Booklet for the RME/FIsme workshop at ICMI Study 22: Task Design

Example tasks and background information

Peter Boon, Michiel Doorman & Paul Drijvers

Fractions and percentages as tools to compare situations

Activity 1

The students will be asked to solve the following problem:

Yesterday some children practiced shooting the basketball.

Below you see how often they scored.

A black dot means the child did score, an open dot means the child missed.

1. **Which child has scored best?**
2. **Can you put all children in order?**

Ernst ●●●●● ●●●●● ●●●●● ●●●●● ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○
Julia ●●●●● ●●●●● ○○○○○ ○○○○○
Tom ●●●●● ●●●●○ ○○○○○ ○○○○○ ○○○○○
Kim ●●●●● ●●○○○ ○○○○○ ○○○○○
Bram ●●●●● ●●●●● ●●●●● ●●●●● ●●●●○ ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○
Paul ●●●●● ●●●●● ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○
Tess ●●●●● ●●●●● ●○○○○ ○○○○○
Liz ●●●●● ●○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○ ○○○○○

Start of a lesson plan

The teacher introduces the problem and makes sure that the students understand the situation and understand what is asked.

A prediction of student responses

Reasoning with absolute numbers: students may say, for example, that Bram is the best, because he has the most black dots.

Referring to proportions: students may say that it is the proportion of the black dots that matter. For example: 'Bram scores 24 times, but he has tried more often than others', or: 'Ernst has scored 20 *out of* 50, but Julia 10 *out of* 20.'

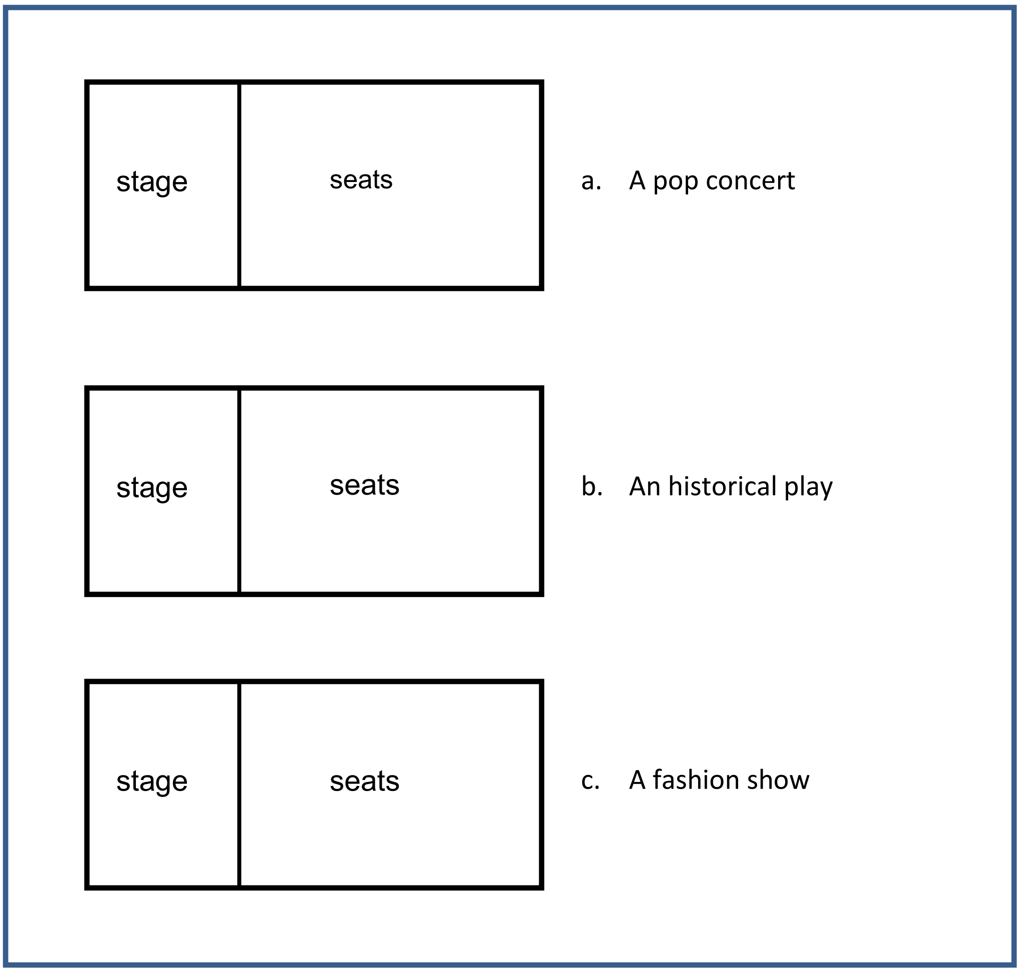
Actions of the teacher

The teacher stimulates a discussion about proportional reasoning. In a basketball game counts how many times someone might score when there is a chance. In other situations a reasoning in absolute terms might be more appropriate.

If all students immediately interpret the situation in proportional terms, the teacher may ask a question like: 'In another group a student said that Bram was the best, because he has scored 24 times, what do you think about that?'

Activity 2

Three performances will take place in the school theater. How busy will the theater be during each performance? Color the part of the hall that is occupied and write down the percentage of the seats that is occupied.



Background Basketball activity

The emphasis in this lesson should *not* be on the calculations, but on the fact that in this situation a proportional comparison is appropriate - not a comparison in absolute numbers - and on the mathematical tools we have for such a comparison: ratio's, fractions and percentages.

The first question - who has scored best - can be answered by remarking that Tess is the only one with more than a half of her trials in. The ordering of all children asks for more extensive calculations.
The score of Ernst - 20 out of 50 - offers an easy step to percentages.

Background Theatre activity

This is an example of an explorative activity to support students in building models (i.e. the bar model) based upon their prior ideas and experiences. With a system of tasks, including open inquiry activities as well as more closed practicing activities, students are guided to reinvent the mathematics of percentages. The bar model initially emerges as a model of specific situations and turns into a (thought)model for reasoning about percentages and connecting them with fractions.

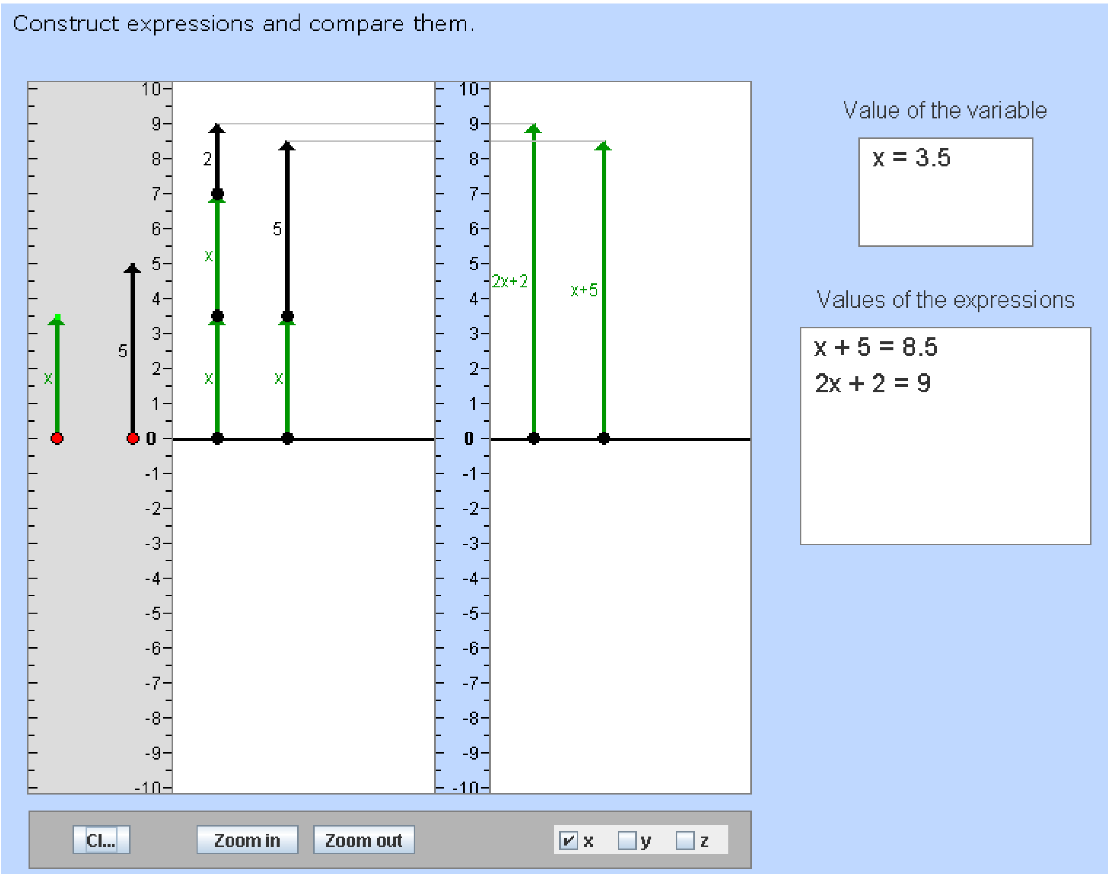
Reference

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics 54*(1), 9-35.

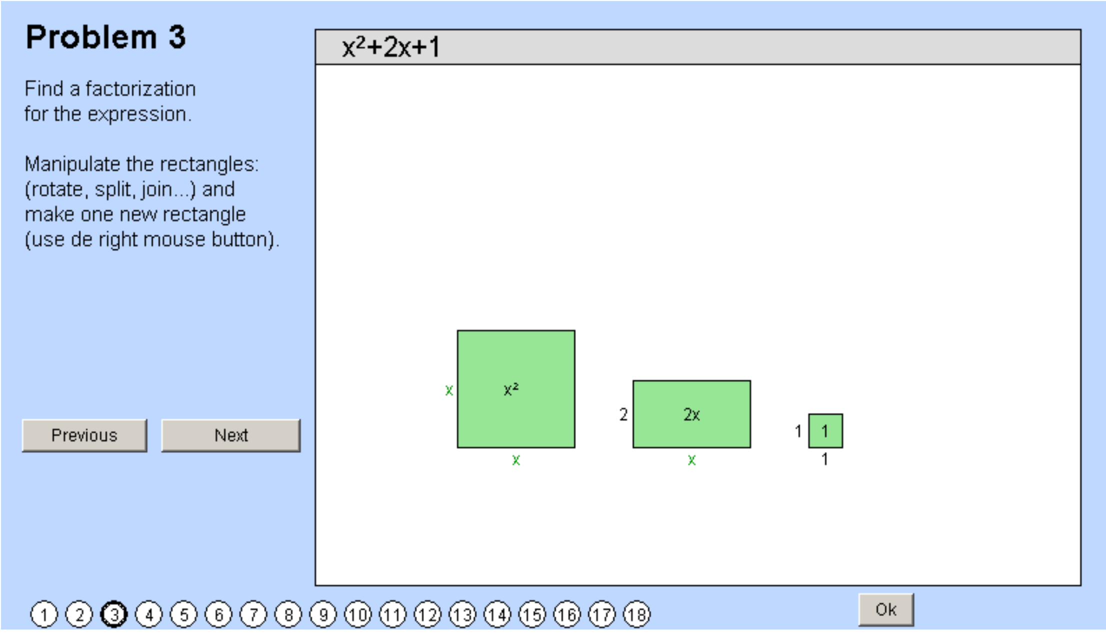
From rope puzzles to algebra

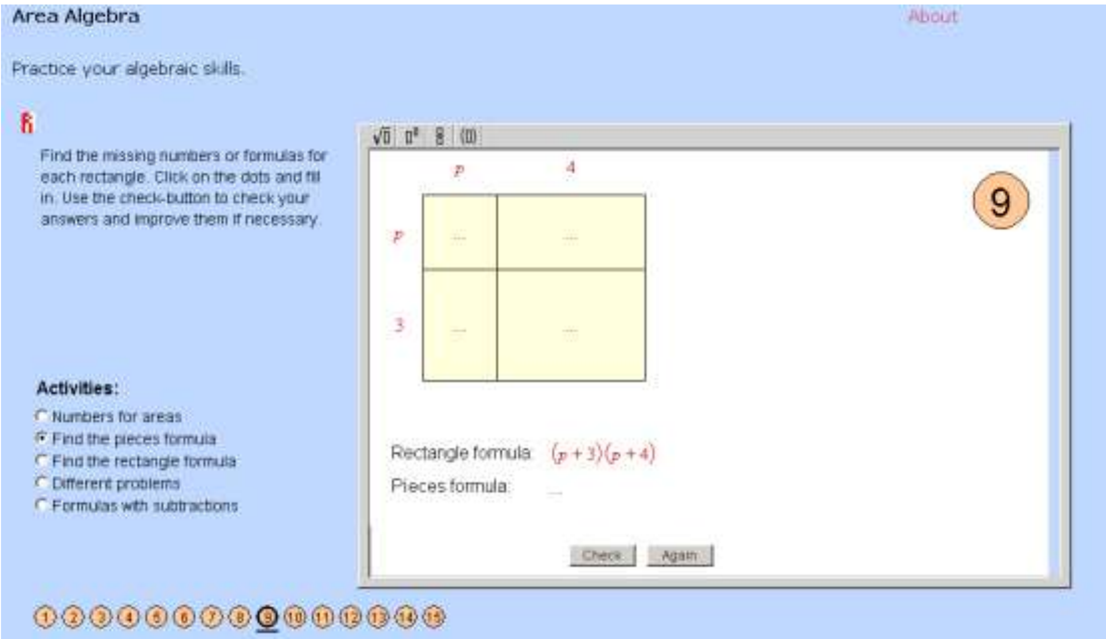
1. A rope of 30 meter is divided in 5 short and 3 long parts. A short and a long part together are 9 meter. How long is a short part?

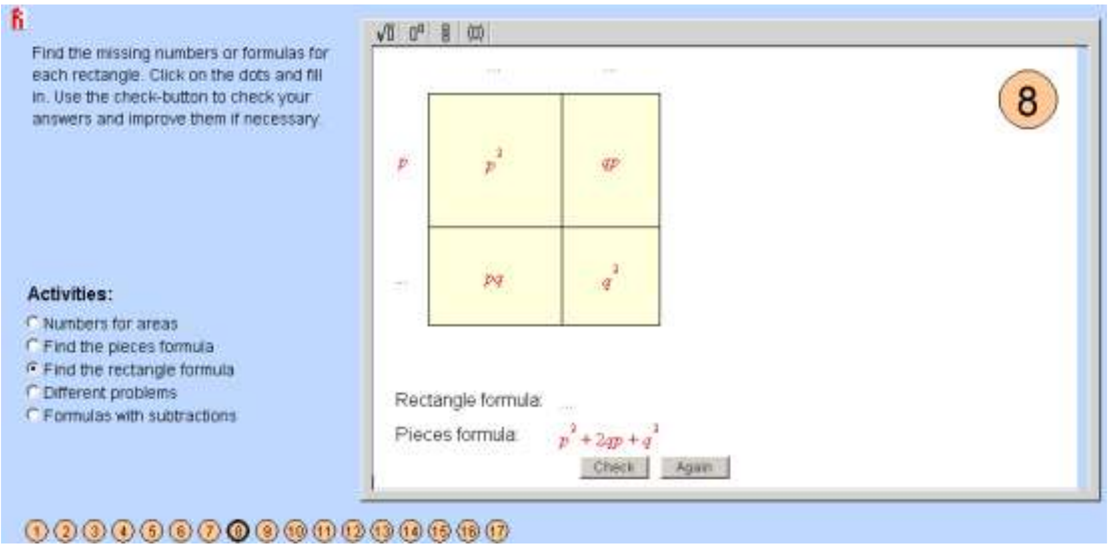
2. Use the applet GeomAlg1D to explore situations. Change the length of the x-arrow. When do the two arrow-stacks have the same height?



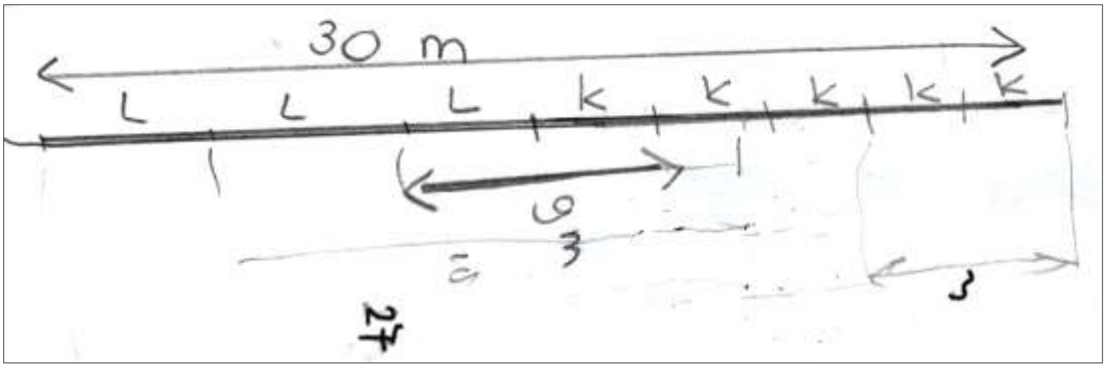
3. Example activities after the step towards GeomAlg2D:

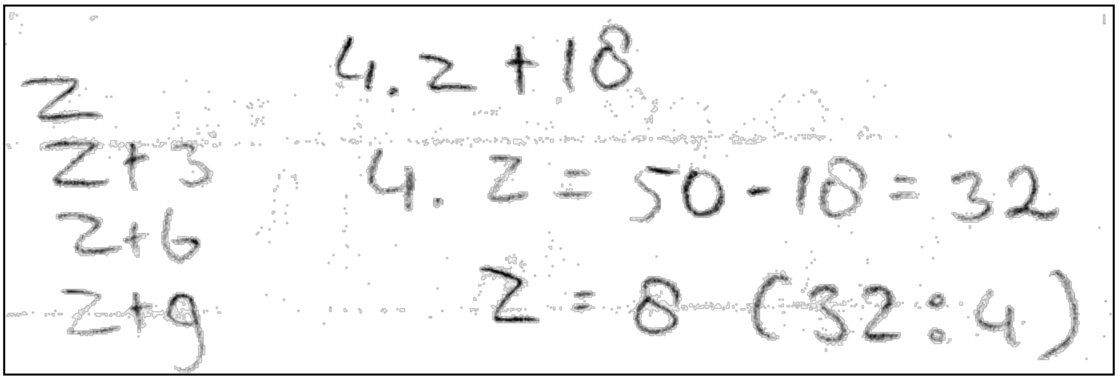


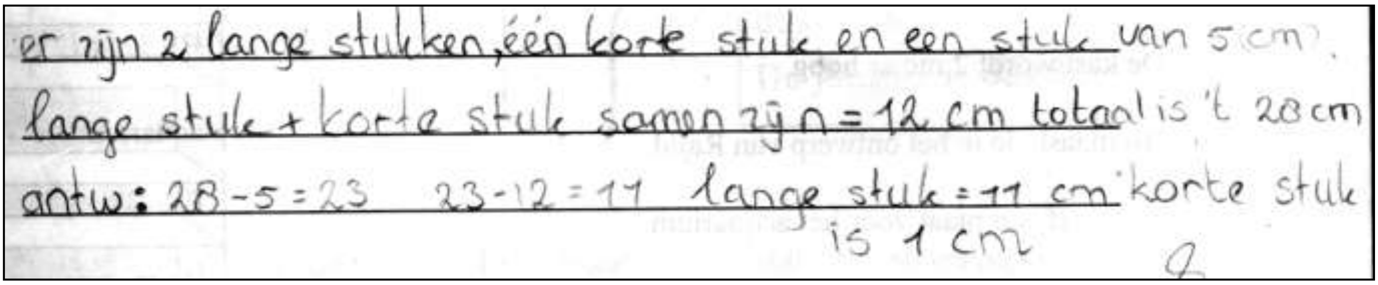


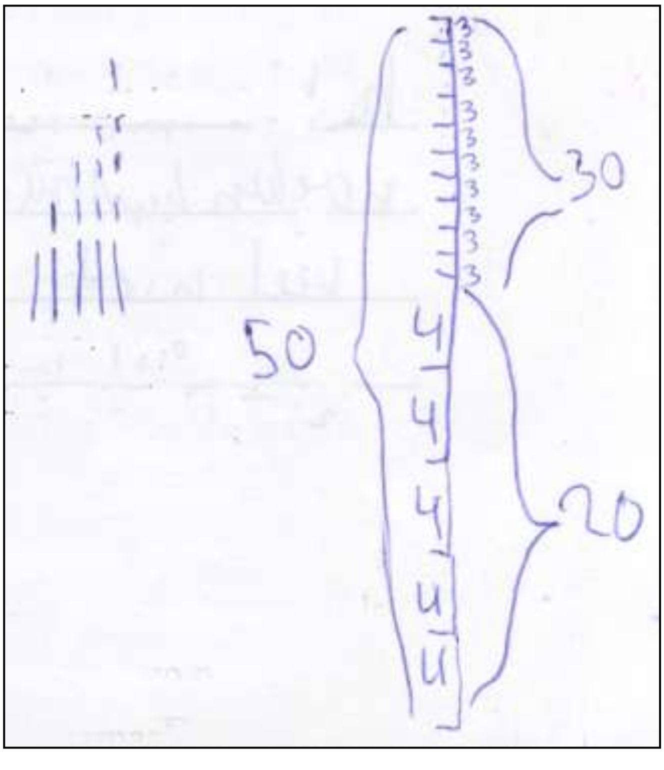


4. Student answers to rope problems offering starting points for constructing and reasoning with algebraic expressions



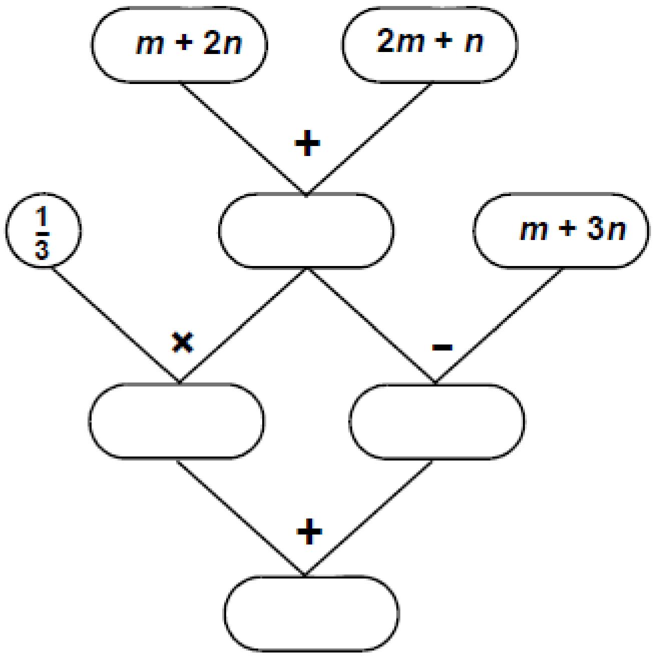






Productive Practice in Algebra

Operating with expressions

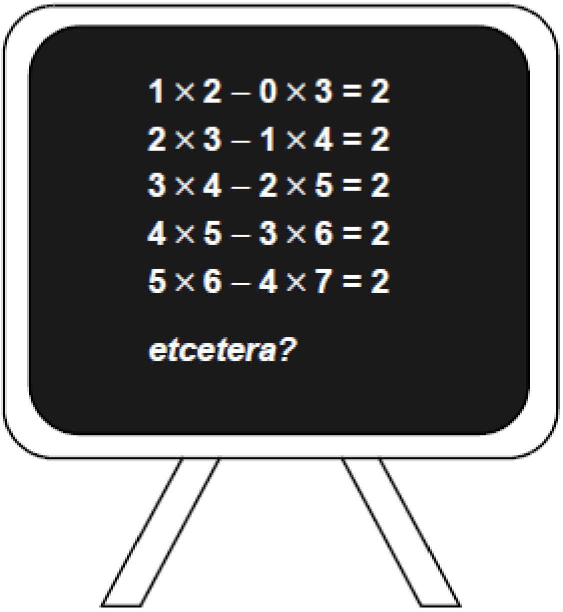


1. Complete the above Algebra Tree

2. Create a similar task for your colleagues or students

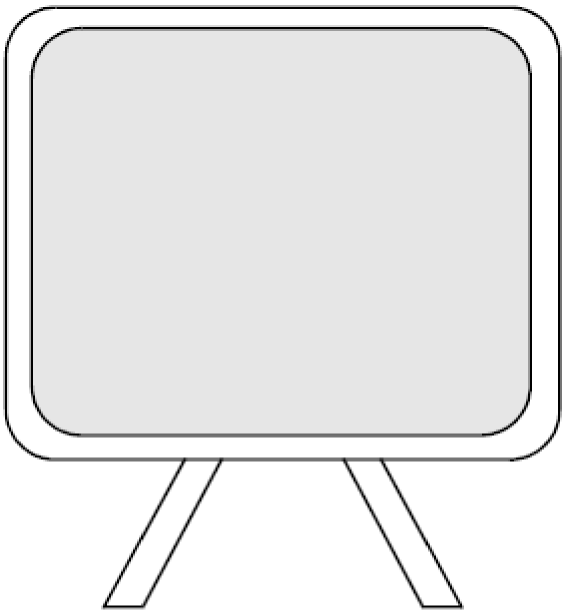
(Kindt, 2004)

You can count on it



3. Check the calculations on the blackboard and add some lines. Which formula reflects the regularity in this sequence of calculations? How can you prove the formula?

4. Design a similar sequence of calculations (with the same result on each line), set up a corresponding formula and prove it.



(Kindt, 2004)

Productive Practice

Practice is essential to anchor skills acquired through insight. For most students, the effect of practice will improve to the extent that the exercises require more thought, elicit more independent contributions from the students and offer more possibilities for reflection. In short, the effect of practice will improve to the extent that the exercises have a more productive character. […]
Here are ten recommendations that have been explicitly or implicitly addressed in these examples.

1. Ask reverse questions to promote mental agility.
2. Vary the practice formats and activities as much as possible.
3. Challenge the students to reason logically (for example, by using coherent strings of problems).
4. Challenge the students to generalize (for example by means of number patterns).
5. Practice the substitution of ‘formulas in formulas’ (formal substitution).
6. Practice the elimination of variables in systems of formulas or equations.
7. Pay attention to the verbal reading and writing of algebra rules or formulas.
8. Challenge the students to create their ‘own productions’.
9. Also practice algebra in geometry.

and more generally

10. Where possible, maintain and strengthen previously acquired computational and algebraic skills.

(Kindt, 2010, p. 175-176)

References

Kindt, M. (2004). *Positive Algebra*. Utrecht: Freudenthal Institute.

Kindt, M. (2010). Principles of Practice. In P. Drijvers (Ed.). *Secondary Algebra Education. Revisiting topics and themes and exploring the unknown* (pp. 137-178). Rotterdam / Boston / Taipe: Sense Publishers.

Expressions and Formulas (MIC)

Home Repairs

Jim is a contractor specializing in small household repairs that require less than a day to complete. For most jobs, he uses a team of three people. For each one of the three people, Jim charges the customer $25 in travel expenses and $37 per hour. Jim usually uses a calculator to calculate the bills. He uses a standard form for each bill.

1. Show the charge for each plumbing repair job.
   a. Replacing pipes for Mr. Ashton: 3 hours
   b. Cleaning out the pipes at Rodriguez and Partners: 21–2 hours
   c. Replacing faucets at the Vander house: 3–4 hour

People often call Jim to ask for a price estimate for a particular job. Because Jim is experienced, he can estimate how long a job will take. He then uses the table to estimate the cost of the job.

Hours	Labor Cost (in dollars)	Travel Cost (in dollars)	Cost per Worker (in dollars)	Total for Three Workers (in dollars)
1	37	25	62	186
2	74	25	99	297
3	111	25	136	408
4	148	25	173	519
5				

2.

a. What do the **entries** in the first row of the table represent?
b. Add the next row for five hours to the table.

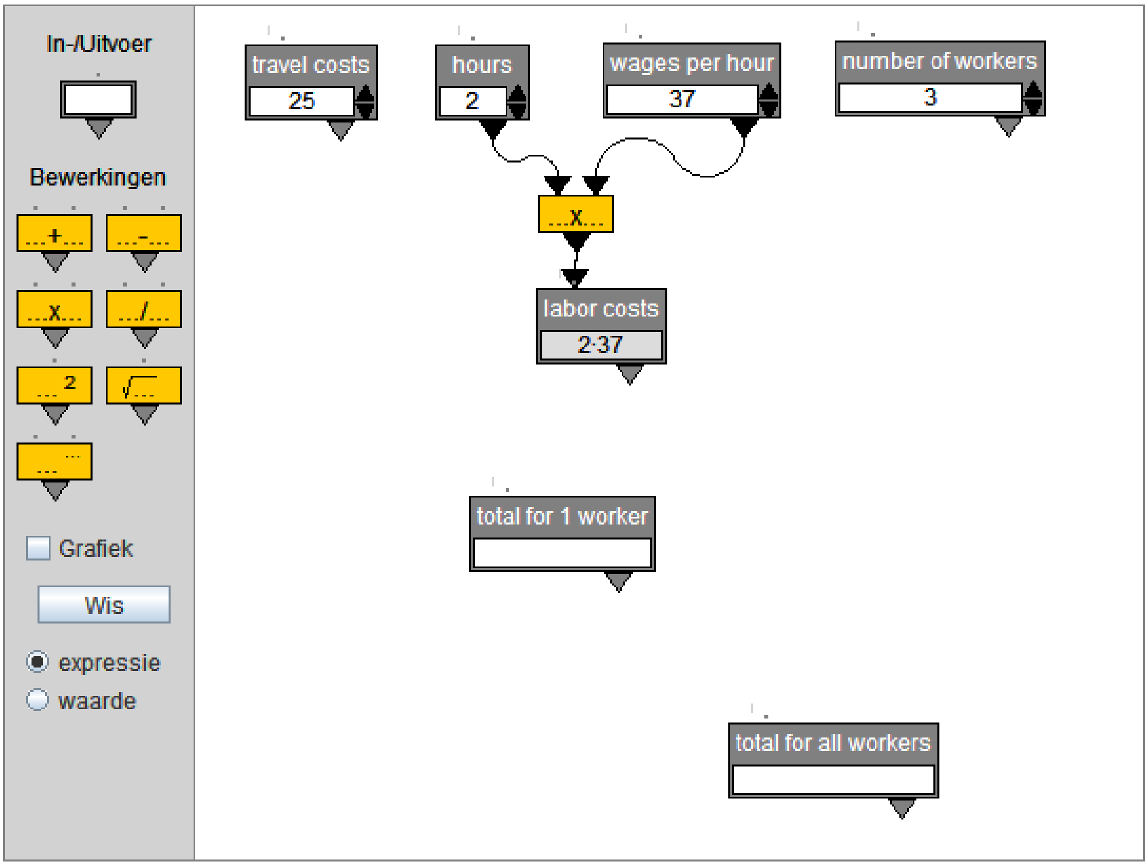
3.

a. Explain the regularity in the column for the labor cost per worker.
b. Study the table. Make a list of all of the regularities you can find. Explain the regularities.

4.

a. Draw an arrow string that Jim could use to make more rows for the table.
b. Use your arrow string to make two more rows (for 6 and7 hours) on the table.

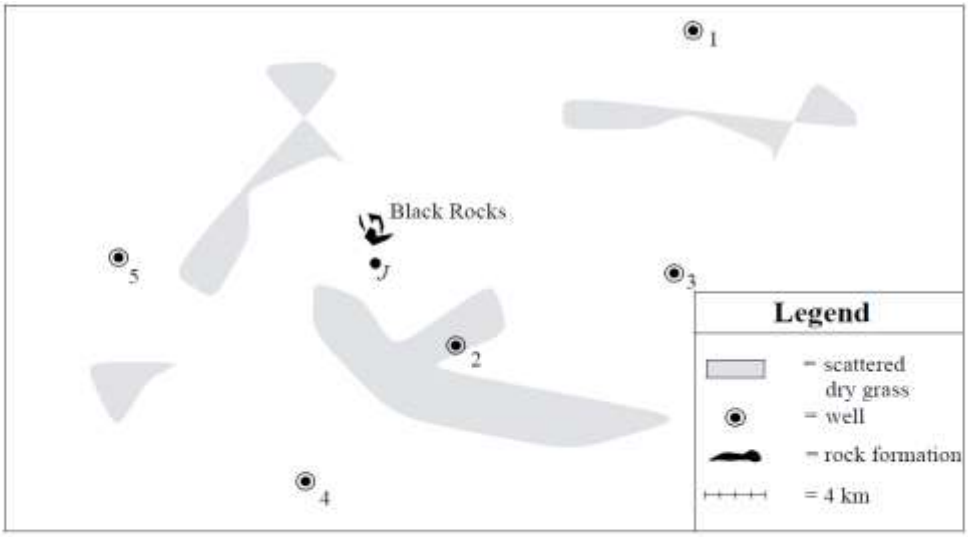
The calculations within this task can be structured in a way that prepares for dealing with functions:



Distances and Voronoi diagrams

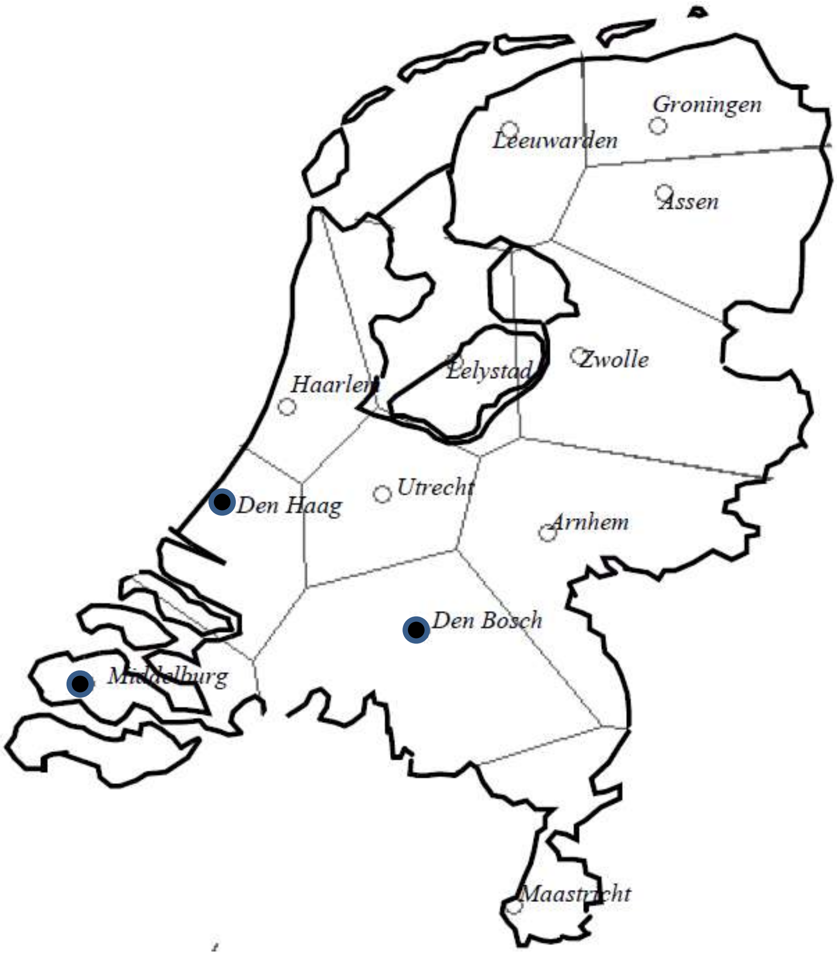
Thirsty in the desert
Below you see part of a map of a desert. There are five wells in this area. Imagine you and your herd of sheep are standing at *J*. You are very thirsty and you only brought this map with you.

1. To which well would you go for water?
2. Colour the region of positions that all have well 2 as the closest place to find water.

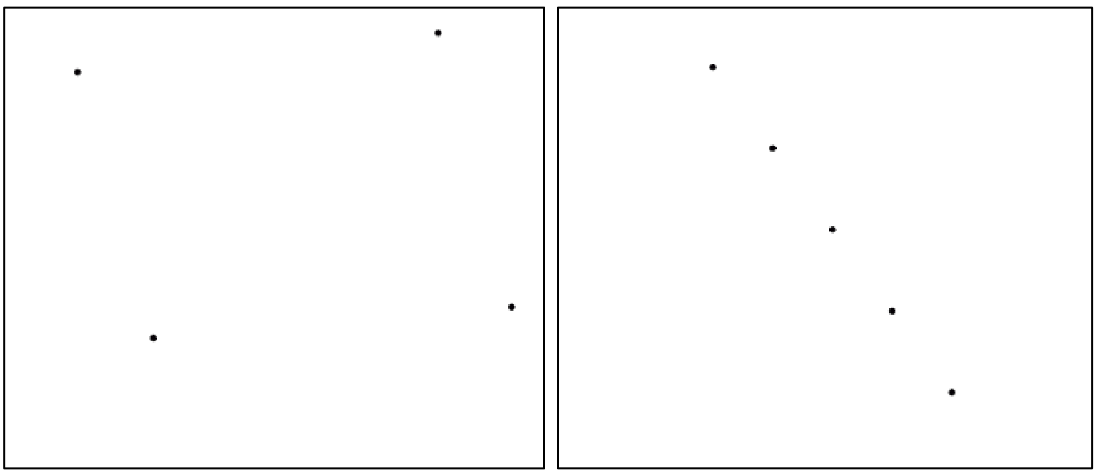


Provinces in the Netherlands
On the next page you see a redivision of the Netherlands in provinces, based on the positions of their capitals.

3. How are the province borders created? How could you find them by folding the map?
4. What property holds for the 'three-countries-point' of the cities of Middelburg, Den Haag and Den Bosch?



Other province capitals



5. For each of the two above windows, find the province borders in case the points represent the capitals.
6. Find an arrangement of points that leads to a 'special' arrangement of 'province borders'.

(adapted from Goddijn, Kindt, & Reuter, 2004)

Voronoi diagrams

Voronoi-diagrams are named after the mathematician Voronoi. He (in 1908) and Dirichlet (in 1850) used these diagrams in a pure mathematical problem, the investigation of positive definite square forms. In 1911, Thiessen used the same sort of diagrams while determining quantities of precipitation in an area, while only measuring at a small number of points. In meteorology, geography and archaeology the term Thiessen-polytope instead of Voronoi-cell became established.
(Goddijn, Kindt, & Reuter, 2004, p. I-9)

Reference

Goddijn, A., Kindt, M., & Reuter, W. (2004). *Geometry with applications and proofs*. Utrecht: Freudenthal Institute.

Drug level

A doctor presents the following details about the use of a specific drug:

- An average of 25% of the drug leaves your body by secretion during a day.
- The drug is effective after a certain level is reached.
  Therefor it takes a few days before the drug that you take every day is effective.
- Do not skip a day.
- It can be unwise to compensate a day when you forgot the drug with a double dose in the next day.

N.B. These details are a simplification of reality.

Activity 1: Investigation

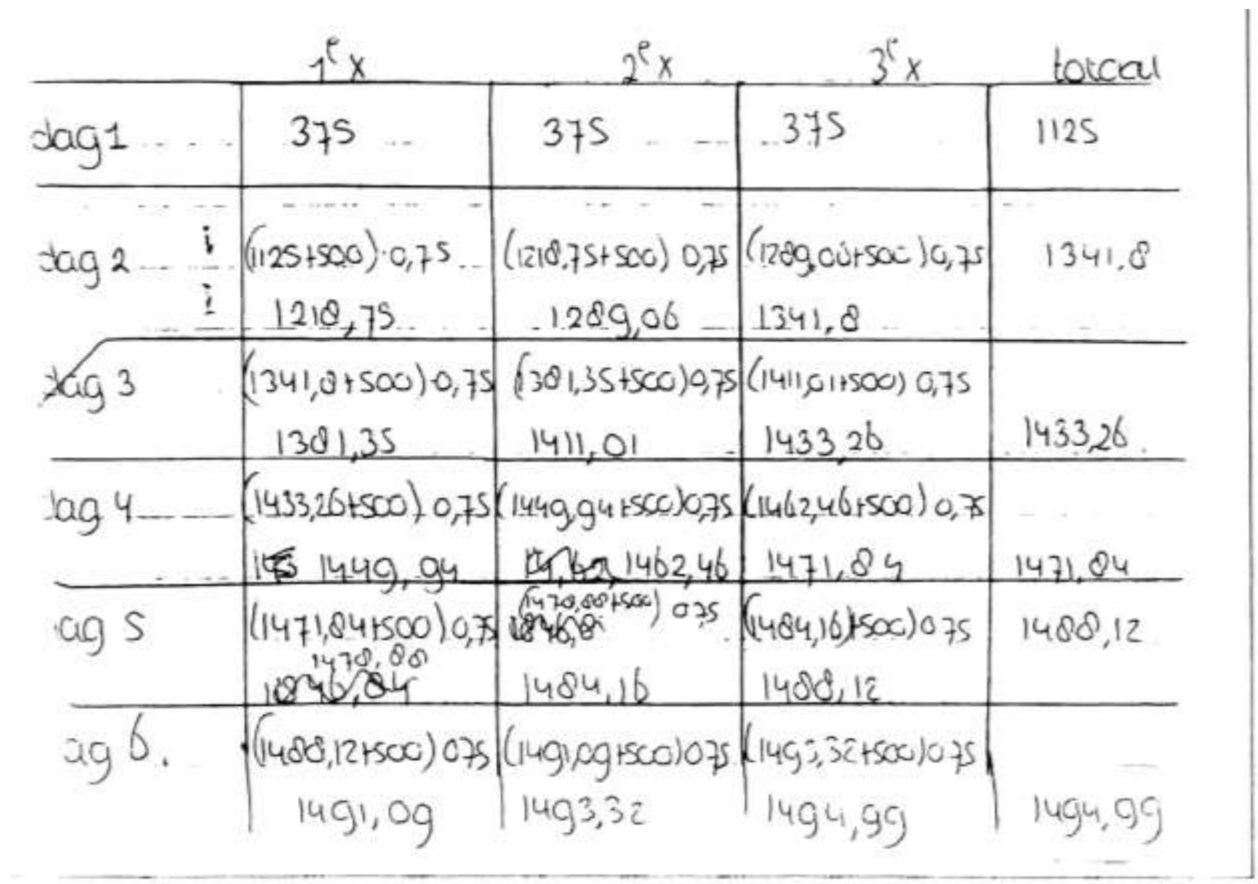
- Use calculations to investigate how the level of the drug changes when a person starts taking in the drug with a daily dose of 1500 mg with for instance three times 500 mg.
- Are the consequences of skipping a day and/or of taking a double dose really so dramatic?
- Can each drug level be reached? Explain your answer.

Design a flyer for patients with answers on the above questions. Include graphs and/or tables to illustrate the progress of the drug level during several days.

Activity 2: Reflection with dynamic models

After the introduction of difference equations ($X_n = aX_{n-1} + b$) students are confronted with their previous results.

You investigated last year the progress of a drug level during several days.
The illustrations below show some solutions. As you can see, with similar information you reached quite different results.
Explain the differences by using formulas for the underlying calculations.



	1e x	2e x	3e x	totaal
dag 1	375	375	375	1125
dag 2	(1125+500)·0,75 1218,75	(1218,75+500) 0,75 1289,06	(1289,06+500) 0,75 1341,8	1341,8
dag 3	(1341,8+500) 0,75 1381,35	(1381,35+500) 0,75 1411,01	(1411,01+500) 0,75 1433,26	1433,26
dag 4	(1433,26+500) 0,75 ~~1?~~ 1449,94	(1449,94+500) 0,75 ~~[illegible]~~ 1462,46	(1462,46+500) 0,75 1471,84	1471,84
dag 5	(1471,84+500) 0,75 ~~1846,84~~ 1478,88	(1478,88+500) 0,75 1484,16	(1484,16+500) 0,75 1488,12	1488,12
dag 6.	(1488,12+500) 0,75 1491,09	(1491,09+500) 0,75 1493,32	(1493,32+500) 0,75 1494,99	1494,99

Solution 1

Solution 2

1e dag werkt 1500 mg
2e dag werkt 1125 mg + 1500 mg = 2625 mg
3e dag werkt 844 mg + 1125 mg + 1500 mg = 3469 mg
4e dag werkt 633 mg + 844 mg + 1125 mg + 1500 mg = 4102 mg
5e dag werkt 475 mg + 633 mg + 844 mg + 1125 mg + 1500 mg = 4576 mg
6e dag werkt 356 mg + 475 mg + 633 mg + 844 mg + 1125 mg 1500 mg = 4932 mg

DAGEN	1	2	3	4	5	6
werkend medicijn (mG)	1500	2625	3469	4102	4576	4932
toename werkend medicijn (mG)	-	1125	844	633	475	356

dus:

$2000 \cdot 0{,}75^{x} = y$

x = dagen
y = toename werkend medicijn (mg)

DAGEN	26	27	28	29	30	31
werkend medicijn (mG)	5996	5996,3	5996,9	5997,4	5997,8	5998,1
toename werkend medicijn (mG)	1	0,8	0,6	0,5	0,4	0,3

Met het slikken van een vaste dagelijks dosis zal een eindpeil 6000 mg niet bereikt worden. Dat blijkt uit de bovenstaande tabel.

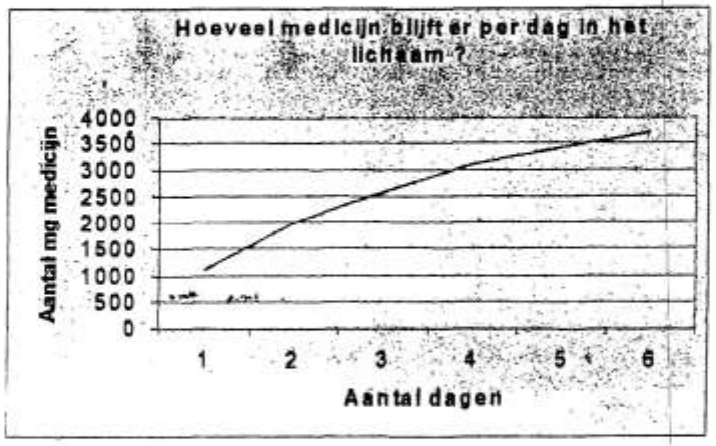
Solution 3

de Medicijnspiegel

Als je eenmaal per dag naar het toilet gaat verlaat 25 % van de door jou ingenomen medicijnen je lichaam
Dat betekent dat als je eerste dag van je medicijngebruik 3 keer 500mg slikt er daarvan 1500 * 0,75 = 1125mg in je lichaam overblijft

Als je elke dag 3 keer 500mg van het medicijn zou slikken krijg je het volgende resultaat

Dag	*totaal (mg)*	*Vn1*	*Vn2*	*Vn3*
1	1125			
2	1986,75	843,75		
3	2601,5625	632,8125	210,9375	
4	3076,2	474,6375	158,175	52,7625
5	3432,1	355,9	118,7375	39,437
6	3699,09	266,99	88,91	29,8275



De verschilrijen worden niet constant dus is het ook niet mogelijk bij deze rij een directe formule te geven. Wel is er een recusieve formule die luidt : *m+1 = (1500 + m) * 0,75*
Dit betekent dat het aantal medicijn in je lichaam geleik is aan het aantal van de vorige dag, daarbij komt 1500 mg en na het plassen blijft er nog 75 % van de totale hoeveelheid over in je lichaam

Het kan gebeuren dat je een dag vergeet je medicijnen in te nemen
Kun je dan zomaar de volgende dag de dubele dosis innemen en heeft dit gevolgen voor het eindpeil ?
Dat is in een tabel duidelijk weer te geven

Dag	*Constant*	*1 keer overslaan*
1	1125	1125
2	1986,75	843,75
3	2601,5625	2882,8125

Tussen de eindhoeveelheden zit niet zo een groot verschil, ongeveer 281,25 mg
Maar als je meerdere dagen overslaat en het later compenseert wordt het verschil steeds groter en krijgt het weldegelijk invloed op het eindpeil. Het is dan ook niet aan te raden dit te doen want hierdoor krijg je een veel te hoog eindpeil

Het kan natuurlijk ook voorkomen dat je een ander eindpeil hebt dan gewenst als je elke dag constant de medicijnen neemt. Dit kan komen doordat je gemiddeld meer of minder dan 25% uitscheid
Maar ook door hoe snel het lichaam de stoffen opneemt e.d

Realistic Mathematics Education[1]

Marja van den Heuvel-Panhuizen & Paul Drijvers

Freudenthal Institute for Science and Mathematics Education
Utrecht University

Keywords: Domain-specific teaching theory; Realistic contexts; Mathematics as a human activity; Mathematization.

What is Realistic Mathematics Education?

Realistic Mathematics Education – hereafter abbreviated as RME – is a domain-specific instruction theory for mathematics, which has been developed in the Netherlands. Characteristic of RME is that rich, 'realistic' situations are given a prominent position in the learning process. These situations serve as a source for initiating the development of mathematical concepts, tools and procedures and as a context in which students can in a later stage apply their mathematical knowledge, which then gradually has become more formal and general, and less context-specific.

Although 'realistic' situations in the meaning of 'real-world' situations are important in RME, 'realistic' has a broader connotation here. It means students are offered problem situations which they can imagine. This interpretation of 'realistic' traces back to the Dutch expression 'zich REALISEren', meaning 'to imagine'. It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to students can come from the real world, but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the student's mind.

The onset of RME

The initial start of RME was the founding in 1968 of the Wiskobas ('mathematics in primary school') project initiated by Edu Wijdeveld and Fred Goffree, and joined not long after by Adri Treffers. In fact, these three mathematics didacticians created the basis for RME. In 1971, when the Wiskobas project became part of the newly-established IOWO Institute, with Hans Freudenthal as its first director, and in 1973 when the IOWO was expanded with the Wiskivon project for secondary mathematics education, this basis received a decisive impulse to reform the prevailing approach to mathematics education.

In the 1960s, mathematics education in the Netherlands was dominated by a mechanistic teaching approach; mathematics was taught directly at a formal level, in an atomized manner, and the mathematical content was derived from the structure of mathematics as a scientific discipline. Students learned procedures step-by-step with the teacher demonstrating how to solve problems. This led to inflexible and reproduction-based knowledge. As an alternative for this mechanistic approach, the 'New Math' movement deemed to flood the Netherlands. Although Freudenthal was a strong proponent of the modernization of mathematics education, it was his merit that Dutch mathematics education was not affected by the formal approach of the New Math movement and that RME could be developed.

---

[1] Van den Heuvel-Panhuizen, M., & Drijvers, P. (in press). Realistic Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. xxx-xxx). Dordrecht, Heidelberg, New York, London: Springer.

Freudenthal's guiding ideas about mathematics and mathematics education

Hans Freudenthal (1905-1990) was a mathematician born in Germany who in 1946 became a professor of pure and applied mathematics and the foundations of mathematics at Utrecht University in the Netherlands. As a mathematician he made substantial contributions to the domains of geometry and topology.

Later in his career, Freudenthal (1973, 1991) became interested in mathematics education and argued for teaching mathematics that is relevant for students and carrying out thought experiments to investigate how students can be offered opportunities for guided re-invention of mathematics.

In addition to empirical sources such as textbooks, discussions with teachers and observations of children, Freudenthal (1983) introduced the method of the didactical phenomenology. By describing mathematical concepts, structures, and ideas in their relation to the phenomena for which they were created, while taking into account students' learning process, he came to theoretical reflections on the constitution of mental mathematical objects, and contributed in this way to the development of the RME theory.

Freudenthal (1973) characterized the then dominant approach to mathematics education in which scientifically structured curricula were used and students were confronted with ready-made mathematics as an 'anti-didactic inversion'. Instead, rather than being receivers of ready-made mathematics, students should be active participants in the educational process, developing mathematical tools and insights by themselves. Freudenthal considered mathematics as a human activity. Therefore, according to him, mathematics should not be learned as a closed system, but rather as an activity of mathematizing reality and if possible even that of mathematizing mathematics.

Later, Freudenthal (1991) took over Treffers' (1987) distinction of horizontal and vertical mathematization. In horizontal mathematization, the students use mathematical tools to organize and solve problems situated in real-life situations. It involves going from the world of life into that of symbols. Vertical mathematization refers to the process of reorganization within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols. The two forms of mathematization are closely related and are considered of equal value. Just stressing RME's 'real-world' perspective too much may lead to neglecting vertical mathematization.

The core teaching principles of RME

RME is undeniable a product of its time and cannot be isolated from the worldwide reform movement in mathematics education that occurred in the last decades. Therefore, RME has much in common with current approaches to mathematics education in other countries. Nevertheless, RME involves a number of core principles for teaching mathematics which are inalienable connected to RME. Most of these core teaching principles were articulated originally by Treffers (1978), but were reformulated over the years, including by Treffers himself.

In total six principles can be distinguished.

- The *activity principle* means that in RME students are treated as active participants in the learning process. It also emphasizes that mathematics is best learned by doing mathematics, which is strongly reflected in Freudenthal's interpretation of mathematics as a human activity, as well as in Freudenthal's and Treffers' idea of mathematization.

- The *reality principle* can be recognized in RME in two ways. First, it expresses the importance that is attached to the goal of mathematics education including students' ability to apply mathematics in solving 'real-life' problems. Second, it means that mathematics education should start from problem situations that are meaningful to students, which offers them opportunities to attach meaning to the mathematical constructs they develop while solving problems. Rather than beginning with teaching abstractions or definitions to be applied later, in RME, teaching starts with problems in rich contexts that require mathematical organization or, in other words, can be mathematized and put students on the track of informal context-related solution strategies as a first step in the learning process.
- The *level principle* underlines that learning mathematics means students pass various levels of understanding: from informal context-related solutions, through creating various levels of shortcuts and schematizations, to acquiring insight into how concepts and strategies are related. Models are important for bridging the gap between the informal, context-related mathematics and the more formal mathematics. To fulfill this bridging function, models have to shift – what Streefland (1993) called – from a 'model of' a particular situation to a 'model for' all kinds of other, but equivalent, situations.

  Particularly for teaching operating with numbers, this level principle is reflected in the didactical method of 'progressive schematization' as it was suggested by Treffers and in which transparent whole-number methods of calculation gradually evolve into digit-based algorithms.
- The *intertwinement principle* means mathematical content domains such as number, geometry, measurement, and data handling are not considered as isolated curriculum chapters, but as heavily integrated. Students are offered rich problems in which they can use various mathematical tools and knowledge. This principle also applies within domains. For example, within the domain of number sense, mental arithmetic, estimation and algorithms are taught in close connection to each other.
- The interactivity principle of RME signifies that learning mathematics is not only an individual activity but also a social activity. Therefore, RME favors whole-class discussions and group work which offer students opportunities to share their strategies and inventions with others. In this way students can get ideas for improving their strategies. Moreover, interaction evokes reflection, which enables students to reach a higher level of understanding.
- The *guidance principle* refers to Freudenthal's idea of 'guided re-invention' of mathematics. It implies that in RME teachers should have a pro-active role in students' learning and that educational programs should contain scenarios which have the potential to work as a lever to reach shifts in students' understanding. To realize this, the teaching and the programs should be based on coherent long-term teaching-learning trajectories.

Various local instruction theories

Based on these general core teaching principles a number of local instruction theories and paradigmatic teaching sequences focusing on specific mathematical topics have been developed over time. Without being exhaustive some of these local theories are mentioned here. For example, Van den Brink (1989) worked out new approaches to addition and subtraction up to twenty. Streefland (1991) developed a prototype for teaching fractions intertwined with ratios and proportions. De Lange (1987) designed a new approach to teaching matrices and discrete calculus. In the last decade, the development of local instruction theories was mostly integrated with the use of digital technology as investigated by

Drijvers (2003) with respect to promoting students' understanding of algebraic concepts and operations. Similarly, Bakker (2004) and Doorman (2005) used dynamic computer software to contribute to an empirically grounded instruction theory for early statistics education and for differential calculus in connection with kinematics respectively.

The basis for arriving at these local instruction theories was formed by design research, as elaborated by Gravemeijer (1994), involving a theory-guided cyclic process of thought experiments, designing a teaching sequence and testing it in a teaching experiment, followed by a retrospective analysis which can lead to necessary adjustments of the design.
Last but not least, RME also led to new approaches to assessment in mathematics education (De Lange 1987; Van den Heuvel-Panhuizen 1996).

Implementation and impact

In the Netherlands, RME had and still has a considerable impact on mathematics education. In the 1980s, the market share of primary education textbooks with a traditional, mechanistic approach was 95% and the textbooks with a reform-oriented approach – based on the idea of learning mathematics in context to encourage insight and understanding – had a market share of only 5%. In 2004, reform-oriented textbooks reached a 100% market share and mechanistic ones disappeared. The implementation of RME was guided by the RME-based curriculum documents including the so-called 'Proeve' publications by Treffers and his colleagues, which were published from the late 1980s, and the TAL teaching-learning trajectories for primary school mathematics, which have been developed from the late 1990s.

A similar development can be seen in secondary education, where the RME approach also influenced textbook series to a large extent. For example, Kindt (2010) showed how practicing algebraic skills can go beyond repetition and be thought-provoking. Goddijn et al. (2004) provided rich resources for realistic geometry education, in which application and proof go hand in hand.

Worldwide, RME is also influential. For example, the RME-based textbook series 'Mathematics in Context' has a considerable market share in the USA. A second example is the RME-based 'Pendidikan Matematika Realistik Indonesia' in Indonesia.

A long-term and ongoing process of development

Although it is now some forty years from the inception of the development of RME as a domain-specific instruction theory, RME can still be seen as work in progress. It is never considered a fixed and finished theory of mathematics education. Moreover, it is also not a unified approach to mathematics education. That means that through the years different emphasis was put on different aspects of this approach and that people who were involved in the development of RME –mostly researchers and developers of mathematics education, and mathematics educators from within or outside the Freudenthal Institute – put various accents in RME. This diversity, however, was never seen as a barrier for the development of RME, but rather as stimulating reflection and revision, and so supporting the maturation of the RME theory. This also applies to the current debate in the Netherlands (see Van den Heuvel-Panhuizen 2010) which voices the return to the mechanistic approach of four decades back. Of course, going back in time is not a 'realistic' option, but this debate has made the proponents of RME more alert to keep deep understanding and basic skills more in balance in future developments of RME and to enhance the methodological robustness of the research that accompanies the development of RME.

References

Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools*. Utrecht: CD-Bèta Press.

De Lange, J. (1987). *Mathematics, Insight and Meaning*. Utrecht: OW & OC, Utrecht University.

Doorman, L.M. (2005). *Modelling motion: from trace graphs to instantaneous change*. Utrecht: CD-Bèta Press.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment*. *Design research on the understanding of the concept of parameter*. Utrecht: CD-Bèta Press.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel Publishing Company.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Reidel Publishing Company.

Freudenthal, H. (1991). *Revisiting Mathematics Education*. *China Lectures*. Dordrecht: Kluwer Academic Publishers.

Goddijn, A., Kindt, M., Reuter, W., & Dullens, D. (2004). *Geometry with Applications and Proofs*. Utrecht: Freudenthal Institute.

Gravemeijer, K.P.E. (1994). *Developing Realistic Mathematics Education*. Utrecht: CD-ß Press / Freudenthal Institute.

Kindt, M. (2010). *Positive algebra*. Utrecht: Freudenthal Institute.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education*. *A Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1993). The design of a mathematics course. A theoretical reflection. *Educational Studies in Mathematics, 25*(1-2), 109-135.

Treffers, A. (1987). *Three dimensions*. *A model of goal and theory description in mathematics instruction – The Wiskobas project*. Dordrecht: D. Reidel Publishing Company.

Van den Brink, F.J. (1989). *Realistisch rekenonderwijs aan jonge kinderen* [Realistic mathematics education for young children]. Utrecht: OW&OC, Universiteit Utrecht.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht: CD-ß Press / Freudenthal Institute, Utrecht University.

Van den Heuvel-Panhuizen, M. (2010). Reform under attack – Forty Years of Working on Better Mathematics Education thrown on the Scrapheap? No Way! In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 1-25). Fremantle: MERGA.