The Laboratory of Mathematical Machines of Modena

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Mathematical objects cannot be easily grasped by our senses. This creates well-known problems in the popularization and in the teaching of mathematics. Yet students (and mathematicians as well) usually make use of external representations of mathematical objects (e.g., words, schemes, symbols, gestures, instruments). In common sense the representation of a mathematics lesson is a blackboard full of formulas. Yet from the very ancient times tangible instruments (like the ruler and the compass) are part of mathematical experience and of the iconography of mathematics.

The Laboratory of Mathematical Machines (the MMLab), at the Department of Mathematics in Modena, contains a collection of geometrical instruments ('mathematical machines'). They have been reconstructed with a didactical aim, according to the design described in historical texts from classical Greece (linked to the theory of conic sections) to the 20th century. The MMLab works for both didactical research and popularisation of mathematics. The research activity has been funded by several agencies, local (the University, Regional Institute for School), national (Ministry of Instruction), international (the European Commission).

The MMLab staff includes academic researchers, university students, teachers and the members of association 'Macchine Matematiche'.(2)

The aim of this paper is to present the MMLab, an unusual case of Math laboratory.

What is a mathematical machine?

A mathematical machine (related to the field of geometry) is an artefact designed and built for the following purpose, that does not depend on the practical use (if any): it aims at forcing a point, a line segment or a plane figure (supported by a material support that makes them visible and touchable) to move or to be transformed according to a mathematical law that has been determined by the designer.

The most well-known mathematical machine is the pair of compasses (to draw circles) that is part of the iconography of mathematicians (Fig. 1). It is the ancestor of many curve drawing devices and pantographs. Another class of mathematical machines is given by perspectographs (Fig. 2) that are related to the ancient three-dimensional theory of conics, on the one hand, and to the roots of projective geometry on the other hand.

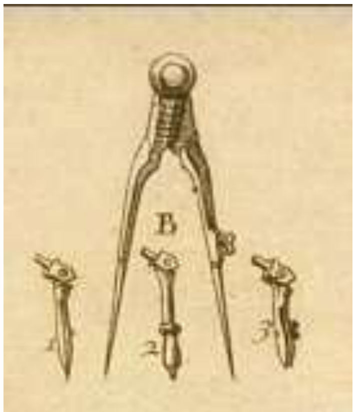


Figure 1: Compass

Figure 2: Perspectograph

In the MMLab there are different kinds of instruments:

- curve drawing devices;

Figure 3: Instrument for hyperbolic lenses (Descartes)

Figure 4: Leaves of Suardi

- pantographs for geometric transformations;

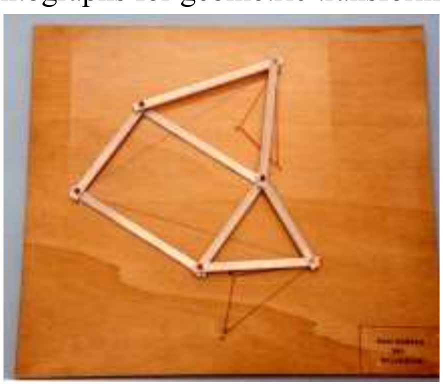


Figure 5: Pantograph of Sylvester (rotation)

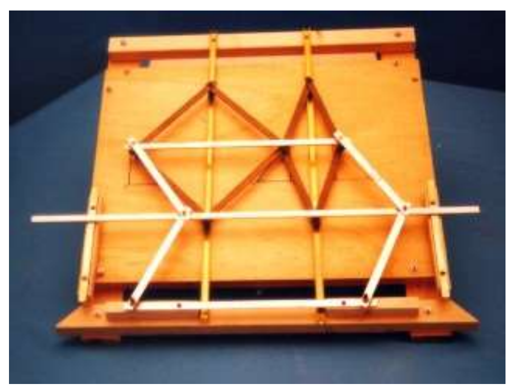


Figure 6: Product of two axial symmetries

- models of conic sections (see also below the orthotome model, Fig. 23);

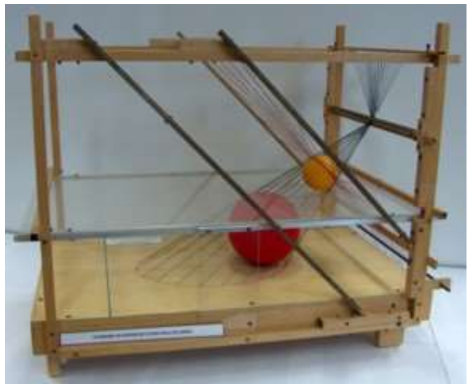


Figure 7: Model showing Dandelin theorem

- models of singular point of algebraic curves;

Figure 8: Salient point

- models of the 3d genesis of plane geometric transformations;

Figure 9: Homothety

• perspectographs;

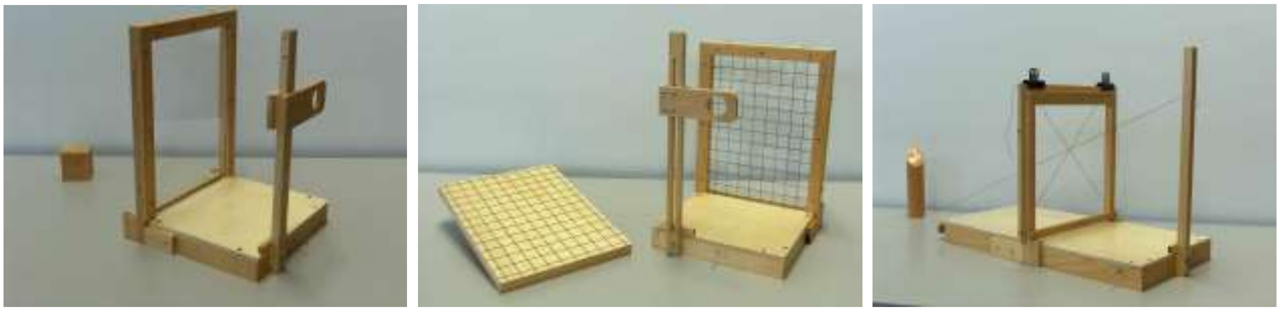


Figure 10: Dürer's perspectographs (glass, grid and 'door')

• anamorphoses;

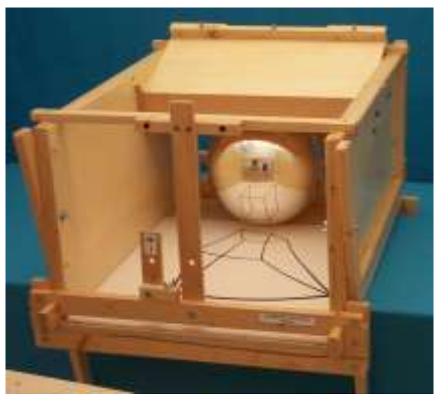


Figure 11: Anamorphosis: sphere

• and so on.

The story of the MMLab

The Laboratory was initiated in the early 80s by a small group of secondary school teachers, who were inspired by the didactical work of Emma Castelnuovo and Lucio Lombardo Radice in Rome. They began to build instruments with poor materials in the basement of a secondary school (the 'Liceo Scientifico Tassoni' in Modena) and they used them in everyday classroom activity. Very soon they established deep links with the team of didacticians at the Department of Mathematics (University of Modena — Reggio Emilia). When they retired from school, they constituted the non-profit Association 'Macchine Matematiche', that has already cooperated with the University and with other Museums, by producing exhibits and preparing exhibitions. In 1996 the Laboratory was moved from the school to the Science Museum of the University, then to the Department of Mathematics in 2002. During that time, dozens of instruments (about 200) have been reconstructed.

Figure 12: The MMLab

In 1992 the expositive activity began with a public exhibition (*Macchine matematiche e altri oggetti*) in the Town Hall of Modena. After that, many exhibitions were realised in Italy and abroad. Just to quote one, we were invited at '5^{e} Salon de la Culture & des Jeux mathématiques' (2004) organized by the 'Comité International des Jeux Mathématiques' in Paris. The two major collections of instruments are *Theatrum Machinarum* and *Perspectiva Artificialis*. Two smaller travelling exhibitions for schools have been prepared. They have been on show at the Science Festival in Genoa (2004 and 2005) and at the Mathematics Festival in Rome (2009). They have met the challenge of the many thousands visitors of a shopping centre and of the interested students who took part in the National Olympic Competitions of Mathematics (2005).

Besides producing working physical models, the MMLab entered the field of multimedia. After the production of motion pictures concerning pencils of conics, in the 90s, the MMLab began to produce 'virtual' copies of instruments, by means of various software from the didactic (e.g., Cabri-géomètre) or the professional fields (e.g., Cinema4d — Fig. 13, or Java — Fig. 20). The aim is both to show the functioning of a physical instrument and to produce 'infinitely many' different instruments by changing parameters. The MMLab produced *Labmat* (http://www.museo.unimo.it/labmat/), *Theatrum Machinarum* and *Perspectiva Artificialis* (http://www.mmlab.unimore.it).

Figure 13: Cinema4d frames

From 1999 to 2003 the MMLab has been involved in the Thematic Network 'Maths Alive', financed within the 5th framework program of the European Commission and coordinated by Albrecht Beuthelspacher (Mathematikum, Germany). In 2004, the project 'Hands on Maths' was finalist for the Altran award for Innovation. In the same year a DEMO of *Perspectiva Artificialis* was shortlisted at the Pirelli INTERNETational award. After the exhibition in Cremona (2008), a new catalogue of *Perspectiva Artificialis* was published (http://archiviomacmat.unimore.it/CR/Copertina.html), with the contribution of the teachers and the students involved in the workshops related to the exhibition.

In parallel, didactical research on the use of instruments in the geometry classroom had been carried out for many years. A product is represented by the two teaching units on curve drawing devices and transformation pantographs for secondary school.(3) In order to foster the personal contact of students with instruments, the MMLab has designed special kits (Fig. 14) for schools. Teaching experiments for primary school have been also carried out.

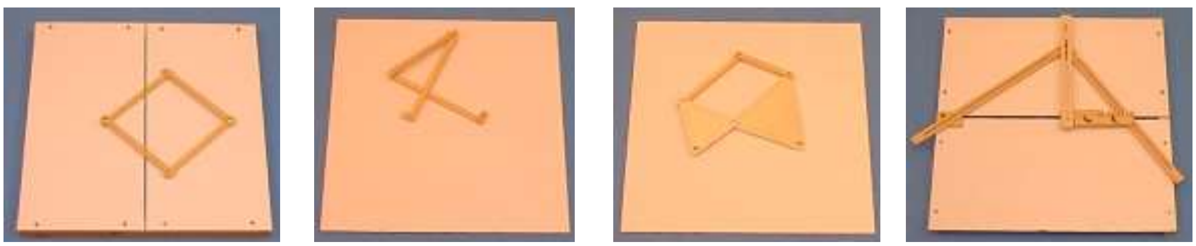


Figure 14: Images from the kit for classroom activity

At the end of 2003, the MMLab hosted the meeting of the International Program Committee of the ICMI Study 16, *Challenging Mathematics in and beyond the Classroom*.

Informal and formal learning of mathematics

The two main fields of activity of the MMLab are related to:

1) raising public awareness of mathematics (exhibitions, free distribution of multimedia on the web) and popularizing the main ideas and processes of mathematics thinking;
2) studying the conditions for effective use of instruments in the teaching and learning of mathematics (design, realization and analysis of controlled teaching experiments; publication of books and working sheets for teachers).

The two activities are often referred to as informal and formal learning: they are related to each other although they do not coincide. Informal learning, by either hands-on or virtual exploration, is meant to raise curiosity, to amuse and to amaze visitors, whilst formal learning, at least in traditional mathematics classrooms, is supposed to be boring and scaring.

We may contrast informal and formal learning by looking at what happens *with the same instruments* in an exhibition situation and in the mathematics classroom.

In the former the attention is focused on the beauty and the wonder of instruments; there is no opportunity to look for the reason that justify the functioning: this does not depend (only) on the children's age but also on the informal situation, with no request to write anything nor to link the experience to what may be known about curve drawing devices. In the latter, there is a very precise working sheet, where the exploration is guided to discover the essential property of a single instrument. For instance, when the session concerns conics, the path goes towards the equation of the curve, that is part of the standard school knowledge about conics.

Figure 15: Children at the Salon CIJM (Paris 2004)

Figure 16: Small group in the maths classroom

The two settings are quite different as regards the time spared (a few minutes in the former vs. one hour in the latter); the exploration (free vs. guided); the aim (to enjoy novelty vs. to discover an old thing in a new setting); the posture (to stand vs. to sit down); the task (to handle and talk vs. to handle, reflect and write); and so on.

In both cases instruments are amazing and amusing, but only in the latter case is the potential astonishment immediately and intentionally directed towards a didactical aim. Mathematics teachers may learn from the strategies used in informal learning: to foster a positive attitude towards mathematics, emphasizing the discovery and the enjoying aspects of mathematical activity; to make people aware that mathematics is a developing part of human culture, connected with art, technology and everyday life. Yet, in formal learning didactical aim must be in the foreground. In our experience we found that, on the one hand, the use of working copies of historical instruments has the potential to address cultural and affective issues, on the other hand match the need to construct the meaning of mathematical objects and to practice mathematical processes such as conjecturing and proving. For these reasons, the activity with the instruments of the MMLab may effectively interface both informal and formal learning.

The MMLab lends copies of instruments to teachers for the use in the mathematics classroom (according to working sheets that have been tested many times) or welcomes teachers with their classes in the large room of the Department of Mathematics in Modena, where the machines are stored. The situation is different because in the first case the teacher herself is responsible for teaching whilst in the second case the teacher is relieved of the responsibility of teaching. In both cases, however, we have a good example of what has been called a mathematics laboratory by Teaching Commission of the Italian Mathematical Society:

> A mathematics laboratory is (...) rather a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. (...) We can imagine the laboratory environment as a renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together (without distinguishing between teacher and students). (4)

This description is quite different from the popular image of a mathematics classroom. Some features of informal learning have been borrowed and introduced in school. This is not completely new: the practice of tangible instruments in mathematics (and in geometry too) was present in the work of mathematicians until the early decades of 20th century, when the Bourbakist program shifted the focus to formal and symbolic aspects. The enactive mode of knowledge (according to J. Bruner) remained in mathematics education, usually limited to young pupils, as if the importance of handling objects and exploring space were decreasing with age. In some cases the confidence in the power of the concrete experience itself was surely excessive, as if it were transparent for the mathematical meanings or procedures embodied in it. Anthropological research, however, has pointed out that the transparency of any technology always exists with respect to some purpose, it is intricately tied to the cultural practice and it cannot be viewed as a feature of the artefact in itself.

Teaching experiments in the MMLab

We have carried out over years many teaching experiments. We are giving some fragments below.

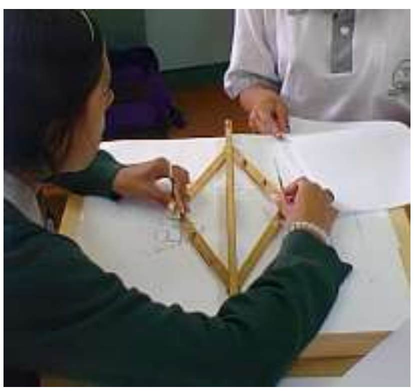
Primary School: exploring a perspectograph

In (Bartolini Bussi et al., 2005), a theoretical framework is given to study the construction of meaning of mathematical objects such as the pyramid and the relationship with the modelling process of perspective drawing. The process is lead by the exploration of a model of the Dürer's glass (Fig. 17 and Fig. 18) in the classroom. In the experiment, however, the introduction of some photorealistic animations of instruments for perspective shows that a deep and long term exploration of a concrete artefact allows the interpretation of a 'new' instrument offered only in the virtual world (Maschietto et al., 2004, 2005).

Figure 17: Dürer's glass ***Figure 18: Drawing of the Dürer's glass***

High School: pantographs for geometric transformation

In Fig. 19 the girl is using a machine (studied by Delaunay in the 19th century) that allows the drawing of two symmetrical figures. This suggests the role of the muscle-perception in the construction of the meaning of symmetry. From the bodily perspective, the experience is not comparable with the experience of the java applet (Fig. 20) where the left point (directing point), controlled by the hand though the mouse, provokes the motion of the right point (tracing point).



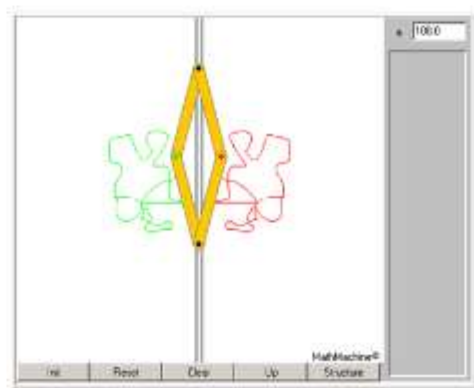


Figure 19: The brass machine ***Figure 20: The virtual machine***

Fig. 19 is taken from a teaching experiment carried out in Mexico by Veronica Hoyos, with instruments built by our MMLab. The muscular perception gives such instruments a big potentiality to be used by blind students (a project is now ongoing for the construction and analysis of a special version of some mathematical machines).

High School: curve drawing devices

With secondary school students, the approach to proof may be pushed beyond. In the experiments, the process of using instrument is analysed. Students have to explore in depth the linkage, in order to produce (and argue for) a conjecture concerning some geometrical properties which are related to all the configurations of the linkage.

For example, with respect to the exploration of the ellipsograph (Fig. 21), the conjecture about the relationship between the articulated anti-parallelogram (HIGF, Fig. 22) and the point E (i.e., the tracing point) is not easy. However, the long process of producing a conjecture is essential in the following process of constructing the proof, which is a crucial point in the teaching of mathematics.

Figure 21: The ellipsograph

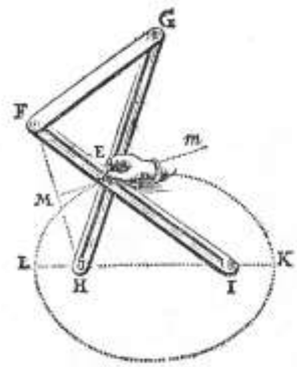


Figure 22: A drawing of the instrument (van Schooten)

Another example concerns the link between the ancient approach to conic sections and the 'organic' approach (i.e., by instruments). Fig. 23 and Fig. 24 illustrate the orthotome, the most ancient model of parabola, obtained by means of a right-angled right cone (i.e., a cone obtained by rotation of a right-angled triangle about a cathetus) cut by a plane orthogonal to a generatrix.

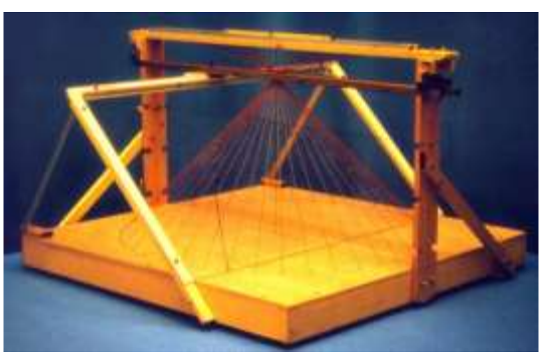


Figure 23: A model in wood, plexiglas and cotton — threads of orthotome

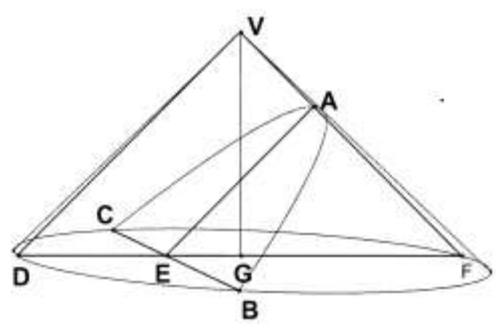


Figure 24: Letters to fix the proof in written form

It is not very difficult for high school students to prove, on the three-dimensional model (Bartolini Bussi, 2005) that

$$CE^2 = 2AV * AE.$$

The proof is based on the comparison of some similar triangles that belong to different planes. This relationship describes the property (i.e., the symptom, according to the word used by Greek geometers) of the point C, that is on the conic section. The same property holds for the point B, and for every point of the section, if one imagines shifting up and down the base plane.

The same relationship, in the 16th century, may be used to produce (and not to describe only) a parabola. In Fig. 25 there is the original drawing by Bonaventura Cavalieri (*Lo specchio ustorio*, 1632) that represents two movable squares (NLM that goes up and down, dragging AIK that is constrained to pass through A — fixed point — and through K — where LK in a fixed distance).

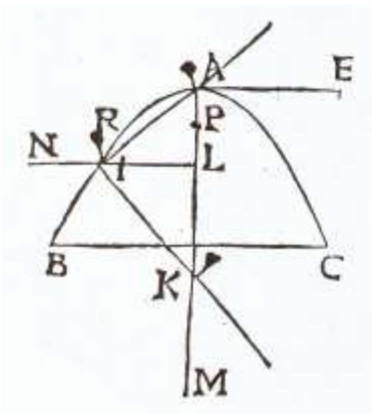


Figure 25: Original drawing by Cavalieri

Figure 26: A working copy of the instrument

The triangle AIK is right-angled. Hence there is the proportion

$$LK : LI = LI : LA.$$

That is equivalent to the symptom, whichever is the position of the two squares. And if we put:

$$LI = x, \qquad LA = y, \qquad LK = 2p,$$

we obtain:

$$x^2 = 2py,$$

that is, the canonical equation of parabola.

Concluding remarks

These are only fragments from our teaching experiments that allow to join intelligent manipulation of instruments with the reading of historical sources and to experience the mathematical processes of conjecturing, arguing and proving, at all school levels. Many examples are discussed in the book by Bartolini Bussi and Maschietto (2006). It was the starting point of the *Sciences and Technology – the Mathematics Laboratory* project for teacher education (2008-2013), in which many teachers constructed and proposed laboratory sessions with mathematical machines to their classes.(5)

Collections of working mathematical instruments are more and more spread in many countries. They may be precious historical instruments, stored in closed windows (e.g., the *Hilbert Raum* of the

Mathematics Institute in Göttingen), or interactive exhibits in mathematics centres (e.g., *Mathematikum* in Giessen; *la Cité des Sciences et de l'Industrie* in Paris; *Atractor* in Oporto; the *Giardino di Archimede* in Florence). The aim of the activity of mathematics centres — that have been created according to the model of science centres — is the popularization of mathematics. The travelling exhibitions organized by our team share this aim. Yet the main aim of our permanent Lab at the University of Modena and Reggio Emilia is didactical research, i.e., research into the teaching and learning of mathematics, within a context that has proven to be really motivating and effective for hundreds of secondary school students.

Notes

(1) An abridged version of this paper, authored by M. Maschietto, has been published in the *EMS Newsletter* 57 (September 2005), 34-37.
(2) http://associazioni.monet.modena.it/macmatem/
(3) http://www5.indire.it:8080/set/set_modelli/modellizzazione.htm
(4) http://www.dm.unibo.it/umi/italiano/didattica/2003/secondaria.pdf
(5) http://www.mmlab.unimore.it/site/home/progetto-regionale-emilia-romagna.html

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