# What Do We Know? And How Do We Know It?

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The International Program Committee of ICME-11 proposed that we launch the academic activities of the congress through a dialogue on issues of crucial interest for mathematics education such as the following:

- *What do we know that we did not know 10 years ago in mathematics education, and how have we come to know it?*
- *What kind of evidence is accessible, and what has to be looked for in mathematics education?*
- *What are the societal expectations regarding our field, and how do we situate ourselves regarding them?*
- *Up to what point can visions of teaching and learning mathematics and evidence in the field transcend the diversity of educational contexts and cultures?*
- *What are the main challenges that mathematics education faces today?*

In our joint plenary, we tried to develop such a dialogue, presenting our respective views of the dynamics of the field and of its outcomes in the last 10 to 15 years, our views of the main challenges we have to face today, and our views regarding how we can address those challenges. This paper, written for the ICME-11 proceedings, reflects that dialogue in both its content and form.

## Preliminary Comments: What Do We Know, and How?

*Michèle:*

As anyone can imagine, this question is very difficult to answer. The difficulties one experiences when trying to answer it are themselves sources of insight for understanding what the field of mathematics education is and how knowledge develops in this field.

The question can be addressed from a variety of positions and perspectives according to the meaning that one gives to *mathematics education* and according to one's personal position and experience in the field. Personally, I am a university academic attached to a mathematics department, teaching both mathematics and the didactics of mathematics. My first field of research was logic, but my current field of research is the didactics of mathematics. And my research experience has been strongly shaped by the educational and didactic culture of the country where I live. I am from France. Moreover, there is no doubt that my vision of mathematics education in the last decade has been substantially affected by my participation in the governance of the International Commission on Mathematical Instruction (ICMI).

In preparing this plenary address, I posed the first question to friends and colleagues from different parts of the world, and I would like to thank them here for their insightful answers. As could be expected, the responses I got were very diverse. Nevertheless, some common trends emerged from that diversity. The responses helped me understand better what I wanted to express and also that even if the question was articulated using the collective *we,* my answer will necessarily be a personal one.

*Jeremy:*

My responses, too, are very much conditioned by my position and my experience. I have been a mathematics educator for 50 years, and I currently teach in a college of education, in a department of mathematics and science education. I teach students from undergraduates preparing to be mathematics teachers to doctoral students in mathematics education, but mostly doctoral students. I have participated in all kinds of research, from early work on problem solving to more recent work on assessment, curriculum, and the history of mathematics education.

My interest in the history of our field, by the way, has made me suspicious. I have often heard mathematics educators say, "We now know . . . ," and then they say what it is we now know. Some of us who have been around for a little longer than 2 or 3 years say, "Well, actually there were people who knew that some time ago." So, it's useful to have some kind of historical perspective on our field. My answer, like that of Michèle, is going to be personal, and probably idiosyncratic. I certainly do not pretend to represent the United States, which is a very diverse place.

## What Do We Know That We Did Not Know 10 Years Ago in Mathematics Education, and How Have We Come To Know It?

*Michèle:*

To the question of what we know in mathematics education that we did not know 10 or even 15 years ago, some people would certainly answer, "Nothing." As you can imagine, that is not my position. Looking back at the past 15 years (I have extended the time a bit), I personally see a field where an evident progression of knowledge has taken place. That progression has been multidimensional: It has concerned not only mathematical topics that research had already addressed extensively, such as number, algebra, or geometry, but also topics that are becoming increasingly important in both mathematics and education, such as probability and statistics. It is not by chance that the ICMI Executive Committee decided some years ago that the time had come to launch an ICMI Study on statistics education. The conference associated with that study took place just last week in Monterrey.

Even when the focus has been put on mathematical topics such as those I have just mentioned, however, the progression of the field has been tightly dependent on the more global evolution of the field, on the constructs and approaches progressively introduced and refined, and on the efforts that have been undertaken in the last decade to understand why educational research has apparently been so ineffective, why it has had such a limited influence on practice. From that global and meta-level perspective, I would like to mention three major sources of progress:

The first is the consolidation of sociocultural and anthropological approaches in mathematics education. I think that this has helped us understand and approach better the systemic dimension of the educational reality we study. It has helped us understand the constraints that shape it at different levels of *determination* (using a term introduced by Yves Chevallard, 1999, 2007), from those situated at the level of mathematical topics to those situated at the highest level of civilization. It offered us new perspectives on the nature of learning processes at a time when the limitations of constructivist perspectives were becoming evident. It has offered new perspectives for approaching how teachers can support and guide those learning processes at a time when some tended to forget that teachers cannot limit their role to organizing the meeting of their students with mathematical knowledge. They have to teach; they have to "show the way," as our Japanese colleagues say.

The second point is the development of research about teachers' beliefs, representations, knowledge, practices, preparation and professional development, attested for instance by the creation of a specific journal *The Journal of Mathematics Teacher Education*, and the recent publication of a specific handbook.[1] In the last 15 years, I would say that the teacher came to be considered the problematic actor in the didactic relationship, as students had been 2 decades before. This led to the investigation of the specific mathematical needs of the teaching profession and differences from the needs of professional mathematicians. It led to the identification of the nature of teachers' professional work and the reasons underlying their didactical choices, to try to understand the rationality underlying teachers' practices. I think that thanks to that research we understand better today the limited impact of research designs on effective practices. We see the evident limitations of many teaching preparation programs, and also we can see better how those programs could be improved. The reader will find in the book resulting from ICMI Study 15 (Even & Ball, 2008) a synthesis of recent advances in that area.

The third point is the increasing attention paid to the semiotic and discursive dimensions of mathematical practices (Saenz-Ludlow & Presmeg, 2006). This is fully coherent with the first point above. This increasing attention has made evident the dialectic relationship existing between the genesis of semiotic representation and conceptualization of mathematical knowledge. It has made us more sensitive to the role that semiotic mediation plays in the development of mathematical knowledge. It has also led us to extend the semiotic systems considered beyond the most traditional ones, and to pay, for instance, due attention to gestures. Research on technology, from the seminal work of Jim Kaput (Kaput, 1992) to the most recent advances synthesized in the book resulting from ICMI Study 17 (Hoyles & Lagrange, in press), has played a substantial role in this evolution, and conversely has benefited from this evolution.

Of course, many germs for this global evolution were already visible 15 years ago, and it is certainly quite impossible to identify an idea that is considered of importance today and existed nowhere at that time. But these germs have developed and disseminated, becoming more

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[1] The *Handbook of Mathematics Teacher Education* published in June 2008 by Sense Publishers contains four volumes respectively edited by Peter Sullivan and Terry Wood (Volume 1), Dina Tirosh and Terry Wood (Volume 2), Konrad Krainer and Terry Wood (Volume 3), Barbara Jaworski and Terry Wood (Volume 4).

central and more consensual. Their scientific discussion has been nourished by an increasing number of experimental studies. This gives me the feeling that today many of us cannot see mathematics education as a field of research and as a field of practice the way we saw it 15 years ago.

*Jeremy*

As you will see, I agree very much with Michèle's analysis. If the organizers expected us to have radically different ideas, I am afraid they will be disappointed. As Michèle has noted, one's response to the question of what we know now depends on the meaning of mathematics education. It is a field of *study* and a field of *practice*. Both the study and the practice can concern either teaching mathematics or teaching mathematics education—the field has a recursive quality regarding the preparation of those who will prepare teachers to teach.

This question of what we know now that we did not know before is an interestingly persistent one at International Congresses of Mathematics Education. I have heard this question asked many times. It is usually asked by people who are thinking about International Congresses of Mathematics and the way that new findings are announced there. Or possibly they are thinking about medicine and the way effective new treatments are announced from time to time by medical researchers.

Mathematics education is regularly compared with mathematics itself and also, perhaps even more often, with medicine. In those fields, the progression of knowledge seems obvious—as long as you do not look too closely. But such a progression is far from the case in mathematics education, where the same questions get raised repeatedly and never seem to get completely satisfactory or final answers. It seems to me that questions in mathematics education are answered only provisionally at best, and they need to be readdressed in each generation. As I have said elsewhere, they are like vampires that repeatedly rise again from the dead, and we never quite manage to get the stake driven through their heart, as you might be able to do in other fields.

Mathematics education is not like other scientific fields. If anything, it is a social science. Felix Klein, in his inaugural address in Erlangen in1872, noted a critical difference between mathematics and other fields that should keep us from trying to make these comparisons. "Each mathematical generation," he said, "builds on the accomplishments of its predecessors, whereas in other fields it often happens that the old buildings are torn down before the new construction can proceed" (English translation from Rowe, 1985, p. 136). It seems certainly the case that in mathematics education we tear down our buildings when they no longer serve our purposes, activities, and values. We do not always start completely from scratch, but we do a lot of demolition as well as construction.

I want to endorse, however, the sources of progress that Michèle has identified by adding some comments and examples of my own. With respect to this question of anthropological approaches and sociocultural approaches to our field, I am not the one at all to elaborate on the contributions of Michèle and others who have developed the construct of *instrumental genesis:* the way in which users shape the artifacts they use, and the artifacts shape the users, and that yields instruments. But I did want to observe that the construct has been very helpful as an example of progress made in our understanding of the interaction between learners and their tools. I keep reading new reports of innovative work done with these ideas, and I expect them to continue to be influential.

I want also to second Michèle's observation that much attention in our field, and probably the majority of attention by researchers, has shifted over the last 10 or 15 years from learners to teachers. There is now considerable research on teachers' knowledge, their beliefs, and their practices. I do not have time to go into the last two, but I did want to point out that quite a few mathematics educators these days are looking at and trying to understand the construct of *pedagogical content knowledge,* which was introduced some years ago by Lee Shulman (1986, 1987). People are trying to figure out how that works in mathematics education. Others are looking at the construct of what has been called *mathematical knowledge for teaching* (MKT), and they are trying to understand what it is. How is it related to other knowledge? How is it related to pedagogical content knowledge? How is it related to all the other types of knowledge that teaching mathematics requires? In particular, I want to cite the work of Deborah Ball and Hyman Bass (Ball & Bass, 2000, 2003), who have been trying to help us understand that MKT is usefully thought of as a special kind of applied mathematics. I think that is a good way to think about it.

Our field has given increased attention to teaching practices over the past decade. Helpful in that process have been video studies of teaching, which have allowed the careful, detailed study of ordinary classrooms where mathematics is being taught and learned. We have had cross-national video studies—not only the TIMSS (Trends in International Mathematics and

Science Study) Video Studies (Hiebert et al., 2003; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999) but also the Learner's Perspective Study (Clarke, Emanuelsson, Jablonka, & Mok, 2006; Clarke, Keitel, & Shimizu, 2006) and other studies comparing the teaching in different countries. These studies have led to claims that there are what some people would call national styles of teaching, claims that I think have been strongly debated over the past few years. There are arguments on both sides of that question. Whether teaching has common characteristics across national borders or whether it is constrained by them is in my view similar to the question of whether there exists a so-called canonical curriculum in school mathematics either internationally, which is what TIMSS and PISA (Programme for International Student Assessment) both assume, or even nationally.

It is very hard to study teaching practices in different countries because we do not have the vocabulary for talking about teaching that we do for talking about the curriculum. When we talk about the curriculum, there are well known and accepted mathematical terms that we use. When we talk about teaching, we tend to fall into a jargon that may not be understood the same way by people in different countries. As an example, consider "learner-centered instruction," which has been interpreted in many ways and can mean a lot of different things.

And finally, I would point to the uses of technology, some of the benefits it confers on us, and some of the problems it presents to teachers. Those uses are being handled better, and we know more about them. But I will let Michèle discuss that topic.

*Michèle*

I will do it, but up to now, my discourse has been rather general. I would thus like to make it more concrete by considering two personal examples. And I would like with these examples to illustrate how the consolidation of sociocultural and anthropological approaches has moved my personal vision. The first example I have called "overcoming false dichotomies." There is no doubt that dichotomies are frequent in the discourse of mathematics education. In general, they are simplistic, and they are dangerous.

One of these is the dichotomy opposing concepts and techniques. Teaching practice is supposed to focus on the first or on the second. Through my research on digital technology, especially computer algebra systems (CAS; e.g., Artigue, 2002), in collaboration with French colleagues Jean-Baptiste Lagrange, Luc Trouche, and many others, I became especially sensitive to it. In the early 1990s, research in that area emphasized the dichotomy by pretending that the use of CAS, by freeing the students from the technical burden, allowed them to focus on conceptual thinking and understanding. But the classroom observations I made at that time did not provide any evidence of this phenomenon. That intrigued us, and we tried to understand it within the framework of the instrumental approach that we had developed.

To understand this dichotomy better, we decided that it was important to attach to techniques both an epistemic and a pragmatic value. A pragmatic value because they are operational; they produce results. And an epistemic value because they contribute to our understanding of the objects they involve. One crucial point from that perspective is that if mathematical techniques are taught, it is not just because of their pragmatic power. It is also because of their epistemic power. Think just for a moment about the technique of long division, which is an object of curriculum debate in mathematics education today between mathematics educators and mathematicians.

This move had an especially insightful result for me. I could no longer see the question posed by the educational use of digital technology as I had before. The resistance to digital technology—in particular, the incredible recurrence of debates on the use of calculators in the elementary grades—could be reinterpreted in terms of a balance broken between the epistemic and pragmatic values of usual techniques. The ordinary use of digital technology plays on the pragmatic power of technology, doing more things more quickly at the expense of its epistemic power. But what makes a technique legitimate at school cannot be its pragmatic power only, which is an essential difference between school and the outside world. Making technology legitimate and mathematically useful at school requires modes of integration that allow a reasonable balance between the pragmatic and the epistemic power of instrumented techniques. This balance, which has been well evidenced by research, requires tasks and situations that cannot be reduced to simple adaptations of paper-and-pencil tasks. And these tasks, which is also evident from research, are not so easy to design when, like many teachers, you enter the technological world with your paper-and-pencil culture (Laborde, 2001).

This is just one particular example too briefly described, but it corresponds to one of those rare moments in my life as a researcher when I had the feeling that I had learned something important that obliged me to look at educational resistances differently, at teachers differently. It also obliged me to question the resources that, as researchers, we provide to teachers and institutions. Moreover, I had the feeling that I could express this knowledge in rather simple terms and make it understandable beyond the community of researchers.

The second example is that of institutional transition. I first became sensitive to it when I was supervising the doctoral thesis of Brigitte Grugeon (1995). She tried to understand the general failure in algebra of students entering high school after successfully completing a vocational program. She wanted to question the usual interpretation of this failure, which was based on the idea that failure was rather normal for these students because everyone knows that vocational students have limited mathematical potential. She showed that, in fact, vocational and high schools convey two different algebraic cultures even if they share the same object and use the same language. This difference made it difficult for the students to understand what was expected from them in the new institution and for teachers to recognize their students' algebraic knowledge and build on it. Once the invisible discontinuities between the two institutions, the two cultures, were identified, new educational strategies became accessible. And they proved to be successful in the experimental setting in which they were implemented.

I then used the same approach in investigating with Frédéric Praslon (2000) the secondary school-to-university transition in the study of analysis. For a long time, students' difficulties with this transition had been approached by investigating the specificities of advanced mathematical thinking, by identifying epistemological obstacles and cognitive difficulties in the transition (see Tall, 1991, 1996, for synthetic views). The development of anthropological and institutional perspectives did not disqualify this view but obliged us to resituate it in a wider perspective: that of the transition between two institutions. The focus of interest thus moves from the student to the institution, with the postulate that students learn is what the institutions to which they are subjected allow them to learn, and that for understanding students' difficulties in the transition, one has first to understand the kind of mathematical practices the student is exposed to in the two institutions, the continuities and discontinuities between them, and the way they are managed. Once more, the shift was very productive and offered new perspectives for addressing the discontinuities of the transition. It was productive for us but also for the other researchers who since that time have been working along the same line. I think, for instance, of the thesis by Analia Bergé (2008) from Argentina and also the work of our Spanish colleagues (Bosch, Fonseca, & Gascón, 2005).

## What Kind of Evidence Is Accessible, and What Has to Be Looked for in Mathematics Education?

*Jeremy*

I begin by observing that it is clear in our field that we do not have enough good evidence on most topics. On very few topics do we have a set of research studies that could be said to provide us with a basis for making strong claims. Any such evidence ought to meet the following criteria: (1) It should be *relevant* to the questions we are asking; (2) it should be *sound,* meaning that it ought to be valid; and (3) to some degree, it ought to be *general*—able to be generalized to a larger context. If we have multiple studies on a given topic, those should *converge* in some sense. They should converge across locations, circumstances, researchers, groups, and methods. And even more important, I think, they should fit within some kind of network that makes both common and theoretical sense. Those were the criteria that we used in the Mathematics Learning Study that produced the volume *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001, pp. 21–24).

What happens if you take narrower criteria? There are a number of attempts to study mathematics education research that have adopted very narrow criteria. You get into problems, for example, when you apply the so-called gold standard of randomized controlled trials. You discover that we have almost no studies that meet that standard, and therefore, you have almost nothing to work with as evidence. There are far too many research questions for which either randomized controlled trials would be impossible or an appropriate study would require so many controls as to make the interventions, whatever they are, unrealistic. Randomized controlled trials, or something approximating them, are strictly speaking required if one is to make causal inferences.[2] We have, however, many issues in our field that do not require such evidence. When narrow criteria are applied, what happens—in the cases I have seen—is that too much is left to untested opinion and individual experience. Not enough use is made of the professional community's judgment and experience.

We need more evidence than we have. It should be evidence, it seems to me, not simply that some intervention works. We need help in understanding when and why it works, and what

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[2] For an expanded view of causal inference, see Maxwell (2004).

it means to "work." We also need descriptive and interpretive evidence regarding mathematics teaching practices even when those practices are not "interventions" but are occurring naturally in some setting. The analogy with medicine is not necessarily a good one for us, but even medicine does not undertake randomized controlled trials that have not been preceded by a lot of exploratory work, including case studies, cohort studies, and clinical trials.

*Michèle:*

I agree with you globally, Jeremy; it is a pity for the audience. You mentioned initially your interest in the history of the field of mathematics education. I think that, regarding this point, it is interesting to have a reflective look at the history of the field. From the 1960s, mathematics education has tried to establish itself as a scientific field, and this has of course shaped the kind of evidence that was looked for. In its early phases, in most countries, the field has tried to reach scientific status through the use of methods inspired by the experimental sciences like, for instance, experimental psychology at that time. Experimental and control groups, pretests and posttests were the norm. We cannot forget the limitations of these methodologies that were observed regarding educational phenomena: On the one hand, pertinent variables were not so easily identified and controlled, and on the other hand, even when these methodologies were able to show differences, they did not give access to the underlying mechanisms. These phenomena led to the development of the methodologies that predominate today, where evidence is mainly sought through the triangulation of multiple sources of data and analysis.

These methodologies have proved to be efficient for identifying didactic phenomena and understanding them, for revealing the rationality underlying students' and teachers' behaviors, and for making sense of classroom dynamics and learning trajectories. In France, for instance, classroom research has always played a central role in the field. This was certainly due to the systemic and situational perspective underlying the theory of didactic situations initiated by Guy Brousseau (1997), which has been and still is very influential in the field. This theory has led to the development from the early 1980s of a specific methodology for classroom research known as didactical engineering, which strongly rejected the experimental-control group paradigm and looked for quantitative and qualitative evidence through the comparison between what we called a priori and a posteriori analysis of didactic situations (Artigue, 1992). There is no doubt that the most important advances of the theory of didactic situations have resulted from the use of that methodology. The development of research on teachers' practices in the 1990s has needed the development of less invasive methodologies and has led to the increasing use of more naturalistic observations. These, of course, use other sources of evidence but still obey the same global philosophy.

These research methodologies have been and are productive due to the strong explanatory and at times predictive power of the knowledge they produce. They are essential tools for fundamental research in mathematics education, but they also have limitations regarding the kind of evidence they provide. In a field so dependent on cultures and contexts, one major issue is that of generality. Most often, evidence in mathematics education results from fine-grained but very local studies. What these strictly provide is some kind of existence theorems. One can suspect that the phenomena identified, the results obtained, are of more general value, but that more general value is not warranted at all by the research itself. As pointed out by Schoenfeld (2007), "Typically authors imply the generality of a phenomenon by tacitly or explicitly suggesting the typicality of the circumstances discussed in the study. Implying generality is one thing, however, and providing solid evidence for it is another" (p. 93). This leads him to introduce a distinction between claimed, implied, potential, and warranted generality as ways to think about the scope of generality of a study. I fully agree with this position.

Can we expect more from educational research? I hope so, as there is no doubt that even if research can go on progressing thanks to the methodologies it has favored up to now, and the kind of evidence these provide, this is not enough for allowing research to meet social expectations. Neither is it enough for developing productive links between research and practice, for scaling up the positive outcomes obtained locally in experimental settings. This leads me to the next question.

## What Are the Societal Expectations Regarding Our Field, and How Do We Situate Ourselves Regarding Them?

*Michèle:*

There is no doubt that mathematics education is considered of critical importance in most countries today. Good quality mathematics education is seen as a condition for scientific and economic development, as well as for inclusion and citizenship in our modern democratic societies. Beyond the transmission of a cultural heritage that is one of the greatest achievements of humankind, what is expected from mathematics education by society is, on the one hand, to ensure a reasonable level of mathematical literacy for all students, making them able to pertinently mobilize mathematical knowledge and thinking in the real world when necessary, and on the other hand, to prepare the mathematically qualified workforce needed by our societies. In the complex and changing world we live in, what is expected from mathematics education changes but does not decrease, far from it, and it is generally acknowledged that most educational systems fail to meet these expectations, as they failed 50 years ago when the new math reform period began.

Even if research in mathematics education has developed for that exact reason, trying to build the kind of knowledge that is required for improving the situation, we have to acknowledge that, whatever be its advances, it has not changed the face of the world. Mathematics education research has played a limited role in supporting decisions regarding curricular content and organization, teaching approaches, assessment modes, and teacher preparation. It is most often considered of little use and offering limited scientific evidence. Today, research is asked to provide the kind of evidence that is the norm in medicine and pharmacology with random trials, objective measures of effects, and large-scale experiments. This is especially the case in the United States but is not specific to that country alone.

The image of objectivity and reliability that is given by international comparisons such as the PISA (Organisation for Economic Co-operation and Development, 2003) occupying the forefront of the media scene tends to make a great impression on our societies. As a community, and ICMI must contribute, we certainly have to develop a critical stance regarding the ideas and values that society tends to impose on us. We are those best equipped to question pretentions to objectivity, reacting to views of evidence that do not make sense in mathematics education and questioning the measuring instruments used to determine what is counted as knowledge and what is not. We have also to stress that mathematical knowledge with the diversity of its facets cannot easily be captured in a one-dimensional structure. But this being said, we cannot ignore the societal demands and the questions they raise about the way we have carried out the research enterprise; the way we have worked at the dissemination of its results beyond the community of researchers in the field. I see two major challenges for the immediate future, which are partly dependent on one another:

- Taking seriously issues of scaling up, considering these as true research questions whose solution requires specific knowledge, the development of specific constructs and methodologies, the contribution of other expertise than those accessible in the field and new partnerships, and new kinds of didactical designs more robust than the sophisticated products usually built by researchers.
- Finding ways to make the results of research in mathematics education understandable and useful for their potential users. ICMI, for instance, tries to contribute to this effort through the ICMI Studies volumes, which will soon be freely accessible online 3 years after their publication. But a lot more remains to be done.

*Jeremy:*

On the question of social expectations, in every society, people expect their children to learn mathematics to a high level, first for themselves as individuals—that is, each child needs to learn mathematics to function in the society—but there is also a societal need to have people educated in mathematics. This set of dual expectations poses many problems for us. One of the major issues we face in attempting to change mathematics in the society is that members of the public tend to define mathematics as what they learned in school, which is often a barrier to change. And changing how mathematics is taught is, in our experience, likely to be even more difficult than changing the subject matter topics that are taught there—although both are difficult enterprises.

Society also expects that research in mathematics education can provide some definitive answers to questions about mathematics teaching and learning. Attempts to synthesize the research on a given topic are almost always disappointing. Policymakers would like to be able to make causal claims about the effectiveness of various instructional interventions, but there is little reason to believe that a single intervention will be equally effective across all topics, teachers, and students. Researchers and policymakers should move away from comparing the mean performance of groups receiving innovative and alternative instructional interventions. They should be looking at variation rather than means. On which topics are there differences? For which teachers? For which groups of students?

There are many different ways to approach the question of distilling research when making policy recommendations. As I mentioned earlier, in the discussion of evidence, the committee that produced *Adding It Up* (Kilpatrick et al., 2001) took a reasonably generous view of what research can tell us and was able to survey a broad spectrum of evidence. The U.S. Academy of Education is currently engaged in a so-called white papers initiative to provide policymakers in the next administration and Congress with the best available evidence on selected education policy issues. The group working on science and mathematics education policy, like the Mathematics Learning Study committee, has cast a fairly wide net to collect information that might inform policy. In contrast, one of the problems faced by the National Mathematics Advisory Panel (2008) in its recent report was its use of very stringent criteria for the quality of evidence examined. That approach left the panel with little to offer, beyond opinion based on personal experience, about what the literature says on various topics. Definitive answers are not possible for most of the questions to which society expects an answer, but researchers need to find better ways to address those questions anyway. There is never sufficient evidence, and I agree with Michèle about the importance of scaling up our research as well as finding ways to report it clearly and usably.

## Up to What Point Can Visions of Teaching and Learning Mathematics and Evidence in the Field Transcend the Diversity of Educational Contexts and Cultures?

*Jeremy:*

Diversity does condition what we can say to each other as members of what we think of as the same community. What do we mean across cultures, and even within cultures, when we say, "algebra"? Or "curriculum"? Or, as I said earlier, "learner centered"? Furthermore, efforts to localize mathematics are not always successful. For example, the important work of Ubi D'Ambrosio in ethnomathematics has penetrated the mathematics curriculum of some countries extensively but has had much less influence elsewhere. People want what they think of as general. And yet somehow we have to find ways for school mathematics to attend seriously to local conditions.

Returning to the question of international comparative studies such as TIMSS and PISA, one should note that such studies depend on there being a sort of canonical school mathematics curriculum that can be used as a template against which the curricula of different countries can be measured. This canonical, or idealized, curriculum is not a curriculum one can find in any single country. Instead, it is a hypothetical construction devised to make possible the use of a common set of assessment items across national borders (Keitel & Kilpatrick, 1999). The organizers of international comparative studies thereby reduce the problem of the diversity of educational contexts and cultures to the question of whether students had an “opportunity to learn” the content presumably assessed by the assessment items, pushing aside thorny questions of how to describe, let alone take into consideration, local curriculum conditions. We still face the problem, identified by Hans Freudenthal (1975) long ago, of constructing assessment instruments that are internationally equivalent, yet take local circumstances into account.

Mathematics educators have long recognized that the so-called implemented curriculum may bear little resemblance to the official curriculum that is issued by the ministry of education. And certainly the classroom door can be a formidable barrier to change. It is also interesting to notice that centralized systems are often not as centralized as we think they are, and decentralized systems are not as decentralized as commonly supposed (Howson, Keitel, & Kilpatrick, 1981/2008). A long time ago, before the English had a national curriculum, a French school inspector once made the following observation, which I think applies not just to France and England:

> In France, every teacher is supposed to be doing the same thing at the same time but nobody is, and in England, where everyone is supposed to be going his own way, nobody is. (p. 58)

*Michèle:*

I agree with you, Jeremy. That was a long time ago, and today no French inspector would dare to say that.

I would like to speak about diversity in another way. As I said at the opening ceremony, diversity can be seen in a negative way, as an obstacle to the kind of general evidence that the field of mathematics education should provide if it were a real scientific field. In my opinion, this is a completely erroneous view. I would like to emphasize the positive side of diversity in mathematics education. In the last 15 years, our field has learned a lot from diversity.

One interesting example indirectly results from the international comparisons I was criticizing a moment ago. TIMSS has attracted interest toward some regional areas such as Asia, where several countries had significantly higher results than most Western countries on the TIMSS achievement tests. And through complementary studies, researchers have tried to identify possible reasons for the observed differences. ICMI has contributed to it through a beautiful volume (Leung, Graf, & Lopez-Real, 2006) associated with ICMI Study 13. It appeared in 2006, and it compared mathematics education in different cultural traditions. More precisely, it compared mathematics education in countries of East Asia belonging to the Confucian tradition with that in some Western countries. What we have learned from this research work is quite interesting because it shows that the difference results mainly neither from the curriculum, nor the number of hours devoted to mathematics at school, nor the students' interest in mathematics. Instead, the difference results more deeply from what being mathematically educated in a Confucian culture means, the relationship to knowledge and to school that it implies, and the ways it shapes student-teacher relationships and their respective institutional positions. The knowledge offered by such studies disqualifies any attempt at improving the situation of mathematics education in a given country by just paying attention to surface and administrative characteristics, whatever their importance may be. It paves the way towards more productive reflection, trying to understand the strength and limitations of our respective educational choices, placing them into a more global structure with strong cultural components, and using this understanding to think about possible transpositions and changes. It is also our responsibility to make policymakers aware of this because they are not very often spontaneously aware and look for a miracle or the cheapest solution to the problem.

What is also interesting in this phenomenon is the fact that foreign eyes have allowed the identification of original designs with important didactic potential that existed as natural objects in these cultures. That was the case, for instance, for the *lesson study* system used for professional development in Japan, which was revealed by these comparative studies and has since become an object of research.

I could give many other examples of learning from diversity. I will just mention briefly another example, more personal. It was through the comparative work on the teaching and learning of algebra developed in the ICMI Study on that topic (Stacey, Chick, & Kendal, 2004) that I became aware of the diversity of educational strategies used worldwide for introducing students to the world of algebra. Thanks to this study, I better understood the respective implications of these different strategies regarding the difficulties of the transition between arithmetic and algebra. There is no doubt that the thesis of Brigitte Grugeon (1995) that I mentioned above had made me sensitive to this implication. But without such comparative studies, I was lacking the kind of evidence that is provided by the analysis of the large-scale use of different educational strategies.

## What Are the Main Challenges That Mathematics Education Faces Today?

*Michèle:*

It is difficult to make a reasonable response in the time available, but we have selected three challenges:

- *Technological challenge*

I have been involved in research and educational activities dealing with the topic of technology for more than 20 years, so I am very sensitive to it. It is evident that educational systems are still struggling with the difficulties they meet in taking advantage of the affordances of technology. These difficulties concern not only the most recent technologies but even those developed more than 2 decades ago such as graphic calculators and dynamic geometry software. But today technological evolution makes us enter a new phase where not only does technology affect mathematical objects, their representations, and the way we can manipulate and connect those representations, it also affects didactic interaction and more generally the way we access information. Digital technologies today can support and foster collaborative work—at a distance or not—between students, between teacher and students, between teachers, and between teachers and researchers. The consequences such work can have on students' learning processes and on the evolution of teachers' practices and their professional development is certainly one of the essential dimensions that educational research has to systematically explore in the future. ICMI Study 17 on digital technology (Hoyles, Lagrange, Son, & Sinclair, 2006), whose study volume (Hoyles & Lagrange, in press) will appear next year, will contribute to addressing this challenge.

*Jeremy:*

- *Coherence challenge*

Coherence is a challenge we both identify as being important. Every challenge that faces us in mathematics education has both internal and external aspects, and that is true of the coherence challenge as well. Internally, the field of mathematics education needs to cope better than it has so far with a growing proliferation of theories and constructs that guide our work. Some of these theories, such as Brousseau's (1997) theory of didactical situations or van Hiele's

(1984) model of geometric thought, have been developed within the field. Other theories, such as those related to the work of Piaget or Vygotsky, have been imported from outside and adapted for our field. We need a greater coherence among the theories we are using.

We also have a proliferation of constructs. I have identified some of them above, such as pedagogical content knowledge, but there are many more I could have listed, such as situated cognition or sociomathematical norms. These constructs are being used today with several different meanings by mathematics educators, and they need analysis, critique, and explication. Internationally, we as a community already have trouble communicating across our native languages. It does not help, and in fact magnifies the problem, when we are using the same term with different meanings.

Another kind of internal challenge arises from the gulf between teachers and researchers in many places, although I think it is fair to say that the movement toward taking teachers more seriously as researchers has begun to reduce that gulf. Still, there are more efforts that each—and especially researchers—might make to develop a more coherent approach to research by listening more attentively to and working more closely with the other.

Externally, people outside our field see it as fragmented. Mathematics educators seldom speak with one voice on matters of consequence in education. Fundamental research in our field is sometimes interpreted as leading to opposite conclusions and as supporting quite different practices. The public often sees mathematicians and mathematics teachers as proposing conflicting ways to resolve issues of mathematics education. Those proposals, in fact, may be in conflict. But I think the field would benefit if conflicts could be worked out amicably before people grab the spotlight and begin making pronouncements.

If mathematics education is to be taken seriously as a field of study and a field of practice, it will need to become more coherent in the discourse it promotes both internally and externally. Michèle reports that European mathematics educators have made a start along these lines in recent years.[3] Different projects are supporting the development of coherent and integrative views, clarifying commonalities and differences. Such work cannot be the work of

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[3] This effort is, for instance, evidenced by the existence of a working group specifically devoted to these questions at the recent conferences of the European Society for Research in Mathematics Education (CERME4, CERME5, CERME6). See CERME proceedings accessible on the ERME website http://ermeweb.free.fr/ for more information.

individual researchers. It needs international collaboration and adequate structures. Those are problems to which an institution such as ICMI can contribute.

*Michèle:*

The experiences I have had in the last 5 years of working in a kind of theoretical integration, or at least networking between theoretical frames, within the CERME Working Group mentioned above or in European projects focusing on technology enhanced learning in mathematics make clear that this effort is highly rewarding. But, I agree with Jeremy that it needs specific organization and international structure in order to be developed. For me, a first evidence resulting from this work is the fact that, just by reading the writings of researchers living in another context, in another culture, and having different approaches, you cannot understand how their approaches functionally affect their research work and the claims they make about practice. For that, you need to develop some kind of collective practice that allows you to enter into this process of operationalization between theoretical approach and practice. In the frame of two European projects called TELMA and ReMath,[4] I had the opportunity to contribute to developing such a practice through a methodology of cross-experiments obeying strict guidelines (Artigue et al., 2007). Now, 5 years later, I have the feeling that I see better where networking is useful, where networking is not useful, what is complementary, and what is not compatible. Such attempts give you another vision of the field, but knowing how to share the knowledge that we have gained collectively in these European projects with a wider audience is still an open problem for me.

- *Equity challenge*

Here, we all certainly share the view that having access to quality mathematics education is a human right, and that mathematics education has to serve the cause of equity. But we all know that this is far from being the case today. Even the idea that mathematics for all and the nurturing of mathematical talents are two conflicting ambitions is not at all a marginal position. Mathematics education in many parts of the world contributes to the social divide and is itself a

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[4] TELMA is a European Research Team of the Kaleidoscope European Network of Excellence focusing on technology enhanced learning in mathematics. TELMA publications are accessible on the TELMA Web site: http://telma.noe-kaleidoscope.org ReMath (Representing Mathematics with Digital Media) is a European project from the Information Society Technologies Programme (IST4-26751). Information about this project is accessible on the ReMath Web site: http:// www.remath.cti.gr

source of inequity. Research has provided extensive evidence of that since the seminal work of Carraher, Carraher, and Schliemann (1985, 1987) with Brazilian street sellers, showing that school was not able to benefit from the knowledge and experience these pupils have gained in their out-of-school activities. Experiments and studies carried out in the last decade—for instance, the research developed by Jo Boaler (2002, 2008) in England and then in the USA, but there are many other examples—show that the current situation is not fatal. Their results provide us today with existence theorems, but we need much more. I sincerely hope that the new study ICMI is launching on the teaching and learning of mathematics in multilingual contexts will substantially contribute to the reflection in that domain.

I will end by quoting the editorial by ICMI Vice-President Jill Adler in the June 2008 ICMI *Newsletter*. She repeated a question that was raised at the Rome Symposium on the ICMI Centennial in March: “In what ways does the work we do contribute to the Millennium [Project] goal of universal primary education by 2015?” In the editorial, she expressed the hope that ICME-11, through its diverse activities and having this challenge in focus, would help us make a decisive step so that in 2012, when we will be in Seoul for the next ICME, we can provide some evidence of significant progress in that direction. I share her hope and am sure that this is also the case for Jeremy.

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