2\sin\pi/6 = 1,$$

and we have a solution.

A more systematic approach involves using the product-sum conversion rules. Since

$$2\cos x\cos 5x = \cos 4x + \cos 6x,$$

the equation becomes

$$4(\cos^2 4x + \cos 4x\cos 6x) = 1,$$

from which $x = \pi/12$ is an easy guess.

Alternatively, we could render the equation as

$$2\cos 2x + 2\cos 8x + 2\cos 10x + 2 = 1$$

and make the substitution $t = \cos 2x$ to obtain

$$0 = (4t^2 - 3)(8t^3 + 4t^2 - 4t - 1)$$

and thus find a solution. Thus, we have a well-constructed problem with a multiplicity of solutions. However, the cubic factor has at least one real root between 0 and 1, so there are evidently other solutions not so easily described.

Challenge 1.5.7 (Ages 15 to 20): Prove the inequality $a^2 + b^2 + c^2 \geq 4S\sqrt{3}$ where a, b and c are the sides and S is the area of a triangle.

Discussion: Like the previous problem, this too involves trigonometry, but in a geometric setting, so that one can look to various areas such as algebra, coordinate geometry or vector geometry for solutions. The solver has a variety of tools and must make a productive selection. Because of this, a class of students may produce many different solutions (one class found nine different ones), so that there is the additional challenge of finding one that is as elegant as possible. Some relied on standard inequalities such as that of the arithmetic and geometric means, while others used some more obscure results from trigonometry. However, students with an elaborate approach may face difficulties in working out the details, while the solutions that are the most basic seem not only more natural but have the bonus of avoiding awkward technicalities.

For example, one of the briefest starts with a rendition of the inequality in vector form

$$\mathbf{a}.\mathbf{a} + \mathbf{c}.\mathbf{c} + (\mathbf{c} - \mathbf{a}).(\mathbf{c} - \mathbf{a}) \geq \mathbf{2}\sqrt{\mathbf{3}}\mathbf{a} \times \mathbf{c},$$

where the origin of vectors is at B while $\mathbf{a}$ and $\mathbf{c}$ represent respectively the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. This leads to the scalar equation

$$a^2 + c^2 - ac \ \cos B \geq \sqrt{3}ac \ \sin B,$$

which is also obtainable using the Law of Cosines and a standard area formula. The difference of the two sides can be manipulated to

$$(a - c)^2 + 2ac(1 - \sin(B + \pi/6)),$$

which is clearly nonnegative. Equality occurs if and only if the triangle is equilateral.

Challenge 1.5.8 (Ages 15 to 20): What digit does the number $M = (\sqrt{2} + 1)^{500}$ have in the 100th position after the decimal point?

Discussion: While this problem may be a challenge to a novice contestant, it has become such a standard type that the experienced solver will know exactly the trick involved. The sum of the number M and its surd conjugate is an integer. As the surd conjugate is less than 1, it becomes merely a technical task to complete the solution.

Challenge 1.5.9 (Ages 13 to 20): Solve the equation $x^2 + 4[x] + 3 = 0$, where $[x]$ denotes the largest integer that does not exceed x.

Discussion: Without the greatest integer function, this would be a standard quadratic equation. The challenge is created by "breaking the pattern". While students may have met the greatest integer function, its appearance in this setting is novel and unbalancing. The challenge comes from the combining of apparently incompatible notions.

How to solve it? Perhaps one might begin by checking for integer solutions: $x = -3$ and $x = -1$ both satisfy the equation. To get other solutions, we need to narrow down the field. If x is not an integer, we have $[x] < x < [x] + 1$, so that

$$x^2 + 4x + 3 > x^2 + 4[x] + 3 > x^2 + 4(x - 1) + 3 = x^2 + 4x - 1.$$

This allows us to deduce that $[x]$ must be either -4 or -5 and obtain that x is either $-\surd 13$ or $-\surd 17$.

Thus there are four solutions. The presence of the greatest integer term breaks the rule of quadratics having but two roots, and a curious student might be challenged to investigate the number of possible solutions equations of this form can have.

Challenge 1.5.10 (Ages 13 to 20): Find the area of the set in the plane determined by the inequality

$$(y^3 - \arcsin x)(x^3 + \arcsin y) \geq 0.$$

Discussion: A student may begin by sketching the graphs of the curves, noting that, because of the functions involved, x and y must both lie between -1 and $+1$ inclusive. The challenge here is that the standard approach is simply unmanageable and another perspective is needed. Denoting the set by M, the crucial observation is that a 90 degree rotation about the origin takes M to its complement in the square, so that its area is half that of the square.

This challenge is similar to Challenge 1.2.13, in which a new way of looking at the situation makes the solution easy. An easier version with the same idea is to determine the area under the graph of $y = \sin^2 x$ for $0 \leq x \leq \pi/2$; in this case, consider a half-turn about the point $(\pi/4, \tfrac{1}{2})$.

Challenge 1.5.11 (Ages 13 to 20): Given a circle with diameter AC, draw a chord BD passing through a given point F in AC so that the quadrilateral $ABCD$ has the largest possible area.

Discussion: Many challenging problems are based on what might be called a "hidden target" principle. Roughly speaking, one is required to find the value, but the only way to solve it is to find a different value and then connect it to the given one. This problem given to the 1980 Moscow Mathematical Olympiad by I.F. Sharygin illustrates this principle. One can see how it might have been created by working backwards from a simple problem, a common technique in the creation of competition problems.

First, note that if BD is a diameter, then F is the center of the circle and the result is straightforward. Henceforth, assume that F lies strictly between the centre of the circle and C.

The crucial idea is to pass from the area of $ABCD$ (which is inconvenient to deal with) to the area of triangle OBD, where O is the centre of the circle. After this "change of target", the problem is almost standard. Observe that

$[ABC]$:$[OBF]$ = AC:OF = $[ADC]$:$[ODF]$, so that $[ABCD]$:$[OBD]$ = AC:OF. Thus, instead of maximizing $[ABCD]$, we maximize $[OBD]$. Since the sides OB and OD are constant, the largest area corresponds to the largest value of the sine of the angle BOD. This is 1 when $OF/OC \geq \sqrt{2}/2$, and is also attained when BD is orthogonal to AC.

1.6 Content and context

A mathematics educator, in giving students a mathematical problem, attempts to solve a pedagogical one. Different examples of mathematical challenges have been presented in this chapter. Such challenges may be used in the most diverse pedagogical situations. Consequently, analysis should and must focus on the interaction between their pedagogical and mathematical aspects, both with respect to their creation and with respect to their application. In other words, it must focus on the interaction between mathematical content and pedagogical context.

In general, considerable attention has been devoted in the scholarly literature to the influence of various parameters that shape and determine the context in which mathematical education takes place. (For example, cultural-historical parameters have been addressed in D'Ambrosio 1990, and Leung et al. 2006, while the effects of the students' range of abilities have been examined in Sheffield 1999.) However, there has been a dearth of discussion of the concrete practices involved in selecting mathematical assignments for various educational situations. Recently, this topic has attracted increasing attention, as evidenced by the work of Arbauch and Brown (2005) and by the publication of the special issue of the *Journal of Mathematics Teacher Education* devoted to mathematical tasks (2007 No. 4). Such studies must be continued and expanded. The aim of what follows is to urge educators to engage in such research and to suggest several topics that deserve to be explored in greater depth.

Usiskin (2000) identifies eight levels in the development of talent, from the levels of the uneducated person and the "ordinary" American schoolchild at the bottom, to the level of Gauss at the top. Such a classification is naturally somewhat arbitrary: Usiskin's levels can be subdivided further, and conversely, several of them can be united into one. Nonetheless, Usiskin poses the significant problem of development to the next level and expresses confidence in the realization of such a program. Our own basic premise is that students need challenges at every level of development. However, the selection of assignments at every level and in each situation must be accomplished in its own distinctive way.

1.6.1 Three groups of requirements for assignments

It may be argued that three groups of nonmathematical requirements must be taken into account in selecting any problem for any challenge:

- the students' cultural-educational level;
- the students' psychological characteristics;
- the pedagogical problem being addressed.

It is undesirable, and often impossible, to offer problems that rely on knowledge or conceptions that the students do not possess. For example, educators putting together problems for middle school students must usually avoid excessive generality and abstraction. Problems involving real-life linguistic details that are unfamiliar to the student are likewise infeasible.

By way of a simple example that confirms these claims, it was once observed that a student teacher gave students a "Problem of the Week" that dealt with similar figures. Wishing to make it more understandable and vivid—and thus more interesting for the students—the teacher formulated the challenge as a problem about making reduced photocopies. It turned out, however, that no one in the class had encountered a photocopier before. Therefore, although the problem was well within the students' mathematical abilities, it was not understood by any of them.

The second group of requirements compels educators to anticipate the influence that, for example, success or failure will have on the students' perception of mathematics and on their involvement in it. Naturally, much depends on a purely pedagogical effort to create an atmosphere in which even those students who fail to solve various problems will feel sufficiently at ease; but a great deal is also contingent upon the way in which the mathematical assignments are selected and structured. The ability to anticipate students' perceptions relies on an analysis of their individual abilities (where this is possible), but also on the consideration of group characteristics: those who have repeatedly taken part in national mathematics Olympiads will react differently to the fact that they fail to solve a problem, than those who are first-time participants in an Olympiad at the school level. Consequently, in the latter instance, the inclusion of excessively unfamiliar problems may turn out to be unacceptable.

Finally, mathematical challenges may be given for various different purposes: to involve students in mathematics, to develop their level of education and to involve them in research, to assess them, and for other aims. One should also not forget about other attributes of different pedagogical situations: the Olympiad problem, for instance, is one that, by definition, students should be able to solve in a few hours (and in reality much faster); in selecting problems aimed at involving elementary school students in mathematics, particular attention must be given to making the formulation engaging. In each case, the specific nature of the situation must be taken into account when assignments are being selected and composed.

In this respect, it is important to study the experience accumulated by educators in different countries. Compare, for example, the aforementioned Challenge 1.2.7 (a) from the Russian (Farkov 2004, p. 45) and Challenge 1.2.7 (b) from the American (Flener 1989, p. 10), problems books which were written as an aid to conducting local mathematics Olympiads in schools.

Both problems are typical "school-level" problems, in the sense that nothing is required to solve them except a certain degree of accuracy in the application of techniques studied in school. At the same time, the combination of these techniques goes beyond the limits of what is demanded in ordinary classes: one cannot immediately remove a factor or immediately make use of the formula of a difference of squares.

In this way, the problems offered for inclusion in local school Olympiads turn out to be challenges that are in some sense closest to school-level problems. Furthermore, in determining their distance from school-level problems, many technical details must be taken into account: the number of steps in the solution, the place in which the grouped terms of an expression appear, and so on; all of these elements are significant. The two problems are quite similar. Yet it is clear that not every problem that could be considered challenging in one country would have the same status in another country, even if it appears in a similar pedagogical situation. Studying which problems are considered challenging (in Olympiads or in a regular pedagogical setting) can facilitate a better understanding of the specifics of mathematics education in different countries.

1.6.2 Challenges in classrooms: identifying patterns in their appearance

Cooney (1985) has described a beginning teacher who believed that genuine problem solving—that is, doing challenging tasks—should take place during every classroom session, but who at the same time was convinced that challenges could be drawn only from recreational mathematics and not from what the students were learning in their textbooks. This is obviously not the case. For example, Challenge 1.2.4 grew out of a perfectly routine school problem taken from a textbook. However, in order for standard textbook problems to be developed into genuine challenges, and in order for teachers—or better yet, students—to pose such problems, a special atmosphere of openness towards construction and exploration must arise in the classroom (Brown and Walter 1990, Watson and Mason 2005). How such an atmosphere may be achieved, how it is achieved in different countries, and what are the basic existing techniques and patterns used for creating such an atmosphere—all these are topics that deserve further investigation.

On the other hand, vast sets of challenging problems have been and continue to be created in different countries precisely on the basis of ordinary school-level mathematics. In effect, new fields have appeared in school-level mathematics, in which active research continues to take place. One salient example is the case of Russia, where problems books for schools offering advanced teaching of mathematics or for elective courses in mathematics are published on a regular basis (Sharygin 1989–1991, Galitsky et al. 2001, Karp 2006). Such problems books reflect both the history of mathematics education, with some problems taken from books of the nineteenth century, and the contemporary creativity of teachers and examination administrators. Sometimes, the content of such problem books is highly specific and adapted to a specific curriculum, while in other cases it can be successfully used in other countries. In either case, it is important to become acquainted with the work that has been done in this respect in different countries.

1.6.3 The psychology of the art of writing problems as a research problem

The process of putting together a new challenge can unfold in many different ways. The general claim can be made, however, that literally all aspects of the task faced by a writer and compiler of problems are connected with the specific context in which the problems will be posed. This concerns even such characteristics as the beauty of the mathematical problem.

Let us return, for example, to the problem on finding the area of the region defined by the inequality $(y^3 - \arcsin x)(x^3 + \arcsin y) \geq 0$ (Challenge 1.5.10). The experienced solver of problems will be attracted by its unexpectedness and beauty. However, this problem is unlikely to be seen as beautiful by someone who has never solved standard problems on finding areas. On the contrary, such a student will see it only as pointlessly cluttered and unwieldy. The writer of this problem had his audience in mind, to some degree, when he wrote the problem, and based on an informal analysis of what the audience generally expected, he formulated a problem that thwarted those expectations. Insofar as the writers of problems play with and manipulate their audiences, the psychology of the art of problem writing resembles the psychology of art (Karp 2004), and as such, it deserves further study.

1.6.4 Using different areas of mathematics in different contexts

The problems of Challenge 1.2.7 are challenging for students with little contact with mathematics. Those of Challenge 1.2.8, also devoted to factorization, will be challenging for far more experienced problem solvers. However, in higher-level Olympiads, problems with such straightforward formulations as "Factor . . ." were quite rare. On the whole, such problems turn out to be too technical; contemporary writers and compilers of high-level Olympiads avoid not only school-level solutions, but school-level formulations as well. Factorization may turn out to be a necessary element in the solution of a problem—or more precisely, a sub-problem, which solvers must pose on their own—but it rarely constitutes the entire problem for the well-prepared student who is interested in mathematics. Paradoxically, factorization problems are probably equally unsuitable for students who are not interested in mathematics.

If we believe that even those who are planning to devote themselves exclusively to the humanities in the future must nonetheless become acquainted with mathematics, then we must offer them challenging problems, both in class and in extracurricular work. However, these problems cannot be completely identical to the problems offered to students who intend to deal with mathematics in the future; if they were, they would fail to win student interest. One promising approach is to use new non-traditional areas of mathematics, for example, the mathematics of elections, which looks at different election strategies and outcomes (COMAP 2006). This can be both engaging for a humanities student and, at the right level, not difficult even for a ninth-grader.

The pedagogical context in this case compels us to look for new areas of mathematics wherein to find challenges. We observe the same process in working with mathematically gifted students, where all new areas of modern mathematics are used to formulate problems (Berlov et al. 1998). The choice of the mathematical field from which challenges are drawn is not quite arbitrary. In most cases, it is directly or indirectly determined by the context. The context spurs us to search for new forms of presentation and new ways of structuring problems.

1.6.5 The structure of problems and the form of their presentation as a means of responding to context and transforming it

In extracurricular activities as well as in classroom practice, problems must not be seen in isolation from one another, but as representing specific groups (blocks or spaces, the terminology varying according to author, Karp 2002, Watson and Mason 2005). Classical examples of problems book (such as Pólya and Szegö 1976), in which problems were structured in order to instigate and help students engage in systematic and deeper investigations, can have an impact on the writing of assignments in areas of mathematics far more elementary than those to which the problems books are devoted. The craft of the problem writer consists in being able, while taking into account an existing context, to alter it gradually in a certain direction and thus enrich the students' mathematical notions.

Consequently, it becomes useful to study the structure of mathematical assignments, the way in which the different parts of a problem set are interrelated. The "morphology" of problem blocks—as they are assembled in different countries—is a worthy topic of investigation.

1.6.6 The issue of mathematics teacher education

The crafting of problems and challenges has a role to play in teacher education as well. One might suppose that a rather large number of teachers do not consider the selection of challenges and indeed of any mathematical assignments to be a part of their work at all (Skott 2001). Meanwhile, it is precisely the teacher who, at least at the beginning, poses challenges for students and encourages them to pose them for themselves. It is therefore vital to understand how to prepare teachers who are capable of doing such work. This topic is discussed in the special issue on mathematical tasks of the *Journal of Mathematics Teacher Education* (2007, No. 4) and in a number of chapters in this volume. The investigation and dissemination of good practice in this area of teacher education is inextricably linked to the degree of success that school teachers will have in the exploitation of problems and mathematical challenges in their classes.

1.6.7 Conclusion

In this section, we have mainly concentrated on listing questions and topics for further research. Indeed, mathematics education research in this area is only beginning. But an enormous amount of experience has been accumulated, internationally. The generalization and dissemination of this experience is a task of great practical and scientific importance.

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Chapter 2
Challenges Beyond the Classroom—Sources and Organizational Issues

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This chapter surveys the existence of many particular types of beyond classroom mathematics challenges from around the world, discusses the value and describes special features of each type and gives a very large number of examples which indicate the wide variety of types of challenge which successfully operate around the world.

2.1 Introduction

The classroom is only one of the "homes" of education. The process of conveying and/or acquiring information and knowledge takes place in many forms and in many places. There are highly efficient ways of learning today which by-pass the classroom. Printed materials (books, journals and newspapers), radio, television and the World Wide Web are powerful information sources that operate parallel to schools. Many students undertake extra learning of mathematics in what are known as clubs or "circles", that is, specially invited groups of students from typically more than one school who meet with an independent teacher to develop their mathematical knowledge.

Tasks given to students as "homework" or "take home" exams also contribute to the "beyond school education" in some countries. One should take into account the educational role of different extracurricular activities which are conducted outside regular classroom hours such as competitions, mathematics houses or camps. In addition, one should not neglect the impact on learning of communication and interaction with more experienced peers, friends, parents, relatives, specially invited friends and other volunteers. All of these factors, taken together, form a kind of "beyond classroom education" which complements, extends and enriches what has been achieved in the classroom.

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The "beyond classroom education" has one specific and important role which deserves to be mentioned explicitly and separately. The school educational system in the majority of countries is designed to serve the needs of students with average abilities because the latter constitute the majority in the class. The educational requirements and standards are set up in such a way that even students with lower abilities could cover them, albeit with significant efforts. This means that many students are learning less in the classroom than their capabilities might allow. Further, students of higher ability do not necessarily attract the same level of attention from their teachers as those who experience difficulties.

The standard school curriculum and syllabus requirements do not represent a real obstacle or challenge for higher achievers. They are not prompted to apply more efforts during the educational process and, as a result, their abilities and even talents can remain undiscovered and undeveloped if they rely purely on what is available in the classroom.

On the other hand, every society needs high level professionals in all areas of sciences, economics and social sciences. This kind of professionalism rests on identification, development and cultivation of higher abilities and talents. It is achieved through continually motivating and challenging those whose talents and abilities can be developed. Unlike other resources, such as mineral deposits, which disappear once discovered and used, the abilities and the talent of a given person will disappear forever if *not* discovered, developed and used.

This is where beyond classroom education has an irreplaceable and prominent role: to challenge the minds of those with higher abilities and talents. This refers especially to mathematics where insight and understanding are obtained as a result of a solid investment of time and effort. This role of beyond classroom education is well recognized today in many countries. It has prompted the existence of an impressive variety of activities designed to challenge the minds of those who could achieve more.

The most popular and earliest developed beyond classroom activities of this type have been the different competitions and competition-like activities. They take various forms in different countries but there are many success stories of whole generations of strong mathematicians who may not have developed as such otherwise, and the vast numbers of students entered in competitions annually now is a mark of success.

These competitions exploit the intrinsic desire of human beings to compete with others and allow students to exhibit their abilities and talents. They also motivate the future participants in competition and competition-like events to work hard in order to be among the winners. This, in turn, deepens mathematical knowledge and improves competition skills.

It is well known that there are many students who dislike mathematics, because they never had a chance to feel mathematics and enjoy it or maybe because they were not fortunate to be exposed to adequate teaching. Beyond classroom education has a role here as well.

In this chapter we describe several types of activities which belong to beyond classroom education and have already been tested and used in many countries.

Taken together these activities form a "Challenging Environment" (on national and international levels) where young people can exhibit and develop their abilities in the field of mathematics or can enjoy mathematics by observing and feeling its essence and beauty. These activities provide the "context", the infrastructure, in which challenging happens.

In Chapter 1, we discussed the mathematical content suited to challenge young minds, while in Chapter 3 we turn our attention to the contribution that information and communication technologies can make to challenging environments.

2.1.1 Working as individuals and in teams

An important issue about mathematics competitions is whether students should work individually or in teams. Arguably there is a role for both. Certainly learning in teams can be valuable and an important foundation for later life. Most international competitions are individually based but some training for this can usefully be done in teams. Some countries, such as Iran, have found that students can perform more successfully in individual competitions if they have had prior training working in teams, and team competitions have increasingly become part of their profile.

Norway has developed another example of a successful team competition. Students work in teams comprising their whole class in the *KappAbel* contest (see Section 2.2.1.2).

2.1.2 Involvement of teachers

Most countries have been concerned about the difficulty of getting strong involvement from other than a few committed teachers. As a result quite innovative methods have been designed to increase this with much success. Teachers are required to score results and are encouraged to propose problems, prepare their students and supervise volunteers. As an example, Iran (Rejali 2003) in particular has a good record of being able to organize events so that teachers become involved without the need for experienced problem creators or university professors to be present. The same has applied also for competitions in statistics. The direct experience for teachers in these activities can add significantly to their professionalism and confidence. Often experiencing the atmosphere has a very positive impact on the teachers.

2.2 Environments for challenging mathematics

Mathematics competitions were the earliest form of challenge in a beyond the classroom context and are still arguably the most popular as measured by participation numbers across the world. One of the Affiliated Study Groups

(ASG) of ICMI is the World Federation of National Mathematics Competitions (WFNMC), which was founded principally on a base of interest in competitions, but has since broadened its interests to encompass such activities as mathematics clubs (or circles), mathematics camps, mathematics days and journals (see WFNMC 2002).

However there is also a number of other beyond the classroom activities which provide challenge and which are outside the stated interests of WFNMC. The best example of this is probably mathematics exhibitions, of which there are a number of notable ones.

Here we provide a list of some of the main sources, together with some indication as to how the writers see their main strengths.

- **Mathematics competitions.** Competitions come in two main categories. Traditionally the main competitions were the exclusive competitions, typified by national and international Olympiads. But the latter half of the 20th century has seen the rise of inclusive competitions, in which entry numbers go into hundreds of thousands, and which provide challenge for the average student. Examples of these are in the USA, Canada, Australia and Europe (the Kangaroo, or *Kangourou*, based on the Australian model, has annual entries in the millions). Exclusive competitions are perceived to be strong in identifying mathematics ability and thinking skills, enriching mathematical knowledge and skills and providing a path for nurturing high achievers, while inclusive competitions have provided recreation and fun through mathematics and raised public awareness of mathematics.
- **Mathematical journals, books and other resources.** These are seen as particularly strong at enriching mathematical knowledge and skills and also in nurturing high achievers.
- **Research-like activities and conferences, projects.** These are also seen as important in nurturing higher achievers, giving them experience of life as a mathematician, as well as identifying and enriching mathematics skills.
- **Mathematics exhibitions, mathematics playgrounds, mathematics rooms, historical displays, mathematics and science centers.** These are seen to be particularly strong at providing recreation and fun through mathematics and in raising public awareness, but they certainly provide a challenging environment, sometimes identifying a different type of student than the one who excels in competitions.
- **Mathematics clubs or circles**. These are generally designed to nurture talented students and develop their knowledge and skills to a higher level.
- **Mathematics houses**. These are environments for enriching mathematical knowledge and skills. They provide recreation and enjoyment, and because of their physical presence they raise public awareness. These houses are also a place for identifying talented students.
- **Mathematics lectures**. In various forms these provide a mechanism for enriching and nurturing talented students.

- **Public lectures, columns in newspapers, magazines, movies, TV, books and general purpose journals.** These events all popularize mathematics and provide recreation and fun through mathematics for the public at large.
- **Mathematics days and open houses at universities and science centers.** Normally conducted in a recreational atmosphere, they raise public awareness but can also identify and enrich talented students.
- **Mathematical modeling programs.** These can at various levels introduce a student to an enjoyment of mathematics and help a talented student to a greater understanding.
- **Correspondence programs.** Particularly useful for students in remote locations, these can provide all students with challenge at their level.
- **Interdisciplinary workshops, games, puzzles, short term exhibitions.** These can have a wide range of values, depending on the actual programs.
- **Web sites**. There is a range of web sites from those which provide Olympiad style enrichment to those from which students at any level can access information and learn.
- **Mathematics camps, summer schools, summer institutes.** Depending on their theme, these can be held for very advanced students, or have broader themes, for example, exploring what mathematicians do in normal life.
- **Family mathematics programs.** These are often in the public domain and take a recreational and fun approach.
- **Presence of mathematics in community fairs and events.** Again these normally take on a recreational and fun approach in the public domain.

This list is indicative rather than exhaustive. We now discuss some of these in more detail.

2.2.1 Mathematics competitions

First, it should be noted that Kenderov (2006) and Kenderov (2007) provide definitive commentary on the role and extent of competitions as they currently apply, and also their history.

It is not easy to trace back the origins of mathematics competitions for school students. According to V. Berinde (2004) a primary school math competition with 70 participants was held in Bucharest, Romania, as early as 1885. There were eleven prizes awarded to two girls and nine boys.

Nevertheless, it is widely accepted today that the Eötvös Competition in Hungary (1894) is the forerunner of contemporary mathematics (and physics) competitions for secondary school students. Its model is still widely used today. The competitors were given four hours to solve three problems individually (no interaction with other students or teachers was allowed). The problems in the Eötvös Competition were specially designed to challenge and check creativity and mathematical thinking, not just the acquired technical skills; the students were often asked not only to give the correct answer, but also to provide the reasons why this was the correct answer. As an illustration, we give here the three problems from the very first Eötvös Competition in 1894.

P1. Show that the set $\{(m, n)$: 17 divides $2m + 3n\}$ coincides with the set $\{(m, n)$: 17 divides $9m + 5n\}$.
P2. Given a circle C, and two points A, B inside it, construct a right-angled triangle PQR with vertices on C and hypotenuse QR such that A lies on the side PQ and B lies on the side PR. For which A, B is this not possible?
P3. A triangle has sides length a, $a + d$, $a + 2d$ and area S. Find its sides and angles in terms of d and S. Give numerical answers for $d = 1$, $S = 6$.

(Complete collections of problems from this competition can be found in Rapaport 1963a, Rapaport 1963b, Liu 2001.)

To compete means to compare your abilities with the abilities of others. Thus, the broader the base of competition, the better would be the comparison. This seems to be the driving force behind the natural transition from school competitions to town competitions, to national and to international competitions. The Eötvös competition was the beginning of a remarkable development. The idea migrated from country to country, each time getting enriched both as to the style of conducting the competition and as to the mathematical content.

In 1934 a mathematical Olympiad was organized in Leningrad, USSR (now St. Petersburg, Russia). This event was unique (and still is) in that students need to submit some of their solutions orally. A Moscow Olympiad was also founded shortly after.

In the middle of the 20th century, the flagship of math competitions, the International Mathematics Olympiad (IMO) , was born. In 1959, the first IMO took place in Romania with participants from seven countries: Bulgaria, Czechoslovakia, German Democratic Republic, Hungary, Poland, Romania, and the Soviet Union (USSR). The second IMO (1960) was organized by Romania as well, and since then it has been hosted by a different country every year (except 1980, when no IMO was held).

Originally, each country had the right to send a team of up to eight high school students guided by a team leader and a deputy team leader. In 1982 the number of students in a national team was reduced to four students. Since 1983 a national team has consisted of up to six students accompanied by a team leader and a deputy team leader. Over the years the number of participating countries has increased and in 2007 the IMO in Vietnam was attended by competitors from 93 countries.

The conduct of the IMO is subject to strict and formal rules which regulate every aspect of the competition: participation, problem selection, assessment of solutions, distribution of medals, and many other essential details. More details and information on the IMO can be found on its official site www.imo-official.org/.

The competition itself occupies two consecutive days. On each day the students have 4.5 hours to solve three problems. The problems are selected by an international jury comprised of team leaders as well as representatives of the host country. There is no official syllabus for the IMO but the problems are accessible to the most talented secondary students. The problems are difficult and their solution requires a significant degree of inventive ingenuity and creativity. The solution to a problem is worth seven points. The perfect score is, therefore, 42 points.

After the competition, there is a social program for participants so that they can get to know each other, discuss different solutions to the problems and share future plans. Team leaders also exchange experience and good practices on such matters as the creation of new "Olympiad" problems and preparation of students.

Officially, IMO is a competition for individuals. Participants are ranked according to the points scored. On this basis, the medals are distributed. Unofficially, as in the Olympic Games, the medals and points obtained by the participants in a certain team are accumulated in order to rank the countries. This provides an opportunity for international comparison.

IMO is the most prestigious international mathematics competition today. To participate successfully in IMO, countries had to develop their own systems of competitions in order to identify students with high abilities and to motivate them for hard work. It is noteworthy that the existence of the IMO led to International Olympiads in other sciences—physics, chemistry and biology. In 1989 under the auspices of UNESCO the first International Olympiad in Informatics (computer science) was organized in Bulgaria. There are also new Olympiads in some narrower disciplines (or sub-disciplines), although these are not as large as the biggest five science Olympiads, and are not recognized by every country. Further, the resources of many countries restrict them to attending only the original five broad scientific Olympiads.

2.2.1.1 Inclusive competitions

Today the world of mathematics competitions encompasses millions of students, teachers, research mathematicians, educational authorities, publishers and parents. Hundreds of competitions and competition-like events with national, regional, and international importance are organized every year. A remarkable international cooperation and collaboration has gradually emerged in this field. How the system works can be seen from the following story.

The Mathematical Association of America first held a national competition, open to all, with multi-choice problems in 1950. The University of Waterloo followed with a Canadian competition of the same kind in 1963 (see cemc.uwaterloo.ca). Such papers, with problems for students of broad ability groups, were marked on computers with answer sheets read by optical readers.

Australian mathematician Peter O'Halloran, while on sabbatical leave at the University of Waterloo over 1972/73, observed these events and identified the potential to hold such a competition in Australia. This event was held for the first time in Canberra in 1976 and was so popular that the competition went national as the Australian Mathematics Competition (AMC) (see www.amt.edu.au/eventsamc.html). The AMC run by the Australian Mathematics Trust (based at the University of Canberra) involves up to half a million participants, larger than the Canadian Mathematics Competition. In turn, the European competition *Le Kangourou des Mathématiques* (modeled, as the name suggests, after the AMC) (see www.mathkang.org/), which started in the nineties, in 2005 involved 3.5 million students from different countries.

Some national competitions are truly impressive too. The Mathematical Olympiad for public school students was initiated in Brazil in 2004 (see www.obmep.org.br/). In 2006 it alone had 12 million participants!

It would not be an exaggeration to say that the rise and the development of mathematics competitions is one of the characteristic phenomena of the 20th century. It deserves to be studied and analyzed. However this is not an easy task. A glance at the World Compendium of Mathematics Competitions (www.amt.edu.au/wfnmccom.html) maintained by the Australian Mathematics Trust and at MathPro Press (www.mathpropress.com/competitions.html) (a list of web sites related to competitions maintained by Stanley Rabinowitz) reveals a variety of competitions which resist any classification attempts.

The terms "inclusive" and "open"

There is a distinction between these terms. The term "inclusive" (WFNMC 2002) in the sense used here means that a large number of students of all standards participate. However there are competitions which are "open" in the sense that anyone can enter, which have wide participation in terms of numbers, but which would not be regarded as "inclusive". A good example of this is the Mathematics Challenge for Young Australians, run by the Australian Mathematics Trust, which is open to all and attracts up to 15,000 students annually.

This is an event in which students have three weeks to discuss problems which have stages, starting normally with an easy question, but building up a theme step by step to something more advanced. Because the student can consult with various people, including their teachers, there are no prizes, and the student is essentially competing with herself, and potentially gaining the satisfaction of solving a challenging problem.

However the participants are intended to be in the top 10 to 20 per cent ability level and so this would not be regarded as "inclusive".

2.2.1.2 Different types of competition

While "inclusive" (intended for all) competitions are intended for students of a wide range of abilities, "exclusive" (by invitation only) events target talented students. (Examples of the second type are the IMO and competitions conducted to select a national team for IMO.) There are "multiple-choice" competitions where each problem is supplied with several possible answers, from which the competitor has to find (or guess, as no justification is required) the correct one. In contrast, "classical style" competitions (like the Eötvös

Competition and IMO) require students to present arguments (proofs) in written form. In "correspondence" competitions like Tournament of Towns (www.amt.canberra.edu.au/imtot.html), the students do not necessarily meet each other, but write the papers under supervision in their home towns (without the need to travel, such competitions are cheaper to organize). Such competitions are also organized by many journals (see Section 2.2.2).

In "presence" competitions the participants are competing together in the presence of other competitors. This enables all students to participate under the same conditions. There are even mixed-style competitions, with a presence-style first stage and correspondence-style subsequent stages. Competitions may also be "individual" (for ranking of students, like IMO), or "team competitions" where the score of the whole team is what matters and students might cooperate in the solution process.

The competitions may differ by age of participants (primary school students, secondary school students, students in colleges and/or universities). Competitions differ also by affiliation of participants: from one school, from several schools or from all schools in a town, nationwide competitions and international competitions. However there are competitions which escape such classifications.

There are also competitions in special topics of mathematics. For instance Iran has a statistics competition and Australia has a statistics poster competition.

Here are some (of the many) examples:

1. **An exclusive competition of interactive style.** The competition Euromath is a European cup of mathematics (www.cijm.org/index.php?option = com_content&task = blogcategory&id = 17&Itemid = 8). Each team is composed of 7 people: students from primary school to university and one adult. The six best teams are chosen to participate in the final competition by the results of their work on logical games. In the final, these teams work in front of spectators. To win, a team needs to be quick and to have good mathematical knowledge but the most important thing is *l'esprit d'équipe*.
2. **Another model of an inclusive competition.** *KappAbel* (www.kappabel.com/index_eng.html) is a Nordic competition for 14-year-olds in which whole classes participate as teams. The first two rounds consist of problems distributed on the Internet and downloaded by the teacher. Within a 90-minute time limit, the class discusses the problems and decides how to answer each problem. The third round is divided into two parts: a class project with a given theme (ending with a report, a presentation and an exhibition), and a problem-solving session run as a relay where two boys and two girls represent the class. Recent themes have been Mathematics and local handicraft traditions (2000), Mathematics in games and play (2001), Mathematics and sports (2002), Mathematics and technology (2003), Mathematics and music (2004), Mathematics and communication (2005) and Mathematics in holidays (2006). The three best teams from the third round meet on the following

day for the final, which is a problem-solving session with an audience consisting of teams that did not make it to the final (www.kappabel.com/overskriftside-eng.html).

3. **Ontario Math Olympics**. This is a team contest for year 7–8 students. First, it runs in each region. There are school teams of 4 students (2 boys, 2 girls, 2 year 7, 2 year 8), which complete various mathematics tasks for the whole day. Usually, there are four 30-minute team activities in the morning and three or four 15-minute activities in the afternoon. The results are announced at the end of the day. During the breaks, there are some mathematics games or fun activities that involve all participants. The best two teams from each region compete in the Provincial Math Olympics hosted in June over two days at some Ontario University.
4. **International Mathematics Tournament of Towns**. The International Mathematics Tournament of Towns is a mathematics problem-solving competition in which towns throughout the world can participate on an equal basis. Students participate in their own towns which involves minimal transport and administrative costs.

   The Tournament is conducted each year in two stages—Autumn and Spring. Each stage has two papers, an "O" level and an "A" level, which are spaced roughly one week apart. The A level paper is more difficult, but offers more points. Students and their towns may participate in either stage or level, or in both levels and both stages.

   The Tournament is open to all high school students. Students are awarded points for their best three questions in each paper, and their annual score is based on their best score in any of the four papers for the year.

   There are two versions of each paper, known as the Senior and Junior papers. Students in Years 10 and 11 (the final two years of high school in the Russian nomenclature) are classified as senior participants and therefore attempt the senior paper. So that Year 10 students are not disadvantaged their scores are multiplied by 5/4. Younger students, in Years 9 and below, attempt the junior paper. To ensure that the scoring is fair to all levels of students, Year 8 students have their scores multiplied by 4/3, Year 7 students have their scores multiplied by 3/2 and Year 6 students and below have their scores multiplied by 2.

   Students who exceed a certain minimum score are awarded a diploma by the Russian Academy of Sciences. Local organizing committees also present their own awards. The Tournament is managed by a central committee in Moscow, which is a subcommittee of the Russian Academy of Sciences.

   The Tournament dates back to the late 1970s in the USSR. At that time, the National (All-Union) Olympiad of the USSR was based on a system that gave relatively little opportunity to students in the larger republics such as Russia and Ukraine.

   The first Tournament was known as the Olympiad of Three Towns (Moscow, Leningrad and Riga) and was held in the 1979–1980 academic year. Participation quickly grew and the Tournament changed to its current name in the following year.

The Tournament had difficulty in obtaining political recognition in its early years, but its popularity grew and it finally won recognition in 1984 when it became a subcommittee of the USSR Academy of Sciences. The support of the USSR Academy of Sciences allowed the Tournament to become international. This attracted entries initially from Eastern Europe, particularly Bulgaria, where a national committee was formed.

The Tournament was confined to Eastern Bloc countries until 1988, when Canberra was invited and participated. Since then the Tournament has continued to grow with over 100 towns participating recently. New towns in recent Tournaments included Buenos Aires and Bahia Blanca (Argentina), Luxembourg (Luxembourg) and Subotitsa (Yugoslavia). Other entries have come from Australia, Canada, Colombia, Germany, Greece, Iran, Israel, New Zealand, Slovenia, Spain, UK and USA.

The problems of the Tournament papers are very challenging and provide a good source of classical mathematics problems at the high school level. Five volumes of Tournament of Towns problems and solutions have been published by Taylor (1992, 1993, 1994), Taylor and Storozhev (1998) and Storozhev (2005). The Tournament's web site is at www.amt.edu.au/imtot.html.

5. **The Mathematics A-Lympiad**. The Freudenthal Institute has established the A-Lympiad competition which has two rounds. In the first round, teams of students compete for a full day at their own school. There is also a second international round in which about 16 teams compete over a whole weekend, in a conference centre in a Dutch national park.

   The main aim of this competition is teaching problem solving or modeling by providing appropriate tasks to practice these skills. It is a mathematics competition for teams of three or four students. The teams work on an open-ended assignment in which problem-solving and higher-order thinking are required to analyze a real-world situation. The result of the assignment is a written report.

   The competition is intended for students in years 11 and 12 (ages 16–18) of secondary schools. The Freudenthal Institute of Utrecht University in the Netherlands (www.fi.uu.nl/alympiad/en/welcome.html) started the competition about ten years ago. It provides a challenging atmosphere for team work, as well as mathematical modeling experience for high school students.

2.2.1.3 Some general comments

Information about mathematics competitions and competition-related activities is regularly published in *Mathematics Competitions*, the journal of the World Federation of National Mathematics Competitions (www.amt.edu.au/wfnmc.html).

There are opponents of competitions and a number of arguments arise. These matters are addressed in the Proceedings of Discussion Group 16 at ICME-10 (Taylor et al. 2004).

One of the problems with the classroom is that a school curriculum is rather restricted and cannot suit all. Competitions enable students to be exposed to other aspects of mathematics and for them to apply the skills they have to new situations. Competitions enrich the learning experience of hundreds of thousands, in fact millions of students who participate in the inclusive competitions.

Competitions and mathematics enrichment activities can be viewed as events that provide impetus for subsequent discussions among students (as well as among their teachers, friends and parents). From the viewpoint of acquiring new mathematical knowledge (facts and techniques) these "after competition discussions" might be as important as the preparation for and the competition itself. Many mathematicians owe a significant part of their knowledge to just such "corridor mathematics". From this point of view, the social programs organized after competitions provide additional importance.

2.2.2 Mathematics journals, books and other published materials (including Internet)

Although they are important for identifying mathematical ability, the competitions themselves can be the culmination of a challenging situation. Their benefits accrue to those students who have undergone a period of mathematical enrichment that involves a lot of time and effort as they improve their knowledge and skills.

To motivate and encourage such students, a variety of supports have emerged: activities such as circles, clubs and camps, attractive educational materials, mentorship through personal contact or correspondence, journals and electronic materials (websites, compact discs, games and software).

In addition to the Eötvös competition the year 1894 was notable also for the birth of the famous math journal *KöMal* (an acronym of the Hungarian name of the journal, which translates to "High School Mathematics and Physics Journal"). Founded by Dániel Arany, a high school teacher in Györ, Hungary, the journal was essential for the preparation of students and teachers for competitions. About one-third of each issue was devoted to problems and problem solving and readers were asked to send in solutions. As noted by G. Berzsenyi in the preface of Oláh et al. (1999), about 120–150 problems were published in *KöMal* each year, and about 2500–3000 solutions were received. The best solutions and the names of their authors were published in subsequent issues.

This type of year-round competition helped many young people discover and develop their mathematical abilities. Many of them later became world-famous scientists. For more information in this respect, see the journal web site (www.komal.hu).

About the same time, a similar development occurred in Hungary's neighbor, Romania. The first issue of the monthly *Gazeta Matematică*, an important journal for Romanian mathematics, was published in September 1895. The journal organized a competition for school students, which improved in format over the years and eventually gave birth to The National Mathematical Olympiad in Romania.

The journal played another important role too. For legal reasons, it was transformed to *Society Gazeta Matematică* in August 1909. The following year, the Romanian Parliament approved the legal status of the new society and this is considered to be the birthday of the Romanian Mathematical Society (Berinde 2004).

There are many examples around the world of journals designed to stimulate student interest in mathematics. These journals contain historical articles, expository articles on contemporary subjects of interest, such as the four color theorem and Fermat's Last Theorem, and problem corners, where new problems are posed, other current problems from Olympiads are discussed and students may submit their own solutions. Examples of such journals in Eastern Europe, where the traditions are older, are *KöMal* (Hungary) and *Kvant* (Russia). In the West outstanding examples are *Crux Mathematicorum* (Canada), *Mathematics Magazine* (USA), *Mathematical Spectrum* (UK), *Parabola* and *Function* (Australia), and *Mathematical Digest* (South Africa).

There are many publications which enrich and challenge the student's interest in mathematics. Which young mathematicians can we find who has not been influenced by expository books on mathematics, such as Courant's *What is mathematics?* or papers or problems in journals like *Mathematical Intelligencer, American Mathematical Monthly* or *Mathematics Magazine*. In the English language the Mathematical Association of America has a massive catalogue and the Australian Mathematics Trust has a significant number of publications.

In Russian, there is also a very rich resource, traditionally published through Mir. In the French language the Kangourou and other publishers have a prodigious catalogue, as does the Chiu Chang Mathematics Education Foundation in the Chinese language.

This list refers to just a few major languages, but there are other mathematics journals in the world that motivate students. For example, in Iran, there is an expository journal called *Yekan*, which interested young high school students for its problems without solutions! But there were also some papers in this journal about the so-called new mathematics and set theory at a time when the school curricula did not contain any set theory.

As a result of observations of the effect of this Iranian journal on the interest of students (Rejali 1989), publications of some expository mathematics journals were proposed and today there are many expository mathematics journals in Iran.

It is impossible to list all possible references with information about printed materials. One could follow the links at www.mathkang.org/ksf/index.html

(France), www.amt.edu.au (Australia), and www.maa.org/ (USA) for a rich variety of printed resources.

A relatively recent development is the publication of the MATHEU Manual. The MATHEU Project was carried out with the support of the European Community within the framework of the Socrates Program. Its complete name is "Identification, Motivation and Support of Mathematical Talents in European Schools" (www.matheu.eu/). The Project networked the efforts and the experience of the different countries in the work with higher ability students in mathematics.

The Manual (MATHEU 2006), the outcome of the MATHEU Project, contains a sequence of "ladders" designed to challenge the minds of students (and teachers). Climbing a ladder is possible only if the person increases his/her knowledge. Each ladder is a self-contained mathematical text, focused on a specific mathematical topic, which could be used by teachers or by students in their work in and beyond the classroom.

In essence the ladder is a succession of mathematical problems, explanations and questions for self-testing, ordered in such a way that the degree of difficulty increases slowly. By working up the ladder-text, the student, as well as the teacher, could elevate their mathematical knowledge. Using the ladder, students, and teachers, can enrich, deepen and test their knowledge on a specific mathematical topic.

The lower part of the ladder is rooted in the normal curriculum material studied in class. As "steps", one has mathematical problems, definitions and explanations, pieces of information and other challenges that the learner has to master in order to acquire the higher level of understanding of the material. Depending on their individual abilities the students advance, that is, "climb", to different heights on the ladder. The degree of advancement will single out higher ability students. Therefore the ladders help identify talented students too.

If the ladder is well designed and consists of interesting and challenging problems, it will attract and motivate the students to apply more time and energy in studying mathematics.

2.2.3 Research-like activities, conferences, mathematics festivals

The majority of contemporary competitions cultivate the ability to answer questions and problems posed by other people. However, the ability to formulate questions relevant to a problem or situation is also important, especially in scientific research.

A drawback of classical competitions is that success depends on the student having, not only a good mind, but a quick one. With a limited time allowance of usually three or four hours, most competitions impose significant stress on participants. Not only do they have to solve the problem correctly, but do so quickly in the presence of other competitors.

However there are many highly creative students who do not perform well under pressure. Such students often come up with new and valuable ideas a mere day (or even just five minutes) after the end of the competition, yet receive no reward or incentive.

Traditional competitions disadvantage such students, even though some of them could become good inventors or scientists. What matters in science is rarely the speed of solving difficult problems posed by other people. More often, what matters is the ability to formulate questions, to generate, evaluate, and reject conjectures, to come up with new and non-standard ideas. All these activities require ample thinking time, access to information resources in libraries or on the Internet, communication with peers and experts working on similar problems, none of which are allowed in traditional competitions.

Obviously, other types of competitions are needed to identify, encourage, and develop such minds. Such competitions should reflect the true nature of research, containing a research-like phase, along with an opportunity to present results to peers—precisely as it is in real science.

As a matter of fact such competitions designed to identify students with an inclination to scientific (not only mathematical) research already exist.

2.2.3.1 Jugend Forscht (youth quests), Germany and Switzerland

Jugend Forscht in Germany celebrated its 40th anniversary in 2005. It is an annual competition for students under the age of 21, who work, alone or in teams, on projects of their own. The projects are presented at special sessions, where the winners are awarded (www.jugend-forscht.de).

Switzerland has a similar competition, which is organized annually by the *Schweizer Jugend Forscht* (Swiss Youth Quests) Foundation, established in 1970. The competition covers all scientific directions, including social sciences and humanities (www.sjf.ch).

2.2.3.2 Research Science Institute (RSI), USA

In fact, *Jugend Forscht* was originally modeled on the many "Science Fairs" in USA. We mention only one such program from USA here, because it emphasizes mathematics, has an international character and was successfully used as a model for similar programs in other countries.

The Virginia-based Center for Excellence in Education (CEE) (www.cee.org) was founded by Admiral H. G. Rickover in 1983. The major CEE event is the Research Science Institute (RSI). Each summer approximately 75 high school students gather for six weeks. They are selected from the United States and other countries and participate in a rigorous academic program which emphasizes mathematics, sciences, and engineering.

Only outstanding students, who are carefully selected, are admitted to this program. The RSI starts with a series of professional lectures in mathematics, biology, physics and chemistry. The students are paired with experienced scientists and mentors, who introduce them to interesting research topics and share with them the joy and excitement of exploring new territories. The RSI days are filled with research, evening lectures, as well as recreations such as sports and ultimate Frisbee. At the end of the program, the students present their own research, both in written and oral form, and awards are given to the best performers (www.cee.org/rsi/).

The RSI is an international program: almost a third of its students come from other countries. It provides a unique environment for talented students from different parts of the world to meet, live and work together for a relatively long period of time. The networking and friendships fostered by the RSI program are important for the future development of the participants. The fact that they know each other could make their future collaboration more fruitful.

2.2.3.3 High School Students' Institute for Mathematics and Informatics, Bulgaria

The Virginian RSI model was adapted to Bulgarian conditions and traditions and in 2000 a High School Students' Institute of Mathematics and Informatics (HSSIMI) was created.

Throughout one academic year, the involved high school students (grade 8–12) work on freely chosen topics (projects) in mathematics and/or informatics (computer science). They work individually or in teams and are supervised by a teacher, a university student, a relative, or just any specialist in the field, willing to help. In fact, some recent HSSIMI projects were successfully supervised by former HSSMI participants, who are now university students.

HSSIMI organizes three major events: two competition-like student conferences and a summer school. The first student conference is held in January and the second one is a stand-alone (but otherwise regular) section at the annual spring conference of the Union of Bulgarian Mathematicians (UBM) in April. The latter section is the most visited event during the spring conference of UBM. It is attended by university professors, researchers, teachers, parents and school peers.

To participate in the HSSIMI events, students submit a written paper (or software product) with the results of their work. Specialists referee the submitted projects, assess the materials, and suggest improvements. Students present their results at the conferences and both content and presentation skills are evaluated by the jury. Winners receive awards. As a special award, two of the winners are sent to USA as participants in the above mentioned RSI.

The authors of the best projects are invited to a three-week summer school. During the first two weeks, eminent specialists from universities, research institutes, and software companies give lectures and practical courses in mathematics and informatics.

As in similar programs, the main goal of this preliminary training is to expand the students' knowledge in topics of their interest and to offer new problems for possible projects. During the third week, students hold a High School Students Workshop, where they briefly present their ideas for new projects.

For the relatively short period of its existence, the HSSIMI has become a valuable addition to the (rather densely populated) system of traditional competitions in Bulgaria. As was expected, the HSSIMI attracted students who were not regular participants in the traditional competitions.

2.2.3.4 Mathematics festivals, Iran

Students who get involved in studying and doing cooperative research in schools or outside their schools during their school years learn to do research and learn mathematics more eagerly. The experience of doing research in school years makes students successful young researchers in their university work.

In Iran there are many festivals for young students to present their papers or discoveries about mathematics. Every year there is a student festival in which students present their achievements. The Kharazmi Festival has different sections. Young researchers can present their discoveries or papers in this prestigious Festival.

One year, a group of high school students presented their discoveries about fractals found in ancient buildings of Isfahan. This work was done as a project in the Isfahan Mathematics House (IMH), with the cooperation and guidance of architects. This group succeeded in gaining a good result at the Kharazmi Festival. After this success, these students were more anxious to learn mathematics. There are several reports in which the benefits of early years' research were presented. Other festivals such as Isfahan Mathematics House Festival and Fars High School Student Festival are all providing a challenging atmosphere for students of mathematics throughout the country.

The experience of the Isfahan Mathematics House (IMH) and many other institutes throughout the world prove that research-like activities and opportunities for students to present their findings and the results of their studies make a challenging atmosphere in which they can learn mathematics.

When a student gets a chance to present a lecture at a national or local conference or is recognized at a festival, it makes him or her excited to learn more and achieve better results in future. All of these activities provide a challenging atmosphere to encourage further learning.

2.2.4 Mathematical exhibitions

Exhibitions, in the sense of gathering material together for people to view or interact with, are becoming increasingly common. These are generally outside the classroom and may be aimed as much at the general public as they are at students. They can also take place in a variety of settings from schools to museums to shopping malls to the open air.

We mention here several examples of these. The idea of a science centre is to present scientific phenomena in a hands-on way. This means that visitors are challenged by a real experiment and then try to understand it. Some countries, for example, Australia (*Questacon,* www.questacon.edu.au/ in Canberra) and Israel (The Israel National Museum of Science, Technology and Space, in Haifa www.MadaTech.org.il) have national science centers which include mathematical experiments.

There are also science centers devoted exclusively to mathematics, for instance the *Mathematikum* (www.mathematikum.de/) in Germany (which is discussed in more detail in Section 2.2.4.2) or *Giardino di Archimede* web.math.unifi.it/archimede/archimede/index.html) in Italy or *Atractor* in Portugal (Chaves 2006). These permanent centers, best visited with a guide, attract tens and hundreds of thousands of visitors per year. Instruments for museums, laboratories or mathematics centers may be very expensive.

There are also annual exhibitions, varying in content from year to year. An example of this which attracts tens of thousands of visitors per day is the *Salon Culture et Jeux Mathématiques* in Paris. Further, there are also occasional exhibitions, such as the international exhibition *Experiencing Mathematics* (www.mathex.org/MathExpo/EnHomePage). Sponsored by UNESCO and ICMI jointly with other bodies it was presented in 2004 at the European Congress of Mathematics and the 10th International Congress on Mathematical Education (ICME-10). This is discussed in Section 2.2.4.2.

Exhibitions can have a special theme, such as the one at the University of Modena and Reggio Emilia featuring mathematical machines (www.mmlab.unimore.it/on-line/Home.html). These machines are copies of historical instruments that include curve drawing devices, instruments for perspective drawing and instruments for solving problems. The mathematical machines exhibition is discussed in more detail as a case study in Chapter 5.

2.2.4.1 Historical background

The root of mathematics exhibitions may be the *Wunderkammern* (small rooms of curiosity) in the 17th century, where mathematics instruments and models were on show together with other “wonderful” scientific artifacts (including improbable stuffed beings made up of pieces of different animals). Famous examples were the perspective rooms (Baltrusaitis 1984). These were not for the general public; kings and nobles organized them to astonish and display their power to their visitors.

Scientists and mathematicians of the time had an ambiguous response (as often happens with any popularization of science). No real scientific intention of popularization was included; yet the cabinets and shelves where the marvels were on show evolved into the first museums of science.

In mathematics, they evolved to the displays of mathematical instruments and models that were common in universities in the nineteenth and early twentieth centuries. Here too, the audience was not the general public, but mathematicians and students. Thus, the exhibitions became linked to education. In the second half of the twentieth century, the number of mathematical exhibitions increased and tended to incorporate hands-on exhibits to involve visitors.

Exhibitions can be classified into one of three types, according to aim.

1. **Exhibitions to illustrate mathematical ideas or processes.** These are aimed at the general public, including teachers and students, and the exhibitors are normally mathematicians.

   The challenge, for exhibitors, is to communicate abstract mathematical ideas without misrepresenting them, while the challenge for visitors is to reconsider their own attitudes towards mathematics.

   When discussing challenge for visitors, it might be considered awkward not to include mathematics challenge in the form of solving a mathematical problem. Yet problem solving scarcely works when somebody stands for a few minutes in front of an exhibit in a noisy room! In exhibitions that take place in shopping centers, the "contact" of visitors is considered even when it lasts a few seconds.

   Examples of such exhibitions include mathematical corners in science centers (e.g. Prague, Boston, San Francisco), *Mathematikum* in Gießen, *Mathematica viva* in Lisbon, *Giardino di Archimede* in Florence, *Matemilano*, *Matetrentino*, and others in Italy, UNESCO Exhibition (Paris), exhibitions for the year 2000 and other special events.
2. **Exhibitions to illustrate some methodology for teaching.** These address the general public, but the focus is specifically directed to teachers and educators. The exhibitors tend to be mathematics educators.

   In these exhibitions, the challenge for exhibitors is to create teaching and learning environments that combine epistemological analysis with cognitive and didactical needs, while the challenge for visitors is to fixed ideas about how mathematics can be taught.

   Examples of such exhibitions include the Mathematical Machines exhibition in Modena and the Mirrors exhibition in Oporto.
3. **Exhibitions to illustrate the product of didactical innovation**. These exhibitions are aimed at the general public (including teachers, students and educators), and particularly towards parents and the local community. The exhibitors can be teachers, students and sometimes parents.

The challenge for exhibitors here is to bridge the gap between school and out-of-school experiences, while the challenge for visitors is to their own views about mathematics in school.

An example is *Matematica nella realtà* by Emma Castelnuovo, held at École Decroly in Belgium.

The reader is also referred to the Study Volume for ICMI Study 5, *Popularization of Mathematics*, particularly the chapters on mathematics education.

In every exhibition (as the exhibition space and the visitors' time are not unlimited!) a choice has to be made between the following:

- to focus on one particular topic; or
- to explore a lot of mathematical ideas.

The aim in each case is different. The former aims to show the depth of mathematical experience and to illustrate at a micro-level the network of mathematical processes. The latter aims to show the width of mathematical experience and to illustrate at a micro-level the network of mathematical processes.

Examples of the former include, *Oltre il compasso* (Beyond the compass, by *Giardino di Archimede*) in Italy, and Mathematical Machines (Modena), and the exhibition on Perspective at *la Villette* in Paris.

Examples of the latter are *Mathematikum* and UNESCO.

Before discussing some particular examples, we refer to the page of the *Il Giardino di Archimede* site web.math.unifi.it/archimede/archimede_NEW_inglese/presentazione2.html , where there is a discussion of what it means to exhibit mathematics, with a focus on philosophy.

2.2.4.2 Examples of exhibitions

We now discuss some well-known exhibitions in more detail.

Example 2.1: Hands-on Mathematics at Mathematikum (in Gießen).

The idea of the science centre *Mathematikum* has roots more than 10 years old. In 1993 Albrecht Beutelspacher (Professor of Mathematics at the University Gießen), together with a group of his mathematics students, started the first activities in hands-on mathematics.

The project had a simple goal: to give all people interested in mathematics, or even the not so interested, the opportunity to discover the beauty of mathematics by performing hands-on experiments.

"Being amazed is the first step to get behind the mathematical secrets", Professor Beutelspacher has said.

Starting with only a few geometrical exhibits in 1994, the project grew to more than 100 hands-on mathematics exhibitions in different German cities, supported by Professor Beutelspacher and his team. The activities of Albrecht Beutelspacher culminated in the opening of the science centre *Mathematikum* in November 2002.

Now *Mathematikum* is a very big hands-on mathematics exhibition in its own building. But *Mathematikum* is much more than a mathematical museum or exhibition in the common or traditional sense. Under the motto "learning by doing" the students, their teachers and their families are invited to learn mathematics using their hands, to think about mathematical experiments and phenomena and to find out mathematical secrets.

This special mathematics centre connects the exhibition of mathematical experiments with a lot of other activities. These include:

- workshops for children or students on special mathematical problems (such as the 2006 workshop "Number summer" on the mathematics of Pascal's triangle);
- "Maths for kids": children's lectures with mathematical topics dealing with daily life, nature and the world around us;
- "The mathematical couch": mathematical talks between students and invited mathematicians;
- science weekends;
- Art in mathematics: special expositions.

Important educational aspects include:

- giving a creative environment for doing mathematics; given a real thematic problem, understanding the situation, trying to find out the solution by experiments and understanding by doing;
- for all students: doing – understanding – loving mathematics;
- accompanying the mathematical experiments by theoretical information.

Thus, *Mathematikum* lives up to its motto. Its web site is at www.mathematikum.de.

Example 2.2: The 2004 Exhibition sponsored by UNESCO and ICMI.
Let us consider an example from this exhibit. In the tiling and symmetries corner of the exhibition *Experiencing Mathematics* (www.mathex.org/MathExpo/TilingsSymmetries), some colored wooden shapes are offered with a very short message on a poster.

Tiling techniques
Can we cover a floor with tiles of any shape without gaps or overlaps?

Many shapes work but not all, as, for example, a regular pentagon. Tiling patterns, which repeat periodically by translations, are well understood and their symmetries allow 17 different types of patterns.

The study of these types and their symmetries is based on group theory as devised by Evariste Galois. If we want to tile more freely—not periodically—the study is far from being finished. So, is it possible to tile using only one shape? It's a mystery! Tiling patterns find applications in mathematics, crystallography, codes, particle physics and other fields.

Challenges are only given. When visitors manipulate the material on their own, they are not expected to solve a mathematical problem, but to broaden their view of mathematics in several respects:

- mathematics may have aesthetic qualities;
- mathematics is not a frozen body as it still contains mysteries;
- mathematics may have applications (in tiling floors);
- mathematics may concern also physical objects.

This is an example of informal (or lifelong) learning. The visitor may find that a negative attitude towards mathematics they might have developed can be reversed. The organizers state their aim clearly:

> *Welcome to MathExhibition. We hope that you will take pleasure to go through this site. This exhibition is aimed particularly at young people, their parents and their teachers, but it is hoped that the ideas will interest all those who want to learn more about Science in general and Mathematics in particular.*

The same material may be used—in the exhibition itself or in an attached laboratory—under the guidance of an operator who can offer hints for reflection. The unprogrammed exploration is replaced by a more directed investigation that suggests paying attention to angles in order to understand why shapes may or may not tile a floor. The manipulation of physical objects may, in some cases, be substituted by movies or animations (as in the many web sites concerning tiling). These contexts are examples of non-formal education, where educational aims are not as strong as in the school system and do not usually reach any assessment step.

Finally, the same exhibit (maybe in multiple copies) may be used in a classroom, under teacher control with an explicit link to the mathematics curriculum. Children may be given challenging problems concerning the discovery of regularities. For instance, they are asked to measure angles by means of a protractor and to look for patterns, when a "perfect" tiling is made.

Example 2.3: Istituto e museo di storia della scienza Firenze (Italy): Exhibition of scientific instruments

This museum (see galileo.imss.firenze.it/) comprises exhibitions and learning materials on the history, the exploration and the use of historical mathematical instruments such as

- materials and possibilities to reproduce the instruments;
- interactive learning materials on the mathematics (geometry) and on the history of the instruments;
- multimedia applications (see brunelleschi.imss.fi.it/esplora/compasso/index.html);
- children's lectures on the instruments, their history and the mathematics behind;
- hand outs for children.

Special features include

- online exhibitions;
- online learning;
- educational activities for school groups (see www.imss.fi.it/espo/index.html).

Example 2.4: Museo della Matematica del Comune di Roma: I Racconti di Numeria
This is an exhibition with learning materials on historical instruments and mathematical models, and includes

- materials to understand the mathematics behind the instruments and models;
- children's lectures (see http://www2.comune.roma.it/museomatematica/).

Example 2.5: Historical mathematical collections in universities or high schools
Examples are the mathematical collection of the universities Göttingen or Halle.

Exhibition and learning materials on the history, the exploration and the use of historical mathematical instruments and models include

- handouts for reproducing the models;
- interactive learning materials;
- multimedia applications;
- children's lectures on special models, on the mathematics of the models, on the history of mathematics;
- solving of historical problems connected with the instruments or models.

2.2.5 Mathematics houses

Since 1999, in Iran, teams of teachers and university staff have established what are called Mathematics Houses throughout the country. The Houses are meant to provide opportunities for students and teachers at all levels to experience team work by being involved in a deeper understanding of mathematics through the use of various media. These include information technology and independent studies, feeling the essence of mathematics and learning about the history and applications of mathematical sciences, playing mathematics games and studying interdisciplinary ideas such as mathematics and art, studying mathematics and the ancient Iranian heritage and buildings, studying mathematics and genetics, mathematics and social sciences, and medical or engineering mathematics.

Team competitions, e-competitions, using mathematics in the real world, studies on the history of mathematics, the connections between mathematics and other subjects such as art and science, general expository lectures, exhibitions, workshops, summer camps and annual festivals are some of the non-classic mathematical activities of these houses. See Appendix 2.4.1 for more information about these houses.

The houses provide opportunities for public mathematics awareness, especially for families of the students. They also present mathematics through recreation and fun by playing with mathematics tools, learning the applications of mathematics and observing the mathematical contribution to art and other areas of science, technology, social or medical aspects of life.

The members of the houses can demonstrate their mathematical ability and thinking skills, and the houses can identify these abilities. Students enjoy the atmosphere of cooperative working and exchanging information in these houses and this helps to enrich their mathematical knowledge and skills.

Teachers and talented students may take part in all the activities or special programs of the houses, and in this way the houses nourish the higher achievers. Mathematics House is a playground and a center for providing good challenging infrastructure and its aim is to answer all the questions on the problems of the challenges in mathematics education. It provides an up-to-date state of art for making necessary challenges for teachers and students.

We do note that some other activities listed in this section are similar to the concept of the Mathematics House, such as the Adam Ries house referred to in Section 2.2.13.2.

The Mathematics organization *Archimedes* (see Appendix 2.4.2), located in Serbia, may be viewed as another similar organization, because of its permanent physical presence, but there are some differences in philosophy between these organizations, probably due to the personalities of the energetic people who founded them.

2.2.6 Mathematics lectures

In many mathematics departments weekly colloquia are designed for presenting new ideas, exchanging knowledge and introducing new discoveries. This is a good tool for enhancing research activities, starting joint research work and introducing new areas. They also provide a challenging atmosphere for the lecturer as well as the audience, prepare the students for presentations of their ideas and promote learning about different branches of mathematics.

In many schools, districts and societies' meetings, mathematics lectures given by appropriate people will help audiences (mostly teachers and students) exchange ideas and experiences and learn more about different methods of teaching, applications and the concepts of mathematics. These lectures may be delivered by invited speakers from universities or schools (teachers or even students), and the follow up questions and works on the subject make a challenging atmosphere for learning and doing mathematics.

Such lectures occur widely; examples are those in the UK by the Royal Society and the Sunday afternoon science lectures in Toronto by the Royal Canadian Institute. These latter lectures have been given each year since 1849 and have recently generally included one or two on mathematics in each season. Many museums also conduct science lectures which include mathematics topics.

By giving mathematics lectures all around the world mathematicians like Paul Erdös inspired many from their audiences to become researchers. As an example of an eminent case one can see in the biography of Kurt Gödel the influence of the lectures given by Furtwängler on Gödel's switch into mathematics (see www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html).

2.2.7 Mentoring mathematical minds

As discussed in Sheffield and Gavin (2006), one aim of mentoring is to narrow the gap between the established level and the potential for each student. Many countries have programs to prepare students for advanced rounds of competitions. For example, members of the national team often train over several months for the IMO. There is evidence that such programs are successful (Carroll and Carroll 2004).

2.2.8 Mathematics camps, summer schools

Camps and schools run for several days, sometimes a week or two where all participants are accommodated for the whole period. Along with mathematical activities, such as lectures, workshops, friendly competitions, small or large research investigations, there are contrasting social and recreational activities.

Benefits include:

- The opportunity to meet and have access to some prominent mathematicians and educators (they usually are in the organizing team or in the academic team) is so appealing and motivating to the students that they feel recognized, respected and privileged, which is a strong motivator for further improvement.
- Students get to know each other better in an informal environment, which helps build connections and friendship for many years to come. Good students keep in touch exchanging ideas on their further work, thus, motivating and supporting each other in future.
- Because the group of students is relatively homogeneous and many hours can be dedicated to a topic or investigation, it is possible to deliver a high-quality rigorous program that can greatly enrich the students in the long run.

Several examples are given in the next sections.

2.2.8.1 International Mathematics Tournament of Towns summer camp

The Tournament of Towns referred to in Section 2.2.1.2 also has a summer camp, to which winners of the competition are invited.

The camp is held in a country setting, a different place each year. The students arrive and there are a few in-depth problems presented to them. It will start with some work which is readily accessible, gradually developing an idea, until the last part is typically unknown even to the problem author. The authors are experienced composers of problems who stay at the camp for the week-long duration of the camp.

Students can work individually but usually work in teams. In the middle of the week they formally report back on their progress and at the end of the week there are prizes for the best solutions. Sometimes students have been known to solve open problems.

The camp has a number of other activities going simultaneously. There can be activities which are not mathematical, and a number of interesting ad hoc lectures are given, not just relating to the set problems.

2.2.8.2 International mathematics kangaroo summer camps

These are organized every year in several European countries, as a reward for the national winners of the European Kangaroo contest. For a week, international groups of students gather together and participate in many activities that involve team work, some friendly competitions, many recreational activities and trips.

2.2.8.3 Summer School Festival UM +

This festival runs every year in July for several days. It is the final round of the national contest for young talents UM +, for students from Years 4 to 7, organized by the Bulgarian mathematics magazine *Mathematics PLUS*.

The participants are selected from the best performing students after the three correspondence rounds of the contest. At the festival, they attend lectures and participate in workshops led by outstanding teachers and mathematicians. The tradition started in 1994, when the first UM + festival was organized within the program of the Second Conference of the WFNMC in Pravetz, Bulgaria. Then the first participants were honored to meet face-to-face legends of mathematics such as Paul Erdös and Peter O'Halloran.

2.2.8.4 The Canadian seminar

This camp, for about 50 students who are the winners of the Canadian national inclusive competition, is held at the University of Waterloo for a week each

June. The students live in a college of the university, and experience a number of activities, including lectures by invited mathematicians, problem-solving sessions, and some fun activities such as mathematics relays.

2.2.8.5 Isfahan summer camps

Students who take part in Isfahan mathematics summer camps (see www.math-house.org) are usually developing interest in other activities of the house and in doing so learn more mathematics. Those who attend these camps or summer schools learn mathematics through playing games, living together and enjoying social life. These camps and summer schools provide useful and efficient tools, not just for learning mathematics, but also teach the participants to hone social skills.

2.2.8.6 The Institute for Advanced Study in USA

This organization has a long history of running these programs (see www.admin.ias.edu/ma/1999/index.html).

2.2.9 *Correspondence programs*

These programs provide access to challenging activities for a wide range of interested students, many who live in remote areas and cannot be taught face-to-face. While the form and details vary, generally students get regular problem sets that have to be solved with a deadline of a week to a month.

These solutions go to professionals who send comments along with a collection of model solutions. Materials from past rounds are available and can be used as a resource for self-directed learning and training. These programs help identify talented students who may lack the opportunity to participate in contests and other challenging activities.

Correspondence schools and programs are very helpful and provide good opportunities for challenging mathematics.

Examples include

- Mathematical Olympiads Correspondence Program (Olymon), Canada (see www.cms.math.ca/Competitions/MOCP/ and www.math.utoronto.ca/barbeau).
- The national contest for young talents UM + , for students from grade 4 to 7, organized by the Bulgarian mathematics magazine *Mathematics PLUS*, Bulgaria. This contest runs in three correspondence rounds, where ten problems are posted. The students who submit the best solutions to these problems are invited to participate in the aforementioned three-day mathematics camp UM + .

- The national competition for students in grade 5 to 7 organized by the Bulgarian mathematics magazine *МАТЕМАТИКА*. This competition runs in several correspondence rounds during the year. The best students participate in the final round, where they compete face-to-face.
- A correspondence school for secondary students was established in Lithuania (Stankus and Kasuba 2006) to reach students across the country. This school, first operating during the Soviet era from 1969 to 1989, was supported by mathematicians at Vilnius University . Students of the school were first given a theoretical background. Problems were published in a Lithuanian daily newspaper, and a selection of these was chosen for students to write up their solutions and submit them for marking. This school has been renewed in the year 1998 on a wider basis with the cooperation of several universities, teachers and students. The Lithuanian Mathematical Society is also involved. The main object of this school is competition which provides a good atmosphere for challenging mathematics.
- In some Australian states, because of the remoteness of some students from the city in which the main IMO training takes place, state directors substitute face-to-face teaching with equivalent correspondence programs, which are often successful in identifying students for the IMO team; in any case, they lift the standard of participating students.
- In Iran, the university student members of the Isfahan Mathematics House correspond with Iranian and foreign scholars who live in the country or abroad, in order to develop their research skills (see www.mathhouse.org).
- In USA, the Gelfand Correspondence Program in Mathematics (GCPM) was established by I. Gelfand in 1990. It is administered by faculty and staff at the mathematics department of Rutgers University and is open to high school students all over the United States. Although it is based on experience from the Moscow Correspondence School, the GCPM program has been designed to be compatible with American education. Participation in GCPM is valuable for students intending to continue the study of mathematics or mathematically based sciences. The goal of GCPM is not just the education of future scientists, but enhancing a student's intellectual abilities, and will be useful for a student no matter what career they ultimately choose. For many students the most important component of a correspondence school is interaction with the mentors, teachers or scientists, which provides a catalyst for challenging mathematics outside the classroom later (see gcpm.rutgers .edu/more.html).
- *Leipziger Schülergesellschaft für Mathematik* (Mathematical Student Society Leipzig) is a correspondence program combined with seminar activities and camps or weekends. The central idea is challenging mathematically interested and gifted children with different activities under the supervision of mathematicians.

  This society offers to mathematically interested and gifted children different programs to enable them to come together and be engaged with mathematics. The main activities are seminars that regularly take place at the

University of Leipzig and are conducted by students and mathematicians from the mathematics department. These seminars address mainly high school students in Years 5 to 12 in the Leipzig region.

Interested pupils who cannot come to the meetings are sent letters with course material and exercises for their year level. The participants prepare answers and send them back. The tutors prepare sample solutions and remarks on the participants' solutions that are returned and discussed in three or four meetings held later in the academic year. Each year during the summer holiday time the society offers a special summer camp with mathematical training.

Additionally, for students in Years 9 to 12 the society organizes seminar weeks and seminar weekends (once per academic year for about 20 participants) with a more academic character. There the tutors (students and mathematicians from the mathematics department of the University) present lectures and discussions at an advanced mathematical level and give introductory classes on higher mathematics.

- Mathematical student journals or materials combined with a mathematical correspondence problem contest. These include: *Die Wurzel* (mathematical student's journal (grade 5 to 12), supported by the University of Jena), and *Monoid* (mathematical student's journal (Years 5 to 12), supported by the University of Mainz, see www.mathematik.uni-mainz/monoid). A further example is *Problems of the Month*, a correspondence program supported by the Hamburger Schülerzirkel Mathematik and given to all schools of Hamburg (www.hh.schule.de/ifl/mathematik/zirkel.htm).
- As another example, in *MATh.en.JEANS* (also discussed in Section 5.3.5) each team works in collaboration with a university researcher who has proposed a problem, ideally connected to their research, on which the students work for a long period (often up to a full school year).

  MATh.en.JEANS is a virtual laboratory of mathematical research (investigation), under the auspices of a scientific committee, open to all whether curious laymen, amateurs or professionals, that allows them

  (a) to learn about and better appreciate problems under investigation by professional mathematicians;
  (b) to follow a research protocol beginning with a question that is at once open (that is, the complete answer is not known) and accessible (it is posed in immediately understandable terms).

  Given mathematical problems of different levels the students are invited to solve them and to send their solutions to mathematical experts. Very good solutions are published in the journal of *MATh.en.JEANS* (see www.mjc-andre.org/pages/amej/accueil.htm). There are some important didactical aspects of the program, including

  (a) learning mathematics by solving interesting problems at home (under a relaxed, in some sense, but not really isolated atmosphere), and

(b) problem solving and solution discussion (via correspondence)

- problem solving;
- formulating the solution, or possibly formulating questions to an expert;
- understanding the answer of an expert;
- thinking about the answer;
- starting to solve the problem with the hints of the expert.

2.2.10 Web sites

ICT (Information and Communication Technology) is a good medium for the propagation of mathematical challenges; this will be treated in detail in Chapter 3. Both teachers and students enjoy the use of the Internet to acquire mathematical information. Especially for younger students, electronic books, competitions and weblogs are important tools for learning. Pozdnyakov (2006) noted that computers can be used to stimulate student research.

The web makes possible chat rooms that enable problem solvers around the globe to communicate, share notes and exchange problems. Significantly, in this environment, students are self-motivated and learn together without the need of teachers.

Many societies, associations and even schools make use of their web sites to teach mathematics. However, the most popular site among students aspiring to compete in national and international Olympiads is the Art of Problem Solving (www.artofproblemsolving.com) founded by Richard Rusczyk.

2.2.11 Public lectures, columns in newspapers, magazines, movies, books, general purpose journals

Mathematics can be interesting, exciting and attractive. Unfortunately, for many, it is no more than arithmetic calculations and simple practical measurement.

One of the first steps toward promoting mathematics challenges is to increase public awareness on what mathematics is and what mathematicians do. This is closely related to improving the publicity of mathematics-related events from all areas (research, education, applications, etc.) in the media and making them accessible to a larger number of people. To date, everything excellent that happens in mathematics institutes and laboratories is isolated and known in the first instance only to a relatively small group of specialists. If something comes via the media to the general attention of the public there can be misunderstandings among journalists. It is helpful to involve public relations specialists in the process of preparing and presenting mathematics news to the public in a popular, attractive and concise way.

Not only will this de-demonize the nature of mathematics as something too abstract to be applicable, but also it will show the variety of problems and themes mathematics comprises. As a result it then has more potential to seem connected to people's everyday life experience.

There has been an increasing trend for mathematics to be a background to movies, plays and books (such as *A Beautiful Mind*, *Proof* and *The Da Vinci Code*) and there is an increasing trend for the proof of theorems (Fermat, Poincare) and news events such as announcements of Fields Medals and the backgrounds of the winners to be reported. There has, for example, been considerable coverage of the Fields Medal announcements for former IMO Gold Medalists Perelman and Tao, and interesting stories about them.

2.2.12 Math days, open houses, promotional events for school students at universities

Examples of such events include:

- Mini-enrichment courses are organized by some universities in Ontario. Every year during the first week of May, high school students participate in a variety of mini-enrichment courses, taught by university professors. They have the opportunity to observe laboratories and research centers, and to get first-hand impressions on the everyday work of scientists and researchers. They increase students' interest and knowledge, and are not restricted to highly motivated and capable students only.
- Within Education week in May, the Department of Mathematics at Ottawa University, together with the Canadian Mathematical Society, organizes Mathematics Horizons Day for senior high school students (grade 11). About 30 schools participate, each with a team of four students. In the morning, the students attend a lecture on a selected math theme, presented by a faculty member. In the afternoon, they participate in a team contest, where they have to solve a sequence of 10 problems, divided into two rounds of five (known as a Math Regatta). The best three teams get trophies. For more information see www.mathstat.uottawa.ca/community/outreach_en.html.
- There are many other examples of successful Mathematics Days designed on similar models: Canberra (Australia) and Ulm (Germany) are two of many sites.

2.2.13 Mathematics fairs

These events are not competition-based and are designed to enable large numbers of students to experience some mathematics, in the form of lectures, displays, and interactive experiences, certainly providing challenge.

2.2.13.1 Canadian Andy Liu model

A very good example is the SNAP Math Fairs, founded by Andy Liu in Edmonton, AB, Canada. The acronym makes its characteristics clear: **S**tudent-centered, **N**on-competitive, **A**ll-inclusive, **P**roblem-based. A description with sample exhibits can be found on the website www.mathfair.com.

An active proponent of such fairs in Ontario is Tanya Thompson, who has organized them in Collingwood, Toronto and Ottawa. We describe an Ottawa example—a fair involving Year 7 students—to illustrate how interdisciplinary projects can be used to create a challenging environment. From about fifty problems and puzzles (some of them classical, such as the one of the farmer crossing the river with a wolf, goat and cabbage) proposed by the teacher, each student (or group of students) selects one.

They are required to write a story using characters from the Wizard of Oz that present the problem and to produce a poster that allows other students to try or demonstrate the problem. After the projects are completed, they are presented at a school exhibition where parents, teachers and other students are invited to participate. The presenting students played the role of authoritative guides who have to give suitable hints to those who attempt their problems.

2.2.13.2 A mathematical house for younger children (Years 1 to 5)

Located in the historical house of the arithmetic teacher Adam Ries in Annaberg, Germany, the arithmetic school gives younger students opportunities to discover the history of mathematics as well as special historical mathematical techniques.

Special activities for children's groups beyond the classroom include:

- children's lectures and workshops on the arithmetic of Adam Ries with and without the abacus;
- children's lectures and workshops on the mathematics of the Adam Ries era;
- workshops on historical mathematical problems of the Adam Ries era;
- workshops on the history of mathematics of the Ries era.

Important educational aspects are:

- learning mathematics in (a special) historical context;
- understanding basic mathematical techniques by using them for historical problems;
- understanding the connection of mathematics and history through special problems;
- opportunities for team work.

The web site is located at www.adam-ries-bund.de.

2.2.13.3 Mathematics day at universities

Examples are at Universities of Kaiserslautern and Hamburg. These are different from the more purely competition-based Mathematics Days such as those in

Canberra (University of Canberra and Australian National University) and Ulm (University of Ulm) (Section 2.2.12). At these events mathematics departments invite students of all grades to visit the university and to discover mathematics with activities such as:

- students' lectures with mathematical topics dealing with actual mathematical research;
- children's lectures with mathematical topics dealing with daily life, nature and the world around us;
- mathematics team contests;
- workshops on special mathematical problems.

2.2.13.4 Long night of mathematics at the high school, Karlsruhe

This activity involves lectures on interesting mathematical problems or topics given by mathematicians of the high school.

Important educational aspects include

- learning mathematics and doing mathematics in a nearly authentic research context;
- gaining experience in problem solving and discussing problems with mathematicians;
- being able to work at different levels.

2.2.13.5 India

The Montessori school in Lucknow has organized a series of mathematics and computer fairs, known as Macfairs. According to its web site (www.cmseducation.org/macfair), their goal is to expose the young "to technology of tomorrow through a series of grueling and interesting competitions". In the belief that competitions are a vital component of education, the events "strive to create a competitive atmosphere that is free of all limitations, prejudices and distinctions".

2.2.14 Mathematical quizzes

The popularization of mathematics and the growth of motivation for learning mathematics can be promoted by mathematical quizzes—a particular form of competition (on an individual or a team basis) that uses questions typically solvable in 10–30 seconds. Answers to such questions require prompt and meticulous thinking, contributing also to the development of mathematical reasoning.

Such quizzes can be stand-alone events, or can be very suitably run in conjunction with other events such as mathematics days and fairs.

2.2.14.1 The mathematical organization *Archimedes*

This organization has arranged more than hundred mathematical quizzes so far, usually during summer and winter mathematical camps and mathematical performances. Among these was the quiz "Sharpen Your Mind" on a regional broadcasting station with seven episodes. These quizzes have been appreciatively received by both the competitors and the audience. *Archimedes*' quiz usually involves three or four teams of 3–5 students from different grades, who solve the majority of problems in 10–30 seconds. The number of questions is usually between 10 and 15. Two quiz questions (one for younger and one for older students), taken from Kadijevich and Marinković (2006), are given here.

- Express 0 using three 3s (by using the digit three times, write an expression that is equal to zero). Time: 10 seconds. **Answer:** $(3 - 3) / 3$ or $(3 - 3) \times 3$.
- A box contains blue, green and red balls (in sufficient numbers for each color). How many balls should be minimally taken from this box in a blind draw to have at least four balls of the same color? Time: 20 seconds. **Answer:** 10 balls.

The experience of the *Archimedes* club suggests that tasks for mathematical quizzes, directed to challenges beyond the classroom, should be used in the classroom as well, to refresh regular mathematics education for the cognitive and affective benefits for all students.

2.3 Concluding remarks: challenging infrastructure—a powerful motivational factor

Young people are natural competitors. They like challenges and contests. As far as mathematics is concerned, participation in any form of competition (or competition-like event) brings a lot of benefits. Here is an incomplete list of them:

- Competitions provide an opportunity to compare to others and to elevate standards.
- Competitions build character and life-long skills such as perseverance, reasoning, communication and independence.
- Competitions motivate students to work hard preparing and practicing, because they are able to see feasible goals and real benefits from making the effort.
- Participating in some competitions, mathematics camps and schools provides interesting opportunities for socializing—students travel, meet new friends, experience exciting moments and build a network for further contacts. In this light, the competitive activities contribute to this very important aspect of their life as adolescents.

- Students willingly push themselves towards learning at a higher level of complexity, improving academically and achieving significant results. In the long run, the process of training and practicing for a contest is often more beneficial than the contest itself. A healthy portion of stress (if that is the right word, as it is free of the stress of normal assessment which has a more official status) is a powerful positive force in this process.
- Students learn to manage stress, they learn how to cope with negative emotions in case they do not win, and how to benefit and learn from their mistakes.
- It is a rewarding activity and brings the joy of success, pride of work well done and recognition by society. It builds self-esteem and motivates for further efforts.
- The benefits of competitions become even more significant when children have already got a sense of mathematics while in elementary school. The number of contests suitable for younger students seems to increase each year, providing further opportunities.

It is important to choose the challenge, regarding form and complexity to suit the age, abilities and training of the participants. Challenges can be adapted to all levels of achievement. Even those with limited abilities can benefit from a challenging environment. They will be involved in the investigation and strategizing from the outset, and so gain an intimacy with the mathematics involved. As a bonus, they may improve their skills at taking tests and managing stress.

It is also important to set achievable goals for each student. In some contests and for some students, the simple fact of participation is a great success. The principle of the Olympic Games is strongly exuded in Mathematics Olympics—let everybody participate in the game and do their best and let the best ones (on the day) win. But the spirit is largely to compete, show respect for your fellow competitors, and use the event to help widen your circle of friends.

2.4 Appendix

We describe here two organizations which, in recent decades, because of the diligence of a small number of committed teachers, have established strong programs with an ongoing physical presence.

Prominent organizations around the world have developed infrastructure for a range of enrichment programs, such as those at the Australian Mathematics Trust and the University of Waterloo in Canada, but the two described here are lesser known, and indicate what might be achieved with limited financial resources.

There are similarities between the organizations discussed here, but there are also some differences in philosophy. However they do form models of what could be achieved elsewhere.

2.4.1 Iran: what is a Mathematics House?

Mathematics Houses have become a distinct institution in Iran, with support of various kinds from municipal councils. We give here some background to the Iranian version, which is a relatively new, but rapidly growing phenomenon.

Although we have mentioned many mathematical centers in Section 2.2, the Mathematical Houses in Iran are distinctive. Their mandate is to provide an infrastructure to explore challenging mathematics outside of the classroom, not only for the benefit of teachers and students with a concomitant effect on the education system, but also for the public at large. Indeed, they are supported in part by municipal councils.

Isfahan Mathematics House, the first of its kind, was established as a project for the World Mathematical Year 2000 with the help of the Isfahan Municipal Council. Its operation in this historical and cultured city is the fruit of cooperation among mathematicians who love young people, mathematics and education.

2.4.1.1 History

To prepare for the World Mathematical Year (WMY), the First Iranian Mathematics Education Council (1st IMEC) was organized in Isfahan in 1996. This led to the establishment of Mathematics Teachers' Societies across Iran, the enrichment of mathematics education and the provision of information technology facilities for teachers and their students. A high commission headed by the President of Iran for the observance of the WMY was set up, and, in 1997, took as a goal the creation of mathematics houses to function in part as centers of research. The first one opened in Isfahan in 1997.

The goals of these were:

1. popularizing mathematics;
2. investigating the history of mathematics;
3. investigating the applications of mathematics, statistics and computer science;
4. developing information technology;
5. expanding mathematical sciences among young students;
6. promoting team working among young students and teachers.

through:

- procuring facilities for non-conventional education;
- introducing new instructional techniques;
- establishing scientific data banks;
- encouraging joint and collaborative research;
- modeling and applying mathematical sciences;
- welcoming relevant novel ideas.

To date, there are mathematics houses in Isfahan, Neishabour, Tabariz, Yazd, Kerman, Khomein, Kashmar, Sabzevar, Babul, Zenjan, Gazvin, Gonbad and Najafabad. To regulate the cooperation among these, a high commission for the chain of houses has been established. The web site for IMH is www.mathhouse.org.

2.4.1.2 Audiences

As the houses serve as a playground for non-conventional education, an information center for the history of mathematics and a place to familiarize young people with various mathematical sciences through observation, collaboration and access to resources, they serve the general public, students of all levels and their families, including gifted and blind students, teachers and even university professors, graduate students, researchers and artists.

2.4.1.3 Activities

1. **Lectures (popular and special topics)**
   One of the main activities is a series of public lectures directed to both professionals and amateurs that are designed to expose the history and scope of mathematics and its significance for their lives. Each year at the Isfahan Mathematics House (IMH) there are five or six expository lectures as well as special talks for special groups of students, teachers and members of the house.

   Especially popular are talks about the mathematics of architecture, as Isfahan is a showpiece for the achievements of ancient scientists and architects.

   Special talks on mathematics and its applications by Iranian and foreign scholars help different groups use ideas in their own investigations. Talks and workshops on mathematics education help teachers become aware of new pedagogical developments and to find ways to achieve a better learning environment in their classes.
2. **Mathematics and information technologies exhibitions**
   Mathematics and statistics lectures and playrooms allow observers, students especially, the opportunity to experiment with mathematical tools to develop a feel for mathematics and its different branches and applications. There are special "days" and "weeks" in which exhibits create a challenging environment for visitors, particularly children and their families.

   The houses provide computer facilities where participants can use and develop software, access the Internet and be taught mathematics electronically. Many of them have their own site with pages for statistics, e-competitions and teaching tips.

2.4.1.4 Activities for high school students

Expository mathematics workshops are held regularly to encourage cooperative research among students and to acquaint them with mathematical concepts.

- **Research groups**
  High school members of the houses join a research group. They work in various fields and present the result of their investigations in annual festivals or in publications.
- **Mathematics team competition**
  IMH organizes the participation of students in the International Tournament of Towns; unlike other cities, Isfahan students participate as teams instead of individually. In October, 2006, a new statistics team competition was organized.
- **The Isfahan school net**
  To establish electronic communication for schools and provide information technology for education and research, IMH established the Isfahan School Net.
- **Robotics workshops**
  Workshops have been instituted to acquaint students with robotics and organize student robotics teams.
- **Camps and problem-solving workshops**
  These activities are designed to popularize mathematics and expand the experience of young people through the provision of challenging situations to spur mathematical learning.

2.4.1.5 Activities for university students

- **Statistics day**
  This is observed annually with the assistance of student scientific societies to foster research work among them.
- **Research groups**
  Research groups of university students are organized to help them participate in collaborative research through electronic communication with Iranian researchers abroad. Interdisciplinary studies and research are their main functions.
- **Entrepreneurship**
  University students have the opportunity to design web pages and software.
- **Introductory workshops**
  Students become versed in using mathematics and statistics software.

2.4.1.6 Activities for teachers

- **Research groups**
  To motivate and support teachers as well as secondary and tertiary students in research, IMH has organized teachers' research groups

in various educational fields. This has also been done by some other houses.

- **Information technology workshops**
  The workshops are held in conjunction with scientific societies for teachers to train them in the use of modern educational devices and to acquaint them with information technology.
- **Teachers' workshops**
  IMH has conducted workshops on goals, standards and concepts of mathematics education in elementary schools for teachers. High school teachers are served by workshops on new secondary courses and information technology.

2.4.1.7 Other activities

Apart from the activities described above, the houses host seminars for studying problems of university entrance examinations with the cooperation of major universities and educational organizations, which lead to modifications in the admissions process. The houses also exchange ideas among themselves, as well as with a number of professional organizations, such as the Adib Astronomy Centre, the Iranian Mathematical Society, the Iranian Statistical Society, the Isfahan Mathematics Teachers' Society, the Iranian Association for Mathematics Teachers' Societies, the Scientific Society for the Development of Modern Iran, the Ababasir Educational Centre and the Science and Art Foundation.

2.4.1.8 Library

IMH and some other houses maintain specialized libraries which provide access to other information resources in the country, particularly the library of the Teachers' Research Centers as well as the I.P.M. library in Tehran.

2.4.1.9 Laboratories

The mathematics, statistics and physical science laboratories provide good facilities for visiting students and teachers of many levels. At IMH, a group of researchers are preparing the blind to invoke their hearing acumen to use standard computer facilities and access the Internet and software packages.

2.4.1.10 Achievements

In a short period, the houses have altered how students can learn mathematics and introduced new methods for teachers. They have presented new ideas, nationally and internationally, for developing mathematical sciences, promoted information technology among students and teachers, and enjoyed success in teaching mathematics and computer science to the blind.

In 2002, the houses received first prize and special recognition in the fifth round of awards of the Society for the Development of Sciences of Iran. IMH teams won the 2001 and 2002 Kharazmi National Festivals. The houses are receiving increased recognition internationally.

2.4.2 Serbia: the mathematics organization Archimedes

In Belgrade, Serbia, there is a fascinating organization, which acts as a high-level base where students are challenged, and is a place to visit much like the Iranian Mathematics Houses. It has an open, democratic structure, but is largely under the inspiration of a very remarkable and dedicated mathematics teacher by the name of Bogoljub Marinković. There is a building in central Belgrade which acts as a mathematics house, catering for students of a range of standards. The club attracts the most skilful teachers and is in effect the centre of training of students who will form the nucleus of the Serbian IMO team. The following information is based on Kadijevich and Marinković (2006).

The organization is called the *Archimedes* Mathematical Club, and has the web site www.arhimedes.co.yu. *Archimedes* is much more than a normal club because of its activities and respectable results concerning challenging mathematics beyond the classroom, and in-service professional development of mathematics teachers. For example, students who have improved their mathematical knowledge and skills in *Archimedes*' mathematical schools and camps have traditionally been among the best solvers at local examinations, and at regional, state and international mathematical competitions. Despite such a continuous success, the financial support from the state and other local authorities has been minimal, with most of the drive behind the club coming from teachers acting in a voluntary capacity.

As regards challenging mathematics beyond the classroom, *Archimedes* has dealt with most questions raised by the ICMI Study 16 Discussion Document for more than thirty years. In order to justify this statement, we summarize below some of the club's activities and the main lessons learned from these activities.

2.4.2.1 Activities

Archimedes was founded in 1973 and its main goals are

- the organization of mathematical schools as well as summer and winter mathematical camps ;
- the popularization of mathematics and science in general;
- the organization of mathematical competitions;
- the organization of in-service professional development of mathematics teachers;

- the formation of a specialized library containing some of the leading national and international mathematical publications;
- the publication of various materials for mathematics education (tests, books, booklets, magazines).

Archimedes' members are mathematics teachers and students at all educational levels as well as mathematical enthusiasts and other interested adults. *Archimedes'* teachers are not only experts from universities and other institutions, but also distinguished teachers from primary and secondary schools. *Archimedes* is governed by an 11-member board.

Basic statistics (1973–2007)

- mathematical schools and camps: more than 22,000 students in schools (33 generations) and about 10,000 students in 95 camps (about 44,000 lessons delivered);
- popularization of mathematics and science: 75 popular lectures or presentations and about 200 other gatherings (quizzes, exhibitions, etc.), about 23,000 persons present;
- mathematics team competition: 62 competitions with more than 3,400 teams (one team per school) comprising 4 or 5 students (about 15,000 students);
- International Mathematics Tournament of Towns: participation of Belgrade team for the regular part and Serbian students for the final summer conference;
- Kangaroo-like competitions in 2006 and 2007 with about 7,000 and 14,000 participants, respectively;
- in-service teacher professional development: 1,330 presentations/lectures for primary and secondary teachers of mathematics and informatics (at seminars and other professional events) with about 55,000 teachers present;
- specialized mathematics library: about 25,000 books and about 5,500 issues of journals and magazines with many rare publications (about three-quarters of these publications are available to teachers and students);
- publication production: about 300 titles of various publications published in more than two million copies;
- registered members: about 27,000 members; more than 90 per cent are pupils and students and 800–1,000 members are active each year.

2.4.2.2 Lessons learned

Challenging mathematics should be arranged in a continuous and well planned way. In each grade, students in the *Archimedes* schools have a 90-minute lesson each week over 25–30 weeks. There are two programs for deepening and extending mathematical knowledge. For students who like mathematics but are not competition minded, there is a standard program. For students with a good record at mathematical competitions who have passed *Archimedes*' entrance examination , there is an advanced program. Each program concludes with a test.

The test's results are very good because not only do students work in homogenous groups, but they are also taught by distinguished teachers (many of whom have developed their expertise within *Archimedes*). A continuous and well-planned use of mathematical challenges is also arranged for candidates for national teams for international mathematical competitions, enabling *Archimedes*' students to be the most successful members of these teams.

Mathematics competitions should require both basic and advanced mathematical knowledge.

A team mathematics competition (an open team championship of Serbian secondary schools) arranged by *Archimedes* has proved a good combination of team and individual competitions at a national level. These competitions are arranged for lower secondary school (Years 4–8) and upper secondary school (Years 9–12). Each team includes one student from each grade.

To help students gradually and successfully deal with challenges, students in each grade work on two groups of tasks: the first group comprises tasks reflecting the official mathematics curriculum, while the second one consists of non-standard tasks from the additional mathematical curriculum (Marinković 2004). Tasks used in *Archimedes* are usually (1) non routine, (2) with good mathematical ideas behind them, (3) interesting with respect to formulation and content, (4) with nice and perhaps unexpected answers, and (5) with short outline solutions that requires students to attend to details.

Challenging mathematics should use mathematically isomorphic tasks that differ contextually.

If not resolved appropriately, the re-use of challenging tasks may be an significant obstacle to their infusion into mathematics education. An appropriate approach may be found in creating and using sets of mathematically isomorphic problems that differ contextually. Such sets may comprise tasks that have mathematical themes that are very similar or isomorphic. Another possibility is to have sets of tasks whose solution methods are very similar or identical. What follows is a sample of tasks with the same mathematical themes taken from Kadijevich and Marinković (2006).

- At a test each student correctly solved at least five tasks. Also, each task was correctly solved exactly by four students. Were there more students or tasks at this test? [Hint: Start with $5s < 4t$.]
- Marabs and Sarabs live in one country. Each Marab knows 9 Marabs and 7 Sarabs, whereas each Sarab knows 6 Marabs and 8 Sarabs. Are there more Marabs or Sarabs in that country? [Hint: Make use of $7m = 6s$.]

Challenging mathematics requires challenging professional development of mathematics teachers.

To help teachers deal with challenging mathematics, *Archimedes* has organized a continuous and well-planned in-service professional development program for mathematics teachers. This development is based upon ten lectures/

presentations per year (one per month except for July and August) as well as 6–8 lectures/presentations arranged during the traditional one-day winter seminar. However, even such continuous and well-planned professional development of mathematics teachers has had a small global impact on the use of mathematical challenges in the classroom.

According to the results from *Archimedes*' mathematical competitions, probably just about 5–10 per cent of all students have met mathematical challenges in the way cultivated by *Archimedes*. A promising way to improve matters requires *Archimedes* to have more challenging professional development of mathematics teachers that makes use of some versatile digital tools as detailed in Chapter 3.

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Chapter 3
Technological Environments beyond the Classroom

Viktor Freiman, Djordje Kadijevich, Gerard Kuntz, Sergey Pozdnyakov, and Ingvill Stedøy

This chapter discusses several options that technology can bring to mathematics teaching and learning, in providing access to many challenging virtual resources that become easily available to all learners. Recent studies point to new challenging learning opportunities that may be enhanced by technology by means of dynamic and interactive tools of modeling and experimenting. Finally, recent developments of flexible and sophisticated communication tools create numerous virtual spaces where people can meet, ask questions, discuss, and work collaboratively on challenging mathematical learning tasks. We present a broad view of existing worldwide practices and analyze concrete examples that deepen our understanding of the advantages and disadvantages related to integrating technology into challenging mathematical activities in and beyond the classroom. We conclude with several research paths that are opening in this relatively new field of study in mathematics education.

3.1 Introduction

Playing different roles in different stages of human development, technology serves as a tool facilitating mathematical computing, application of mathematical knowledge, model building and investigating and learning mathematics. Inspired by the image of mathematical challenge as an idea to revitalize discourse about the role of mathematics in education culture, in classrooms and beyond, we can emphasize the potential benefits of technology in promoting and/or providing of mathematical challenge as a tool to develop curiosity, imagination, inventiveness and creativity.

As educators, we can also see a close connection between technology and mathematics throughout history and in all cultures, especially in the recent development of technology based on the new generations of computers and the

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Internet. These push our perception of mathematical activity beyond the limits of traditional educational forms and settings, offering new challenging opportunities of doing and learning mathematics for all.

Chapter 7 of this Study Volume discusses the role of technology in making challenging tasks in the classroom accessible at the lower school levels. At the same time, technology gives also numerous challenging opportunities to "break the walls of the classroom" and to bring everyone into the wonderful and fascinating world of mathematics through the freedom of entertainment and freedom of learning, freedom of fun and freedom of challenge. ("Technology breaks the walls of the classroom" is an expression used by Jean-Marie De Koninck, the author of the 29-episode TV series *C'est mathématique* and the more recent project ShowMath (www.smac.ulaval.ca/showmath/) at his opening speech of the 2004 winter meeting of the Canadian Mathematical Society (www.math.ca/).)

Moreover, this freedom of choice in technology offers a unique source for intrinsic motivation that helps to combine in one activity entertainment with learning and fun with challenge. This particular role of technology, or ICT (Information and Communication Technology), the term used in a much broader sense, has been underlined by Kennewell (2004, p. 96) as a catalyst of challenging activities:

"ICT can bring difficult tasks within the ZPD (Zone of Proximal Development, in the Vygotskian sense, discussed in more detail in Section 6.2.2.3 and also discussed in Sections 4.5.4 and 7.3.2) of more pupils through screen cues that provide scaffolding not available from the teacher. Tasks can be motivating and enjoyable because of the interactivity. Conjecture and risk taking are encouraged by the provisionality so that pupils can learn constructively from their mistakes. This fosters perseverance which enables pupils to meet the high expectations placed on them and appreciate the role of personal effort in achieving success." (Vygotsky 1978)

An unprecedented growth of web-based educational resources allows Klotz (2003) to affirm that "in mathematics, as in other disciplines, the Web is expanding our concept of the classroom itself, changing what gets learned and how, affecting student-teacher relationship, and providing access to new types of mathematical activities and resources that can be used by teachers".

Mathematical challenges can be differentiated to meet the educational needs of all learners, by giving them access to tools unavailable in the classroom. Learning environments enhanced by technology offer access to traditional mathematics outside the classroom as well as to new forms of mathematics unrealizable in classroom settings.

Chapter 2 of this Study Volume analyzes a number of types of beyond the classroom activities that generate different outcomes such as public awareness, recreation and fun, identification of mathematical abilities and thinking skills, enriching mathematical knowledge and skills, and helping to nurture high achievers with more challenging mathematics.

In this chapter, we will analyze how technology may contribute to the realization of these outcomes and look at its advantages and disadvantages. We will study in depth how technology can become a catalyst of challenging mathematical activity beyond the classroom. We will discuss what kind of challenging mathematics can be supported by technology, analyze different technological tools that help to create challenging learning environments, as well as discuss the diversity of forms of challenging mathematics supported by these tools.

At the same time, we will study several examples of new opportunities of challenging mathematics that can be created by use of technology. We will also analyze pedagogical and technological issues that mathematics educators may face if they want to integrate technology into challenging mathematical activity in and beyond the classroom.

3.2 Technology and challenging mathematics beyond the classroom: tool of learning and fun

We can categorize the activities offered through technology into four groups:

- mathematical publications and media addressing the general public and professional mathematicians (TV programs, books, magazines);
- mathematical events addressed to the individuals, community and family (conferences, fairs, competitions);
- general, mathematics-specific, and interdisciplinary activities (puzzles, games, clubs, circles, correspondence schools);
- web-based resources in mathematics (encyclopedias, dictionaries, discussion forums, applets, learning scenarios, lessons).

Activities from the first can be presented in electronic form (text and multimedia), provide users with information about activities and give immediate feedback through discussion, blogs and wikis. With a user-friendly system of management, technology gives access that is both flexible and adaptable to user needs any time, anywhere and for everyone.

The activities from the second group can be supported by virtual resources (Internet or CD-ROMs) giving information about the events and allowing registration (online) and communication with other participants. The third group can be enhanced by all resources mentioned as well as online and CD-ROM-based interactive resources (in the form of applets). The last group contains all kinds of resources including those that can only be used with the help of technology (programming, multimedia design, modeling, investigations, etc.).

Mariotti (2002) sees technology as a catalyst transforming socio-mathematical norms. More specifically, she stresses a new kind of relationship between problems and knowledge regarding the type of problem as well as the process of solution. It might be a direct relationship in the case of symbolic manipulators such as DERIVE or an indirect one when professional software like EXCEL or AUTOCAD can be used to solve mathematical problems. The reference Mariotti (2002) made to the classical work of Papert remains valid in terms of unprecedented opportunities that information technology separately and in all combinations opens in terms of the quality of learning environments at work, at school, and in recreation.

Technology can enhance the mathematical development of young children offering them attractive challenging environments. Parents can provide children with these activities in the form of educational software (CD-ROMs) and multiple resources on the Internet. These activities can stimulate logical thinking (puzzles, patterns, card games, and many others), number and spatial skills (adjusting to the level of children), categorization, ordering, classification and grouping abilities (leading to the higher levels of conceptual thinking in the Piagetian sense).

There are three major types of educational software that provide children with mathematical challenge: sets of learning activities of different kinds, animated stories that allow children to explore a situation while engaged in mathematically challenging activities, and finally, all kinds of creativity software that allow children to build their own world by using, among many options, mathematical structures such as models, shapes, and transformations.

As examples of the first type of activities, we can name Millie's Math House (www.kidsclick.com/descrip/millies_math.htm), Mighty Math Number Heroes (www.kidsclick.com/descrip/mm_numbhero.htm), Thinking Things collection 1–3 (www.kidsclick.com/descrip/tt3.htm), JumpStart Learning System (www.knowledgeadventure.com/), ADIBOU (www.adibou.com/), Math Rabbit's collection (www.learningcompany.com/), and many others. The second type can be illustrated with animated story books like Disney's Mulan, Arthur's Computer Adventures, The Cat in the Hat, Winnie the Pooh and Tiger Too (www.superkids.com/aweb/pages/reviews/e_read/3/sw_sum1.shtml). The third type of activities can be done with all kinds of creative artist's software like KidPix, Read, Write and Type (www.learningcompany.com/).

All kinds of electronic mind-boosting games can also be useful to nurture mathematical curiosity in very young children by challenges: LEGO, chess, checkers, cards, TIC-TAC-TOE, puzzles, labyrinths, and others (www.funbrain.com/; www.crocodilus.org/; mzone.mweb.co.za/residents/profmd/challenges.html; www2.toulouse.iufm.fr/rallye/; www.legoeducation.com/store/SearchResult.aspx?pl = 7&pt = 8&bhcp = 1; www.chessmaster.com/us/; compgeom.cs.uiuc.edu/~jeffe/mathgames.html).

All these computer-based activities are very attractive to young learners because of multimedia features (colors and animations), high level of adaptability giving children free choice of activity, friendly feedback and easy-to-use interfaces that they can personalize. Interactivity is another factor attracting young learners.

Much of this technology-supported educational and recreational material may help children sharpen their minds as they move on through their schooling. They complement the regular classroom.

During school years, learners can benefit also from all kinds of online mathematical challenging activities, such as math clubs and problem-solving contests. They can become members of informal online mathematical communities building strong interpersonal ties. We will analyze some examples of these activities in later sections. These kind of mathematical activities are very important since they can give everyone access to the variety of mathematical resources and activities which can be offered only by technology.

It is very important to mention that this variety of resources supported by technology, and the high level of their adaptability to the needs of different categories of learners are key elements that can provide appropriate challenges to mathematically gifted children and children with special needs or disabilities. Some of these children may, for various reasons, not be able to attend school. For them, technology becomes the only window into the world of mathematics. Kennewell (2004) analyzes four different roles that ICT can play in providing differentiation:

- by task: different pupils are working on different aspects of the problem during project-based activity;
- by response: interactivity of ICT allows able pupils go beyond the basic learning objectives making and testing their own conjectures during investigation;
- by support: ICT resources are generally adaptable to the level of pupil's understanding guiding her with appropriate cues;
- by resource: ICT supported educational resources can be created in different formats (images, words, layouts) that can be suitable to pupil's learning capacity.

We stress the crucial role of ICT resources in compensating for disabilities to enable pupils with vision or hearing problems to function at a higher level than they would otherwise be able to attain.

The multiple intelligences theory also supports these particular roles of ICT (Campbell et al. 2006). Regarding development of logical mathematical intelligence it puts an emphasis on modeling and simulation as methods of mathematical activity that combine mathematical challenges with various multimedia tools that give an immediate retroaction to the learner.

It allows the latter to move further than simple use 'kill-and-drill' computer programs in order to reach superior levels of Bloom's taxonomy. For example, the Quebec-based site *Récréomath* (www.recreomath.qc.ca) proposes enrichment activities that go beyond traditional classroom mathematical practices.

As students move to the higher grades of their schooling, they might face new challenges such as a choice of profession that may require a higher level of mathematical knowledge than might be provided by schools. Many technology resources are available for self-learners, helping them to prepare for all kinds of exams and competitions.

Adults can also find many useful and interesting ways to deal with challenging mathematics through many mathematical entertainments, popular mathematical e-literature, and lifetime learning opportunities. Finally, everyone who uses mathematics of any kind at work can find in technology some personal challenges.

3.3 What kind of challenging mathematics beyond the classroom can be supported by technology?

Information and communication technology (ICT) has radically modified our working space. Educational technology (ET) , on the contrary, has rather marginally contributed to changing ways of teaching and learning so far. However, there is a considerable potential for ET to be meaningfully integrated into the process of teaching and learning in general (Jonassen 2004) and mathematics in particular (King et al. 2001, Guin et al. 2005). Today's level of ET development allows use of powerful digital tools and learning environments that can help teachers extend their work to the new "going beyond the classroom" mathematics practice. For example, an automated theorem prover can be integrated with a geometric microworld in such a way that properties, suggested by experiments on constructed objects, can be not only formally verified (e.g. WinGCLC at www.emis.de/misc/software/gclc/), but also added with their illustrations and proofs to a knowledge repository for later uses (Janicic and Quaresma 2006).

3.3.1 Types of digital tools and their support for challenging mathematics

3.3.1.1 A context for thinking skills and digital tools

The ISTE (International Society for Technology in Education) Educational Technology Standards for students require students to use educational technology as versatile tools for creativity and innovation, communication and collaboration, research and information fluency, as well as for critical thinking, problem solving and decision making (www.iste.org). As one digital tool (e.g. Casio ClassPad) can be used for creativity, research fluency and decision making, types of digital tools should rather be defined with respect to relevant aspects of knowledge such as development, presentation, and sharing.

By doing this, we can, following Jonassen (2000), make a distinction between *semantic organization tools* (e.g. databases and concept maps tools), *dynamic* *modeling tools* (e.g. spreadsheets and microworlds), *interpretation tools* (e.g. search tools and visualization tools), *knowledge construction tools* (e.g. hypermedia and multimedia tools), and *conversation tools* (e.g. asynchronous and synchronous conferencing tools). Furthermore, as computers should be used in constructivist ways that "engage learners in representing, manipulating, and reflecting on what they know, not reproducing what someone tells them" (Jonassen 2000, p. 10), learners should try to utilize these tools primarily as mindtools.

According to Jonassen (2000), a degree to which a particular type of digital tool is used as a mindtool depends on whether, and to what extent, this tool type promotes different thinking skills, among which he underlines the following:

- *critical thinking skills*, which involve major skills of evaluating, analyzing, and connecting;
- *creative thinking skills*, which call for general skills of elaborating, synthesizing, and imagining;
- *complex thinking skills*, which comprise three major types of skills: designing, problem solving and decision making.

Although the degree to which one digital tool is used as a mindtool depends upon the features of learner, assigned task, and utilized learning environment, different types of digital tools may provide support for different major thinking skills.

Our analysis of Jonassen (2000) reveals that while all nine types of major thinking skills reported can be promoted by tools for semantic organization, dynamic modeling and knowledge construction, interpretative tools can mostly promote skills of evaluating, connecting and imagining. Additionally, although conversation tools mostly promote skills of evaluating, elaborating, and synthesizing, they can also promote skills of designing, problem solving and decision making, especially when asynchronous conferencing tools are used.

3.3.1.2 Challenging mathematics in this context

The expression *challenging mathematics* is typically used to describe mathematics or a mathematical task that is interesting and perhaps enjoyable, but not easy to deal with or attain. Although the degree of challenge of a particular task, which can be described in terms of a range and depth of major thinking skills required, obviously depends upon the features of the learner and learning environment, many tasks that can require the learner to transfer, justify, check, compare, extend, integrate or apply some mathematical results can be challenging.

Challenging tasks may particularly be those that require the learner to relate mathematical entities (concepts or procedures), by considering, for example, their different representations, views or applications (Kadijevich 2007). Solutions to such connecting tasks appropriate for learning projects beyond the classroom are primarily supported by a dynamic modeling tool, which, for example, enables simultaneously-updated displays in algebraic and geometric windows (e.g. Casio ClassPad and TI-Nspire). In addition, a semantic organization, interpretation or knowledge construction tool may be used.

Challenging mathematics entails going beyond the traditional presentation, understanding and use of mathematics. A challenging task is thus, for example, one that requires mathematics teachers to present the historical, structural and applicative issues of the examined topic, enabling different learning paths within that topic. Such a task is called Historical Structural Applicative (HSA) (Kadijevich 2004). What digital tools are needed to support the challenges of this task?

Being an instructional design task, the principal tool is a knowledge construction tool (e.g. hypertext writing environment StorySpace, multimedia authoring software Opus or web development package MS Front Page). Types of digital tools that support meeting challenges of its underlying subtasks are:

- interpretation tools for historical and structural issues (a search tool like Google for historical issues and a visualization tool such as JavaView for structural ones);
- dynamic modeling tools for structural and applicative issues (e.g. GeoGebra, Fathom and TI-Nspire).

Solutions of task HSA may be attained through a collaborative work supported by a conversation tool (e.g. ClearWiki). Of course, at a simple level, tools for knowledge construction and conversation may be attained by widely used software such as MS Word (use its option File/Save as web page) and Mozilla Firefox (explore your e-mail account by using web-mail service).

The development and use of solutions to task HSA, due to its complexity, are more suitable for mathematics learning beyond the classroom. In order to make this learning more successful, learners should be, for mathematically complex tools, scaffolded through a process that transforms digital tools they use (i.e. impersonal devices) into their digital instruments (i.e. personal devices). This process, called instrumental genesis, is usually hindered by some features of the applied tools, which should be recognized and adequately pedagogically treated (Guin et al. 2005).

3.3.2 Two approaches to challenging mathematics by hypermedia learning

3.3.2.1 Learning through design

Hypermedia is multimedia hypertext, that is, hypertext expressed in several media such as text, graphic, sound and animation. As already underlined, hypermedia and multimedia tools are used for knowledge construction. As developers of instructional material learn more from these materials than their users, environments for developing hypermedia, multimedia or web applications (e.g. Opus, StorySpace or MS Front Page) can be powerful tools for knowledge construction (Jonassen 2000), especially when learning units are based upon sound multimedia learning principles like those given in Mayer (2001).

Mathematics teachers may develop multimedia mathematical lessons that respect the following six requirements:

1) use at least words and pictures to develop multimedia lessons;
2) achieve a good technical realization of the developed artefact;
3) present the mathematical structure of the examined topic;
4) show the application(s) of this topic;
5) make possible different learning paths within the topic;
6) deal with relevant conceptual and procedural mathematical knowledge and their relation.

A recent study examined the implementation of the six requirements for future Serbian teachers of mathematics (Kadijevich 2004). The teacher first made the Serbian designers aware of the six requirements and some web sites where suitable, applet-based lessons could be found. A five-week group project was then set with no technical support given. Suggestions for improvements were sent to students who asked for feedback using e-mail.

HTML pages based on applets or other kinds of animations (simple hypermedia) were successfully developed for various topics despite missing technical skills at the beginning (using software such as Euklid, developing and using Java applets within web pages). It was much easier for the designers to fulfill requirements 3) and 4) than 5), which was in turn much easier than 6). Many students liked the project because it encouraged teamwork and creativity.

The outcomes of this study show that mathematics teachers can cope with HSA-like tasks with some success, which would be higher when pedagogy is appropriately linked to technology to support the use of various types of major thinking skills. By using a detailed approach to hypermedia design with various digital tools, further research may thoughtfully examine the learning difficulties and opportunities of this kind of learning.

3.3.2.2 Learner-tailored instruction

According to Cassarino (2003), e-learning should be based upon the delivery of interactive content according to the user's choice of the implemented states of learning and the built-in navigational modes supporting the chosen state that differs from state to state. To attain this end, a hypermedia/multimedia solution to task HSA may be, developed, for example, in a general way that can be personalized for particular learners according to their preferences for learning path, targeted competence, and instructional mode, such as:

- *learning path*: from applicative issues, to structural and epistemological issues, to historical issues (defined by available sequencing of the implemented global issues);

- *targeted competence*: making connections (chosen from those available of the implemented competences);
- *instructional mode*: exploring and examining different standpoints and their consequences (selected from a list of available strategies or components of the implemented instructional design).

Of course, available options for learning path, targeted competence and instructional mode (possibly constraining each other) may differ from topic to topic.

Such a conceptualized generator of personalized solutions to the HSA task, which should also help the learner challenge the applied learning process, appears as a tool of the future. However, there is already a platform that is close to this generator called *ActiveMath* (www.activemath.org).

This web-based, multi-lingual platform combines cutting edge approaches to e-learning and intelligent tutoring systems: individual courses are assembled according to learning goal, learning scenario, competences, learning content and preferences specified by the learner; during tutoring, realized according to the learner's model, the learner can take the initiative; the learner can explore and negotiate with his/her model generated by the system; the structures of the examined mathematical domains can be visualized; and the content of ActiveMath can be searched by using both text search and semantic search.

Such a platform clearly helps learners engage more explicitly in the process of knowledge construction (Melis et al. 2006). As the use of several Computer Algebra systems is supported by ActiveMath (the work on a collaborative discussion tool is under way), the user can indeed make use of various tools for semantic organization, dynamic modeling, interpretation and knowledge construction to ease meeting challenges of mathematical content and its presentation, as well as challenges of the management of their e-learning.

Further research may thus examine how different aspects of the learner's work with ActiveMath (designing instruction and learning from it with its possible modifications) influence his/her cognitive, metacognitive and affective e-learning outcomes.

3.4 Making mathematics beyond the classroom more challenging

Here we discuss what kind of technological environments might make mathematics beyond the classroom more challenging and we investigate related technological, pedagogical and social issues.

The opportunity to manage an environment is a powerful internal stimulus for human activity. In other words it means use of an environment as a tool for realization of mental plans (Vygotsky). Constructive activity is a prime example of such a possibility. Using computer programs allows constructive activities such as the creation of geometrical or mechanical virtual objects. That gives a new challenging dimension to such a productive class of problems as "problems with straightedge and compasses". It expands this area by "problems in geometrical transformations", "problems in geometrical measuring" and "problems in relation to analytical and geometrical representations of objects", and can foster mathematical investigations in and beyond the classroom.

In this context one can ask if such well-known mind-boosting tools as LOGO can still be used in providing young learners with mathematical challenges. As it has been argued by Maddux et al. (1997), self-correcting, highly motivating LOGO can provide practice in spatial relationships as well as using mathematics in an arena free of negative associations and experiences. Building special technology-supported environments for distractible children, who do not often succeed at school, can help them be more successful in mathematics and thus have positive mathematical experiences. They may, at the same time, change their attitude towards mathematics and improve social skills and peer relations.

3.4.1 Software to support mathematical investigation

Dynamic geometry software, formal calculation tools, spreadsheets, graphic design software and calculators are extraordinary tools of mathematical investigation. One who masters these instruments won't take long to use them to search for patterns and invariants making conjectures (which itself is an important piece of mathematical competence to be acquired) and trying to test them and prove those which seem to be plausible.

Utilization of these tools helps the user get more autonomy and power to test different hypotheses and study particular in-depth cases. By intelligent and inventive use of different software, modifying parameters and changing registers, students get more insight into problems. The more the user becomes creative with these tools, the better performing they can become. Moreover, technology tools favor development of such important qualities of human activity in general and mathematical activity in particular, as initiative, invention and creativity. With the help of technology, mathematical activity can be enriched adding an experimental dimension that can be somehow lost in traditional learning environments. This dimension, in its turn, can enhance mathematical questioning.

Use of this kind of software also brings a new domain into extracurricular mathematical activities like statistics (using spreadsheets to generate random data upon certain criteria) and mathematical modeling. These two domains are significant examples of how technology might enhance mathematical experiment.

Today's mathematicians have to develop, as early as possible, a competence in mathematical experimentation with technology. Therefore, there is a need for more problems, in and beyond the classroom activities, that integrate fully all the technological advantages and challenges we have discussed.

At the same time, technology can play the role of a Trojan horse due to the pixel-type nature of the computer screen. Critical evaluation and reflection of results obtained with technology become an important part of the investigation process. Students must be aware of situations such as ill-conditioning related to approximations and graphs obtained with calculation tools that might not yield accurate answers. This awareness has to be part of their learning so that their discoveries and control of technology have a solid mathematical base.

Technology lets one deal with new kinds of mathematical activities which are not part of the traditional school curriculum. According to Rubin (1999), increasing the number of tools for dealing with complexity is one of the most important developments in technology for mathematicians. We stress that accessibility for mathematicians, as for students, of tools of mathematical modeling like Mathematica, AgentSheets, StarLOGO and MAPLE make it possible to investigate by simulation. Examples of such activities include models of natural systems (most popularly, predator/prey systems).

Using a tool to explore these patterns not only gives students the opportunity to learn about a biological interaction, but teaches them about functions, variables, cyclical functions and sensitivity analysis. Exploring these concepts by hand is practically impossible, since the number of calculations necessary to yield any kind of pattern is astronomical. A whole new area of mathematics becomes suddenly available to middle and high school students.

Another example is the rise of the field of chaos. Previously the field of study for a few mathematicians, chaos is now an example of the connection between mathematics and art facilitated by technology. Even middle school students can create tessellations with Tessellmania or fractal images with simple programming languages. As has been mentioned by Rubin (1999), “the implications of the rise of chaos in the mathematical community have yet to filter through the educational system, but the addition of such an engrossing and artistic topic may turn out to be an opportunity to engage and challenge more students”.

Due to their very universal nature, the technology tools of mathematical simulations and modeling can play an essential role in various challenging activities beyond the classroom, like mathematical competitions, Olympiads, mathematical recreations, expositions addressed to general public. They can also stimulate more challenging mathematical activities within the classroom.

Use of computer tools changes the nature of mathematical activities. Now we can assert the existence of experimental mathematics. The experimental work helps to search for mathematical properties and verify conjectures preceding theoretical proof. The same tendency can be seen in the process of teaching mathematics. For example, creators of problems for Olympiad competitions use the tools of dynamic geometry to search for geometrical invariants to be proved by contestants. This represents a significant shift in a new direction using ideas of experimental mathematics with technology to enhance challenging mathematics beyond the classroom.

The essence of many interesting mathematical facts can be shown by means of visualization tools, which are provided by modern computer technologies. Use of such material in the pedagogical process of teaching mathematics will nurture the interest of the students to study important mathematical ideas.

The understanding of some difficult mathematical laws becomes accessible to students, who had difficulties in understanding them previously.

We must take account, not only of the stimulus, but also of pedagogical obstacles in introducing challenging mathematics. An interesting and challenging problem may appear to be inconsequential to a student lacking knowledge and experience. Accordingly, the student needs an opportunity for discussion in order to appreciate the subtleties and difficulties. Textbook problems should be augmented by tools for verifying partial solutions. This will give a second life to these books and expand the spectrum of problem types, helping to facilitate or even replace communication between student and teacher. Computer tools provide alternative ways to verify students' assertions. Such verification may be partial to check the correspondence between the solution and the necessary conditions for it. But it could suffice in leading the student to understand some mistake or mistaking of the problem. Further development of computer instruments will make it possible to use tools for the verification of theorem proving in and beyond the school settings.

3.4.2 Numerical working spaces

New opportunities for challenging mathematics are related to the Internet-based virtual so-called Numerical Working Spaces (NWS) . They may offer the learner a unique opportunity to benefit from all these tools at the same time, namely software for experimentation and modeling, diverse databases (such as mathematical, pedagogical and didactical knowledge bases, problem banks, banks of clearly explained problem solutions, philosophical and conceptual issues) and communication and discussion space for all users. They are thus adaptable to all challenges independently of their form, allowing the public at large to discover and to deepen their mathematical knowledge and expertise.

The NWS is therefore a powerful tool for preparation of contenders (groups or individuals) for mathematical competitions (Rallies, Kangourou, Olympiads) (www.cijm.org/; www.mathkang.org/default.html; cemc.uwaterloo.ca/; www.amt.edu.au/mcya.html; www.math.toronto.edu/oz/turgor/archives.php).

They also provide an opportunity to extend the realm of mathematical exposition to virtual reality and to continue a dialogue with the public. A NWS is also a tool that makes possible online meetings for different mathematical groups and clubs. It can also be used inside the classroom giving a further taste of mathematics for beginners.

For example, Math Forum (mathforum.org) is one such numerical working space that contains the following elements:

- The Problem of the Week which offers a new challenging non-standard math problem every other week that can be answered online or offline, with the opportunity for feedback from mentors.

- Ask Dr. Math in which all participants can get advice from professional and expert volunteers.
- Math Tools allows exploration and discussion about the use and rating of the variety of interactive tools for understanding mathematical concepts.
- Teacher2Teacher invites educators from around the world to work together on many challenges of teaching and learning mathematics.
- Other options include online mentoring and teacher professional development, face-to-face workshops and collaboration with practicing teachers.

Most of these services at Math Forum were developed with research funding and volunteer support. Some of them now charge a nominal fee for operating costs.

There exist a number of NWSs that give access to all through a public Internet site. But it is also possible to develop a spirit of a virtual community, by restricting access to those with a username and password that can be obtained through electronic registration. More information can be required in the case of groups (mathematics clubs) whose aim is preparing for mathematical rallies and competitions. Very often, NWSs offer different levels of access adapting to every user's need, which reflects the very nature of the NWS.

To construct an efficient NWS, one needs to create the highest possible level of flexibility for opening or locking access to particular tools based upon pedagogical objectives, tasks and activities, and categories of users.

3.4.2.1 NWS1. Goal adaptable databases

Depending on the goals of the mathematical activities within a NWS, databases can accomplish several functions: knowledge base (facts and formulas needed to solve particular problems), problem base (problems from past years for mathematical competitions with solutions), discussion forums, comments from teachers and experts. Data can be organized in different forms: texts, multimedia applets, videos of conferences, and so on.

For example, in order to construct a site with mathematical problems, we need to create mechanisms to present and discuss solutions, to classify and store problems and solutions. The site devoted to research in a concrete area of mathematics must have a system of references to other works in this area, special journals, information about conferences, a mechanism for publication of results, and so on. Sites which aim to invite the broader public must be grounded in brilliant and highly innovative ideas, which would attract new people to do challenging mathematics. One such site is www.websudoku.com/ .

The content of databases is constantly updated by offering a larger number and variety of electronic documents and activities. It becomes a kind of collective memory of competitions, expositions and all kinds of mathematical activities whose aim is to provide every member with mathematical challenge and entertainment, thus deepening their knowledge, in particular, of complex areas of mathematics as well as nurturing public interest in mathematics by means of puzzles, games and recreation.

The immediate, diverse and flexible access to these electronic documents changes radically the very nature of mathematical learning beyond the classroom, giving more power, and giving the user more autonomy in choosing the time, the place and the type of challenging mathematical activity.

3.4.2.2 NWS2. Communication and exchange

There are different communication tools that can also be used in these virtual spaces to enable direct chat, and file and video exchange between members.

A group of people who participate in a common activity via the common information space build a so-called community of practice. The main feature of such a community of practice is to create a common information space as a necessary mechanism for the achievement of goals which are common for this group.

Students who join such communities may belong to different classes or schools; they may have different ages and different goals of participation for activity beyond the classroom. Some of them would be interested in participation in Olympiad-related activities, others would be involved in some kind of research activity, but all of them find an additional source of motivation in the opportunity to communicate with peers and experts. Anyone might find a partner for the purpose of discussing questions, propositions and conjectures.

Each contribution is carefully kept in the computer memory, organized in a form that allows users to follow the development of discussion and to interact with other users (publimath.irem.univ-mrs.fr/biblio/AAA05010.htm). The virtual communication space becomes thus a vehicle of co-construction and sharing of knowledge in which everyone can contribute. The discussion is monitored and directed by teachers and experts. They can intervene, bringing new ideas and asking new questions.

The main advantage of the Internet is that it provides an opportunity for people to communicate with each other from different parts of the world. Another advantage is that the role of teachers can be fulfilled by people who do not necessarily have direct educational training (such as scientists or specialists in specific areas). But there are corresponding disadvantages. People in virtual space groups may have no experience of working together, in contrast to the traditional classroom where students may study together over a long period of time.

So participants of virtual common space must learn to use common language, which can be inconvenient for some participants (the term "language" here means all avenues of communication) and may affect the quality of exchange and productivity of work. Finally, group participants may have different interests and background regarding the context of the activity. So we must take into account all these factors in implementing challenging mathematics beyond the classroom by Internet-based learning environments.

Several examples of existing communication spaces prove their high educational potential to stimulate different categories of learners to engage with more challenging mathematics in and beyond the classroom, increasing their motivation and engagement (publimath.irem.univ-mrs.fr/biblio/AAA05010.htm).

In such learning environments teachers and experts would have to adjust to new roles and functions as moderators and guides. They would have more freedom to take charge of the most difficult questions by reformulating them, giving hints and clues. The community would free them of other easier issues handling them by themselves. The activity helps to create a network of members in which everyone's input would be welcomed and appreciated.

3.4.2.3 NWS3. Freedom for learning mathematics

If one compares features of an activity beyond the classroom with those in the classroom, the main difference would be "mental freedom". In activity beyond the classroom, many restrictions, which are typical for classroom activity, are eliminated. There are no restrictions in choosing content, no limitations of time and age.

So the main question in the creation of a technological environment is how to support the mental freedom of students to assist them in studying mathematics. Technological environments for supporting activity beyond the classroom can be distributed between two big groups:

- technological environments for common use; they will support any common activity without restrictions on content;
- technological environments for supporting special mathematical needs.

The first group is good for supporting "near mathematics" activities where special tools for organizing mathematical information are not needed.

The next group is necessary for discussing mathematical results by writing formulas, doing algebraic transformations, drawing pictures. For such purposes there are special tools such as "virtual mathematical blackboards" or dynamic geometry software, which give a great variety of communication means. An interactive technological environment can test a solution of a problem and eventually point to some mistakes, helping students in understanding of the problem and in discussing productive ideas on how to solve it.

Communication via Internet may be supported by pedagogical assistance, which would articulate the mathematical ideas of participants or give hints for open-ended problems.

A NWS gives its members full freedom to choose the time and duration of their connections to the space. All members are free to participate in the discussion, ask questions and share their ideas, and be in full and efficient control of a chosen activity. This freedom breaks down hierarchy and dependence on teachers, supervisors and other members. Everyone becomes a partner in the search for solutions and answers in a context where trial and error is a normal and well accepted strategy that frees the spirit of discovery and creativity, allowing full expression of ideas and development of a collective mind.

The new learning tools enhanced by technology modify essentially the ways of dealing with all kinds of mathematical challenges. They change the well-known teaching-learning routine where the teacher plays the important role of knowledge holder and translator, and students are passively consuming information and methods, which gives them a certain feeling of comfort. It is not surprising that both teachers and students may meet such profound change with some controversy. But those who accept and adapt to the new rules of a NWS could find pleasure of freedom and autonomy of teaching and learning.

3.4.2.4 NWS4. Local, regional and worldwide expansion

The flexibility and adaptability of NWSs (along with required resources) expand rapidly, attracting experienced and gifted mathematicians, and at the same time, others who just like to taste the beauty of mathematics. Sites which can be particularly useful are language specific for languages other than English, like Spanish and Russian, which are languages common to many countries.

For a common information space to be effective, its participants must share common features and goals.

In the next section, we will analyze some examples of NWS and their possible contribution to mathematical challenges.

3.4.3 Issues related to cost and maintenance

In order to develop and run a NWS, we need to take into account several issues related to the technical and financial components, because of the complexity of such systems. For example, we need not only find optimal technical solutions but also constantly upgrade and replace some of their components or restructure them completely. There are costs of programming, hosting, content development, management and monitoring by experts, which require stable and continuous financing. This issue has to be carefully studied from the very beginning of the project. A modular structure would facilitate the development, and much experimentation is needed to find the most efficient format and optimize costs.

3.4.4 Psychological difficulties

We have alluded to psychological discomfort as another obstacle to be considered with technical and financial issues. We might expect resistance from teachers concerned with losing control over the development of their students. They might fear being challenged by learners who might go beyond their expertise. In these new environments, teachers have to accept the same uncomfortable role as learners as their students (in all its technical, pedagogical, didactical and cognitive aspects).

We can say that the real challenge of bringing more challenging mathematics *beyond* the classroom is related to the willingness of teachers to be challenged. They have to realize that it is a real pleasure to move from unknown to new knowledge, even if it comes along with a hard, sometimes even destabilizing, search for solutions to difficult problems. But how would a teacher challenge students without being challenged? Accordingly, it is important to cultivate the culture of challenge in the process of initial teacher training, ensuring that new generations of teachers become more receptive to challenges in NWSs, in teaching and learning.

3.5 Challenging mathematics beyond the classroom using Internet-based learning environments

In their analysis of issues related to online collaborative learning in mathematics, Nason and Woodruff (2004) reveal two obstacles in establishing and maintaining computer-supported knowledge building communities (CSCL) in mathematics: the inability of most "textbook" mathematics problems to elicit fruitful discussion (a content-related issue), and the limitations of the representational tools to be used (a technology-related issue). They thus opt for innovative approaches to be used in mathematical CSCL, allowing for the creation of authentic mathematical problems that involve students in model-eliciting mathematical experiences and in developing new tools to represent mathematical problems, to allow more constructive hypermedia-mediated discourse (Nason and Woodruff 2004, p.104). In the following section, we will discuss how these innovations can help to build challenging mathematical virtual environments and analyze several promising pedagogical and technological solutions.

3.5.1 Problem of the week challenge

We have known for many years that problem-solving abilities play an important part in the development of mathematical understanding. Recent work (Lampert and Cobb 2003, Sfard 2003) has also revealed that the role of communication should be more emphasized in the learning processes used to construct an understanding of mathematics concepts. "When students are given the opportunity to communicate about mathematics, they engage thinking skills and processes that are crucial in developing mathematical literacy" (Pugalee 2001, p.296). We can then say that communicating mathematical ideas is part of building mathematical understanding. However, those communication activities need to be authentic and they have to make sense for the students. The use of information and communication technologies (ICT) can become a great tool to create significant activities. As a matter of fact, the National Council of Teachers of Mathematics (2002) asserts that technology is an essential component in the learning and teaching of mathematics. Not only can technology guide the mathematical concepts that are being taught but it could also increase the learning potential of students by offering a variety of challenging problems.

In her model for the development of mathematically promising children, Sheffield (1999) opted for the use of multidimensional tasks within the scope of a heuristic and open model, which would contribute to the development of abilities to create, make links, investigate, communicate, and evaluate. In short, in order to succeed in teaching mathematics, it is necessary to create a good equilibrium between the routine and well-structured tasks, and those more creative and innovative. The students need to create a solid repertoire of positive experiences in problem solving, which would allow them to develop their self-confidence and potential (Klein 2003).

Several recent studies report a positive effect of virtual problem-based environments on the pupils' motivation toward mathematics. For example, the NRICH Project (www.nrich.math.org) was created in 1996 with the original idea to provide the most talented students with enriched resources online that are not otherwise available in schools. In their framework for mathematics enrichment, the authors of the project share their research findings:

- Such on-line resources are not suitable solely for the most able, but have something to offer pupils of nearly all abilities.
- Enrichment is not only an issue of content, but a teaching approach that offers opportunities for exploration, discovery and communication.
- Effective mediation offers a key to unlocking the barriers to engagement and learning (Piggot 2004).

An example of a pedagogically powerful virtual environment has been created within the project Math Forum (mathforum.org). Members of a virtual community interact around the services and resources participants generate together. These interactions provide a basis for participant knowledge building about mathematics, pedagogy, and/or technology. The interactions also contribute to what might be described as a Math Forum culture that encourages collaboration on problem posing and problem solving (Renninger and Shumar 2002).

Seeing a problem-solving communication process as a teaching approach brings yet another dimension to such projects. The project *Solving Mathematical Problems within a Virtual Environment* was started by a group of educators from several Quebec universities (Charbonneau 2000). It promotes a connection between teacher training programs and schools, setting up a virtual bank of mathematical problems and a computer environment for discussion and communication. The Internet-based project CASMI (www.umoncton.ca/casmi) was created in 2000 on the following three assumptions:

- Challenging problem solving is an essential part of learning mathematics.
- Communication plays a significant role in reaching a higher level of understanding of mathematics.
- ICT could help create authentic learning situations in mathematics attracting a larger population of students to rich mathematical resources.

CASMI became one of the more informal educational resources developed to motivate school students from all school levels to solve challenging problems, and thus contribute to the deepening of their understanding of mathematical concepts and the improvement of their abilities to reason and to communicate mathematically. Pursuing a long-time tradition of mathematicians challenging each other with difficult problems by normal mail, it turned out to be a valuable example of meaningful integration of technology into a mathematics classroom of the 21st century.

As a result of a close collaboration between the university and the schools, the project had three original goals:

- to be a tool to help francophone schoolchildren develop problem-solving and communication skills;
- to be a tool for pre-service teacher training in mathematics education, helping students to learn how to integrate technology in a mathematics classroom, how to evaluate a solution of mathematical problem in a formative way, and how to understand children's mathematical reasoning and ideas beyond it;
- to be a resource for teachers for challenging activities in and beyond the classroom (Vézina and Langlais 2002).

The functioning of CASMI is quite simple. Every Monday morning (for 30 weeks during the school year), four problems are posted on the web site. Problems are divided in the following way: one problem intended for students at the level Kindergarten to Year 2, one problem for students at the elementary level (Year 3 to Year 5), one problem for students at the middle school level (Year 6 to Year 8), and finally one problem for students at the high school level (Year 9 to Year 12). Students have the week to send their solutions, which should include sufficient explanation to meet the requirements on communicating mathematical information, using a web-based form or e-mail. Once the solutions are received, the pre-service teachers' work begins.

Grouped in teams, they have to assess the work done by the students. First, they look at all the solutions that were sent. Then they write constructive messages aimed at improving students' problem solving abilities. The messages are personally sent to each student who has given an e-mail address for receiving feedback. They also select from all the solutions received some that can be defined as exemplary for featuring on the web site. Some pre-service teachers were also occasionally involved in creating problems for the web site. The dynamic between the pre-service teachers and the students from the school system seems to be a positive one. It meets the different needs of both groups of participants: helping to improve problem solving and communication abilities for some and to develop better teaching and assessment strategies for others.

The CASMI virtual community is built upon a few basic principles:

- Friendly welcome: Every student from Kindergarten to Grade 12 is welcome to join the community at any time; it is free of charge and obligation.
- Math challenge: Everyone can submit a solution to any challenging math problem and send it to the CASMI team.
- Formative feedback: School students who send their solution get a personal comment from a university student; this comment is always positive and encouraging; it aims to encourage each participant to be persistent and continue to participate.
- Variety: The CASMI environment is open to a variety of styles and strategies; they attempt to understand different ways of thinking.
- Open communication: Communication is a vehicle of the community as it promotes knowledge sharing and knowledge building through collaboration and discussion.

In fact, the importance of challenge in mathematics through problem solving and communication turns out to be an important discovery pre-service teachers make while undertaking a CASMI project within mathematics education courses. This evidence is confirmed by schoolchildren who say (although in fewer numbers) that this feedback helps them to improve their problem-solving skills.

Research data also show that the university students find CASMI beneficial for the development of mathematical communication with young children (Freiman et al. 2005). This communication is fruitful when it is bidirectional. This means that pupils are encouraged to communicate their mathematical ideas using a variety of tools. Pre-service teachers should be able to understand this variety, to appreciate it and to guide children through their problem-solving process.

As an informal resource, CASMI gives pupils a chance to choose an appropriate problem and solve it at their own pace using their own strategy and communication tools. The fact that each participant gets a personal comment from a university student can be seen as a motivating and challenging factor for schoolchildren. They see that their work attracts the attention of other people and is being socially validated by personal attention or even public recognition. Children become directly aware of this recognition when their solution is posted as interesting or their name is placed on the congratulation list.

3.5.2 Collaborative problem solving: challenge and historical context

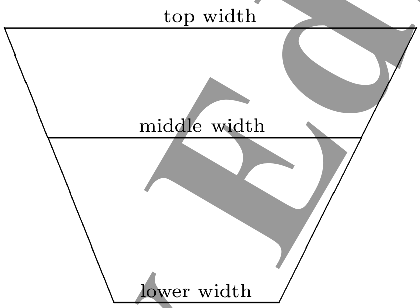
What would happen if we gave the same open-ended problem to learners from 20 different classrooms and asked them to work together using the same virtual collaborative space? Such observations can be made by readers of the magazine *Activités mathématiques et scientifiques* through the remote access given by the IREM of Montpellier to the collaborative learning space Plei@d (sudest.pleiad.net) that records all activity.

Here is the text of one Babylonian problem.

3.5.2.1 A Babylonian problem

In Mesopotamia fields are trapezium-shaped.

A land surveyor is to share a field equally between two brothers. The field is a trapezium. The two bases are 7 and 17; the two shared areas are both trapeziums.



We observe the following Babylonian vocabulary:

- 17 is the "top width".
- 7 is the "lower width".
- The fair share line which is parallel to the two bases is called the "middle width".

Question: Find the middle width.

Twenty classes were split into six groups of three or four. Exchanges were recorded within each group by a tutor, and a project team coordinated all activities. The tutor led discussion, helped to settle some issues and gave suggestions. One of these suggestions was to use CABRI dynamic software (which was not used by students spontaneously to test their conjectures).

The problem was addressed initially to older students (2e–Terminale, in French, or Years 10 to 12). However, it also attracted some groups of students from Grades 6e–5e (in French, or Years 6 and 7) to whom it presented additional challenge. In fact, the younger students were surprisingly ingenious and creative.

One of the teachers of these classes realized this and they decided to use this problem in the beginning of the school year in order to install an atmosphere of investigation into the classrooms. This investigation helped to introduce later important concepts of area, Thales theorem and equations. So children could see the utility of what has to be learnt. In addition, the use of CABRI software was seen by students not as an external tool imposed by teachers but rather accepted it as a meaningful and important tool for problem solving and investigation.

3.5.2.2 *L'Agora de Pythagore*: virtual communityof young mathematical philosophers

L'Agora de Pythagore (euler.cyberscol.qc.ca/pythagore/nav/index.html) is a virtual learning community that invites several groups of schoolchildren to participate in discussion forums linked to mathematics, its nature and its learning.

The discussion starts in the classroom where the teacher takes up with the students some philosophical-mathematical questions. Students will then pursue this discussion with peers around the world using an Internet-based environment. Pallascio (2003) analyzes an example of such discussion proposed by a group of students around the question of what geometric shapes can be constructed only by straightedge and compasses, and why the ancient philosophers became so interested in this question. Following the discussion, the researchers made interesting observations. For example, contrary to the fact that mathematicians of ancient times believed that well-known constructability problems like duplication of a cube could be somehow solved, today's children would right away believe that it is non-solvable.

In the conclusion to his analysis Pallascio (2003) notes that in such communities there is a need for an expert who can play the role of a scientific guide. They also ask what could be a source of motivation for children to participate actively in the discussion. It seems that the vehicle for this participation would be the role of the student not only as an actor in the process of learning but also as the *author*, creator of their knowledge. The environment supported by technology creates thus an educational situation in which students can interact, interpret, build on conjectures, and produce reflective arguments.

3.6 Effect on practices of teachers

The advent of technology has led to reconsideration of the function of the teacher. Instead of working alone in the classroom, she can exchange ideas and collaborate with others. Her role changes from that of knowledge provider to that of learner and guide in a challenging technological environment.

It has to be stressed that with technology not only do the classroom's physical boundaries tend to disappear but so also do boundaries between teachers and learners. In fact, everyone becomes a learner independent of the chosen activity. According to Putnam (2003), for students, computers and calculators provide new ways to represent abstract mathematical ideas, rich sites with realistic data for mathematical problem solving, and computational tools that can enable them to focus on "higher level" aspects of problem solving. For teachers, technologies give unprecedented opportunities to access information and interactions with others.

Video and multimedia hold great potential for representing teaching and learning in rich ways that can serve as examples for others and support self-study and improvement. On another side, the immense Internet-based source of mathematical activities of all kinds seems to bring new dimensions to mathematics education, if we as educators would "focus on the expressive possibilities of the Internet and engage distributed population in challenging, constructing, and discussing knowledge" (Iseke-Barnes 2001, p. 302).

Furthermore, Hollebrands and Zbiek (2004) report a new possible role of teacher as collaborator. This opportunity arises in a situation when technology allows students to explore roads unfamiliar to the teacher or not part of the activity the teacher had planned. In such situations, for example, students can get an opportunity to pose a novel problem and to enter a different area of mathematics. Teachers can then make their way with the student as collaborator by testing together conjectures leading to the development of a new solution that yields interesting mathematical connections.

This shift of teacher-learner roles also puts emphasis on the development of communities of teachers in order to co-create and to share resources, ideas and experiences. The creation of sites and the remote exchange about emblematic software (dynamic geometry, computer algebra, spreadsheets, etc.) help the teacher to come out from isolation and from doubt related to difficulties of the teaching job. The information gets circulated, training is permanent, experiments, whether successful or disappointing, are circulated. Everyone's ideas are made available for all. The individualistic teacher discovers collective intelligence.

We give four examples which illustrate the evolution of choice.

3.6.1 Mathenpoche

The program *Mathenpoche* ("Maths in the pocket") (mathenpoche.sesamath.net/) virtually speaks for itself. A small number of French teachers, forming an association, have given themselves the mission of creating online resources to cover all of the mathematics taught in *Collège* (Years 6–11). Those resources are to be available to teachers without cost, with parents also having free access. They asked many colleagues, over various sites, to suggest ideas for exercises. Scripts were written and put online. They were tested in classes on a large scale, and improved according to the criticisms that were received. A vast exchange occurred between hundreds of teachers, leading to the evolution from basic multiple-choice items to more delicate and elaborate forms, including open problems.

Groups were constituted in various IREMs so as to create resources, which integrate into their collection all the ideas and experiences accumulated over thirty years of educational and didactic research. Supported by its numerous participants, the collection evolves, becomes richer and diversifies. No doubt, it is the one most commonly used in French *Collèges* (secondary school, Years 6 to 11). An official experiment was lead by the Ministry of Education in the *département* of "Seine et Marne" (*département* here means one of the 95 French administrative districts). Some 80 per cent of the teachers in charge of *sixième* (Years 6 to 7) participated on a volunteer basis. Teachers, pupils and parents unanimously approved the experience. It is to be carried further in *cinquième* (Years 7 to 8) in the following year. *Mathenpoche* will probably expand considerably in the coming years and become a standard for the teaching of mathematics in France. Extensions to primary schools and *lycée* (Years 11 to 13) are being studied.

3.6.2 WIMS

WIMS (Web Interactive Mathematics (now Multipurpose) Server: wims.univ-mrs.fr/ with the English version available at wims.univ-mrs.fr/wims/wims.cgi? lang = en& + session = 69F0DBD8AC.1& + module = home), a virtual tool for sharing, has many points in common with *Mathenpoche*.

It was created by just one man, Gang Xiao (a Professor at Nice University), and it is now involves many colleagues from a number of departments. Among those are many teachers of mathematics from *collège*, *lycée* or *université*. Though it was created in order to teach mathematics, there is no reason why it could not be used to teach other subjects like chemistry, electricity, French and English. That interesting prospect would also apply to *Mathenpoche* which could be diverted to fields other than mathematics. The powerful tools of ICT have broad application, so that even the Ministry of Education might find possibilities in economies of scale.

Unlike *Mathenpoche*, whose creators provide the content, WIMS is dependent on the users to develop content. Each user is invited to become a creator of exercises and activities in turn. Being tested and improved by use, online resources gradually gain in importance. But the fields covered by that basis depend on the conceivers' centers of interest. A data base of mathematics exercises is implemented (unlike *Mathenpoche*, which tries to cover all the fields included in the programs of secondary schools, WIMS only deals with the themes suggested by its creators) and offered, free of charge, to everyone. That work is collective and not exhaustive.

Like *Mathenpoche*, WIMS allows the creation of virtual classes, which makes it a tool well adapted to the work of a teacher with a class or group of students.

3.6.3 PUBLIREM

PUBLIREM (www.univ-irem.fr/index.php?module = Publirem&func = view) is a search engine for the various online resources of the French IREM. Most of the resources evolved in various research groups over several years. They provide rich and complex situations. These resources are not directly meant for the students. They require the mediation of a teacher who draws inspiration from them and adapts them to the needs of his class. Teachers find it useful for planning, and can use the source collaboratively. PUBLIREM is scarcely used in comparison with *Mathenpoche*. (At first sight students can be set to work on *Mathenpoche* without any special preparation. But there is a risk of poor results and of loss of control of the class.)

Many other sites could be mentioned, which are close in spirit to the three sites briefly described in this section. They are a collective work and they leave, in their trail, an important collection of exchange, criticisms, and improvements. Teachers who take part benefit from the intense intellectual activity thus generated and in which they eventually take part. In France this has been a considerable revolution.

3.6.4 La main à la pâte

It is with the site *La main à la pâte* (meaning "Hands on", www.lamap.fr/), often referred to as LAMAP, that we can fully measure the power of transformation brought by a quality site with ideas on the teaching of science and on teachers' practices.

Like PUBLIREM and unlike *Mathenpoche*, LAMAP is not directly meant for student use. It offers scientific resources (not particularly mathematics-oriented, but promoting an experimental attitude in scientific matters) in great quantity and quality to teachers of the elementary school (Years 1 to 6) to adapt to the level of their class. They can complete their own scientific training, which may be insufficient. They can find scientists and educationalists with whom they can exchange virtually or by personal contact.

Concurrently with the teaching of science, LAMAP insists on the importance of language for the purposes of description, debate or explanation: each pupil expresses one's scientific observations and conclusions in one's own words, in one's notebook of experiences.

An internal LAMAP seminar took place in September 2005 to re-evaluate the place of mathematics on the site. A fundamental question was: can mathematics be included in the experimental process which characterizes LAMAP? It was concluded that it can if teams of teachers adapt to that experimental spirit some situations that were created within the framework of mathematics. That approach is yet to be completed. Teams must be set to work in the LAMAP spirit to imagine new situations using primary schools mathematics. (A site in statistics is in preparation. It will cover a field ranging from elementary school to *lycée*, that is, the end of secondary education. The experimental aspect will be very important in it.)

The French Ministry of Education, as a result of the quality of the site and the support of Georges Chapak (www.biocrawler.com/encyclopedia/Georges_Charpak), has profoundly changed the programs and methods of teaching science in elementary schools. The way of thinking and process of LAMAP have been largely adopted.

Within a few years, LAMAP has become globalized. Seminars and sites multiply from Brazil to China, from Germany to Africa, thus continuing to improve the access to science to learners from an early age.

3.7 General discussion

The process of ICT integration in educational practices is relatively new. It has neither a long history nor traditionally accepted theoretical frameworks. It is however a booming field of study and a principal driver of innovative educational practices. An important body of research has proved it generally to be of high educational potential and offers a positive impact of technology on the learning and teaching of all subjects in and beyond the classroom.

Unlimited access to resources generated or supported by many types of technology creates many opportunities for challenging mathematics in and beyond the classroom. In previous sections we critically analyzed underlying theoretical principles of development and implementation of challenging mathematical environments enhanced by technology, and also several practical applications of these principles that confirm the role of ICT as a catalyst of expansion of challenge beyond the classroom. In particular this looked at:

- context-sensitive new learning opportunities adaptable to particular needs of individuals and groups and promoting mathematics among the general public (attractions, competitions, discussions, recreations);
- tools that support traditional mathematical activities and create new opportunities in computation, modeling, visualization, search for information, communication allowing dynamic virtual mathematical investigations and complex problem solving;
- virtual spaces that integrate several tools and expand social and cognitive activities to the community level allowing common data (resources) collection and sharing, and knowledge building and sharing;
- roles of individuals and groups in mathematical activity when the learner becomes more active, autonomous and intrinsically motivated, the teachers in their turn become learners and collaborators, as well as guides and facilitators; specialists and experts play new roles of software and content developers together with educators; many human resources are involved in management and marketing activities.

With our concept of using technology to provide challenges beyond the classroom, we can enunciate several principles for an ICT-enhanced educational space. First, there is freedom to choose the activity. Secondly, there is great potential to elevate mathematical thinking through the use of appropriate content, not necessarily tied to curriculum, with multiple resources created and animated by experts, with no restrictions on time of access and open to all social and cognitive characteristics, as well as theories of learning and pedagogical interaction. In this environment, the computer serves to empower the mind in the solution of challenging problems.

Our survey shows that pedagogically and technologically efficient integration of ICT supported resources in beyond the classroom practices depends highly on the successful resolution of several issues.

3.7.1 Conception

Issues include:

- how to produce challenging educational material with the help of technology—all questions related to the development and maintenance which includes factors of time, costs and human and technological resources;
- how to make it known to potential users (learners, educators, and general public), promotion, access, search engines (possibility of informational overflow—too many resources);
- how to construct and manage common information space and challenging mathematical activities—financial needs, learning theory based on technical and pedagogical support (Vygotsky, instrumental and environmental genesis), organization and control, access, security and ethical issues, reliability of the content, interactivity, adaptability, speed;
- how to provide different groups of users with access to rich resources (we need tools for collecting and cataloguing, creating a place on the Internet and building a network of challenging mathematics and challenging information systems).

3.7.2 Forms

There are different ways of relating ICT and challenge in mathematics, through

- providing challenging mathematical activities of all kinds (such as puzzles, games);
- investigating different mathematical structures;
- creating challenging mathematical communities (such as math clubs);
- establishing forums for discussion;
- providing competitions.

(ICT tools are not normally used in competitions, but the forms of mathematics competitions are often diversified. A problem might be posed in a very different way. For example, candidates might be directed to find a property of a certain figure or to discover an equation.)

Participation in these activities may have a positive impact on all participants in many aspects:

- fun (emotional and social pleasure);
- deeper understanding of mathematics;
- boosting and empowering of the mind;
- new ways of learning (more creative, divergent thinking);
- use of a variety of learning tools;
- adjustment to individual needs (such as adapting for pace and style)—personal challenging environments;
- stimulation and motivation to pursue mathematical activities;
- evaluation of accomplishments (publication);
- access to challenging, special instruments for experimental mathematics environments;
- adaptation of resources to a particular population of learners;
- provision of the pleasure of performing difficult intellectual work;
- provision of a rich context for mathematical work;
- discovery of new interdisciplinary links;
- discovery of new faces of mathematics (showing that mathematics can be more than solving routine problems and doing exercises);
- appreciation of the usefulness of mathematics;
- development of a social identity—being a part of the community.

3.7.3 Content

In order to ensure the realization of the opportunities discussed above, we need to re-create mathematical environments to which technology could add more challenging components:

- programming contests: informatics contests without formal programming;
- construction algorithms: how to construct concrete figures, new pictures with other data;
- computer algebra systems;
- statistical multifunctional software;
- Geogebra (an algebra and geometry package used in schools);
- experiencing data: using Excel one can experiment with different data, looking for patterns not only to solve problems but find questions;
- real data collected from the Internet (sometimes requiring one to complete an electronic form);
- mathematics clubs on the Internet;

- interactive games with mathematical content (e.g. save a community of mathematicians, magic squares, Sudoku);
- tools of construction and investigation (LOGO, dynamic geometry software (Geometry Sketchpad, CABRI)).

3.7.4 Implementation

Practical questions on implementation will include:

- to whom resources are directed (ICT for specialists and for teachers);
- how to start and to boost challenging learning ICT activities beyond the classroom;
- how to keep learners challenged: will they continue? will they change their attitude?
- how to evaluate resources;
- how to improve the quality of a site: obtaining feedback from visitors, monitoring activity;
- how to maintain statistics of visitors (number of users may decrease with increasing difficulty of mathematical content).

3.7.5 Research

We have pointed out how ICT can have many positive impacts on mathematical teaching, learning, and recreations. There is an important body of research that has studied these impacts in formal educational settings.

For example, the study of the efficacy of online teaching conducted by Taylor and Maor (2000) used the Constructivist On-Line Learning Environment Survey (COLLES) to assess university students enrolled in online courses with respect to perceptions, professional relevance, reflective thinking, interactivity, cognitive demand, affective support, and interpretation of meaning.

While preliminary results report that students have high expectations in relationship to reflective thinking and affective support, relevance and interpretation, the study raises questions about cognitive support and interactivity. The former is related to the perception that the role of the tutor in challenging their assumptions, stimulating their thinking and modeling good discourse and reflective thinking is important, but not always. Exceptions would be cases of monologue-like interaction rather than dialogues and discussions.

However, there are relatively few studies that involve challenging mathematical activities beyond the classroom. One of these studies was conducted by the NRICH.team (Jones and Simons 2000). Research data collected from students' questionnaires revealed some important characteristics of such activities:

- The main source of information about web sites came from browsing the net and recommendation from parents to teachers.
- Access to such resources can be sometimes from the school, sometimes from home, and in a few cases from a public library or other public access location.
- For most young participants there was no access to other extracurricular facilities (such as “mathematics clubs”) at their school.
- The majority of pupils accessed NRICH about once a month. They mostly accessed the pages of mathematics problems and puzzles.
- There was little difference between the relative usage of NRICH by girls and boys.
- Pupils had positive views about the activities organized by the NRICH project and almost half of them said that NRICH was better than the mathematics they did in school.
- Most pupils argued that NRICH had made them more interested in mathematics and more likely to continue studying mathematics.

One of the important characteristics of ICT integration as a research field consists in a constant development of technological tools based on new and innovative pedagogical ideas. For example, the NWS, mathforum.org, has developed a new project called VMT (Virtual Math Team, vmt.mathforum.org/vmt/TheVMTProject.pdf).

It gives members of this Internet community an opportunity to form different groups using three types of virtual environment: open rooms for anyone, restricted rooms for people invited by the person who created them, and limited rooms allowing someone who was not originally invited to ask for permission to join.

It also creates opportunities for various types of challenging mathematical activities like posing and solving problems together, exploration of open-ended mathematics situations in order to develop interesting questions related to studying structural properties of the world, and finally, organization of discussion on an open topic related to mathematics, such as perplexing questions or difficult homework problems.

The immense information available on the Web along with stimulating digital or computational resources gives the potential of the Internet to link learners with sources of knowledge around the world. The VMT project explores these advantages of technological innovations offering unique opportunities for engrossing mathematical discussions, which are rarely found in traditional classrooms that depend on one teacher, one textbook and one set of exercises, to engage and train a room full of individual students over a long period of time.

The experience documented by the Math Forum research team shows that the traditional forms of learning can now be supplemented through small-group experiences of VMT chats, incorporating a variety of adaptable and tailored personal interactions.

It is also important to underline that as with many other challenging mathematical activities, those which are enhanced by technology have to be created and maintained by countless enthusiasts who like to share their ideas and expertise, devote their time and energy and thus contribute to the promotion of mathematics.

As examples of such combinations of collective and individual work, we can cite the Interactive Mathematics Miscellany and Puzzles site for teachers, parents and students who seek engaging mathematics (www.cut-the-knot.org/index.shtml) and the site of the Moscow center of continuous mathematics education (www.mccme.ru/). These examples illustrate the limitless nature of resources that can be created or collected to educate and entertain everyone who loves mathematical challenge.

Summarizing our findings, we can say that study of challenging mathematics enhanced by technology is an important and promising field of research and development and has an enormous potential to influence both innovative technological and educational development and real "in and beyond the classroom" practice.

3.8 Conclusion

Technology can bring new dimensions in challenging mathematical activities in and beyond the classroom. ICT, when used appropriately, supports teaching and learning in terms of resources, tools, and contexts. But it has to be made clear that technology cannot completely replace standard methods of mathematical work and play.

Sometimes, the use of technology can even lead to the incoherent representation of mathematical objects and relationships, and thus to incorrect conclusions or solutions. So some critical thinking and logical mindtools should be applied to analyzing all outcomes. Moreover, challenging experimental mathematical activities done by means of technology do not eliminate the necessity of proving discovered results (www.lettredelapreuve.it/PreuveAvertFR.html). In many cases, a paper-crayon environment thus keeps its value and even proves to be more efficient on many occasions.

It is also a fact, validated by research, that the most important advantage of using technology is the diversification of teaching and learning approaches, rediscovery of dynamic aspects of mathematics, and, especially, learning through communication with others. All these aspects have to be taken into account by educational systems in mathematics whether or not technology is being used. However technology can help indirectly to give a new boost to mathematical and non-mathematical activities, providing new challenges to everybody.

> Technology . . . increases the range and nature of experiences that can be provided for the learning of [complex and abstract] subject matter... The interactive character of modern technology can support reasoning by amplifying the nature and boundaries of

> scientific models of objects and events. But the full realization of the potential of such experiences will still rely on students' access to conversation partners who carry on discussions in which these models and concepts are validated. Technology should not be seen as replacing such communication but rather as providing a resource for supporting it. (Säljö 1999)

We summarize what technology can bring to all members of the worldwide community in terms of mathematical challenge beyond the classroom.

1. Technology can give access to the resources that cannot be otherwise accessed.
2. Technology can provide a free choice of resources based upon the level and the particular needs.
3. Technology can provide dynamic tools of mathematical investigation giving a chance to modify parameters of an activity in an interactive way.
4. Technology is a valuable tool of communication about mathematics with other people.
5. Technology empowers the people with the instruments, facilitating routine operations and more sophisticated mindtools.

Technological tools finally help to build more complex and, at the same time, more adequate mental representations of mathematical objects, structures, and operations through model building and experimentation, thus contributing to the promise of more positive experiences in mathematics for everyone.

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Chapter 4
Challenging Tasks and Mathematics Learning

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In this chapter, we present a view of didactical goals of challenging mathematical problems and the cognitive importance of problem-solving schemas. We distinguish between mathematical tasks, exercises and challenging problems and discuss how challenging problems promote the construction of problem-solving schemas. Similar in purpose to the nine case studies presented in Chapter 5, we offer six diverse examples of challenging mathematics problems from varied cultural and instructional contexts. For each example, we examine issues related to its mathematical, cognitive and didactical aspects. Two examples are research-based and accompanied by analysis and discussion of students' work, while the other examples are informed by considered reflection on their use in practice. In the aggregate, the examples illustrate how challenging mathematics problems are suitable for a range of learners and diverse didactical situations; how such problems can be instruments to stimulate creativity, to encourage collaboration, and support the formation of problem-solving schemas; and, finally, how the use of challenging problems invite educators to study learners' emergent mathematical ideas, reasoning and schemas.

4.1 Introduction

4.1.1 A goal for challenging mathematical problems

In many countries, students have come to experience school mathematics as cold, hard, and unapproachable, a mysterious activity quite distinct from their everyday lives and reserved for people with special talents. After repeated failure in school mathematics and estrangement from the discipline, students often assume a view similar to what a student once expressed to the first author: "mathematics is something that you do, not something that you understand."

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Similar views emerge from other students' school experiences. A considerable proportion of such students become excluded from meaningful participation in academic mathematics. This is particularly true of students who are members of socially excluded sectors of their societies, lacking in privileged economic or social capital, to use Bourdieu's (1986) categories. As Zevenbergen (2000) notes, "aspects of pedagogy and curriculum…can exclude students … [since] patterns of language, work, and power are implicated in the construction of mathematics, it becomes [important] to understand how we can change our practices in order that they become more accessible and equitable for our students" (p. 219).

To contribute toward making mathematics more accessible and equitable or less exclusionary and, thereby, more inclusive, this chapter posits the use of mathematical tasks that have particular characteristics. Even further, in addition to the social function of inclusion, such tasks have important psychological and cognitive consequences. The chapter will explicate how engaging students in solving challenging mathematical problems can lead them to construct effective and important problem-solving schemas. The pedagogical goal is to engage students with different mathematical backgrounds in different settings so that they can further develop their mathematical ideas, reasoning and problem-solving strategies, as well as enjoy being mathematical problem solvers.

4.1.2 Importance of schemas in mathematical problem solving

A paramount goal of mathematics education is to promote among learners effective problem solving. Mathematics teaching strives to enhance students' ability to solve individually and collaboratively problems that they have not previously encountered. To discuss the role of schemas in achieving this goal, we first discuss our understanding of problem solving and then that of schemas.

The meaning of mathematical "problem solving" is neither unique nor universal. Its meaning depends on ontological and epistemological stances, on philosophical views of mathematics and mathematics education. For the purposes of this chapter, we subscribe to how Mayer and Wittrock (1996) define problem solving and its psychological characteristics:

> Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver (Mayer 1992). According to this definition, problem solving has four main characteristics. First, problem solving is cognitive—it occurs within the problem solver's cognitive system and can be inferred indirectly from changes in the problem solver's behavior. Second, problem solving is a process—it involves representing and manipulating knowledge in the problem solver's cognitive system. Third, problem solving is directed—the problem solver's thoughts are motivated by goals. Fourth, problem solving is personal—the individual knowledge and skills of the problem solver help determine the difficulty or ease with which obstacles to solutions can be overcome. (p. 47)

Coupled with these cognitive and other psychological characteristics, problem solving also has social and cultural features. Some features include what an individual or cultural group considers to be a mathematical problem (D'Ambrosio 2001, Powell and Frankenstein 1997), the context in which an individual may prefer to engage in mathematical problem solving, and how problem solvers understand a given problem as well as what they consider to be adequate responses (Lakatos 1976). In instructional settings, students' problem solving activities are strongly influenced by teachers' representational strategies, which are constrained by cultural and social factors (Cai and Lester 2005).

An attribute that distinguishes expert mathematical problem solvers from less successful problem solvers is that experts have and use schemas—or abstract knowledge about the underlying, similar mathematical structure of common classes of problems—to form solutions to problems. In general terms a problem schema, as Hayes (1989) characterizes it "is a package of information about the properties of a particular problem type" (p. 11).

The role of schemas in mathematical problem solving has been investigated by psychologists and cognitive scientists, as well as mathematics education researchers. Below is a summary of this research (Schoenfeld 1992):

- Experts can categorize problems into types based on their underlying mathematical structure, sometimes after reading only the first few words of the problem (Hinsley et al. 1977, Schoenfeld and Hermann 1982).
- Schemas suggest to experts what aspects of the problem are likely to be important. This allows experts to focus on important aspects of the problem while they are reading it, and to form sub-goals of what quantities need to be found during the problem-solving process (Chi et al. 1981, Hinsley et al. 1977).
- Schemas are often equipped with techniques (e.g. procedures, equations) that are useful for formulating solutions to classes of problems (Weber 2001).

To illustrate the notion and utility of schemas for problem solving, consider the following problem: Two men start at the same spot. The first man walks 10 miles north and 4 miles east. The second man walks 4 miles west and 4 miles north. How far apart are the two men? In discussing a similar problem, Hayes (1989) notes that when experienced mathematical problem solvers read this statement, it will evoke a "right triangle schema" (problems in which individuals walk in parallel or orthogonal directions to one another can often be solved by constructing an appropriate right triangle and finding the lengths of all of its sides). A technique for solving such problems involves framing the problems in terms of finding the missing length of a right triangle, setting as a sub-goal finding the lengths of two of the sides of the triangle, and using the Pythagorean theorem to deduce the length of the unknown side.

4.1.3 *Mathematical tasks, exercises, and challenging problems*

In the mathematics and mathematics education literature, no universally accepted definition exists for the mathematical terms "task", "problem", or "exercise" and for the appellation "challenging" when describing a mathematical task or problem. In this chapter, as a starting point, we use Hayes's (1989) sense of what a problem is: "Whenever there is a gap between where you are now [an initial situation] and where you want to be [an adequate response], and you don't know how to find a way [a sequence of actions] to cross that gap, you have a problem" (p. xii).

In other words, "a problem occurs when a problem solver wants to transform a problem situation from the given state into the goal state but lacks an obvious method for accomplishing the transformation" (Mayer and Wittrock 1996, p. 47). For something that may or may not be a problem, to talk about it, we use the generic term "task". To complete a mathematical task, a problem solver needs to apply a sequence of mathematical actions to the initial situation to arrive at an adequate response. Even before applying mathematical actions, the problem solver will have to represent the gap virtually or physically—which is to say, to understand the nature of the problem (Hayes 1989).

The definition provided by Hayes as well as that provided by Mayer and Wittrock suggest grounds to distinguish between two closely related tasks: exercises and problems. Distinguishing these terms cannot be done without consideration of the problem solver. A mathematical task is an exercise to an individual learner if, due to the individual's experience, the learner knows what sequence of mathematical actions should be applied to achieve the task (such as knowing what equation into which to insert givens). In contrast, solving a mathematical problem involves understanding the task, formulating an appropriate sequence of actions or strategy, applying the strategy to produce a solution, and then reflecting on the solution to ensure that it produced an appropriate response.

A mathematical problem may present several plausible actions from which to choose (Schoenfeld 1992, Weber 2005). We call a mathematical problem challenging if the individual is not aware of procedural or algorithmic tools that are critical for solving the problem and, therefore, will have to build or otherwise invent a subset of mathematical actions to solve the problem.

For instance, most proofs in high school geometry are problems, and sometimes difficult ones, since the prover needs to decide which theorems and rules of inference to apply from many alternatives (Weber 2001). However, proofs that require the prover to create new mathematical concepts or derive novel theorems would make these proofs challenging problems. To solve challenging mathematics problems, learners build what are for them new mathematical ideas and go beyond their previous knowledge.

4.1.4 Use of challenging problems to promote schema construction

In mathematics education, challenging mathematical problems have psychological and cognitive importance. Since "problem-solving expertise is dependent upon the acquisition of domain-specific schemas" (Owen and Sweller 1985, p. 274), many researchers argue that an important goal of the mathematics curricula should be to provide students with the opportunities to construct problem-solving schemas (De Corte et al. 1996, Nunokawa 2005, Reed 1999). What is less clear is how this goal should be achieved. Marshall (1996) argues that the issues of how students construct problem-solving schemas and what types of environments or instructional techniques might foster these constructions are open questions in need of research.

Some psychologists and mathematics educators have suggested that students construct schemas by transferring the solution of one problem to another superficially different but structurally analogous problem (Novick and Holyoak 1991, Owen and Sweller 1985). Unfortunately, students often have difficulty seeing the deep structure of problems and transferring the solution of one problem situation to another (Lobato and Siebert 2002, Novick and Holyoak 1991). Accordingly, it is suggested that schema construction can be facilitated by providing students with basic problems to which that schema applies, both to increase the likelihood of successful transfer and to minimize the cognitive load that students use to solve these problems, thus leaving more resources available for learning (Owen and Sweller 1985). Contrary to these findings, discussing a long-term research project on the development of students' mathematical reasoning, Francisco and Maher (2005) report evidence that students often develop a rich understanding of essential ideas in the context of solving complex, challenging problems. In this chapter, in one specific example among others, we will illustrate how students developed a powerful combinatorial schema while solving strands of problems that were challenging (in the sense described earlier in this chapter).

4.2 Categories of challenging mathematics problems

There are many different categories of mathematics problems that are suitable as challenges for a learner or a group of learners. This diversity is also discussed and illustrated in Chapter 5 of this volume, and Chapter 3 has treated the issue of challenging mathematics and the use of information and communication technologies. Whether a specific mathematics problem is a challenge depends on the mathematical experience of an individual learner. Nevertheless, appropriate challenges can be given to mathematically talented students as well as to socially excluded and struggling students, be they children, teenagers, or adults.

Moreover, as will be discussed later in this chapter, there are important pedagogical, psychological and social reasons that all students should be engaged with challenging mathematics problems. In this chapter, we present different types of challenging problems, some of which are about paradoxes, counterintuitive propositions, patterns and sequences, geometry, combinatorics and probability. It goes without saying that the categories of challenging problems that we present are neither comprehensive nor exhaustive: there are many areas of elementary and advanced mathematics that our examples do not include.

Not only is it important to consider the type of problems to use but also to contemplate the physical setting and pedagogical climate in which they are used. For instance, the setting might be formal as a school classroom or informal as an afterschool program or with street kids or adults learners in a public space. The pedagogical approach may include collaborative or cooperative learning with an instructor as a facilitator or involve groups of learners presenting their solutions. The actual mathematical challenge may be selected by students or be a sequence of challenging problems that contribute to students building problem-solving schemas.

4.3 Challenging mathematics problems and schema development

We present six diverse examples of challenging mathematics problems from varied contexts, one in this section, four in Section 4.4, and a final one in Section 4.5. Some of the examples that we present contain several challenging problems. The first and third examples are empirically based, while the remaining four are informed by reflection on practice. Following the presentation of each example, we provide three types of analysis: mathematical, cognitive, and didactical. The two research-based examples are each also accompanied by an analysis of students' work and a discussion.

4.3.1 Strands of challenging mathematical tasks

In this section, based on analysis of Weber et al. (2006), we exemplify how over time students can develop an important and effective combinatorial schema from their work solving a strand of challenging problems. The students' building of problem-solving schemas related to combinatorics occurred within the context of a longitudinal study, now in its 20th year, tracing the mathematical development of students while they solve open-ended but well-defined mathematical problems (Maher 2005).

The problems are challenging in the sense that students often initially are not aware of procedural or algorithmic tools to solve the problems but are asked to develop them in the problem-solving context. The strand of problems presented here are used in an environment in which collaboration and justification are encouraged, and teachers and researchers do not provide explicit guidance on how problems should be solved or whether the solution that students develop is correct or not, that judgment being left to the students.

One aspect of this study was that students worked on strands of challenging tasks—or sequences of related tasks that may differ superficially but designed to pertain to identified mathematical concepts. The use of a strand of challenging problems allows teachers and researchers to trace the development of students' reasoning about a particular mathematical idea over long periods of time (Maher and Martino 1996).

This study in which the challenging mathematics problems were used has an important distinguishing feature. Most studies examining schema construction or transfer take place over a short period of time in conceptual domains in which students have limited experience (Lobato and Siebert 2002). However, meaningful mathematical schemas are likely constructed over significant stretches of time after students become accustomed to the domain being studied.

Hence, Anderson et al. (1996) argue such studies seek evidence of schema usage and transfer in places where one is least likely to find it. We are not aware of long-term studies in mathematics education that address schema acquisition. Hence, the longitudinal and empirical nature of the study that Maher (2005) describes has the potential to offer unique research findings in an important area.

The following set of mathematical challenges is an example of the problems in a strand of combinatorial tasks. Working of the problems in the strand allowed students to develop mathematical ideas and reasoning strategies within a particular domain.

To provide a comprehensive sense of the possibility that students can develop problem-solving schemas within a specific mathematical domain, we detail a case of five students from a research project at Rutgers University (Weber et al. 2006).

First, we present three problems—challenging for the particular group of students in the study—a brief mathematical analysis of the problems, and indicate cognitive, mathematical structures that learners can build from engaging with these problems. Next, we will provide results and a discussion of how a group of five students solved the three problems. Following this presentation, in the next section, we present other examples of mathematics problems, challenging for the context in which they have been used.

4.3.2 Examples from a strand of challenging mathematical tasks

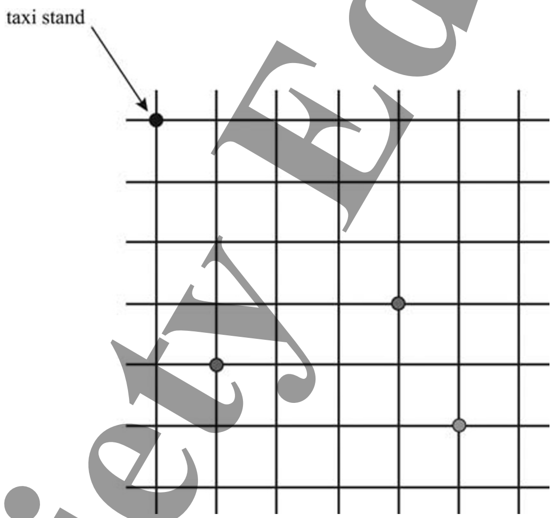
The Four-Topping Pizza Problem
A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni. (No halves!) How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

A Towers Five-Tall Problem
Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as possible, each with three red and two yellow

cubes. See if you and your partner can plan a good way to find all the towers four cubes tall.

The Taxicab Problem
A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



4.3.3 Mathematical analysis

The answer to the first task is $\sum_{r=0}^{4}\binom{4}{r} = 2^4$. The second has as an answer $\binom{5}{2} = \frac{5!}{2!3!} = 10$ or, equivalently, $\binom{5}{3}$. The answer to the third task is $\binom{5}{1} = \frac{5!}{1!4!} = 5$, $\binom{7}{4} = \frac{7!}{4!3!} = 35$ and $\binom{10}{5} = \frac{10!}{5!5!} = 252$ for the three pickup points. However, these problems all have the same underlying mathematical structure that can be associated as a "Pascal's triangle schema." The students in the research project had not studied combinatorics, were not familiar with the standard notation for permutations or combinations, and yet, as we will show, they correctly solved the three problems by other means.

4.3.4 *Cognitive analysis*

Based on teaching and research experiences, the mathematical ideas and reasoning strategies that students are likely to develop or engage include the following:

1. counting without omission or repetition;
2. symmetry;
3. powers of 2;
4. Pascal's triangle;
5. counting the number of distinct subsets, combinations $\binom{n}{r} = {}_nC_r$;
6. reasoning by controlling variables (determining which independent variable to change and manipulating this independent variable to determine changes in the dependent variable);
7. reasoning about isomorphism (see Table 4.1).

Table 4.1 Taxonomy of isomorphisms among three mathematical tasks

	Taxicab	Towers	Pizzas
Objects	East and south vectors	Red and blue Unifix cubes	Different toppings
Actions	Go east or south	Affix red or blue Unifix cube	Add a topping or no topping
Products	Different shortest taxicab routes	Different Towers	Different Pizzas

4.3.5 *Students' work on problems from a strand of challenging tasks*

Weber et al. (2006) prepared the work excerpted in this section for the ICMI Study 16. The students whose work is analyzed are participants in the long-term study described by Maher (2005). Here, Weber et al. (2006) examine how a group of five students (Ankur, Jeff, Brian, Michael and Romina) solved the three problems presented above when they were in 10th and 12th grades.

4.3.5.1 How many pizzas are there with four different toppings?

In a 10th grade session, Ankur, Jeff, Brian and Romina used case-based reasoning and various counting strategies to obtain the correct answer—fifteen pizzas with toppings plus one pizza with only cheese and tomato sauce. Michael developed a binary representation to create each of the pizzas. Each of the pizzas was represented using a four-digit binary number, where each topping was associated with a place in that number, where a one signified that the topping was present on the pizza and a 0 signified that the topping was absent. For instance, with the four toppings—pepperoni, sausage, onion and mushroom—the binary number 0010 would refer to a pizza with only onions. Michael was able to use this notation to explain why 16 pizzas could be formed when there were four toppings available and convinced his group that there would be 32 pizzas if there were five toppings available (the other group members believed that there would be 31, not 32 pizzas).

At the end of the session, the researcher asked the group if this problem reminded them of any other problems. Brian responded "towers"—referring to the problem of forming four-tall towers from red and yellow cubes. However, Ankur noted the problems were "similar, but not exactly the same", since more than one yellow could appear in an acceptable tower, but you couldn't list mushroom more than once on the toppings of the pizza. All of the students at this time accepted Ankur's explanation. The following week, Michael represented the towers problem using binary notation—the *n*th digit in the notation refers to the *n*th cube in the tower, with a 0 signifying a yellow cube and a 1 a red cube. For example, 0010 would represent a four-tall tower in which the third block was red but the other three were yellow. Hence, using this binary notation, Michael was able to show his group a correspondence between the towers and the pizzas. (For an elaborated analysis of Michael's binary representation and how he used it to indicate an isomorphism between the towers and pizza problems, see Kiczek et al. 2001.)

There are two things worth noting about these problem-solving episodes. First, when students were initially comparing the pizza and towers problems to one another, they did not seem to see the deep structure between the problems. In fact, Ankur argued the problems differed significantly. The connections between the problems were not immediately perceived but were only constructed by Michael after reflection. Secondly, the notational system that Michael developed while working on the pizza problem was critical for the construction of his correspondence.

4.3.5.2 Linking the pizza problem, the towers problem, and Pascal's triangle

One month later, students were invited to further explore the relationship between pizza problems and tower problems. They were asked to determine how many five-tall towers could be formed with three yellow blocks and two red blocks. Using Michael's binary representation, they translated this problem to determine how many five-digit binary numbers with three 0s and two 1s could be formed. By controlling for where the first one in this sequence occurred, the students were able to deduce that 10 such towers could be formed. Note that the methods Michael developed to cope with the previous pizza problems were now a scheme that the students used to make sense of a new pizza problem (see Uptegrove 2004). After obtaining their solution, a researcher introduced students to Pascal's triangle, explained how the *n*th row of Pascal's triangle were the coefficients of the expression $(a + b)^n$, and that the terms in Pascal's triangle were often represented using combinatorial notation. For instance, the fourth row—1, 4, 6, 4, 1—can be written as $\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$. She then asked the students to try to understand what these coefficients might mean in terms of what they've just done. After thinking about these problems, the students were able to make these links. They noticed the 10 that appears in the fifth row in Pascal's triangle corresponding to the expression $\binom{5}{2}$ also corresponded to five-tall towers with two red blocks (and three yellow blocks). Further investigations led these students to describe the relationship between Pascal's triangle and the pizza problem—namely, that the $\binom{n}{i}$ entry in Pascal's triangle corresponds to the number of pizzas that could be formed with *i* toppings if there were *n* to choose from. These students could also explain why $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ and $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$ (Pascal's Identity) were true by using the towers problem and the pizza problem.

4.3.5.3 Solving the taxicab problem

Two years later, Michael, Romina, Jeff and Brian (now in 12th grade) were given a version of the taxicab problem. In essence, they were asked how many ways that a taxi could take a shortest route along a grid to go four blocks down, one block right; three blocks down, four blocks right; and five blocks down, five blocks right. This qualified as a challenging problem for the students. The solution to this problem more or less requires the application and use of combinatorial techniques, yet the students solving this problem had not used such techniques before to solve novel problems. The initial stages of the students' activity were exploratory in nature. They worked to make sense of the problem, posed some initial conjectures that turned out to be incorrect (for example, the distance from the starting point to the endpoint would tell you the number of shortest routes), and tried to answer the question by explicitly drawing and counting the routes.

Romina asks if it would be possible to "do towers" to the problem. Michael and Romina note that the distance to one of the points is 10 and wonder if the total number of shortest routes to that point is 2^{10}. Later, the students attempted to solve the problem by finding the number of shortest routes to corners close to the point of origin (e.g. there are two shortest routes to go one down, one right; three shortest routes to go two down, one right). They produced a table like the following:

1	1	1	1	1
1	1	3	4	5
1	1	6	10	12
1	4	10	15	
1				

In the table, the *m*th by *n*th cell represents the number of shortest paths to go *m* units to the east, *n* units south.

Romina notices that the fourth diagonal of this table is the sequence 1, 4, 6, 4, 1 and declares, "it's Pascal's triangle", where the diagonals in the table correspond to the rows of Pascal's triangle. Jeff notes that the 12 and the 15 in the next diagonal would not be correct if this was the case and asks Brian to re-evaluate the number of routes it takes to go four over and two down. When Brian announces that he found 15 routes, Michael comments, "it means that it is the triangle." A little later, Romina writes a 20 in the box for three right, three down while Brian worked on re-computing this value. At this point, Michael asks his colleagues how they knew it was 20. Jeff responds that if they can show the triangle works, they don't need to verify that it's 20.

To understand why Pascal's triangle would provide the number of shortest routes to any points on the grid, Romina announces that she will try and relate the triangle back to the towers and focuses on the 1 2 1 diagonal. She notes that all of the points on this diagonal are two away from the starting point and this also forms the second row of Pascal's triangle. Further, she notes a connection between the middle entry in that column—with towers, the middle entry would refer to a two-tall tower with one yellow and one red block; with taxicabs, this refers to a trip with one across and one down. Likewise, the entry two down, one right, would refer to a tower that was three-tall, with two yellow and one red blocks, or the taxicab location three away, with two down and one across. The students filled in the rest of their grid in accordance with Pascal's triangle. For instance, when they filled in the cell for five down, two over, they reasoned that the number of routes would correspond to the fifth entry of the seventh row of Pascal's triangle (not counting the beginning 1) since it would be "five of one thing and two of another thing." At a researcher's request, Michael also explains the connection between Pascal's triangle and the pizza by using his binary number notation. For the taxicab geometry problem, a 0 would indicate going down and a 1 would indicate going across. Hence, using the example of going two down and one across, one would need to find the number of binary strings that have two 0s and one 1. In their work relating Pascal's triangle to the pizza problem, the group had already established that this would be the first entry (ignoring the first 1) of the third row of Pascal's triangle. Finally, the group was able to use these constructions to answer the given questions, for instance, the number of shortest routes to the point that was five right and five down would correspond to the fifth entry of the tenth row of Pascal's triangle.

4.3.5.4 Discussion—strand and schema

In the first two excerpts above, we illustrated how students constructed a powerful problem-solving schema for solving combinatorial problems. We then illustrated how students applied that schema to solve the challenging taxicab geometry problem. The application of this schema not only allowed them to construct the solution to the problem, but it also provided them with a deep understanding of their solution and enriched the schema that they had constructed. In this section, we will discuss four aspects of our problem-solving environment that enabled students to make these constructions.

First, students were asked to work on *challenging* problems. If students were asked to work on problems for which they had already had strategies, they may have attempted to see whether various techniques that they had learned would be applicable to the problem. As the students needed to *develop* techniques to make progress on these problems, this was not an option for these students. A particularly important precursor toward developing the schema that these students constructed was the development of useful ways of representing the problem. Michael's binary representation of the towers and the pizza problem, in particular, paved the way for students to see the deep structure that these problems shared. One general finding from the longitudinal study was that students developed powerful representations in response to addressing challenging problems (Davis and Maher 1997, Maher 2005).

Secondly, students were asked to work on *strands* of challenging problems that were superficially different but shared the same mathematical structure. This provided students with the environments in which schemas could be constructed. Researchers also fostered this construction by encouraging students to think about how the problems they were solving might be related to problems that they had solved in the past. However, we believe that having students work on strands of challenging tasks is a necessary but not sufficient condition for schema construction and usage. Students also need time to explore the task and benefit from heuristics that guide their explorations in productive directions.

Thirdly, students were given *sufficient time* to explore the problems and were also given the opportunities to revisit the problems that they explored. The students did not instantly see the connections between the towers and pizza problems, nor did they see how the taxicab problem was related to either of these problems. It is especially noteworthy that students initially believed that the towers and pizza problems were similar, but also differed significantly, and that Romina's initial suggestion to relate the taxicab problem to the towers was not immediately pursued. Further, as the students revisited problems, their representations of the problems became increasingly more sophisticated, enabling them to see links between the problem being solved and previous problems on which they had worked. As Uptegrove (2004) illustrates, many of the connections students made could be traced back to problem-solving sessions on which they worked months or years before.

Finally, as Powell (2003) emphasizes, the heuristics that students used in their problem solving enabled them to relate the problem situation to their schema. Among the heuristics that the students used were the following: solve a difficult problem by solving easier ones (before finding the number of shortest routes to a location ten blocks away, find the number of shortest routes to a location two blocks away); generate data and look for patterns; and see if there is an analogy between this problem and a familiar one (Powell 2003). Without the use of these heuristics, the links to an existing schema may not have been made. However, the disposition to use such heuristics was likely developed during the students' years of solving challenging problems (Powell 2003, Uptegrove 2004). Moreover, these students' co-constructed schemas through a process that Powell (2006, p. 33) terms *socially emergent cognition*.

4.4 Other examples and contexts for challenging mathematics problems

In this section, we present four other examples of challenging mathematics problems and describe the context in which each has been used. In the fifth section, we present another category of challenging tasks: paradoxes. As noted earlier, the second example in this section is empirically based, while the remaining three are informed by reflection on practice.

4.4.1 Example: Number producer

The Context

The problem we will discuss as an example of a challenging mathematical task is called the Number Producer, and we consider two different settings where it has been used. In the first setting, the participants were students taking part in an entrance interview for the University of Oxford, UK. The student, while alone with the interviewer, was given problems on a piece of paper and had paper available for calculations. The Number Producer was given as one of the problems the student should attempt to solve in front of the interviewer, to provide information on his or her potential as a mathematics student, and hence on whether to offer this student a place. The student was given some time to think about the problems before the discussion with the interviewer started.

(The Number Producer was suggested as a problem by Juliette White of the Open University, UK, and has its origin with Smullyan (1982)).

In the second setting, the participants were third- and fourth-year mathematics students at a teacher training college in Vestfold, Norway. They were presented with the Number Producer problem in class where it was talked through. They were then given the problem as an assignment to hand in after five days. Some worked in groups, others individually. Some asked for and received hints and clarification via e-mail. After the solutions had been handed in, there was a discussion of the process.

The Number Producer

In this problem, a number means a positive integer written in decimal notation with all its digits non-zero. If A and B are numbers, by AB we mean the number formed when the digits of A are followed by the digits of B, and not the product of A and B. For any number X, the number $X2X$ is called the associate of the number X.

There exists a machine. When you put a number into the machine, after a while a number comes out of the machine. However, the machine does not accept all such numbers, only some. Those numbers accepted by the machine are called acceptable. We say that a number X produces a number Y if X is acceptable and when X is put into the machine, Y comes out of the machine.

The machine obeys three rules:

R1. For any number X, the number $2X$ is acceptable, and $2X$ produces X.
R2. If a number X is acceptable and produces Y, then $3X$ is acceptable and produces the associate of Y.
R3. If you cannot decide that a number is acceptable from R1 and R2, then it is not.

Questions:

1. What is the associate of 594?
2. For each of the numbers listed below, find whether or not it is acceptable. If it is acceptable, find the number it produces.

   (a) 27482
   (b) 435
   (c) 25
   (d) 325
   (e) 3325
   (f) 33325
   (g) 345
   (h) 333
   (i) 32586

3. Can you describe the numbers that are acceptable?
4. Can you think of a number that produces itself?
5. Can you think of a number that produces its associate?

Mathematical Analysis
We note that the presentation of the problem is as it was given to the Norwegian participants (translated). In the interview setting, questions 2 and 3 were grouped together as one question, as were questions 4 and 5.

Those who know about functions might think "function" instead of "machine" when they read through the Number Producer. The first question is posed in order to build up the mathematical action that whenever one sees the symbol AB, one should think concatenation, and not multiplication, of the numbers A and B, and also posed to assist students to grasp the definition of the associate of a number. Hence, one should find that the associate of 594 is 5942594.

Questions 2 and 3 help the problem solver understand how the machine works: the numbers in (b), (g) and (h) are examples of not acceptable numbers, whereas the others are acceptable. Also note the order of the given numbers in question 2 (a) is acceptable (and produces 7482); (b) is not acceptable; (c–f) should help the problem solver to see and create a pattern (with answers 5, 525, 5252525, and 525252525252525, respectively); followed by (g) and (h) which are not acceptable; and finally we have (i) as a "check" for the understanding (which produces 5862586). From this, the problem solver might have created the algorithmic tool for answering question 3: the acceptable numbers are of the form "(possibly 3s)2(a number)".

Once the problem solver has been able to do these questions, he or she can try to solve the final two questions, using what he or she now knows. (The answers to questions 4 and 5 are "yes, 323 produces itself", and "yes, there is also a number that produces its associate. . .," which we leave for the reader to find!)

Cognitive Analysis
For a challenge to have a positive effect on learning, it should not be too difficult, but "just out of reach". That is, it should be within the zone of proximal development, as it is referred to in Vygotskian terms (Vygotsky 1978, defined in this Study Volume in Section 6.2.2.3 and also discussed in Sections 3.1 and 7.3.2). For a particular group of learners, an appropriate challenge has to have the possibility of being mastered. In the Number Producer, there are mathematical and cognitive challenges, where the definitions and the notation must be understood and accepted. For example, it might help to think of the numbers as an alphabet in this problem. In any case, a learner also has to accept the rules the machine obeys, which in turn creates new mathematics for the learner.

In the first setting (the interview), all the students accepted the challenge immediately and some quickly started talking while others thought for a few minutes. The interviewer started asking questions to see whether the student had understood. Through various degrees of hints, they all managed to answer the questions given in the Number Producer. This particular setting forced the students to be extremely focused. They all said it was an interesting problem, and managed to give most answers very quickly. Through the discussion the interviewer could follow the process the students went through to understand how the machine worked, hence learning new mathematics. It was obvious that none of them had seen this problem before. Seeing that they could talk through the problems with the interviewer rather than just presenting the answers they could come up with gave a positive feel to a stressful situation. Being able to discuss a mathematical challenge in such a setting is a valuable experience.

In the second setting (that of the teacher training students), the Number Producer was given as one of several problems to illustrate mathematical thinking as part of the history of mathematics. One idea was to make the students give some thought to how new mathematics develops. Since this was the first assignment of the course, there were no immediate complaints, and all the students went away to make an attempt at the problem.

However, it turned out the students spent a lot of time on it and found it very hard. Most of them got frustrated with it and thought it was too difficult for them. They tried seeking help from others. Most of them managed to hand in partial solutions, while a few didn't hand in anything at all. As one of the students said, "This is a problem where we had to think for ourselves and couldn't look up a formula." Many students had searched for help in textbooks without luck. To these students the problem was challenging, in the sense we described in Section 4.1.3—the problem solver is not aware of procedural or algorithmic tools that are critical for solving the problem, and therefore will have to build or otherwise invent a subset of mathematical actions to solve the problem.

As for the solutions, some students just wrote the answer, whereas others elaborated so that one could follow their process. From this, it was clear that they asked themselves good questions in order to figure out how the machine worked, and hence learned something new. And so they built what are for them new mathematical ideas and went beyond what they previously knew.

In the discussion that followed, several points were made. Some felt that this sort of problem could be destructive in the sense that some students lose confidence when they cannot produce an answer at all. Further, they spent a lot of time on the problem, and some felt it was a waste of time when they could not find any answers. However, it turned out that several students had started thinking about why this challenge was given, and one said, "I didn't interpret it as traditional maths, but thinking back I realize that maybe it was." Also, they learned what it was like to not always be able to solve a problem completely; they were obviously used to handing in almost perfect solutions to assignments.

Didactical Analysis

The background of the participants and the setting in which the challenge is given has implications for learning. For example, in the case of the Number Producer, the teacher candidates were not used to and hence didn't expect to be challenged the way they were, whereas the interviewed students were certainly expecting a challenge. Another difference was that the teacher candidates had more time to think about the problem, but they did not have the interviewer with whom to discuss their insights. This makes the learning processes and the outcomes very different, and hence influences the effect of the challenge on learning.

Another point to be made about the effect on learning from this example is motivation. The person presenting the challenge and the people receiving it must have some sort of agreement beforehand: Why do this? The interviewed students receiving the Number Producer were very clear on their motivation (in a non-typical classroom situation). The teacher training students (who were in a typical classroom situation), it turned out, were not.

Variation to the curriculum requires interesting and useful challenges in order to have a positive effect on learning. For example, one of the teacher candidates faced with the Number Producer said, "It wasn't an interesting exercise, but one can't expect all exercises to be interesting to everyone."

Still, the Number Producer is an example of a challenge that can be an addition to the curriculum. For one thing, it doesn't require a lot of background theory. All the ingredients are explained in the problem statement. The first few questions helped the students find out how the machine works, whereas the final few questions were themselves new challenges. These new challenges are easier to accept if you have done the first questions, then you want to apply the things you have learned. Learning something new and being challenged on it immediately enhances learning. "Now that you have understood how the machine works, can you find a number which produces itself?" It is in human nature to learn, and we all need to be challenged on what we learn, otherwise we lose interest.

4.4.2 Example: Pattern sequence

The Context

The turmoil following the 2001 crisis in Argentina led to many students dropping out of school. The severity of the situation is indicated by the following statistics: 35 per cent of youth between the ages of 15 and 24 neither study nor work; 13 per cent of teenagers abandon school; the unemployment rate among those under 29 years old is 13 per cent, among whom 54 per cent live in poor households.

In 2004, the government of Buenos Aires started the Back-to-School program for secondary students, in line with the Zero Dropout Plan (for more information, see www.buenosaires.gov.ar/areas/educacion/desercioncero), which targets adolescent school leavers living on the margins of society. Its aim is to provide a curriculum equivalent to that of compulsory secondary education that will lead them to gain the necessary official certificates and grades to get a dignified job. Since 2002, in the city of Buenos Aires, education has been compulsory until the students are 16 (Law no. 898 on compulsory secondary education).

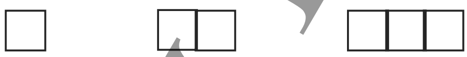
The Zero Dropout Plan targets an important sector of the young population. The participants in the Back-to-School program must be at least 19 years old and have interrupted their education for at least a year, but be interested in and committed towards completing their secondary education and show commitment towards it. A significant number of these students have experienced failure in both their primary and secondary schooling, and many of them combine their education with family and work responsibility.

Some students dropped out of school many years ago, while others have poor primary education. Many of them do not even know the multiplication and division algorithms nor can they use basic procedures for subtraction. Some have a criminal record or suffer from drug addiction. Attendance is poor and a high rate of absenteeism interferes with continuity. Particularly in first year, many have difficulty in adapting to the school environment.

Pattern and Sequence Problems

The initial problems given to the students required them to describe the general step or the result of a regular process, such as the addition of the first n natural numbers or the calculation of the number of elements of a certain geometric configuration. The geometrical context helped students recognize the equivalence of different descriptions of the pattern.

The teacher presented the following sequence of figures built with matches and explained how they should be further assembled.



(a) Determine the number of matches needed to form the sixth figure in the sequence.
(b) How many matches would be needed to build the 100th figure in the sequence?
(c) Find a formula for the number of matches in the nth figure.
(d) Is it possible for one of the figures to be composed of 1549 matches? 1500 matches?

Mathematical Analysis

Algebra can be understood as a tool to model and handle problems of a certain type. The process that students go through to obtain a formula for the number of elements of a collection is reflected in the form of the expression found. At the same time, this process helps students to appreciate the meaning of a "letter" used as a variable and get a feel for the correct use of algebraic expressions. Moreover, different approaches to the same problem may illuminate a discussion on the equivalence of different expressions and how algebraic expressions can be transformed.

From this perspective, for the students in the project, a challenging activity is the production and validation of formulas using natural numbers. We intend that students look for patterns, find formulas to describe them and produce arguments to validate them. The teacher is not expected to "teach" the formulas nor the students to "apply" them; rather the students have a chance to speculate, create, test and validate them. The problems are designed to admit multiple approaches and formulas for the same process.

Cognitive Analysis
The work of the students illustrates how equivalent formulas are found for the number of matches required for n squares. For example, students who saw each new square as resulting from the addition of three matches produced the formula $n.3 + 1$. Another formula $2.n + n + 1$ came from students who counted the horizontal matches in pairs, added in the vertical matches completing the squares and finally the initial vertical match. Other students gave the formula $4 + 3(n - 1)$, noting the four matches for the first square and the three additional matches for each new square. Finally, some students gave $4.n - (n - 1)$, counting four matches for each square and then subtracting the number of vertical matches that were double-counted.

Harmonizing the equivalent expressions provided a basis for introducing the notions of common factor and distributivity. Thus, in showing the equivalence of $4+3(n-1)$ and $3n+1$ many students used the concept of multiplication as repeated addition. They considered $3(n - 1)$ as $(n - 1) + (n - 1) + (n - 1)$ and recorded the sum vertically, as for natural numbers:

$$\begin{array}{r} n-1 \\ n-1 \\ \underline{n-1} \\ 3n-3 \end{array}$$

Then they added by associating on the one hand the ns and on the other hand the -1s leading to establish the equation $3(n - 1) = 3n - 3$. We observed that the students implicitly made use of the commutative and associative properties in connection with addition although they had not learned the symbolic formulation. From this, the common factor and the distributive property, which the students had not yet worked with, could be formalized. Eventually, students would write such equations as $4 + 3(n - 1) = 4 + 3n - 3 = (4 - 3) + 3n = 1 + 3n = 3n + 1$.

Proving the equivalence of two formulas is a gateway to algebraic manipulation. When a formula involving a variable arises in some context, students can check special cases numerically.

We conclude with an examination of the work of some students who answered the questions as to whether the sequence contained a diagram requiring 1549 or 1500 matches.

While some calculations were tentative, the following one led to a correct answer:

$$\begin{array}{r} 301 = 100 \\ \times 5 \\ \hline 1501 = 500 \end{array}$$

By this, the students wished to express that if 301 matches are needed for the 100th figure, then for the 500th figure, they would need $301 \times 5 - 4$, the subtraction for the number of matches that repeat when concatenating five series of 301. They multiplied 16 by 3 for the matches needed for an additional 16 squares, and these added to the 1501 yielded 1549 matches, the number required for the 516th figure in the sequence.

Didactical Analysis
The problem was developed in one of the reinsertion schools in Villa Lugano, a neighborhood of the city of Buenos Aires. The teacher we collaborated with had a strong commitment to the project, positive expectations of her students and a sound mathematical education. We collaborated in designing problems that were to challenge her students mathematically.

We expect that the performance of the students on this problem would help provide a benchmark for suitable mathematical challenges. We plan to formulate what is a challenge from a theoretical perspective as well as from the perspective of the teacher and students. Together with them, we will study from a socio-cultural perspective how mathematically challenging activities can motivate students to participate in the mathematics classroom and how a particular way of handling interactions among the participants can contribute to a classroom culture that facilitates participation as a step towards learning.

4.4.3 Examples: Probability

The following two examples demonstrate how challenging mathematical problems can be used to engage students in a post-secondary, introductory probability course. In such a course, students have difficulties in seeing connections between basic probability models and word problems of a varying verbal content, that are based on these models. Furthermore, a typical dilemma for students in this course is to combine, in a proper way, intuitive and strictly mathematical approaches to problem solving. In order to stimulate students' creativity, the following course project was offered to students at Community College of Philadelphia. The students have the option of selecting a challenging problem from external sources or attempting one suggested by their instructor. In the examples below, students selected the first problem, and the instructor offered the second. Both problems were solved and presented to the class by students.

Foot-and-Mouth Disease Problem
- One person per hundred people has the infectious Foot-and-Mouth disease.
- The probability of a person with this disease testing positive is 0.9, and the probability of a person who does not have this disease testing positive is 0.2.
- What is the probability that a person who tests positive has the disease?

Three Cards Problem

- Suppose you have three cards: a black card that is black on both sides, a white card that is white on both sides, and a mixed card that is black on one side and white on the other.
- You put all the cards in a hat, pull one out at random, and place it on a table. The side facing up is black.
- What is the probability that the other side is also black?

Mathematical Analysis

The level of difficulty of both problems is higher than that of standard problems in this course. The solution of the Foot-and-Mouth disease problem involves such notions as conditional probability, complete probability and Bayes' formula.

This is one solution presented by a student:

We define events as "*Yes*"**:** a person has the disease; "*No*"**:** a person has no disease; "*Pos*"**:** a person is tested positively; "*Neg*": a person is tested negatively.

We can see $P(Yes) = 0.01$ and $P(No) = 0.99$.

Then

$$P(Pos/Yes) = 0.9 \quad P(Pos/No) = 0.2.$$

What is $P(Yes/Pos)$?

Now the "branch probability"

$$P(Yes/Pos) \times P(Pos) = P(Pos/Yes) \times P(Yes) = P(Pos \cap Yes).$$

Using Bayes' formula:

$$P(Yes/Pos) = \frac{P(Pos/Yes) \times P(Yes)}{P(Pos)},$$

that is,

$$P(Yes/Pos) = \frac{P(Pos/Yes) \times P(Yes)}{P(Pos/Yes) \times P(Yes) + P(Pos/No) \times P(No)},$$

that is,

$$P(Yes/Pos) = \frac{0.9 \times 0.1}{0.9 \times 0.01 + 0.2 \times 0.99} = \frac{9}{207} = \frac{1}{23} = 0.043 \approx 4\%.$$

An alternative solution was also presented based on a tree diagram and evaluating branch probabilities.

The Three Cards problem is a well-known example of a counterintuitive problem. This problem is discussed broadly in the literature (Nickerson 2004) and on the Internet (en.wikipedia.org/wiki/Three_cards_problem). There are various solutions of this problem based on notions of reduced sample space, conditional probability, and multiplication of probabilities.

This problem contains two types of challenge: mathematical and psychological. While the mathematical challenge is to find a solution, the psychological one is to be confident of one's solution even if it may disagree with one's intuition. An effective approach to solving the problem is to consider six faces, three black and three white, with probability 1/6 for each face.

One of the solutions, based on Bayes' theorem, is as follows: if event E is to draw a card black on both sides, and event F is to see a black face, then

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F/E) \times P(E)}{P(F)} = \frac{1 \times 1/3}{1/2} = \frac{2}{3}.$$

Another possible solution is based on the formula for conditional probability and the reduced sample space for faces.

A quite popular approach employs the idea of labeling faces of three cards. If B1 and B2 are black faces on the black card, and B3 is a black face on the mixed card, then one can see that the probability of a black face being B1 or B2 is 2/3.

The Three Cards problem was offered in two introductory probability courses and a calculus-based probability and statistics course, 50 per cent of whose students were undergraduate majors in mathematics. In all three classes, 25 students in each, approximately 60 per cent of students gave the same wrong, "intuitive" answer (it was ½), and three students in each class presented the correct answer and solution.

Cognitive Analysis

In the introductory probability course, many students are future elementary school teachers. However, some of them may be placed in a category of "remedial and struggling" mathematics students with a strong math anxiety. The goal of this course project is to combine methods of challenging and collaborative learning to help students develop logical thinking and creativity, both of which are critical for future teachers.

In the course project, there was a general opinion among students that a word problem with an appealing content, even if it is difficult, is more stimulating than an easier but boring, non-contextualized problem. For instance, the Foot-and-Mouth disease problem was uptodate in its content since this disease was discussed widely in the press at that time. In addition, the result, showing that the probability that a person who tested positively has the disease is as low as 4 per cent, stimulated an active discussion about reliable interpretation of test results.

During the work on the project, students appreciated having independence and a stress-free atmosphere. All students, including individuals with a weak background, found that they benefit from solving and presenting challenging problems. Capable students, not previously identified in class, can be recognized as informal group leaders in this process.

Didactical Analysis
In the project, students were randomly divided into study groups and were asked to find challenging, curriculum-related problems, satisfying certain criteria, and present their solutions to the class. Each group was also encouraged to solve and present a second problem from the set of challenging problems offered by the instructor.

The project started with class discussion about its purpose and possible outcomes. Students had four weeks to prepare the assignment. At the end, students completed a questionnaire and evaluated the course, choice of problems, quality of presentations and effectiveness of the project.

Importantly, most of the groups selected interesting, amusing problems no matter how difficult they were, and prepared their presentations carefully. Students clearly described ideas and concepts related to each problem, accurately explained methods and formulas applied and operated validly with mathematical notions many of which they found difficult in routine class studies. The audience met presentations with a great interest; every presentation generated a number of questions and was accompanied by animated discussion. In evaluating the project overall, students found this activity stimulating and helpful for their success in the class. We believe that carefully selected challenging problems can be incorporated into the introductory probability curriculum and may be used dynamically throughout the entire course.

4.4.4 Examples: weekly problems

The problems discussed below are different from the ones discussed so far, as they were given in a class taught in English to students with a different first language. For the students in Papua New Guinea's University of Technology in Lae, English is a second or third language, so learning mathematics may require multiple translations. They may find it hard to interpret a problem, but can normally solve it once it is explained to them. The use of mathematical logic, the conversion of word problems to mathematical ones and their solution is very challenging for them (Sukthankar 1999).

In 1992, Sukthankar and her colleagues started a feature in the University of Technology weekly publication, *Reporter*, called "Fun With Mathematics," which contained mathematical quizzes. The problems were designed to create interest in mathematics and to encourage maximum participation by the appropriate provision of clues. They tried to add problems which would not only challenge students to improve their mathematical skills but also teach them how to translate word problems symbolically with proper mathematical interpretation and correct use of technical English words.

They found that a little help from the lecturers made a big difference in the number of participants who used the clues to research a particular topic and arrive at a solution. There were prizes awarded every week to the winning students. The student response to these quizzes was excellent, and consequently, the Department of Mathematics and Computer Science decided to extend this feature of mathematical quizzes to the weekly publications of other universities and tertiary institutions in Papua New Guinea. This idea led to the establishment of the Annual Mathematics Competition for all tertiary institutions in the country.

Below are five sample problems chosen from the weekly quizzes:

Problem 1: If three dice are thrown, what is the probability that the sum of numbers on the top faces is not more than 15?

Problem 2: Tim and John celebrate their birthdays today. In three years, Tim will be four times as old as John was when Tim was two years older than John is today. If Tim is a teenager, what is his age?

Problem 3: In a test given to a large group of people, the scores were normally distributed with mean 70 and standard deviation 10. What is the least whole number score that a person could get and yet score in about the top 15 per cent?

Problem 4: The numbers p, q, r, s and t are consecutive positive integers, arranged in increasing order. If $p + q + r + s + t$ is a perfect cube and $q + r + s$ is a perfect square, then what is the smallest possible value of r?

Problem 5: Sheep cost \$40 each, cows \$65 each and hens \$2 each. If a farmer bought a total of 100 of these animals for a total cost of \$3279, then how many sheep, cows and hens did he buy?

Mathematical Analysis

Methods for solving all these problems are different. The solutions involve knowledge of counting elements in the sample spaces, solutions of simultaneous equations, normal distributions, properties of prime numbers and mathematical logic.

In Problem 1, the first thing that the students needed to note was that it was much easier to count the number of sums greater than 15 than the number of sums less than or equal to 15. Then they had to ensure that no arrangement was missed or repeated while calculating the number of ways. This is a good problem in which to learn how to calculate the sample space systematically. The students always had problems understanding phrases like "more than," "less than," "not more than," "not less than," "at least," "at most," and so on. The solution of Problem 1 was a double challenge since it required correct interpretation and then a mathematical solution.

Students found it hard to translate the apparently confusing wording of Problem 2 into a mathematical equation in two variables. Let t and j denote the ages of Tim and John. When Tim was two years older than John is today, John's age was less than John's present age j by the difference between Tim's present age and his age when he was two years older than John, namely $t - (j + 2)$. Thus, John's earlier age was $j - (t - j - 2) = 2j - t + 2$. From the given conditions we get $t + 3 = 4\ (2j - t + 2)$, which gives us $5t = 8j + 5$. Hence $5(t - 1) = 8j$ and since Tim is a teenager, this implies that $t = 17$.

Problem 3 is a typical example of a straightforward probability problem, involving conversion of a normal distribution to a standard normal distribution and finding the probabilities using the standard normal distribution probabilities table. Although it is not so much mathematically challenging, it was a challenge for the students to interpret the problem correctly.

The solution for Problem 4 is based on properties of prime numbers. If we denote the five consecutive numbers as $n-2$, $n-1$, n, $n+1$ and $n+2$, then from the given conditions, we have to find the least n such that $5n$ is a perfect cube and $3n$ is a perfect square. The smallest n that satisfies both of these conditions has to be divisible by 5^2 and also, since 3 divides n, must be divisible by 3^3.

Hence $n = 5^2 \times 3^3$. Most students started by writing the consecutive numbers as $n, n+1, \ldots, n+4$. They soon realized that it was getting too complicated to derive from the given information $5n+10$ as a perfect cube and $3n+6$ as a perfect square, the smallest possible $n+2$. Then of course, they chose the sequence $n-2, n-1, n, n+1, n+2$ and arrived at the solution.

Problem 5 deals with properties of integers and logical elimination process which are used in simple number theoretical problems very often. A sheep costs \$40, a cow costs \$65 and a hen costs \$2. Let s, c and h be, respectively, the number of sheep, cows and hens. We note in passing that c must be odd and that the units digit of h must be 2 or 7. These facts can be used to help narrow down the search, or as a check on the answer. We have two equations

$$40s + 65c + 2h = 3279$$

and

$$s + c + h = 100.$$

Subtracting twice the second from the first gives $38s + 63c = 3079$. Since $19(2s + 3c) + 6c = 19(162) + 1$, we see that $6c - 1$ must be divisible by 19. Hence, c must have a remainder 16 when divided by 19. Since c is odd and $63c < 3079$, we must have $c = 35$. This quickly leads to $s = 23$ and $h = 42$. The same problem can also be solved in many ways using congruence and divisibility properties.

Cognitive Analysis

The students knew how to calculate the number of ways to get a particular sum with a two dice problem, but to extend to three dice was challenging for most of them. To find the number of ways to get a sum equal to 8 on two dice A and B, some students took the following systematic approach of listing combinations like (6,2), (5,3), (4,4), (3,5) and (2,6). This method uses an approach of starting with a 6 on the first die and then decreasing the numbers to 5, 4, 3 and 2 and getting the appropriate numbers on the second die to make the sum 8. By this method, no combination is missed and all possible combinations are counted.

The same idea is extended for the three dice problem. By the same method, it was easy to calculate the 10 combinations that give the sum 15 on three dice: (6,6,3), (6,5,4), (6,4,5), …. Not all students could think of this combinatorial method. Instead, some tried to pick up the combinations giving the sum 15 at random and had no reliable way of checking whether they had considered all the possibilities. Once shown how to arrive at a definite answer by a systematic counting approach, they appreciated the combinatorial method.

The phrase "not more than 15" was confusing for some students. We normally do not notice this problem with students who have English as their mother tongue. They were not sure whether the expression, "the sum on the top faces of three dice not more than 15," meant $3 \leq \text{sum} \leq 14$ or 15!

In Problem 2, one needs to interpret carefully the verbal expression into an equation. Students needed help to get the equation $5(t-1) = 8j$. Some could easily derive the answer $t = 17$ from the equation since Tim is a teenager and 8 has to divide $t-1$.

For solving problems on normal distribution, it should be noted that a problem of the following type was easier for students to solve:

"If X is normally distributed with mean $\mu = 16$ and standard deviation $\sigma = 4$, find the probability P $(X < 10)$."

However, it would take them longer to solve if it was worded as follows:

"The weekly salaries of 5000 employees of a large corporation are assumed to be normally distributed with mean \$640 and standard deviation \$56. How many employees earn less than \$570 per week?"

The students had been exposed to solving quizzes involving elementary number theory, geometry and mathematical logic. Therefore more than half of them could solve Problem 4 correctly. For the rest, it was challenging but within their reach!

Students solved problem 5 in different ways. It is a good exercise to find the properties of numbers using mathematical logic. By observing carefully the costs of each sheep, cow and hen, the number of animals and the total cost, the method reduces the number of choices for integers to be the number of sheep, cows and hens bought totaling to 100 with total cost \$3279.

Didactical Analysis

An important outcome of these efforts was that students realized that solving math problems could be fun. They were involved in small study groups and had personal consultations with lecturers. They felt that they were enjoying mathematics as a subject and it is not as intimidating as they had earlier thought.

We also used this opportunity to concentrate a bit more on our female students and find out the reasons for their lack of active involvement in classroom mathematics learning. Mathematics is still regarded as a male subject, especially in Papua New Guinea. Boys always dominated classroom discussions and were expected to do better in education than girls. Girls have almost never taken part in any mathematical discussion and for most of the time were silent listeners. They seemed to lack the ability to initiate any mathematical activity (Sukthankar and Wilkins 1998).

During an academic semester in 1997, ten first-year female students from the University of Technology who identified themselves as having low self-concept in their ability to learn mathematics were studied (Sukthankar and Wilkins 1998). During the first half of the semester, they were interviewed and their performance was closely monitored as well as their manner of study and classroom participation.

Then in the second half, they were especially encouraged to participate in the weekly mathematics quizzes; their lecturers also gave them additional help. We learnt from the interviews that the use of computer algebra systems for their course work and understanding of mathematical concepts was beneficial. They were also given extra help to prepare for the term tests. After the tests, their strength and weaknesses were discussed and they were appropriately tutored. They were also given special problem solving sessions and were encouraged to enroll for the Annual Mathematics Competition. We found that over the period of almost six months, there were some positive changes in their attitude towards mathematics. They participated in the Annual Mathematics Competition. This time we found that almost all of them were very enthusiastic to compete, there was an urge to do better and their final results were very considerably beyond their expectations. During the interviews, we found that the main cause of their inability to do mathematics was deeply rooted in the social and cultural factors of their society. Overall, they felt incompetent and had a low self-esteem, and could not see the relevance of studying higher mathematics once they could do basic mathematics. A change in attitude improved their performance and as a result they felt more confident to take up higher studies.

4.5 Example: the challenge of a contradiction and schema adjustment

As for the examples presented in the previous section, the presentation of the following example results from considered reflection on practice.

4.5.1 Inconsistency, contradiction and cognitive development

In addition to developing schemas, it is important to ensure a certain flexibility and richness in a learner's overall schema system. A poor or rigid schema system may force a problem solver to use a very specific representation, and as a consequence, to choose a non-optimal or inadequate solution method or approach. One example of this is the so-called *Einstellung* effect or mechanization of thought, when a solver, based on her repetitive practices, forms a certain stereotype and tends to use the same method again and again without noticing a novel element that critically changed the situation.

For instance, when asked to find the area of a right triangle with hypotenuse equal to 12 units and altitude drawn to the hypotenuse equal to 7 units, a solver uses the usual area formula 12(7)/2 without noticing that a right triangle with such measurements simply does not exist! (Applebaum and Leikin 2007). Familiarity with inconsistent questions can cause a solver to focus in the future on making sense of the given information before proceeding towards a response or conclusion. Checking data for consistency should be completed prior to selecting a formula or solution method.

A flexible schema can lead to efficiency. For example, it is inefficient to solve the quadratic equation, $x^2 - 123456790x + 123456789 = 0$, by calculating the discriminant and using the standard formula for roots. If one observes that the constant term is just one less than the middle coefficient, one can use the Viete theorem or the factor theorem to obtain the answer immediately without calculation.

A novice problem solver can easily overlook a trap offered by a problem. In contrast, an expert's schema system includes, besides methods and procedures, possible error and verification techniques that make use of multiple representations and often prevent the solver from using flawed reasoning and making false statements.

In the rest of this section, we illustrate how inconsistent and contradictory propositions can be used for further development of a learner's cognitive system.

Based on her practices, a learner forms a set of domain-specific expectations about the nature of problems and statements. She develops ways to judge and form an opinion about what is likely to be true and what is not. Often, a statement that surprises a learner or challenges her expectations will stimulate the whole process of understanding the subject. It may also help to break the learner's stereotypes and uncover clarity in the realm of explicit rules and formal theories.

Say, for instance, one is able to illustrate that $1 = 2$ by certain mathematical manipulations and reasoning. The problem then becomes one of locating the error (logical, algebraic or arithmetic) that leads one to the impossible outcome. Notice that the main psychological feature of the situation that distinguishes it from other forms of intellectual inquiry is the presence of the appealing voice of the problem, the voice that essentially passes the ownership of the question directly to the learner. The very fact of the impossibility of the conclusion forces the learner to search for an inconsistency in the reasoning apparently accepted as truthful just a moment ago. The following problems illustrate the situation.

4.5.2 *What do you do if you have to prove that 1 = 2? and other paradoxes*

This section gives four paradoxical problems of different nature. They are followed by a short comment about mathematical reasons and instructional implications.

Problem 1: Consider the following algebraic derivation:

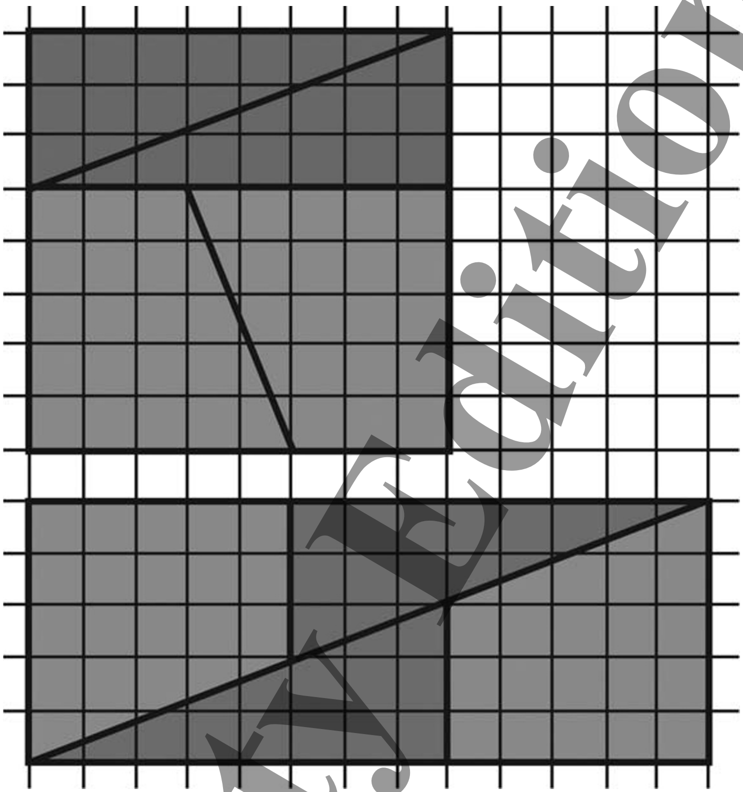
1. Let $a = b$.
2. Then $a^2 = ab$.
3. Then $a^2 - b^2 = ab - b^2$ or, equivalently, $(a + b)(a - b) = b(a - b)$.
4. Then $a + b = b$.
5. Since $a = b$ due to step 1, we have $2b = b$.
6. Thus, $2 = 1$.

Problem 2: Draw a semicircle of diameter 2. Then draw two semicircles, one on each of the halves of the diameter. Then draw four semicircles, one on each quarter of the new diameters, and so on.

Note that the length of the very first semicircle is π, and so is the sum of the lengths of the next two semicircles, as well as the sum of the next four. One can reason then in fact it will remain true for any positive power, n, of 2. On the other hand, when the power n is getting larger and larger, the curve consisting of 2^n semicircles gradually approaches the segment of length 2. This apparently proves that $\pi = 2$.

Problem 3: Three traveling salesmen have car trouble and are forced to spend the night at a small town inn. They go in and the innkeeper tells them, "The cost of the room is \$30". Each man pays ten dollars and they go up to the room. The husband of the innkeeper says to her, "Did you charge them the full amount? Why not give them five bucks back since their car is broken and they hadn't planned to stay here." She then brings the men five one-dollar bills and each man takes one while the other two dollars rest on the table. Originally each man paid ten dollars: $10 \times 3 = 30$; now each man has paid nine dollars $9 \times 3 = 27$ and there are two dollars sitting on the counter: $27 + 2 = 29$. The last dollar has disappeared. (Note that this problem was also used as an example in Chapter 1.)

Problem 4: Areas paradox: this figure "proves" that $64 = 65$.



The tasks presented in this section may be viewed as illustrations of challenging conceptual tasks in the sense that Kadijevich (1999) describes.

4.5.3 Brief comments on the paradoxes in Problems 2 to 4

Problem 2 is deeper and trickier than Problem 1. It appears in the framework of real analysis, and leads to the old philosophical questions. Does a segment consist of a collection of points? Is a point just a circle with radius zero? How do we justify the limit whenever such an operation appears in our reasoning? What is convergence and why do we talk about different types of convergence?

Now, Problem 3 perfectly illustrates the joke about the existence of three kinds of people: those who can count and those who can't. The resolution turns on properly allocating the amounts. The \$27 consists of the \$25 kept by the innkeeper and the \$2 not returned to the men. It does not include the \$3 given back to the men.

Finally, Problem 4 is an interesting visual illusion. If one looks at the side lengths of the triangles and rectangles involved in the figure, one notices that they are 3, 5, 8, 13, some of the Fibonacci numbers $F_{k+1} = F_{\mathrm{k}} + F_{k-1}, k > 1$. It is remarkable that the illusion is based on the property of Fibonacci numbers $\left|F_k^{\,2} - F_{k+1}F_{k-1} = 1,\right|$ which implies smallness of the difference of the slopes, $\left|\dfrac{F_k}{F_{k-1}} - \dfrac{F_{k+1}}{F_k}\right|$, especially if one pick large values of k.

4.5.4 Analysis of Problem 1

Mathematical Analysis

The algebraic expressions $AX = BX$ and $A = B$ are equivalent only if X is not 0. In our example, $X = a - b$ is zero since $a = b$. Thus reduction from $AX = BX$ to $A = B$ is not possible. Since a forbidden step was made (passing from line 3 to 4) a contradictory conclusion occurs. Note that the reduction from line 3 to 4 still makes sense if $a = b = 0$. But then the reduction from line 5 to 6 is not possible for the same reason.

Cognitive Analysis

If $A = B$ then $AX = BX$ for all X. Students often tend to mistakenly treat an implication (if-then statement) as an if-and-only-if statement. Thus the reduction from $AX = BX$ to $A = B$ could be taken mistakenly as an equivalent to the initial one.

The reduction $AX = BX$ to $A = B$ works in the majority of cases (all but $X = 0$). Students tend to ignore this special case and proceed formally. If an algebraic example is relatively long and the student is relatively new to the activity, she would tend to follow the main route and ignore the rare case. Her working memory would be occupied by other tasks such as factoring and the assumption that this case could be temporarily put aside. At the moment of reduction the joy of finding similar factors on both sides of the equation dominates the fact that this factor is equal to zero.

Technically, the students know about the rule that division by zero is not allowed. However, it is often a dead rule, one on a list of other rules. Students may accept it formally and easily overlook it in practice. When the paradox is demonstrated and the contradiction in line 6 reveals itself, the student tries to find why the contradiction occurs. She knows that 2 is not equal to 1, and that forces her to resolve the contradiction, to find where something went wrong. The fact that there are only six lines supports her hope for success. A paradox presents a kind of self-appealing (self-contained) challenge.

Compared to an algebraic exercise that just requires simplifying an expression, this one, which leads to a contradiction, provides a motivation to check the derivation and locate the mistake. When a student finds that violation of a certain rule leads to a contradiction, the student gets to understand the reason behind why the rule is worth remembering and obeying. This example illustrates how a paradox serves as a disequilibrator of learner's schemas, and how understanding of an algebraic rule develops from rethinking (restructuring) a schema.

Didactical Analysis
The mathematical challenge of Problem 1 may be given to students familiar with algebraic derivations. Students with experience and success in similar algebraic problems are expected to be able to resolve the contradiction. The fact of the contradiction is obvious. To ensure that the whole problem belongs to the ZPD (Zone of Proximal Development (Vygotsky 1978, also see Section 6.2.2.3)) of a learner, the teacher provides sufficient training in algebraic reductions and makes sure that students can do and check their algebra. Some students will tend to substitute numbers in place of letters to check the derivations. This is a possible approach as long as the student does algebraic substitution consistently.

In the experimental setting (Kondratieva 2007), the paradoxes from Problems 1 and 4 were given to first- and second-year university students to be resolved in class during a ten-minute period. The students were not tested nor taught algebra or geometry immediately prior to the task since they had all passed a placement test, and therefore, it was implicitly assumed that they had already been trained in the subjects. The experiment showed that:

1. Everyone was intrigued and motivated by the contradictions.
2. Not everyone was able to find the reasons for the contradictions. Some students were able to locate the wrong line in Problem 1, but no clear explanation was given. Even fewer students were successful with Problem 4.
3. Good students found that the problems were not difficult but were nevertheless interesting. They said that they learned to stay alert while doing formal derivations or trusting a pictorial proof.
4. Some students composed their own examples of paradoxes using similar ideas. Such a task was not assigned, and the fact that they did so voluntarily illustrates an important human tendency to mimic-and-create during the process of acquisition of new knowledge.

4.5.5 Concluding remarks

While there are different levels of difficulties in the apparent contradictions we have considered, they all have in common the intrinsic call for a resolution, when, rephrasing Aristotle's metaphorical idea, the mind experiences itself in the act of making a mistake. And then it makes sense.

The role and place of paradoxes in the process of cognitive development can be identified within Piaget's theory of equilibration, which refers to the Kantian epistemological proposition that the knower constructs her knowledge of the world. Paradoxes disequilibrate a learner's schemas, and that is the starting point of the process of accommodation of a portion of new information. Then the learner will go through stages ranging from "beyond belief" to acquisition of knowledge with justification.

If we want students to learn how to verify and validate their solutions and to critically read others' work, we need to familiarize them with situations involving contradictions and paradoxes. They then need to know how to handle such situations and how to analyze and arrive at possible resolutions and explanations. That is why an exposition of paradoxes is so valuable.

We conjecture that the phenomenon of paradoxes drives the whole of human intellectual development because the challenge of a contradiction is the mainspring of learning on both individual and historical levels. Therefore, these types of challenges cannot be ignored but instead need to be carefully analyzed and promoted as instructional tools inside and beyond the classroom.

4.6 Conclusion

The preceding examples of challenging mathematics problems together with the descriptions and analyses of students' responses to them illustrate that students benefit socially and cognitively from engagement with challenging problems. The qualitative analyses suggest that the gains are evident in the short term and are intellectually important over time. Students build adequate and sophisticated strategies to solve challenges.

From a cognitive perspective, through meaningful engagement over time with problems within a strand of mathematics, students build effective and important problem-solving schemas. They develop insights into the mathematical structure of related problems and this knowledge becomes schematized. Moreover, students need to develop flexible schemas since rigid ones may inadvertently cause a problem solver to choose a non-optimal or inadequate solution method or approach. Resolving inconsistent and contradictory propositions or paradoxes can support the development of flexible schemas.

Most research on schema construction has been done using traditional psychological paradigms, investigating how and (more often) to what extent individuals can construct and apply schemas in a short period of time. The research of Weber et al. (2006), which forms the basis of the examples in Section 4.3, differs from this paradigm significantly, looking at how students developed schemas over time, all the while solving challenging problems. They believe that this change in perspective radically altered the nature of their findings. If their research participants were given straightforward problems, they would not have had the need to develop the useful representations for these problems that were critical for their schema construction.

If they were only given a short period of time to explore these problems, the schemas also would likely not have been constructed. In fact, students initially did not see the deep connections between the various problems on which they worked. Hence, looking at the processes that individuals use to form and use schemas in relatively short periods of time is looking at only a subset of the processes used in this regard. The work of Weber et al. (2006) demonstrates that studying the way that students solve challenging strands of problems over longer periods of time provides a more comprehensive and useful look at how students can construct and use problem-solving schemas.

As this chapter has illustrated, challenging mathematics problems are suitable for a range of audiences and didactical situations. They are apt as interview questions for entrance into university mathematics programs to obtain windows into how students think mathematically; as investigations for teacher candidates to further develop their own mathematical understanding and to acquire insight into how learners learn mathematics; as supplements to or material integrated throughout a course; as a means to reinsert marginalized students into mathematics, providing them with a context with which to entertain their minds; and, by placing mathematical challenges in a university's daily or weekly newspaper, as vehicles to popularize and create interest in mathematics among students studying the subject in a language other than their own.

Challenging mathematics problems can be instruments to stimulate creativity, to encourage collaboration and to study learners' untutored, emergent ideas. We have also shown that they are appropriate for secondary and post-secondary students as well as for high-achieving and low-achieving learners. From a didactical perspective, it is important that the problems require little specific background and generally can be attempted successfully by students of varying mathematical backgrounds.

Economic and social capital need not be markers of who can participate in mathematics. In Fioriti and Gorgorió (2006), from which the example in Section 4.4.2 is excerpted, the authors indicate how it is possible to engage socially excluded youngsters with challenging mathematics problems so that they are reinserted into school settings and thereby widen their possible social and academic participation in their society. Clearly, there are a host of socio-economic realities that need to be addressed to truly democratize academic and social participation. However, engaging students of diverse backgrounds in challenging mathematics problems contributes to this larger goal.

Making mathematics less exclusionary and more inclusive depends on shifting from traditional pedagogies and procedural views of mathematics learning (Boaler and Greeno 2000). It requires reversing a common belief among teachers that higher-order thinking is not appropriate in the instruction of low-achieving students (Zohar et al. 2001). If challenging mathematics problems were used in settings such as formal classrooms and other informal arenas, learners might begin to recognize mathematics as accessible and attractive (cf. Zohar and Dori 2003). They would have opportunities to build mathematical ideas and reasoning over time, develop flexible schemas and inventive problem-solving approaches, and become socialized into thinking mathematically.

As Resnick (1988) suggests: "If we want students to treat mathematics as an ill-structured discipline [that is, one that invites more than one rigidly defined interpretation of a task]—making sense of it, arguing about it, and creating it, rather than merely doing it according to prescribed rules—we will have to socialize them as much as to instruct them. This means that we cannot expect any brief or encapsulated program on problem solving to do the job. Instead, we must seek the kind of long-term engagement in mathematical thinking that the concept of socialization implies". (p. 58)

If mathematics educators and teachers adopt a long-term perspective on the development of problem-solving schemas, then a paramount goal of mathematics education—to further learners' effective problem solving—would be more achievable.

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Chapter 5
Mathematics in Context: Focusing on Students

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This chapter presents nine case studies in which school students engage in challenging mathematics outside their immediate classroom environment. In each case, students are encouraged to collaborate in investigations that go beyond the standard curriculum and creatively use the ingredients of the particular context. In Italy, students visit a mathematical laboratory to understand and utilize mathematical machines. Morning assembly at an Indian school brings students from many classes together in the solution of mathematical problems. Four of the projects are from France: students analyze the configuration of a heap of sand, pursue astronomical investigations with software, obtain a flavor of research by having secondary school teams investigate interesting problems, and are presented at all levels with open-ended research problems. There are three programs from the United States, the first, an advanced geometry sequence for secondary students completing the regular syllabus early, the second, activities arising from exhibits in an art museum, and the third, using the school lawn to deepen student understanding of geometric constructions. All such activities need to be evaluated for their effectiveness, so that they move from just being initiatives of dynamic individuals to serve as the foundation for systemic improvements in the way in which students learn, understand and use mathematics. In the early part of this chapter, we briefly mention how research into such activities might be approached.

5.1 Introductory comments

For centuries, teachers such as John Comenius (Jan Komensky, 1592–1670), a Czech educational reformer, and Maria Montessori (1870–1952) have appreciated the benefits of actively involving learners in their education, consigning the teacher to the role of a wise guide.

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Accordingly, we can expect that challenges that invite and engage students would be effective in and beyond the mathematics classroom. Some questions emerge naturally. For example, we can ask

- whether challenges can be used to motivate students for study in mathematics and science;
- how mathematics can be linked to popular culture;
- how challenges can stimulate the enjoyment of mathematics.

More intrinsically, we can enquire also

- whether challenges lead to a deeper understanding of concepts and the nature of mathematics;
- whether challenges improve retention and help better correlate ideas and techniques;
- whether challenges foster more facility and resourcefulness in the use of mathematical ideas and techniques.

The design of a general framework in which to study, compare, and contrast the implementation of challenging experiences in mathematics becomes crucial. Several aspects must be considered: the diversity of contexts and the use of various artifacts; the distinguishing characteristics of challenging mathematics; the cooperation of teachers and researchers; obstacles to a successful implementation of mathematical challenges in classrooms, mathematics laboratories, scientific museums, and popularization events; and the categorization and evaluation of challenges. We include in this chapter a description of long-running programs to help us identify the salient features of such a framework.

The philosophy of learning through being challenged evokes two types of questions. On the one hand, there are psychological ones dealing with expectations, motivation, disequilibrium, Zone of Proximal Development, and short-term processes, for example. On the other, there are organizational issues , such as stability, consistency, didactical content, and suitable contexts.

Following a general presentation of the position of the learner in a challenging environment, several case studies will be described. They will illustrate how educators in diverse settings have successfully implemented mathematical challenges in and beyond the classroom. Here is a brief summary (consistent with elsewhere in this Study Volume, authors of the individual case studies are identified in the Acknowledgements):

- **5.3.1 Mathematics laboratory:** In a mathematics laboratory, students work in groups with mechanical devices designed to produce conic sections. Their study is guided by worksheets (depending on the grade level). Students present the results of their explorations.
- **5.3.2 Indian school:** Mathematical challenges are periodically given to students at the morning assembly in school. Occasionally these challenges take on a life of their own, being discussed by students and modified into other

challenges. In these cases, a special bulletin board is created so that all students may see the problem and its solution.

- **5.3.3 Sand pouring:** Inspired by an exhibition in the Jardin des Plantes in Paris, students who participated in the *Rallye mathématique de Paris* (2000) and the *Rallye mathématique de la Sarthe* (2003) experimented with pouring sand on a flat surface. They discovered that the angle between the sides of the conical heap and the surface is always the same. This problem can also be taken to the classroom in an extended form.
- **5.3.4 Hands-on Universe, Europe:** Interactive astronomy is brought to classrooms from primary school to university. Begun at the University of California at Berkeley, this collaborative project between researchers and educators is spreading globally. The European group has developed special software for students to practice science and mathematics.
- **5.3.5 *MATh.en.JEANS*:** Young research mathematicians develop a list of interesting problems, which are then selected by teams of high school students. Each problem is selected by two teams from different schools. Teams explore the problems, sometimes with the aid of the proposers, and then present their results. *MATh.en.JEANS* is the organization responsible for coordinating these activities.
- **5.3.6 Applied geometry:** Many high school students complete calculus at the end of their eleventh year. For such advanced students at Quincy High School (Illinois), a special course has been designed. The focus is on various topics in two-, three-, and four-dimensional geometry, with an emphasis on hands-on work.
- **5.3.7 Open-ended problems:** Students and teachers of all levels, primary school through university, are presented with open-ended research problems in mathematics (such as polyomino exclusion problems). The context may be the classroom, teacher training, or a special event. Participants design their own problems, and are assisted by a professional mathematician. Results of their work are presented both orally and through poster presentations.
- **5.3.8 Mathematics and art:** Students visit a local art museum and see how symmetry and topology are involved in artwork. In teams, they go on a geometrical "treasure hunt".
- **5.3.9 Lawn constructions:** Using sticks and brightly colored yarn, students perform geometrical constructions on a large, flat lawn. They then analyze their constructions using pictures of their work taken from a nearby tall building.

5.2 Discussion of contexts for challenges

Of course the feature common to all mathematical challenges is mathematics itself. While setting challenges mathematical facts are not presented by an authority. Rather, through explorations, students arrive at results and modes of reasoning on their own. This process of discovery allows them to learn mathematics in a more meaningful way. New cognitive processes are developed.

We discuss in turn different aspects of the context for mathematical challenges:

1) the social learning environment;
2) the time element;
3) instruments and objects;
4) pedagogical methods.

Clearly, there is a great diversity in social learning environments for mathematical challenges. While the case studies highlight many of these, many educators spend much of their time in classrooms. Thus they need to accept the challenge of making the classroom both engaging and challenging. Many of the challenges described here and elsewhere in this Study Volume may be adapted for classroom use. For example, many teachers could use problems from mathematics contests or Olympiads in the classroom. Teachers may also recruit parents to introduce and discuss challenges at home; we do not deal with this possibility here.

The duration of mathematical challenges varies greatly. While an exhibition visit may consume only a single afternoon, programs such as *MATh.en.JEANS* take place over the course of an entire year. Some classroom challenges may be met within a single classroom session, while in a course for advanced students, they may last for the entire school year. Often, the challenge must be adapted to suit a given time frame; many allow this sort of flexibility.

Even greater diversity occurs with the instruments and objects used in the design and implementation of mathematical challenges: special instruments may be constructed for use in a mathematical laboratory; an innovative use may be found for an ordinary material like sand; one classroom project used three thousand donated compact discs.

Educators use many pedagogical methods to engage their students in challenges. An effective strategy is to design these so that they can experiment and discover mathematical principles. This is usually followed up by recording their thoughts in a mathematical journal or making a presentation to classmates, family members, or a group of researchers; not only is this exciting for the students, but a different kind of learning takes place when they discover and formulate results for themselves. Journals are especially important when challenges are spread over a longer period of time, as they enable students to review their thoughts whenever the challenge is revisited.

The relationship between student and teacher is implicit in the use of pedagogical method. For with a challenge, a student may pose questions whose answers might not be clear to the teacher. Indeed, the teacher might not even know the outcome of the challenge. Thus the didactical contract, in the sense of Brousseau (1997), evolves as the challenge gains momentum.

Students and teachers also have relationships with others, such as parents, the popular press, granting bodies, and the professional community. In France, there are several mathematics publications directed to different age groups. Their goal is to popularize mathematics among students, teachers and the public at large. They generate discussion in the classroom and in the home. In some countries, such as the United States, professional mathematicians are encouraged to take a more active role in primary and secondary education so that students are encouraged to proceed to a higher level.

The evaluation of mathematical challenges is both important and complex. Their very nature forces a significant qualitative component. The challenges themselves must be assessed for difficulty, appropriateness and ability to support particular curricular and affective goals. The degree of engagement and the consequent long-term effects among students must be evaluated. Students may judge how a challenge induced them to change their thinking and take stock of what they had already learned. Teachers may reflect on how a challenge has altered their perspective on their pedagogy and on the mathematics. How can challenges be assessed externally? Reliable positive assessments of challenges may lead to administrators and designers of syllabi becoming convinced of the need for more mathematical challenges in the learning environment.

It is evident that contexts for challenge in mathematics are many and varied. However, even with a well-designed challenge, there may be obstacles to introducing it to a learning environment. The need to adhere to a prescribed syllabus may not allow time to consider additional topics. Money may be lacking to obtain the necessary materials. The attitude of administrators towards mathematics may be counterproductive. There may be students whose mother tongue is not the language of instruction. Students may use the Internet which may show solutions to challenges that one might otherwise use.

5.2.1 Highlight on long-term studies

It is important to recognize that, in order to be effective, the use of challenges should not be an occasional diversion but should become part of one's philosophy of teaching. Many of the challenges described in this chapter take place over a long period. This makes the processes more complex with a greater chance of obstacles arising over the longer time frame. Thus, the wise use of class time is an important consideration. In some countries, such as Italy, teachers teach students for more than one year. Accordingly, they must plan in the long term, especially as many feel responsible for the development of the personalities of their students.

In the past, it has been easier for educational research to focus on short-term studies. However, new research techniques developed in Europe and North America make the complexity of long-term projects more accessible to study. It is beyond the scope of this chapter to review all of this work, so we limit our discussion to two examples and provide some references to the literature.

5.2.1.1 Ingénierie didactique

Drawing on the theory of didactical situations (Brousseau 1997), French researchers in mathematics education created a concept known as "didactical engineering" (Artigue 1989). This theory takes into account the complexity of the classroom, and considers the relationship between research and practice in learning environments. The focus is on the "situation", a system of conditions enabling a group of students or an institution to solve a problem. Each situation incorporates the "didactical triangle" of teacher, learner, and knowledge, and the relationships among the parts of this triangle. Ideally, the system and its evolution can be modeled.

Just as the idea of "situation" extends that of "problem," the concept of "didactical engineering" extends that of "didactical design". The object of didactical engineering is to create and test a sequence of didactical situations as tools for the teacher, to make explicit the options available in using these situations, and to justify the choice among these options theoretically and experimentally.

Each option is viewed from two perspectives: a global perspective concerning the consistency of the entire sequence, and a local perspective concerning the individual situations. The engineer specifies the various stages and didactical variables of each situation, always keeping in mind the motivation of the student and the objectives of the sequence. Interactions between students and situations should be characterized, if possible.

Now the sequence is ready for experimentation, followed by an analysis based on data collected during each session. Data may take the form of observations, video, or written protocols. This comparison between theory and practice is crucial, as it validates using the sequence as a tool for learning and teaching, or suggests possible improvements to the design. Most French researchers use this model in studying the application of theoretical ideas in the classroom.

5.2.1.2 Activity theory

In this approach, individual processes are understood only in the context of social processes. This is because people do not passively absorb and react to stimuli, but rather actively explore and transform their physical and social environments. Mellin-Olsen (1987) analyzed the design and implementation of long-term studies from the perspective of activity theory, and in Nordic and other countries, some scholars are using a similar approach.

In this view, mathematical ideas are considered historical "artifacts" of human culture (Vygotsky 1998). Therefore, the process of learning to think mathematically is a process of enculturation. This is clearly an interpersonal activity involving the student, teachers, peers, and others.

Artifacts may also include instruments such as straightedge or compasses. They may be used by individuals or groups to solve mathematical problems.

Students working in a group may, over time, formulate abstract principles as a result of their concrete experiences.

This process has been observed in groups of students solving the following problem (Bartolini Bussi 1996): Students are given a perspective drawing of a table, and are asked to draw a small ball in the center of the table. They may use instruments, and must explain their reasoning.

Students from Year 2 through to university level have found this problem challenging. Most students, including adults, try connecting the midpoints of opposite sides of the quadrilateral representing the top of the table. However, since the table is drawn in perspective, this procedure does not lead to a correct solution.

Working together, students formulate postulates of perspective drawing, such as "if three points line on a straight line, their perspective images also lie on a straight line". Once they formulate enough postulates, the problem shifts from the concrete to the abstract. Instead of working with concrete drawings, students begin to reason using the abstract principles they have developed. This process moves them closer to the solution: place the ball at the intersection of the diagonals of the quadrilateral. Students then justify this solution using the abstract postulates they have formulated.

5.2.2 Conclusion

Creating and implementing mathematical challenges is ambitious and demanding. It is hard to create a learning environment that engages students in a challenge and stimulates the development of their mathematical reasoning abilities. In this regard, it is not just the student who is challenged. We as educators are also challenged.

It is hoped that the case studies included in this chapter will serve as inspiration. Interested readers may contact any of the authors for more details on their work. We must work together to insure that mathematical challenges are available to all students everywhere.

5.3 Case studies

5.3.1 The Laboratory of Mathematical Machines

The Laboratory of Mathematical Machines (the MMLab), in the Department of Mathematics in Modena (Italy), is involved in both didactical research and the popularization of mathematics (such as exhibitions). It contains a collection of instruments (mathematical machines) for the teaching and learning of geometry. The MMLab is visited by both teachers and students (www.mmlab.unimore.it).

5.3.1.1 Learning environment

The MMLab is open to classes during the whole school year. When teachers decide to come to the MMLab, they reserve it and choose to study either conic sections or geometrical transformations. In this section, we would like to draw attention to some aspects which characterize MMLab activity.

1. A session at the MMLab is structured differently from a visit to an exhibition or a classical mathematical classroom—only the beginning can be compared to a “standard” visit, where a presenter introduces the exhibits.
2. In the MMLab, the “rules of the game” are made known to teachers (even if it is the first time that they accompany the classes), but are not imparted to the students who know only the chosen topic.
3. The three subjects involved in the MMLab session (i.e. the animator, students and teacher) play different roles. The animator is the voice of history and mathematical culture. He or she supports and controls the working groups, guides the pupils’ presentations, and validates their answers. The pupils are visitors and manipulators. They sketch, formulate conjectures, and discuss, and they present their work. Usually, students do not play so many different roles during a mathematics class. Only the teacher’s role is not well defined in our laboratory session: he/she is a kind of “joker” and is relieved of the responsibility of teaching during a MMLab session. Moreover, the teacher can, on one hand, be an observer of pupils’ exploration processes; on the other hand, he/she can experiment with a different kind of mathematical session without being directly involved.

Generally, this new situation can determine different relationships among the pupils, between the pupils and teacher, and between pupils and mathematics.

In order to analyze the laboratory format and compare this with other situations such as the classroom and “standard” visits to exhibits, we distinguish among formal education, non-formal education and informal learning (EC Communication 2001, Rogers 2004, Rodari et al. 2005, SEEQUEL Project 2004).

Briefly, “formal education” takes place in a planned way at recognized institutions such as schools, colleges, and universities. Teachers mediate the learning in a prescribed setting. “Non-formal education” shares with formal education the characteristic of being mediated, but the motivation for learning may be wholly intrinsic to the learner. It typically does not lead to certification. “Informal learning” results from personal exploration and discourse and may occur spontaneously in situations in everyday life. These distinctions are noted in Chapter 2 where mathematics exhibitions are treated. Because it is difficult to determine the boundaries between them and the importance of the context, recent approaches suggest considering a continuum between formal and informal learning.

Thus, the MMLab’s activity may include both non-formal and formal education or, in other words, is between non-formal and formal education.

Indeed, a MMLab session does not entirely correspond to a mathematics classroom concerning a didactical long-term project (formal education): time, objectives and handling are different. Nevertheless, the session does not fit non-formal education, because of the presence of the teacher and the animators and the management of the session.

5.3.1.2 Duration

A laboratory session lasts an hour and a half. Three stages constitute each session.

Stage 1: The chosen topic is introduced. This stage is conducted by the MMLab personnel, using both physical instruments present at the MMLab and interactive/non-interactive simulations (Cabri II Plus, Cinema 4D).

Stage 2: Pupils are invited to form small groups (four or five pupils), which receive a mathematical machine and a worksheet to guide their exploration.

Stage 3: Each group presents the studied mathematical machine to their fellow pupils. It is an important institutionalization moment (Brousseau 1997), because the results of each group work session are shared by the class and the teacher. In this way, those results belong to the class's repertoire and they can be recalled by the teacher after the session.

5.3.1.3 Instruments

The practice of using tangible instruments in mathematics (especially in geometry) was historically included in the work of mathematicians. In the MMLab, there are many different kinds of mathematical machines (Bartolini Bussi and Maschietto 2006). A mathematical machine (in geometry) is an artifact that is designed to force a point, a line segment, or a plane figure (supported in a way as to make it visible and touchable) to move according to a determined mathematical law. Two hundred such machines (examples in Figures 5.1 and 5.2) have been reconstructed with a didactical aim, according to the design described in historical texts from classical Greece to the 20th century.

5.3.1.4 Pedagogical methods

The use of instruments in mathematics education is supported by the Vygotskian construct of semiotic mediation (Bartolini Bussi 2000, Bartolini Bussi and Mariotti 2008, Bartolini Bussi et al. 2005, Maschietto 2005, Maschietto and Bartolini Bussi 2005), which is built around three different poles:

1) the cultural-historical pole, to describe the features of technical and psychological tools which have the potentiality of creating "new forms of a culturally-based psychological process" (Vygotsky 1987);

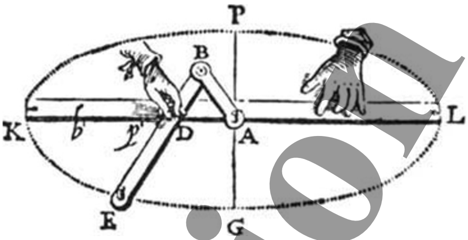


Figure 5.1: Ellipsograph

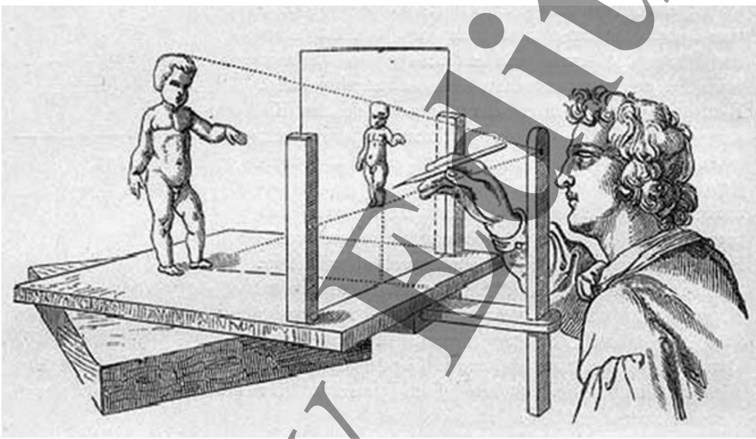


Figure 5.2: Perspectograph

2) the didactic pole, to describe the way of designing, implementing and analyzing processes of semiotic mediation;
3) the cognitive pole, to describe the process of internalization of interpsychological activity, that creates the plane of individual consciousness.

The MMLab is a good example of what has been called a "mathematics laboratory" by the Teaching Commission of the Italian Mathematical Society (UMI-CIIM).

We can imagine the laboratory environment as a renaissance workshop, in which the apprentices learned by doing, seeing, imitating, and communicating with each other, in one word: practicing. In the laboratory activities, the construction of meanings is strictly bounded, on one hand, to the use of tools, and on the other, to the interactions between people working together (without distinguishing between teacher and students).

The enactive mode of knowledge (Bruner 1967) is usually limited to young people in mathematics education, as if the importance of handling objects and exploring space decreased with age. In some cases the confidence in the power of the concrete experience itself was surely excessive, as if the mathematical meanings were transparent from the procedures embodied in it.

5.3.2 *Seeding mathematical challenges at morning assembly at a school in India*

Morning assemblies are common in Indian schools. Students and teachers gather to be led in prayer, sing the school song, hear invited speakers on specific topics, or commemorate special occasions. The event is not tied to the curriculum and varies from school to school. At Vidyaranya High School, Hyderabad, the twenty minutes occupied by morning assembly were used creatively. Teachers had the opportunity of presenting any interesting topic for discussion with students and fellow teachers. This provided the opportunity to seed mathematical challenges.

Sharada Gade taught mathematics at the middle grades, having the same students for two or three years in a row. Sometimes, at the assembly, the school combined high school and middle school students; sometimes, they were kept separate. She had the opportunity to set mathematical challenges for students she had taught, was currently teaching, or would teach in the coming years. In setting a challenge, there were three considerations. First, what would the challenge be about? This depended upon who would be present in the assembly and what she thought would interest them. She could build upon a current topic in the curriculum, or offer something she had learnt herself elsewhere in the discipline of mathematics. Secondly, discussion of challenges in this setting provided an open invitation to everyone to participate. Thirdly, could she communicate the challenge in the time available?

Not all students accepted the challenge. The provision of the challenge took the form of an invitation, and those students accepting the challenge were encouraged to get in touch with her again during the day. At times they shared their solutions; at other times they shared the excitement about what one, a pair, or a group of them experienced in attempting the challenge. As a teacher, this provided valuable feedback as well as further opportunity. First, students' responses gave her an idea of what kinds of problems challenged students, combined with the nature of the mathematics that challenged them. Secondly, she established contact with the students interested in the mathematical challenges she set.

Access to interested students allowed her to put students across different grades in touch with each other. At times this created a kind of friendship or kinship that she initiated but did not pursue further. At other times interest in a particular challenge was taken a step further. Could the challenge be attempted using other skills and techniques? Was the use of an equation necessary? Could the present challenge be a background upon which to set another? At times the challenge that was seeded in the morning resulted in students bringing forth challenges from their home, set by a friend, aunt, or uncle. Some of these were also shared with other students at a pinup board, either in the classroom or in common areas between classrooms.

Seeding challenges and keeping interest in them alive, beyond the teaching-learning of the curriculum in the classroom, called for an alert disposition in her role as a teacher. The processes she initiated demanded time and attention to dynamical situations which were not envisaged or anticipated. However, the very setting forth of a dynamic process had benefits for routine curriculum teaching-learning as well, since the students were some of the same individuals inside her classroom. With challenges at morning assemblies as background, it was possible to create, without additional effort, the elusive aspect of interest in the subject of mathematics. The attitude of questioning, seeking solutions, conjecturing, and attempting questions became commonplace in the learning environment for mathematics in the classes she taught.

In the nature of seeding challenges described above, the time element for each varied. Some students were not even attentive for the twenty minutes; some were interested only as long as their friends were; but for some, the challenges lasted the bus ride home, for some the weekend. The flexibility of such an approach on behalf of the students was intentional, since by the process she wished to appeal to their voluntary disposition and interest, not making challenges part of either regimen or routine. The combination of creating interest in a mathematical challenge, her authority as a teacher at the morning assembly, accompanied by a voluntary disposition, led to and demanded a plurality of pedagogical methods.

Sometimes the twenty minutes of discussion across a hundred students resolved the challenge. At other times the challenge was pursued in a more focused manner in the mathematics class that followed the morning assembly, where a solution was debated among thirty students. At still other times a group of students from different classes met to discuss the challenge in the intervals between classes. On occasion a group of students presented a challenge at the morning assembly themselves.

Sharada's goal in setting challenges was generally a simple one: that of questioning the present knowledge that any student displayed, thereby extending their current knowledge of mathematics and in mathematics. Since she became fairly well acquainted with the interests and abilities of her students, she came to the obvious realization that a challenge for one student was not a challenge for another.

Mathematics usually considered routine was in itself challenging for many. For example, finding the lowest common multiple (LCM) of the algebraic terms $3x^2y$ and $4xy^2z$ was challenging for some, just as finding the LCM of 15 and 18 was challenging for others. Finding the volume of a toothpaste carton through application of the algebraic formula was easier for some than making, with paper and scissors, one with a certain volume, or even making one with twice the dimensions of a given carton. Examples of challenges follow.

Example 5.3.2.1
Observe the pattern which emerges on conducting the following three steps.

Step 1. Take any number to start with.
Step 2. If the number obtained is odd, triple it and add one; if the number obtained is even, halve it.
Step 3. Repeat Step 2 with the number which results from performing Step 2.

For some students finding the 4, 2, 1 pattern that emerges was a challenge, where for others it was the use of the number 27 in the first step. For the more adept, the challenge was to find out if a regular pattern resulted from any other sequence of steps or stated conditions.

Example 5.3.2.2
Find the difference between 825 and its reverse. Is it a multiple of 9?
Find the difference between the two-digit number '*ab*' and its reverse. How can you say whether it is or not a multiple of 9?

The extension of this challenge into related challenges was not very difficult for Sharada's students.

In summary, the practice of seeding mathematical challenges was enabled and supported by the way the morning assembly was conceptualized in the school, and how Sharada was able to capitalize on the opportunity to make mathematics challenging and exciting. The seeding of challenges and nurturing of them in the school spawned a welcome culture of teaching-learning in mathematics.

The practice provided an opportunity not only for the setting of a particular mathematical challenge, but also for the development of a particular challenge both between peers and across time. The seeding of mathematical challenges and making the related practice "common knowledge" enabled a greater sense of awareness of the creativity and cultural inheritance of mathematics in the students and thereby also in the school. For more details see Gade (2004).

5.3.3 Heaps of sand: what we can do with sand in and beyond the classroom with a mathematical aim

In 2000 an exhibition was constructed in the Jardin des Plantes in Paris by the artist, Jean Bernard Métais (www.jbmetais.com/), with the assistance of scientists from Jussieu University in Paris. It provides the inspiration for a challenge for the participants of the *Rallye mathématique de Paris*.

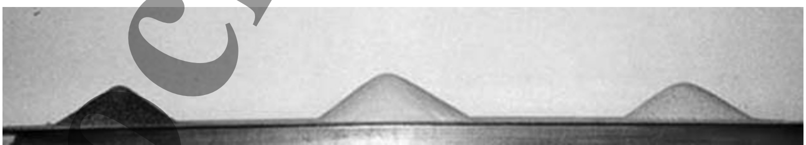
On similar lines, students of a scientific workshop (a multidisciplinary learning environment where students are free to study as they choose) in a school at Nantes (in the west of France) studied the form of some heaps of sand. Four pupils and their teacher explained their work at the *Salon Culture et Jeux Mathématiques* in Paris.

Sand is a special material, and a scientific workshop at the Château d'Olonnes in the west of France, has been studying the substance since 1990 (www.chez.com/sable/).

In collaboration with other sciences, we can engage our students in many experiences that involve observation and measurement, along with the challenge of accounting for what is seen. For example, one discovers that regardless of the type of sand, when poured upon a flat plane, sand gathers into a heap that is always a cone with a constant base angle equal to about 35 degrees.

We describe the tasks that were presented in the first *Rallye mathématique de Paris* (family teams) and also in the *Rallye mathématique de la Sarthe* (competition for whole classrooms). In both contests, the questions were similar; they are given here in simplified form.

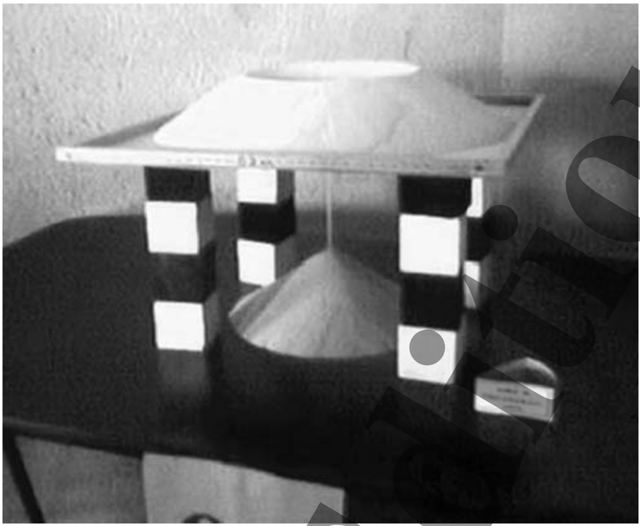
Example 5.3.3.1
Look at the heaps of sand in front of you.



Using pictures of these heaps, take several measurements of the base angle θ. You will note that the angle appears to be constant. Determine the average of your measurements.

What is the geometrical form of the heap? Measure the base diameter of the heap and determine the volume of sand.

Now we open a small hole exactly at the centre of the base of the heap.



Sand pours through it to form another heap underneath the plane of the first.

When does sand stop pouring from the top heap? We note that this happens when the base angle is equal to θ. Describe the form of the upper heap of sand at the end of the experiment. Draw the vertical cross-section of the upper heap through a diameter. What is the volume of sand in each part?

For longer durations, this activity can be extended to last for perhaps six hours in the final year of middle school. The pupils of a class are broken into small teams and presented with a situation that requires them to observe, model and calculate. The role of the teacher is just to set the experiment in motion. Each team is evaluated on the basis of the observations and conclusions recorded in journals and on an oral presentation.

There are several levels of challenge for the students to account for in their observations. The most basic is that the heap of sand is symmetrical, so what kind of symmetry is exhibited and why? Moreover, the shape of the heap seems to be independent of the amount of sand involved, which raises the whole issue of scale. Whether the sand fills a cup or the back of a dump truck, it appears that the shape of the heap is invariant. When sand from a top heap is allowed to leak into a bottom heap, students might relate the shape of that part of the top heap vacated by the sand to the shape of the lower part; the two parts of course have the same volume, but will they also be congruent? Is this to be expected? Such questions have both mathematical and physical components.

5.3.4 SalsaJ: astronomical software

“Hands-on Universe” is an international project for teaching astronomy from primary school to university. This project, started at the University of California at Berkeley, is thriving and growing actively in many other places in the world, for instance in Europe (EU-HOU). It is the result of collaboration between researchers in astrophysics and teachers of mathematics and science.

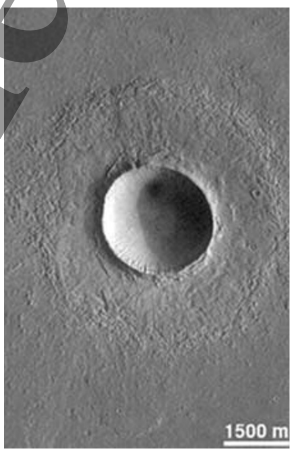
The projects are based on real observations, possibly acquired by the students themselves through a worldwide network of automatic telescopes linked to the Web. The observations can be manipulated with special software, such as SalsaJ, developed by the European component, and integrated into pedagogical resources.

Using this software, students can practice science and engage in challenging mathematical activities. A full description of the program along with descriptions of many projects and software can be obtained from the website www.euhou.net.

Here you can see a very small example of the activities developed in a scientific workshop. The title of this workshop was “Cratères et volcans dans le Système Solaire”.

Example 5.3.4.1 Study of a shock crater on Mars

1) What is the shape of a pixel?
2) Describe this image—not only the main crater, but also other disturbances on the ground of Mars (compare with other examples of craterisation on the Moon or on the Earth). Can you say where the Sun is?
3) Using the scale indication, calculate the value of one pixel (in meters).
4) Infer the diameter of this crater. Explain your method.

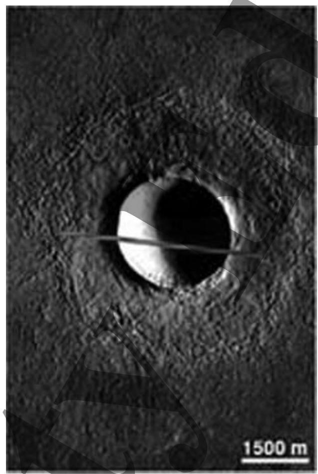


Using the tools *line*, and afterward *analyze* and then *crop*, we can discover the number of pixels on the 1500 meters, sub-picture.

We can know that, in this case, one pixel is approximately 20 meters.

Using again the same tools (line, analyze and crop), we can discover that the diameter of this crater is 2500 meters.

Furthermore, we can do the same operations again and again to obtain more details about the disturbances resulting from the shock. For example, we can measure the crown around the crater where the ground is modified.



5.3.5 Mathematical challenges around Orsay

MATh.en.JEANS is a network of mathematics clubs operating across France that was founded approximately two decades ago. A description of its scope and activities can be found on the website mathenjeans.free.fr/amej/accueil.htm. A secondary school teacher may decide to create a club, and, with the permission of the headmaster, can contact the *MATh.en.JEANS* association which provides access to consultants and mathematical problems and activities. Each year there is a national congress that focuses on particular topics.

L'Université Paris-Sud at Orsay is one centre of activity. Apart from its participation in *MATh.en.JEANS*, there are numerous contacts between its mathematicians and secondary students. Challenging activities for students are built around visits to the laboratory of mathematics at Orsay and to the *Institut des Etudes Scientifiques* in Bures/Yvette by classes of secondary school students, in particular at the occasion of the so-called *Fête de la Sciences*, held every year in the fall.

Longer term activities take place as part of *MATh.en.JEANS*, whose local coordinator is Professor Pierre Pansu in the Department of Mathematics. A sample of one of their activities, on Bezier curves, is described at the site www.math.u-psud.fr/~pansu.courbes.html. Young research mathematicians in the department propose a list of problems. Each subject can be chosen by two teams of high school students at different schools, who explore the problems by themselves, sometimes with the help of the proposers. A sample of themes may be found at the webpage www.math.u-psud.fr/~pansu/sujets_0708.html.

In any case, there is contact between teams and the proposers during the whole process, which lasts several months. The process ends with a national event, where teams come from the whole of France to present their reports during an afternoon. In 2006 the congress took place at La Cité des Sciences et de l'Industrie in Paris, where the theme was *Pourquoi fait-on des mathématiques*?

The subjects in 2006 included combinatorics, number theory, probability, magic squares, theory of automata, graph theory and econometrics. Here is one of the tasks, a generalization of a chessboard's knight's tour problem:

Example 5.3.5.1
ABCD is a rectangular board, with $AB = 20$ and $BC = 12$. It is partitioned into 20×12 unit squares. A positive integer r is given. One is allowed to go from one square to another if and only if the distance between their centers is exactly $\sqrt{r}$. Is it possible to go by steps from the square with vertex A to the square with vertex B?

The teams that worked on this problem consisted of young people in the final grade of high school (*lycée*) who were interested in mathematics.

This is a problem in number theory. First, it has to be decided what values of r actually permit one to go from one square to any of the other squares. In other words, for the problem to be solvable, it is necessary that $r = (a^2 + b^2)$, where a and b are nonnegative integers representing the horizontal and vertical distances between the two squares.

However, it is not obvious which values of this form actually work; there is a system of Diophantine equalities and inequalities to explore.

The students could have used complex numbers in approaching this problem, but none did. Because the possible values of r were limited by the size of the board, no general criterion for the values for r was needed. Nevertheless, the general result was guessed by both teams.

From a didactical perspective, it is significant that students and teachers were in the same position with respect to this problem. This is an unusual situation for the teacher, one appreciated by the students. The atmosphere of the whole class benefited. Not all teachers can face a situation where they appear as learners along with the students, but this is typical of a true research situation.

The fact that the proposers were young research mathematicians helped to establish a good relationship.

The student reports to an audience consisting of members of other teams, their teachers and mathematicians from the university, were both serious and enthusiastic. They enjoyed the subject and explained their results, using computers as well as the blackboard, in an almost professional way.

An interesting feature of the final event was a debate that followed the lecture, “Why do we do mathematics?” The audience was faced with opposing opinions of two famous mathematicians, Fourier and Jacobi. Shortly after Fourier died, in 1830, Jacobi wrote a letter to Legendre saying that Fourier had been wrong in criticizing Abel and himself for avoiding work on questions of public interest or problems raised by the natural sciences. Fourier, he wrote, should have known that “le but unique de la science est l’honneur de l’esprit humain” (“the sole purpose of science is to honour the human spirit”). Actually, Fourier had expressed his opinion a few years earlier: “L’étude approfondie de la nature est la source la plus féconde des découvertes mathématiques” (“the thorough investigation of nature is the most prolific source of mathematical discovery”). Of course, these are very different opinions.

The audience was invited to express its opinion by a vote. The votes for Jacobi were cast almost unanimously from students and teachers, while the votes for Fourier, together with the refusal to choose, overwhelming came from university people who were there. It can be guessed that the vote for Jacobi would have been almost unanimous among university people also, some thirty years ago.

5.3.6 Challenging gifted high school students

At Quincy Senior High School in Quincy, Illinois, USA, an increasing number of eleventh graders complete calculus. The Creative Problem Solving in Mathematics course (CPSM) was designed to challenge these students in the twelfth grade. Developed under the leadership of Dr. Sandra Spalt-Fulte of the Quincy Public Schools in 1996, CPSM is now in its tenth year and is team taught by Todd Klauser of Quincy High School and Dr. Vince Matsko of Quincy University.

The text for the course is an innovative geometry manuscript, “Polyhedra and Geodesic Structures” (Matsko 2005). A detailed outline of the course is provided below.

Students have found the course exciting and valuable. Former students occasionally return as guest speakers to talk about careers in mathematics-related fields. One former student remarked that CPSM was the most beneficial course in preparing for college. Another, referring to her individual project, said, “it has been the most wonderful experience for me.” Others said, “I [was] challenged and learn[ed] about new areas of math that I never knew existed”, and, “because of the class size we [were] able to enjoy the learning in an environment unlike that of any of my other classes.”

Below is the current topic-by-topic syllabus of CPSM, with the approximate length of time spent on each topic. Class meets five days a week for 47 minutes. One day every two weeks is Problem Day, which consists of presentations of solutions to two problems assigned over the two-week period. These problems are chosen to expose students to various topics in mathematics and to develop technical writing ability. Problem areas include number theory and Diophantine equations, combinatorics, calculus and probability.

Students select individual topics for a research project sometime in the third quarter. Occasional class days are devoted to work on these projects. The three-week project period at the end of the year allows for students to give twenty-minute presentations on their projects. They must also write a ten-page summary paper. Students sometimes work in pairs on larger projects. Past projects include building stellations of an irregular dodecahedron, constructing a harmonograph, writing programs to render three-dimensional computer graphics, cryptography, and designing a geodesic house.

In addition to the individual projects, students undertake a more extensive class project. For a recent example, see the Q-Ball at www.vincematsko.com/.

Hands-on work, whether in the form of drawing mathematical envelopes or building polyhedra, is an integral part of the course. A "B" next to a topic indicates that individual or class building projects are a part of that unit. A "G" next to a topic indicates that students use Geometer's Sketchpad during the unit.

Chapter numbers refer to the draft manuscript "Polyhedra and Geodesic Structures" (Matsko 2005). Some chapters are not covered in class but are handed out for self-study and possible use for individual projects. Material for other topics is given as class notes.

1. G: Basic Constructions (Appendix A, 1 week). Basic compasses and straightedge constructions are reviewed.
2. Trigonometry (Chapter 0, 1 week). A review of important trigonometric relationships is given.
3. G: Angles and Constructions (Chapter 1, 1 week). The construction of regular figures and the trigonometric functions of 36 degree and 72 degree angles are introduced.
4. BG: The Platonic Solids (Chapter 2, 1 week). Geometric and algebraic enumerations of the Platonic solids are given.
5. BG: Spherical Trigonometry (Chapter 3, 2 weeks). Basic formulas are derived and used to calculate the edge and dihedral angles of the Platonic solids. Non-Euclidean considerations are emphasized.
6. BG: Taxicab Geometry (2 weeks). Students explore the geometry of the "taxicab" metric (Krause 1986).
7. BG: Geodesic Structures (Chapter 4, 2 weeks). Spherical trigonometry is applied to the design and construction of geodesic spheres.
8. BG: The Archimedean Solids (Chapter 5, 1 week). The Archimedean solids are enumerated both geometrically and algebraically.

9. Angles and Archimedeans (Chapter 6, 2 weeks). Spherical trigonometry is applied to calculating the edge and dihedral angles of the Archimedean solids.
10. G: Geometrical Inversion (2 weeks). Inversion in a circle is presented, including extending the plane by adding a point at infinity.
11. BG: Geodesic Structures, II (Chapter 7, 1 week). Further techniques for creating geodesic spheres are derived using spherical trigonometry.
12. Antiprisms and Snub Polyhedra (Chapter 8, handout only).
13. B: Duality (Chapter 9, 1 week). Duals of the Platonic and Archimedean solids are discussed. Edge and dihedral angles are calculated using spherical trigonometry.
14. Geodesic Structures, III (Chapter 10, handout only).
15. B: Deltahedra (Chapter 11, 1 week). The deltahedra—convex polyhedra with equilateral triangular faces—are enumerated. Dihedral angles are found using spherical trigonometry.
16. Kepler-Poinsot Polyhedra (Chapter 12, handout only).
17. Euler's Formula (Chapter 13, handout only).
18. Coordinates of Polyhedra (Chapter 14, 2 weeks). Cartesian coordinates in three dimensions are found for the vertices of the Platonic solids and some Archimedean solids.
19. G: Mathematical Envelopes (2 weeks). A parameterized family of lines gives the illusion of curvature; the apparent curve is the envelope. Calculus is used to find a Cartesian equation of an envelope given a parameterization of lines (Boltyanskii 1964).
20. Matrices and Symmetry Groups (Chapter 15, 2 weeks). The symmetry groups of some of the Platonic solids are represented as groups of matrices.
21. Graph Theory and Polyhedra (Chapter 16, 2 weeks). The adjacency of vertices on a polyhedron may be represented as a graph. Various properties of such graphs are discussed.
22. Projects (3 weeks).

Creative Problem Solving in Mathematics is a stimulating, challenging course for talented students. Its demands on the instructors are perhaps greater than for a typical high school course; the teacher may need to learn new topics and devise ways to present them at an appropriate level. In our case, the involvement of a university faculty member (Vince Matsko) as mentor to a high school teacher (Todd Klauser) was especially valuable. Currently, Matsko visits the CPSM classroom once or twice weekly as time permits. It is important to consider either release time from the usual course load or other forms of compensation when involving a university faculty mentor.

Also crucial is the support of school administrators. The enthusiasm of the coordinator for the mathematics curriculum in the public schools (Dr. Sandra Spalt-Fulte) cannot be overstated. Without Spalt-Fulte's vision, dedication, and ability to coordinate diverse groups of stakeholders, the development of CPSM would not have been possible.

5.3.7 *Maths à Modeler: research situations for teaching mathematics*

The context of research and the solving of open questions form the breeding ground of scientific knowledge. We shall assume that an epistemological and didactical study of "real mathematical research situations" (that is, situations which are crucial to the core of ongoing mathematical research, still partially unsolved) is promising and has an innovative teaching and learning potential.

Research Situations for the Classroom (RSC) can be considered as the transferring of mathematical research to the classroom, at different levels and in different contexts (for example, in primary and secondary schools, universities, and teacher training programs, but also for the public at large, as in scientific museums or popularization events). They give access to the study of mathematical growth.

Since 1991, we have devised and studied several RSC from a theoretical and experimental point of view. They have proved to be interesting in many ways: in the type of situations and tasks to be achieved, in the knowledge and skills developed, and in the teacher's role in the classroom.

5.3.7.1 Research situations for the classroom (RSC): a definition

From our viewpoint, a RSC must fulfill the following criteria (Godot and Grenier 2004).

1. A RSC is akin to a professional research strategy. It must be related in some way to unsolved questions for the following reason: the student will be confronted with tough questions, putting him/her in a real research situation. Both teacher and student are in the same position as the researcher.
2. The initial question should be easily accessible. In particular, the question should be easily understood by students, and the problem should not demand heavily formalized mathematics.
3. Possible initial strategies are in view, and can be considered without requiring specific prerequisites.
4. Several research strategies and several developments are possible, from the point of view of mathematical activity (construction, proof, calculation) as well as from the point of view of the mathematical concepts involved.
5. A solved question can possibly lead to other fresh questions.

The main incentive in studying and implementing such situations in the classroom lies in the fact that these RSC offer an opportunity to grasp specific transversal knowledge and skills (i.e. skills and knowledge which straddle mathematics, used in a variety of mathematical contexts). These include proving, conjecturing, refuting, creating, modeling, defining, extending but also transforming a questioning process, being able to mobilize non-linear reasoning, experimenting, decomposing-recomposing, and having a scientific responsibility.

All these points may have a place in French curricula under the heading key word “scientific activity”. Of course this has only partially to do with problem solving, but the way in which we consider RSC leads us to go beyond the frame of problem solving and heuristics as defined by Schoenfeld, for instance (Schoenfeld 1985).

The RSC we implement in classroom have been used for a long time in various workshops (like *MATh.en.JEANS* described in Section 5.3.5) from elementary school to university, and have been studied from a theoretical point of view both by teachers and by a group of researchers from various departments. Since 2003, a project has been underway: *Maths à Modeler* (www.mathsamodeler.net). The current team is composed of researchers both in discrete mathematics and mathematics education. The field of discrete mathematics is actually a good environment for learning, training and popularizing mathematics and mathematical heuristics. The core problems of researchers in mathematics education concern the nature of the relationship between the knowledge and the management of RSC.

To develop this point, note that we can distinguish between knowledge (*savoir* in French) and knowing (*connaissance* in French), as is done by Balacheff and, later, by Brousseau. Brousseau (1997) finds it important not to confuse these two notions because they have different meanings. He makes explicit the relationship in these words:

> The distinction between knowledge and knowing depends primarily on their cultural status: a piece of knowledge is an institutionalized knowing. (Brousseau 1997, p. 62)

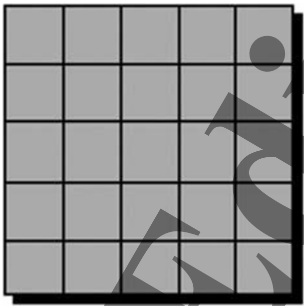
From our perspective, students dealing with RSC are supposed to construct and mobilize mathematical heuristics such as modeling, proving, refuting and defining. For the teacher who guides students during their research, it is not easy to manage a research process involving several heuristics. How can she help students? What kind of knowing are the students able to mobilize and construct? How will the teacher institutionalize knowing in order to contribute to the construction of pieces of mathematical knowledge which are as transversal and undefinable as modeling and proving?

This is a crucial problem. Teachers should have skills to use RSC in their classrooms, feel comfortable using them and also have confidence in such situations. Actually, this does not seem obvious because these Research Situations come from current mathematical research and are still partially unsolved. (This may frighten teachers who usually possess knowledge.) So, the researchers of *Maths à Modeler* try to explain to teachers how to guide RSC and how the knowledge may be attained by the students in order to convey to them the essence of RSC.

In a general way, the theoretical background uses the theory of didactical situations (Brousseau 1997). The designing of a full theoretical framework is actually at stake. It concerns the design, the development, and the management of RSC as a whole (Godot and Grenier 2004, Knoll and Ouvrier-Buffet 2006, Ouvrier-Buffet 2006). Let us now present such a RSC and the guidance we can propose to a teacher.

5.3.7.2 Hunting the beast!

Let us consider a given territory (a rectangle on the grid in the present case).



A beast is a given polyomino constituted by a few squares, (see below)

 or . A trap is a single square . 

We have to position traps on the territory in such a way that no beast can be placed. The aim is to position the smallest number of traps.

This problem resembles optimization problems. Indeed, in order to prove the optimal value, it is necessary to produce a solution with this value on the one hand, and to prove that we cannot do better on the other hand. The problem, which has given us food for thought for this RSC, is due to Golomb (1994).

In the classroom

We shall now describe the chronology of an experiment consisting of five sessions (one hour each). Hunting the beast! was used experimentally and observed by didacticians and mathematicians in several contexts (at the end of the elementary level, with psycho-pathologic pupils with mental and behavior disorders, at the secondary level, and in teacher education). We would like to point out that students work in groups (3 to 5 per group) on RSC for the usual cognitive and didactical reasons. A teacher and an educational researcher are present. All the participants are in the same role: researcher.

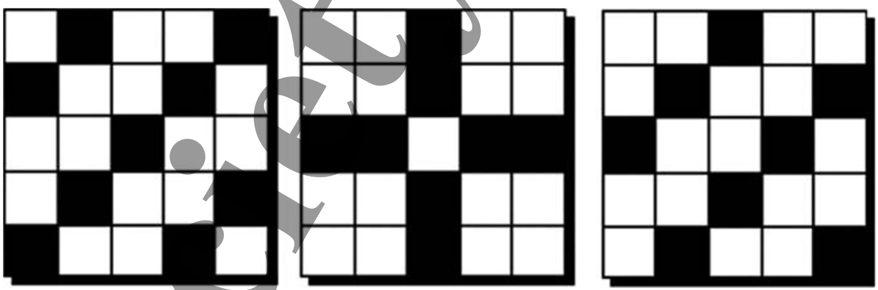
Let us describe some important features about the roles of each protagonist. Students are both in the position of researcher and project manager. They are not expected to demonstrate notional knowledge. The instructor, called Manager-Observer (MO), is also in the position of researcher, but he is also a project manager in an unusual position. His functions are to identify transversal knowledge learning opportunities and bring assistance without leading and/or hinting. The MO is not a custodian of knowledge but a catalyst. The MO may be the teacher and/or the researcher.

The next presentation about Hunting the beast! is conducted through mathematical and didactical problematics. The progress of the RSC Hunting the beast! is as follows (we can obviously generalize such a process to another RSC).

1) **Devolution of the situation and the problem of optimization :** each group of students chooses a beast (consisting of at most 5 squares) to hunt on an 8 by 8 grid. We note that mathematicians do not necessarily know the optimal solution in each case.

   Different groups propose solutions. The Manager-Observer removes one trap from the students' solution to see if a beast can be introduced. If this is not possible, then it may be that the optimization problem is solved: we have excluded the beast with the fewest traps. However, this raises a subtle issue. Is it in fact the configuration with the fewest traps? In other words, is such a "local" optimum a "global" optimum? The group may believe so, until another group produces a better solution.
2) **Research on a particular case:** the next session concerns the hunt for the 3 by 1 beast  on a 5 by 5 grid.

   After solutions have been found for 11, 10 or 9 traps, students produce solutions using only 8 traps.

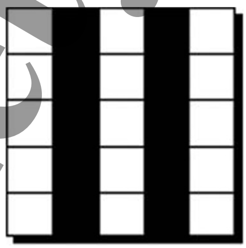


Students are usually persuaded that the optimum is 8 traps because they tried and failed several times with 7 traps. A crucial point appears then: the necessity of a proof. The position of the MO becomes important to engage students in a rational proof problematic. He can refer to simplest cases (to prove that it is impossible with 1, 2 or 3 traps, for example).

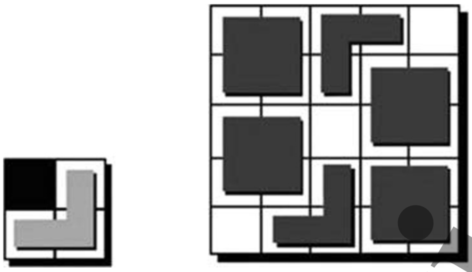
A "tiling" argument may appear during the proving process: if one can put 5 disjoint beasts on the territory, then at least 5 traps are required. In our case, one can prove that 8 traps are necessary.

3) **Research on a particular case:** the next session concerns the hunt for the beast

on a 5 by 5 grid.
The organization of this session is similar to the previous one: the students build upon the previous proof process. Nevertheless, the tiling argument for this beast is not enough because one can put at most 8 beasts on a 5 by 5 grid. In the diagram below, we see that we can catch the beast with 10 traps.



Thus, we know that the optimum number of traps lies between 8 and 10 inclusive. Such results are common in mathematical research. However, we can refine the tiling proof in order to prove that 10 traps are necessary. We have obtained this proof from some groups of students. The following figure shows a sketch of the proof. Note that each square area requires two traps.



4) **Application of students' strategies:** students apply their ideas and/or methods to larger territories (7 by 7 squares, for instance).
5) **Realization of a poster and an oral presentation:** students describe their own research processes: ideas, results, methods, and also conjectures. Students present these in a seminar called *Maths à Modeler* Junior in our research laboratory.

Excerpts (pupils, 10 years old)

Let us bring to the reader some pupils' conclusions (their mathematical production is very close to the previous development). They underline that they began to learn to:

- work together;
- listen to each other;
- not say "it's impossible" right away;
- simplify a problem to examine it more closely;
- try to prove and find arguments;
- discuss these proofs with each other;
- understand flaws in reasoning.

These sentences speak for themselves.

Conclusion and didactical perspectives

These RSC have been inspired by current mathematical research. They have been designed to train participants to use concepts and modes of reasoning more attuned and in keeping with the practice of active research mathematicians. RSC give us a chance to study the cognitive processes of students going through doubting, conjecturing, refuting, generating new counter-examples, and testing. A window is thereby opening on the educational potential of those studies. We are now planning to engage in a collaborative effort to work out tools fine-tuned to suit the different grades of students for the evaluation and design of such experiments in the classroom. In order to promote the use of research situations in teacher education and in the classroom, we must consider the management and practical implementation of such situations.

We try to arrive at a theoretical framework. But how can this be achieved in the absence of a researcher in the classroom? In fact, the presence of a researcher in the classroom has two advantages: the researcher is not the "knowledge holder", contrary to the classical position of the teacher. It makes the devolution (in Brousseau's sense) of the research process easier for the students. Moreover, the researcher can explain and show concretely how to manage such RSC, that is, how to transmit the research process to the students and how to deal with the students' procedures and issues. Besides, the researcher can underline how to identify the students' processes. This last point cannot be easily and concisely expressed.

In the absence of the researcher, the teacher should take charge of the devolution of the research process and the guidance of the situation. It requires that the teacher has been trained previously. In other words, the teacher must have experienced the situation in order to understand both the research process involved and the position of "being a researcher". The teacher must also have witnessed a RSC. That is the reason that researchers of *Maths à Modeler* regularly go into classrooms in order to convey RSC to teachers and to create a researcher-teacher network for further experimentations.

It is particularly desirable that our experiments are realized also with underachievers. Ultimately, experiments in standard classes should then be easier. To work and experiment with several audiences allows the emergence of invariants in the use of Research Situations in the Classroom. With underachievers for instance, we notice that their relation to mathematics upgrades from a bad one to a more confident one. This "psychological" fact (i.e. the students' relation to mathematics is modified when they work on RSC) has to be studied too in order to describe all the stakes and potentialities offered by Research Situations.

Maths à Modeler now proposes concrete support for teachers with a researcher in the classroom and didactical cards that provide the teacher with a basis for further use of Research Situations. The experiments of these last few years have emphasized a structure for the guidance of the RSC which is easy to convey to teachers (such as the foregoing script for Hunting the beast!). Now, we have to go beyond this stage, in mathematics education, to renew the relationship of a teacher to knowledge within the didactical contract. We should also develop a more precise characterization of the MO's behavioral and cognitive style. Such an attitude is essential for a good transmission of RSC from the researchers to the teachers. Several theoretical frameworks have been used in this perspective but the design of such a useful framework is still at stake.

5.3.8 Mathematics and art

This challenge involves a visit to a local art museum. Students engage in two interesting challenge activities.

Challenge 5.3.8.1

The study of symmetry is a subtext of much of the algebra and geometry we teach. Sometimes it is useful to keep the symmetry below the surface, as intuitive motivation for the techniques we discuss. But sometimes it is important to make symmetry a subject of direct inquiry.

In the latter case, geometric symmetry is perhaps the easiest form to approach. Students can observe and exploit symmetries in geometric figures more quickly than in algebraic expressions. For some students, merely observing the symmetries and distinguishing the different types is a challenge. Others may need a greater challenge, one that involves a more formal study of the composition of symmetries and the group structures to which this operation gives rise.

Whatever the level of challenge sought, it is important to start the work with concrete objects: diagrams, paper folding, pictures, and so on. The following is an example of one adventurous way of approaching this subject.

New York City's Metropolitan Museum of Art has a fine collection of objects from Islamic cultures. These objects offer wonderful examples of a large variety of finite symmetry groups. For students living in Western European cultures, the viewing of these objects is, quite literally, a different way of looking at the world.

If we examine artifacts from most European cultures, we distinguish mostly finite symmetry groups of even order, usually of order 2, 4, or 8, and including both rotations and line reflections (i.e. the dihedral group of order 2, 4, or 8). Islamic art, however, exploits many more types of reflection. Students often find it interesting that some objects can be "right- or left-handed"; that is, their symmetry groups do not contain line reflections. They rarely see, in other contexts, symmetry groups of odd order.

Every year, the students of Mark Saul in New York take a trip to the Metropolitan Museum of Art to go on a "treasure hunt" to find objects with these types of symmetry. They first get a quick tour of the collection, including a brief introduction to the cultures which produced these artifacts and a few examples of objects with symmetry groups of odd order, and symmetry groups which do not include line reflections.

They are then divided into teams. Each team is challenged to find ten objects with interesting symmetries. They are rewarded for finding symmetry groups of odd order: a group of order $2n$ receives n points, while one of order $2n+1$ receives a full $2n+1$ points. They are also rewarded for finding symmetry groups which don't contain reflections: a group without reflections receives double the number of points it would, had it included reflections (so that such a group of order $2n$ receives $2n$ points, and such a group of order $2n+1$ receives $4n+2$ points). Only groups of order 12 or less are considered lest students spend too much time counting the 100-fold symmetry of a large object.

(Some of the most interesting mathematics in Islamic art involves infinite groups of symmetry: those that include translations as well as rotations and line reflections. Unfortunately, the Metropolitan Museum of Art does not have enough examples of these in the collection to use for study. Perhaps in some other localities the activity could be significantly extended.)

Students get 'bonus' points (5) for finding examples of two-dimensional designs which form knots or links. Since they can submit only ten objects, they must choose carefully which objects will give them the highest score. They write down the acquisition number of each object, a mathematical description, and its score. Then teams exchange papers, and check each other's work.

Challenge 5.3.8.2

While not central to the mathematics curriculum, topological properties of objects can make for interesting student investigations, particularly on an intuitive level. The notion of the genus of an object (roughly, the number of 'holes' it has) is one that students find intriguing. Simple as it may seem, it is sometimes a significant challenge to perform a topological transformation of one object into another to see that they have the same genus.

African artists, carving in wood or working in ceramics, have created objects which are interesting aesthetically and also topologically. On their annual trip to the Metropolitan Museum of Art in New York City, my students visit the extensive African collection, and answer questions about the genus of various objects.

Students study objects selected in advance, and try to find pairs of objects with the same genus. They are asked to look at the design of the object, not its execution. (So, for example, holes used to attach a mask to the wearer's body, which are not usually part of the design of the mask, are not counted.) They end up looking at the objects in a new and unusual way.

As it builds only minimally on previous mathematical experience, this activity offers a fresh start for students. This is a particular benefit to remedial students, or students who have not otherwise experienced success in the classroom.

Often, ordinary visitors to the museum are intrigued by the students' conversation and will ask them to explain the mathematics. For remedial students, this is sometimes the first occasion they have ever been considered experts in anything mathematical.

5.3.9 Lawn constructions

The topic of geometric construction is often misunderstood. The classic (Euclidean) development of geometry involves construction using unmarked straightedge and collapsible compasses. The latter is usually not discussed in curricula: there is no simple tool which will not allow the student to "transfer" distances, and Euclid himself, in his first two propositions, shows that any construction which can be done with "rigid" compasses can also be done with collapsible compasses.

Indeed, the importance of constructions lies not in the particular instruments selected to do the constructions, but rather in the restriction to the particular tools chosen, and the resulting analysis of the problem at hand. So it can be useful, in the classroom, to consider construction using folded paper, or construction using both sides of a straightedge, or two linked straightedges, or a device which performs line reflections. This challenge provides another example of a construction environment.

For many years, co-author Saul was lucky enough to teach in a school with a large, flat lawn. On nice days in June or September, he would use this resource to work with students on construction problems. He got a box of medical tongue depressors (sticks about 10 cm long) to use as stakes, and balls of thick, bright, colored yarn, and asked students to do large-scale constructions on the lawn. The points were indicated by the stakes and the lines by the yarn. The allowable operations were developed as students worked. They could stretch yarn between two stakes, measure one length of yarn against another, or describe circles by swinging the yarn in an arc, with a fixed point as center. Their intersection points with lines could be indicated, and circular arcs could be approximated with segments.

To begin, students were asked to perform many lawn constructions that they had learned with straightedge and compasses: dropping and erecting perpendiculars, bisecting segments or angles, and so on. They found that they could bisect a segment simply by halving the length of yarn representing that segment (measuring one half against another). They could create a right angle by forming lengths of yarn in the ratio 3:4:5, then constructing a triangle out of them. Sometimes students invented new operations with yarn and stakes. If they could justify them to others, they were allowed to use them in their constructions. These added possibilities gave them new ways to think about construction problems.

They were then asked to perform some new constructions. For example, they were asked to construct a rectangle that is not a square, and then find its center. In this construction, they had to understand that it was not enough to make the opposite sides of the figure equal: they had to be sure it contained a right angle (some students figured out that they could do this by drawing two segments—the diagonals—that were equal and bisected each other). They also had to know that the center of the rectangle was the intersection of its diagonals. All these insights gave new meaning to the theorems they had studied back in the classroom.

In trigonometry, students were challenged to construct a regular dodecagon, then its largest diagonal (a diameter of its circumscribing circle), then the segments perpendicular to this diagonal from each of the vertices. Having done this, they could then approximate a sine curve by copying these segments at equal intervals along a line, perpendicular to that line. This activity gave them insights into what sine actually measured.

In pre-calculus, students used yarn and stakes to construct conic sections. They constructed parabolas with a given focus and directrix, and hyperbolas and ellipses with given points as foci. The construction of the ellipse usually ends up to be a large-scale version of the classic string-and-thumbtack construction. Sometimes, students enjoyed "acting out" Kepler's law: planets in orbit sweep out equal areas (from the focal point) in equal times. That is, they walked around the ellipse, going slower the further they were from the sun. While there was no quantitative lesson to be had here, and perhaps no real challenge, the opportunity is a good one to fix in the students' minds this interesting, and somewhat counterintuitive, phenomenon.

In all these cases, after the completion of the construction, and before the students dismantled it (for clean-up), they climbed to a high floor of a nearby building to observe their construction for accuracy. Sometimes they took photographs of their constructions. An examination of the photographs often led to new discussions about the nature of projections. They could recognize the rectangle they constructed in the photograph, but could also recognize that its image was actually a parallelogram because of the angle at which the photograph was taken.

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Chapter 6
Teacher Development and Mathematical Challenge

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In this chapter we look at the issues around teacher professional development as they relate to teaching using mathematical challenges. We look at what mathematics is; discuss why challenging mathematics problems are important in school classrooms; give some examples of problems that can provide a challenge in a classroom situation; and suggest some barriers that might inhibit the use of challenging problems. We then look at the mathematics education research that is relevant to the theme of the chapter. This is followed by effective pedagogy and teacher preparation, which includes both theoretical and practical aspects and suggestions for someone who may want to design a professional development project using challenging mathematics.

6.1 Introduction

6.1.1 What is mathematics and what are mathematical challenges?

At the heart of the discussion on challenges in the mathematics classroom that is taken up over this chapter and the next two, is the notion of what mathematics is. We consider the subject to be made up of two interconnecting parts. One of these is the content that has developed over centuries. This covers everything from basic facts to calculus and beyond. This part of mathematics contains most of what is commonly taught and examined in school systems. It contains the rules and algorithms with which we are all familiar.

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However there is more to mathematics than this. There is also the creative side, the processes that lead to the solution of open problems and to the generation and consolidation of new mathematical knowledge. This creative side is a key ingredient in research, in the history of the discipline, and in the development of the mathematical thinking of each individual student. If we define problem solving as solving problems when the way to a solution is not immediately clear, this side of mathematics harbors the processes used in solving challenging mathematical problems. This part of mathematics is where experimentation may be needed in order to make conjectures. It is where conjectures may either fall to refutations or promote proofs, both of which may need to be achieved by example or by argument. Further, it is where problems are extended and generalized. A more complete discussion of this side of the subject can be found, for example, in Hadamard (1945), Hardy (1969), Holton et al. (2001), Lakatos (1976), Pólya (2004), Tall (1980) and Thomas and Holton (2003).

Mathematical challenges are not just difficult problems (see Chapter 1). Furthermore a mathematical problem is only a challenge with respect to an individual or a group at a given time. The same problem may be a challenge for one student and a routine problem for another. However, challenging problems for able students are not necessarily different in nature from challenging problems for regular students. The solution for the same problem may also be scaffolded differently to provide challenges on several levels, as we shall see with the problems in Section 6.1.3 below (and see, for example, Sheffield 2003).

Of course the teacher is central in promoting the mathematical understanding and learning of students by choosing appropriate tasks and providing expert assistance. Many mathematics educators (for example, Simon 1997 and Steinbring 1998) suggest that learning may be facilitated by using a cyclic model of teaching in which teachers first provide their class with challenging learning opportunities. By then noting and reflecting on the results of the interactions with their students, teachers can adjust their initial plans. In this context, handling a mathematical challenge should be a part of the didactical contract—a set of implicit rules that determine students' expectations from their teacher and teachers' expectations from their students (Brousseau 1997, Sierpinska 2007).

Throughout this chapter and the next two chapters we will assume that it is important that teachers' pedagogical content knowledge contains four aspects. These are

1. a recognition of the two interlocking aspects of mathematics content and process described above;
2. an awareness of the cognitive and social processes of learners;
3. the ability to adjust teachers' learning agendas in the light of their interactions with students;
4. an awareness of the nature and importance of mathematical challenges.

6.1.2 Why are challenging mathematics problems important in school?

We will focus here on five aspects of this complex question. It goes without saying that we want students to develop a robust conceptual image of mathematics as a discipline and to enjoy the mathematical experience. Following on from the first section then, one answer to the question posed in the section title is that challenging problems help students to have a better understanding of what mathematics is and how mathematics develops.

This aspect concerns students' impressions of mathematics and the beliefs that are generated in them and developed by them in the process of problem solving, beliefs that subsequently exert a considerable influence on their behavior (Schoenfeld 1991). As Pólya (2004, p. 172) wrote, "Teaching the mechanical performance of routine mathematical operations and nothing else is well under the level of the cookbook." Within such a mechanical framework, it is impossible to develop an interest in mathematics, an understanding of its significance, the ability to apply it in even slightly nontrivial situations, and above all, even the ability to conceive of any such application (Cooney 2001, Cooney and Krainer 1996).

But there is a second consideration that is no less important. The memorization of basic techniques, which is precisely the object of solving myriads of nearly identical exercises at the cost of avoiding more substantive assignments, in fact is also achieved most effectively through routine problem solving. The studies of memory conducted by Luria (2004) support the conclusion that "the more difficult an intellectual activity is, the more conducive it is to the memorization of the materials to which it is devoted" (p. 220). The crucial fact regarding problem solving is that the operations which it requires the students to perform can only be made sense of—can only be seen as "meaningful structures"—when they are contextualized within a broader framework; in other words, only through solving more challenging problems.

Practicing teachers are familiar with situations in which, having worked partly through the exercises given on a worksheet, students seem to begin to perform the operations that are demanded of them correctly. A short while later, however, everything has been forgotten. This is hardly surprising: there has been no interpretation, no "making sense of", and no intellectual work, not to mention the absence of any positive emotional reaction, the importance of which in the process of memorization was also emphasized by Luria (Karp 2006).

This also raises the third matter, that of developing and interweaving adequate mathematical processes and mental images to get an appropriate mental concept of a problem, a *Vorstellung* in German (for details see Meissner 2002). From Dual Process Theory we know that two types of *Vorstellungen* develop, partially independent of, and sometimes even conflicting with, each other. Type 1 is mainly intuitive and spontaneous and based on "common sense".

On the other hand, type 2 is reflective, analytical and logical (Kahneman and Frederick 2005, Leron and Hazzan 2006).

In mathematics education largely it is the second type that is developed in regular schoolwork. But to work successfully on a challenging problem both are essential, a sound and mainly intuitive "common sense" and a conscious knowledge of rules and facts. To produce strong mental representations of mathematics both types of *Vorstellungen* must develop and work together. The challenging problems are needed in order to nurture balanced *Vorstellungen*.

The first three aspects of the importance of challenging mathematics apply to all students no matter what their ability. A fourth aspect that we wish to propose is that challenging problems are of vital importance for mathematically able students. These students can become unmotivated and bored very easily in "routine" classrooms unless they are challenged and yet it is common to hold back our brightest students. This continues to be the case today in the United States, over twenty-five years after the National Council of Teachers of Mathematics noted that, "The student most neglected, in terms of realizing full potential, is the gifted student of mathematics" (NCTM 1980, p. 18).

Much more recently, in a study of the effects of teachers and schools on student learning, William Sanders and his staff at the Tennessee Value-Added Assessment System found: "Student achievement level was the second most important predictor of student learning. The higher the achievement level, the less growth a student was likely to have" (DeLacy 2004, p. 40). Challenging problems in school appear to be a step in the direction of addressing this problem of the more able students and may help them to develop more quickly.

Finally, in their comprehensive analysis of mathematics lessons in the US, Germany and Japan, Stigler and Hiebert (1999) pointed out the importance of the kind of mathematics that is taught. They say, "If the content is rich and challenging, it is more likely that the students have the opportunity to learn more mathematics and to learn it more deeply" (p. 57).

The researchers consider the quality of school mathematics as a function of content elaboration, content coherence, and making connections, and state that the quality of mathematics at each lesson contributes to the development of students' mathematical understanding. Mathematical tasks that the teachers select as well as the settings in which the students are presented with them determine the quality of mathematical instruction (Leikin 2004b).

6.1.3 *What do challenging mathematical problems for school classrooms look like?*

Here we present four examples of challenging problems or situations that have been used in school settings. They illustrate a range of uses of challenging problems. The first one, the Six Circles problem, is not explicitly connected to normal curriculum content (though it does need a knowledge of arithmetic and algebra), but it does provide a method of introducing students to the mathematical processes of experimenting, conjecturing and proving. It is also a problem where a variety of approaches is possible.

The second problem, the Decimal Grid, is a challenge that has been specifically designed to make students aware that the product of two numbers is not always bigger than either of the two original numbers. The Triangle of Odd Numbers is used in a professional development course for teachers and also allows a range and depth of approaches. Finally the Dirichlet Principle is specifically designed for mathematically able students. It is not linked to most curricula and can be undertaken by students outside the classroom. However, it can be used by teachers to allow exploration and to connect different mathematical fields.

Before continuing, we note that it is well known that a crossword or a Sudoku puzzle loses its “challenge” when we look at the solution before we have tried to solve it ourselves. The same is true for working on a mathematical challenge. Therefore we will ask the reader to attempt the problems given here before reading the next parts of this section. This will help to get a better understanding and relevance of the different solutions that we describe later.

At this point too we should note that all of these problems are challenging in some way for all students, not just for the very able mathematicians in the class.

6.1.3.1 The Six Circles problem

This problem can be used with a class of students where there is a range of abilities. The Six Circles problem (Holton 2003) has a number of points where students might stop while still having engaged in a challenging activity. The problem asks if the numbers 1 through 6 can be put into the six circles (see Figure 6.1) so that the sum of the numbers on each side of the triangle is the same. (Each number can only be used once.)

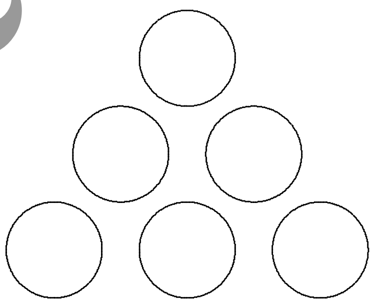


Figure 6.1: The Six Circles problem

Note that if there is an answer, it can quickly generate five others by use of the symmetries of the equilateral triangle. For simplicity we will consider these to be the same answer.

There are at least six potential challenges or staging posts here (Holton 2006). After each staging post, given a student's ability, they or their teacher may decide that they have gone as far as they can. A first challenge is to find one answer; a second is to see how many answers there are; a third is to show that there are only four answers; a fourth is to extend the problem from the set $\{1, 2, 3, 4, 5, 6\}$ to other sets of six numbers; a fifth is to conjecture which sets of six numbers will work here; and a sixth is to prove the conjecture. Further challenges can be introduced by asking the students to find another proof, to formulate additional "what if" questions, and so on.

This investigation has the advantage of being able to be tackled by all students, even those in elementary school, because in its early stages it only requires a knowledge of arithmetic and the application of logical arguments. It also has the advantage of giving elementary students a great deal of practice with number sense and addition facts in a challenging manner that will facilitate their internalization of and fluency with these basic facts. Further, it will also give secondary students a chance to practice algebraic skills in an open situation.

With help, bright elementary students can show that there are only four answers. This can be done without algebra by noting that, because 1 has to be on one side, the highest sum it can be involved in is 12 ($= 1 + 6 + 5$). A similar argument with 6 shows that the side sum has to be at least 9. By looking at how 9, 10, 11 and 12 can be made up using the numbers 1 to 6, it can be shown that there are only four answers to the original problem.

Because it seems to need algebra, a complete proof of which sets of six different numbers can be put into the six circles to produce equal side sums is not accessible until upper secondary school. However students can learn to make convincing arguments using techniques such as making exhaustive, organized lists and analyzing the possibility that the numbers that must be used in the corners must add to a multiple of three. In the process of using such problems a teacher can see where a student's talents lie and help to develop them.

6.1.3.2 Decimal grid task

In the grid of Figure 6.2, select a path from A to B. Change the direction after each numbered step. Multiply together (with a calculator) the decimal numbers on each step you go along. Find the path with the smallest product.

You have 4 trials. What is your answer? What is your smallest product?

The problem solvers start working, individually or in small groups. The reader should do the same. Please answer the questions before you continue reading. Be warned, there is more to this problem than you might think.

After about 10 to 15 minutes the class might be asked for solutions. Who got 62.121… as their smallest product? Who got 1.89? Who got 0.3564?

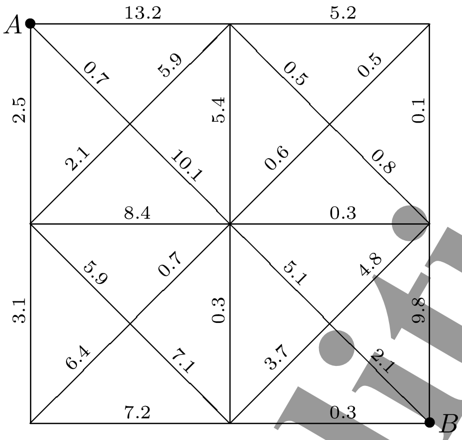


Figure 6.2: The Decimal Grid

Who else got a product between 1 and 0.1? Who got a product between 0.1 and 0.01? Who got one between 0.01 and 0.001? Is there a product smaller than 0.001?

Then the challenge continues: Find a better path than you had. Try to explain the paths to others. Are there rules for finding better paths? Is there a chance of finding the best path?

Before you continue reading, please make three or four notes about possible strategies or arguments which could come up in the discussion.

Now we will summarize strategies and experiences that have been reported in the past:

(a) I do not remember why I chose a specific path;
(b) I always took the smallest number at each point (this strategy gives 62.121...);
(c) I took as few steps as possible ($2.5 \times 8.4 \times 0.3 \times 0.3 = 1.89$);
(d) I looked for as many "zeros" as I could ($13.2 \times 0.5 \times 0.6 \times 0.3 \times 0.3 = 0.3564$);
(e) After 5.9 (second step) we can continue with smaller factors ($0.7 \times 5.9 \times 0.5 \times 0.6 \times 0.3 \times 0.3 = 0.11151$);
(f) I was looking for small "detours" (instead of 13.2 go 0.7×5.9; instead of 5.4 go 0.5×0.6; instead of 8.4 go 5.9×0.7; ...);
(g) I discovered a "cycle" ($\ldots \times 0.6 \times 0.8 \times 0.3 \times \ldots$);
(h) I tried to use a "small path" more than once ("small path" means decimal less than 1);
(i) I ran through the cycle (see (g)) many times.

Analyzing the different paths we also find some assumptions that were made but that were not mentioned:

(j) multiplication produces bigger numbers;
(k) make no "detours";
(l) reach the goal B as quickly as possible;
(m) do not go "backwards".

In this problem several cognitive jumps can occur. For example:

(n) an unconsciously existing notion, "multiplication makes bigger", may be destroyed;
(o) a new experience can become conscious: "multiplication does not always give a bigger result";
(p) a more concrete realization is "for multiplication, decimals bigger than 1 are quite different from decimals less than 1";
(q) another new experience might be "more factors may lead to a smaller product";
(r) repeating the "cycle" ($\ldots \times 0.6 \times 0.8 \times 0.3 \times \ldots$) several times makes the product smaller and smaller;
(s) repeating the "cycle" ($\ldots \times 0.6 \times 0.8 \times 0.3 \times \ldots$) again and again, we might reach zero.

Discussion is very important in the completion of problems like the above because there are often conflicts between intuitive and analytical *Vorstellungen*. Speaking about their experiences makes students' unconscious *Vorstellungen* conscious and leads to revelations, as for example, in (n) to (s).

6.1.3.3 Triangle of odd numbers

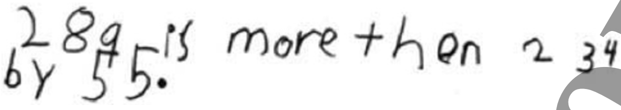
Teachers need to experience solving challenging problems themselves to better appreciate student problem-solving experiences.

Each week, teachers enrolled in a graduate course at Northern Kentucky University on Assessment Techniques in K–12 Mathematics tackled a different problem themselves and then took the problems back to their own students. The students of these teachers ranged in age from 6–17, and teachers were consistently amazed at the depth of understanding exhibited by their students.

Teachers often noted that students found enjoyment in challenges that had sometimes stumped the teachers, and frequently found solutions that the teachers had not thought of. One such problem asked students to find the next three lines in the diagram below and to find where the number 289 was. Work from Tyler, aged 7, and Dan, aged 16, is shown below. Again we note that there is more here than first meets the eye.

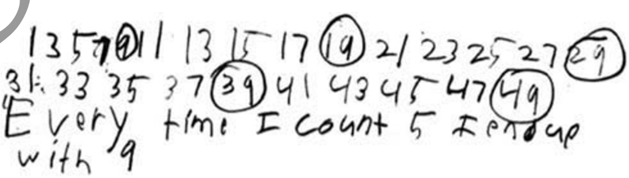
1
3 5
7 9 11
13 15 17 19
21 23 25 27 29

Tyler's first response was:



As a second grade student, Tyler was using subtraction to find the answer. Next Tyler was asked to find the row that contained 289 and Tyler responded as follows:

Tyler first noticed the pattern of finite differences increasing by 2 each time in the numbers at the end of each row on the right-hand side. When he ran out of space on the right, he noticed a similar pattern in the first number in each row. When he got to 307, he realized that he had passed 289 and noted on the previous line that he needed one-half of the row. When asked to find another pattern, he noted the following:



Dan, a high school student, used algebraic reasoning to explain why the middle number in each row was a perfect square.

A.

1
3 5
7 9 11
13 15 17 19
21 23 25 27 29
31 33 35 37 39 41

B. The number 289 can be found in the centre of row 17. 289 is equal to 17^2. By looking at the chart, one notices that the mean and median (middle number) are equal to the row number squared. For example, the median of the third row is nine (3^2). Therefore, it can be inferred that the median of the seventeenth row is 289 (17^2). The proof behind this is as follows:

Since each row has the number of numbers equal to the row number, one can add the row numbers to determine the number of numbers in total. Therefore, by the end of row n, this equals $1 + 2 + 3 + \ldots + n$. Then, the fact that the sum of a series is equal to $n(n + 1)/2$ combined with the fact that each term equals its number doubled minus one, one finds that the last term is equal to $(n(n + 1) - 1)$. Therefore, to find the first term, one merely needs to find the last for the previous row and add two, making $(n(n - 1) - 1 + 2)$.

Representing the first number in the row by $(n(n - 1) - 1 + 2)$ and the last by $(n(n + 1) - 1)$, the mean is therefore

$$\frac{(n(n-1)-1+2)+(n(n+1)-1)}{2}=\frac{(n^2-n+1)+(n^2+n-1)}{2}=\frac{2n^2}{2}=n^2.$$

C. The sums of the consecutive rows are consecutive perfect cubes: 1, 8, 27, 64, 125, … The sum of all numbers from one to n gives us the mean of row n. For example $1+3=4$, mean of row 2, $1+3+5=9$, mean of row 3, $1+3+5+7=16$, mean of row 4. The diagonal columns increase by 2 more each time as one goes along, 2,4,6,8,… from one left diagonal, and 4,6,8,10,… from one right diagonal, and so on. This pattern also holds true for the even-numbered triangle

2
4 6
8 10 12
14 16 18 20

The pattern in part B doesn't hold true in this even-numbered triangle. In this case, the squared row number is one less than the mean of the row whether it is even or odd.

6.1.3.4 The Dirichlet Principle

The project MATHEU (www.matheu.eu) is designed to produce methods and supporting materials for the identification, motivation and development of students with higher abilities in mathematics. The kernel of these supporting materials is a set of "ladders". Each ladder is a self-contained mathematical text, focused on a specific mathematical topic, which can be used by teachers or students both inside and outside the classroom. Ladders are sequences of mathematical problems, explanations and questions for self-testing, ordered so that the degree of difficulty, and amount of content, increases slowly.

The ladder entitled the DIRICHLET PRINCIPLE was prepared by Sava Grozdev (2006) of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences. This ladder involves a series of graded questions on the Dirichlet Principle (also known as the pigeonhole principle). After an introduction the principle is stated as: if m objects are distributed into n groups and $m > n$, then at least two of the objects are in one and the same group. This is followed by a series of problems.

Problem 1. There are 367 pupils in a school. Show that at least two of them celebrate their birthdays on one and the same day.

Problems 4 and 5 lead to a theorem:

Theorem If n is a natural number, then from arbitrary $n + 1$ natural numbers one could chose two such that their difference is divisible by n.

We give here two of the problems from the next sequence that demonstrate the use of the principle applied to points in a square.

Problem 10. Given is a 5×5 square, which is divided into 25 unit squares. In an arbitrary way 26 points are marked on the square. Prove that at least 2 points fall on one of the unit squares.

Problem 14. Given is a 3×3 square with 9 unit squares. One of the numbers -1, 0 or 1 is written on each of the unit squares. Prove that at least two of the sums of all the rows, columns and diagonals are equal.

Problem 15 then leads to a more general version of the Dirichlet Principle which we state after the problem.

Problem 15. A glutton ate 10 sweets from a candy box with 3 kinds of sweets. Find the greatest possible value of n to be sure that the glutton has eaten at least n sweets of the same kind.

The more general form of the Dirichlet Principle is: if m objects are distributed into n groups and $m > nk$, where k is a natural number, then at least $k + 1$ objects fall into one of the groups.

This leads on to a number of applications in Problems 16 to 22. Among the final three problems, the principle is used not to consider the distribution of objects but rather the number of groups.

We will say more about the pedagogy of these and other challenging problems in Section 6.3. The Dirichlet Principle is also discussed in Chapters 1 and 7.

6.1.4 What barriers might prevent teachers from using challenging problems?

There are several reasons that teachers may not use challenging problems. Not the least of these is the lack of an overview of mathematics that sees the process side as being significant. Two possible reasons are that this view has never been presented to them or that they have failed to grasp its importance. Furthermore the latter may be due to limitations in their subject-matter knowledge and pedagogical content knowledge so that they are not able to take their class further than the material of their current textbook.

Another reason might be that a straight top–down teaching of algorithms and procedures requires less ability and knowledge than organizing cooperative processes of discovery learning or teamwork. This might also be linked with a teacher's view that their students are low achievers and so teachers have low expectations for them. Low achievers customarily learn mathematics through pedagogical strategies that are more procedural and simplified than knowledge generating (Boaler et al. 2000, Houssart 2002, Siber 2003, Zevenbergen 2003).

It is also possible that teachers have not been motivated to undertake challenges in their classes. This may simply be the result of mistakenly believing that only the more able students can benefit from such an approach (Leikin and Levav-Waynberg 2007). Further, they may believe that they do not have adequate resources at their disposal.

Finally the teachers' lack of use of challenges may have systemic foundations. Various social or educational policies or economic conditions may be at the heart of the problem. Teachers' understanding of mathematics and pedagogy within the community of practice is bounded by socially constructed webs of beliefs, which determine teachers' perception of what should be done (Brown et al. 1998). Thus, appropriate use of challenging mathematics by teachers is not a simple task and is sometimes impossible given that contemporary school mathematics is strongly result-oriented and mostly topic-centered (Schoenfeld 1991). These social conditions may lead to a curriculum that holds no place for challenges. This is especially so if teachers perceive the school curriculum and available instructional materials as the prescribed sources of knowledge (Leikin and Levav-Waynberg 2007).

One further factor, especially in the upper levels of secondary school, may be assessment. Especially where there are high stakes examinations, teachers may not feel that they can stray too far from the standard problems that are to be found in the examination because of pressure from parents, students and the school administration.

6.2 Research

6.2.1 What do we know about the effect of teachers' knowledge and beliefs on the teaching and learning of challenging mathematics?

In this section we look at the literature on teaching and teachers' knowledge and beliefs. This leads to a model of teachers' knowledge.

Epistemological analysis of teachers' knowledge reveals significant complexities in its structure (Scheffler 1965, Shulman 1986, Wilson et al. 1987). We now introduce a three-dimensional model of teacher's knowledge, which describes this complexity (Leikin 2006, based on Scheffler 1965, Fischbein 1987, Kennedy 2002, Shulman 1986). In the context of this ICMI Study, we present this model in relation to the teachers' role in promoting challenging mathematics (see Figure 6.3).

Dimension 1 The axis "Types of knowledge" is based on Shulman's (1986) components of knowledge. Teachers' Subject-Matter Content Knowledge (SM in Figure 6.3) comprises their own knowledge of challenging mathematics, and their ability to tackle mathematical problems themselves using both critical and creative thinking. It is important that teachers have a broad view of

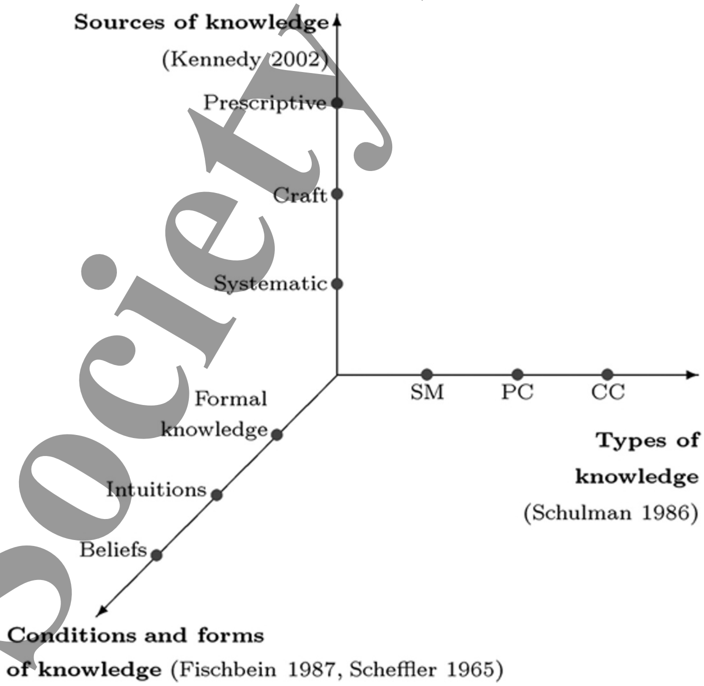


Figure 6.3: Dimensions of teachers' knowledge (from Leikin 2006)

mathematics that encompasses both content and process and they should realize the value of challenges in the subject that have been noted above. (Note this is also frequently called Subject Content Knowledge and given the acronym SCK, which is used in Section 7.3.4.)

Teachers' Pedagogical Content Knowledge (PC in Figure 6.3, frequently also given the acronym PCK, as used in Section 7.3.4) includes

- knowledge of how students deal with challenging mathematics;
- teachers' knowledge of appropriate learning settings that can be matched to particular mathematical content and to particular mathematical classrooms;
- identifying, developing and supporting each student's potential;
- helping students to communicate their mathematical discoveries;
- motivating students to engage in increasingly complex mathematical challenges.

Teachers should have a number of challenges as part of their pedagogical repertoire that are appropriate both for different parts of the curriculum and for a range of students and they should renew these challenges constantly.

Teachers' Curricular Content Knowledge (CC in Figure 6.3) includes knowledge of mathematical challenges in different types of curricula and the understanding of different approaches to teaching challenging mathematics.

Dimension 2 "Sources of knowledge" is based on Kennedy's (2002) classification of teachers' knowledge according to the sources of its development.

Teachers' Systematic Knowledge is acquired mainly through systematic studies of challenging mathematics and related pedagogy in colleges and universities, and through reading research articles, journals and professional books. This knowledge tends to be theoretical, codified and abstract, and concerns the teachers' sense of responsibility.

Craft Knowledge is largely developed through classroom experiences with challenging mathematics. Kennedy argues that teacher knowledge of this type is mainly intuitive and makes an incremental and cumulative impact on teachers.

Prescriptive Knowledge is acquired through institutional policies, which are transparent in tests, accountability systems, and texts of diverse nature. It is motivated mainly by teachers' sense of responsibility to students and to community and will have the strongest influence on teachers' decisions concerning the implementation of challenging mathematics in their classes.

Dimension 3 "Conditions and forms of knowledge" differentiates between teachers' Formal Knowledge, which is mostly connected to teachers' planned actions, Intuitive Knowledge as determined by teachers' actions that are not premeditated (Atkinson and Claxton 2000), and teachers' Beliefs, which are expressed in teachers' concepts of teaching. Intuition is a form of knowledge distinct from the formal (deductive) form, in which the person does not feel the need to prove formally or factually an interpretation or representation. Beliefs are "abstract things, in the nature of a habit or readiness" that express "a disposition to act in a certain way under certain circumstances" (Scheffler 1965, p. 76).

Thus, professional development programs and/or courses that are aimed at the development of teachers' awareness of, and favorable disposition toward, offering students the opportunity to experience different types of challenges in school mathematics have to take into account the complexity of teachers' knowledge. Such courses consider teachers' intuition as the basis for the development of their formal knowledge. On the other hand, they should consider teachers' beliefs and be aimed at developing teachers' enthusiasm for the introduction of challenging mathematics in school.

6.2.2 What other factors are important to teaching and learning challenging mathematics?

Teachers also need to be introduced to, and involved in, research in this area so that they begin to understand better the practical side of using challenging situations. Below we consider aspects of motivation and the work of Vygotsky. This is followed by the need for more research on challenging mathematics.

The main reason for teachers to study this research is to learn how students reason. In a professional development course on this material, for example, the reasoning of gifted and talented students may be compared with the reasoning of regular students and that of the teachers themselves. Mathematical challenge is the context for this research on students' mathematical reasoning. The courses may include reading studies by well-known researchers, individual research projects or collaborative research with students.

6.2.2.1 Motivation

Motivated teachers, as described by the Oregon School Boards Association (2006) (see www.naen.org), are ones who not only feel satisfied with their jobs, but who also are empowered to strive for excellence and growth in instructional practice. Teacher motivation is one of the driving forces and determinants of how a teacher functions, particularly so, in the area of challenging mathematics. Here, more attention should be paid to the work content factors (NCES 1997) as these are associated with intrinsic motivation. These factors concern professional development, recognition, challenging and varied work, increased responsibility, achievement, empowerment, authority, and so on (Frase 1992). These are simply about the things that would enable and motivate a teacher to achieve the goal of ensuring that learning occurs.

Intrinsic motivation is perceived to have the potential for sustaining lifelong learning and professional development for both prospective and in-service teachers. Creating a mind-set in which lifelong learning is appreciated, promoted and enabled forms a very useful base for attracting teachers towards work on challenging mathematics. They would then view challenge as a learning opportunity rather than an obstacle to be avoided. Teachers who are intrinsically motivated to tackle challenging problems themselves are much more likely to present challenging problems to their students. Since this is related to the development of positive attitudes, efforts should be made in teacher development to engage teachers in activities known to have the potential for changing attitudes. Teachers with a positive mind-set towards working with challenging mathematics will not only be in a position to employ challenging mathematics in their instruction but may also take the initiative to develop this area in their spheres of professional influence, be these their school, their locality or even their own country.

6.2.2.2 Brain development

Research has shown that the brain changes structurally as well as functionally depending on learning and experience. Regular opportunities for tackling challenging mathematical problems have the potential to change the human brain for life. This has tremendous implications for all levels of education (see www.newhorizons.org/neuro/front_neuro.html).

6.2.2.3 Zone of proximal development

The power of Vygotsky's ideas lies in his explanation of the dynamic interdependence of social and individual processes. In contrast to those approaches which focus on internal or subjective experience and behaviorist approaches which focus on the external, Vygotsky conceptualized development as the transformation of socially shared activities into internalized processes. In this way he rejected the Cartesian dichotomy between the internal and the external. By way of contrast, Vygotsky (1978) looked at the unity and interdependence of the internal and external, starting from birth. To address the way in which this social and participatory learning took place, Vygotsky (1978) developed the concept of the zone of proximal development (ZPD) , which he defined as "… the distance between the actual developmental level as determined through independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). Teachers need to be able to find the appropriate ZPD for each student in the presentation of challenging problems.

The notion of ZPD is central both to teachers' knowledge and to the knowledge of teacher educators. Learners, students and teachers, face difficulties when coping with mathematical tasks. The principles of "developing education" (Davydov 1996), which integrate Vygotsky's (1978) notion of ZPD, and Leontiev's (1983) theory of activity, claim that to develop students' mathematical reasoning the tasks should neither be too easy nor too difficult. Challenging teachers with "powerful tasks" is fundamental for teacher development (Krainer 1993).

The ZPD is also discussed elsewhere in this Study Volume, particularly in Sections 3.1, 4.5.4 and 7.3.2.

6.2.3 What other research is needed?

There is considerable room for research into the effect of teaching and learning through challenges both with teachers and with students. Research is necessary in challenging mathematics classrooms to see what changes might be expected in both students' and teachers'

- content knowledge;
- attitudes toward mathematics;
- willingness to continue in mathematical pursuits;
- risk-taking and up-taking of other challenges.

6.3 Effective pedagogy

6.3.1 What is the role of the teacher in a class where challenging problems are used?

Jaworski (1992, 1994) offers a teaching triad, which is consistent with constructivist perspectives of learning and teaching. The triad synthesizes three elements, which are involved in the creation of opportunities for students to learn mathematics:

- the management of learning;
- sensitivity to students;
- mathematical challenge.

Although apparently quite distinct, these elements are often inseparable. According to Jaworski (1992, p. 8) "this triad forms a powerful tool for making sense of the practice of teaching mathematics". According to this triad, in any classroom situation the teacher should include mathematical challenges and know how to manage the learning process in accordance with students' reasoning and needs.

In this section we list a number of roles that we think a teacher in a challenging mathematical classroom should have. These are:

1. enjoying solving problems;
2. promoting challenging mathematics;
3. fostering enjoyment of mathematics;
4. assisting students to communicate their ideas;
5. motivating and encouraging the use of a variety of techniques and approaches;
6. celebrating students' achievements;
7. allowing students to try their own ideas.

Generally, these are in no particular order, but the first and most fundamental of these is a necessary attribute to establishing a challenging environment in the classroom. Above all, teachers themselves should enjoy solving mathematical problems. Unless teachers are fluent in solving problems and are able to think critically and creatively, it will be difficult for them to engender these abilities in their students.

Steinbring (1998) has a model in which teachers offer learning opportunities to their students in which the students operate and construct knowledge of school mathematics in an autonomous way. This occurs by subjective interpretations of the tasks in which they engage and by ongoing reflection on their work. The teachers are responsible for the design of learning situations in which students deal with challenging mathematics. By observing the students' work and reflecting on their learning processes the teacher constructs an understanding, which enables variation of the learning opportunities in ways that are more appropriate for the students.

Consequently the learning opportunities that teachers provide for their students are a function of that teachers' knowledge. (Similar arguments about the importance of teachers' knowledge for students' learning may be found in many places, for example, Artzt and Armour-Thomas 2002, Ball 1992, Ball and Cohen 1999, Simon 1997). To be able to increase their own knowledge and so design appropriate learning situations to teach challenging mathematics, teachers should be continually looking for problems for themselves to solve as well as for their students. Teachers might also think about creating problems for their students and encouraging students to create problems for themselves and their peers. With this background, teachers will be better able to support and encourage students to tackle new problems.

Such teachers would have to be very motivated to engage in challenging mathematics since research shows that not many teachers display the qualities of perseverance required in dealing with mathematical challenges (Shroyer and Hancock 1997). Additionally it would seem that one of the reasons for education is to support students for both lifelong learning and to solve new problems as they arise in everyday life. By using challenging questions in mathematics, the subject can support this type of learning.

Secondly, and perhaps obviously, the teacher should promote challenging mathematics in school. Not all learning, even mathematical learning, takes place in class, as we have seen in Chapters 1, 2 and 3. Teachers should be aware of, if not involved in, extracurricular opportunities for challenging mathematics such as individual and group competitions, math clubs/circles, problems, mentors, weekend and summer math programs and exhibitions. In addition, they should be aware of what is available on the web through such sites as www.nrich.maths.org.uk and www.nzmaths.co.nz . All of these sources provide fresh challenges for teachers and students alike.

A third role for the teacher, and one of the aims of teaching generally, is to identify each student's potential and develop and support their learning. In addition, their critical and creative abilities should be developed and their enjoyment of math should be encouraged and fostered.

In the process of solving problems, it is not sufficient to simply get an answer. This is because it is not clear that the student knows how to solve the problem until an argument has been given. Further by writing, or otherwise communicating the reason for their results, the student's understanding is developed. Consequently a fourth role for the teacher is that of assisting students in communicating their ideas and solutions.

The use of ladders and rich mathematical problems (see Section 6.1.3) encourages students to engage in increasingly more complex mathematical challenges. This is a fifth role for the teacher—to motivate students to go deeper into mathematics in contexts that are both pure and applied, and both are part of and outside the required curriculum.

Often a class will find a number of different ways to solve a given problem. For instance, to prove that there are precisely four answers to the Six Circles problem one can use several approaches. The teacher should encourage the use of a variety of techniques, resources and technologies, and ensure that the students realize that many problems can be approached from different angles.

The sixth role of the teacher is to explicitly recognize a student's successful achievement in mathematics. This may range from simply commenting to the student when a good job has been done to public recognition when a good result has been achieved in an external math competition, say.

The last role is being able to let go. There are many anecdotal examples of teachers who have suggested to students that they should stop the approach that they are using and work on the teacher's suggestion. Students who will not take the teacher's advice or who stick to their own method may well solve the problem their own way.

Teachers might be open to learn from their students. And perhaps more importantly, they must realize the importance of giving control to students. After all, education is for life and students will not always have a teacher to turn to. It is important that students develop their own ways of tackling problems.

So it is important for teachers to realize that their responses to certain situations should be more meta-cognitive. Questions like "Have you seen something like this before?" and "What if you tried a simpler case?" are more valuable for students in the long run, than "Why don't you write that as $x + 2 = 7$?" Meta-cognitive scaffolding such as this prepares students for the next problem as well as for independent learning (Holton and Clarke 2006).

6.3.2 What is effective pedagogy for classrooms using challenging mathematical problems?

From the start we should say that the literature shows that there is no general teaching method that will generate high performances in all students (Hiebert et al. 2003). In fact, Watson and De Geest (2005, p. 223) note that, after studying teachers who seemed to make a difference to their students' learning, some practices "would have comfortably fitted into a typical 'reform' classroom; some would have comfortably fitted into a classroom in which silent textbook work was the norm".

However, there were some common themes that appeared in Watson and De Geest's study. They found that the kinds of teachers' practices that lead to active and purposeful learning include:

- developing routines of meaningful interaction;
- choosing how to react to correct and incorrect answers;
- giving students time to think and learn;
- working explicitly or implicitly on memory;
- using visualization;
- relating students' writing and learning;
- helping students to be aware of progress;
- giving a range of choice;
- being explicit about connections and differences in mathematics;
- offering, retaining and dealing with mathematical complexity;
- developing extended work on mathematics;
- providing tasks which generate concentration and participation.

This list is reinforced in a review of research on numeracy practices by Brown et al. (1998), who found that good teaching practices include:

- the use of higher order questions, statements and tasks which require thought rather than practice;
- an emphasis on establishing, through dialogue, meaning and connections between different mathematical ideas and contexts;
- more autonomy for students to develop and discuss their own methods and ideas.

Clarke and Clarke (2004) have a similar list from their work with teachers of students in the early years of school. However, they feel that the features listed below may apply across all levels of schooling:

- focusing on important mathematical ideas;
- providing structured and purposeful tasks that engage children;
- using a range of materials/representations/contexts;
- making connections between mathematical ideas;
- engaging students' mathematical thinking through a variety of organizational structures;
- establishing an effective learning community;
- having high but realistic mathematical expectations of all learners;
- encouraging mathematical reflection;
- using assessment effectively for learning and teaching.

In his examination of the video lessons that accompanied the Trends in International Mathematics and Science Study (TIMSS) Watson (2004) noted two main ideas: the presentation of mathematical concepts and the level of student engagement.

In lessons from Japan and Hong Kong, the difficulties of the problems that students were exposed to were not played down. In fact, instead of simplifying problems down to their technical level, classes focused on the relationships and complexities within the mathematics.

So what seems more important than teaching style or classroom organization for enhancing student performance is the creation of a community of "apprentice mathematicians" who are all actively involved in the cognitive demands of the problems with which they are presented. To achieve this community, the teacher's role can be thought of as developing the kind of challenges that contribute to what mathematicians actually do. Watson et al. (2003), after their work on the Improving Attainment in Mathematics Project (IAMP) , list among these practices:

- choosing appropriate techniques;
- generating a student's own enquiry;
- contributing examples;
- predicting problems;
- describing connections with prior knowledge, giving reasons;
- finding underlying similarities or differences;
- generalizing structure from diagrams or examples;
- identifying what can be changed;
- making something more difficult;
- making comparisons;
- posing their own questions, giving reasons;
- working on extended tasks over time;
- creating and sharing their own methods;
- using prior knowledge, dealing with unfamiliar problems;
- changing their minds;
- initiating their own mathematics.

(This extends and reinforces the concept of mathematics that we outlined in Section 6.1.1.)

These aspects are not readily easy to achieve either for teachers or students. The IAMP project showed that there was student progress in their ability to work with more difficult and extended problems over a period of two years. Students also became better and more confident at tackling more complex and unfamiliar tasks.

As far as teachers are concerned, those who are self-sustaining, generative learners (Thomas and Tagg 2005) are the ones who learn most from professional development. They are able to connect ideas from professional development to their own classrooms. As they teach, they continue to reflect on and adapt what they have learned (Bicknell and Anthony 2004, Higgins et al. 2004).

Teachers' pedagogical competence appears to be strongly related to their personal beliefs and their desire to learn continually.

Similarly, Chambers and Hankes (1994, pp. 286–7) refer to the intent of the Cognitively Guided Instruction Project as "to help teachers understand children's thinking, give the teachers an opportunity to use this knowledge in their classrooms, and give them time to reflect on what happens as a result of using this knowledge".

Although many of the studies that we have quoted here do not explicitly mention challenging mathematics, it is clear that to achieve the high mathematical performance that they are capable of, students have to be challenged.

We acknowledge here our indebtedness to the best evidence synthesis of characteristics of effective pedagogy in Mathematics/Pāngarau by Anthony and Walshaw (2007) on which this section was based. They used a series of international studies to "identify a set of fundamental principles that inform the work of the effective classroom teacher. In the studies met thus far, these principles were enacted in various ways, by different teachers, and in different classroom settings. We have made use of the following principles to guide our evidence-based synthesis:

- acknowledgement that all students, irrespective of age, have the capacity to become powerful mathematical learners;
- commitment to maximize access to mathematics;
- empowerment of all to develop mathematical identities and knowledge;
- relationships and connectedness of both people and ideas;
- holistic development for productive citizenship through mathematics;
- interpersonal respect and sensitivity;
- fairness and consistency."

6.4 Teacher preparation

6.4.1 What is the role of professional development in encouraging classes with challenging mathematical problems?

In this section we consider what role professional development has in encouraging and aiding teachers to use challenging mathematics in their classrooms. Throughout we make little distinction between the part played by pre-service or in-service education. Consequently when we refer to "teachers" we generally mean both teachers involved in in-service activities and prospective teachers in pre-service education.

6.4.1.1 Modeling

First we underline here the importance of modeling with teachers the challenging approach to teaching that we would like them to use in their classroom.

Two premises underlie our comments in this section.

1. Many teachers fail to grasp the relationship between university undergraduate mathematics and school mathematics (Franks and Tuncali 2004, Moreira and David 2004).
2. Teachers tend to organize their classroom and inform their teaching in much the same way as they were taught (Brown et al. 1990, Cobb et al. 1990, Scharm and Lappan 1988, Shulman 1987).

With respect to the first of these, the objective is that teachers develop a profound understanding of fundamental mathematics as an essential feature of their mathematical education. In this way they should see how to provide rich, challenging and creative opportunities in their classes by exploring the way higher mathematics relates to school mathematics (Ma 1999).

In working towards this end, the following practices of effective pedagogy in mathematics that we have singled out from previous sections should be modeled in teacher development programs. These are:

- establishing meaning and connections;
- being explicit about connections;
- describing connections with prior knowledge;
- making connections between mathematical ideas.

One of the ways to model the incorporation of challenges into every learning situation is to provide mathematical textbooks specifically designed for teachers that relate higher mathematics directly to school mathematics—by tracing the progressive construction of meaning for mathematical concepts—and explore challenging problems in each context (Acevedo and de Losada 1997, 2000). Challenges, that explore in breadth and depth the richness of fundamental mathematics, are a key element of this process.

The perspective of "challenging mathematics for the teacher" that includes full treatment of appropriate higher mathematics and clear ties to fundamental mathematics can be especially fruitful for understanding the nature of mathematical challenge (English 1997, Sfard 1994, Lakoff and Núñez 2000).

In this respect, the history of mathematics shows clearly that new mathematics can be created even when based on partially inadequate or incomplete mathematical concepts. (Thus Euler produced many beautiful results in mathematical analysis although the meaning he gave to the concepts of infinity and even of the real number system was seriously flawed). Properly oriented, this will encourage the teacher to present challenging problems that deal with a particular concept or concepts, even when students have yet to construct a fully adequate meaning for the concept.

Additionally, this approach allows the teacher to understand the original challenging nature of problems that have become routine, and to extract extra mathematical mileage even from that part of mathematics described above as content.

Furthermore, turning to the second premise, if specific preparation in school mathematics and challenges for school mathematics is missing in teacher development, teachers will resort to their own school experience for guidance in presenting school mathematics to their students, potentially perpetuating the emphasis on content over process and the absence of challenging material in mathematics classrooms.

Thus in all teacher development activities the facilitator should serve as a model, incorporating challenge as a fundamental aspect of the mathematics to be treated both in extended pre-service courses (Blanton 2002, McNeal and Simon 2000) and in short-term, self-contained enrichment experiences for the teacher. (For a recent example, see McGatha and Sheffield 2006.)

Leikin and Winicky-Landman (2001) consider a course that focuses on challenging tasks, types of tasks and types of mathematical challenges. The teachers cope with challenging mathematics as learners and so undergo the same experiences that their students will face. Although the development of Subject-Matter Knowledge (Section 6.2.1) is the explicit purpose of the course, Pedagogical Content Knowledge and Curricular Content Knowledge are developed implicitly through facing different learning experiences.

6.4.1.2 Didactical content

It is of fundamental importance also to consider didactical content. Professional development in the area of challenging mathematics must include work on actual challenging mathematical problems. This is especially so for teachers in elementary school as they may not have had sufficient background in mathematics to see the possibilities that problem solving can provide. But it should not be overlooked with secondary teachers. In either case it will offer chances to model pedagogical methods with them and provide examples for their future classroom use.

The challenging situations considered here should involve examples that are appropriate for the level at which the teacher is teaching and at the teacher's level of ability. In the former case they will be useful for the future. In the latter case they will enable teachers to experience the learning process that their students will experience, in particular the thrill of success.

Classroom management techniques should be part of the didactical content of professional development. Matters such as the presentation of problems, individual and group work, questioning techniques, the use of oral mathematical discussion and time management must be considered. Assessment in this situation is clearly important too but we leave that discussion until Chapter 8.

Project M3: Mentoring Mathematical Minds (Gavin et al. 2006), a five-year collaborative research effort in the United States, emphasizes the use of a student-centered inquiry approach that encourages students to think like mathematicians, asking questions that enable them to make sense of mathematics. Student study units have been developed to add depth and complexity to the typical elementary mathematics curriculum.

Each lesson has "Think Deeply" questions and a Mathematician's Journal that students use to develop and organize their mathematical reasoning. These questions generally follow an investigation where students are asked to delve deeply into a "big idea" in mathematics, and are designed to assist students in organizing their thinking and making sense of the concept. Students who are ready for more challenge are presented with "Think Beyond" questions that encourage them to probe further into the mathematics. Students frequently work with a partner and in small groups that provide stimulating and necessary dialogue to foster conceptual understanding.

This is often followed by whole class discourse, giving students an opportunity to further develop and consolidate their own mathematical reasoning and questioning skills as they work with classmates to develop complex skills and analyze concepts. Students in the program have had significant gains over a comparison group of like ability on standardized tests as well as on an open response assessment that contains released items from the National Assessment of Educational Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS) . (See www.projectm3.org.)

In addition, it is important that time should be spent on technology (including graphic calculators, individual computers and math labs), visual and physical models, manipulatives, and (external) visualization and (internal) imaging. One advantage of this is to enable students to produce a number of examples quickly and so have access to a wider range of experience.

The new technological environment, therefore, can change the nature of mathematical tasks from proof to inquiry. Inquiry dialog (Wells 1999) has been seen as a way of increasing the quality of school mathematics. Inquiry tasks are usually challenging, cognitively demanding, and stimulate highly motivated students. In such an environment students are encouraged to conjecture, debate the conjectures, search for explanations and proofs and discuss their preferences regarding different ways of solution (Yerushalmy et al. 1990).

The use of technological tools allows children to enjoy engaging in geometry, pattern and number sense activities. Among these technological tools are games, dynamic geometry environments (DGE) and graphical algebra tools. Games allow both structured practice and free exploration as well as the development of strategies based on analysis and inference, and are linked to real-life experiences. Use of these tools can include suggestions for relating the software with hands-on classroom activities and home practices.

Using DGE students can investigate geometric concepts inductively, make conjectures, and then frame deductive arguments to prove or refute their conjectures. Teachers can guide their students by asking scaffolding questions that challenge students and change the quality of mathematics in the classroom (Rasmussen 1996). Technology may transform the content of the algebra curriculum.

The learning sequence applied with the support of a technological tool can help to make mathematics that was once assumed to be difficult for students become "natural" (for example, recursive thinking, visualizing equations in two unknowns in three dimensions). Technologically-supported curricula can lead to change in students' cognitive hierarchies, though such change may have as much to do with the curriculum as it has to do with technology (Yerushalmy 2004).

6.4.1.3 Practica

Even for teachers who have taught for several years, much can be gained by working in a classroom with an experienced teacher (see Section 6.4.2.3). Having the chance to experiment with new ideas under guidance is invaluable for changing teaching practice. Here we look at an example of this use of practica. We also consider the use of outstanding student work and look at ways of assessing professional development.

6.4.1.4 Mathematical coaching

To be effective, professional development must be ongoing, deeply embedded in teachers' classroom work, specific to grade levels or academic content and focused on research-based approaches. The Kentucky Center for Mathematics housed at Northern Kentucky University designs and offers professional development for state-wide mathematics coaches who work with other teachers in their school buildings or districts. This professional development is designed to build teacher's knowledge of mathematical content, pedagogy, and students, and to create a state-wide electronic (see www.kentuckymathematics.org) and face-to-face support network where teachers can share and critique ideas about teaching and learning mathematics.

Mathematics coaches begin a program in the summer with two weeks of intensive training in pedagogical content knowledge as well as coaching strategies and techniques for working with adults. During the school year, coaches work with other teachers for at least half of their time and meet online with other coaches weekly to discuss challenges and share tips and successful strategies.

6.4.1.5 Outstanding student work

Analyzing student work is an effective strategy for improving teacher understanding and appreciation of student thinking and ability. Once teachers see examples of outstanding student responses to challenging problems, they are more willing to continue to offer additional problems of this type. (See Looking at Student Work: www.lasw.org or the results of competitions on the local, national or international level.)

6.4.1.6 Teacher-innovator model

One way to investigate the effects of professional development activities on teachers' propensity to include challenge in the mathematics classroom is by using the 4-I Model. This model is adapted from the teacher-innovator model developed to study teachers' responses to an educational reform (Yeap 2006).

The model comprises four stages—Ignoring, Imitating, Integrating and Internalizing.

Teachers at Level 0 (Ignoring) are indifferent to the need to provide challenge in the mathematics classroom even though they are aware of the curriculum requirements for its inclusion and have attended professional development courses that encourage the inclusion of challenge.

Teachers at Level 1 (Imitating) design tasks that are structurally the same as those that they have encountered during the professional development courses. Any differences though are superficial.

Teachers at Level 2 (Integrating) are able to make structural changes to the tasks they have used during professional development courses.

Teachers at Level 3 (Internalizing) believe in the use of challenge in the mathematics classroom. They may not even use any of the tasks used during the professional development course. However, their actions indicate that the culture of the classroom is one where challenge is present and valued.

6.4.2 Some in-service and pre-service programs

There are many projects reported in the literature that provide professional development for teachers. We present four of these in some depth below. However, two other useful contributions of the same type that were presented at the ICMI Study are the CASMI Project (see www2.umoncton.ca/cfdocs/casmi/casmi/index.cfm, Section 3.5.1, Freiman and Vézina 2006, Freiman et al. 2005a, Freiman et al. 2005b) and the *Association Rallye Mathématique Transalpin* (Grugnetti and Jaquet 2006). Both of these essentially provide problem-solving activities for students but they also provide valuable professional development for teachers. Another example from Singapore can be found in Chapter 8.

6.4.2.1 A Chinese experience

The essence of the new Chinese Mathematics Curriculum Standards is the promotion of understanding and the use of challenging problems. After the release of these standards, the Shanghai Research Institute of Educational Science embarked on a study entitled "Teacher Action Education" to help teachers in Shanghai familiarize themselves with the rationale and logistics of the standards. The research team identified two key aspects of a model of teacher professional development as a result of a questionnaire survey (Gu and Yang 2004). These two aspects are:

1. Teachers need exemplary lessons to guide their professional development. The guidance should be facilitated by curriculum reform specialists and

experienced teachers who would like to make a shift toward the new curriculum standards.

2. Teachers need to work together with the curriculum reform specialists and other experienced teachers as a team to prepare lessons, carry out the teaching experiments as planned, reflect on all aspects of the teaching experiments, and come up with revised teaching experiments as follow-up actions.

The two steps of reflection are important characteristics of the basic model of Teacher Action Education (see Figure 6.4). First, a teacher conducts a lesson in the usual manner. This lesson is then discussed with the curriculum reform specialists and other experienced teachers in the project team. The gaps between the teacher's conceptions and those espoused in the curriculum standards (e.g. ideas of mathematization) are then identified. The original lesson is then redesigned and taught to a new class as a teaching experiment by the same teacher. This lesson is reflected upon again.

This second time of reflection will focus on whether students acquire the desired knowledge with understanding and the desired skills with proficiency. The lesson is subsequently redesigned again and experimented with by the same teacher in a third class of students. Those lessons that succeed through these teaching experiments may be used as exemplary lessons for documentation and dissemination both to local schools and to schools in other parts of China.

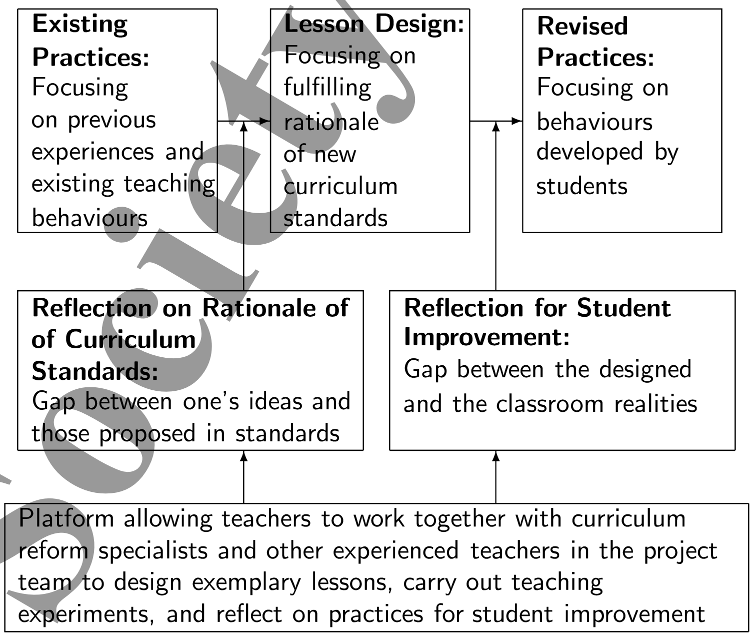


Figure 6.4: Basic Model of Teacher Action Education

6.4.2.2 A German experience

Pre-service experiences from Münster help prepare student teachers to become open to using mathematical challenges in their future classrooms. (Some of this material can be found in Meissner 1995 and 1996). First of all they themselves must experience work on mathematical challenges. They must also learn team work and they must learn to learn by doing. Learning by listening and being told is not sufficient. On the other hand, in addition to their own and often unconscious experiences, they must consciously learn about the mental processes which may be happening when working on a mathematical challenge. They must also experience how to bring that theoretical knowledge into practice. To summarize, they must gain experiences and they must become able to reflect on them. Only then might they be able to guide and help their future students.

Three different learning environments help pre-service teacher trainees in Münster to reach these goals and the other goals of Section 6.3.1.

First, teamwork is encouraged. In many mathematics faculties the students just "listen to" or "follow" the mathematical lectures, and parallel to these lectures they must solve homework problems and participate in small groups (10 to 30 students) where the best solutions of the homework problems then get presented. A system that was practiced in the Netherlands by Hans Freudenthal was introduced in Münster and has been used for more than 20 years.

The students form three-person teams to solve their homework problems. Every week each team works together first individually, then all the teams come together into one big seminar room to discuss questions that have not yet been completely resolved. Here an exchange of views and experiences between different teams begins. When there is no team nearby with a hint or a successful idea of how to continue with the problem, students can raise their hands to get help from one of the tutors or staff members.

The goal of this session is that each team should actively find solutions that can be written up by the team after this session. For many students this is the first time that they have experienced speaking and discussing mathematics. As a result, they learn to formulate and to answer questions and their existing knowledge is actively expanded.

Secondly, video is used extensively. To work on challenges needs flexibility and creativity. This is necessary for both mathematical content and for the process of learning mathematics. But flexibility and creativity cannot be taught theoretically, neither as a list of algorithms, procedures or activities, nor as a list of theories, properties and rules. Flexibility and creativity must be experienced personally and then be reflected on later.

Video can assist in several ways. When pre-service teachers are practicing teaching in schools, an experienced university staff member videotapes the lessons. After the lesson the pre-service teachers discuss their individual observations of the lesson and then study the videotape. The tapes provide a valuable learning experience.

Another use of video is to record groups of student teachers solving a mathematical challenge. They are encouraged to argue and discuss the problem and can work with objects or manipulatives. These videos are then analyzed in a teacher pre-service seminar.

In seminars on mathematics education, students are often asked to deliver an oral presentation together with a written report. In some of the Münster seminars the students get a choice, they also can produce a video to present the topic. Many creative video spots have been produced which are later used in other mathematics education lectures or seminars.

Finally, pre-service teachers learn as researchers. There are many similarities between solving a mathematical challenge and doing successful research. The pre-service students in Münster are involved in empirical research projects where they produce materials and tests, prepare and teach lessons, observe and analyze teaching sessions, and evaluate. Finally they write a report related to a partial aspect of the project. The topics of this research vary widely from topics that concentrate on curriculum development, to teaching methods, to topics that further a deeper and better understanding of the psychological aspects of teaching and learning mathematics.

6.4.2.3 A New Zealand experience

Motivated by the poor 1995 TIMSS results (Garden 1997), New Zealand developed their Numeracy Development Projects (NDP) on the basis of research into children's number understanding and learning. (For the basic material and some understanding of the NDP, see www.nzmaths.co.nz/Numeracy/index.aspx).

For New Zealand teachers, the NDP was novel in both the approach to the delivery of basic arithmetic and the method of professional development. The delivery centers on problems where students are encouraged to use their own solution method. Challenges are considered to be important in all students' understanding and learning (Wright 2006).

As this was a new approach to teaching arithmetic for almost all primary teachers, professional development was required on a national basis. So facilitators were trained in the delivery methods and based in the country's six major institutions of teacher education. The outline program was available to teachers in print and on the Ministry's web site. However, the facilitators were the key people in the professional development process. They first introduced the project to teachers from local clusters of schools using workshops. They also went into schools and worked with individual teachers in their own classrooms. This work consisted of giving sample lessons, co-teaching with the classroom teachers, and observing their lessons. The facilitators' pedagogical approach in workshops was the same as was expected from teachers in their classrooms.

This last observation is of fundamental importance both in the NDP and in professional development generally. Professional development that mirrors what is expected of teachers in their own practice appears to be very effective

(Leikin 2003, 2004a). Hence we would advocate this approach in any professional development around challenging mathematics. We note that this approach is time intensive and therefore expensive to operate. However, for a national change in approach to teaching basic number it appears to have been fairly successful, at least as judged by student performance (Ell et al. 2006).

6.4.2.4 An American experience

Each state in the United States has its own laws governing teacher certification and within each state, different universities prepare teachers slightly differently. In most instances, students must take an exam and meet other standards to enter a teacher certification program.

Upon completion of all courses, they must take another exam to receive initial certification or licensure in addition to receiving a Bachelors or Masters degree. Teacher education programs might be on either the graduate or undergraduate level and almost always require practical courses and student teaching, where the prospective teachers work with students at the appropriate grade levels and subject areas.

At Northern Kentucky University pre-service teachers at the elementary, middle or secondary level take one three-credit mathematics education class (45 contact hours) focusing on teaching and learning mathematics. This is combined with a field-based practicum class where they work with an experienced teacher to teach mathematics to students at the appropriate level. In this mathematics education class, as well as in the previous 3–12 mathematics content classes, challenging and creative mathematics is stressed.

Pre-service teachers work on challenging mathematics problems individually and in small groups, share a variety of solutions, often using technology and concrete hands-on and/or visual models, and then use similar problems with elementary, middle or high school students.

Lesson plans for the mathematics they will teach to K–12 students, following state guidelines, must include a description of how the lesson will be differentiated for special education students who may have difficulty learning mathematics, as well as how to cater for gifted and talented students who may have already mastered the mathematical concept that is the topic for the day. Pre-service teachers study theories of learning mathematics, curriculum design, analysis, and evaluation and the effects of assessment of various types. They learn to guide students to ask questions, such as those beginning with "why, why not, what if, when does that work, and how many ways might I . . .", that lead students to focus on making sense of more complex concepts and to dig more deeply into related mathematical ideas.

Professional development for practicing teachers might take place in graduate university classes, in sessions offered by school districts, embedded in everyday practice, online and through formal or informal lessons or topic study groups.

6.5 Summary

6.5.1 Overview

In this chapter we have looked at the various aspects of professional development and the theoretical basis that is required to enable and assist teachers to teach using a challenging mathematics approach. Here we summarize the basic aspects of professional development that we consider to be valuable in any professional development program.

6.5.1.1 Fundamental principles

The list from Anthony and Walshaw (2007) that we quoted at the end of Section 6.3.2 provides a possible framework for professional development in this area. In summary, we want to give students a better learning, understanding and appreciation of mathematics as a subject in its own right and a tool for solving real problems. In doing so we acknowledge that all students, irrespective of age, have the capacity to become powerful mathematical learners. This does not mean that we expect all students to want to complete PhDs in mathematics but it does mean that all learners can be guided to see relationships between apparently different parts of mathematics. They can also be empowered to see for themselves how mathematics develops. In the process they can be encultured into mathematics and become apprentice mathematicians.

6.5.1.2 Aims

Clarke and Clarke (2004) noted that the effective teachers involved in their Australian study believed that mathematical learning should be an enjoyable experience, had a good knowledge of the subject, and made known their pride and pleasure when their students were successful. These would seem to be overall aims to which all professional development might aspire.

But fundamental to student gains is teacher knowledge (Hill et al. 2005). This has two aspects: the teachers' own mathematical knowledge and their pedagogical knowledge. As far as their personal mathematical knowledge goes, teachers should acquire:

- a deeper knowledge of mathematics content and processes;
- an understanding of the relationship between higher mathematics and school mathematics;
- an enthusiasm for solving challenging problems;
- an ability to tackle challenging situations;
- a knowledge of where to find suitable challenges;
- a knowledge of challenges outside the classroom.

On the pedagogical side, in order to develop a classroom of apprentice mathematicians they should:

- have a theoretical knowledge of how students' learn;
- be able to use appropriate scaffolding;
- be able to develop cooperation between students;
- encourage both written and oral communication between students.

6.5.1.3 Modeling

Professional development can range from direct delivery in person or by the web for a relatively few students, up to a national project. In all cases we feel that it is important to model what it is that teachers will be expected to develop in their classrooms. As part of the program, it is also valuable to see teaching in action. Even better, teachers value and are more easily persuaded, if someone involved in the development process can actually work with their class and show how the process they are proposing can produce gains by their students.

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Chapter 7
Challenging Mathematics: Classroom Practices

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In this chapter we examine classroom practice issues related to teachers providing mathematical challenges in their everyday classrooms. We examine how challenging mathematics can become the essence of mathematics classrooms, how challenging mathematics can be designed for the everyday classroom and how classroom artifacts and practices can be designed for mathematical challenges. Finally, the question of suitable research designs for research into classroom practices associated with the use of challenging mathematics in everyday classrooms is addressed and illustrated.

7.1 Challenging mathematics—the essence of mathematics classrooms

Our challenge as educators is not just to make challenging mathematics available in school. It is to enable, invite and scaffold students to accept and exploit the challenge that richer mathematical understandings can offer them (Mason and Janzen Roth 2004). Consider the following task used by Mason and Janzen Roth in one of a series of design experiments (Cobb et al. 2003, English 2003) which will be overviewed later in this chapter.

The tennis ball problem: Consider the following problem for which you are given a set of resources you might use. For each use of a resource, there is a point-cost that reduces the total for your team. Up to 100 points will be awarded for a clear and accurate solution. The resources are:

- a tube container just large enough for the 3 tennis balls it holds: 60 points;
- a tennis ball: 30 points;
- a cloth tape measure: 20 points;
- a meter stick: 10 points;
- a 1-meter string: 5 points.

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You have 10 minutes to decide on the resources you might use. You then have a further 10 minutes to answer the question.

Question: By what percentage does the height of the tube container for the tennis balls exceed the distance around the tube (or vice versa)?

This task has been used to introduce a unit of mathematics that engages students in looking more deeply into the formulae for the circumference and area of circles. The students who participated in the design experiment had learnt in previous years that $C = 2\pi r$, $C = \pi d$ and $A = \pi r^2$, but it was a repeated source of disappointment for the researchers how few students actually considered using the relationships summarized by the formulae to answer the problem, and so answer the question without buying any resources. Why so few? Why have the capable students we teach chosen to develop a purely procedural understanding of the mathematical relationships they are studying? Mason and Janzen Roth posit that this is a consequence of students not perceiving the potential benefits, including the intrinsic satisfaction, available to them if they accept the challenge of developing richer understandings. What is it about classroom practices that foster this shallow understanding of the power of mathematical relationships?

We could, of course, be pessimistic and take Vinner's line that "there are two essential conflicting elements in the human psychology which are active in the domain of teaching and learning mathematics: the need for meaning and the ritual schema", and "there is no chance that one tendency will take over the other. The educators will continue their call for meaningful learning, whereas the masses of students will prefer the ritual (procedural) approach" (Vinner 2000, p. 121).

Alternatively, we could throw up our hands and say perhaps this lack of depth in student understanding of concepts underpinning formulae, say, is an artifact of classroom practices which result from different teaching styles. The King's College Project (Askew et al. 1997), for example, which studied teaching styles in the UK found three that were prevalent: transmission, discovery learning and connectionist.

Teachers adopting a connectionist viewpoint, for example, would use challenging activities such as problem solving in the classroom, by building on students' current knowledge and existing connections between mathematical ideas to look flexibly at a problem to make sense of it and generate new connections in innovative ways. This is a far cry from the transmission of procedures that work in familiar contexts but are quickly discarded by students as useless in slightly novel situations (Stillman and Galbraith 2003, p. 183).

In contrast to the transmission method of instruction, which is often seen as the traditional approach in Western mathematics classrooms, a traditional method in Japanese elementary school is to solve a problem through full-class discussion. With a skilful teacher, the children can learn more than the curriculum intends.

The Japanese open approach (Hino 2007, Nohda 2000, Shimada 1977, Tejima 2000) "offers opportunities for especially bright students to exercise their creative abilities and devise insightful ways to deal with mathematical topics and problems" (Hashimoto and Becker 1999, p. 101). When using the open process aspect of this approach, the focus is "on different ways of solving a problem when the answer is unique" (Hashimoto and Becker 1999, p. 101). As an example of this practice, suppose the class is given the problem of dividing 4/5 by 2/3. One student might observe that 6 is the least common multiple of 2 and 3, and write

$$\frac{4}{5} \div \frac{2}{3} = \frac{4 \times \frac{6}{2}}{5 \times \frac{6}{3}} = \frac{4 \times 3}{5 \times 2} = \frac{12}{10}.$$

The children can then come to realize that this method is equivalent to the standard algorithm and can be used with other choices of fractions. "It is important how [the] teacher leads students to make relationships on the basis of different conceptions" (Tejima 2000, p. 252) of the process, as in this case. However, the different conceptions could also be of the problem formulation or what counts as a solution, if it is an open-ended problem. From the teacher's point of view, this dynamic is unpredictable. Consequently, the teacher requires deep mathematical understanding and sure skills in order to handle the situation. However, when the challenge of such an alternative solution process is taken up rather than dismissed and the approach succeeds, the children deepen their mathematical experience. What is it about the classroom practices in this approach that ensures this "meta-learning" (Nohda 2000, p. 30) occurs?

7.1.1 Why do we need challenges in regular classrooms?

Organizing mathematical challenges in overloaded mathematics lessons, whether they be short open or closed problem-solving tasks (Sriraman and English 2004), or investigations (Ponte 2007), mathematical modeling tasks (Galbraith et al. in press, Kadijevich 2006) or projects involving extended challenging tasks particularly those involving real-world contexts, can be time-consuming. Therefore, there must be valid reasons for doing so. There are indeed pay-offs for teachers using challenging mathematical tasks in regular classrooms.

First, when well-planned in accordance with a theory of knowing, such as Gardner's Theory of Multiple Intelligences (1983, 1999) or Tall's Theory of Mathematical Growth (2006), challenging mathematics classroom practices enable students to develop a fuller understanding of the many aspects of a mathematical concept, enriching their concept image. A student's concept image is "the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (Vinner and Dreyfus 1989, p. 356).

Teaching the same concept using multiple channels and perspectives (that is, teaching with and through multiple intelligences) is one promising way to allow students to learn concepts with deep understanding (Cheung 2003). Students' concept images become enriched through challenging tasks if, for example, the tasks allow students to see that there are more transparent forms of the same concepts in different situations. This happens when the Chickens and Pigs problem ("There are pigs and chickens in a farmyard. Altogether, there are 23 heads and 68 legs. How many chickens and how many pigs are there?") is solved using a spreadsheet rather than by hand. Thus finding a solution to a particular task is of little benefit in itself, rather the enrichment comes when the students are able to look across several situations or several solutions and see the different manifestations of the concept, some of which are more easily recognizable and separable from the task context than others.

Secondly, setting the mathematical tasks in real-life situations not only makes the challenges more personally relevant to the daily life of students (Kadijevich 2006), but also affords students opportunities to approach the challenges at different levels of mathematization (Freudenthal 1973). Julie (2007) is one of the few researchers who have investigated the real-life contexts that learners prefer to learn about in mathematics class or through mathematics. He confirmed that students show "strong interest in issues of direct personal appeal bolstered by a high media visibility" (p. 201). However, in agreement with Skovsmose (1998), he points out that schooling is also about "foregrounding" issues that "learners do not as yet perceive as interesting" (p. 201) so such real contexts for challenges need not be restricted to current interests if a long-term perspective is taken. Students will have greater opportunities to encounter similar challenges in their everyday lives as they grow older. Their experiences will accumulate through these encounters.

According to Freudenthal, there are four levels of mathematization which could be the basis for real-world tasks: (1) the situation level where knowledge and strategies specific to the domain in which the task is set are used in the task context; (2) the referential level where models and strategies refer to the task context; (3) the general level where the focus is on mathematical strategies and models for the task context; and (4) the formal level involving working with formal mathematical notation and procedures. (See Gravemeijer (1999) and Cheung (2005) for an explanation and illustrative examples.) Such tasks need not be at a higher level of mathematization to be challenging, though. In the research of Mason and Janzen Roth (2005), for example, tasks such as whether it is possible to draw a square of area 10 cm^2 have proved to be challenging for Year 9 secondary students.

Thirdly, when students engage in solving challenging mathematical tasks, they are placed at a psychological boundary between their comfort zone and risk-taking. Challenges teach students how to sustain themselves in uncertainty—a skill relevant for lifelong learning—and successes with challenges prepare students for real life. As real life is often messy and may not be easily reduced to simple mathematical forms, challenges help students become aware of the intricate details and the significance of the roles these details may play for solving the challenges. Therefore, it is essential that mathematical challenges should be presented at different forms of mathematization so that students at all grade levels can experience how open-ended real-life problems may be approached by mathematicians and their teachers.

7.1.2 How often should challenges be used and for whom?

It is essential that challenging experiences be provided regularly (Kadijevich and Marinković 2006, Silver and Stein 1996). Students need the opportunity to engage with such tasks on many occasions. They may not always engage with a particular challenging task but encountering several over time will give multiple opportunities for them to access such tasks and bring them to the realization that it is an expectation of all students to be able to do so.

The regular use of challenges in classrooms with appropriate structuring or scaffolding of the task as necessary (Mason et al. 2005, Sheffield 2003, Stillman et al. 2004) indicates to all that solving mathematical challenges is applicable for all students of various learning abilities (Woodward and Brown 2006) and experiential backgrounds, and as such it is one viable way to address inclusion and intellectual diversity (Sriraman 2006).

All students should be challenged and should challenge themselves to learn deeper mathematics in our classrooms, as challenge is one of the characteristics of academic tasks that motivate learning, according to Paris and Turner (1994). Amit et al. (2007) see it as a "social obligation" that we ensure "no child be denied the materials, conditions, and kinds of teaching necessary for developing good mathematical thinking and the social and economic benefits deriving from it" (p. 75). Williams (2003a, b, 2006) suggests this is possible through the development of resilience.

"Resilience relates to how a child explains occurrences in their day-to-day encounters with the world" (Williams 2003b, p. 374). Resilience is "an 'optimistic orientation' to the world characterized by a positive explanatory style where successes are perceived as permanent, pervasive, and personal, and failures as temporary, specific, and external (Seligman 1995)" (Williams 2003b, p. 752). According to Williams (2003a), "resilience, and inclination to pursue novel mathematical ideas, appear to be mutually sustaining (overcoming a mathematical challenge conditions an optimistic orientation and an optimistic orientation increases student inclination to pursue the next challenge)" (p. 758).

Not all students display resilience but rather can be seen as displaying pessimism or somewhere in between these two. However, Seligman (1995) has found that the resilience of a child can be altered over time. Using Csikszentmihalyi's (1992) concept of flow as a framework, he found "that in overcoming small challenges to gain successes, the child's inclination to undertake future challenges was increased" (Williams 2003b, p. 378). Williams describes flow and its effects as "an optimal learning condition that may occur when a person works just above their present skill level on a challenge almost out of reach. Individuals or groups in flow become so engrossed with the task at hand that they lose awareness of self, time and the world" (p. 378). Those challenges that develop resilience are self-set as individuals or groups spontaneously decide to explore unfamiliar mathematics encountered in a challenging task set by the teacher (Williams 2006).

Thus, what is being advocated here is regular use of challenging problems within which groups or individuals have opportunities to set their own challenges at a level of difficulty that is appropriate for them. Through teachers providing opportunities for mathematical challenges for all students, there is the potential for students to develop resilience, if they are not already displaying it, and an inclination to desire to pursue more mathematical challenges.

7.2 Designing challenging mathematics for classrooms

7.2.1 Setting the scene

Before we can consider how challenging mathematical activities and tasks might be designed for the everyday classroom there are three issues that must be addressed. First, it is necessary to consider the nature of the mathematical understandings that we expect to be deepened by use of these challenges. Secondly, teachers need to be aware of the nature and extent of the gap between what they are currently doing in mathematics classrooms and what is possible using mathematical challenges. Thirdly, it is necessary to point out that when mathematical challenges are being used, students need to be made aware that the rules of the didactical game (Brousseau 1997, Mercier et al. 1999) have altered and so the didactic contract (Brousseau 1997) needs to be renegotiated.

7.2.1.1 Nature of mathematical understandings expected to be deepened

In designing challenging activities and tasks for the regular classroom, our purpose is not to restrict and confine students with activities or tasks that students find palatable and the mathematical payoff is limited, rather we want an invitational package that not only draws students towards the activity but also has the qualities of mathematical inquiry that will sustain it.

Thus, "we must do more than invite people to our mathematical game—we will have to play host to ensure all the guests feel the pleasure of addressing an intellectual challenge" (Mason 2000, p. 111) through engaging in meaningful and rewarding mathematical thinking. The processes that are required to solve challenges and the meta-cognitive knowledge and strategies that can be brought to bear during application of those processes are also considered content, when it comes to the understanding necessary to engage with challenges fruitfully in the classroom.

7.2.1.2 The gap between what is being proposed and present practice

Teachers are often not aware that their own teaching practices contribute to the students in their classes not experiencing or not desiring to be challenged in their mathematics classrooms. However, the gap between present practice and what is being proposed may not be all that wide as the following vignette from Japan illustrates.

Primary teachers in Japan use the area of a rectangle as a starting point for finding the area of a circle. In Years 4 and 5, 10 and 11 year olds already know the formula for area of a rectangle (Takahashi 2006) and are given many small problems about area for complicated configurations such as those in Figures 7.1 and 7.2. Students are asked questions such as: Which has the bigger area, the area in Figure 7.1(a) or that in Figure 7.1(b)? If the perimeter of a figure is longer, is the area bigger?

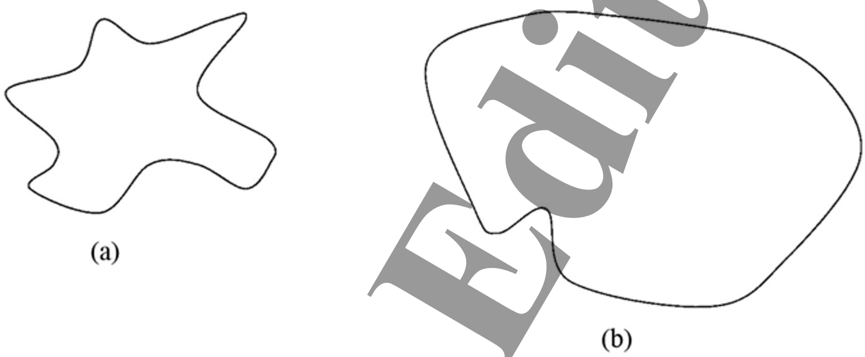

Figure 7.1: Complicated configurations for comparing area and perimeter for Year 4 and 5 students

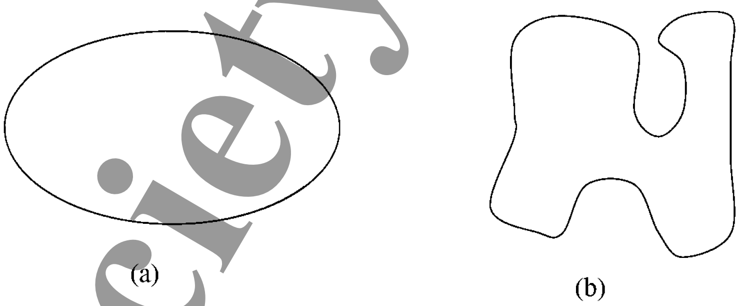

Figure 7.2: Complicated configurations for finding area for Year 4 and 5 students

How can you find the area for a complicated configuration such as in Figure 7.2(a) or (b)?

Students begin by cutting out rectangles then progress to drawing rectangles to find answers to these questions. Then students are asked to compare the areas of the following figures (Figure 7.3).

One student might propose to compare these figures by regarding them as the bases of boxes and then filling them with small balls represented by circles as shown in Figure 7.4. So the area of Figure 7.3(b) is bigger than that of Figure 7.3(a).

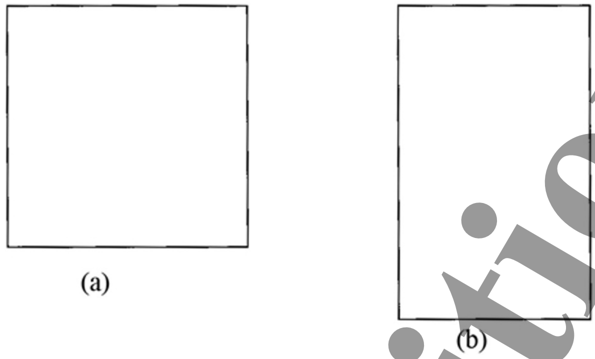

Figure 7.3: Comparing area of rectangles task for Year 4 and 5 students

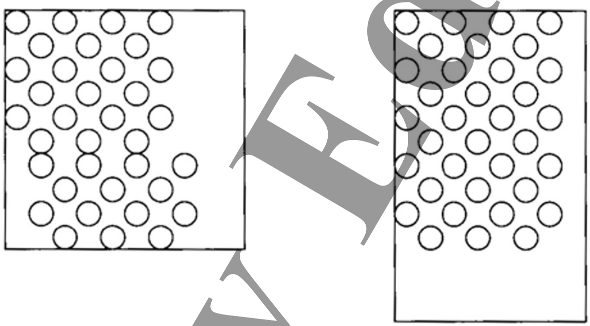

Figure 7.4: Using balls to compare areas

Another student raises a doubt about the applicability of the proposed method pointing out that circles will never fill up the rectangles. At this point, there are several ways for the teacher to handle the challenge to the first solution method. One possibility is to use circles of different radii, progressively making the radii smaller and smaller. In this case students will see the essence of the idea that a certain infinite process is necessary. Then the teacher has the opportunity to explain, or students can be given the opportunity, to find for themselves that to discover the area of configurations such as Figures 7.2(a) and 7.2(b), even if rectangles are used, an infinite process is still needed. This would be a good prelude to considering the area of a circle. Another possibility is to ask what happens if we use dice to cover the bases of the boxes instead of balls.

However, if teachers think these questions will merely make the students puzzled, they might take the point of view that it would be better just to skip the question (and so skip the challenge) saying that it would result in the same answer by using rectangles or circles anyway. In this instance, the teacher just wants to make things simple and ignores the challenge of using circles, which is much more difficult: an opportunity lost.

After these preparations, students are asked to find the area of a triangle themselves by using rectangles and known formulae for these (Figure 7.5).

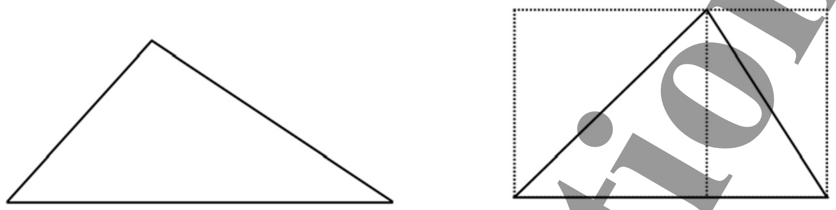

Figure 7.5: Finding area of triangle from rectangles solution by Year 4 and 5 students

Students then find areas for a parallelogram, trapezium and a circle (Figure 7.6). However, unless the challenge is taken up in the teaching moment, rather than avoided, the link to the deeper notion of infinite processes (Figure 7.7) is lost.

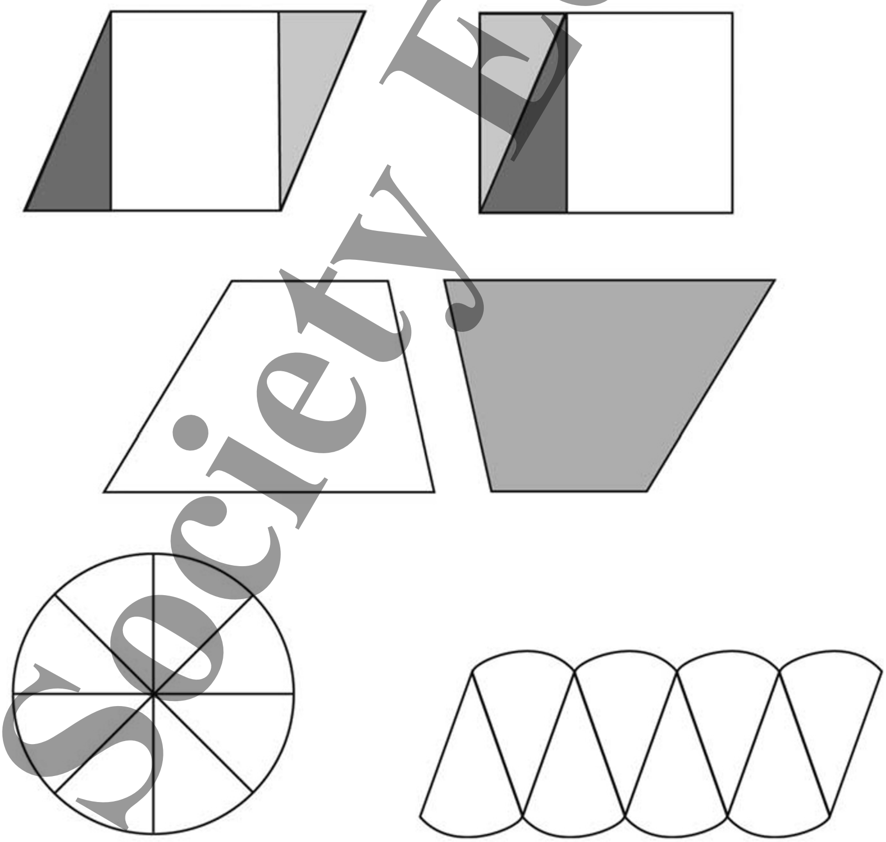

Figure 7.6: Finding areas of standard figures by Year 4 and 5 students

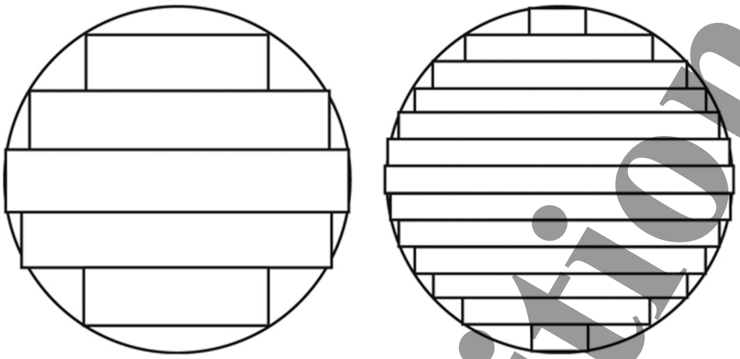

Figure 7.7: Finding the area of a circle by the infinite process of using rectangles of decreasing width

Thus, teachers are the key to effecting change in the classroom. However, they need to be convinced themselves that it is necessary to go in a new direction (Pehkonen 2007) and to engender students with confidence that challenges are essential in the classroom. Continually avoiding the addressing of challenges denies students the opportunities to learn from engaging in them. It is imperative therefore that the teacher suspend judgment for a while when these serendipitous challenges arise and see what prevails.

7.2.1.3 Clarifying changes in expectations

According to Mercier et al. (1999), "Brousseau defines the didactical contract as a system of reciprocal expectations between teacher and pupils, concerning knowledge, which contract is setting both pupil's and teacher's acts and can explain them thereby" (p. 343). Furthermore, "Modeling a teaching situation consists of producing a game specific to the target knowledge among different subsystems: the educational system, the student system, the milieu, etc." (Brousseau 1997, p. 47). But what happens when the implicit rules for this didactical game have to be changed? If students are used to doing several short quick tasks in every lesson where the mathematization of such tasks has been fully laid out before them, then introducing a task that now requires, say, that they work out the mathematization themselves and that they are expected to persist for several lessons working on the same task is a contravention of the usual didactical game. Students need to be alerted to the changes in the teacher's intentions and expectations in the new didactical situation.

Thus, when introducing mathematical challenges into the classroom, whether they are short or long tasks, abstract or set in a real-world context, attention must be given to this issue of clarifying expectations. Failure to do so can result in few outcomes related to the mathematical purpose of the lesson or organizational difficulties and frustrations for both teachers and students (see Cheung 2006, for an example).

7.2.2 Task design

Kadijevich and Marinković (2006) in making a plea that regular curriculum content be the source of challenging tasks in the regular classroom, suggest that mathematical quizzes involving questions which are solvable in 10 to 30 seconds be used. These quiz items should "require a prompt and meticulous [form of] thinking, contributing to the development of mathematical reasoning" (p. 34). Sample items are:

- A mouse's body is 12 cm in length and this is one-third of the length of its tail. How long is the tail? (15 seconds are allowed for primary students to answer.)
- Write down an expression for 100 by using (a) 5 six times; (b) 3 seven times. (30 seconds are allowed for Years 7 to 9 students to answer.)

The development and use in regular classrooms of sets of challenging tasks, which are isomorphic with respect to the mathematical content or solution method, are also advocated by Kadijevich and Marinković (2006). Examples of the former types of tasks are:

Marking pens: Two marking pens cost more than three pencils. Do 5 marking pens cost more than 7 pencils (no discount offered)?

Parking stations: Two parking stations in a town are competing for customers. At present, more cars can be parked in one parking station that has two levels than in another one with three levels. As a result, the parking station with three levels will be extended to five levels. Will an extension of the first mentioned station from two levels to three levels enable it to continue to have more parking space than the extended second station?

Examples of tasks that are isomorphic with respect to method would be non-routine area tasks, which are all solvable using translation of geometric figures to gain an insight into a simple method of solution. Exemplar tasks can be found in Kadijevich and Marinković (2006). Design features that should be borne in mind in developing such tasks are: "(1) there is a good mathematical idea behind the task; (2) the task is not routine; (3) the task is interesting with respect to formulation and content; (4) the task has a nice and perhaps unexpected solution(s); (5) the task requires its solver to stretch his/her mind; and (6) the solution of the task is usually short and not complicated, enabling the solver to use knowledge and skills traditionally learned in the classroom" (p. 35).

Another source of challenging tasks for the regular classroom is real-world tasks. Designing extended real-world investigative tasks that present manageable and engaging challenges for lower secondary students is not without challenges as was shown in the RITEMATHS project, (extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/). (RITEMATHS was a collaborative research project, funded by the Australian Research Council Linkage Scheme, involving the Universities of Melbourne and Ballarat, six schools and Texas Instruments as industry partners.)

Despite thoughtful consideration by the teacher in both designing the tasks used in one part of the project, and providing timely task scaffolding at points during task implementation when students were expected to be challenged by the cognitive demand of tasks, there were always differences between the moves expected of students, the challenges and what transpired. Some students took up the challenges as expected but for others these same challenges did not eventuate as the significance of particular requirements of the tasks were missed, or the mathematical implications of results produced during the task, which should generate challenge, were not realized. At other times, unforeseen challenges arose for individual students as they discovered different complexities in their unanticipated interpretation of tasks (Stillman 2006).

As an example of the former, in the task, Shot on Goal, one pair of students were challenged by their interpretation of the task when they tried to mathematize the run line. The teacher expected the students to consider a player approaching the goal on a run line parallel to the sideline. Instead of advancing down their specified run line in 1 m intervals which would have kept the line of the path parallel to the sideline, this pair of students took a stepped trajectory towards the goal considerably increasing the challenge beyond that expected of them.

The major elements of Shot on Goal Task from RITEMATHS research project are:

> Many ball games have the task of putting a ball between goal posts. The shot on the goal has only a narrow angle in which to travel if it is to score a goal. In field hockey or soccer when a player is running along a particular line (a run line parallel to the side line) the angle appears to change with the distance from the goal line. At what point on the run line, has the attacking player opened up the goal to maximize the possibility of scoring the goal?
> Assume you are not running in the GOAL-to-GOAL corridor. Find the position for the maximum goal opening if the run line is a given distance from the side line. As the run line moves closer or further from the side line, how does the location of the position for the widest view of the goal change?

7.2.2.1 Rephrasing as a means of tweaking a task for different grade levels

Another illustrative example of a challenging mathematical task set in an everyday context applicable to Years 7 to 12 is presented below. This task will be further modified in order to demonstrate how mathematical challenges might be introduced into regular classrooms at all grade levels through a process of rephrasing. This task has been used by Cheung (2006) in teaching experiments in Macao, but it is a common task in secondary schools in many contexts worldwide although not all versions or implementations are challenges (French 2002, Harvey et al. 1995, Pierce and Stacey 2006). As will be discussed later in this chapter, whether or not a task is challenging for a particular student is very much dependent on the actions of the teacher and other students and the student's own abilities and experiences.

Open box problem (initial version): Use a sheet of 12 cm × 12 cm cardboard, cut away four corners, fold and glue to form an open box to hold as many things or as much as possible.

When this problem is introduced for student project work, individually or as a group, it is advisable for the teacher to phrase it in a manner that establishes a better connection with everyday life and appears more personally relevant to the students. For example, the problem may be stated as:

Open box problem (rephrased in an everyday context): Using the cardboard (12 cm × 12 cm) provided, form any kind of open box to contain as many paper clips as possible (or contain as much rice as possible).

The rephrased problem provides students with the opportunity to change a realistic practical problem into a mathematical problem for solution. If this happens, the student is encouraged to approach the problem at the lowest level of mathematization, that is, the situation level (Freudenthal 1973, Gravemeijer 1999). Before assigning the problem to the students, the teacher needs to consider carefully what background knowledge and skills are available for the students in order to arrive at a solution. At the situation level, students engage in both hands-on and minds-on activities. They need to make an open box and then find the one box that contains the largest number of clips. Teaching experiments conducted in Macao with this problem (Cheung 2006) revealed that some younger students do not know how to make an open box. They simply treat the task as a paper-folding exercise to come up with a container and then see how many clips it can contain without spilling. Their attempts to make such containers are by no means systematic. They simply treat the task as a mathematical game where the container holding the most wins, not realizing they might not have stumbled upon the one that contains the most possible. They do not see that the mathematical purpose is to find the relationship between volume and height of the open box. They fail to see the relationship between the height of the box and the length of the side of the square cut from the corners of the cardboard.

Another interesting observation is that some students are reluctant to cut away corners from the cardboard when the open box is made. Interviews conducted after the teaching experiments revealed that there is a misconception preventing them from doing so, namely, students possess an intuition that the more that is cut from the cardboard the lower the volume of the open box enclosed from the remaining cardboard will be. This illustrates that teachers should always be ready to learn from their students, and teachers should treat their not knowing an unexpected phenomenon such as this as an asset, not a shortcoming. This is because these moments give direction for future teaching that can be based on these misconceptions or alternative conceptions.

In this task, as in all mathematical challenges, teachers need to know what the gap is between what is proposed for them to accomplish mathematically and what the students may do.

Students in Macao learn how to conceptualize and calculate the volume of a cuboid before Primary 6. Cuboid and capacity are two mathematical terms that may prompt students to adopt a mathematical approach for a solution. The problem could be rephrased accordingly.

Open box problem (rephrased in an everyday context with mathematical hints): Using the cardboard (12 cm × 12 cm) provided, form an open box in the form of a cuboid with an internal capacity as large as possible. (Paper clips, rice, ruler, calculator, etc. may be provided if students opt to solve the problem using an experimental approach, or a combination of experimental and mathematical approaches.)

Students need to form a net in the form of a cross in order to have it folded and glued into an open box. Four identical square corners with equal lengths need to be cut from the cardboard in order to form the cross. Students may experiment with how much rice or how many paper clips the open box is able to contain, and/or go straight to use of the volume formula of the cuboid to calculate the internal capacity of the open box. For those students who adopt an experimental approach, they need to continue making more boxes of different sizes and make the necessary size comparisons accordingly. For those students who adopt a mathematical approach, they need not make more boxes once they can make sense of the calculations done to the first box.

Instead, they can try different heights of the cuboids and calculate the corresponding internal capacity to come up with an optimal solution. Since the mathematical approach adopted makes references to a concrete case in order to help reasoning, this solution process is at the referential level of mathematization (Freudenthal 1973, Gravemeijer 1999). In order to reach the optimal solution, students may make tabulations to find relationships between the height of the cuboid and the internal capacity of the open box.

However, those students who insist on adopting an experimental approach will not know whether their solutions are optimal or not. What they are targeting are better and better solutions by making more and more boxes. The level of mathematization still remains at the situation level of mathematization. However, realizing that mathematical challenges are essential to develop reasoning, the teacher should suspend judgment for a while and should not usher students into a higher level of mathematization right away, "as it is the extent to which the locus of knowledge generation is with the learners which makes the difference" (Watson 2004, p. 366).

This task can be assigned as a group project so that students not only learn with each other but also learn from each other. When students of a heterogeneous group engage in collaborative exchanges, they can contribute their intellectual strengths and at the same time have their weaknesses scaffolded by their peers. The effectiveness of peer scaffolding is, however, mediated by the appropriateness of task-related questions framed by their peers "and the extent to which one learner attends to the questions (and other contributions) of the other" (Clarke 2001, p. 310).

For a project the teacher could, for example, ask the group to guess which of the following five cases, that is, those with 1 cm, 2 cm, 3 cm, 4 cm, and 5 cm square corners, should be cut to form the required net. Research done in Macau classrooms (Cheung 2006) shows that the 1 cm, 2 cm and 4 cm cases are students' favorite choices. Some students choose the 1 cm case because it entails the least amount of paper cut from the cardboard and therefore they think this results in an open box of the largest internal capacity. Some students choose the 4 cm case because they believe that if the cuboid resembles a cube then the volume of this cube should be the largest. However, they forget that the net is to be folded into an open, not closed, box.

Students choose the 2 cm case because they perceive the open box to have a broad base area and yet have considerable height to produce a large enclosed volume. In this task, the mathematical challenge becomes one of seeking to find the relationship between the height of the open box and the internal capacity of the box formed. Even senior primary (Cheung 2006) and junior secondary students without knowledge of advanced mathematics, for example, inequalities and calculus, can generate some solutions using a variety of approaches.

At the referential level of mathematization, one has to be aware that it is still very difficult for the teacher to explain to students why the 2 cm case produces an open box with maximum capacity. The teacher can only point out that it is the largest amongst the five cases under consideration. In this regard, reformulating the mathematical challenge to higher levels of mathematization, such as the general or formal levels (Freudenthal 1973, Gravemeijer 1999), is required for a convincing explanation. This case will be considered in the context of answering our next question.

7.2.2.2 Does the nature of the task change with increasing grade level?

With increasing grade level the sophistication and the breadth of mathematics that students can bring to solving a challenge increases. Returning to the example of the Open Box problem, the problem can easily be rephrased to afford higher levels of mathematization.

Open box problem (rephrased in mathematical context affording higher levels of mathematization): Given a sheet of cardboard of dimensions a cm $\times$ a cm, what is the maximum capacity of the open box that can be formed from it?

This statement of the problem may be regarded as a typical textbook problem for senior secondary students. At the general level of mathematization, no open box needs to be made by the students and the length of the sides of the cardboard needs not be specified numerically. Students can simply make sketches of the net of an open box and use the sketch to formulate a non-linear equation relating capacity of the open box V cm^3 with length of the side of the corner cut x cm. Using the non-linear equation, students find the maximum value of V without recourse to the original problem situation.

$V = f(x) = x.(a - 2x)^2$, where a is the length of the side of a piece of square paper, is therefore a model used for relating variables having non-linear relationships of third degree. Students may solve this problem if they possess knowledge of inequalities. At the formal level, using $V = f(x)$, students solve the maximization problem using formal mathematical methods such as differentiation.

One difficulty faced by students at the general and formal level of mathematization is that their attention is often exhausted on the correct formulation of $V = f(x)$, and success in tackling the task depends on whether they can make use of the mathematical knowledge and skills taught to them during formal lessons. If they cannot formulate an equation and if they are unwilling to investigate the problem at a lower level of mathematization, they cannot proceed further. In this sense, such textbook problems may not be considered as mathematical challenges at all. In contrast, relaxing problem conditions and constraints can make problems more challenging than can requiring the use of sophisticated formulae and techniques.

Open box problem (rephrased with a relaxation of problem conditions and constraints): Given a 12 cm $\times$ 12 cm square piece of cardboard, construct an open box in the form of a cuboid with an internal capacity as large as possible. You are not allowed to waste any of the cardboard. Any cardboard that is cut away should be taped back to the net.

This problem is open-ended and can be assigned to both junior and senior secondary students. Students generally need to start from the situation level of mathematization to come up with one tentative solution first. After that, they need to attempt alternative methods and find out if other solutions exist. Since it is often time-consuming to experiment with more cases by hand, students may simulate the problem situations using a computer or graphing calculator.

If a maximal solution is being sought, then the problem needs to be solved at the general or formal levels using sophisticated mathematical methods, for example, partial differentiation, or alternatively, graphical or symbolic manipulation techniques assisted by technology. The ultimate solution relies on the application of the Lagrange multiplier method. The constraint is that the area of the net equals 144 cm^2 because no paper is to be wasted. After derivation, V can be shown approximately equal to 166.28 cm^3.

7.2.2.3 Do we need new topics for challenges or can we find them within the existing curriculum?

Although the topics in existing curricula have not been exhausted as sources, there are some areas that could prove fertile ground for a source of challenges. Sriraman and English (2004) make a case for combinatorics as the topic is "accessible to students starting at the elementary levels because it builds from simple enumerative techniques" (p. 183). In Sriraman and English (2004) examples are given of the use of combinatorics problems as challenges in primary school. Sriraman (2006) suggests that in addition the area of number theory is a good source of challenges. Sriraman and Strzelecki (2004) offer suggestions for challenges in number theory. In Section 7.4, Sriraman's use of challenges from both these areas in secondary classrooms will be reviewed.

7.2.2.4 How can technology be incorporated into task design to facilitate the use of mathematical challenges?

Use of electronic technologies such as calculators and image digitizers can reduce the cognitive demand of tasks through "supplementation" and/or "reorganization" of human thought (Borba and Villarreal 2005) by carrying out routine arithmetic calculations, algebraic manipulations, or graph sketching; acting as an external store of interim results; or overlaying visual images within an interactive coordinate system to facilitate analysis. However, these technologies also have potential to influence the complexity of what students do as they transform classroom activity and allow new forms of activity to occur.

Regulation of this complexity is a further opportunity for teachers to mediate cognitive demand, and therefore the challenge, of tasks through careful crafting of tasks and management during implementation. In particular, use of multiple representations, easily accessible with graphing calculators and tasks amenable to electronic technology use, harness opportunities for students to use technology to stimulate higher order thinking in investigating real-world situations. Within tasks, diagrammatic, numerical, symbolic, graphical and algebraic representations can be intentionally employed to support bridge-making from one representation to another and to provide opportunities for interpretation across representations, as well as from each representation back to the situation being investigated.

As Dede (2004) points out, several projects implementing well-formulated technology-based designs have demonstrated that "typical middle years students [are capable of] mastering science and mathematics previously thought appropriate to teach only" (p. 111) to students at higher schooling levels. However, two challenges middle years students face when engaging in extended investigations for the first time (Loh et al. 2001) are the inability to recognize when to keep records and failure to plan and monitor progress effectively. It is thus prudent for teachers designing extended tasks for the lower secondary years, initially at least, to provide timely instructions throughout task statements supporting recording of key information, a planned solution, checking and verification of results. As student task expertise and familiarity with technology grow, some "fading" of this scaffolding (Guzdial 1994) should occur, particularly that related to task structuring and technological tool selection and instructions. This is not to say mathematical analysis tools need be withdrawn. On the contrary, "learning to 'work smart'" in a technology-rich learning environment may involve "learning to establish one's own scaffolds for performance, and fading these may be beside the point" (Pea 2004, p. 443).

7.3 Designing classrooms for mathematics challenges

7.3.1 How do we teach students strategically to address a challenge?

To help students not only to address the mathematical challenges with which they have been presented but also to deepen their mathematical reasoning and develop their mathematical creativity, students can be encouraged to question the answers to the challenges, not just answer the questions. The use of a student-centered inquiry approach that encourages students to think like mathematicians, asking questions that enable them to make sense of mathematics, is one of the critical aspects of the Project M3: Mentoring Mathematical Minds (Gavin et al. 2006). This curriculum and research study in the United States is designed to nurture mathematical talent and creativity in elementary students by creating challenging, creative and motivational curriculum units for students. The deeper mathematical reasoning occurs when students begin to create and solve their own challenges, realizing that each solution is just the beginning of a new investigation.

Just as students answer questions of "who, what, when, where, why, and how" in writing an article for the school newspaper, they can learn to ask and answer these same questions as they investigate, create and extend mathematical challenges. All students can and should challenge themselves to deepen and extend their mathematical reasoning and abilities.

Some suggested questions for students as they create and solve mathematical challenges are:

- **Who**? Who has a new or different idea? Who is right?
- **What or what if**? What sense can I make of this problem? What is the answer? What are the essential elements of this problem? What is the important mathematics? What patterns do I see in these data? What generalizations might I make from the patterns? What proof do I have? What if I change one or more parts of the problem?
- **When**? When does this work? When does this not work?
- **Where**? Where did that come from? Where should I start? Where might I go next? Where might I find additional information?
- **Why or why not**? Why does that work? If it does not work, why not?
- **How**? How is this like other mathematical problems or patterns that I have seen? How does it differ? How does this relate to real-life situations or models? How many solutions are possible? How do you know you have found all the possible solutions? How many ways might I use to represent, simulate, model or visualize these ideas? How many ways might I sort, organize and present this information?

Students of all ages can learn to deepen their mathematical reasoning and enjoyment by asking themselves these questions. The questions might be asked in any order as they fit the problem under consideration. The use of these questions is developed further in *Extending the Challenge in Mathematics: Developing Mathematical Promise in K–8 Students* by Sheffield (2003) where a variety of challenges with samples of student work are presented.

Students are using the heuristic shown in Figure 7.8 as they work on problems. Students may start at any point on the diagram and proceed in any order that makes sense to them. They might do the following:

- relate the problem to other problems that they have solved;
- investigate the problem, think deeply and ask questions;
- evaluate their findings;
- communicate their results;
- create new questions to explore.

As they begin to learn to think like mathematicians, students might change the order of the steps and of the questions that they ask once they begin the exploration of the problem. Throughout the problem solving, students should be evaluating their work, making connections, asking questions, communicating results and creating new problems to investigate.

For example, in one problem in *Extending the Challenge in Mathematics*, students begin by investigating which of the following numbers can be the sum of two consecutive numbers: 15, 18, 57, 58, 228 and 229. They relate this to earlier problems that they have done with odd and even numbers. They then investigate numbers that can be the sum of three consecutive numbers and relate this to work with multiples of three and finding the mean of a set of consecutive numbers. Students are then encouraged to ask related, sometimes divergent questions.

Some students extend their investigations to sums of four or more consecutive numbers and investigate why numbers that are a sum of four consecutive numbers are not multiples of 4. Some students do more work with mean and median while others investigate other patterns with sequences and series. One very interesting investigation involves trying to find all numbers that cannot be

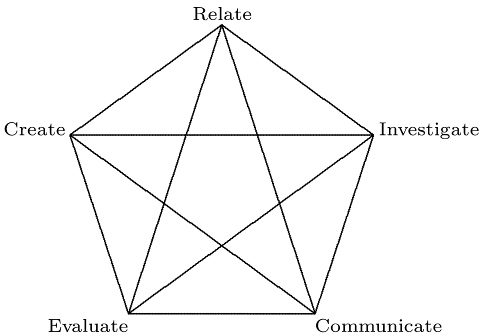

Figure 7.8: Heuristic used whilst solving challenges

written as the sum of any number of consecutive counting numbers. Finding these numbers and proving that they can never be the sum of consecutive numbers challenges even the most outstanding mathematics students.

Another challenge that lends itself to this graded treatment is Challenge 1.2.6 in Chapter 1.

In this way, the challenges for the students are differentiated according to their background and interests, and all students develop a deeper understanding of the topic under consideration. The teacher observes and supervises, challenges students who are ready to move to a higher level, gives hints to students who might be frustrated and ready to give up on a difficult question, and decides when to bring students together as pairs, small groups, or as a whole class to discuss their findings and probe possible misconceptions.

7.3.2 How can we make sense of the pedagogical challenge of having students appreciate challenge in mathematics?

Perhaps the giants of twentieth century pedagogy offer valid starting points. Piaget (1950) offers us a way to think about students' changing orientations to their subject matter. He suggests that as learners we are all more comfortable adding on to our current knowledge and understandings, a process he calls assimilation. Yet it is not by assimilation that students come to change their conceptual orientations. Students are reluctant to change how they perceive themselves in relation to their environment (accommodation), and will do so only in the face of relatively persistent discomfort with the fit between their orientation and some aspect of their environment. It is not difficult for teachers to offer discomfort, that is, cognitive dissonance (Festinger 1957), although some teachers may be reluctant to do so (Pierce and Stacey 2006). (Consult www.learningandteaching.info/learning/dissonance.htm for a brief primer on cognitive dissonance.) The challenge lies in having students accept the discomfort as an invitation to change or grow, rather than rejecting the discomfort as a source of frustration. Here, it is Vygotsky's (1978) idea of the zone of proximal development (ZPD) that helps us organize our pedagogy when we offer students opportunities to grow beyond their current capabilities. (This is defined in this Study Volume in Section 6.2.2.3 and discussed in Sections 3.1 and 4.5.4.)

We must fit what we offer to students into this region beyond what they can already do independently, and sense what they can do with the scaffolding we can offer them in their relationship with us and the classroom in which they are learning. Scaffolding, a term first introduced by Wood et al. (1976), is the interpersonal and strategic supports that teachers and classmates can offer learners to enable them to learn beyond what they can do on their own. "Each task has a relative cognitive value for an individual. Tasks that are too easy or too hard have limited cognitive value" (Diezmann and Watters 2002, p. 78).

However, if students engage in tasks of high relative cognitive value for them, potentially they can explore the cognitive challenge of engaging with mathematically challenging tasks and enhance their learning (Diezmann and Watters 2000, 2002). This suggests that we can offer significant invitations to grow or change (dissonance) (Neighbour 1992) only when we also offer significant scaffolding that enables students to see their engagement beyond their comfort zone as likely to generate success for them.

It must be borne in mind, however, that these are starting points, not ending points. If this is truly scaffolding in the sense of Wood et al. (1976), it needs to be accompanied by cycles of diagnosis of the student's level of performance and the need for scaffolding which then results in an adjustment of the level of scaffolding required (Pea 2004, Stone 1993). Thus, the level of scaffolding fades (Collins et al. 1989, Guzdial 1994) over time. Pervasive forms of support where the scaffolding is not dismantled enable only what Pea (2004, p. 431) calls "distributed intelligence" with the conquering of the challenges, in this case, being "accomplished" not achieved. Scaffolding also needs to be targeted to the needs of individuals and not be "provided to the whole class on the pretext that all students will benefit" (Diezmann and Watters 2002, p. 78). As Diezmann and Watters add, "the gifted students are likely to be most adversely affected by unnecessary scaffolding" (p. 78).

7.3.3 The role of textbooks

We should not underestimate how textbooks might affect the classroom practice of teachers providing challenges. Both their content and the way they are used have an impact.

Textbooks could be written so that challenging activities are the philosophy and leading idea behind them, and not merely fragmented parts of the content of the book. Sadly, this is rarely the case in practice. There is research evidence that many mathematics textbooks in various countries contain few challenging tasks and often the level of challenge is not as high as expected. Haggarty and Pepin (2001, 2002), in a study conducted in 15 lower secondary schools in England, France and Germany found that French textbooks provided students with more challenging tasks than did their English and German counterparts. Furthermore, in Sweden where mathematics textbook tasks are grouped into strands according to difficulty to assist the teacher in differentiation of learning to suit learners' differing abilities, Brändström (2005) found when she examined Year 7 mathematics textbooks "the level of challenge is low in almost all strands, even those intended to be higher" (p. 4).

However, there is some recent evidence that the situation is improving in some countries (e.g. Germany) where there has been "a shift in mathematics textbooks for all grades from rather algorithmically oriented tasks to more demanding problems" (Reiss and Törner 2007, p. 440). It would appear, though, that in many countries teachers might need to look for specialist publications rather than the chosen textbook for sources of challenges.

Even when textbooks have a more than adequate supply of challenging tasks, how these tasks are implemented in the classroom by teachers can affect the cognitive demand placed on students as they engage with the task. Henningsen and Stein (1997) and Stein et al. (1996) have found that the high level of cognitive demand of textbook tasks can be reduced by teachers who remove the challenging aspects of such tasks. Teachers often diffuse challenge in textbook tasks when students start to struggle, whereas struggle is what challenges are all about.

This practice is fueled by a belief that "all but the most able pupils [need] routine and relatively low level demands made of them" (Haggarty and Pepin 2001, p. 124). On other occasions teachers pre-empt difficulties that students might have with challenging tasks and intervene unnecessarily (Diezmann and Watters 2002, Stillman et al. 2007). One of the practices teachers must foster when using challenging tasks is the patience to allow struggle and not to intervene too early.

7.3.4 Managing the challenge

One of the reasons that is often given for a reluctance by teachers to use challenges in the classroom is a lack of knowledge of effective ways to manage all the divergent processes arising when challenging projects or tasks are used. Several strategies are offered that teachers who use challenges in their classroom employ to cope with this diversity.

The use of open questions by teachers (Sullivan and Clarke 1991) is often advocated in mathematics classrooms as a means of facilitating the deeper thinking required when using mathematical challenges. However, as Herbel-Eisenmann and Breyfogle (2005) point out "merely using open questions is not sufficient" (p. 484) to ensure this occurs. Often teachers use a questioning technique called "funneling" (for examples, see Herbel-Eisenmann and Breyfogle 2005, Goos et al. 2007, pp. 51–54) where questioning is used in such a manner that group or "classroom discussion [converges] to the thinking pattern of the teacher" rather than that of the students (Goos et al. 2007, p. 54). This can be desirable when students are early in their experiencing of challenging tasks in the classroom, particularly extended challenging tasks, as the teacher's intention may be to scaffold students along a particular solution pathway, so that they all experience the processes involved in solving such a task and have some insight into the complexity of their management of strategic resources in this process. If funneling questions are being used for this purpose, it is necessary that the teacher brings to the foreground the "meta-cognitive purpose of the questions" being used "and explicitly encourages students to start asking themselves these same questions. As students take responsibility for doing this, the teacher then fades the scaffolding" (Goos et al. 2007, p. 54). The levels of scaffolding provided and how long these are sustained are dependent on the level of schooling of the students undertaking the activity, the abilities of individuals (Diezmann and Watters 2002) and the previous experience of the students with challenging tasks.

Another questioning technique called "focusing" (for examples, see Herbel-Eisenmann and Breyfogle 2005, Goos et al. 2007, pp. 54–58) allows students to "articulate their thinking" (Goos et al. 2007, p. 58) about the challenge. During classroom discussion of a challenge, "the teacher asks clarifying questions and restates aspects of the solution to keep attention focused on the discriminating aspects of the particular student's solution. However, for this to be used effectively, the teacher must be able to see the essence of a mathematical task and, on a moment-by-moment basis, the essence of a task solution preferred by a student" (Goos et al. 2007, p. 58).

The guiding basis for group or class discussion is the lines of thought of the students, not the teacher (Doerr and English 2006). Use of this technique effectively requires that teachers have both well developed PCK and SCK (specifically Mathematical Content Knowledge (MCK)) with respect to the use of challenges. (See Chapter 6, particularly Section 6.2.1, and Leikin 2006, for an explanation of these terms.)

It is not unusual that students faced with challenges in the classroom, no matter what form they take, will "seek to reduce the task complexity by seeking specific input from the teacher" (Doerr and English 2006, p. 9). Often, however, teachers using extended challenges find themselves under sustained pressure as many students simultaneously seek their help with different parts of the task. In this situation a technique observed in the RITEMATHS research project (see Section 7.4.2.1) has proved useful. One or two students who the teacher knows has expertise in the part of the task in question are designated "experts" for the other students to consult for a limited period of time. The student experts are only to be consulted in the same way as a student would ask for assistance from the teacher with his or her own solution. They are not meant to tell or impose their solution on the student asking for assistance. For example, a student might be using a spreadsheet for a numerical solution to a task and decide to graph the data but not know how to do this. An announced spreadsheet expert can then be consulted instead of the teacher.

7.3.5 How can teachers introduce mathematical challenges into the regular classroom?

When challenging tasks are used with middle school students, teachers use a variety of methods to introduce the tasks to students. In Section 7.4.4.2, Olga Medvedeva's approach to introducing abstract mathematical challenges in the regular classroom is outlined. Her approach is based on the ideas of Davydov (1972/1990) and is basically a guided deep analysis of an abstract mathematical task beginning with the heuristic, solving a simpler problem in order to facilitate students' identification of the essential relationships and underlying structure of the problem, and to enable them to generalize to structurally similar problems.

When teachers use extended real-world challenging tasks in lower secondary school (Years 8 to 10), the cognitive demand required for task formulation by middle school students is high, potentially leading to a blockage in this early phase of the solution if students were to find the level of challenge too high for them to engage with the task. To help overcome this, teachers use a variety of methods to ensure students do not have difficulty interpreting the situation (Stillman and Brown 2007). These include physical demonstrations often involving concrete props in which students participate or observe; writing activities such as stating the aim of the task or the goal they have to reach; debating; dynamic computer simulations; and scale drawing or other forms of diagram drawing by both students and teacher. These activities serve to bridge the enactive and iconic worlds as well as being a means of introducing some structure, which reduces some of the cognitive load when there are no cues as to how to deal with the information in the situation presented.

7.4 Designing research for challenging mathematics classroom practices

7.4.1 Fruitful research designs for examining challenges

In the following sections, we will suggest and illustrate three types of classroom-based research designs that we believe are fruitful for exploring and researching classroom practices related to the role of challenging tasks in everyday mathematics classrooms. These are design-based research, Japanese lesson study and teaching experiments conducted by teacher researchers.

7.4.2 Design-based research

Design-based research (Collins et al. 2004) where iterative cycles of design, implementation, evaluation and refinement are used to improve educational practice has potential for researching classroom practices related to the use of challenges and is already being used for this purpose (Mason and Janzen Roth 2004, 2005, 2006, Stillman 2006). The purpose of design experiment research as a form of educational research is to explore the qualities of student understandings and their development of further understanding as the development of instructional resources progresses through these cycles (Lobato 2003).

Researchers and teachers work collaboratively to test theories in everyday classroom settings. Both theory and practice inform the design phases and are informed by what transpires during each teaching experiment (Palinscar 2005) as shown in Figure 7.9.

This example is from an Australian research project where the final implementation cycle for a set of extended tasks developed by teachers and researchers in the RITEMATHS project (extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/) is shown to be informed by a theory about the mediation of cognitive demand of the tasks by teacher actions, as well as wisdom of practice documents prepared by teachers about previous experiences with the tasks, and conditions for success identified by classroom observation in the previous two cycles.

In addition, during this third and final implementation it is indicated that data will be collected about student perspectives on practice. The project will be described more fully in the next section.

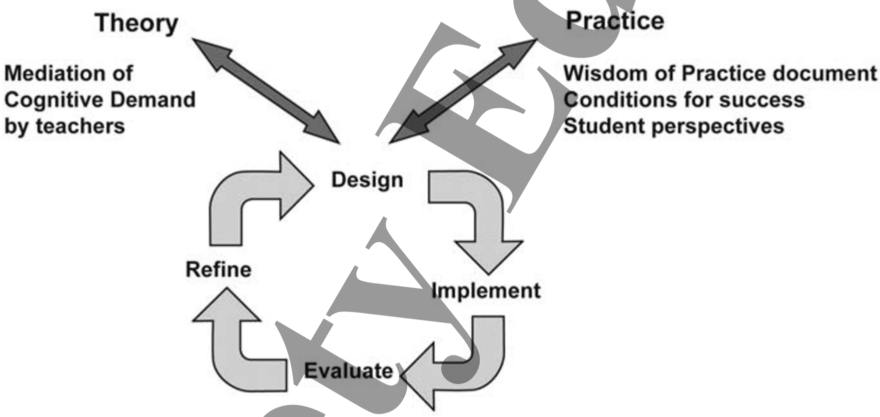

Figure 7.9: Design research cycle used for teacher/researcher meeting in the RITEMATHS Research Project

7.4.2.1 An Australian Example

An Australian research project investigated how teachers engineer learning environments in their classrooms to accommodate increased cognitive demands of tasks involving real-world applications and how students negotiate such challenges.

Teachers at this level of schooling make use of challenges of this form in order to facilitate students' development of an integrated view of necessary competencies to approach challenging tasks, to deepen knowledge of mathematical concepts and procedures they have already encountered in the classroom through application in novel and complex situations and the use of a wide variety of representations, sometimes simultaneously in the same phase of the solution and in different phases of the solution.

According to Kadijevich (2007), “the degree to which mathematical learning is successfully attained depends on the degree to which learner[s] can successfully cope with the coordination of different mathematical entities (competencies, activity, knowledge types, representations, etc.)” (p. 7). It was not the intention of the teachers who were designing and using extended tasks in this project to focus on just one of these aspects at a time, a common failing of mathematics teachers according to Kadijevich (2007).

The design and sequencing of extended investigative tasks so the cognitive demand matches students’ needs at a particular stage in the development of their mathematical, technological, and investigative procedure knowledge were issues of interest to teachers in the project. At the beginning of the project, it was hypothesized that management of cognitive demand of teaching tasks in technology-rich teaching and learning environments is mediated through careful tuning by the teacher of the interplay between (a) task scaffolding, (b) task complexity and (c) complexity of technology use (Stillman et al. 2004).

Task scaffolding is the degree of cognitive processing support provided by the task setter that enables task solvers to solve complex tasks beyond their capabilities if they depended on their cognitive resources alone. Task structure (e.g. carefully sequenced steps or a bald task statement), type of technology chosen (e.g. a real-world interface tool such as a data logger or a mathematical analysis tool such as a calculator), and whether technological assistance rather than by-hand calculation is privileged, all contribute to task scaffolding. Whose choice it is to decide all of these also contributes to the level of task scaffolding.

The complexity of a real-world task can be characterized by identifying and assessing the level of those attributes of the task that contribute to its overall complexity. These are potentially numerous aspects contributing via the mathematical, linguistic, intellectual, representational, conceptual or contextual complexities of the task (Stillman and Galbraith 2003).

Overall task complexity also varies along a continuum from simple to complex with the latter presenting a challenge for many students. For a particular task, students focus on only a subset of attributes when assessing overall task complexity (Stillman and Galbraith 2003) but these indicative cues contribute to their sense of challenge with the task.

One project school developed a lower secondary mathematics curriculum (Years 8–10) providing opportunities for engagement in extended investigation and problem-solving tasks, set in real-world contexts considered meaningful for students by the teachers. A major focus was in Year 9. During the Year 9 program, in keeping with local curriculum requirements, students were introduced to a mathematical model being used to describe the relationship between variables in a real situation, and then being used to predict an outcome in terms of a response variable when a control variable is altered. A series of extended real-world tasks was designed by one teacher, and the implementation and refinement of these tasks was studied in depth over the lifetime of the project.

Adoption and implementation by other classroom teachers who have different motivations for the use of real-world tasks and/or electronic technologies in the lower secondary years, means the integrity of the task is not guaranteed even if the design can be shown to be worthwhile. Some of the tasks from this first school were modified by members of the research team and teachers at other project schools where they were then implemented to fit the different conditions existing at that school.

One research question investigated during the project was how can tasks be implemented in different contexts (e.g. shorter time frame, teachers and students used to more highly structured investigations) but the level of challenge and engagement be retained. As Blum and Leiss (2007) point out, there is a lack of research as well as knowledge amongst teachers "of appropriate ways for teachers to act when diagnosing students' solution processes and when intervening in cases of students' difficulties" (p. 223). In particular, they highlight a lack of knowledge of strategies for "independence-supporting" interventions in demanding mathematical tasks (p. 230).

The difficulty for teachers is to decide when it is necessary to intervene and the nature of that intervention. In several implementations of the tasks in the project, it was observed that blockages to students' progress differed in type and cognitive demand (Stillman et al. 2007). When blockages were induced by a lack of reflection on interim results, or incorrect or incomplete knowledge, students were observed overcoming these blockages without teacher intervention when allowed to continue to struggle and resolve the situation themselves. Students appeared to do this by genuinely reflecting on their mental image of the problem and their approach. This reflection, sometimes stimulated by reflective questions built into the task booklet included the potential to recognize and hence rectify the application of incorrect or incomplete knowledge.

However, there were also instances of blockages observed where students were engaged in cognitive dissonance that prevented them from activating procedures to unblock their progress. These students persisted in attempting to assimilate, rather than accommodate (Piaget 1950) new contradictory information into their chosen structure for the task.

In this instance, successful teacher intervention that supported independent progression on the task involved the promotion of reflective learning where the teacher first tried to alter the students' current mental model through reflection and then the actions of the student. Thus, for example, rather than say a group's model was wrong, the teacher used the group's model to produce an incongruity that the students themselves were able to perceive before focusing attention on what actions might be employed to rectify the situation.

Being able to recognize when students are facing mere lower intensity blockages, which they should be able to resolve themselves if they engage in genuine reflection (self-initiated or orchestrated by the teacher or task sheet), is a critical teaching competency when using challenging tasks in the classroom. Students do learn from resolving these situations themselves, so by allowing them to persist rather than pre-empting when intervention is necessary, seems a pre-requisite for task implementation that does not reduce the challenge intended in the task. Likewise, being able to recognize and intervene in a manner that promotes reflective learning when students are experiencing cognitive dissonance beyond that which they can resolve themselves will also ensure the level of challenge and engagement with such tasks is retained.

7.4.2.2 A Canadian example: can students think like Archimedes?

Mason and Janzen Roth (2004, 2005, 2006, 2007) have used three cycles of design experimentation using instructional resources and strategies that deliberately and aggressively attempt to reorient students' approaches to learning mathematics toward the challenge of developing conceptual understandings rather than the stockpiling of additional procedures. This teaching program uses challenging mathematical activities (e.g. the Tennis Ball problem, Section 7.1) drawn from an academic study of the history of mathematics to accomplish its educational goals.

The curriculum design for these teaching experiments began with a study of the history of mathematicians' inquiries into circle relationships, especially those of Archimedes (Cuomo 2000, Eves 1960) and early Chinese mathematicians (Liu 2003). The history of mathematics provides a context for presenting a narrative of mathematics as an ongoing process developing, through thoughtful effort, our communal mathematical understandings (Arons 1988, Mason 1999). Historical framings of mathematics provide a way to put a human face on the mathematics that students encounter, offering context and story-lines to enliven the content, and role or process models for students to view as examples (positive and negative) for their own mathematical efforts (Mason 2003, Stinner and Williams 1998).

Although individuals' understandings do not necessarily follow the historical order in which mathematics developed, the historical structure of the discipline offers a framework within which educators can think about the educational sequence and structure of topics (Mason 2001, Rudge and Howe 2004). The history of mathematics gives us the mathematical version of the inquiry processes behind the content. Design experiment research may give us the mechanism, over time, to develop instructional versions of those processes that preserve the spirit, challenge and intrinsic rewards of the mathematician's original inquiries.

The research has now completed three full cycles of design, implementation and redesign. The first cycle of curriculum development incorporated the responses of mathematicians and educators to instructional activities attempting to represent the cognition of Archimedes that is summarized by the pi-based circle formulae all students learn to use. It is difficult to reconstruct the cognition of ancient mathematicians in producing their results in mathematics as often just the result with justification, not the thinking that produced it, is all that is recorded in surviving treatises. However, in the case of Archimedes, a copy of a letter written by himself entitled *The Method* and preserved on the surface of a palimpsest was found in 1906. This discovery and technological advances since its restoration to the scrutiny of academics in 2001 provide us with an insight into his thinking about the inquiry processes he carried out in investigating the relationships among the measures of circles, not just the final result (Hoffman 1988, Netz and Noel 2007). "Specifically, in the case of Archimedes' work with circles and with pi, *The Method* shows his thinking to be: (a) geometric: inscribing, circumscribing; (b) empirical: specific examples, specific quantities; (c) algebraic: general quantities, relationships; and (d) conceptual: extending sequences to infinity (early calculus)" (Mason and Janzen Roth 2005, p. 2).

For the students who were to be the audience for the curriculum activities designed by Mason and Janzen Roth, the qualities of the thinking of Archimedes that were desired were: "(a) tangible—practical, tactile, visible; (b) exploratory—questing, noticing, connecting; and (c) thoughtful—abstracting, wondering, generalizing" (Mason and Janzen Roth 2005, p. 2).

In the second cycle (Mason and Janzen Roth 2005), academic Year 9 students interacted with a prototype unit of instruction, including a sequence of six guided instruction student inquiries built around historical vignettes. In the final cycle (Mason and Janzen Roth 2006, 2007), the unit was adapted to challenge the understandings of the nature of mathematics held by a group of academic Year 12 students.

Each cycle has provided opportunities to better understand how to engage students in the challenge of deep understanding of algebraic formulae. First and foremost, students hold a wide range of beliefs and values about the nature of academic mathematics and its rightful place in school. Some held an instrumentalist view of mathematics (Ernest 1989, cf. traditional view, Dionne 1984) as a set of ideas and formulae to be learned for use in applications and in further study. Others held a conceptual view of mathematics as a collection of ideas with internal and interconnected coherence (cf. Platonist view, Ernest 1989, formalist view, Dionne 1984). Some saw mathematics as a field of present-tense inquiry in which they could participate through problem solving and inquiry in the kinds of thinking that are part of our culture and history (cf. problem-solving view, Ernest 1989, constructivist view, Dionne 1984); others saw mathematics as a collection of artifacts from past inquiries to be apprehended and remembered. When probed, students portrayed their beliefs as deeply embedded in their lived histories as learners of mathematics and as products of their personal experiences and their social environments.

Students' initial understandings of the functions and relationships that the circle formulae summarize (for mathematicians) were disappointingly shallow. Yet, the shallowness is clearly remediable, through engaging students in challenging mathematical inquiries related to those relationships (Mason and Janzen Roth 2005, 2006).

It is known that "beliefs have a powerful impact on our thinking and action, and they work for the rationality of our decisions. Thus to know students' beliefs is vitally important" (Furinghetti and Pehkonen 2000, p. 23) for those of us attempting to bring change through curriculum development. It is thus crucially important to find that students of all orientations towards mathematics, including instrumentally orientated students, accepted the challenge presented to them and were open to perceiving mathematics as Mason and Janzen Roth's historically grounded curriculum presents it, as a complex human enterprise available to their abilities.

7.4.3 Japanese lesson study

Japanese lesson study (Fernandez and Yoshida 2004, Isoda et al. 2007) "refers to a process in which teachers progressively strive to improve their teaching methods by working with other teachers to examine and critique one another's teaching techniques. . . .[It] functions as a means of enabling teachers to develop and study their own teaching practices" (Baba 2007, p. 2). It appears to be an ideal method to ensure classroom practices related to using challenges in mathematics classrooms can be improved in the long term by generating, accumulating and sharing "practitioner knowledge . . . within a system [that ensures the transformation of] such knowledge into a professional knowledge base" (Hiebert et al. 2002, p. 10).

Indeed, in sketching a brief history of lesson study in mathematics education in Japan, Isoda (2007) points out that it was the vehicle "for the emergence of teaching methods that focus on problem solving, which today are globally recognized as models of constructivist teaching" (pp. 13–14). This has led to "the problem-solving approach [becoming] well known as a major way of teaching mathematics in Japan" (p. 14).

Although lesson study methodology for research and professional development has spread to other countries, a particular feature that should be replicated if it is to be used in connection with challenges in the classroom is the pivotal role played by researchers and supervisors interested in this area. These "university researchers are expected to have accumulated deep knowledge of teaching practice . . . so that they can provide constructive and well-informed comments on lessons observed and the ensuing discussions" (Stephens and Isoda 2007, p. xx).

7.4.4 Teaching experiments or teaching-research

Teaching experiments where the researcher is also the teacher, and the primary purpose is to improve the teaching in particular classrooms where the research is being conducted, have proved a valuable source of insight into practices related to the use of mathematical challenges in the classroom. This type of teaching-research has its roots in action research and some of the promises and challenges of the design are discussed in Czarnocha and Prabhu (2004).

7.4.4.1 A North American example

Sriraman conducted several teaching experiments (Sriraman 2002, 2003a, b, 2004a, b, 2006) when he was the class teacher at a rural, mid-western high school in the United States of America with a heterogeneous group of Year 9 students enrolled in a beginning algebra course. The goal of these teaching experiments was to offer mathematical challenges not provided by the regular curriculum and to study how students abstract and generalize. In these teaching experiments students were given a series of combinatorics problems that they were to work on independently in their journals over an extended period of time.

In the first teaching experiment (Sriraman 2004b, Sriraman and English 2004) students worked on problems over a four-month period that included four Steiner triple arrangement problems. The problems were framed in the context of recreational arrangement problems such as inviting people over for dinner, schoolchildren on a walk and prisoners chained in triplets (see Gardner 1997 for examples). A Steiner triple system is an arrangement of n objects in triplets such that every pair of objects appears in a triplet exactly once. The students worked on the problems independent of other students and explicit instruction from the teacher. More than 50 per cent of the students were able to devise strategies that required a high level of abstraction and systematization to count all possible arrangements.

In the second teaching experiment (Sriraman 2002, 2003a, 2004a, 2004c, Sriraman and English 2004) students worked on a series of five problems of increasing complexity assigned every second week over a three-month period. The problems were all based on the pigeonhole principle, which states that if m pigeons are put into m pigeonholes, there is an empty pigeonhole if, and only if, there is a hole with more than one pigeon (also discussed in Chapters 1 and 6 of this volume).

The principle is believed to have first been stated by Dirichlet in 1834 under the name *Schubfachprinzip* ("drawer principle" or "shelf principle"). Almost 50 per cent of students were able to use the pigeonhole principle intuitively, by focusing "on understanding the structure of a given problem, in addition to engaging in reflective abstraction" (Sriraman 2004c, Sriraman and English 2004, p. 184). However, all students engaged in thinking mathematically, that is, constructing mathematical representations, reasoning, abstraction and generalization through the use of these problems.

A third teaching experiment (Sriraman 2003b, 2006) focused on Diophantine n-tuples. Students were introduced to elementary Diophantine equations as recreational journal problems. The problem chosen for investigation was the classic n-tuple Diophantine problem supposedly posed by Diophantus himself for solutions in the rationals. A Diophantine n-tuple is a set of n positive integers such that the product of any two is one less than a square integer. It was hoped that a very elementary version of the problem would kindle student interest and eventually result in an attempt to tackle the as yet unsolved 5-tuple problem in integers: does there exist a Diophantine 5-tuple?

The author initiated this problem by simply mentioning in class the 3-tuple problem: if one considers the integers 1, 3, and 8, then it is always the case that the product of any two is always one less than a perfect square. Indeed $1 \times 3 = 2 \times 2 - 1$; $1 \times 8 = 3 \times 3 - 1$; and $3 \times 8 = 5 \times 5 - 1$. This remark led students to wonder if other such 3-tuples existed. This problem was then assigned as a recreational journal problem. Students in the class found different 3-tuples, which led to the following questions naturally: What is the pattern for 3-tuples? Are there 4-tuples? These questions were the catalyst for an investigation of the unsolved 5-tuple problem over the course of the school year.

In this experiment students started working on problems independently but once they had received written feedback in their journals about their solution they were allowed to work with others. In one lesson each week there was a time for presenting solutions and defending strategies.

Many students were surprised at the difficulty of solving these seemingly easy problems as they began writing algorithms and computer programs to check for integer solutions. The progress of the problem depended completely on the "will" of the students. Mathematical notation was created as a class only after every student had expressed the particular idea in their own words. It was crucial that students initiated the process of conjecture, proof and refutation out of their need to resolve the unimagined difficulties that arose from a seemingly easy problem.

All students in this class willingly engaged in trying to solve one of the unresolved conjectures of our time over a seven-month time period through the process of conjecture, proof and refutation. The mathematics created by the students in trying to solve the classic 5-tuple Diophantine problem clearly indicates that students are capable of original thought that goes beyond mimicry and application of procedures taught in the classroom. The students' efforts did not resolve the 5-tuple problem by any means but these fourteen-year-old students persisted over an extended time period in trying to solve this challenging problem.

Sriraman (2006) concluded that the use of journals to nurture the process of conjecture-proof-refutation was invaluable to the teacher, as it was in the other teaching experiments. It allows for constant communication between the individual student and the teacher, and allows room for reluctant students to express themselves. Journals also allow the teacher insight into the affective drives of the students, as well as their capacity for originality and creativity.

Journal problems also allow for extended investigations that are student driven and convey that mathematics is an evolving process of discovery leading to generalities that are either proved or disproved. In Sriraman's experience, "journal problems of varying levels of difficulty, which are characterized by an overarching mathematical generality, is a novel and non-intrusive way of differentiating the curriculum for all students, and not simply for the able students" (2006, p. 7).

7.4.4.2 A Russian example

The work of Olga Medvedeva in researching her own classroom is described in Sriraman and English (2004). Her teaching approach when using challenging problems in the classroom is based on an implementation of Davydov's ideas (1972/1990) about different types of generalization in instruction. The approach is illustrated using a combinatorial problem, The Walking problem.

The walking problem: Consider the problem of walking in a 6 × 4 "rectangular city" (see Figure 7.10). In how many possible ways can a person move from a point A to a point X travelling only up and right along the edge of each grid? (Medvedeva 2002, cited in Sriraman and English 2004) (This problem in another setting is treated in Section 4.3.3.)

First, the students would be asked to work on a simpler problem such as the finding of a path in a smaller "rectangular city", say 2 × 3. Secondly, in order to facilitate abstraction, several students are then asked to read aloud the directions for their path using the words "right" and "up". These paths are then represented using the letters R and U and students write several such strings for different paths they discover. The idea is that the teacher will help students associate path length with string length. Thirdly, the problem of finding all possible paths in the smaller rectangular city can be restated in a generalized form as a combinatorial problem such as determining all possible five-letter strings with 3 Rs and 2 Us that represent a valid path from A to X. Fourthly, students then predict and enumerate paths for other dimensions of the "rectangular city". Finally, students are encouraged to conjecture and test a formula for an $n \times m$ city through specializing (i.e. using specific cases). According to Davydov, "scientific knowledge . . . requires the cultivation of particular means of abstracting, a particular analysis, and generalization, which permits the internal connections of things, their essence, and particular ways of idealizing the objects of cognition to be established" (1972/1990, p. 86). "The essence of a thing is none other than the basis (included in itself) for all of the changes that

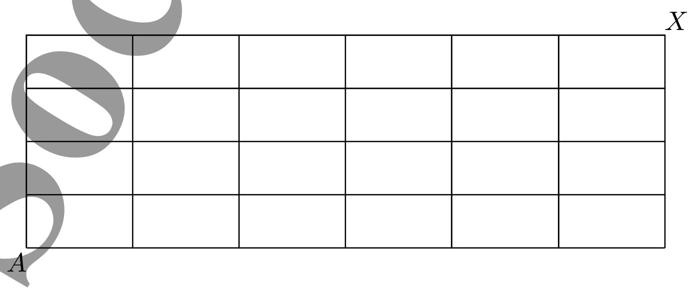

Figure 7.10: 6 × 4 "rectangular city"

occur with it in interaction with other things" (Rubinstein, cited by Davydov 1972/1990, p. 194).

Sriraman and English (2004) add that a further step in keeping with this notion of establishing the essence of the task could be that students be asked to pose problems that extend or are of a similar structure to the given problem. Thus, what Medvedeva is suggesting for teachers who follow this type of classroom practice with students is the use of challenging problems to foster "theoretical abstraction" in everyday mathematics classrooms (Mitchelmore and White 2007).

7.5 Conclusion

Classroom practice issues related to teachers providing mathematical challenges in their regular classrooms are addressed in this chapter. The regular use of challenges for all students in the everyday classroom is advocated. Design features of challenging tasks and how these might be varied for different levels of schooling and degrees of contextualization are addressed. Whilst it is pointed out that the intended curricula in most countries have not been exhausted as sources of challenges, combinatorics and number theory are fertile topics to explore for more challenges. Technology is suggested as a means of mediating the cognitive demand of some challenging tasks. Questions by students of themselves in analyzing a challenging task and the questioning engaged in by teachers in interacting with individuals, groups or the whole class during problem solution are promoted as the means by which students and teachers effectively manage the demand of mathematical challenges. It appears that the ideal of textbooks being written with the perspective of challenging activities as the motivation rather than an add-in is still a long way from being realized. However, we point out that challenging tasks in the classroom do not necessarily challenge students, as this depends on a number of factors, such as their implementation. These tasks are by no means "teacher proof".

In the final section, it is suggested that design-based research, Japanese lesson study and teaching experiments conducted by teacher researchers in their own classrooms are potentially fruitful research designs for the study of classroom practice issues related to the use of challenging tasks in the mathematics classroom. A number of case studies using these research designs are included showing how these research designs proved fruitful in practice: investigating such issues as the type of teacher interventions to be used when students are blocked in their progress so as not to remove the challenge, and how topics outside the normal curriculum can be used to promote student abstraction and generalization through challenges worked mainly independently.

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Chapter 8
Curriculum and Assessment that Provide Challenge in Mathematics

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In this chapter, we use selected case studies of assessment that provide challenge in mathematics to frame a discussion of assessment issues in relation to the provision of mathematical challenge. In the first part of the chapter, case studies from Singapore, Norway, Brazil and Iran are presented. In the second part, the relationship between the conception of the role of assessment, the features of assessment tasks and the provision of challenge, as well as how this relationship may affect issues of curriculum and may vary under different conditions, are discussed. Possible research questions in this area are included in the final part of the chapter.

8.1 Introduction

The influence of assessment on any facet of mathematics teaching and learning is difficult to ignore. In particular, it bears on the implemented curriculum. In many countries, the need to provide challenges for all students, rather than just the most able, is evidenced by the shift in focus from skills and procedures to problem solving. However, such changes will not be effective unless they are accompanied by assessment systems consistent with their goals. In addition, there have been calls to identify those students who are capable of meeting mathematical challenges. Thus, in several countries, curriculum revisions have been accompanied by new assessment practices, some on a national scale and others in the form of small-scale pilot projects.

This chapter documents four cases of assessment practices and discusses if and how providing challenges is related to variables associated with mathematics assessment tasks for students in different grades and of different levels of achievement and ability.

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We first describe how the studies documented in this chapter were selected. The bulk of the chapter is devoted to describing these studies. There follows a discussion on the relationship between the features and practices of assessment tasks and how these tasks encourage or discourage challenge. Does this relationship change under different conditions? Finally, we identify salient knowledge gaps in this area of study and suggest a potentially productive research agenda to fill these gaps.

8.2 The case studies

Assessment practices in Singapore, Norway, Brazil and Iran are given as case studies of practices at the primary, lower secondary and secondary levels respectively. Singapore and Norway implemented their practice for the whole school population, while Brazil targeted a selected group of students. Brazil's case is interesting in that, while its system is not part of the education system, it has a significant impact on all schools. The discussion on Iran is on a number of upper secondary assessments, not the whole school population, but the cases discussed cover a broad part of the population.

8.2.1 The case of Singapore: primary school level

8.2.1.1 Background and curriculum

Singapore has possessed a problem-solving curriculum since 1992 (Ministry of Education Singapore 1990). In 1997, the Ministry of Education made a call for the teaching of thinking skills in key subjects including mathematics. The initiative *Thinking Schools, Learning Nation* encouraged the explicit teaching of thinking skills and heuristics (Goh 1997). The mathematics curriculum was revised in 2001 to align it better with this initiative (Ministry of Education Singapore 2000).

In 2003, the Ministry of Education introduced another initiative to build upon the *Thinking Schools, Learning Nation* initiative. Schools were asked to help pupils develop good thinking habits or habits of mind under the initiative *Innovation and Enterprise* (Tharman 2003). In 2004, the Prime Minister of Singapore made a call for teachers to teach less in order to allow pupils to learn more (Lee 2004). This call underlines fundamental changes that are required to help pupils acquire a set of competencies that are valuable in a knowledge-based economy. Teachers are encouraged to focus on fundamental concepts and use the available time to motivate pupils in the learning process and to require them to figure things out.

More than a decade after the implementation of the problem-solving curriculum, schools have been encouraged to develop strategies to help every pupil learn competencies that are important for the 21st century. Alternative strategies are encouraged in order for pupils who have not done well in schools to also acquire the ability to solve problems. The education system placed an emphasis on every pupil having an opportunity to engage in some challenging situations.

In the latest revised curriculum (Ministry of Education Singapore 2007), pupils are encouraged to use calculators in the last two years of primary school mathematics. This signals a further de-emphasis on computational skills and an increased attention to solving a wider range of problems. Pupils now have the opportunity to engage in more challenging tasks.

8.2.1.2 National examination for primary schools

In Singapore, there are national examinations for students in Grades 6, 10 and 12. In this case study, we focus on the national mathematics examination for Grade 6 students. The items released from the primary school national examination (called the Primary School Leaving Examination or PSLE) over a period of five years (2000 to 2004) were analyzed to identify items that assess competencies beyond procedural knowledge.

The PSLE Mathematics is a two-and-a-quarter hour paper-and-pencil test that comprises forty-eight items including thirty-three constructed-response items. In thirteen of these items, students are required to communicate their solution methods. These thirteen items make up 50 per cent of the total score. For a small percentage of pupils who have not done well academically, an alternate examination called PSLE Foundation Mathematics is offered.

8.2.1.3 Test items

Items released from the PSLE Mathematics corresponding to the years 2000 to 2004 were selected for analysis. A total of 196 of the 250 items were released. About 80 per cent of all the items were released each year. The examination is in English, the language of instruction, although not necessarily the language used at home.

The released items were classified as procedural items or challenging items. Procedural items assess knowledge, basic skills, routine procedures and the solving of familiar word problems. Challenging items require competencies that are beyond routine procedures.

Here we give some examples of items classified as procedural items. The first item assesses knowledge. The second one assesses basic computation skills. The third one assesses a routine procedure to find area. Although the last item involves several steps in the solution, this type of word problem is familiar to the pupils. Such word problems are typically solved in a linear manner by identifying suitable operations and carrying out those operations.

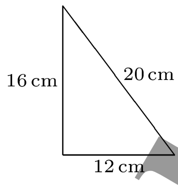
Procedural Item 1
What is the value of the digit 4 in 854 013?
(1) 4000 (2) 400 (3) 40 (4) 4

(SEAB 2005, p.1)

Procedural Item 2
Find the value of $\frac{3}{4} \div 6$.

(SEAB 2005, p.11)

Procedural Item 3
A piece of wire is bent to form the right-angled triangle shown below. Find the area of the triangle.



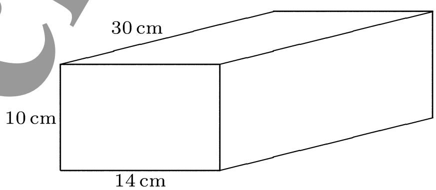
(SEAB 2005, p.24)

Procedural Item 4
Lynn joins Tang Fitness Club. She pays a membership fee of \$45. She also pays \$5.50 each time she books a badminton court. She books the court 30 times. How much does she pay the club altogether?

(SEAB 2005, p.24)

We now present some examples of items classified as challenging items. The first item requires pupils to select the appropriate computation to perform. An inappropriate computation includes dividing the volume of the block by the volume of a 3-cm cube. In the second item, pupils are unable to succeed by only applying the necessary computation skills, even if they possess them.

Challenging Item 1
A toymaker has a rectangular block of wood 30 cm by 14 cm by 10 cm. He wants to cut as many 3 cm cubes as possible. How many such cubes can he cut?



(SEAB 2005, p.25)

Challenging Item 2
Peter, James and Ruth had some stamps. James and Ruth together had 3 times as many stamps as Peter. The ratio of the number of stamps James had to the

number of stamps Ruth had was 3:7. Peter and Ruth had 310 stamps altogether. How many stamps did Peter have?

(SEAB 2005, p.46)

Among the 196 released items, about a quarter of them were classified as challenging. With such a significant proportion of the items considered to be challenging, it is not difficult to understand why the culture of challenge develops in a typical mathematics classroom.

In a study involving 38 pupils solving a collection of these problems, it was found that the ability to perform basic computations and follow procedures were not sufficient for pupils to be successful in problem solving (Yeap 2005a). "Big math ideas" were used together with basic computations and procedures in cases where pupils solved the problems successfully. The four "big math ideas" used by pupils in the study to solve challenging problems are classified as number sense, visualization, patterning and modeling.

8.2.1.4 Discussion

A significant proportion of the items in the national examination were challenging. In other words, all pupils are provided with opportunities and are expected to be able to handle challenge.

The national examination for pupils who are considered to be academically weaker also contained a small number of such challenging tasks. Here we give an example of a challenging item from the Foundation Mathematics examination. The inclusion of challenging items in the examination of the weakest pupils in the system clearly suggests that every pupil is expected to have some opportunity to engage in challenging tasks in mathematics.

Foundation Mathematics Challenging Item 1
Yong gave away a total of 42 marbles to Paul and Rashid. Rashid received twice as many marbles as Paul. How many marbles did Rashid receive?

(SEAB 2007, p.13)

Foundation Mathematics Challenging Item 2
The prices of two sizes of candles at a shop are shown below.

Small candles $2 each
Large candles $3 each

Alice bought an equal number of small candles and large candles. She spent $60 altogether. How many candles did she buy altogether?

(SEAB 2007, p.13)

Foundation Mathematics Challenging Item 3
Meili gave 1/8 of her salary to her mother and another 1/8 of her salary to her father. She kept the remaining $1680.

(a) What fraction of her salary did she keep?
(b) How much money did she give her mother?

(SEAB 2007, p.15)

This leads to the question of how average students, and even weaker ones, can be helped to handle challenging situations without losing their confidence and interest in the subject. In a study on the mathematics textbooks used in primary schools in Singapore, it was found that pictorial approaches were common in two of the most popular textbook series used (Yeap 2005b). The emphasis on the use of pictorial heuristics seems to have provided many pupils with a platform to handle challenging problems.

A study conducted on a typical primary six class in Singapore focused on the heuristics used by the pupils to solve challenging problems. In the next few paragraphs, heuristics used by pupils in the study that have the potential to help many pupils, not just the mathematically inclined ones, to solve challenging mathematics problems are described.

Race Problem
Tom and Gary ran in a race. When Gary had completed the run in 20 minutes, Tom had only run 5/8 of the distance. Tom's average speed for the race was 75 m/min less than Gary's.

(a) Find the distance of the race.
(b) What was Tom's speed in meters per minute?

Indra solved the Race problem by drawing a simple sketch shown in Figure 8.1. Using the basic idea of speed, Indra knew that every minute, the gap between Gary and Tom increases by 75 m. Thus, after 20 minutes, the gap between Gary and Tom is 1500 m, which is 3/8 of the distance of the race. The subsequent computations that Indra did to answer the first question were $1500 \div 3 = 500$ and $500 \times 8 = 4000$. The computations were simple because a diagram was used. A similarly simple computation was done to answer the second question.

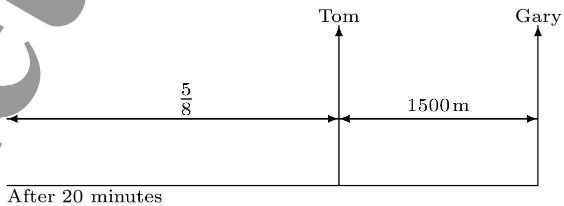


Figure 8.1: Indra's solution

Marbles Problem
At first Sara had 4/7 of the number of marbles Jack had. When Sara received 36 marbles from Jack, both had the same number of marbles.

(a) How many more marbles did Jack have than Sara at first?
(b) How many marbles were there altogether?

(SEAB 2005, p.17)

Janice solved the Marbles problem by using a method that is known as the "model method" in the Singapore textbooks. She used rectangles to represent unknown amounts. Her initial diagram is shown in Figure 8.2.

Figure 8.2: Janice's solution

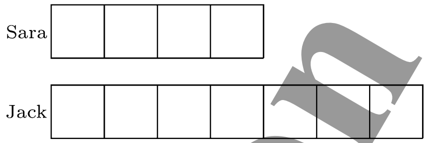
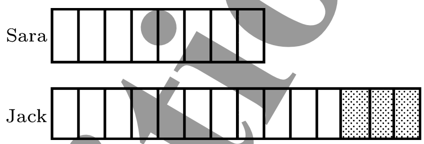


Figure 8.3: Janice's solution



Subsequently, Janice modified her diagram as shown in Figure 8.3 and did the computation $36 \div 3 = 12$ and $6 \times 12 = 72$ to answer the first question and $22 \times 12 = 264$ to answer the second question. She did not have to do cumbersome computations involving fractions because she used the diagram.

These pictorial approaches used by the pupils are the common ones found in the textbooks (Yeap 2005b).

In this line of thought, the role of graphic representation in empowering school children to confront more challenging activities of problem solving is important. For almost two decades competitions such as the "Future Olympian" competition for primary schools in Colombia has generated visual representations of concepts, methods, problems and solutions, as shown, for example, in Figure 8.4, which enable young primary students to think about and successfully solve problems generally associated with more advanced strategies and representations such as those of algebra.

8.2.2 The case of Norway: lower secondary education

8.2.2.1 Background and curriculum

The present system of lower secondary education in Norway was introduced in 1997. Basic schooling includes a ten-year block of compulsory education, the last three grades of which make up lower secondary education. Whereas the class teacher is most common in primary school, the subject teacher is more dominant in lower secondary school. The basic school is not streamed, and this is the situation for all subjects.

In primary school there is no formal grading, and it is very rare for a student to repeat a year, since there is more emphasis on the social functioning of the group. Until 2004 there was no national testing in the first 7 years. National tests were introduced in grades 4, 7 and 10 and the first year of upper secondary in 2004 and 2005. There was no national testing in 2006. Starting in 2007 there are national tests at the beginning of years 5 and 8.

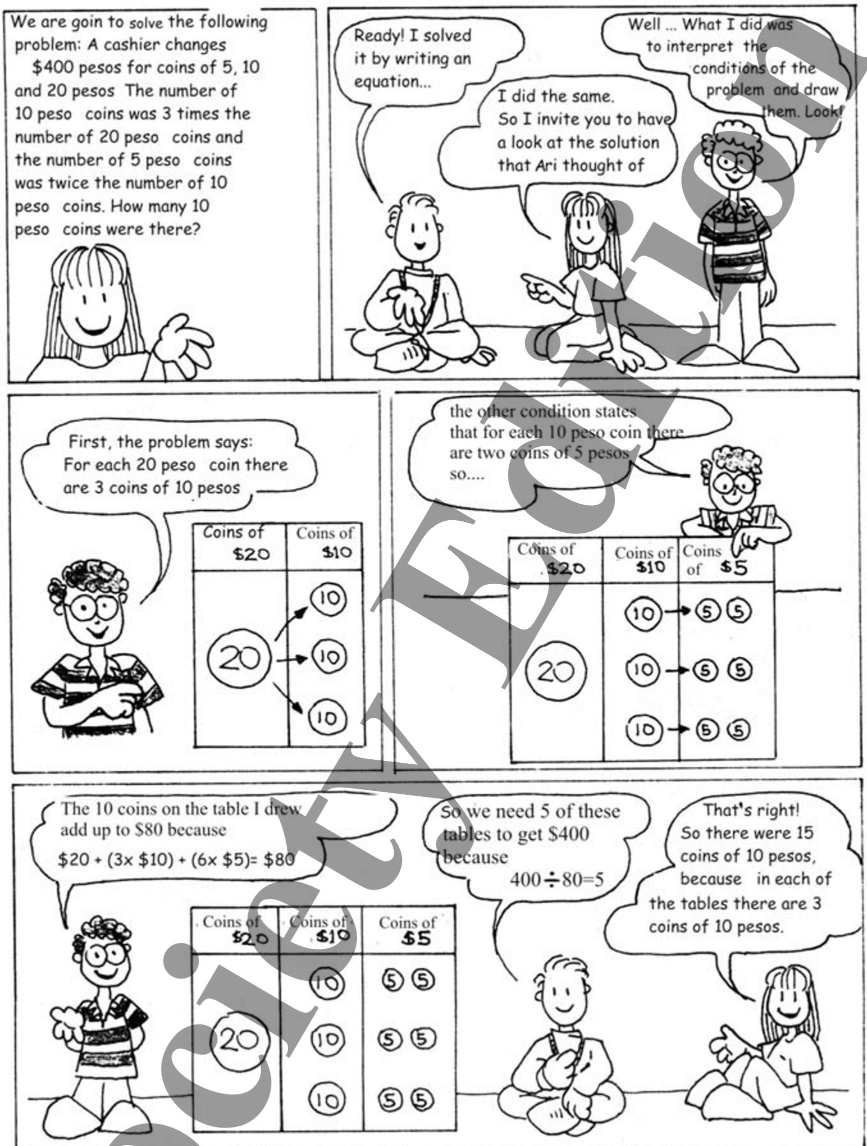


Figure 8.4: A Colombian cartoon representation

In lower secondary school formal grading is introduced. A 6-point scale is used. It should be noted that all grades are passing grades. At the end of the tenth grade there are also national exams; there are written national exams in the subjects English, mathematics and Norwegian, and possibly oral exams in these as well as other subjects.

Traditionally Norway has been a very homogeneous society. Education since the mid-20th century has been dominated by social democratic ideas, as well as Christian (Lutheran) ideas. As a consequence it is felt that compulsory education should give the same opportunity to all. Important choices for the students will come at a later age; no important choice has to be made until the students have completed basic compulsory school. Selection for the different streams of upper secondary education has been based on grades from lower secondary. It should also be noted that every student has the right to three years of upper secondary education. This right, however, the student will lose after some years.

Many teachers feel that the impact of exams and testing should be kept to a minimum, at least during the years of compulsory education, and in fact, in the mid-1970s there was even a government-appointed committee that suggested there should be no formal grades in the then newly introduced 9-year compulsory school. However, this was not to become official policy because of strong opposition.

Another basic principle is that the selection of students should be handled by the lower school level. Hence there are leaving exams, but no entrance exams in "mainstream" education. The type of upper secondary education available to a student is based on performance in lower secondary, and the type of tertiary education is based on performance in upper secondary school. Hence the leaving exams have high stakes for the students.

8.2.2.2 Traditional examination for lower secondary schools

Norway introduced lower secondary schooling as part of compulsory education with the reforms of the mid-1970s. Before this reform lower secondary education consisted of different types of schools, that is, different kinds of vocational schools. The traditional exams in the more academic schools were written exams; oral exams were not usually given to students at this stage.

Written exams in the Norwegian tradition were national exams, and quite extensive—five- or six-hour exams that covered a large part of the curriculum were the rule. Before the reform the student's grade on an exam determined the final grade in the subject. There were also course grades given by the teachers. With the reforms of the mid-1970s different ideas were introduced. The grade given for a course should be a combination of the course grade given by a teacher and the grade given on the exam. Moreover, oral exams became more common. The underlying philosophy was that the grade should reflect different parts of the pupils' competence. Moreover, any test—written or oral—should focus on what the students' know, not on what they do not know. There is no tradition of multiple-choice problems.

During the 1980s and up to the reforms of 1997, there was experimentation with different exam formats. One reason for this was the introduction of calculators in lower secondary education. These were introduced by the revised curriculum in 1987.

8.2.2.3 The written exams

There have been different developments concerning the exam format. With the introduction of technology the exam was divided in two parts, one without and one with the use of technology. The students would receive the whole test at the beginning, and then get a calculator when they handed in the first part. This system is no longer in effect—the calculator is now allowed on the whole test.

However, the test has been divided into three parts having different foci. One part is what we might call “basic skills”, one is “problem-solving” and one is more of a “project” task.

In recent years, the practice has been introduced whereby students are given “general” information to be used in the solution of a problem. They will then have to find the relevant information themselves. An example from the Grade 10 examination given in the spring of 2006 follows.

Example: Norwegian Metrology Service

Controlling weights and measures has been important historically for many centuries. Regulations for weights and measures were included in Magnus Lagabøter’s Law for Norway in 1276. It specified that any trader should use controlled instruments for measuring. Cheats should be punished. Today, the Norwegian Metrology Service is the enforcement agency. The following three examples indicate the importance that measuring instruments have for our economy.

- On average, a household buys electricity for NOK 20000 per year. The measuring instrument used to calculate the amount of consumption has not been subject to quality control by the authorities. For example, a 5 per cent error will represent a significant amount during a year.
- On average we pay NOK 15000 for gas for our car each year. Then it is good to know that the gas pumps are required to be accurate within 0.5 per cent. This is a requirement to which the metrology service pays close attention.
- A household on average buys fruit, green vegetables and meat which are weighed directly in the store for about NOK 5000 per year. This is done by weights with an accuracy of 0.2 per cent (a requirement of the metrology service).

TASK

Look at page 6 in the booklet about the Norwegian Metrology Service. Choose data from one of the three examples and construct a text for a mathematical problem. In the task, you can in addition use your own figures.

Solve the problem that you have constructed.

The student is challenged to assimilate the information, decide what is mathematically significant and select the ingredients for a proper mathematical problem for which a correct solution must be constructed.

Another factor which should be mentioned is that there is a certain system for arranging written exams. Due mainly to the cost of having three written exams for all students, a student presents for only one of the exams, in English, mathematics or Norwegian. The students are given notice of which one of these subjects they will be examined in only two weeks prior to the examination and the choice itself is made in a national draw.

Another line of development is the introduction of what we might call “preparation time”. Two days before the exam, information on possible topics covered by the project part of the exam is given to the students. This development must be seen as a step to make the whole exam more comprehensive.

8.2.2.4 The oral exams

The traditional oral exams were the responsibility of the school, or a group of schools in a region. This is now being changed; the school authorities on a national level are beginning to require that the exams follow some standard. They are also publishing sample tasks for teachers. However, the examination is still the responsibility of the teacher. The school authorities provide guidelines.

As for the written exam, the students for the oral exam are selected through a national draw. Unlike the written exams where a whole class (group) is drawn, the draw for the oral exam is for individual students.

The teachers are given information about the subject for the examination one week in advance. They are not supposed to give any information on the subject to the students at this time. The teacher selects a topic and prepares for the examination that she will be conducting. The students are then given two days notice (exclusive of holidays) of the subject as well as the topic. It is expected that the students make a presentation at the beginning of the examination, after which the teachers will ask questions. There is an external grader present during the examination who has the opportunity to ask the student questions as well. Each student is examined for 20 minutes.

A sample topic as given by the school authorities is presented below. It is up to the teacher to decide how detailed her instructions to the students will be.

Example: Building a house

You have a property of 800 m^2 and want to build a house there. The house should cover the area 100 m^2.

What might the property look like, and where on the property would you locate the house? (Use the scale 1:250.)

You will buy materials for the house, and therefore will need to know how the house will look from all sides.

Make a sketch of all sides of the house with doors and windows. (Remember to include measures.)

You will construct the interior of the house. What type of rooms do you want and how will they be placed in the house?

Draw a plan of the house in a suitable scale; include the doors and the windows.

To get an overview of the materials used and price, make a list of what you need (windows, doors, outside paneling, inside paneling, roof, insulation, etc.).

How much do you need of each type? Make a list of necessary materials and prices with and without tax. Use a spreadsheet.

8.2.2.5 Course grade and final grade

The student also receives a course grade from his teacher. This grade could be based on tests (oral or written) during the year. The final grade for the course is a combination of grades for the different components. The final grade might be only the course grade if the student is not drawn to have any exam in mathematics at the end of compulsory education. The student could perhaps have a written exam in one subject and an oral in another.

8.2.2.6 Discussion

The examination system in lower secondary education in Norway is now changing with the introduction of the new curriculum. What is seen are plans for a more comprehensive system of assessment with several components such as national testing, diagnostic tests and examinations.

However, the basic principles mentioned are still in effect, that is, no formal grades for the first 7 years, all grades are passing grades. There are changes in the test construction as well: multiple-choice questions are slowly making their way into the various forms of testing. Another issue is the effect of having technology available for the exams. Since more and more computers are going into schools, beginning in lower secondary, new forms of testing are being developed.

8.2.3 The case of Brazil: upper primary and lower secondary levels

8.2.3.1 Curriculum and background

No member of the writing team is Brazilian, but Brazil is a country of interest for this Study. The material of this section is obtained from personal contact and the Brazilian site portal.mec.gov.br/seb/.

National Curricular Guidelines in Mathematics in Brazil
In the national curricular guidelines for secondary schools in Brazil it is stated that the guidelines have their roots in a "wide discussion with the technical teams of the state systems of education, with professors and students from the public school network and with representatives of the academic community. The goal of the materials produced is that of contributing to a permanent dialogue between teachers and schools concerning teaching practice. The quality of schools is essential to the inclusiveness and the democratization of opportunities in Brazil; it is the task of all to confront the challenge of offering quality basic education that will enable the student to be included in the development of the country and in his/her consolidation as a citizen." (portal.mec.gov.br/seb/arquivos/pdf/book_volume_02_internet.pdf)

The school system is divided into nine years of basic schooling and three years of middle schooling, corresponding to secondary or high school education.

Mathematical thinking

What are the general goals of the school system? The understanding of the logical structure of mathematics by secondary students must be broadened and deepened. The resources of mathematical thinking should be emphasized: imagination, intuition, inductive reasoning and logical-deductive reasoning. Students should distinguish between mathematical and empirical validation, and progressively become adept with the deductive method.

The guidelines also state that it is pertinent to make a special recommendation in the implementation of public policies that give priority to the continual training of mathematics teachers working in secondary school in order to build up autonomy in their teaching and professional decisions.

Content

Secondary school mathematics is divided into four large categories: numbers and operations, functions, geometry and analysis of data and probability. A description is given of the content related to each of these areas.

It is furthermore stated that sometimes, intentionally, topics not treated fully in elementary and basic schooling are taken up once again in secondary school. It is the moment to consolidate certain topics of elementary school mathematics that require explanations whose comprehension needs more maturity than that possessed by basic school students.

Technology is seen as beneficial. In developing the curriculum of secondary school, both teachers and schools are encouraged to work toward the integration of knowledge, especially interdisciplinary work, which requires cooperation and is a challenge for teachers.

An extract stressing problem solving

The following extract shows clearly the importance given to the development of students' problem-solving abilities in the secondary curriculum.

> The greater part of the content of mathematics in secondary school is devoted to mathematical models of a continuous nature, the real numbers and geometric spaces. The study of the geometry of functions of a real variable fits in this context, reflecting the fundamental role of the calculus in science.
>
> However, in the course of the twentieth century, new technological necessities pertaining to the introduction of computers, which rely upon discrete mathematics, have forwarded much development in this field and incited important development of discrete mathematical models. In this process a significant development of the area of combinatorics and the mathematics of finite sets has taken place. In secondary school, the term "combinatorics" is usually restricted to the study of problems of counting, but this is only one of its aspects. Other types of problems could be the subject of study in school. For example, those related to finite sets with statements that are easily understood but not necessarily easy to solve, such as the Konigsberg bridge problem.
>
> Problems of this type can be used to develop a series of important abilities: modeling a problem using the structure of a graph, identification of situations that do not have a

solution, convergence toward the discovery of a general condition for the existence of a solution. Many other examples of combinatorial problems can be treated in a way similar to the Königsberg bridge problem; examples such as determining the shortest path in a transportation network or determining an efficient trajectory for collecting garbage in a city are given in the guidelines. (portal.mec.gov.br/seb/arquivos/pdf/book_volume_02_internet.pdf)

8.2.3.2 The Brazilian Mathematics Olympiad for public schools

In Brazil the *Olimpiada Brasilera de Matematicas para la Escuela Publica* engages students, not in the form of a required examination but using incentives to have all students in grades 5 through 8 of the public schools take part in an Olympiad. Each school is encouraged to register all its students in the event and will be responsible for their participation.

The objectives of the Olympiad are to:

- stimulate and promote the study of mathematics in the public schools;
- identify young people with talent and give them incentives to go into scientific and technological studies;
- give incentives for the in-service training of teachers in the public schools contributing to their professional development;
- contribute to the integration of the public schools with the public universities, institutes of research and scientific societies;
- promote social inclusion through promoting the dissemination of knowledge.

The incentives for participation cover four levels of interest. First, the students can win a medal and the right to attend a special enrichment course offered by the organizers of the Brazilian Mathematics Olympiads.

Secondly, the teachers—according to the distinction obtained by their students—may win as a prize the right to attend an in-service course. This might enable them to create better challenging environments or give them the opportunity to work on challenging problems for their students.

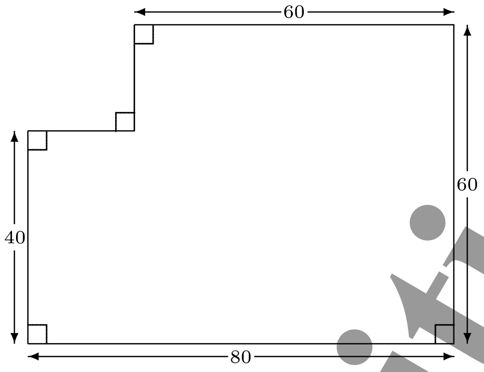
Thirdly, schools are awarded prizes, such as laptop computers and mathematics books for their libraries, according to the number of students who receive medals in the Olympiad from that school.

Finally, a town can win government funding for sports facilities and trophies, according to the achievements of its schools in the Olympiad.

There are two rounds of the Olympiad. In the first, the problems are engaging and set in a multiple-choice or short-answer format. Students have one hour to solve 10 problems. Examples are given below.

Level 1, Grades 5 and 6

Problem #8 Daniela wants to enclose the property shown in the diagram. In the diagram all pairs of consecutive sides are perpendicular and the lengths of some of the sides are shown in meters. How many meters of fencing will Daniela have to buy?



(A) 140 (B) 280 (C) 320 (D) 1800 (E) 4800

Problem #10 A team earns 3 points for each win, 1 point for each draw, and no points for a loss. Up to now, each team has played 20 games. If one of the teams won 8 of their games and lost 8, how many points does that team have right now?

(A) 23 (B) 25 (C) 26 (D) 27 (E) 28

Level 2, Grades 7 and 8

Problem # 6 This is the same as problem number 10 of Level 1. This illustrates that many problems will be of interest and challenge students of different grade levels, and in fact this is common in many competitions.

Problem #7 Twenty people decide to rent a boat for \$200.00 for an outing. This amount is to be divided equally among all of them. On the day of the outing, some of the people decide not to go. On account of this, each person who does go on the outing has to pay \$15.00 more. How many people decided not to go on the outing?

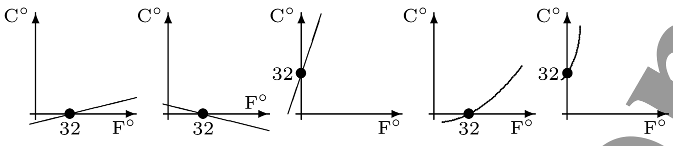
(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Level 3, Grades 9 and 10

Problem # 2 A rectangular sheet of paper of length 10 cm and width 24 cm is folded in half to make a double sheet of length 10 cm and width 12 cm. Then the folded sheet was cut down the middle parallel to the fold making three rectangular pieces. What is the area of the largest of these three pieces?

(A) 30 cm^2 (B) 60 cm^2 (C) 120 cm^2 (D) 180 cm^2 (E) 240 cm^2

Problem #10 In Brazil the Celsius scale is used for measuring temperature while in some other countries the Fahrenheit scale is used. To convert temperatures from the Fahrenheit scale to the Celsius scale, 32 is subtracted from the temperature in Fahrenheit degrees and the result is multiplied by 5/9. Which of the following graphs represents the relation between the measure of the same temperature in Fahrenheit degrees (indicated by $^\circ$F) and in Celsius degrees (indicated by $^\circ$C)?



(A) (B) (C) (D) (E)

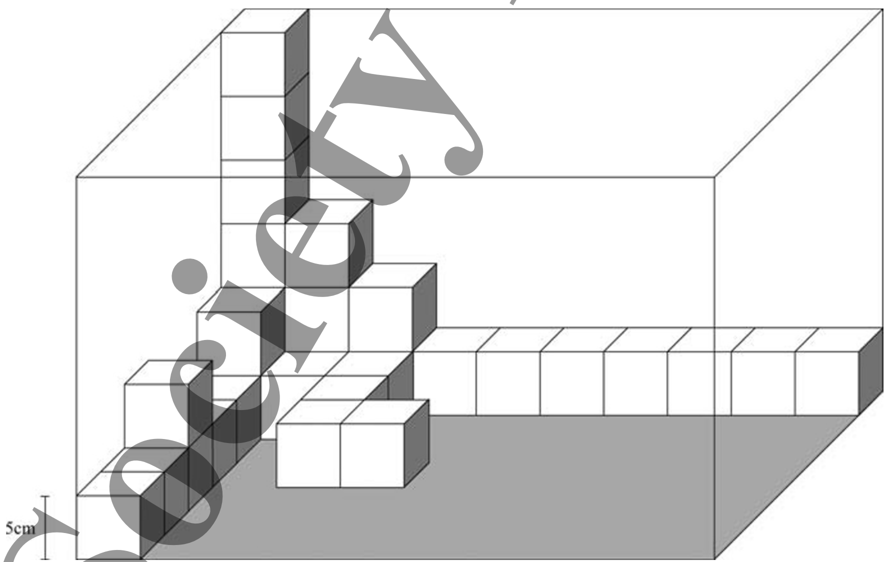
Under these conditions of voluntary participation schools are nevertheless encouraged to register all of their students, coupled with important incentives. In the first version of the Olympiad held in 2005, there were about 12 million students in the first round. Indeed, the coverage resembled that of a required national exam.

A second round is set for the top 5 per cent of the students from each school. It consists of problems for which a full written solution was required. The papers of the 600,000 students taking part in this round of the Olympiad are graded by professors of the public universities of Brazil.

Examples of the problems are shown below.

Level 1, Grades 5 and 6

Problem #3 (of six problems) Emilia wants to fill a box with wooden cubes with edge length 5 cm. As shown in the figure the box has the shape of a rectangular block and some cubes have already been placed inside.



(A) How many cubes has Emilia already put in the box?
(B) Calculate the length, width and height of the box.
(C) How many cubes are required in order to fill the box if Emilia continues to place them around those shown in the figure?

Level 2, Grades 7 and 8

Problem #3 (of six) Jeremy's pick-up can hold loads of up to 2000 kg. He accepts a job of transporting 150 sacks of sugar that weigh 60 kg each and 100 sacks of flour that weigh 25 kg each.

(A) Will Jeremy be able to do the job in five trips? Why?
(B) Describe a way he can do the job in six trips.

Level 3, Grades 9 and 10

Problem #3 (of six) In a certain city there are only two different fleets of taxis, Dona Leopoldina and Don Pedro II. Dona Leopoldina charges a fixed base fare of \$3.00 plus \$0.50 for each kilometer travelled and Don Pedro II charges a fixed base fare of \$1.00 plus \$0.75 for each kilometer travelled. Three friends, Bento, Sofia and Helena work in that city and always take a taxi when returning home from work. To pay less, Helena always takes taxis belonging to Dona Leopoldina's fleet and for the same reason, Bento always takes taxis from Don Pedro II's fleet. Sofia takes taxis from both fleets because she pays the same price for either one.

(A) How much does Sofia pay to take a taxi home from work?
(B) Which of the three friends travels the shortest distance between work and home?

8.2.3.3 Discussion

This can be seen as an interesting alternative to compulsory national testing, and one in which more ambitious objectives with respect to exposing a great many school children—and their teachers—to challenging mathematical tasks can be realized. Notice that the high stakes of outcomes regarding future educational opportunities, such as streaming or university entrance, are absent from this experience. However, incentives strictly pertinent to further opportunities to confront more challenging mathematical learning seem to put the emphasis exactly where it should be. The fact that teachers and schools are encouraged to have all students take part reflects both the belief that some challenging mathematical situations can be handled by all students, and the fact learned from experience that more routine mathematical learning does not always enable a teacher to gauge the mathematical potential of each student. It is a creative testing design that deserves attention and it has great possibilities for adaptation in many countries.

8.2.4 The case of Iran: upper secondary education and beyond

8.2.4.1 Curriculum and background

The last half century has witnessed three phases in the educational program of Iran. In the first period (1956–1973), apart from pre-school education, there were six years each of primary and secondary school. Each level concluded with uniform national final examinations. However, to accommodate newer topics such as set theory, modern algebra, linear algebra and solid analytic geometry, the Ministry of Education revised the secondary curriculum in 1973 and changed the regime to five years of primary school, three years of "guidance school" (middle school) and four years of high school, concluding with a national final examination to qualify for diplomas.

The teachers were not familiar with the new topics and at first the syllabus of the mathematics-physics branch was limited in scope. This was ameliorated during the next few years by omitting some parts of the books and providing teacher professional development. After 1993, the authorities decided to change the curriculum. They reduced the high school period from four years to three years and, more importantly, they changed the yearly system of education to the unit system with each educational year split into two terms. At the end of the three-year period, some students are admitted to a pre-university program of two semesters. Mathematics is now taught in a series of four "books" followed by simple calculus. Differential and integral calculus, previously studied in a freshman course at university, is now studied.

8.2.4.2 The range of assessment in upper secondary level

Assessment of mathematics is given special importance in Iran. There are five important levels of assessment of mathematics for students in the upper secondary level and beyond, each of which provides opportunities for students to deal with challenging mathematics:

1. National University Entrance Examination;
2. High School Students Mathematics Olympiad;
3. Special School entrance assessment;
4. Iranian Mathematical Society Mathematics Competition for University Students;
5. International Scientific Olympiad on Mathematics for University Students.

These assessment experiences have implications for the teaching of mathematics to high school and university students, as well as for the teachers and lecturers, and created challenges for students to learn mathematics. In Pourkaziemi (2006), teaching challenges in mathematics is explained. In the following section we look briefly at these assessment processes and the challenges that are created for non-mathematics majors.

8.2.4.3 The national university entrance examination

Passing this examination, in which mathematics is the most important part, is necessary for university admission. It tests students on the work of the last four years of school, a syllabus that includes geometry, algebra, trigonometry, analytic geometry, differential and integral calculus, discrete mathematics and statistics.

Despite extensive preparation by teachers using previous examinations, only about 25,000 of 400,000 participants are successful. There are 55 multiple-choice questions based on the material in the school books to be answered in 85 minutes. They are partitioned into three groups according to difficulty.

The first group can be answered by almost all the students who have prepared for the examination. The second harder group can be answered by fewer students. Only creative and careful students can answer the third group of questions. These questions are new to the students and difficult, particularly in the time available, and success with them can make a huge difference in the result of the examination.

Asking creative questions allows for the identification of gifted students who can solve unfamiliar questions efficiently and fast. Once a new type of question is used, it becomes part of the stock of questions prepared by future candidates who try to find the shortest solution. These questions become a challenge for both student and teacher to attempt them. The National University Entrance Examination has been used since 1964 (www.sanjesh.org). Here are two questions from the first group:

Problem 8.2.4.1 The equation $mx^2 + 5x + m^2 - 6 = 0$ is given; find the parameter m in which this equation would has two real roots with inverse value.

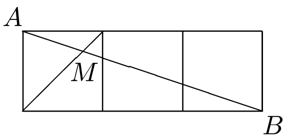
(1) -3 (2) -2 (3) 2 (4) 3

Problem 8.2.4.2 If a, b and c were the roots of equation $2x^3 - x^2 - 5x - 2 = 0$, what is the value of $a^2 + b^2 + c^2 + {}^3/_4\, abc$?

(1) 5 (2) 6 (3) 7 (4) 8

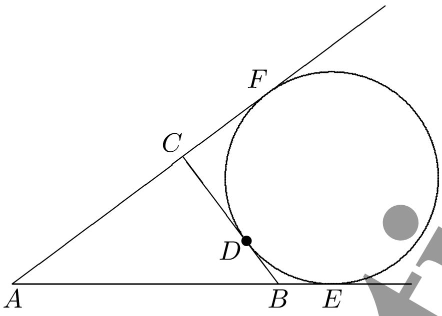
Students in high school have two years of Euclidian Geometry. The teaching of geometry is very important, especially because it familiarizes students with inferential reasoning. On the other hand, it enables students to observe and perceive the relationship between pure and applied mathematics through utilizing the coordinates. Some examples of questions in geometry are given below. Students are expected to answer each question within one minute.

Problem 8.2.4.3 In this figure three squares with same length (1 cm) are shown. What fraction of the length of BM is the length of AM?



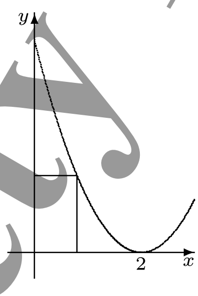
(1) 1/3 (2) 1/4 (3) 2/9 (4) 1/5

Problem 8.2.4.4 In the figure, the point D of tangency between BC and the circle ranges between the two fixed points E and F. What happens to the circumference and the area of the triangle ABC?



(1) The circumference changes; the area changes.
(2) The circumference changes; the area is fixed.
(3) The circumference is fixed; the area changes.
(4) The circumference is fixed; the area is fixed.

Problem 8.2.4.5 Two sides of a rectangle lie along the coordinate axes; the fourth vertex lies on the curve $y = (x-2)^2$, with x lying in [0, 2]. What is the maximum area of this rectangle?



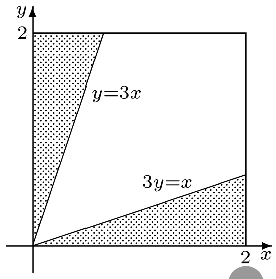
(1) 11/9 (2) 32/27 (3) 10/9 (4) 28/27

Finally, we have two questions from the third group.

Problem 8.2.4.6 Two numbers are chosen randomly in the interval (0, 2). What is the probability that their quotient is less than 1/3?
(1) 1/9 (2) 1/6 (3) 1/4 (4) 1/3

Solution: The sample space is
$S = \{(x, y) | \ 0 < x < 2 \text{ and } 0 < y < 2\}$.
The given outcome is
$A = \{(x, y) | \ x/y < 1/3 \text{ or } y/x < 1/3\} = \{(x, y) | \ y > 3x \text{ or } y < 1/3\}$.

The sample space S and outcome A (shaded area) are shown in the figure.



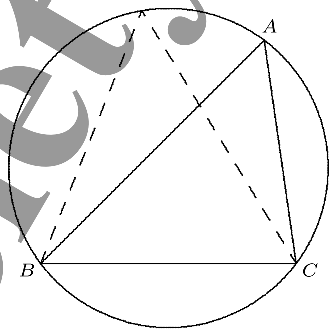
The desired probability is the ratio of the area of A to the area of the square:
$(2/3 + 2/3)/4 = 1/3$.

Problem 8.2.4.7 In triangle ABC, the two vertices B and C are fixed with $BC = 6$ cm, while A moves in such a way that angle A is equal to 60°. The bisector of angle A meets the circumcircle of ABC again in D.

What is the length of the segment BD?

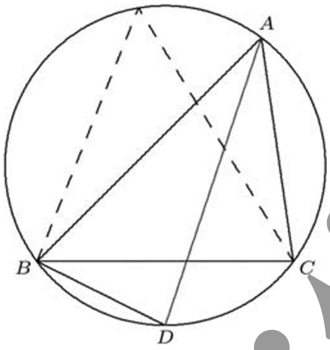
(1) $6^{1/2}$ (2) $3^{1/2}$ (3) 2 (3)$^{1/2}$ (4) 4

First Approach: Students are challenged to come to grips with why all angle bisectors have a point in common; why is this so? The clue comes from the constancy of the angle A, which students should recognize as lying on an arc of a circle.



From here it is then a matter of realizing that the bisector of angle A must then always pass through the midpoint of the minor arc BC. This allows the student to construct the appropriate diagram.

Once this is done, getting the length of BD is standard. For example, the student might recognize that the radius of the circle from B and the segment BD are two sides of an equilateral triangle. Or else, they might realize that it is now enough to look at a particular position of A, in particular, when AD is a diameter. In either case, we are led to the answer (3) as correct.



Second Approach: As vertex A moves on the circumcircle, B and D are fixed, thus length of BD is invariant. When A reaches the perpendicular bisector of BC, the triangle ABC becomes equilateral, and then $AD = 2R$, and angles $DAB = 30°$, $BDA = 60°$ and $ABD = 90°$, therefore $BD = 1/2$ and $AD = R$.

From Pythagoras' theorem in triangle ABD, $R = 2(3)^{1/2}$. Thus answer (3) is correct.

8.2.4.4 High school students mathematical Olympiad

A high school students mathematical Olympiad has operated in Iran since 1984. This is a three-round competition open to year 9 students with an average above 17 out of 20. Publications and special classes in many schools help candidates prepare. The first round is multiple-choice with four options for each question. Out of 16000 candidates in 2007, 800 were invited to participate in the second round. Finally, 40 students competed in the final round (with essay-style questions) for bronze, silver and gold medals. The winners of the gold medals will make up the Iranian IMO team, which has earned many gold medals in the international competition. The winners of the third stage examination also have preferred entrance to Iranian universities (olympiad.sanjesh.org/).

8.2.4.5 Assessment of the gifted high school students to enter special schools

After the 1979 revolution in Iran, special schools were established for gifted students at intermediate and high school levels. Every year there are entrance exams for each one of these schools. Each one is a multiple-choice test with four options and its most important subject is mathematics.

Less than one per cent of Iranian high school students study in these special schools for gifted students. The great desire of students to be accepted in these special schools has led to the previous year's questions of the entrance exam to these schools being discussed in the math classes of other schools.

8.2.4.6 University students mathematics competition

Since 1974, every year there is a math competition among the university math students in the areas of integral and differential calculus, analysis, and applied mathematics. The organizer of this competition is the Mathematical Society of Iran and the questions are in written form (www.ims.ir).

8.2.4.7 University Students International Scientific Olympiad in Mathematics

The University Students International Scientific Olympiad in Mathematics is a two-stage assessment that has been organized every year since 1996. In the first stage a five-member team is selected by each university. These teams compete with each other in the areas of integral and differential calculus, algebra, linear algebra and analysis.

There are around 150 participants from which 30 persons are selected. In the next round, there is an international competition with the participants from other countries like Armenia, Ukraine, Germany, Russia, China, Azerbaijan, Oman, Bahrain, Indonesia and Kirgizstan. The exam is in written form and it has encouraged many students to study mathematics.

8.3 Assessment and learning, assessment and challenge, assessment and curriculum

In this section the relationship between the conception of the role of assessment, the features of assessment tasks and the provision of challenge, as well as how this relationship may affect issues of curriculum and may vary under different conditions, are discussed.

8.3.1 The role of assessment: assessment and learning

Assessment must be viewed as part of the learning process, not as separate from it. The lessons learned about objectivity from natural science apply as well to social science; the measurement itself is an essential part of the situation measured. As part of a more challenging mathematics curriculum, assessment must challenge what students know and can do in new ways that allow them the possibility to learn more—and to do new things with what they have learned—while being assessed, not simply to show what they have already learned and practiced. An important aspect of assessment might be that it induces the student to review a body of mathematics, in the expectation that it will become more coherent and lead to a firmer understanding.

Addressing the role of assessment in a learning culture, Shepard (2000, p. 4) speaks about classroom assessment in the following terms. “This article is about classroom assessment—not the kind of assessment used to give grades or satisfy the accountability demands of an external authority, but rather the kind of assessment that can be used as part of instruction to support and enhance learning."

Shepard's starting point is the contrast that she draws between "social efficiency curricula, behaviorist learning theories and 'scientific measurement' " and a "social constructivist conceptual framework that blends key ideas from cognitive, constructivist, and socio-cultural theories." She elaborates on the ways assessment practices should be consistent with and support social-constructivist pedagogy.

Shepard goes on to detail the use of assessment in the process of learning, stating that improving the content of assessment is important (a mathematician would say necessary) but not sufficient to ensure that assessment will be used to enhance learning and be part of the learning process.

8.3.2 The role of assessment: assessment and challenge

The Norwegian case study sets itself apart from many widely accepted concepts of assessment and standardized assessment practices, in a way that is similar to most out-of-school challenges. Almost hidden in the discussion of the case study presented we find the following phrase, "Moreover, any test—written or oral—should focus on what the students know, not what they do not know."

A slight variation that assessment should focus on what students can do and not on what they cannot do, leads us to consider that the inclusion of more challenging mathematics in the curriculum should imply a change in the very concept of assessment.

The previous two paragraphs have a nice ring to them, but it is not clear how much content they have. One would hope that any decent assessment will focus on what students can do. The point of assessment is to make a judgment as to what students are capable of, presumably with a view to further action. Is an assessment item valid without the ability to discriminate? And if it discriminates, are not some students bound to perform better than others? In particular, might it not turn up what students cannot do? What sort of "change in the very concept of assessment" is envisaged here?

On this point as well Shepard has some interesting contributions to make. She states that "there is a close relationship between truly understanding a concept and being able to transfer knowledge and use it in new situations. In contrast to memorization—and in contrast to the behaviorist assumption that each application must be taught as a separate learning objective—true understanding is flexible, connected and generalizable. Not surprisingly research studies demonstrate that learning is more likely to transfer if students have the opportunity to practice with a variety of applications while learning (Bransford 1979, p. 11).

"To support generalization and ensure transfer, that is, to support robust understandings, good teaching constantly asks about old understandings in new ways, calls for new applications, and draws new connections" (Shepard 1997, p. 27). "And good assessment does the same. We should not, for example, agree to a contract with our students, which says that the only fair test is one with familiar and well-rehearsed problems." (Shepard 2000, p. 11).

Chappuis and Stiggins (2002) echo these ideas in the following terms: "Assessment for learning occurs during the teaching and learning process rather than after it and has as its primary focus the ongoing improvement of learning for all students. Teachers who assess for learning use day-to-day classroom assessment activities to involve students directly and deeply in their own learning, increasing their confidence and motivation to learn by emphasizing progress and achievement rather than failure and defeat."

Considerations such as these are meant for a broad audience, frequently going under the name of formative assessment. However their pertinence for our discussion about the introduction of more challenging mathematics into the curriculum, not as some sort of decorative "enrichment" but as a fundamental and integral component, and about mirroring that core role with the introduction of challenging assessment activities as well, is patent.

For example, Stiggins (2002) speaks about looking for high quality responses to an open-ended math problem, supports student self-assessment and accompanies his remarks with a novel assessment scheme.

As an aside, it is indeed true that in many marking schemes used in challenging problem-solving competitions this same element is present. Even on multiple-choice type competitions it is common that points be given for what a student has accomplished, marks are given for correct answers and for problems left unanswered (a sign the student self-assesses that he does not know), and nothing is subtracted for incorrect ones.

Major challenges of mathematics education are thus related to developing and linking different competencies in solving problems through relating the underlying conceptual and procedural knowledge. This seems to get at the essence of the present chapter. Whatever we do in the classroom is governed by what we regard as effective learning of mathematics. If this involves resourcefulness and making connections, ability and confidence to tackle all sorts of problems, then this provides an incentive to incorporate something that will achieve this in our examinations and classroom practice.

In sum, emphasizing and analyzing, eliciting and bringing to the fore what students can do, rather than focusing on what they might not be able to do, is the vantage point which makes possible the incorporation of challenging and worthwhile assessment into the learning process.

8.3.3 The role of assessment: assessment and curriculum

The case of Singapore shows that challenging mathematics can be a significant part of testing in a standard nationwide setting. Such inclusion broadcasts an important message to teachers, to parents, to students and to the society as a whole. Furthermore, the crux of the "challenge" facing contemporary mathematics education itself is clearly delineated in the Singapore case. Even test items for struggling students required some level of challenge. This shows that the education system placed an emphasis on every pupil having an opportunity to engage in some challenging situations.

There is a call worldwide for more challenging curricula, and the Singapore and Norway experiences address the implications for testing and assessment practices.

Among many others, Oakes and Wells (1998) have focused on this point, and several years later the *Harvard Education Letter* of 2005 continues to find it necessary to underline not just the desirability, but the necessity of greater challenges within the curriculum for all students.

Although general acceptance of this fact may be lagging, with the apparent reason of accountability testing by external authorities, commitment to more challenging curricula for all students is clear in undergraduate mathematics; a recent international conference took ample notice of the need for challenges, and indeed in the literature many of the references to formative assessment speak to the undergraduate level. It is necessary that some control of the situation shift from a lecturer who delivers a body of results to the students who under the guidance of their professor wrestle with significant mathematics. In the words of Paul Halmos, "the hardest part of teaching by challenging is to keep your mouth shut."

Perusing the abstracts from the Second International Conference on the Teaching of Mathematics (at the undergraduate level) held in Hersonissos, Crete, in 2002, we found an interesting spread of references to challenge in the undergraduate mathematics curriculum.

Georgieva (Pacific University) speaks about a course in which a solution manual for problems given as homework was distributed, but that in fact there had been no blind copying, stating that "we attribute this fact to making clear early in the semester that the tests are extensive both in content and level of intellectual challenge. We observe a big jump in the students' motivation." (p. 142)

Guineo & Martínez from Uruguay propose projects to their students based on the motivation given by real-life challenges (p. 153).

Iqbal & Tahir speak of their objectives in a course dealing with algorithms (p. 182), among them to "equip students with the necessary tools and techniques, and above all the confidence, required in solving non-textbook problems." This objective is "essentially a creative effort containing all the ingredients of a thriller: adventure, excitement, challenge and suspense."

Po-Hung Liu, speaking on developing views on mathematical thinking, describes challenges given to students when asking them to discover inadequacies in concepts of the calculus found in historical texts and contexts (p. 236).

Nolan, studying pre-service teachers' conceptions when faced with challenging mathematics, states clearly (p. 281) that: "Factors other than ability influence students' approaches to challenges" and cites factors such as their persistence (or withdrawal) and their use of cognitive skills.

8.3.4 Competitions and curriculum

Out-of-school activities, such as competitions on all levels, can be seen as modeling for teachers the sorts of challenging, intriguing and entertaining mathematics that are central to a deep and broad understanding of the subject, as well as to giving opportunities to develop and enhance students' mathematical thinking.

Liu (2004) describes this possibility as follows: "Teachers can benefit from mathematics competitions because they are a fertile ground for hunting down good problems as classroom examples or homework assignments."

Competitions such as the Hunter Primary Competition state among their aims such things as to "encourage the broad and in-depth coverage" of the mathematics curriculum (Bishop 2002).

Speaking about the UK Primary Mathematics Challenge Peter Bailey (2005) offers this view: "It is not the intention to set problems on secondary topics, but rather to set problems on primary topics that can lead to discovery and further work at this level."

Rounding out this idea with the aims of the Mathematics Challenge for Young Australians, Dowsey and Henry (2000, p. 39) give a prominent place to "providing teachers with some excellent problems as resources for their own classrooms."

8.4 Knowledge gaps and future research questions in this domain

In this section we look at some possible research questions for challenging mathematics in the classroom. In some cases we consider extensions of research already being carried out in the general curriculum.

It seems clear that research into assessment practices in the classroom involving mathematical challenges is scanty. Therefore, the questions to be addressed are abundant.

8.4.1 Examining opposing views of assessment and their relationship to challenging mathematics

Research must shed further light on the opposing views of assessment that can be found in the mathematics education community.

For example, compare Shepard's remarks, "If we wish to pursue seriously the use of assessment for learning, . . ., it is important to recognize the pervasive negative effects of accountability tests and the extent to which externally imposed testing programs prevent and drive out thoughtful classroom practices" (Shepard 2000, p. 9), with a more traditional view of assessment (NCTM 2000): "Assessment should support the learning of important mathematics and furnish useful information to both teachers and students."

This second stance is usually interpreted to mean that assessment should "rank" students, for this is the meaning most commonly given to the phrase "furnish useful information to both teachers and students". This ranking involves setting a standard—which may be high—and then seeing how each student compares to the standard, which traditionally means by how much he or she falls short of the standard.

It is not possible to properly analyze these views without going into the issue of the purpose of the assessment and the pertinence of the standard to it. Where an assessment is made, for example, in order to decide on admission to college or university, we need to study how judgments based on the assessment are in fact borne out by the future progress of the student. Are the criteria authentic enough that no students are spuriously held back or admitted into a situation that they cannot handle? If the assessment is made for certification, for example, of graduation from secondary school, can we improve the mechanisms to ensure that the students indeed have the specified competencies?

How can studies involving the introduction of challenges into the mathematical experience of every student shed light on the relative benefits of each of these views?

8.4.2 What is the role of challenging mathematics in the relationship between assessment and learning?

In their review of research on formative assessment Black and Wiliam (1998) state that "One of the outstanding features of studies in assessment in recent years has been the shift in the focus of attention, towards greater interest in interactions between assessment and classroom learning and away from concentration on the properties of restricted forms of test which are only weakly linked to the learning experiences of students."

As related by Black and Wiliam (1998), "Ames (1992) started from the evidence that 'mastery' (i.e. task-related) goals can secure and review the salient features of learning environments that can help secure these advantages. She concludes that evaluation of students should focus on individual improvement and mastery, but before this the tasks proposed should help students to establish their own self-referenced goals by offering a meaningful, interesting and reasonably demanding challenge." Some key words in their on-going analysis are self-evaluation and self-directed learning.

Furthermore, when examining links to learning theories, Black and Wiliam (1998) tell us: "The arguments given by Zessoules and Gardner (1991) show how any assessment changes of the types described above [relating to the ideas of Ames and similar ideas] might be expected to enhance learning if they help students to develop reflective habits of mind. They further argue that such development should be an essential component in programs for the implementation of authentic assessment in classroom practice. Assessment is to be seen as a moment of learning and students have to be active in their own assessment...."

What kinds of reflective habits are elicited by challenging mathematics in the classroom and in classroom assessment?

In particular, can the use of challenging mathematics in assessment better help the student pull together the threads of mathematics previously encountered to obtain a more coherent view of a mathematical area and a better sense of their growing mathematical power?

What kinds of student activity are triggered by challenging mathematics and challenging assessment tasks?

Furthermore, Black and Wiliam (1998) relate that: "In their detailed qualitative study of the classroom characteristics of two outstanding high school science teachers, Garnett and Tobin (1989) concluded that the key to their success was the way they were able to monitor for understanding. A common feature was the diversity of classroom activities with an emphasis on frequent questioning in which 60 per cent of the questions were asked by the students."

One research question might focus on precisely the kinds of mathematics that elicit deep questioning by the students. Elwood and Klenowski (2002) have this to say: "Research suggests that to improve learning and indeed teaching, educational assessment must be formative in both function and purpose and must put the student in the centre of the assessment process."

How can this be done in relation to challenging mathematics?

8.4.3 What is the nature of good classroom assessment?

Cochairman of this Study Peter Taylor expressed the following position at the Study Conference, worthy of exploring: "A good assessment is one which has a balance with some questions requiring mathematical reasoning which tests their [students'] ability to progress and research unknown areas."

This quotation puts the spotlight on the idea that assessment is not a termination, but a stage in the mathematical development of the student. This pulls together some of the ideas raised in earlier sections about deeper understanding and the ability to move on to a more advanced stage.

8.4.4 What are the differences in focus for challenging mathematics in the light of assessment to determine ability and assessment to determine achievement?

Ecclestone (2002) makes the distinction between behaviorist–based assessment as extrinsic, focused on rewards and short-term goals, and humanist-constructivist based assessment as intrinsic, focused on higher levels of creativity. She makes the further contrast that constructivists look at performance and achievement as well as effort while behaviorists consider ability as fundamental. In her view, formative assessment is linked with critical reflection and engagement.

Implanted in the classroom is a model of assessment that is sometimes friendly when it comes to designing difficult tasks, but is inimical to challenge. How can the difference for the student between difficult and challenging mathematics be delineated by assessment practices? How has challenge been included in tests designed to measure achievement? What are the effects as measured by these tests? What are the implications for challenging mathematics of the difference between achievement tests in the sense expressed by Ecclestone and achievement tests as defined in the dictionary of psychology?

Achievement test: A collection of tests that measure the student's proficiency and accumulated knowledge of specific subject areas. An achievement test is a standardized test that is designed to measure an individual's level of knowledge in a particular area. Unlike an aptitude test which measures a person's ability to learn something, an achievement test focuses specifically on how much a person knows about a specific topic or area such as math, geography, or science.

8.4.5 What are the pedagogical differences and effects attained by enrichment and challenge?

Another possible research question is: What is the difference between enrichment and challenge? What does each mean with regard to the aims of the curriculum, the belief in the promise or capacity of each individual student? This is a question which invites further exploration, however we believe that enrichment is a process which does not necessarily include challenge.

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Concluding Remarks

Many educational developments have led to the situation that engendered this ICMI Study. The typical classroom regime is evolving towards a less authoritarian more child-centred approach, with increasing attention to the present benefits conferred on the pupil by schooling. Teachers are encouraged to look beyond the acquisition of skills to ensure that their charges gain understanding and appreciation of mathematical topics in the syllabus. Further, there has been enormous growth not only in mathematical contests of many types but of clubs, exhibitions, participatory fairs and events; these have generated a wealth of examples of mathematical problems, investigations and projects.

Further, access to resources and ease of communication made possible by technology provides an environment in which it becomes realistic and desirable to introduce challenge into the mathematical edusphere.

In this Study Volume we present the culmination of many years' work by many notable practitioners from a variety of educational and mathematical backgrounds, work that explored the growing opportunity for teachers to provide challenges for their students. A freer ideological atmosphere along with the wealth of resources available through courses, conferences and the Internet have allowed many educators to be innovative both within and without the classroom.

The authors of this Study feel that the scope and value of challenge is not adequately understood. Compared with other fields of educational study, the amount of research and literature directly dealing with challenge is sparse.

The authors also hope that the reader will conclude that challenges can not only take various forms in and beyond the classroom but are equally applicable to students of all standards. Indeed the authors are convinced of the value of challenging students and lay people in various ways. Considerable research is needed to test this belief as well as to find the best ways to achieve the goals of making mathematics intelligible, enjoyable and enriching to those who must or will encounter it.

This project depends on the generous collaboration of mathematicians and educationists. The task of the former is to suggest suitable material, ensure its integrity and to provide the professional edge in monitoring its propagation and use. The task of the latter is to deal with psychological and cognitive issues, to suggest and analyze regimes that will allow teachers and expositors to deploy challenges and assess their effectiveness.

Our hope is that this Study Volume highlights the main issues and inspires the ongoing research that this topic deserves.

June 2008

Edward J. Barbeau
Peter J. Taylor

Acknowledgements

The Study Volume evolved as a set of eight chapters, each authored by a team of about six members formed at the Study Conference. Thus, the volume takes the form of joint-authored papers. Some parts are truly collective work, while others have clearly evolved from individual contributions. This is obvious especially when they treat case studies attributable to a particular author in the team.

Chapter contributions

We acknowledge here specifically some of these contributions.

The translation of the passage from *Paradise* of Dante Alighieri in Chapter 0 is adapted from those of Rev. H.F. Cory, Henry Wadsworth Longfellow and Allen Mandelbaum.

Chapter 1 began as five individual contributions but they evolved and expanded in the process of editing. Section 1.4 was largely contributed by Peter Taylor and Section 1.6 by Alexander Karp, while the remaining parts of the chapter are a reorganization of individual contributions. Vladimir Protasov was the coordinating author for Chapter 1.

Chapter 2 had as coordinating authors Petar Kenderov and Ali Rejali.

The two appendices, on significant enrichment structures, were contributed by Ali Rejali (Mathematics Houses) and Djordje Kadijevich (Archimedes organization). Peter Taylor has in fact visited both the Mathematics Houses in Iran and the Archimedes premises in Belgrade and agrees that these are prime examples of highly useful organizations, supporting their communities in remarkably useful ways.

Chapter 3 retains much of its original structure. Its coordinating author was Victor Freiman.

The coordinating author of Chapter 4 was Arthur Powell. This chapter contained a number of case studies submitted by

- 4.3 Arthur Powell (USA)
- 4.4.1 Inger Christin Borge (Norway)
- 4.4.2 Gema Inés Fioriti (Argentina)
- 4.4.3 Elena Koublanova (USA)

- 4.4.4 Neela Sukthankar (Canada/PNG)
- 4.5 Margo Kondratieva (Canada)

The group which developed Chapter 5 was chaired at the Study Conference by Maria Bartolini Bussi. Vince Matsko coordinated the material into a unified chapter. This chapter has as a fundamental component a number of case studies. These case studies were contributed by

- 5.3.1 Maria G. Bartolini Bussi and Michela Maschietto (Italy)
- 5.3.2 Sharada Gade (India)
- 5.3.3 Martine Janvier (France)
- 5.3.4 Martine Janvier (France)
- 5.3.5 Jean-Pierre Kahane (France)
- 5.3.6 Vince Matsko (USA)
- 5.3.7 Cécile Ouvrier-Buffet (France)
- 5.3.8 Mark Saul (USA)
- 5.3.9 Mark Saul (USA)

The coordinating authors for Chapters 6, 7 and 8 were Derek Holton, Gloria Stillman and Maria Falk de Losada, respectively.

Chapter 8 contains four case studies. These were submitted by

- Singapore: Ban-Har Yeap (Singapore)
- Norway: Gunnar Gjone (Norway)
- Brazil: Maria de Losada (Colombia)
- Iran: Hossein Pourkaziemi (Iran)

Of course many other coauthors of the original submitted papers had important roles without attending the Study Conference or engaging in direct writing of the Study Volume. All such authors are acknowledged on the web site of the Study, www.amt.edu.au/icmis16.html, which published all the pre-conference papers that were accepted.

Organization support

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In Chapter 7 the RITEMATHS research project is cited. This research was supported under Australian Research Council's Linkage-Projects funding scheme (project LP0453701). The views expressed herein are those of the authors and are not necessarily those of the Australian Research Council.

Plenary speakers

In addition to the participants who qualified to attend the Study Conference via pre-conference papers, which are published on the Study's web site, we invited two outstanding mathematicians to deliver plenary lectures. These were Jean-Pierre Kahane (France) and Alexey Sossinsky (Russia). Both delivered memorable lectures which helped to establish the outcomes framework, and both continued to participate in author teams to contribute significant sections of the Study Volume.

IPC conference

This conference was held in Modena, Italy, from 28 November to 01 December 2003. The meetings were held at the University of Modena & Reggio Emilia. We wish to acknowledge Professor Maria Bartolini Bussi for having facilitated this meeting, with the provision of such impressive facilities and other arrangements that made the meeting so effective.

Study Conference

This conference was held in Trondheim, Norway, from 28 June to 02 July 2006. The meetings were held at the Norwegian Center for Mathematics Education,

Norwegian University of Science and Technology (NTNU). We wish to acknowledge the strong support of Professor Ingvill Merete Stedøy and her staff for their significant, dedicated effort in the various arrangements made for such a large number of visitors. The facilities were in all respects impressive.

In particular, we wish to acknowledge the generous financial support from the Center which enabled the attendance of Jean-Pierre Kahane, Alexey Sossinsky, Vladimir Protassov, Djordje Kadijevich, Sesutho Kesianye, Sergey Pozdnyakov, Ali Rejali, Petar Kenderov and Romualdas Kasuba.

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During the last two weeks of May 2008, Ed visited Canberra to work with Peter and various technical supporters to prepare the final material to hand over to Springer. At a meeting during this period are, from left, Ed, Peter, Heather Sommariva, Glenn Andrews and Bernadette Webster.

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Edward J. Barbeau
Peter J. Taylor

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