MAKING SENSE OF STUDENTS' TALK AND ACTION FOR TEACHING MATHEMATICS

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Research and professional rhetoric suggest that understanding students' ways of learning and thinking are central to good teaching of mathematics (e.g., Even & Tirosh, 2002, 2008). This is the focus of this talk. Drawing on research and development studies I conducted in collaboration with colleagues and students in the last two decades, this talk discusses central aspects associated with the process of making sense of students' talk and action for teaching mathematics. First, I attend to problematic characteristics of teacher sense making of students' talk and action. Next, I present an approach we developed for improving teachers' ability to make sense of students' talk and action. Finally, I discuss the problem of the "nest step", i.e., of using understanding of students' mathematics learning and thinking to make instructional decisions – that is, connecting teacher knowledge and practice. In this context I introduce the construct *knowtice* as a lens for capturing the essence of what teachers need to develop.

Difficulties in Attending to, and Making Sense of, Students' Talk and Action[1]

Making sense of students' talk and action is critical for assessing students' understanding, knowledge and learning of mathematics. Yet, the process of teacher interpretation of students' talk and action involves difficulties. In the following I present four episodes of teacher-students classroom interactions where teachers did not understand things said or done by students. For each episode I suggest one resource that could contribute to teachers' misunderstanding of their students. The choice to highlight only one resource is for the purpose of demonstration only and does not imply that other resources could not contribute to the nature of the teacher's understanding. I conclude this section with a discussion of the complexity associated with understanding what students are saying and doing.

Episode 1: Having a plan

In an early algebra lesson, a 7th grade teacher's plan was to motivate the learning of simplifying algebraic expressions by providing his students with experiences of substituting numerical values into complicated and simple equivalent expressions. Such experiences, he had hoped, would demonstrate the advantage of substituting numerical values into simplified expressions. Thus, the teacher asked his students to substitute ½ in each of the following two equivalent expressions:

[1] This part draws on Even (2005a), Even and Wallach (2004), Even and Tirosh (2002, 2008), Tirosh, Even, and Robinson (2002).

$4a+3$ and $\frac{3a+6+5a}{2}$

Yet, the teacher forgot to state that the two expressions are equivalent. Below is an excerpt from the class discourse that emerged:

T: Substitute a = ½

S_1: You get the same result.

T: Are the algebraic expressions equivalent?

S_2: No, because we substituted only one number.

S_1: Yes.

S_3: It is impossible to know. We need all the numbers.

S_4: One example is not enough.

As can be seen, the students were not engaged in the task as the teacher had planned. Instead, they debated how one could decide whether two algebraic expressions are equivalent. Still, the teacher adhered to his original plan, and concluded:

T: We can conclude: It is difficult to substitute numbers in a complicated expression and therefore we should find a simpler equivalent expression.

This episode illustrates how having a plan contributed to preventing this teacher from attending to his students' thinking, as he later remarked: "I prepare my objective and the exercises I want to give the students, and it is very confusing for me when they suddenly ask something not according to my planning." Rather than listening *to* the students' discussion this teacher was engaged in listening *for* something, in "evaluating listening" (Davis, 1997).

Episode 2: Lacking knowledge about students' ways of learning mathematics

Consider the following examples of students' conjoining algebraic expressions:

$10 + 2b = 12b$

$5t + 3t + t + 2 = 11t$

$3m + 2 + 2m = 5m + 2 = 7m$

$3 + 4x = 7x = 7$

Most math teachers would not be surprised by such responses. They are all too familiar. Research-based explanations for the sources of students' tendency to conjoin algebraic expressions include:

- Conventions in natural language (Tall & Thomas, 1991). For instance, in natural language 'and' and 'plus' have similar meanings. Thus, 'ab' is read as 'a and b' and interpreted as 'a + b'.
- Previous learning from other areas that do not differentiate between conjoining and adding (Stacey & MacGregor, 1994). For example, in chemistry adding oxygen to carbon produces CO_2.
- Previous learning in mathematics: the ' behavior' of algebraic expressions is expected to be similar to that of arithmetic expressions (Booth, 1988; Davis, 1975; Tall & Thomas, 1991). For example, students expect a final, single-termed answer or interpret symbols such as '+' only in terms of actions to be performed.
- The dual nature of mathematical notations: process and object (Davis, 1975; Sfard, 1991; Tall & Thomas, 1991). For instance, the symbol 5 x + 8 stands both for the *process* 'add five times x and eight' and for an *object* that can be manipulated.

Let us look now at the 7th grade teacher's attempt to teach his students how to simplify algebraic expressions. The teacher wrote the expression 3m + 2 + 2m on the board and explained:

T: What does this equal to? Add the numbers separately and add the letters separately. Let us color the numbers [3m+**2**+2m]. We get 5m+2.

S_1: And what now?

S_2: 7m.

T: [Rather surprised] No! 5m+2 does not equal 7m. The rule is 'add the numbers separately and add the letters separately'. Here is another example: 4a+5-2a+7. We color the numbers [4a+**5**-2a+**7**]. What do we get? 2a+12. Let us write the rule [dictates]: In an expression in which both numbers and letters appear, we add the numbers separately and add the letters separately. Repeat out loud.

S's: [Repeat the rule out loud.]

T: Let's take another example: 6x+2+3x+5 = . We add according to the rule and get 9x+7.

Later, when working on simplifying the expression: 3+2b+7 the problem continued:

S1: 12b.

T: No!

S2: I got 10+2b. Why isn't it 12b?

As can be seen, the students tended to conjoin or "finish" algebraic expressions. Yet, as the teacher later explained, he sensed there was a problem but did not understand its sources; he did not understand what his students' difficulties were. This episode illustrates how not being aware of students' tendency to conjoin algebraic expressions or of possible sources for it – lacking knowledge about students' ways of learning mathematics – contributed to prohibiting this teacher from making sense of what his students said and confined his understanding of their ways of thinking.

Episode 3: Not valuing students' ways of thinking

A 5th grade teacher gave her class a quiz that included the following problem: "3/5 of a number is 12. Calculate the number. Explain your solution."

The teacher expected a conventional solution:

$$12 : \frac{3}{5} = \frac{12 * 5}{3} = 20$$

Yet, Ron's solution was different:

$$12 * 2 = 24$$
$$24 : 6 = 4$$
$$24 - 4 = 20$$

Assessing Ron's solution, the teacher explained: "He reached a correct answer but I didn't understand what he did. It didn't seem right… Ron is an average-good student who usually has difficulties with homework." And she decided to mark Ron's solution as wrong. When Ron protested and insisted that his solution was correct, the teacher did not think it could be, but eventually asked Ron to explain his solution. And so he did:

<u>Solution</u>	<u>Explanation</u>
$12 * 2 = 24$	If $\frac{3}{5}$ is 12 then 24 is $\frac{6}{5}$
$24 : 6 = 4$	The value of $\frac{1}{5}$ is 24:6 = 4
$24 - 4 = 20$	The number is 24 – 4 = 20

This episode illustrates how not valuing a student's ways of thinking, and not believing that there was something to understand, contributed to preventing this teacher from being tuned to understand her student's way of thinking.

Episode 4: Having a specific mathematics solution in mind

A 4th grade teacher presented the following problem to her class:

> The following task does not have a solution:
> *Divide 15 players into two teams, so that in one team there are 4 players less than in the other team.*
> Change the number of players, so that there will be a solution.

A student solved the problem by building up two groups, using a drawing of 15 "players" lined up. First she circled four "players" (of the 15). Then she added three "players" and marked one team of seven players. Finally, she built the other team by marking three more "players", and explained: "Here are seven players and here are three [pause] so [pause] ten players; seven and three makes ten. And seven minus three is four, so ten players."

The teacher's interpretation of this solution disregarded the building of the two groups: "The solution just came out of the blue", "She just said 10 off the top of her head". Analysis of the teacher's own solution of the problem shed light on this teacher's interpretation of the student's work, illustrating how having a specific mathematics solution in mind contributed to the teacher's difficulties to understand what the student did. The teacher used the strategy of *removing* a *minimal* number of players to reach an even number of players, and changed the number of players to 14. This strategy is very different from the student's strategy of actually *building up* two groups that satisfy the requirement. This discrepancy between the two strategies contributed to preventing the teacher from understanding the student's way of thinking, as she "heard" the student's solution through her own solution: "I was really surprised that they changed to 10, [that they] removed 5... Remove one [pause]… I don't know, it seemed to me that you need to remove one and try."

Hearing students' talk and action "through"…

The four episodes above illustrate how teachers hear (i.e., understand) students' talk and action "through":

- the teacher's plan for the lesson,
- the limited knowledge about the nature and possible sources of students' tendency to "finish" algebraic expressions,
- the teacher's low expectation of a specific student
- the teacher's own way of solving the mathematics problem she presented to her students.

Understanding students, like any other kind of understanding, *cannot* be an accurate reflection of what actually was said or done, not only because the teacher is listening *for* something rather than *to* the students' discussion. Teachers hear students "through" various factors, such as, the teachers' own knowledge of mathematics, their beliefs about mathematics learning and knowing, understandings of mathematics teaching, dispositions toward the teacher's role, feelings about students, expectations from students, the context in which the hearing takes place, and so on.

Moreover, attention to and understanding what students are saying and doing is not necessarily associated with what teachers *do not* do: do not listen to students, do not change their plans, do not know about students' common conceptions, do not understand the mathematics, etc. Attention to and understanding what students are saying and doing is also associated with what teachers *do*: make lesson plans, work out the mathematics, anticipate students' answers, assess their students' learning, and so on.

As can be seen, teacher sense-making of students' talk and action is an active process of interpretation that draws on rich base of knowledge, beliefs, and attitudes. Consequently, this process involves ambiguity and difficulties. Thus, understanding what students are saying and doing should neither be regarded as an unproblematic task nor as something that can be certain. Yet, it can be improved.

Improving Attention to, and Sense Making of, Students' Talk and Action[2]

Teachers can improve their understanding of what their students say, write or do by learning to be open to unexpected events in the classroom, by learning about students' common misconceptions, and by learning to attribute more value to students' original solutions and to pay attention to their processes of solving problems (Even & Tirosh, 2008). In the following I present a multi-stage activity of "replicating a mini study" that

[2] This part draws on Even (1999a, 1999b, 2005b).

we designed as part of the MANOR Program in order to improve secondary school mathematics teachers' attention to, and sense making of, students' talk and action.

The first stage of the activity centred on developing academic knowledge. This stage included reading, presentations and discussions of research articles on student conceptions, classroom culture, and ways of learning mathematics in the domains of real numbers, algebra, analysis, geometry, probability and statistics. The participating teachers found this stage interesting and important. They were astonished to learn that students "are able to think this way", and developed appreciation of the idea that students construct their knowledge in ways that are not necessarily identical to the instruction. The development of academic knowledge contributed also to conceptualizing and making explicit naive and implicit knowledge.

In the second stage of the activity, the teachers conducted a mini-study. They were asked to choose one of the studies presented in the program and replicate it (or a variation of it) with students. They then wrote a reflective report describing the students' ways of thinking and difficulties, comparing their results with the original study. Finally, the teachers presented their mini-studies to their colleagues and to other mathematics educators. The participating teachers felt that they learned a great deal from doing the mini-study, referring to two kinds of benefit. One kind was academic. Doing a mini-study with real students provided the teachers with opportunities to examine theoretical matters (e.g., constructivism) by particularizing them in a specific context. Replicating a study expanded the teachers' theoretical knowledge, and helped them develop better understanding of the issues presented and discussed in the articles they read, as a teacher explained:

> When you read a research article, it is one level of depth. When you have to re-do it and implement it again, it is another level. I mean, what I know now about the study, about its hypothesis, its findings, and the theoretical material, I certainly wouldn't know after reading it once or twice or even if I had summarized it – it is much more. It became mine.

The other kind of benefit reflects an integration of knowledge learned in the academy with knowledge learned in practice. It involved learning about real students in a situation relevant to the teachers' practice, encouraging them to examine their experience-based knowledge in light of their theoretical knowledge, as a teacher described:

> In a mini-research, in contrast to an article that is completely theoretical, you have question marks about the findings. Could it be like this? Is it only a coincidence that this happened? Will it happen to students I know? My students? It is very interesting to see what really happens. To duplicate the study and see, to support the original findings or refute them . . .

The results of this examination were surprising for many – but for two contrasting reasons. One kind of surprise was that students could do more than expected. For example, an experienced teacher remarked:

> Even though I have worked for 30 years as a teacher, I was surprised by some of the things that we found in the group of students we studied. The students reached much higher levels of thinking than what I would have given them credit for. So it was very interesting.

Another kind of surprise was that students could do less than expected. For example, a teacher who expected her students to do better than the ones in the original study was amazed:

> Simply, I was amazed by the results. I said, well, this is a topic [irrational numbers] that we deal with in grade 9. It was several months after we had taught the material. And I said, OK, no problem. Our students, for sure, would know better than those students at the university. And we were shocked that actually with us it was the same as there.

As illustrated above, the "replicating a mini study" activity offered teachers the opportunity to improve their understanding of students' learning processes. The activity challenged teachers' existing conceptions and beliefs about student learning of mathematics, and encouraged the development of an appreciation of the need to attend to students' understanding, as it is not a simple reflection of the instruction. By replicating a mini-study with real students the teachers learned that what they thought they knew about their students was not necessarily a good representation of the students' knowledge. Additionally, by connecting academic and practical knowledge the activity provided teachers with lenses to support sense making of students' talk and action.

Knowledge for Teaching Mathematics[3]

Improving teachers' understanding of what their students say or do is not an aim in itself. It is expected that teachers use this knowledge to make informed instructional decisions. The following case demonstrates how improved knowledge about student learning resulted in making changes in the curriculum.

From general knowledge about student learning to changing the curriculum

As part of the MANOR program mentioned above, the participating teachers learned about students' tendency to "finish" algebraic expressions and its possible sources. As a result, one of the teachers decided to direct the attention of the teachers in her school to the issue of process-object nature of algebraic expressions. Based on the theoretical knowledge she developed, the teacher designed and conducted several meetings with the teachers in her school. She helped the teachers to become aware of the phenomenon by examining their students' understanding. They then analyzed the textbooks they were using with respect to their potential to develop a structural approach to algebraic expressions. Finally, the teachers planned their teaching in this direction, making changes in the curriculum by changing their traditional choices of textbook tasks. The following is the teacher's description of the work:

> I was exposed to it last year and I brought it [to the staff]. And it was simply amazing, the students' responses. And then we gave these things to the class and we raised additional questions, which are actually already in the textbook. But [this time] we concentrated on them and therefore the students gained some more.

Whereas the above case demonstrates how general knowledge about student learning led to changes in the practice of teaching mathematics in school, the following case demonstrates that attention to what students do is not easily connected to making instructional decisions.

From attention to what students do to making instructional decisions?

A teacher assessed the solutions of two of her students (Udi and Anat) to the following "dogs and cats problem":

[3] This part draws on Even (1999b, 2005a).

In March the number of dogs in an animal shelter was 15 more than the number of cats. Suggest at least two possibilities for the number of dogs and the number of cats that were in the shelter in March. Explain your thought process.

Udi's solution

I constructed a table and thought there would be 18 cats. Then I saw that I didn't get a whole number of dogs. I tried with 20 and with 23 and saw that with 23 it is also not a whole number. So I thought that the number of dogs should be a number with a zero at the end.

Cats	Dogs
18	$18 + 3\frac{3}{5} = 21\frac{3}{5}$
20	$20 + 4 = 24$
23	$23 + 4\frac{3}{5} = 27\frac{3}{5}$

The teacher's assessment of Udi's solution

Udi has mastered the operations of addition and multiplication of fractions, knows how to find the value of the part when the whole is known … He was asked to suggest two possibilities for the number of dogs and cats. Didn't do it but reached a generalization. Although not correct, or better said a partial generalization: 'the number of dogs should be a number with a zero at the end.' Actually it could also be a number that ends with the digit five.

The teacher's instructional decision

"If Udi forgot the divisibility rule for 5, it seems that there is a need to repeat this topic in class."

Questions

- Is it a good instructional decision?
- Is Udi's problem indeed that he forgot the divisibility rule for five?
- How could it be useful to repeat the topic in class?

Anat's solution

To the number of cats you add 1/5 and get the number of dogs. For example, 3 cats and $3\frac{1}{5}$ dogs. 10 cats and $10\frac{1}{5}$ dogs.

The teacher's assessment of Anat's solution

Anat understood that the number of dogs is greater than the number of cats... She does not understand the meaning of '1/5 more than the number of cats.' She also does not pay attention to the results she reached, the number of dogs must be a whole number, the number $3\frac{1}{5}$ does not have a meaning when we talk about the number of dogs.

The teacher's instructional decision

"I will have to work in class on the issue of checking and the reasonability of the answer."

Questions

- What could be effective ways to teach this strategy to Anat who "is a student with difficulties who lacks motivation"?
- Is it indeed the main difficulty for Anat?

As can be seen, the teacher attended to Udi's and Anat's ways of thinking and made sense of what they said and did. Yet, the "next step", i.e., using her knowledge and understanding of the students' thinking to make instructional decisions was rather simplistic: In both instances when the students exhibited a difficulty, the teacher decided to work on this in class.

Conclusion

Awareness to, and understanding of, students' mathematics learning and thinking are central to good teaching. Consequently, the development of such awareness and understanding has become part of the curriculum of teacher education for both prospective and practicing teachers in recent years (e.g., Even, 1999a, 2005a; Markovits & Even, 1999; Fennema et al., 1996; Tirosh, 2000). As was illustrated earlier, such professional development can contribute to improving teacher knowledge and disposition so that they advance their ability to make sense of student talk and action. Yet, improving teachers' understanding of what their students say and do still leaves the problem of how to use this understanding to make better instructional decisions. Using this knowledge to make instructional decisions is commonly treated as unproblematic, as if there is a simple connection between understanding what students know and knowing how to use this knowledge in instruction. What is often missing is an integration of knowledge and practice; integration that I term *knowtice* (Even, 2008) to signify that this integration is related to the elements that create it (*know*ledge and prac*tice*), but that the product is a new object. In other words, *knowtice* is the essence of what needs to be learned and developed in order for teacher sense making of students' talk and action indeed be for teaching mathematics.

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